Inverse Trigonometric Function Derivatives

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Abstract

The derivatives of inverse trigonometric derivatives are very different than their basic trigonometric counterparts. The following document will describe the process of getting such a visually unorthodox solution to $\frac{d}{dx}f$, where f is an inverse trigonometric function.

1 Declarations

x; parameter passed into inverse trigonometric function, measured in radians;
For $\sin^{-1}x, x \in [-1, 1]$
For $\cos^{-1}x, x \in [-1, 1]$
For $\tan^{-1}x, x \in [-\infty, \infty]$
For $\cot^{-1}x, x \in [-\infty, \infty]$
For $\sec^{-1}x, x \in (-\infty, -1] \cup [1, \infty)$
For $\csc^{-1}x, x \in (-\infty, -1] \cup [1, \infty)$
$\theta \triangleq x$; used in place of x at times to not confused reader;
arc$f \triangleq f^{-1}$; arc and -1 convention equivalence; $\arcsin x \equiv \sin^{-1}x$
hypotenuse;Latin for "stretching under", the line connecting both catheti; Short: *hyp*
adjacent; Cathetus closest or next to the observed angle; Short: *adj*
opposite Cathetus furthest from or opposite of the observed angle;Short: *opp*

2 Rule

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \tag{1}$$

$$\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}} \tag{2}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{x^2+1} \tag{3}$$

$$\frac{d}{dx}\cot^{-1} x = \frac{-1}{x^2+1} \tag{4}$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}} \tag{5}$$

$$\frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}} \tag{6}$$

3 Pre-Derivation

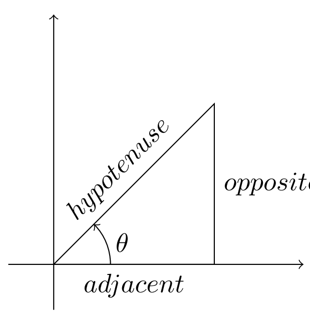
The following notation is the inverse of its subscribed function:

$$\sin^{-1}(\sin\theta) = \theta$$

Pythagorean Trigonometric Identity:

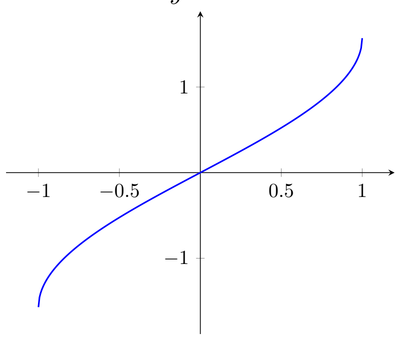
$$sin^2\theta + cos^2\theta = 1$$

The following defines sides of triangle given angle measure θ.

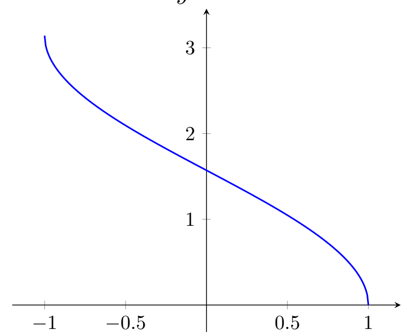


The following are the graphs of all six inverse trigonometric functions for reference:

