0. Introduction

> "There are no real one-particle systems in nature, not even few-particle systems. The existence of virtual pairs and of pair fluctuations shows that the days of fixed particle numbers are over."
>
> *Viki Weisskopf*

The concept of wave-particle duality tells us that the properties of electrons and photons are fundamentally very similar. Despite obvious differences in their mass and charge, under the right circumstances both suffer wave-like diffraction and both can pack a particle-like punch.

Yet the appearance of these objects in classical physics is very different. Electrons and other matter particles are postulated to be elementary constituents of Nature. In contrast, light is a derived concept: it arises as a ripple of the electromagnetic field. If photons and particles are truly to be placed on equal footing, how should we reconcile this difference in the quantum world? Should we view the particle as fundamental, with the electromagnetic field arising only in some classical limit from a collection of quantum photons? Or should we instead view the field as fundamental, with the photon appearing only when we correctly treat the field in a manner consistent with quantum theory? And, if this latter view is correct, should we also introduce an "electron field", whose ripples give rise to particles with mass and charge? But why then didn't Faraday, Maxwell and other classical physicists find it useful to introduce the concept of matter fields, analogous to the electromagnetic field?

The purpose of this course is to answer these questions. We shall see that the second viewpoint above is the most useful: the field is primary and particles are derived concepts, appearing only after quantization. We will show how photons arise from the quantization of the electromagnetic field and how massive, charged particles such as electrons arise from the quantization of matter fields. We will learn that in order to describe the fundamental laws of Nature, we must not only introduce electron fields, but also quark fields, neutrino fields, gluon fields, W and Z-boson fields, Higgs fields and a whole slew of others. There is a field associated to each type of fundamental particle that appears in Nature.

Why Quantum Field Theory?

In classical physics, the primary reason for introducing the concept of the field is to construct laws of Nature that are *local*. The old laws of Coulomb and Newton involve "action at a distance". This means that the force felt by an electron (or planet) changes

immediately if a distant proton (or star) moves. This situation is philosophically unsatisfactory. More importantly, it is also experimentally wrong. The field theories of Maxwell and Einstein remedy the situation, with all interactions mediated in a local fashion by the field.

The requirement of locality remains a strong motivation for studying field theories in the quantum world. However, there are further reasons for treating the quantum field as fundamental[1]. Here I'll give two answers to the question: Why quantum field theory?

Answer 1: Because the combination of quantum mechanics and special relativity implies that particle number is not conserved.

Particles are not indestructible objects, made at the beginning of the universe and here for good. They can be created and destroyed. They are, in fact, mostly ephemeral and fleeting. This experimentally verified fact was first predicted by Dirac who understood how relativity implies the necessity of anti-particles. An extreme demonstration of particle creation is shown in the picture, which comes from the Relativistic Heavy Ion Collider (RHIC) at Brookhaven, Long Island. This machine crashes gold nuclei together, each containing 197 nucleons. The resulting explosion contains up to 10,000 particles, captured here in all their beauty by the STAR detector.

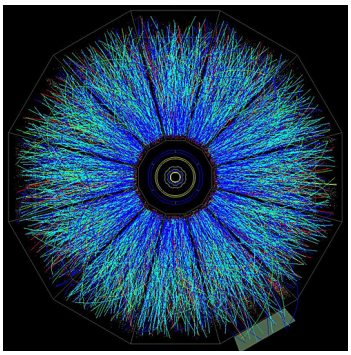

Figure 1:

We will review Dirac's argument for anti-particles later in this course, together with the better understanding that we get from viewing particles in the framework of quantum field theory. For now, we'll quickly sketch the circumstances in which we expect the number of particles to change. Consider a particle of mass m trapped in a box of size L. Heisenberg tells us that the uncertainty in the momentum is $\Delta p \geq \hbar/L$. In a relativistic setting, momentum and energy are on an equivalent footing, so we should also have an uncertainty in the energy of order $\Delta E \geq \hbar c/L$. However, when the uncertainty in the energy exceeds $\Delta E = 2mc^2$, then we cross the barrier to pop particle anti-particle pairs out of the vacuum. We learn that particle-anti-particle pairs are expected to be important when a particle of mass m is localized within a distance of order

$$\lambda = \frac{\hbar}{mc}$$

[1]A concise review of the underlying principles and major successes of quantum field theory can be found in the article by Frank Wilczek, http://arxiv.org/abs/hep-th/9803075

