Linear Differential Equations

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Abstract

Strategy works with first order, linear differential equations

1 Declarations

variable; variable description; *variable domain and range, if applicable*

2 Rule

$$\frac{dy}{dx} + p(x)y = f(x)$$
$$\implies \frac{d}{dx}[\mu(x)y] = \mu(x)f(x)$$
$$\implies y = \frac{\int \mu(x)f(x)}{\mu(x)}dx$$
$$\text{Where } \mu(x) = e^{\int p(x)dx}$$

3 Pre-Derivation

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Must first get equation into standard form, with the derivative of y alone...

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

$$\text{Let } \frac{a_0(x)}{a_1(x)} = p(x), \frac{g(x)}{a_1(x)} = f(x)$$

$$\implies \frac{dy}{dx} + p(x)y = f(x)$$

4 Derivation

$$\frac{dy}{dx} + p(x)y = f(x)$$
$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)f(x)$$

Finding the integration coeffecient, $\mu(x)$...

$$\exists\mu(x)|\frac{d}{dx}[\mu(x)y] = \mu(x)\frac{dy}{dx} + \mu(x)p(x)y$$
$$Product\,rule\,of\,derivatives...$$
$$\frac{d}{dx}[\mu(x)y] = \mu(x)\frac{dy}{dx} + \frac{d}{dx}[\mu(x)]y$$
$$\implies \frac{d}{dx}[\mu(x)]y = \mu(x)p(x)y$$
$$\frac{d}{dx}\mu(x) = \mu(x)p(x)$$
$$\frac{1}{\mu(x)}\frac{d}{dx}\mu(x) = p(x)$$
$$\int \frac{1}{\mu(x)}\frac{d}{dx}\mu(x)dx = \int p(x)dx$$
$$substituion \implies \ln|\mu(x)| = \int p(x)dx$$
$$\mu(x) = \pm e^{\int p(x)dx + C_0}$$
$$\mu(x) = \pm C_0 e^{\int p(x)dx}$$
$$\text{Let } \pm C_0 = C_1 = 1$$
$$\mu(x) = e^{\int p(x)dx}$$

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x)$$
$$\int \frac{d}{dx}[\mu(x)y]dx = \int \mu(x)f(x)dx$$
$$\mu(x)y = \int \mu(x)f(x)dx$$
$$y = \frac{\int \mu(x)f(x)}{\mu(x)}$$

5 Exempli Gratia

Examples of important instances

5.1 Transient and steady state solution

Get in standard form and indentify p(x)...

$$\begin{aligned} y' + 3x^2 y &= x^2 \\ \implies p(x) &= 3x^2 \end{aligned}$$

Find the integrating factor...

$$\begin{aligned} \text{We know, } \mu(x) &= e^{\int p(x)dx} \\ \implies \mu(x) &= e^{\int 3x^2 dx} = e^{x^3} \end{aligned}$$

Multiply through by the integrating factor...

$$e^{x^3} y' + e^{x^3} 3x^2 y = e^{x^3} x^2$$

Apply the product rule of derivatives...

$$\begin{aligned} e^{x^3} y' + e^{x^3} 3x^2 y &= \frac{d}{dx}[e^{x^3} y] \\ \implies \frac{d}{dx}[e^{x^3} y] &= e^{x^3} x^2 \end{aligned}$$

Integrate

$$\begin{aligned} \int \frac{d}{dx}[e^{x^3} y] dx &= \int e^{x^3} x^2 dx \\ e^{x^3} y &= \frac{e^{x^3}}{3} + C \end{aligned}$$

Solve for y

$$y = \frac{1}{3} + C\, e^{x^{-3}}$$

$$\frac{1}{3} \implies \text{Steady state solution}$$
$$C\,e^{x^{-3}} \implies \text{Transient (approaches zero)}$$