UNIVERSITY OF WISCONSIN-LA CROSSE
Department of Computer Science

CS 225 Discrete Computational Structures Fall 2014
Final Exam (Practice) 16 December 2014

- **Do not turn the page until instructed to do so. This booklet contains 12 pages including the cover page.**
- **This is a closed-book exam. All you need is the exam and a writing utensil.**
- **You have exactly 120 minutes.**
- **You will not necessarily finish all questions, so do your best ones first. The maximum possible is 100.**
- **Good luck!**

PROBLEM	SCORE
1	10
2	10
3	10
4	10
5	15
6	20
7	5
8	10
9	10
TOTAL	100

NAME: ***ANSWER KEY***

1. **(10 pts.)** **LOGICAL PRINCIPLES.**

(a) (2 pts.) Write the *negation* of the following expression, using *only* the connectives $\sim$ and $\wedge$ in your final answer:

$$(p \wedge q) \to (r \vee \sim s)$$

Answer: $(p \wedge q) \wedge (\sim r \wedge s)$ (*also acceptable*: $(p \wedge q) \wedge (\sim r \wedge \sim\sim s)$)

(b) (2 pts.) Write the *negation* of the following expression, *without using* the connective $\to$ in your final answer:

$$\exists x \in D,\ \forall y \in D,\ P(y) \to (Q(x,y) \to R(x))$$

Answer: $\forall x \in D,\ \exists y \in D,\ P(y) \wedge (Q(x,y) \wedge \sim R(x))$

(c) (6 pts.) For each of the following, assume we have a finite domain:

$$D = \{a,\ b\}$$

i. Write the following expression in *pure propositional form, without any quantifiers*:

$$\forall x \in D,\ \exists y \in D,\ F(x) \to G(y)$$

Answer: $((F(a) \to G(a)) \vee (F(a) \to G(b))) \wedge ((F(b) \to G(a)) \vee (F(b) \to G(b)))$

ii. Write a *quantificational statement* that is equivalent to the following expression:

$$(P(a,a) \wedge P(a,b)) \vee (P(b,a) \wedge P(b,b))$$

Answer: $\exists x \in D,\ \forall y \in D,\ P(x,y)$

2. **(10 pts.)** **LOGICAL VALIDITY**

(a) (5 pts.) Is the following a tautology, a contradiction, or neither? Use a **full** truth-table to answer the question, and indicate which of the three possibilities is true, and why.

$$(p \lor q) \rightarrow ((\sim p \rightarrow q) \land (\sim q \rightarrow p))$$

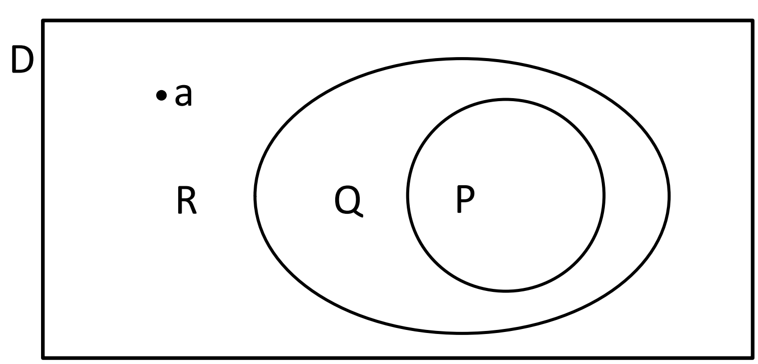
Answer: The expression is a **tautology**, as shown by the truth-table below, where the final column for the main connective is entirely True:

p	q	p ∨ q	~p	~q	~p → q	~q → p	(~p → q) ∧ (~q → p)	(p ∨ q) → ((~p → q) ∧ (~q → p))
T	T	T	F	F	T	T	T	**T**
T	F	T	F	T	T	T	T	**T**
F	T	T	T	F	T	T	T	**T**
F	F	F	T	T	F	F	F	**T**

(b) (5 pts.) Show that the argument form below is *invalid*, by giving a diagram for the truth-sets of the predicates contained in it such that the argument fails.

$$\begin{aligned} &\forall x,\ P(x) \rightarrow Q(x) \\ &\forall x,\ Q(x) \rightarrow R(x) \\ &\exists x,\ \sim P(x) \land \sim Q(x) \\ \therefore\ &\exists x,\ \sim R(x) \end{aligned}$$