At distances shorter than this, there is a high probability that we will detect particle-anti-particle pairs swarming around the original particle that we put in. The distance λ is called the *Compton wavelength*. It is always smaller than the de Broglie wavelength $\lambda_{dB} = h/|\vec{p}|$. If you like, the de Broglie wavelength is the distance at which the wavelike nature of particles becomes apparent; the Compton wavelength is the distance at which the concept of a single pointlike particle breaks down completely.

The presence of a multitude of particles and antiparticles at short distances tells us that any attempt to write down a relativistic version of the one-particle Schrödinger equation (or, indeed, an equation for any *fixed* number of particles) is doomed to failure. There is no mechanism in standard non-relativistic quantum mechanics to deal with changes in the particle number. Indeed, any attempt to naively construct a relativistic version of the one-particle Schrödinger equation meets with serious problems. (Negative probabilities, infinite towers of negative energy states, or a breakdown in causality are the common issues that arise). In each case, this failure is telling us that once we enter the relativistic regime we need a new formalism in order to treat states with an unspecified number of particles. This formalism is quantum field theory (QFT).

Answer 2: Because all particles of the same type are the same

This sound rather dumb. But it's not! What I mean by this is that two electrons are identical in every way, regardless of where they came from and what they've been through. The same is true of every other fundamental particle. Let me illustrate this through a rather prosaic story. Suppose we capture a proton from a cosmic ray which we identify as coming from a supernova lying 8 billion lightyears away. We compare this proton with one freshly minted in a particle accelerator here on Earth. And the two are exactly the same! How is this possible? Why aren't there errors in proton production? How can two objects, manufactured so far apart in space and time, be identical in all respects? One explanation that might be offered is that there's a sea of proton "stuff" filling the universe and when we make a proton we somehow dip our hand into this stuff and from it mould a proton. Then it's not surprising that protons produced in different parts of the universe are identical: they're made of the same stuff. It turns out that this is roughly what happens. The "stuff" is the proton field or, if you look closely enough, the quark and gluon fields.

In fact, there's more to this tale. Being the "same" in the quantum world is not like being the "same" in the classical world: quantum particles that are the same are truly indistinguishable. Swapping two particles around leaves the state completely unchanged — apart from a possible minus sign. This minus sign determines the statistics of the particle. In quantum mechanics you have to put these statistics in by hand and,

to agree with experiment, should choose Bose statistics (no minus sign) for integer spin particles, and Fermi statistics (yes minus sign) for half-integer spin particles. In quantum field theory, this relationship between spin and statistics is not something that you have to put in by hand. Rather, it is a consequence of the framework.

What is Quantum Field Theory?

Having told you why QFT is necessary, I should really tell you what it is. The clue is in the name: it is the quantization of a classical field, the most familiar example of which is the electromagnetic field. In standard quantum mechanics, we're taught to take the classical degrees of freedom and promote them to operators acting on a Hilbert space. The rules for quantizing a field are no different. Thus the basic degrees of freedom in quantum field theory are *operator valued functions of space and time*. This means that we are dealing with an infinite number of degrees of freedom — at least one for every point in space. This infinity will come back to bite on several occasions.

It will turn out that the possible interactions in quantum field theory are governed by a few basic principles: locality, symmetry and renormalization group flow (the decoupling of short distance phenomena from physics at larger scales). These ideas make QFT a very robust framework: given a set of fields there is very often an almost unique way to couple them together.

What is Quantum Field Theory Good For?

The answer is: almost everything. As I have stressed above, for any relativistic system it is a necessity. But it is also a very useful tool in non-relativistic systems with many particles. Quantum field theory has had a major impact in condensed matter, high-energy physics, cosmology, quantum gravity and pure mathematics. It is literally the language in which the laws of Nature are written.

