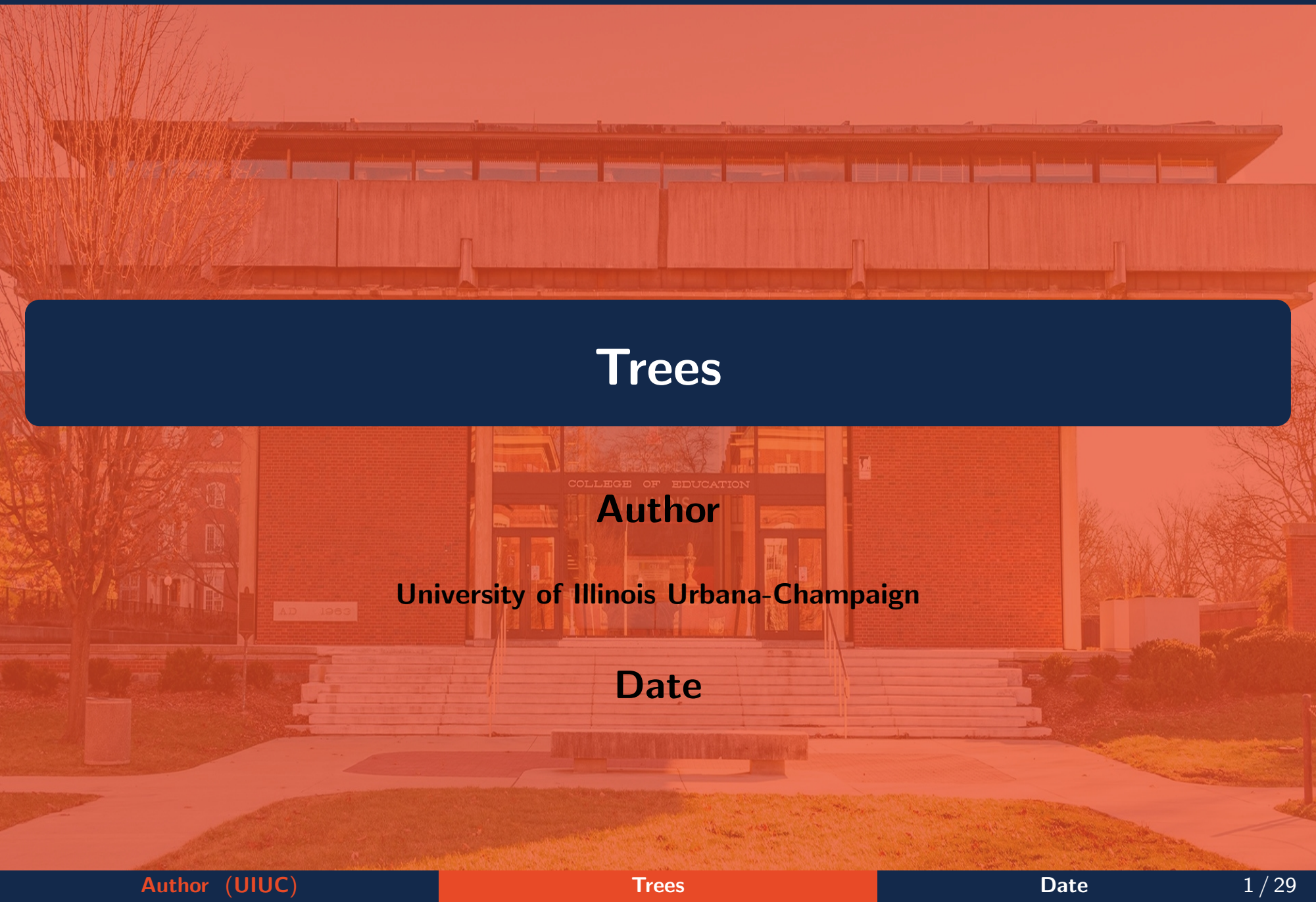


Trees

Author

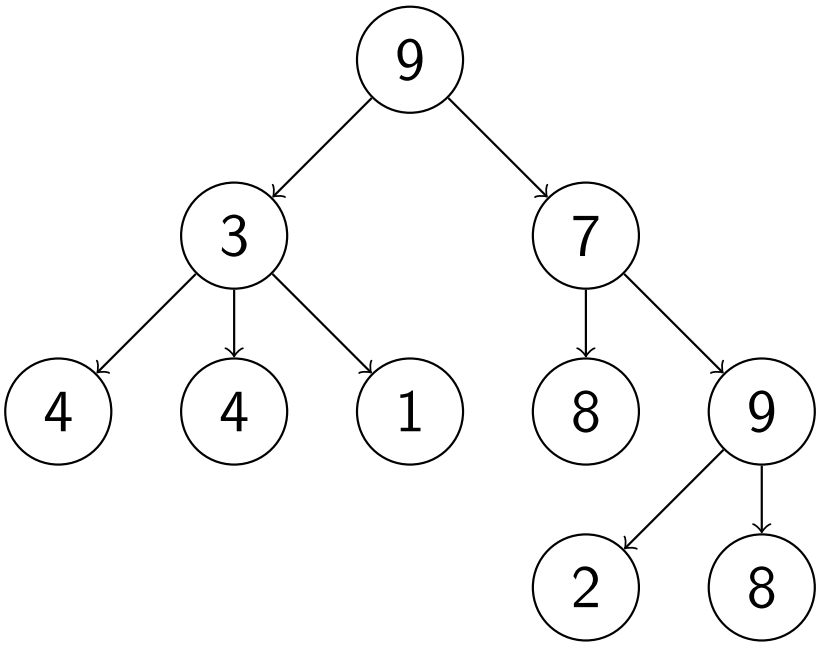
University of Illinois Urbana-Champaign

Date

Objectives

- Identify the core componenets of trees in general
- Cover the implementation of Binary Tree nodes and traversal algorithms.
- Identify the motivations for using Binary Search Trees (BST)
- Cover the implementation of the following BST methods:
  - Search
  - Adding a node
  - Removing a node

Unifying Structure

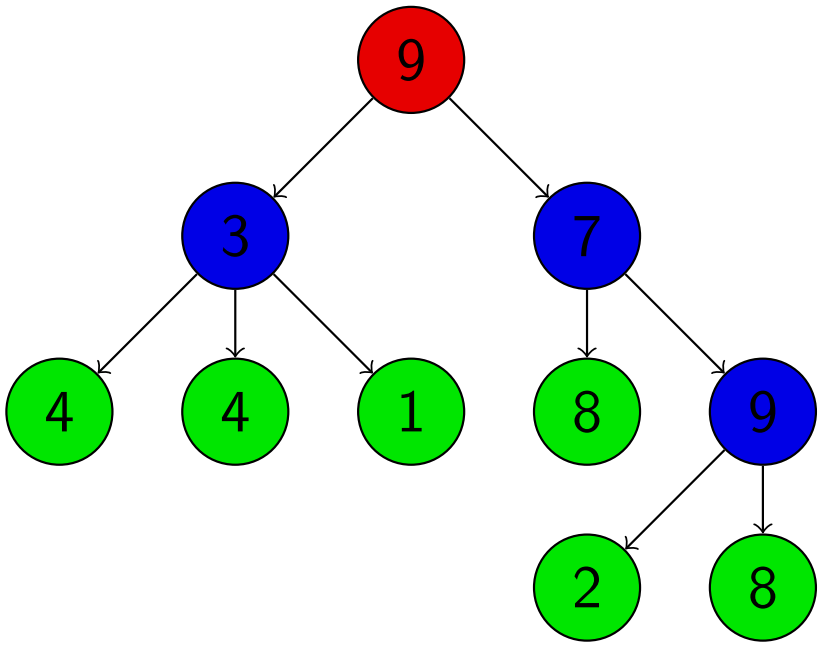


Trees have the following general, properties:

- *Directed:* Like a singly linked list a node does not have a reference to its parent.
- *Acyclic:* There are no "circles" of reference in the tree.
- A collection of nodes with:
  - Some data (generally)
  - The ability to reference child nodes

Tree Terminology: Structural

- *root* (red)
- *branch* (blue)
- *leaf* (green)

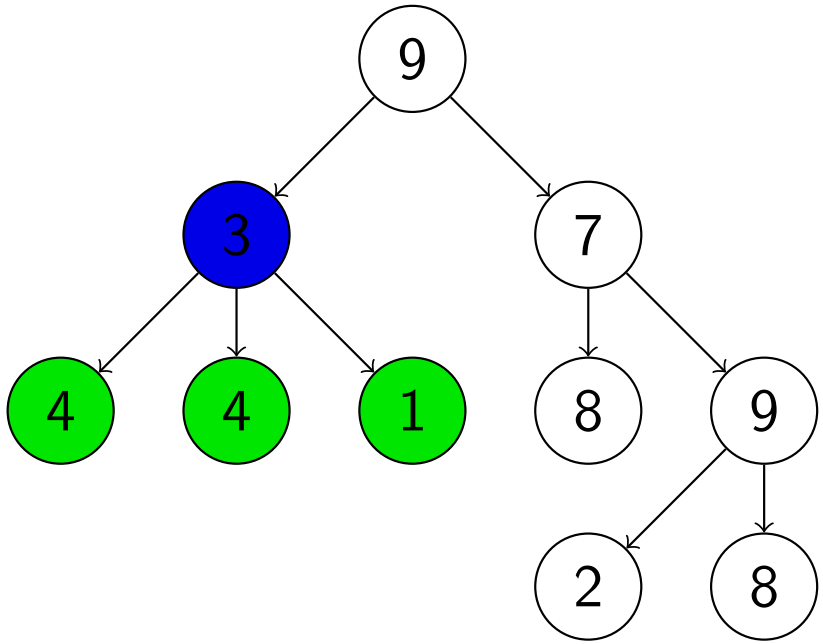


Tree Terminology: Relationship Based

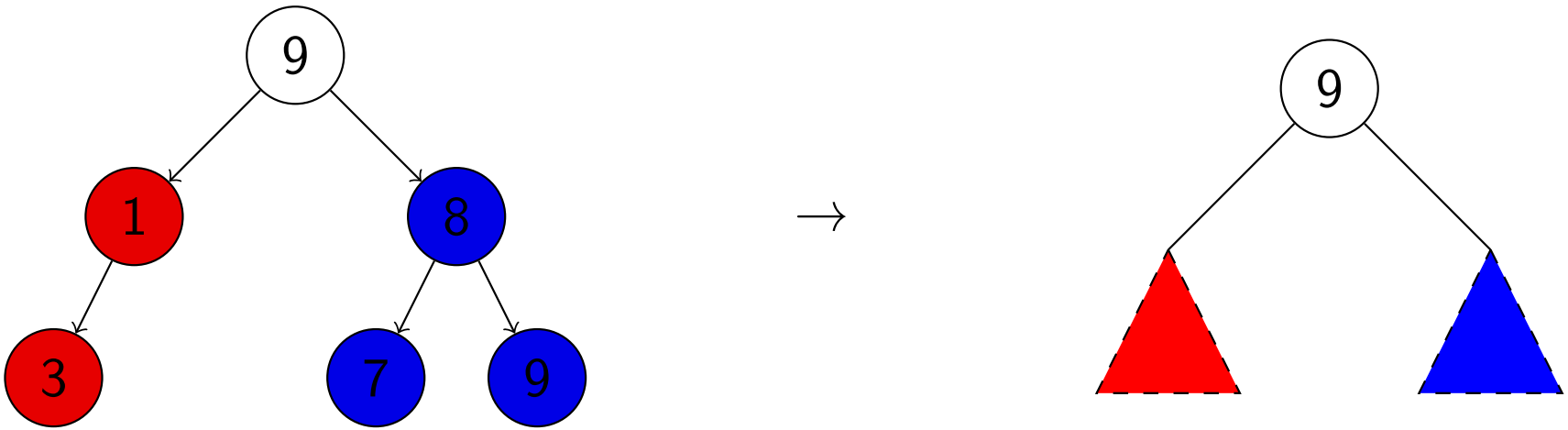
Terms based on relationships between nodes