Answer: The first two premises require that the truth set of P be a subset of the truth set of Q, and the set for Q be a subset of that for R. Then, in order to make the third premise true, but the conclusion false, we must ensure that some object exists that has neither property P nor property Q (the object a), while also guaranteeing that *every* object has property R. Such a diagram is as follows, where the truth set for R is the entire domain D:



3. **(10 pts.) LOGICAL PROOFS.**

(a) (5 pts.) Prove the conclusion $(q \wedge m)$ from the set of premises below. **Each line of your proof should either be a premise, or justified using one of the logical equivalences or proof rules included on the final page of the exam.**

$$\mathbf{P} = \{r \vee (t \wedge s),\ r \to q,\ t \to q,\ r \to m,\ s \to m\}$$

1.	$r \vee (t \wedge s)$	[Premise]
2.	$(r \vee t) \wedge (r \vee s)$	[1, Distributive Law]
3.	$r \vee t$	[2, Specialization]
4.	$r \to q$	[Premise]
5.	$t \to q$	[Premise]
6.	q	[3, 4, 5, Division into Cases]
7.	$r \vee s$	[2, Specialization]
8.	$r \to m$	[Premise]
9.	$s \to m$	[Premise]
10.	m	[7, 8, 9, Division into Cases]
11.	$q \wedge m$	[6, 10, Conjunction]

(b) (5 pts.) Prove the conclusion $R(a)$ from the premises below. **Each step of your proof should be justified by some logical principle.**

i. $\forall x,\ Q(x) \to R(x) \vee Z(x)$

ii. $\forall x,\ S(x) \to Q(x)$

iii. $S(a) \wedge {\sim}Z(a)$

1.	$S(a)$	[Premise iii, Specialization]
2.	$Q(a)$	[Premise ii, 1, $\forall$MP]
3.	$R(a) \vee Z(a)$	[Premise i, 2, $\forall$MP]
4.	${\sim}Z(a)$	[Premise iii, Specialization]
5.	$R(a)$	[3, 4, Elimination]

4. **(10 pts.)** **LOGICAL TRANSLATION.**

Translate the following into symbolic logical form. Use only standard connectives ($\sim$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$) and quantifiers ($\exists$, $\forall$), along with standard set theoretic and mathematical notation.

If you use any non-standard predicates, you must define them precisely.

(a) Every negative integer is less than every positive integer.

$$\forall x \in \mathbf{Z}^{-},\, \forall y \in \mathbf{Z}^{+},\, x < y$$

(b) Every integer is the sum of two integers.

$$\forall x \in \mathbf{Z},\, \exists y \in \mathbf{Z},\, \exists z \in \mathbf{Z},\, x = y + z$$

(c) The sum of any two integers is an integer.

$$\forall x \in \mathbf{Z},\, \forall y \in \mathbf{Z},\, \exists z \in \mathbf{Z},\, x + y = z$$

(d) Every positive real number has a square root that is also a positive real number.

$$\forall x \in \mathbf{R}^{+},\, \exists y \in \mathbf{R}^{+},\, y = \sqrt{x}$$

(e) The square root of any negative real number is not a real number.

$$\sim\exists x \in \mathbf{R}^{-},\, \exists y \in \mathbf{R},\, y = \sqrt{x}$$

5. **(15 pts.)** **LOGIC, SETS, AND PREDICATES**

(a) (10 pts.) Write out the definitions of each of the following sets, in set-builder form, assuming that all sets referenced are subsets of a common domain D.
For full points, you should use logical expressions to define the sets.

i. $(A \cup B)^c$

$$\{x \in D \mid x \notin A \wedge x \notin B\}$$

ii. $A - (B \cup C)$

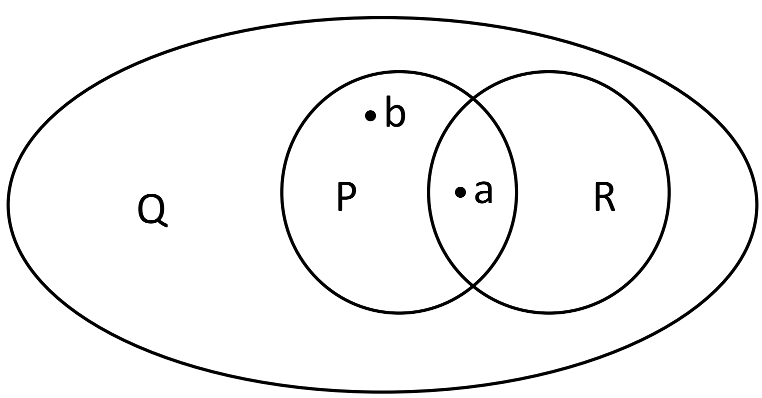
$$\{x \in D \mid x \in A \wedge (x \notin B \wedge x \notin C)\}$$