0.1 Units and Scales

Nature presents us with three fundamental dimensionful constants; the speed of light c, Planck's constant (divided by 2π) $\hbar$ and Newton's constant G. They have dimensions

$$\begin{aligned}[c] &= LT^{-1} \\ [\hbar] &= L^2MT^{-1} \\ [G] &= L^3M^{-1}T^{-2}\end{aligned}$$

Throughout this course we will work with "natural" units, defined by

$$c = \hbar = 1 \tag{0.1}$$

which allows us to express all dimensionful quantities in terms of a single scale which we choose to be mass or, equivalently, energy (since $E = mc^2$ has become $E = m$). The usual choice of energy unit is eV, the electron volt or, more often $GeV = 10^9 eV$ or $TeV = 10^{12} eV$. To convert the unit of energy back to a unit of length or time, we need to insert the relevant powers of c and $\hbar$. For example, the length scale λ associated to a mass m is the Compton wavelength

$$\lambda = \frac{\hbar}{mc}$$

With this conversion factor, the electron mass $m_e = 10^6 eV$ translates to a length scale $\lambda_e \sim 10^{-12} m$. (The Compton wavelength is also defined with an extra factor of 2π: $\lambda = 2\pi\hbar/mc$.)

Throughout this course we will refer to the *dimension* of a quantity, meaning the mass dimension. If X has dimensions of (mass)^d we will write $[X] = d$. In particular, the surviving natural quantity G has dimensions $[G] = -2$ and defines a mass scale,

$$G = \frac{\hbar c}{M_p^2} = \frac{1}{M_p^2} \tag{0.2}$$

where $M_p \approx 10^{19} GeV$ is the *Planck scale*. It corresponds to a length $l_p \approx 10^{-33} cm$. The Planck scale is thought to be the smallest length scale that makes sense: beyond this quantum gravity effects become important and it's no longer clear that the concept of spacetime makes sense. The largest length scale we can talk of is the size of the cosmological horizon, roughly $10^{60} l_p$.

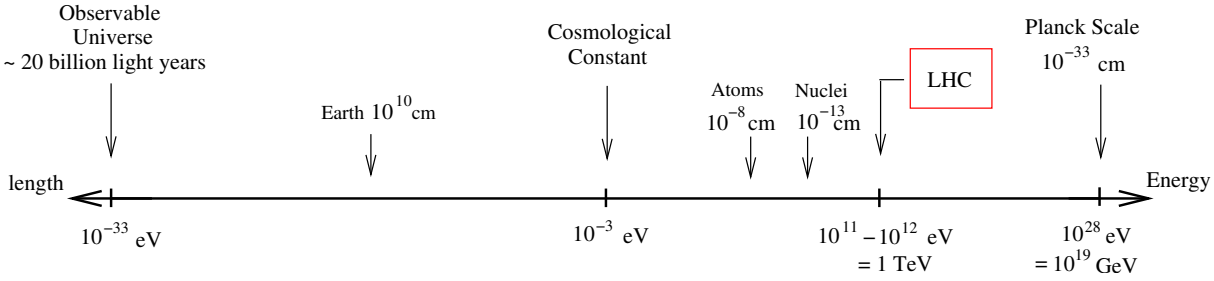

Figure 2: Energy and Distance Scales in the Universe

Some useful scales in the universe are shown in the figure. This is a logarithmic plot, with energy increasing to the right and, correspondingly, length increasing to the left. The smallest and largest scales known are shown on the figure, together with other

relevant energy scales. The standard model of particle physics is expected to hold up to about the TeV. This is precisely the regime that is currently being probed by the Large Hadron Collider (LHC) at CERN. There is a general belief that the framework of quantum field theory will continue to hold to energy scales only slightly below the Planck scale — for example, there are experimental hints that the coupling constants of electromagnetism, and the weak and strong forces unify at around 10^{18} GeV.

