Trapezoidal Rule

Logan Grosz

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Abstract

The trapezoidal rule is a method of approximating an integral numerically. Because functions often is often curved, a trapezoid is more effective at approximating area under the curve than a rectangle, as found in a rectangular Riemann Sum.

1 Declarations

a; Upper limit of integral; $a \in \mathbb{R}$
b; Lower limit of integral; $b \in \mathbb{R}$
f; Function that's integral is being approximated;
n; Number of subintervals for $\int f$ is being approximated; $n \in \mathbb{Z}^+$
Δx; x per n, also known as step size; $\Delta x \in \mathbb{R}, \quad \Delta x = \frac{b-a}{n}$
x_n; x value at a certain n; $x_n = \Delta x\, n$

2 Rule

Trapezoidal Rule:

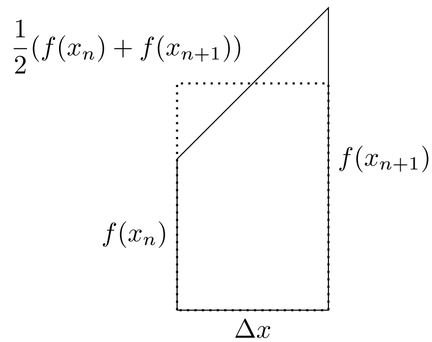
$$T_n = \frac{1}{2}\Delta x(f(x_1) + 2\,f(x_2) +\ f(x_3)\cdots + 2\,f(x_{n-2}) + 2\,f(x_{n-1}) + f(x_n)) \quad (1)$$

3 Pre-Derivation

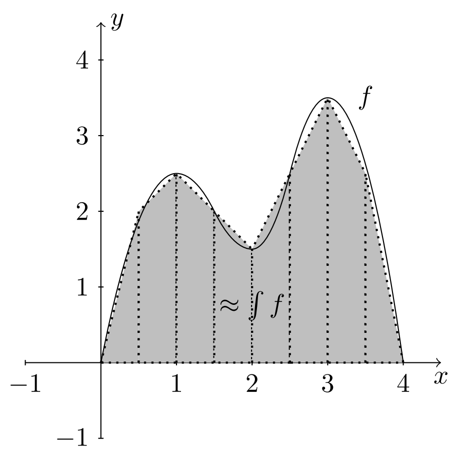
A trapezoid is a quadrilateral with two congruent side lengths and two variable side lengths.

The area of a trapezoid can be determined by finding the area of the subscribed rectangle via averaging the variable side lengths.

Id Est



These trapezoids can be used to approximate the area under the curve of a function. See below.



The image above is a quartic function with eight subscribed trapezoids. Notice the error between the approximation and the actual area under the curve.

4 Derivation

The trapezoidal sum of the above image can be written as:

$$\begin{aligned}
T_8 = &\frac{1}{2}\Delta x\left(f(x_1) + f(x_2)\right) + \frac{1}{2}\Delta x\left(f(x_2) + f(x_3)\right) \\
&+\frac{1}{2}\Delta x\left(f(x_3) + f(x_4)\right) + \frac{1}{2}\Delta x\left(f(x_4) + f(x_5)\right) \\
&+\frac{1}{2}\Delta x\left(f(x_5) + f(x_6)\right) + \frac{1}{2}\Delta x\left(f(x_6) + f(x_7)\right) \\
&+\frac{1}{2}\Delta x\left(f(x_7) + f(x_8)\right)
\end{aligned}$$

A generalized form of this equation can be written as:

$$
\begin{aligned}
T_n = \frac{1}{2}\Delta x\left(f(x_1) + f(x_2)\right) + \frac{1}{2}\Delta x\left(f(x_2) + f(x_3)\right) \\
+ \frac{1}{2}\Delta x\left(f(x_3) + f(x_4)\right) + \frac{1}{2}\Delta x\left(f(x_4) + f(x_5)\right) \\
\vdots \\
+ \frac{1}{2}\Delta x\left(f(x_{n-3}) + f(x_{n-2})\right) + \frac{1}{2}\Delta x\left(f(x_{n-2}) + f(x_{n-1})\right) \\
+ \frac{1}{2}\Delta x\left(f(x_{n-1}) + f(x_n)\right)
\end{aligned}
$$

By diving out common factors:

$$
\begin{aligned}
T_n = \frac{1}{2}\Delta x(f(x_1) + f(x_2) + f(x_2) + f(x_3) \\
+ f(x_3) + f(x_4) + f(x_4) + f(x_5) \\
\vdots \\
+ f(x_{n-3}) + f(x_{n-2}) + f(x_{n-2}) \\
+ f(x_{n-1}) + f(x_{n-1}) + f(x_n))
\end{aligned}
$$

By combining like terms:

$$
T_n = \frac{1}{2}\Delta x(f(x_1) + 2\,f(x_2) + 2f(x_3) \cdots + 2\,f(x_{n-2}) + 2\,f(x_{n-1}) + f(x_n))
$$

5 Exempli Gratia

Examples of important instances