Separable Equations

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Abstract

Abstract goes here...

1 Declarations

$\frac{dy}{dx} \triangleq y'$; Differential and prime notation equivalence;
$p(y) = \frac{1}{h(y)}$;P is the reciprocal of H, for neatness' sake;

2 Rule

$$\frac{dy}{dx} = g(x)\,h(y)$$

$$\implies \int p(y)\,dy = \int g(x)\,dx$$

3 Pre-Derivation

This rule only holds true if a differential equation is of first order and holds the following form, that being the product of two functions, one of x and one of y. See below...

$$\frac{dy}{dx} = g(x)\,h(y)$$

4 Derivation

$$\begin{aligned}
\frac{dy}{dx} &= g(x)\,h(y) \\
\frac{1}{h(y)}\frac{dy}{dx} &= g(x) \\
\text{Let } p(y) &= \frac{1}{h(y)},\ h(y) \neq 0 \\
\therefore \frac{dy}{dx}p(y) &= g(x)
\end{aligned}$$

$$\begin{aligned}
\text{Assume } y &= \phi(x),\ \text{the solution.} \\
\frac{dy}{dx} &= \phi'(x)
\end{aligned}$$

$$\begin{aligned}
p(\phi(x))\phi'(x) &= g(x) \\
\int p(\phi(x))\phi'(x)dx &= \int g(x)dx
\end{aligned}$$

$$\begin{aligned}
\text{But since } y &= \phi(x) \\
\implies dy &= \phi'(x)dx
\end{aligned}$$

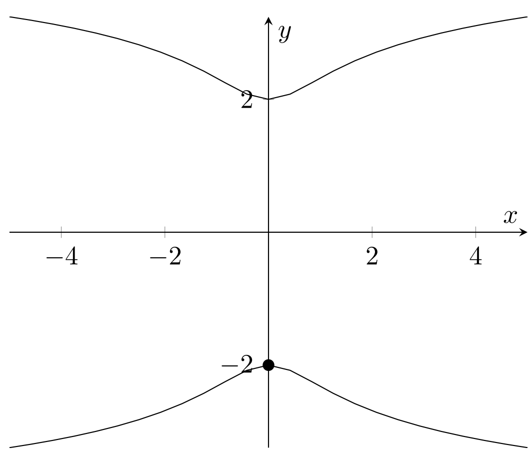
$$\therefore \int p(y)\,dy = \int g(x)\,dx$$

5 Exempli Gratia

5.1 Solution branches with initial value

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2x}{y + x^2 y},\ y(0) = -2 \\
y\,dy &= \frac{2x}{x^2}dx \\
\int y\,dy &= \int \frac{2x}{1+x^2}dx \\
\frac{y^2}{2} &= ln|1+x^2| + C \\
\implies \frac{y^2}{2} &= ln(1+x^2) + C \\
y &= \pm\sqrt{2\,ln(1+x^2) + C}
\end{aligned}$$

$$\begin{aligned}
y(0) = 2 &\implies x = 0,\ y = -2 \\
\text{using } y^2 &= 2\,ln(1+x^2) + C \\
4 &= 2\,ln(1+0) + C \\
\implies C &= 4
\end{aligned}$$



The bottom branch is the solution to the given differential equation because it contains the point. No two discontinuous branches can be solutions at the same time.

5.2 Newton's Law of Heating and Cooling

A thermometer reads 70°F before it's placed in an oven. At $\frac{1}{2}$ minute the thermometer reads 110°F. It reads 145°F at 1 minute. What is the temperature inside the oven?

$$\text{Given...}$$
$$\frac{dT}{dt} \propto T_m - T(t)$$
$$T(0) = 70\,^{\circ}F$$
$$T(\frac{1}{2}) = 110\,^{\circ}F$$
$$T(1) = 145\,^{\circ}F$$

Application of proportionality...

$$\frac{dT}{dt} \propto T_m - T(t) \implies \frac{dT}{dt} = k\,T(t)$$

Separation of Variables...

$$\int \frac{dT}{T_m - T(t)} = \int k\,dt$$
$$ln(T_m - T(t)) = k\,t + C$$
$$T_m - T(t) = C\,e^{k\,t}$$
$$T(t) = C\,e^{k\,t} + T_m$$

Determine T_m. Find all variables in terms of T.

Find C in terms of T_m

$$T(0) = 70\,^{\circ}F = C\,e^{0\,k} + T_m = C + T_m$$
$$\implies C = 70\,^{\circ}F - T_m$$

Determine k in terms of T_m

$$
\begin{aligned}
T(\frac{1}{2}) = 110\,^\circ F = C\,e^{\frac{1}{2}k} + T_m \\
\implies 110\,^\circ F - T_m = (70\,^\circ F - T_m)e^{\frac{1}{2}k} \\
\implies \frac{110\,^\circ F - T_m}{70\,^\circ F - T_m} = e^{\frac{1}{2}k} \\
\implies k = 2\,ln(\frac{110\,^\circ F - T_m}{70\,^\circ F - T_m})
\end{aligned}
$$

Solve for T_m

$$
\begin{aligned}
145\,^\circ F = C\,e^{1k} + T_m \\
145\,^\circ F - T_m = (70\,^\circ F - T_m)e^{2\,ln(\frac{110\,^\circ F - T_m}{70\,^\circ F - T_m})} \\
\implies \frac{145\,^\circ F - T_m}{70\,^\circ F - T_m} = (\frac{110\,^\circ F - T_m}{70\,^\circ F - T_m})^2 \\
145\,^\circ F - T_m = \frac{(110\,^\circ F - T_m)^2}{70\,^\circ F - T_m} \\
T_m^2 - 215\,T_m + 10,150 = T_m^2 - 220\,T_m + 12,100 \\
5\,T_m = 1,950 \\
T_m = 390\,^\circ F
\end{aligned}
$$