For comparison, the rough masses of some elementary (and not so elementary) particles are shown in the table,

Particle	Mass
Neutrinos	$\sim 10^{-2}$ eV
Electron	0.5 MeV
Muon	100 MeV
Pions	140 MeV
Proton, Neutron	1 GeV
Tau	2 GeV
W,Z Bosons	80-90 GeV
Higgs Boson	125 GeV

1. Classical Field Theory

In this first section we will discuss various aspects of classical fields. We will cover only the bare minimum ground necessary before turning to the quantum theory, and will return to classical field theory at several later stages in the course when we need to introduce new ideas.

1.1 The Dynamics of Fields

A *field* is a quantity defined at every point of space and time $(\vec{x}, t)$. While classical particle mechanics deals with a finite number of generalized coordinates $q_a(t)$, indexed by a label a, in field theory we are interested in the dynamics of fields

$$\phi_a(\vec{x}, t) \tag{1.1}$$

where both a and $\vec{x}$ are considered as labels. Thus we are dealing with a system with an infinite number of degrees of freedom — at least one for each point $\vec{x}$ in space. Notice that the concept of position has been relegated from a dynamical variable in particle mechanics to a mere label in field theory.

An Example: The Electromagnetic Field

The most familiar examples of fields from classical physics are the electric and magnetic fields, $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$. Both of these fields are spatial 3-vectors. In a more sophisticated treatement of electromagnetism, we derive these two 3-vectors from a single 4-component field $A^\mu(\vec{x}, t) = (\phi, \vec{A})$ where $\mu = 0, 1, 2, 3$ shows that this field is a vector in *spacetime*. The electric and magnetic fields are given by

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A} \tag{1.2}$$

which ensure that two of Maxwell's equations, $\nabla \cdot \vec{B} = 0$ and $d\vec{B}/dt = -\nabla \times \vec{E}$, hold immediately as identities.

The Lagrangian

The dynamics of the field is governed by a Lagrangian which is a function of $\phi(\vec{x}, t)$, $\dot{\phi}(\vec{x}, t)$ and $\nabla\phi(\vec{x}, t)$. In all the systems we study in this course, the Lagrangian is of the form,

$$L(t) = \int d^3x \; \mathcal{L}(\phi_a, \partial_\mu \phi_a) \tag{1.3}$$

where the official name for $\mathcal{L}$ is the *Lagrangian density*, although everyone simply calls it the Lagrangian. The action is,

$$S = \int_{t_1}^{t_2} dt \int d^3x \; \mathcal{L} \; = \int d^4x \; \mathcal{L} \tag{1.4}$$

Recall that in particle mechanics L depends on q and $\dot{q}$, but not $\ddot{q}$. In field theory we similarly restrict to Lagrangians $\mathcal{L}$ depending on ϕ and $\dot{\phi}$, and not $\ddot{\phi}$. In principle, there's nothing to stop $\mathcal{L}$ depending on $\nabla\phi$, $\nabla^2\phi$, $\nabla^3\phi$, etc. However, with an eye to later Lorentz invariance, we will only consider Lagrangians depending on $\nabla\phi$ and not higher derivatives. Also we will not consider Lagrangians with explicit dependence on x^μ; all such dependence only comes through ϕ and its derivatives.

We can determine the equations of motion by the principle of least action. We vary the path, keeping the end points fixed and require $\delta S = 0$,

$$\begin{aligned} \delta S &= \int d^4x \; \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \, \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \, \delta(\partial_\mu \phi_a) \right] \\ &= \int d^4x \; \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right] \delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \, \delta\phi_a \right) \end{aligned} \tag{1.5}$$

The last term is a total derivative and vanishes for any $\delta\phi_a(\vec{x}, t)$ that decays at spatial infinity and obeys $\delta\phi_a(\vec{x}, t_1) = \delta\phi_a(\vec{x}, t_2) = 0$. Requiring $\delta S = 0$ for all such paths yields the Euler-Lagrange equations of motion for the fields ϕ_a,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0 \tag{1.6}$$