- *parent* (blue)
- *child/sibling* (green)

The blue node has three *child* nodes (green) and, as a result, the green nodes are siblings.



Tree Terminology: Subtrees



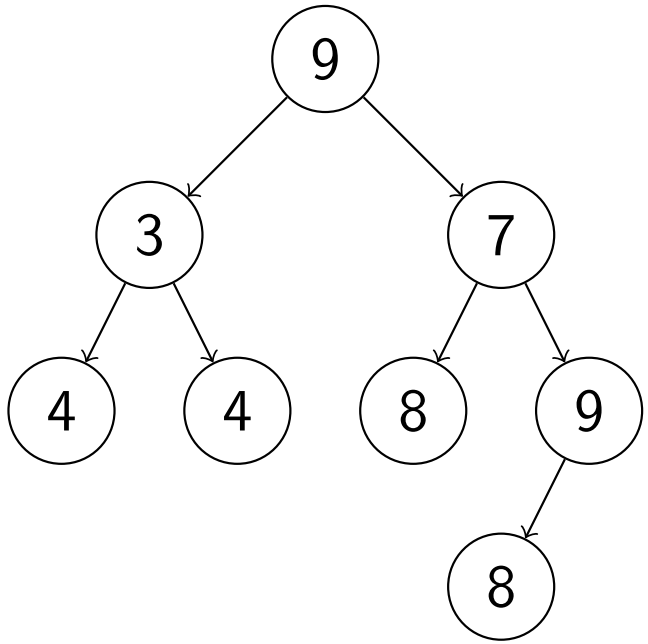
Binary Tree Structure

Binary Tree: Just a tree where each node has *at most two children*.

```
class IntTree{
    static class IntTreeNode{
        int data;
        IntTreeNode left;
        IntTreeNode right;

        IntTreeNode(int data){
            //..
        }
    }

    IntTreeNode root;
    //..
}
```



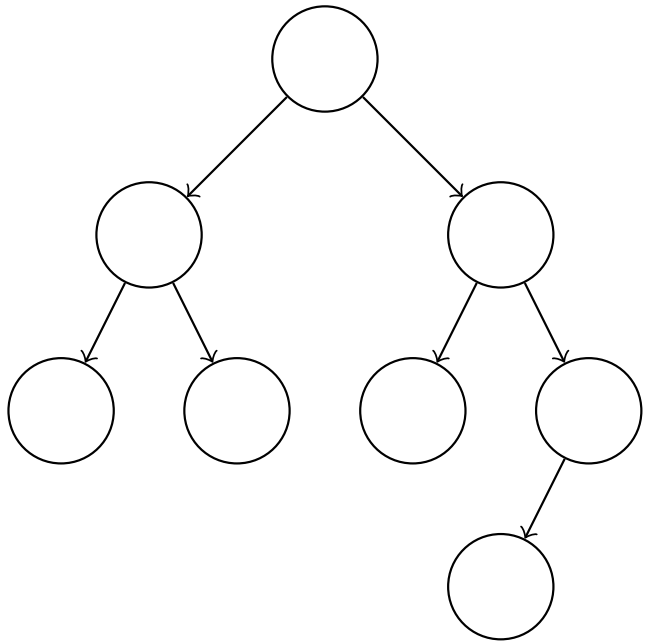
Binary Tree Structure

Binary Tree: Just a tree where each node has *at most two children*.

```
class GenericTree<E>
    class GenericTreeNode<E>{
        E data;
        GenericTreeNode<E> left;
        GenericTreeNode<E> right;

        GenericTreeNode(E data){
            //..
        }
    }

    GenericTreeNode<E> root;
    //..
}
```



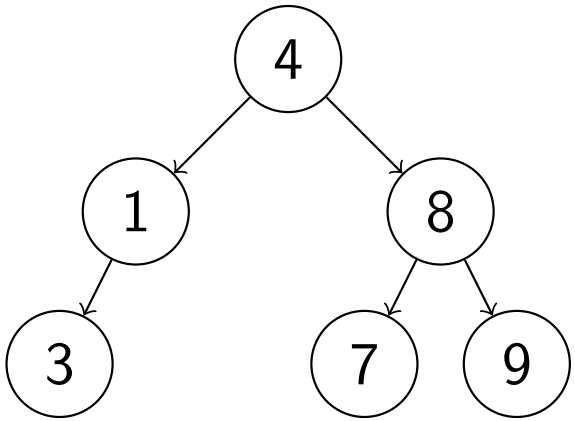
Binary Tree: Manual Construction

```
class Tree{

    IntTreeNode root;

    Tree(){
        root = new IntTreeNode(4);
        root.left = new IntTreeNode(1);
        root.left.left = new IntTreeNode(3);
        root.right = new IntTreeNode(8);
        root.right.left = new IntTreeNode(7);
        root.right.right = new IntTreeNode(9);
    }

    public static void main(String[] args){
        Tree myTree = new Tree();
    }
}
```



Recursive Traversal Algorithms

```
Function Inorder(curr)
    if curr is null then
        return
    end
    Inorder(node.left)
    print(node)
    Inorder(node.right)
return
```

Inorder: left, visit, right

Recursive Traversal Algorithms

```
Function Inorder(curr)
    if curr is null then
        return
    end
    Inorder(node.left)
    print(node)
    Inorder(node.right)
return
```

Inorder: left, visit, right

```
Function Preorder(curr)
    if curr is null then
        return
    end
    print(node)
    Preorder(node.left)
    Preorder(node.right)
return
```

Preorder: visit, left, right

Recursive Traversal Algorithms

```
Function Inorder(curr)
    if curr is null then
        return
    end
    Inorder(node.left)
    print(node)
    Inorder(node.right)
return
```

Inorder: left, visit, right

```
Function Preorder(curr)
    if curr is null then
        return
    end
    print(node)
    Preorder(node.left)
    Preorder(node.right)
return
```

Preorder: visit, left, right

```
Function Postorder(curr)
    if curr is null then
        return
    end
    Postorder(node.left)
    Postorder(node.right)
    print(node)
return
```

Postorder: left, right, visit

Recursive Traversal Algorithms

```
Function Inorder(curr)
    if curr is null then
        return
    end
    Inorder(node.left)
    print(node)
    Inorder(node.right)
return
```

Inorder: left, visit, right

```
Function Preorder(curr)
    if curr is null then
        return
    end
    print(node)
    Preorder(node.left)
    Preorder(node.right)
return
```

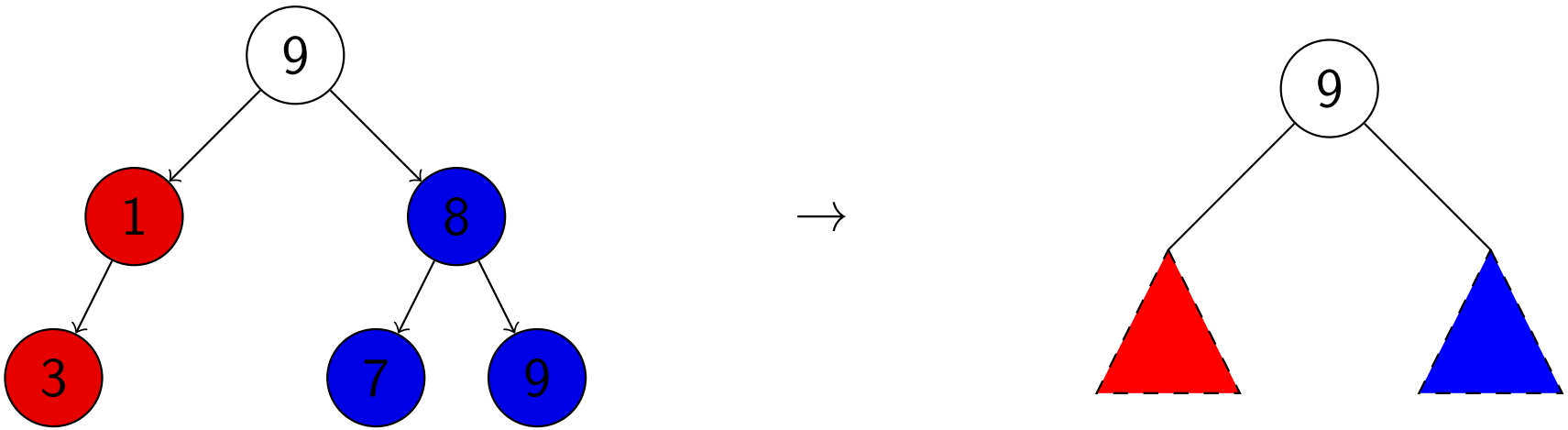
Preorder: visit, left, right

```
Function Postorder(curr)
    if curr is null then
        return
    end
    Postorder(node.left)
    Postorder(node.right)
    print(node)
return
```

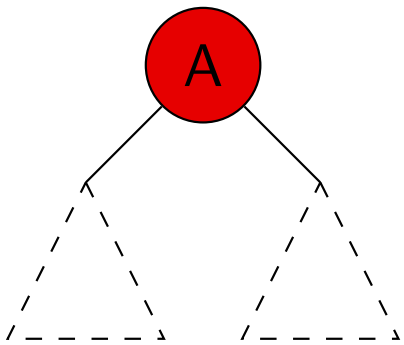
Postorder: left, right, visit

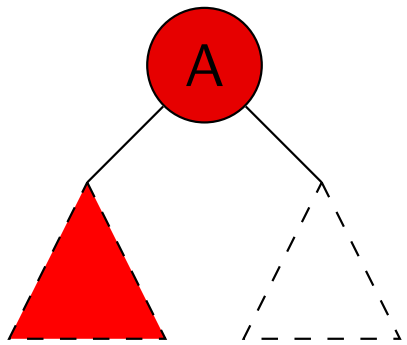
Left, right, visit!? What does it mean!?

