2nd Order, Linear, Homogeneous Differential Equations

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Abstract

Abstract goes here...

1 Declarations

Δ; Discriminant of the quadratic formula; $variable \quad domain \quad and \quad range, \quad if \quad applicable$

2 Rule

$$a\,y'' + b\,y' + c\,y = 0$$
$$\implies a\,m^2 + b\,m + c = 0$$

$$\Delta > 0 \implies y = c_2\,e^{m_1 x} + c_2\,e^{m_2 x} \tag{1}$$
$$\Delta = 0 \implies y = c_2\,e^{m\,x} + c_2\,x\,e^{m\,x} \tag{2}$$
$$\Delta < 0 \implies y = c_1\,e^{\alpha x} cos(\beta x) + c_2\,e^{\alpha x} sin(\beta x) \tag{3}$$

Note: This method works for homogeneous linear equations of any order. Just need to find roots and substitute for m.

3 Pre-Derivation

Graphs of sine and cosine (for the sin x =sin-x and cos -x = -cos x)

4 Derivation

4.1 Solving the Differential Equation

$$a\,y'' + b\,y' + c\,y = 0$$
$$\text{Where } a, b, y \in k$$

$$\text{Let } y = e^{m\,x}$$
$$\implies y' = m\,e^{m\,x}$$
$$\implies y'' = m^2\,e^{m\,x}$$

$$a\,m^2\,e^{m\,x} + b\,m\,e^{m\,x} + c\,e^{m\,x} = 0 \qquad \text{Substitution}$$
$$e^{m\,x}(a\,m^2 + b\,m + c) = 0 \qquad \text{Distribution}$$

$$\forall x(e^{m\,x}) \neq 0$$
$$\implies a\,m^2 + b\,m + c = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a} \qquad \text{Quadratic Formula}$$
$$\text{Let } \Delta = b^2 - 4\,a\,c \qquad \text{Discriminant}$$

$$\text{Case 1: } \Delta > 0 \implies m_1, m_2 \in \mathbb{R}$$
$$\text{Case 2: } \Delta = 0 \implies m_1 = m_2 = \frac{-b}{2\,a}$$
$$\text{Case 3: } \Delta < 0 \implies m_1, m_2 \in \mathbb{C}$$

4.2 Cases 1 & 2

General solution for cases 1 & 2:

$$y = c_1\,e^{m_1\,x} + c_2\,e^{m_2\,x}$$

$$\text{Note: Case 2} \implies y_2 = y_1 \int \frac{e^{\int P(x)dx}}{{y_1}^2} dx$$

4.3 Case 3

$$\begin{aligned} m_1, m_2 &\in \mathbb{C} \\ m_1 &= \alpha + i\,\beta \\ m_2 &= \alpha - i\,\beta \end{aligned}$$

$$\begin{aligned} y(x) =& c_1\, e^{(\alpha+i\,\beta)x} + c_2\, e^{(\alpha-\,\beta)x} && \text{Complex-valued solutions} \\ =& c_1\, e^{\alpha x+i\,\beta x} + c_2\, e^{\alpha x-\,\beta x} \\ =& c_1\, e^{\alpha x}\, e^{i\,\beta x} + c_2\, e^{\alpha x}\, e^{-\beta x} \\ e^{i\Theta} =& cos\,\Theta + i\,sin\,\Theta,\, \Theta \in \mathbb{R} && \text{Euler's Formula} \end{aligned}$$

$$\begin{aligned} &\text{Let } c_1 = c_2 = 1 \\ &\implies e^{\alpha x}\, e^{\beta x} + e^{\alpha x}\, e^{-\beta x} \\ &\quad = e^{\alpha x}(cos(\beta x) + i\,sin(\beta x) + cos(\beta x) - i\,sin(\beta x)) \\ &\quad = e^{\alpha x}(2\,cos(\beta x)) \\ &\implies y_1 = e^{\alpha x}\,cos(\beta x) \end{aligned}$$

$$\begin{aligned} &\text{Let } c_1 = 1\, c_2 = -1 \\ &\implies e^{\alpha x}\, e^{\beta x} - e^{\alpha x}\, e^{-\beta x} \\ &\quad = e^{\alpha x}(cos(\beta x) + i\,sin(\beta x) - cos(\beta x) - i\,sin(-\beta x)) \\ &\quad = e^{\alpha x}(cos(\beta x) + i\,sin(\beta x) - cos(\beta x) + i\,sin(\beta x)) \\ &\quad = 2\,i\,e^{\alpha x} sin(\beta x) \\ &\implies y_2 = c_2\,sin(\beta x) \end{aligned}$$

General solution for case 3:

$$y = c_1\, e^{\alpha x} cos(\beta x) + c_2\, e^{\alpha x} sin(\beta x)$$

5 Exempli Gratia

Examples of important instances