1.1.1 An Example: The Klein-Gordon Equation

Consider the Lagrangian for a real scalar field $\phi(\vec{x}, t)$,

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2} \, \eta^{\mu\nu} \, \partial_\mu \phi \partial_\nu \phi - \tfrac{1}{2} m^2 \phi^2 \\ &= \tfrac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla\phi)^2 - \tfrac{1}{2} m^2 \phi^2 \end{aligned} \tag{1.7}$$

where we are using the Minkowski space metric

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1.8}$$

Comparing (1.7) to the usual expression for the Lagrangian $L = T - V$, we identify the kinetic energy of the field as

$$T = \int d^3x \; \tfrac{1}{2} \dot{\phi}^2 \tag{1.9}$$

and the potential energy of the field as

$$V = \int d^3x \ \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 \tag{1.10}$$

The first term in this expression is called the gradient energy, while the phrase "potential energy", or just "potential", is usually reserved for the last term.

To determine the equations of motion arising from (1.7), we compute

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi \equiv (\dot{\phi}, -\nabla\phi) \tag{1.11}$$

The Euler-Lagrange equation is then

$$\ddot{\phi} - \nabla^2\phi + m^2\phi = 0 \tag{1.12}$$

which we can write in relativistic form as

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0 \tag{1.13}$$

This is the *Klein-Gordon Equation*. The Laplacian in Minkowski space is sometimes denoted by $\Box$. In this notation, the Klein-Gordon equation reads $\Box\phi + m^2\phi = 0$.

An obvious generalization of the Klein-Gordon equation comes from considering the Lagrangian with arbitrary potential $V(\phi)$,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \quad \Rightarrow \quad \partial_\mu\partial^\mu\phi + \frac{\partial V}{\partial\phi} = 0 \tag{1.14}$$

1.1.2 Another Example: First Order Lagrangians

We could also consider a Lagrangian that is linear in time derivatives, rather than quadratic. Take a complex scalar field ψ whose dynamics is defined by the real Lagrangian

$$\mathcal{L} = \frac{i}{2}(\psi^\star\dot{\psi} - \dot{\psi}^\star\psi) - \nabla\psi^\star\cdot\nabla\psi - m\psi^\star\psi \tag{1.15}$$

We can determine the equations of motion by treating ψ and $\psi^\star$ as independent objects, so that

$$\frac{\partial\mathcal{L}}{\partial\psi^\star} = \frac{i}{2}\dot{\psi} - m\psi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\dot{\psi}^\star} = -\frac{i}{2}\psi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\nabla\psi^\star} = -\nabla\psi \tag{1.16}$$

This gives us the equation of motion

$$i\frac{\partial\psi}{\partial t} = -\nabla^2\psi + m\psi \tag{1.17}$$

This looks very much like the Schrödinger equation. Except it isn't! Or, at least, the interpretation of this equation is very different: the field ψ is a classical field with none of the probability interpretation of the wavefunction. We'll come back to this point in Section 2.8.

The initial data required on a Cauchy surface differs for the two examples above. When $\mathcal{L} \sim \dot{\phi}^2$, both ϕ and $\dot{\phi}$ must be specified to determine the future evolution; however when $\mathcal{L} \sim \psi^\star \dot{\psi}$, only ψ and $\psi^\star$ are needed.

1.1.3 A Final Example: Maxwell's Equations

We may derive Maxwell's equations in the vacuum from the Lagrangian,

$$\mathcal{L} = -\tfrac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right) + \tfrac{1}{2}(\partial_\mu A^\mu)^2 \tag{1.18}$$

Notice the funny minus signs! This is to ensure that the kinetic terms for A_i are positive using the Minkowski space metric (1.8), so $\mathcal{L} \sim \frac{1}{2}\dot{A}_i^2$. The Lagrangian (1.18) has no kinetic term $\dot{A}_0^2$ for A_0. We will see the consequences of this in Section 6. To see that Maxwell's equations indeed follow from (1.18), we compute