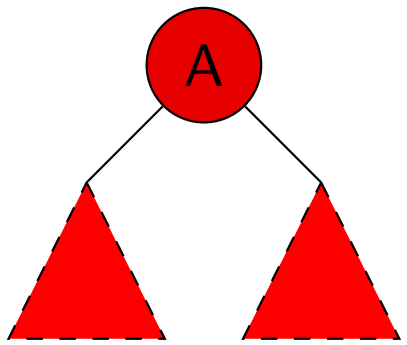
Traversals: Recall Subtrees



Traversals: Preorder Intuition



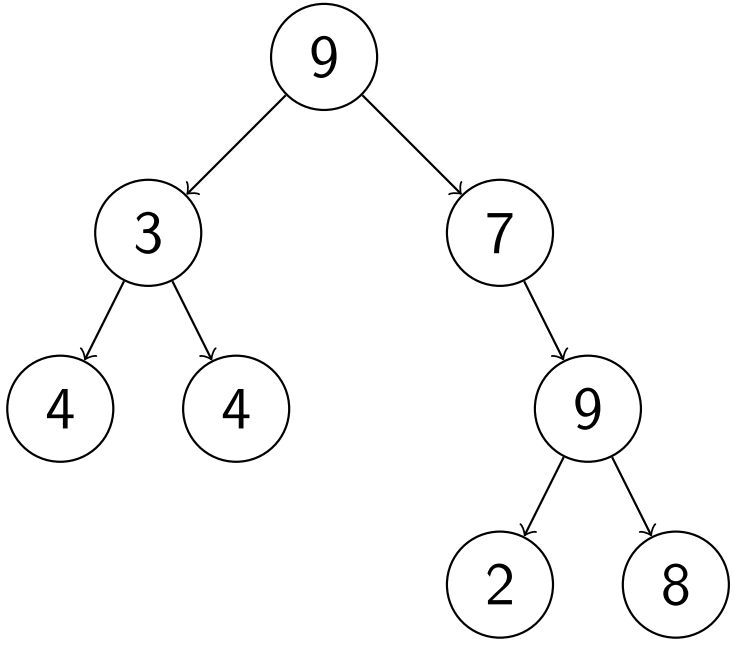




For a given node A visit that node immediatly upon encountering it and then traverse to it's left and right subtrees

Traversals

```
Function Preorder(curr)
    if curr is null then
        return
    end
    print(curr)
    Preorder(curr.left)
    Preorder(curr.right)
return
```



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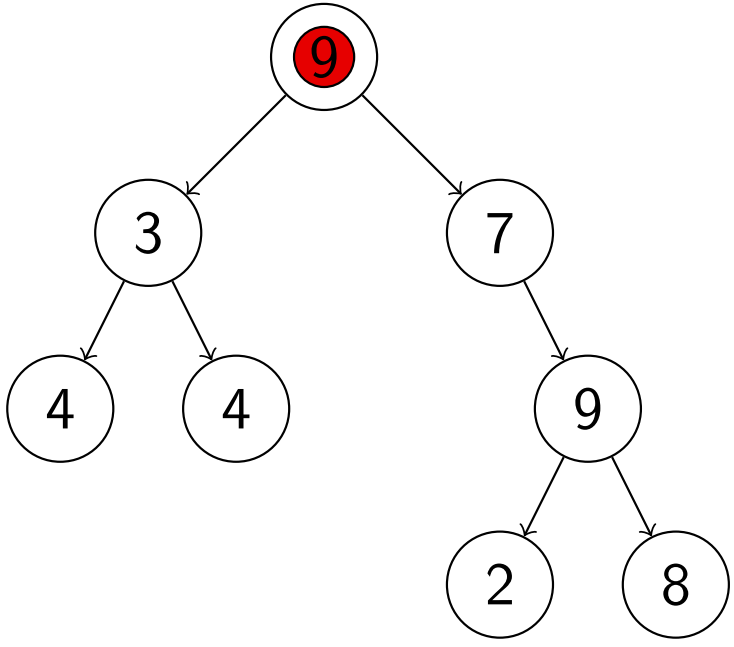

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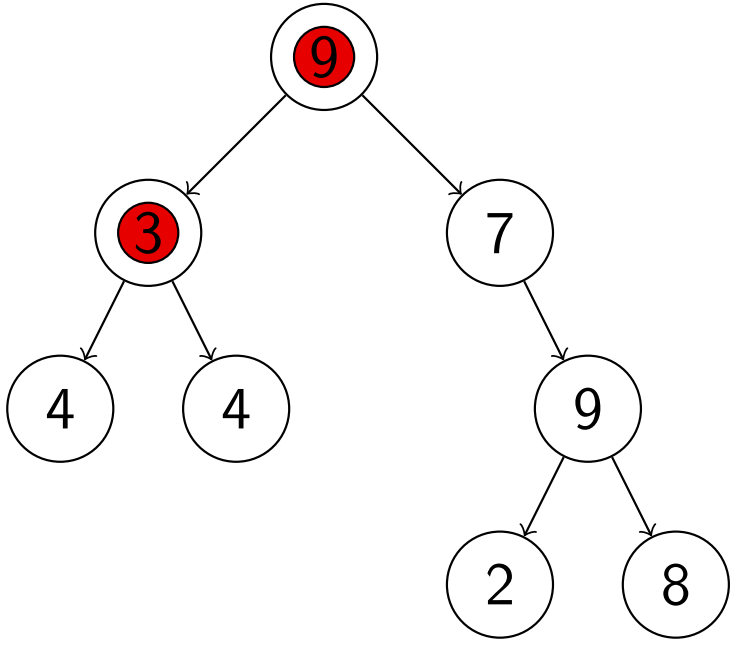

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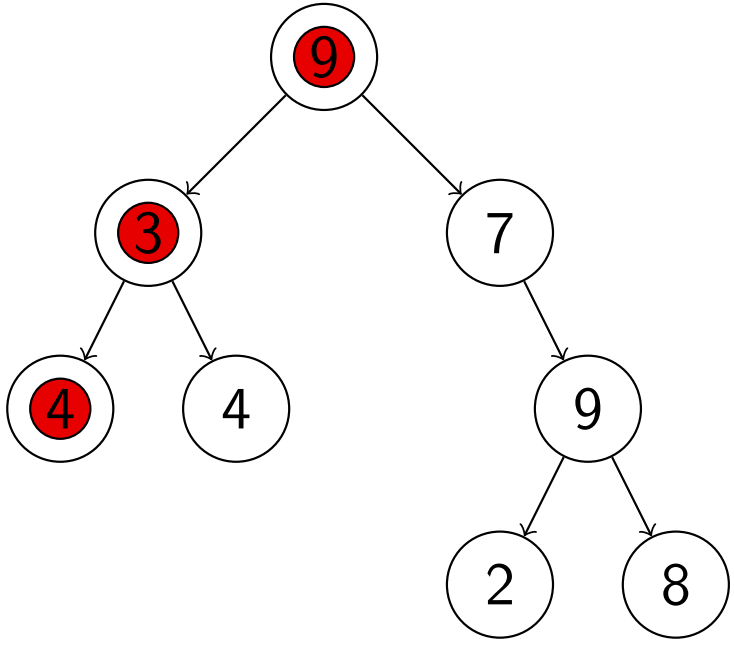

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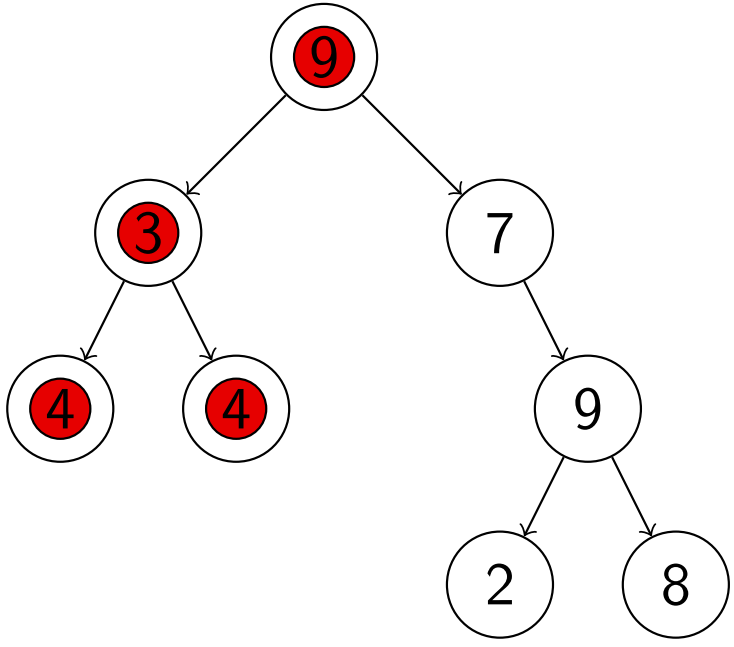

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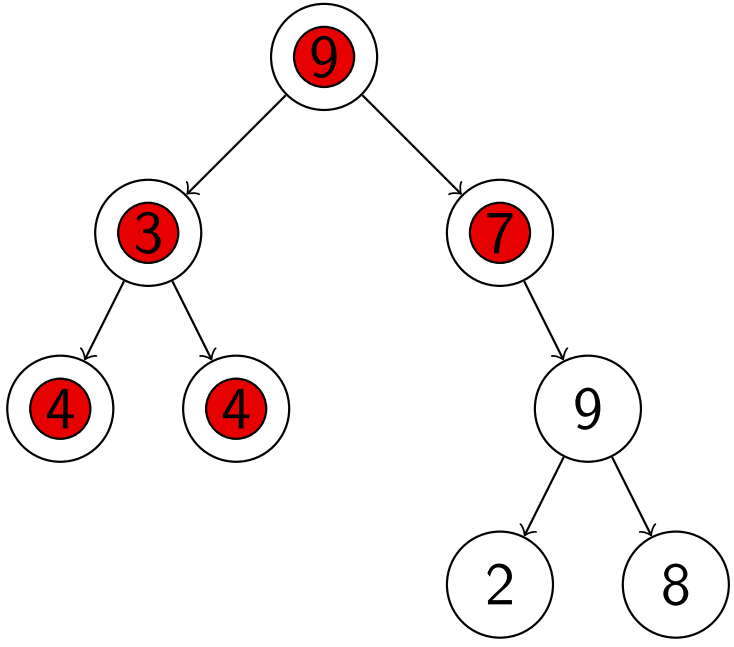