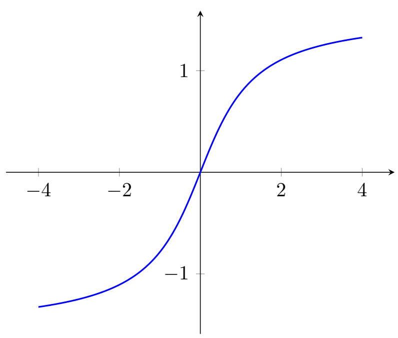
$y = sin^{-1}x$



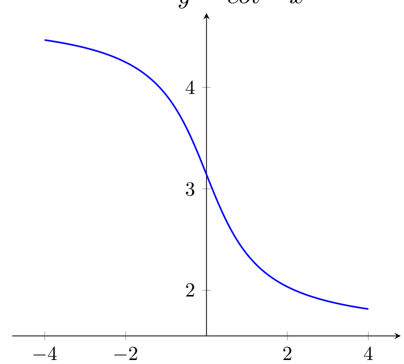
$y = cos^{-1}x$



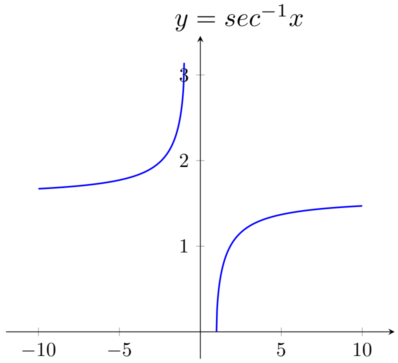
$y = tan^{-1}x$



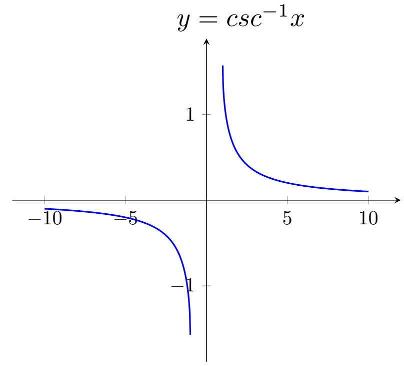
$y = cot^{-1}x$



$y = sec^{-1}x$



$y = csc^{-1}x$



4 Derivation

4.1 Inverse Sine

$$y = \sin^{-1} x$$
$$\sin y = x$$
$$\frac{d}{dx}\sin y = \frac{d}{dx}x$$
$$\frac{dy}{dx}\cos y = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Using $\sin^2 y + cos^2 y = 1$...

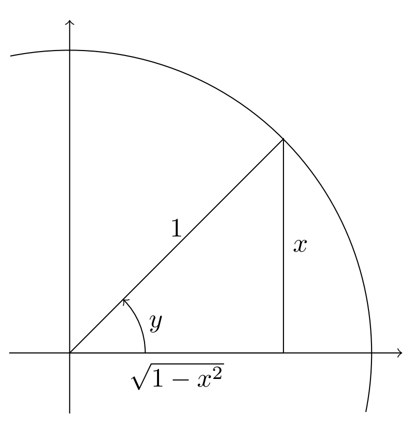
$$\cos y = \sqrt{1 - \sin^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - sin^2 y}}$$

Recall $\sin y = x$...

$$\therefore \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

Id Est



$$\sin y = x \Rightarrow \frac{opp}{hyp} = \frac{x}{1}$$

See above that...

$$\frac{dy}{dx} = \frac{1}{\cos x}$$

See figure at left...

$$\cos y = \sqrt{1 - x^2}$$

$$\therefore \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

4.2 Inverse Cosine

$$
\begin{aligned}
y &= \cos^{-1} x \\
\cos y &= x \\
\frac{d}{dx}\cos y &= \frac{d}{dx}x \\
\frac{dy}{dx} - \sin y &= 1 \\
\frac{dy}{dx} &= \frac{-1}{\sin y}
\end{aligned}
$$

Using $\sin^2 y + cos^2 y = 1$...

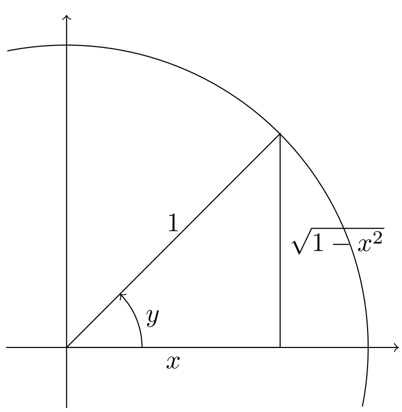
$$\sin y = \sqrt{1 - \cos^2 y}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - cos^2 y}}$$

Recall $\cos y = x$...

$$\therefore \frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}}$$

Id Est



$$\cos y = x \Rightarrow \frac{adj}{hyp} = \frac{x}{1}$$

See above that...

$$\frac{dy}{dx} = \frac{-1}{\sin y}$$

See figure at left...

$$\sin y = \sqrt{1 - x^2}$$

$$\therefore \frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}}$$

4.3 Inverse Tangent

$$y = \tan^{-1} x$$
$$\tan y = x$$
$$\frac{d}{dx} \tan y = \frac{d}{dx} x$$
$$\frac{dy}{dx} \sec^2 y = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

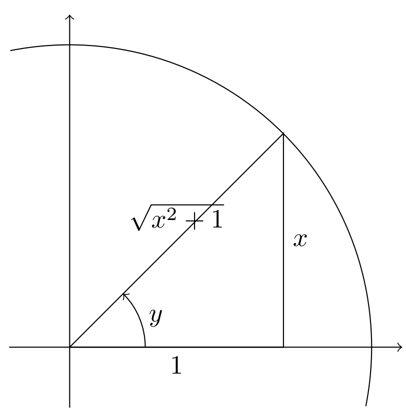
Using $\sin^2 y + cos^2 y = 1$...

$$\sec^2 y = \tan^2 y + 1$$
$$\frac{dy}{dx} = \frac{1}{\tan^2 y + 1}$$

Recall $\tan y = x$...

$$\therefore \frac{d}{dx} \tan^{-1} x = \frac{1}{x^2 + 1}$$

Id Est



$$\tan y = x \Rightarrow \frac{opp}{adj} = \frac{x}{1}$$

See above that...