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + (\partial_\rho A^\rho)\,\eta^{\mu\nu} \tag{1.19}$$

from which we may derive the equations of motion,

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\partial^2 A^\nu + \partial^\nu(\partial_\rho A^\rho) = -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) \equiv -\partial_\mu F^{\mu\nu} \tag{1.20}$$

where the *field strength* is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. You can check using (1.2) that this reproduces the remaining two Maxwell's equations in a vacuum: $\nabla \cdot \vec{E} = 0$ and $\partial\vec{E}/\partial t = \nabla \times \vec{B}$. Using the notation of the field strength, we may rewrite the Maxwell Lagrangian (up to an integration by parts) in the compact form

$$\mathcal{L} = -\tfrac{1}{4}\, F_{\mu\nu}F^{\mu\nu} \tag{1.21}$$

1.1.4 Locality, Locality, Locality

In each of the examples above, the Lagrangian is *local*. This means that there are no terms in the Lagrangian coupling $\phi(\vec{x}, t)$ directly to $\phi(\vec{y}, t)$ with $\vec{x} \neq \vec{y}$. For example, there are no terms that look like

$$L = \int d^3x d^3y\ \phi(\vec{x})\phi(\vec{y}) \tag{1.22}$$

A priori, there's no reason for this. After all, $\vec{x}$ is merely a label, and we're quite happy to couple other labels together (for example, the term $\partial_3 A_0\, \partial_0 A_3$ in the Maxwell Lagrangian couples the $\mu = 0$ field to the $\mu = 3$ field). But the closest we get for the $\vec{x}$ label is a coupling between $\phi(\vec{x})$ and $\phi(\vec{x} + \delta\vec{x})$ through the gradient term $(\nabla\phi)^2$. This property of locality is, as far as we know, a key feature of *all* theories of Nature. Indeed, one of the main reasons for introducing field theories in classical physics is to implement locality. In this course, we will only consider local Lagrangians.

1.2 Lorentz Invariance

The laws of Nature are relativistic, and one of the main motivations to develop quantum field theory is to reconcile quantum mechanics with special relativity. To this end, we want to construct field theories in which space and time are placed on an equal footing and the theory is invariant under Lorentz transformations,

$$x^\mu \longrightarrow (x')^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.23}$$

where $\Lambda^\mu{}_\nu$ satisfies

$$\Lambda^\mu{}_\sigma \, \eta^{\sigma\tau} \, \Lambda^\nu{}_\tau = \eta^{\mu\nu} \tag{1.24}$$

For example, a rotation by θ about the x^3-axis, and a boost by $v < 1$ along the x^1-axis are respectively described by the Lorentz transformations

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.25}$$

with $\gamma = 1/\sqrt{1-v^2}$. The Lorentz transformations form a Lie group under matrix multiplication. You'll learn more about this in the "Symmetries and Particle Physics" course.

The Lorentz transformations have a *representation* on the fields. The simplest example is the scalar field which, under the Lorentz transformation $x \to \Lambda x$, transforms as

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) \tag{1.26}$$

The inverse Λ^{-1} appears in the argument because we are dealing with an *active* transformation in which the field is truly shifted. To see why this means that the inverse appears, it will suffice to consider a non-relativistic example such as a temperature field. Suppose we start with an initial field $\phi(\vec{x})$ which has a hotspot at, say, $\vec{x} = (1, 0, 0)$. After a rotation $\vec{x} \to R\vec{x}$ about the z-axis, the new field $\phi'(\vec{x})$ will have the hotspot at $\vec{x} = (0, 1, 0)$. If we want to express $\phi'(\vec{x})$ in terms of the old field ϕ, we need to place ourselves at $\vec{x} = (0, 1, 0)$ and ask what the old field looked like where we've come from at $R^{-1}(0, 1, 0) = (1, 0, 0)$. This R^{-1} is the origin of the inverse transformation. (If we were instead dealing with a passive transformation in which we relabel our choice of coordinates, we would have instead $\phi(x) \to \phi'(x) = \phi(\Lambda x)$).