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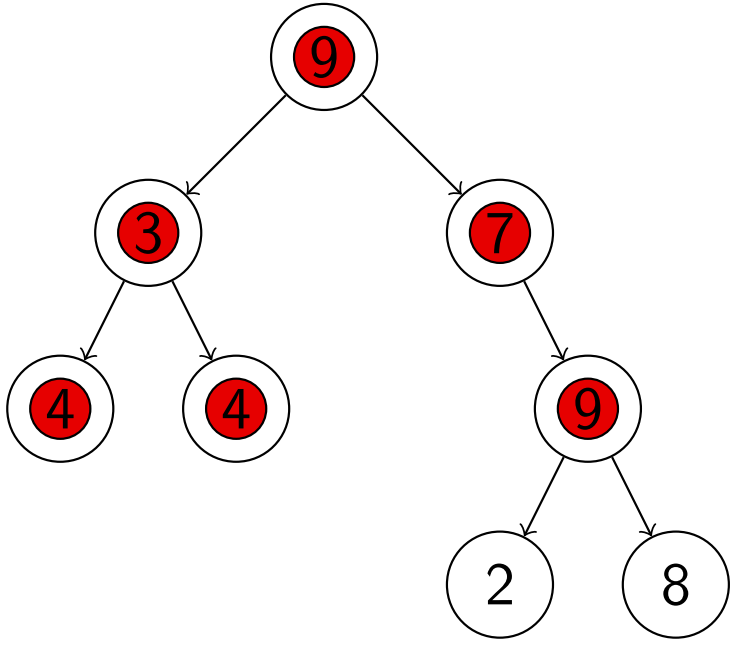

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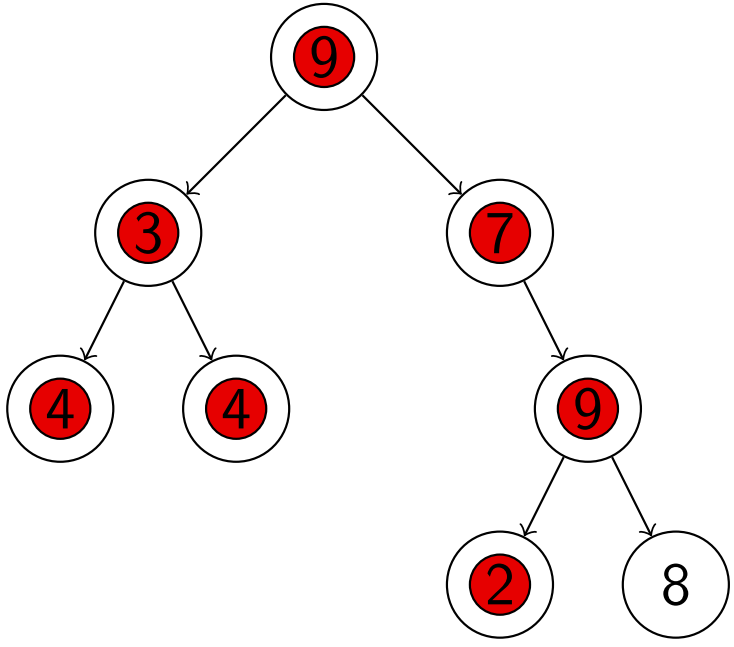

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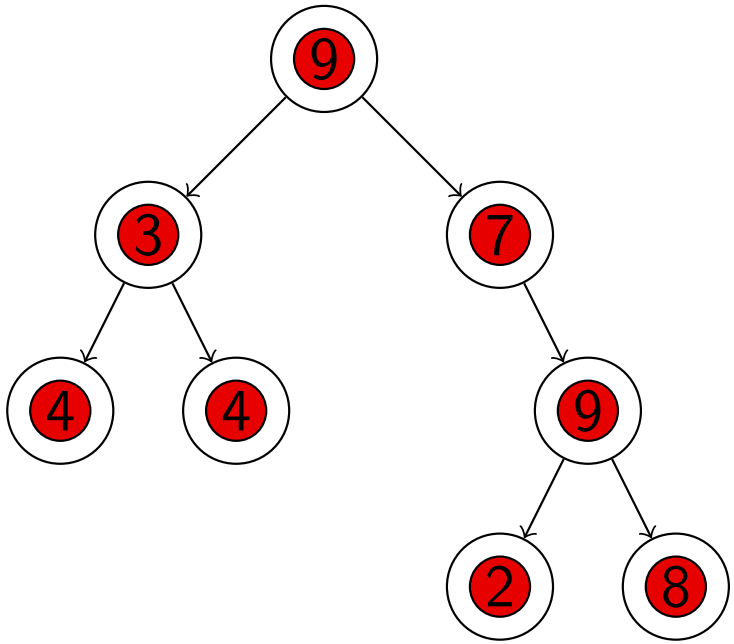
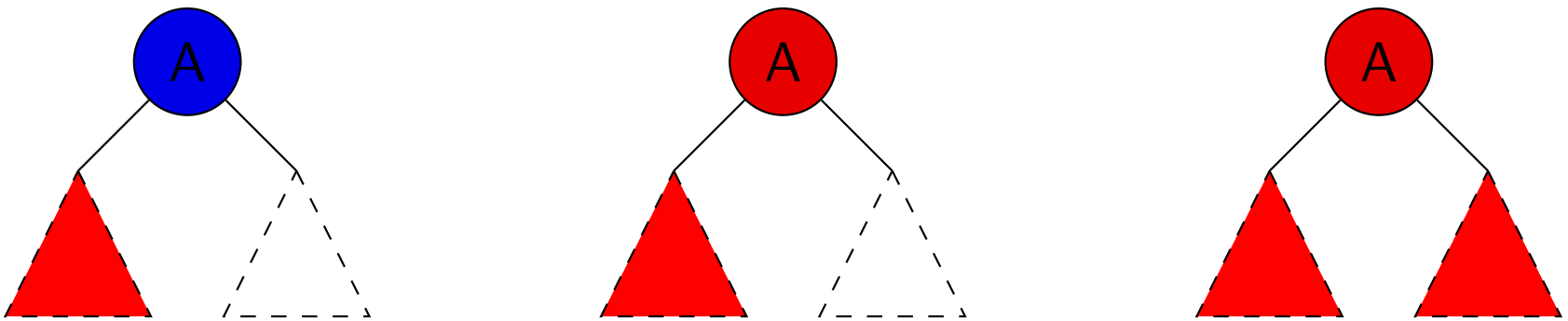


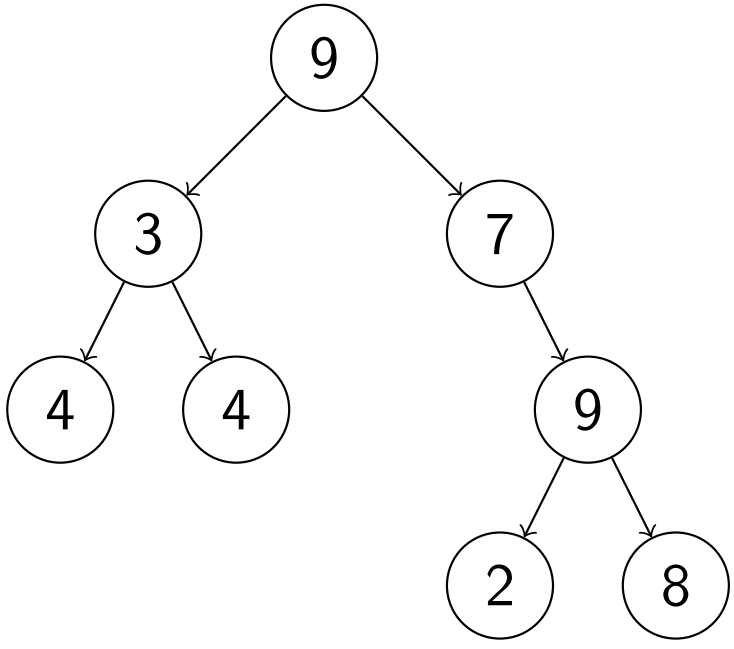
Traversals: Inorder Intuition



For a given node A, all nodes in A's left subtree must be visited before we visit A.

Traversals

```
Function Inorder(curr)
    if curr is null then
        return
    end
    Inorder(curr.left)
    print(curr)
    Inorder(curr.right)
return
```



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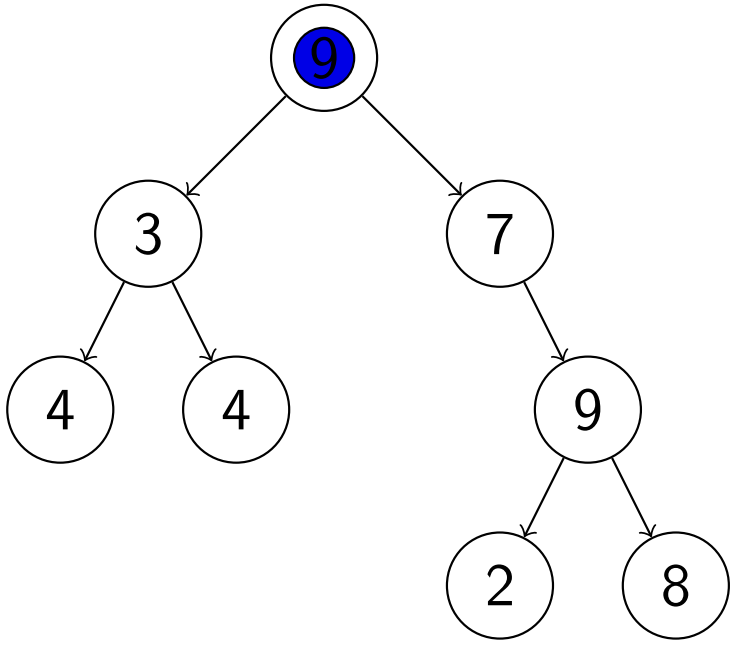