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

See figure at left...

$$\sec^2 y = x^2 + 1$$

$$\therefore \frac{d}{dx} \tan^{-1} x = \frac{1}{x^2 + 1}$$

4.4 Inverse Cotangent

$$y = \cot^{-1} x$$
$$\cot y = x$$
$$\frac{d}{dx} \cot y = \frac{d}{dx} x$$
$$\frac{dy}{dx} - \csc^2 y = 1$$
$$\frac{dy}{dx} = \frac{-1}{\csc^2 y}$$

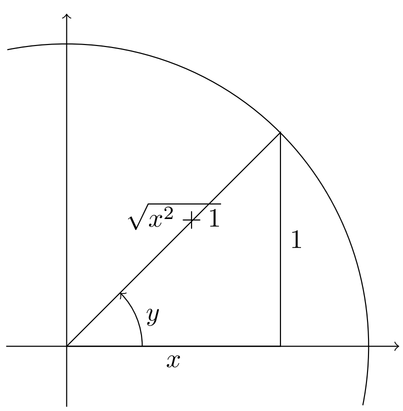
Using $\sin^2 y + cos^2 y = 1$...

$$\csc^2 y = \cot^2 y + 1$$
$$\frac{dy}{dx} = \frac{-1}{\cot^2 y + 1}$$

Recall $\cot y = x$...

$$\therefore \frac{d}{dx} \cot^{-1} x = \frac{-1}{x^2 + 1}$$

Id Est



$$\cot y = x \Rightarrow \frac{adj}{opp} = \frac{x}{1}$$

See above that...

$$\frac{dy}{dx} = \frac{-1}{\csc^2 y}$$

See figure at left...

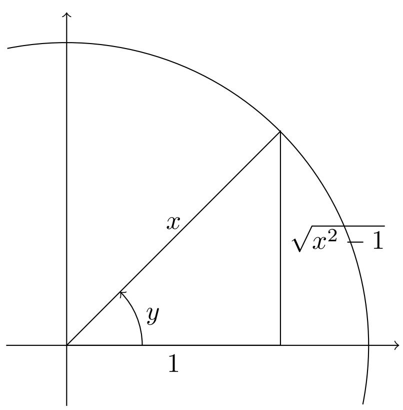
$$\csc^2 y = x^2 + 1$$

$$\therefore \frac{d}{dx} \cot^{-1} x = \frac{-1}{x^2 + 1}$$

4.5 Inverse Secant

$$y = \sec^{-1} x$$
$$\sec y = x$$
$$\frac{d}{dx} \sec y = \frac{d}{dx} x$$
$$\frac{dy}{dx} \sec y \tan y = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

Id Est



$$\sec y = x \Rightarrow \frac{hyp}{adj} = \frac{x}{1}$$

See above that...

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

See figure at left...

$$\tan y = \sqrt{x^2 + 1}$$

Recall $\sec y = x$

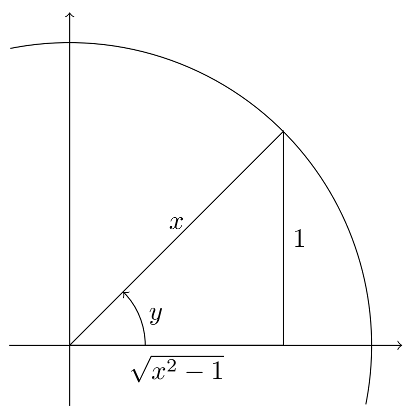
And note $\frac{d}{dx} \sec^{-1} \theta > 0$, so, $|x| = \{x | x > 0\}$

$$\therefore \frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}$$

4.6 Inverse Cosecant

$$
\begin{aligned}
y &= \csc^{-1} x \\
\csc y &= x \\
\frac{d}{dx}\csc y &= \frac{d}{dx} x \\
\frac{dy}{dx} - \csc y \cot y &= 1 \\
\frac{dy}{dx} &= \frac{1}{\csc y \cot y}
\end{aligned}
$$

Id Est



$$\csc y = x \Rightarrow \frac{hyp}{opp} = \frac{x}{1}$$

See above that...

$$\frac{dy}{dx} = \frac{-1}{\csc y \cot y}$$

See figure at left...

$$\cot y = \sqrt{x^2 + 1}$$

Recall $\csc y = x$

And note, $\frac{d}{dx}\csc^{-1}\theta < 0$, so, $|x| = \{x|x > 0\}$

$$\therefore \frac{d}{dx}\csc^{-1} x = \frac{-1}{|x|\sqrt{x^2 - 1}}$$

5 Exempli Gratia

Examples of important instances