iii. $(A \times B)^c$

$$\{(x, y) \in D \times D \mid x \notin A \vee y \notin B\}$$

(b) (5 pts.) Draw a diagram for the truth-sets of the contained predicates on which all of the following expressions are **true**:

$$\forall x \in D,\ P(x) \to Q(x)$$
$$\forall x \in D,\ {\sim}Q(x) \to {\sim}R(x)$$
$$\exists x \in D,\ P(x) \wedge R(x)$$
$$\exists x \in D,\ P(x) \wedge {\sim}R(x)$$



6. **(20 pts.)** **RELATIONS BETWEEN SETS**

(a) (5 pts.) Use an element argument to show that, for any sets A, B, and C:

$$\text{if } B \subseteq C, \text{ then } A \times B \subseteq A \times C$$

Proof: Assume that $B \subseteq C$, and consider an arbitrary pair $(x, y) \in A \times B$. By definition of the cross-product, $x \in A \wedge y \in B$. But then, since $y \in B$ and $B \subseteq C$, we have $y \in C$. Therefore, $x \in A \wedge y \in C$, and by definition of the cross-product again, $(x, y) \in A \times C$.

Thus, since (x, y) was an arbitrary element of $A \times B$, we can conclude that every such pair is an element of $A \times C$, which means that $A \times B \subseteq A \times C$, if $B \subseteq C$, as required. $\square$

(b) (5 pts.) Use an element argument to show that, for any sets A and B:

$$\text{if } A \subseteq B, \text{ then } B^c \subseteq A^c$$

Proof: Assume that $A \subseteq B$ and consider an arbitrary element $x \in B^c$. By definition of set complement, $x \notin B$, and therefore, by definition of subset, $x \notin A$. Thus, $x \in A^c$, by definition of set complement.

Thus, since x was an arbitrary element of B^c, we can conclude that every such object is an element of A^c, which means that $B^c \subseteq A^c$, if $A \subseteq B$, as required. $\square$

(c) (5 pts.) Give a counter-example showing that the following is *not always* true for sets A and B:

$$\text{if } A - B = \emptyset, \text{ then } A = B$$

Answer: Whenever A is a proper subset of B, we can produce a counter-example. For instance, let our sets be:

$$A = \{1,\ 2\}$$
$$B = \{1,\ 2,\ 3\}$$

In such a case, $A - B = \emptyset$, but $A \neq B$.

(d) (5 pts.) Give a counter-example showing that the following is *not always* true for sets A, B, and C:

$$\text{if } A \subseteq B, \text{ then } C - A \subseteq C - B$$

Answer: Again, we can produce a counter-example whenever A is a proper subset of B, as long as there exists some element in $C - A$ that is not in $C - B$. For example:

$$A = \{1,\ 2\}$$
$$B = \{1,\ 2,\ 3\}$$
$$C = \{1,\ 2,\ 3,\ 4\}$$

In such a case, we have the set differences:

$$C - A = \{3,\ 4\}$$
$$C - B = \{4\}$$

Clearly, $A \subseteq B$, but $C - A \not\subseteq C - B$.

7. **(10 pts.)** **USING INDUCTION.**

Prove that for all integers $n \geq 1$, $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

Proof: We first prove that the property holds for base case $n = 1$:

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1}$$

For our inductive hypothesis, we assume the property holds for any value $k \geq 1$, that is:

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

Finally, we show that the property holds for value $(k+1)$, proving that:

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$$

To see this, we can reason as follows:

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)}$$

So, by the inductive hypothesis:

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

$\square$

8. **(15 pts.)** **FUNCTIONS AND RELATIONS**

(a) (2 pts.) Write out the definition of the following function in set-builder form:

$$f : \mathbf{R} \to \mathbf{R} \text{ such that } \forall x \in \mathbf{R},\ f(x) = x^2$$

Answer: $\{(x, y) \in \mathbf{R} \times \mathbf{R} \,|\, y = x^2\}$

(b) (2 pts.) Circle the true statements below:

i. The function f of the last question is $1 : 1$.

ii. The function f of the last question is onto.