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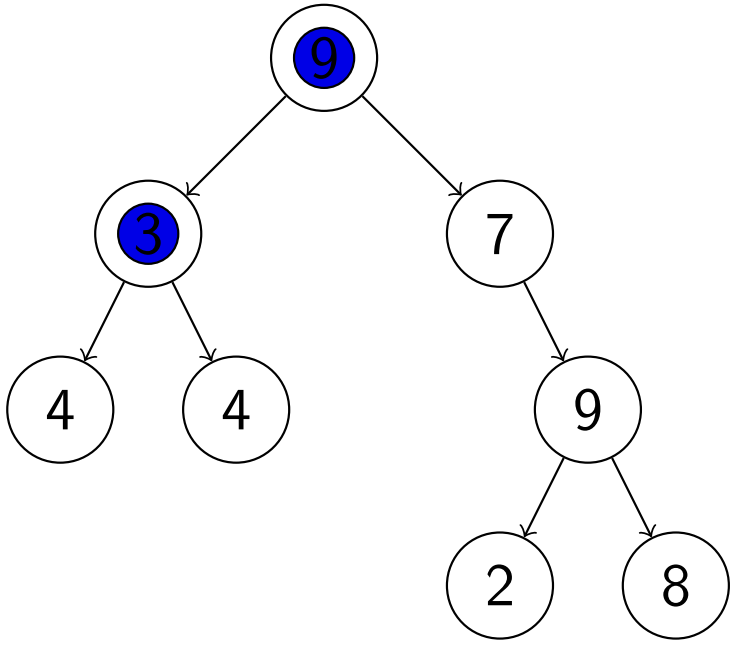

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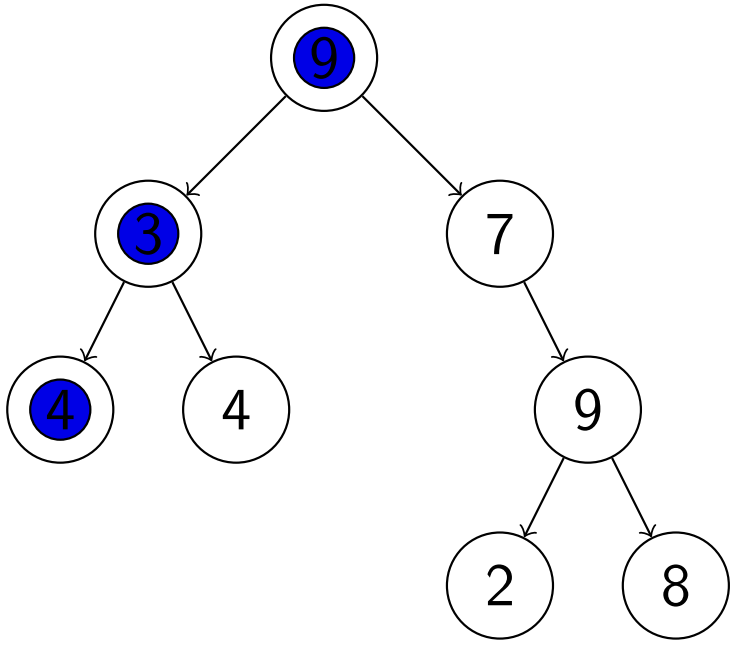

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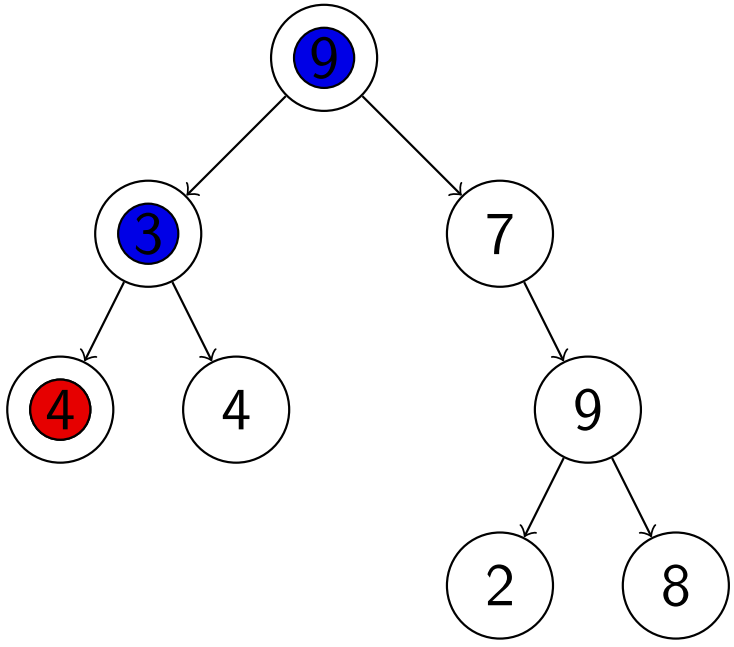

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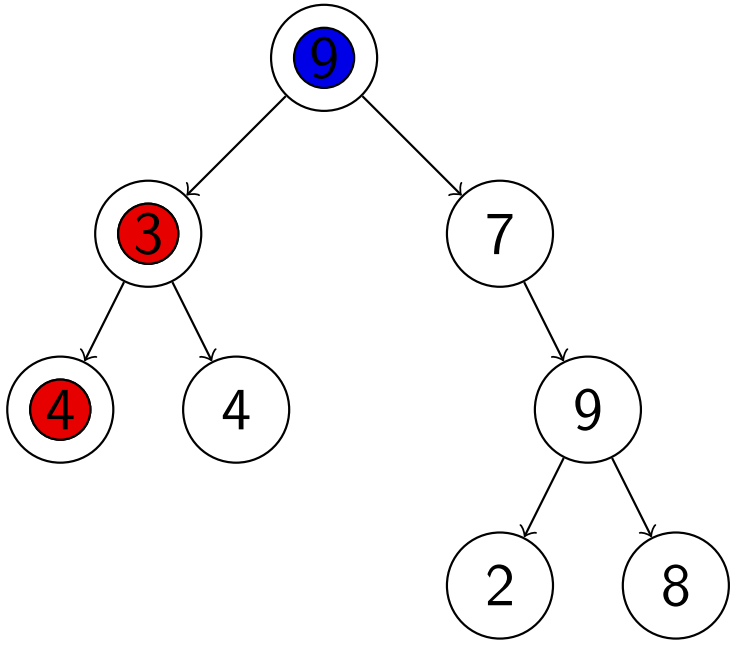

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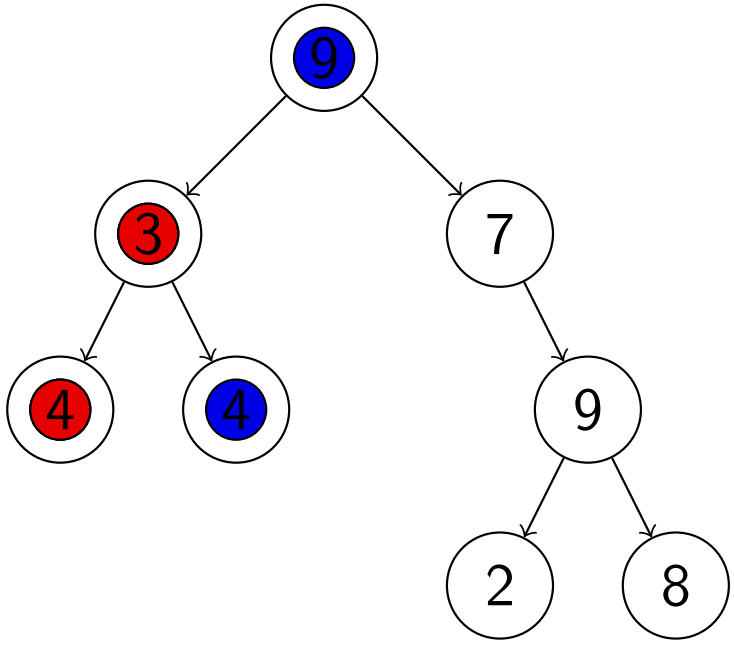

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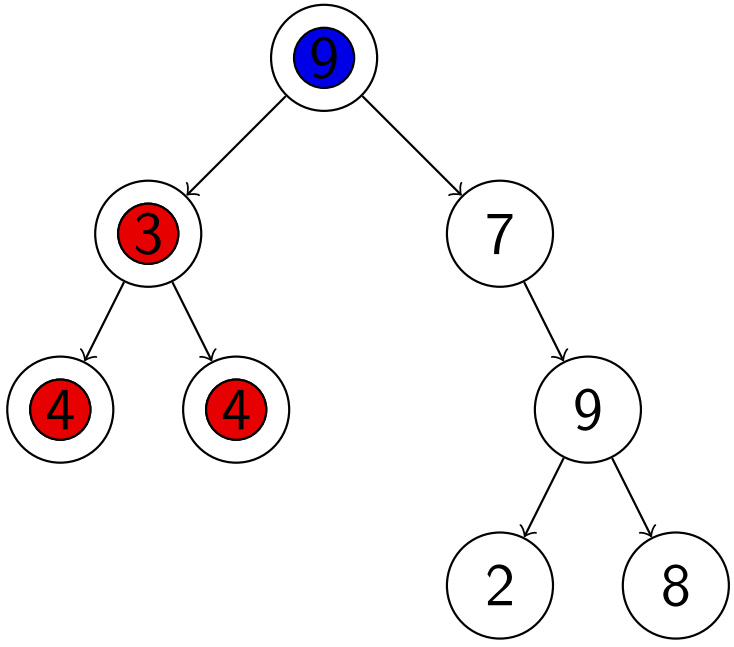

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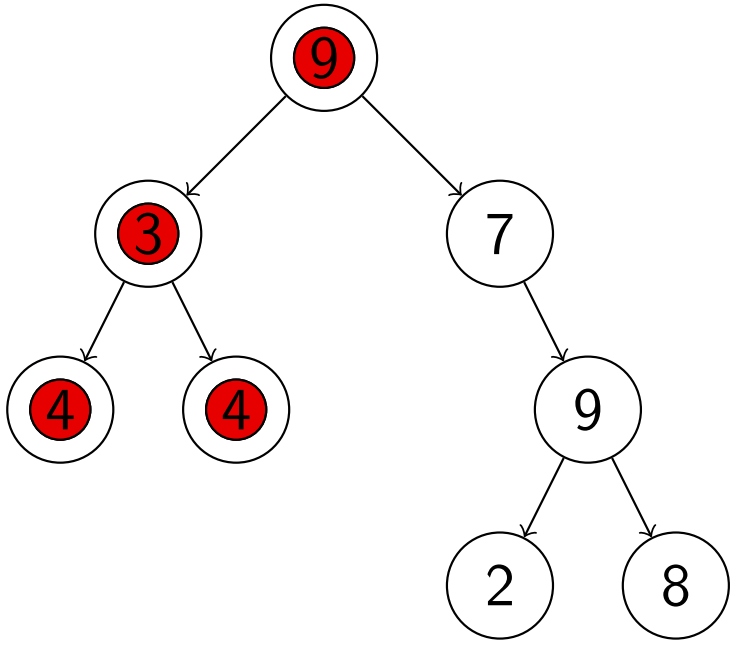

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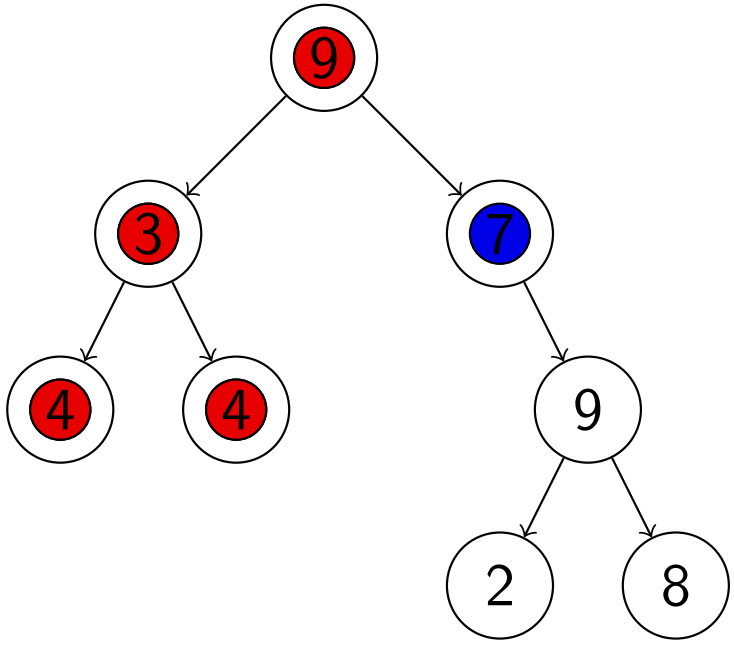

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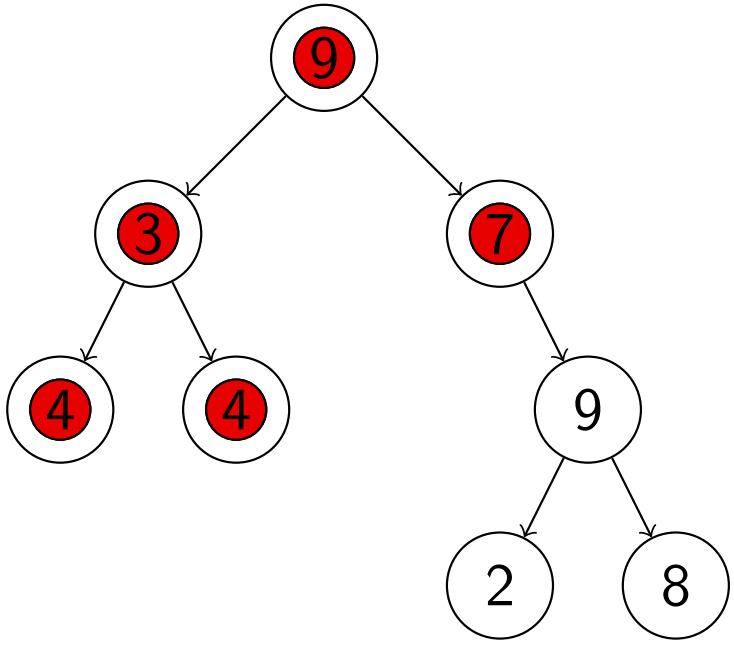

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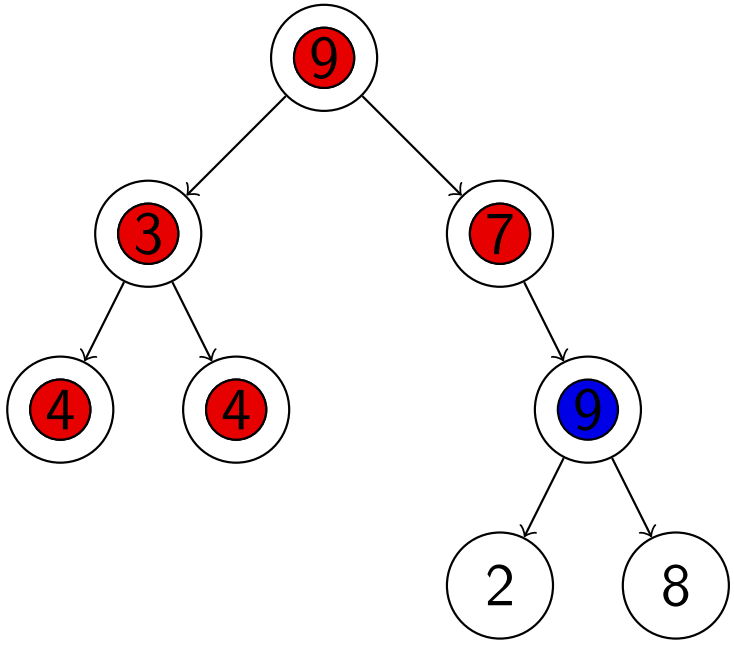

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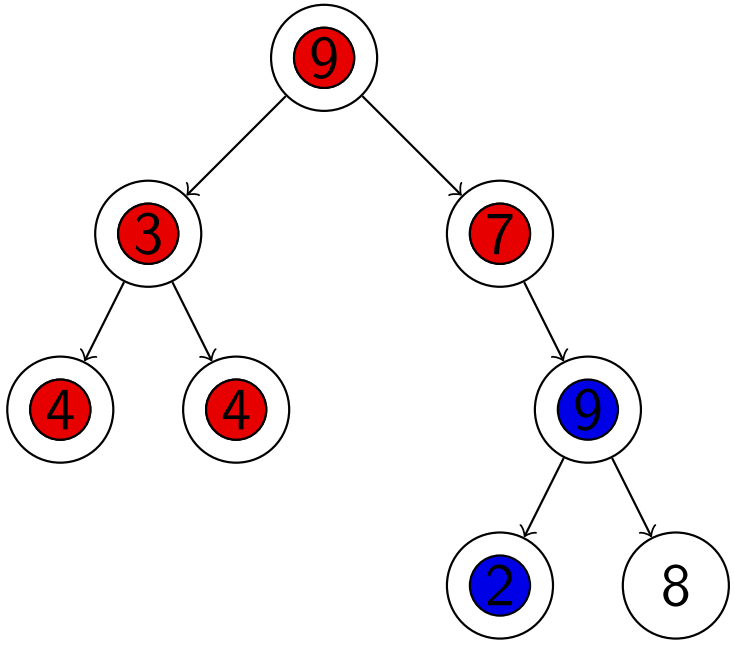

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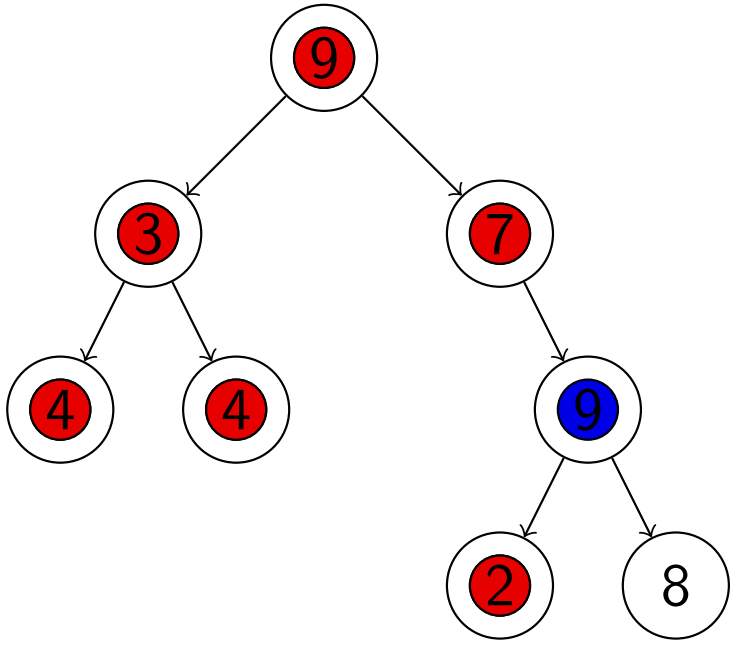

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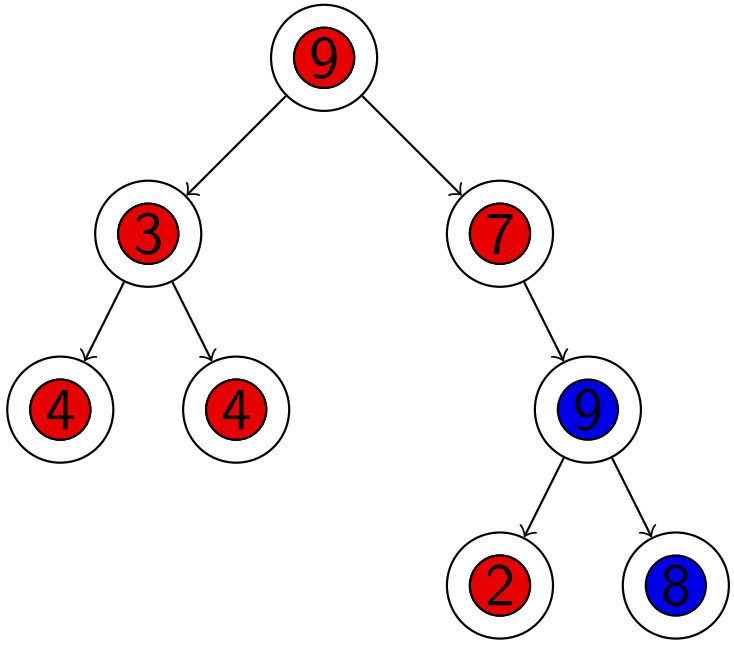