Answer: Neither of the properties are true. The function is not $1 : 1$ because, e.g., $f(1) = 1 = f(-1)$. It is not onto, since there are no real inputs that generate negative real outputs.

(c) (4 pts.) Let S be the set of all binary strings, i.e. those strings formed using '`0`' and '`1`'. Let c be the function that concatenates strings; i.e., for any two strings, it works like the '`+`' operator in Java, creating a new string that is the concatenation of its inputs.

i. What is the domain of the function c?

Answer: The domain is the set of all possible pairs of binary strings: $S \times S$

ii. What is the co-domain of the function c?

Answer: The co-domain is just the set of binary strings again: S

(d) (2 pts.) Circle the true statements below:

i. The function c of the last question is $1 : 1$.

ii. The function c of the last question is onto.

Answer: The function is not $1 : 1$ because, e.g., $c(\texttt{0}, \texttt{01}) = \texttt{001} = c(\texttt{00}, \texttt{1})$. If we allow the empty string, then the function is onto; if there is no empty string allowed, then it is not (since we cannot form a single-digit string by concatenating two others).

9. **(10 pts.) FUNCTIONS AND RELATIONS, II**

Define the relation R on pairs of positive integers as follows:

$$\forall (x_1, y_1),\ (x_2, y_2) \in \mathbf{Z}^+ \times \mathbf{Z}^+,\ (x_1, y_1)\ R\ (x_2, y_2) \text{ if and only if: } x_1 y_2 = x_2 y_1$$

Prove that R is an equivalence relation.

Answer: We prove that the relation R is reflexive, symmetric, and transitive in turn.

Reflexive: for any arbitrary single pair $(x_1, y_1) \in \mathbf{Z}^+ \times \mathbf{Z}^+$, it is obvious that $x_1 y_1 = x_1 y_1$, and so $(x_1, y_1)\ R\ (x_1, y_1)$. Since the pair is arbitrary, this means the relation R is reflexive.

Symmetric: Consider arbitrary pairs (x_1, y_1), $(x_2, y_2) \in \mathbf{Z}^+ \times \mathbf{Z}^+$ such that $(x_1, y_1)\ R\ (x_2, y_2)$; by definition, $x_1 y_2 = x_2 y_1$. But therefore $x_2 y_1 = x_1 y_2$, and so $(x_2, y_2)\ R\ (x_1, y_1)$. Since the pairs were arbitrary, this means relation R is symmetric.

Transitive: Consider arbitrary pairs (x_1, y_1), (x_2, y_2), $(x_3, y_3) \in \mathbf{Z}^+ \times \mathbf{Z}^+$ such that $(x_1, y_1)\ R\ (x_2, y_2)$ and $(x_2, y_2)\ R\ (x_3, y_3)$; by definition, $x_1 y_2 = x_2 y_1$ and $x_2 y_3 = x_3 y_2$. Thus, we have:

$$x_1 y_2 = x_2 y_1$$

$$\frac{x_1 y_2}{y_1} = x_2$$

Therefore, we have

$$x_2 y_3 = x_3 y_2$$

$$\frac{x_1 y_2}{y_1} y_3 = x_3 y_2$$

$$x_1 y_2 y_3 = x_3 y_2 y_1$$

$$x_1 y_3 = x_3 y_1$$

and so $(x_1, y_1)\ R\ (x_3, y_3)$. Since the pairs were arbitrary, this means relation R is transitive.

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* **t** *and* **c**:	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

Modus Ponens	$p \to q$ p • q	**Elimination**	**a.** $p \vee q$ $\sim q$ • p **b.** $p \vee q$ $\sim p$ • q
Modus Tollens	$p \to q$ $\sim q$ • $\sim p$	**Transitivity**	$p \to q$ $q \to r$ • $p \to r$
Generalization	**a.** p • $p \vee q$ **b.** q • $p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ • r
Specialization	**a.** $p \wedge q$ • p **b.** $p \wedge q$ • q		
Conjunction	p q • $p \wedge q$	**Contradiction Rule**	$\sim p \to c$ • p