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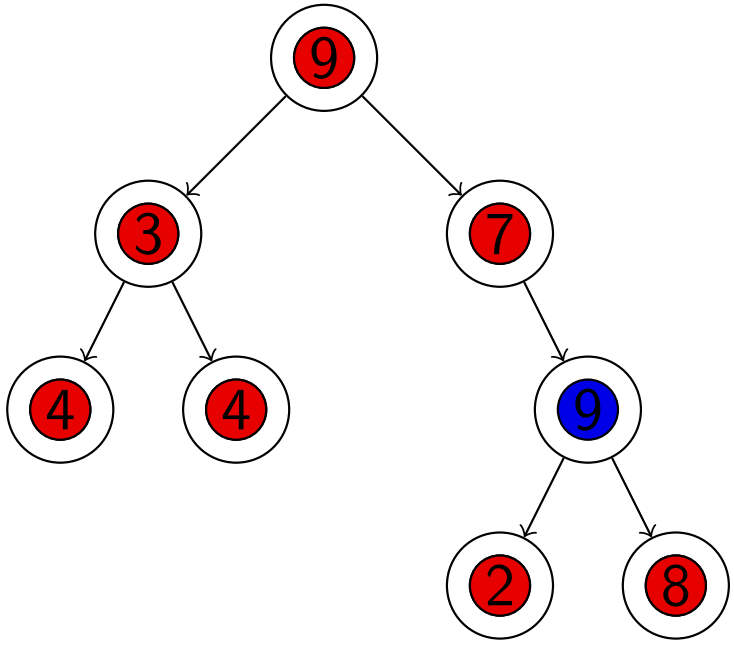

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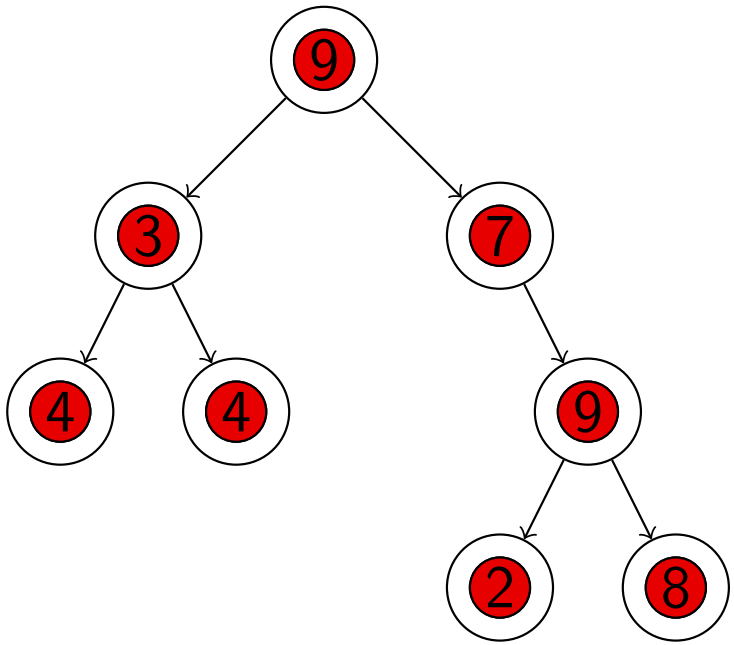
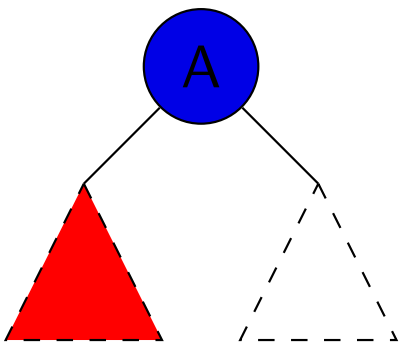
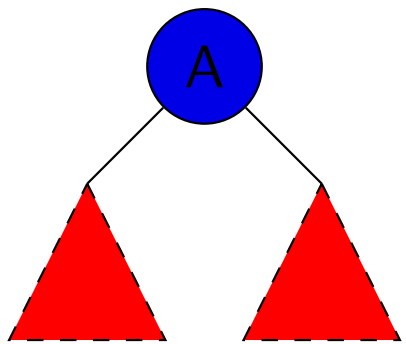
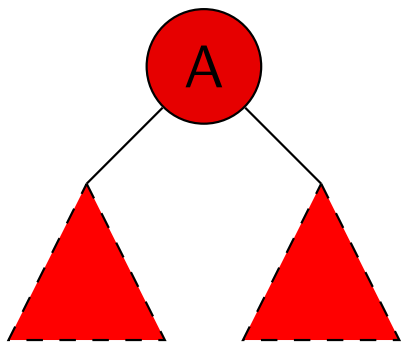


Traversals: Postorder Intuition



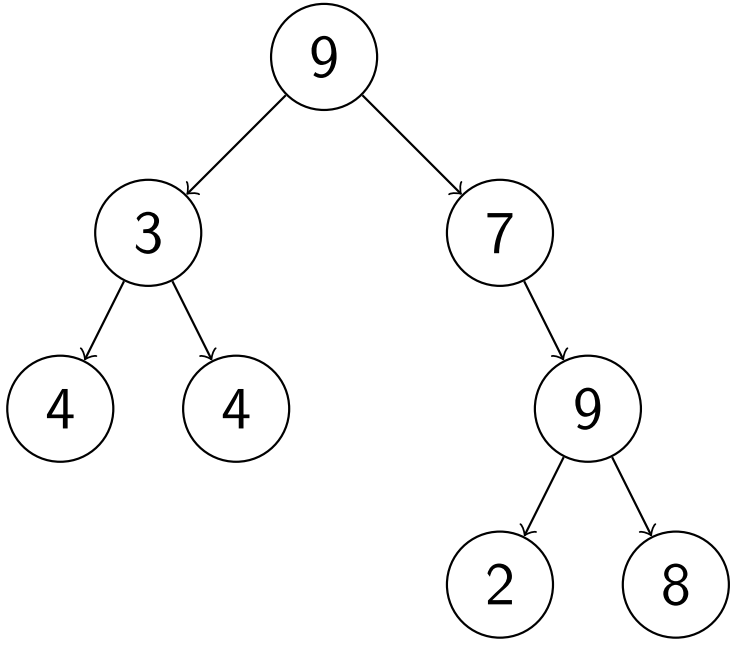




For a given node A, all of the nodes in A's left and right subtrees must be visited before A can be visited.

Traversals: Postorder

```
Function Postorder(curr)
    if curr is null then
        return
    end
    Postorder(curr.left)
    Postorder(curr.right)
    print(node)
return
```



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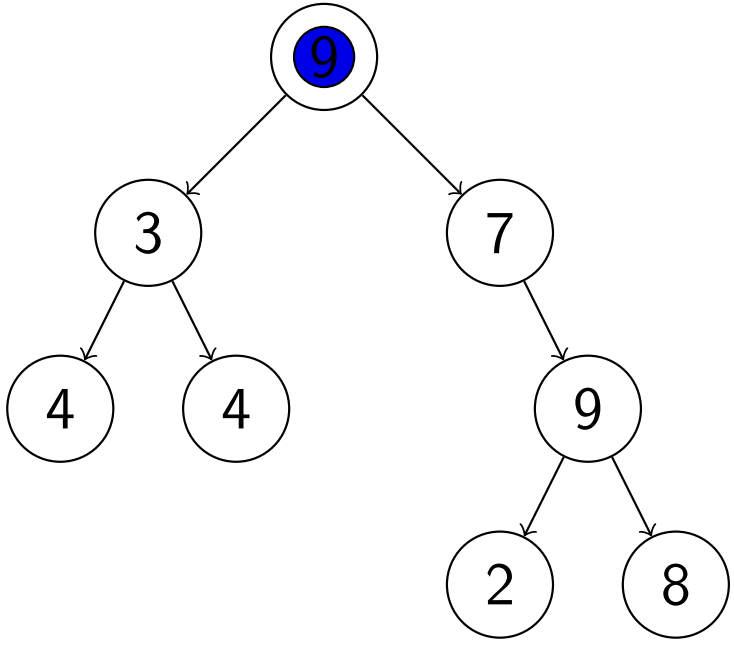

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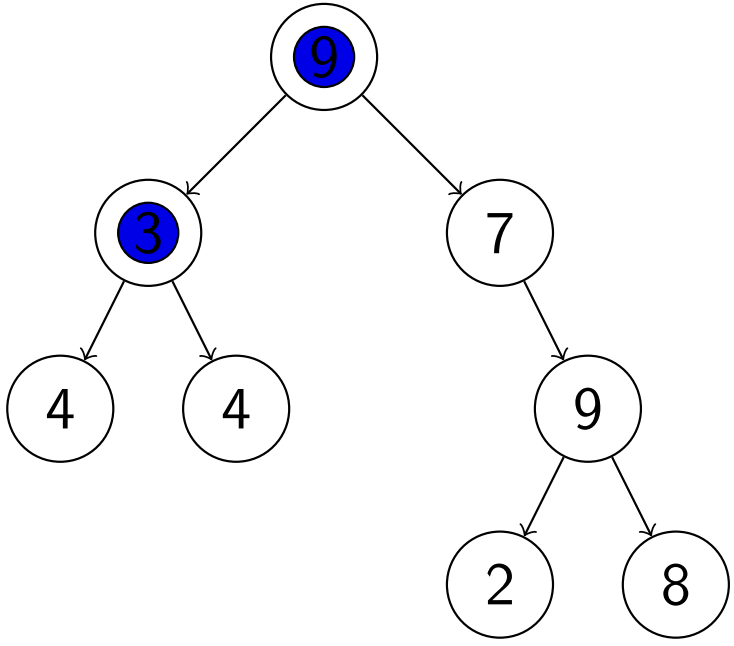

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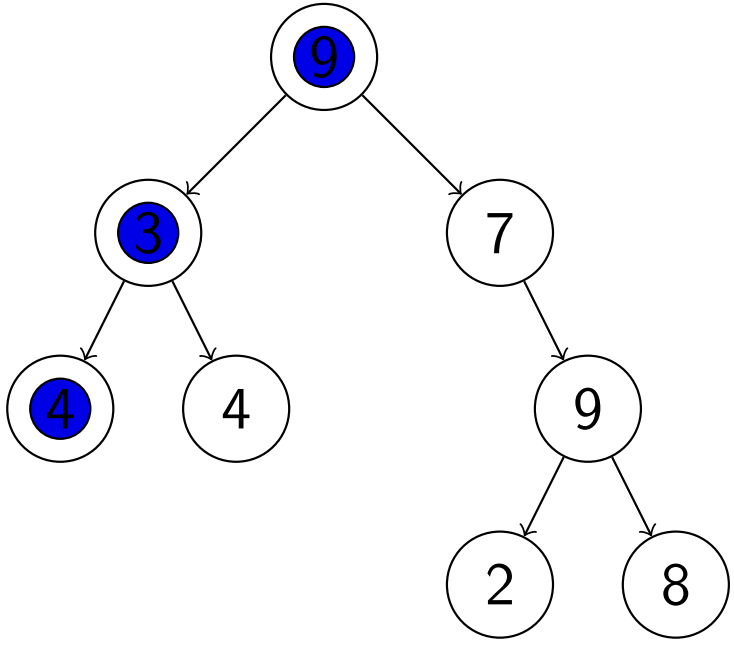

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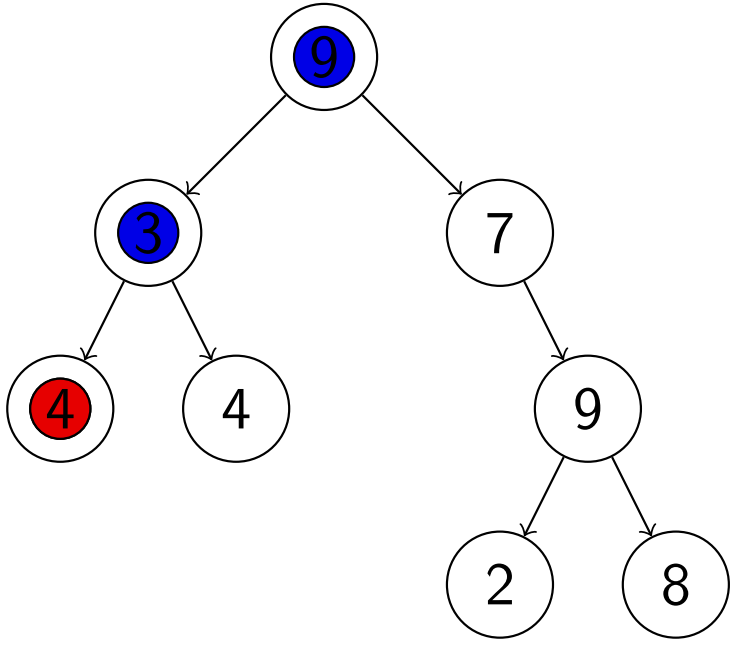

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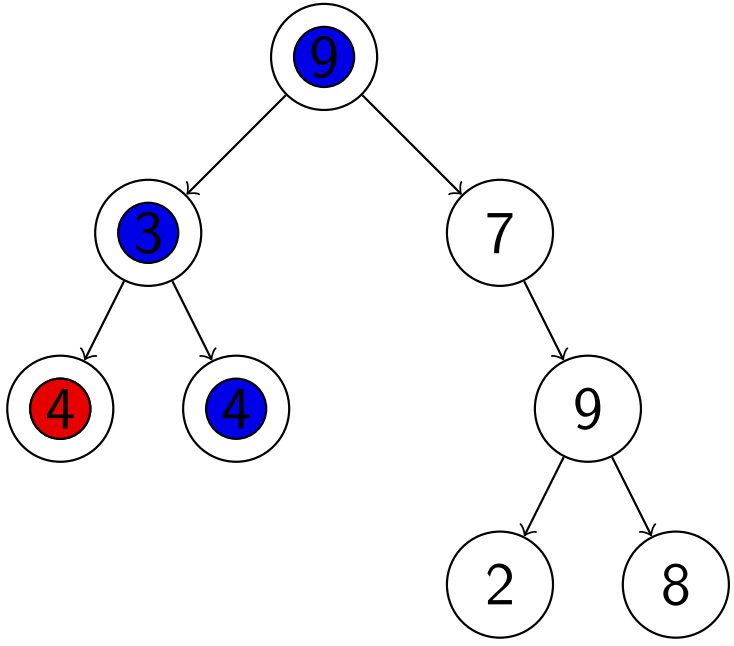

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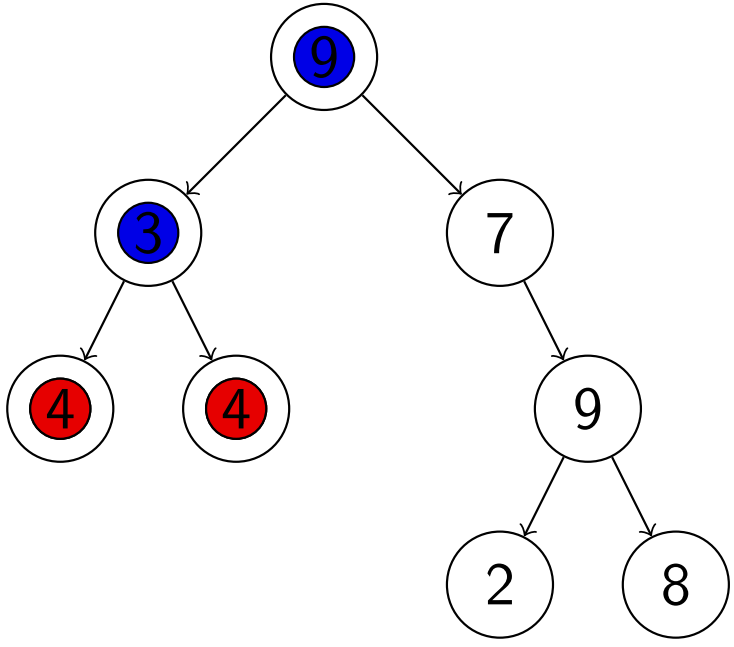

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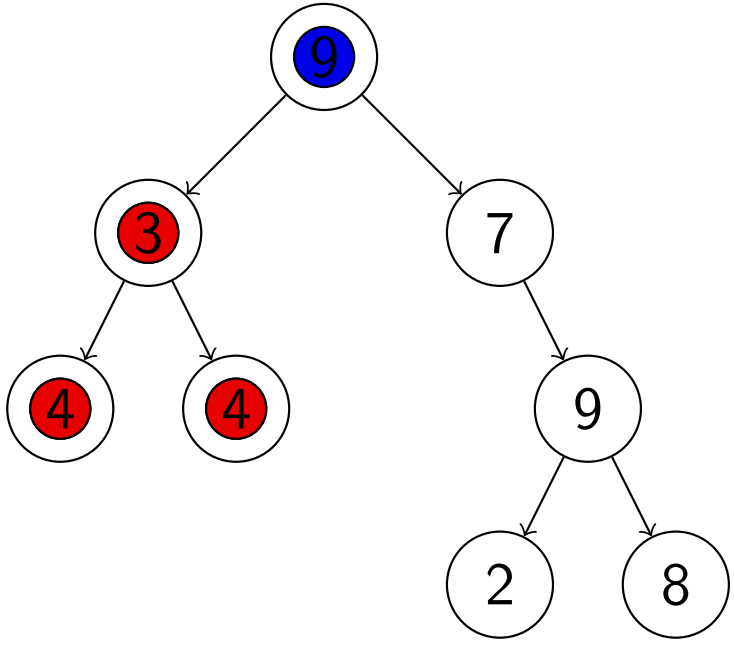

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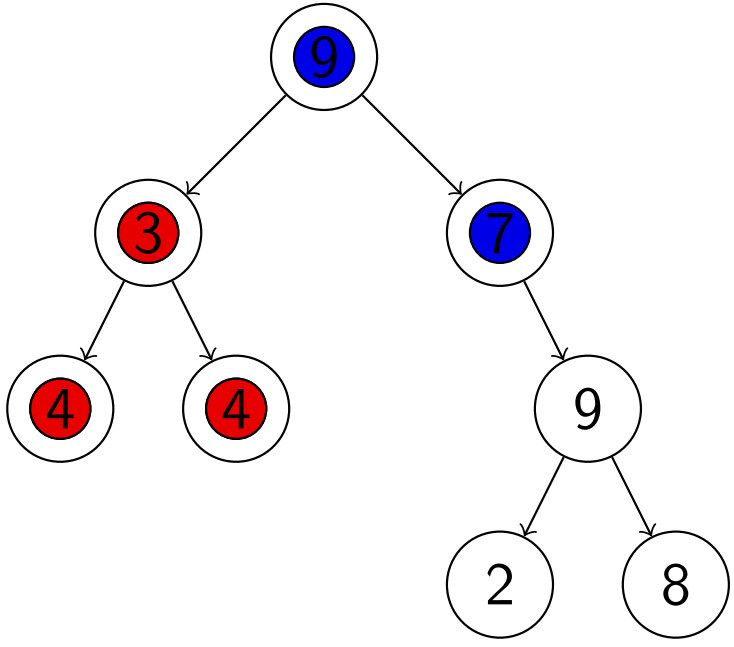

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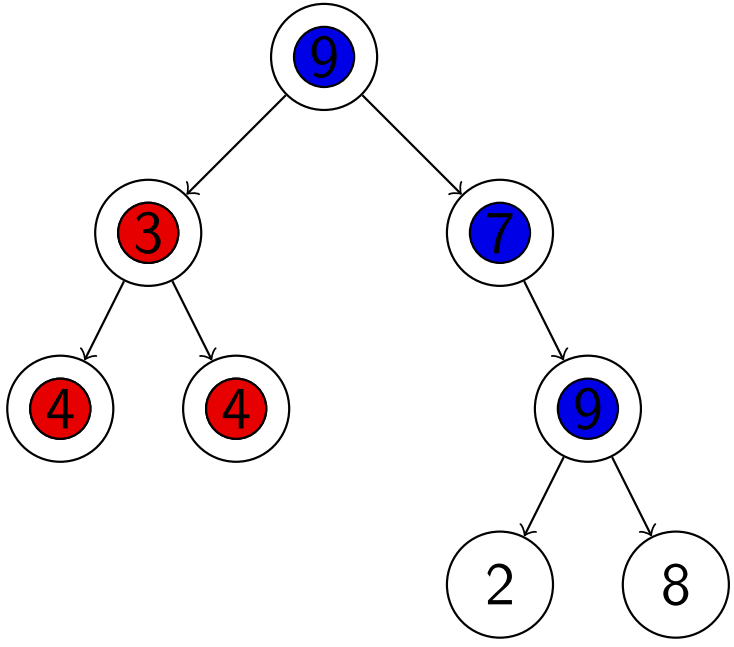

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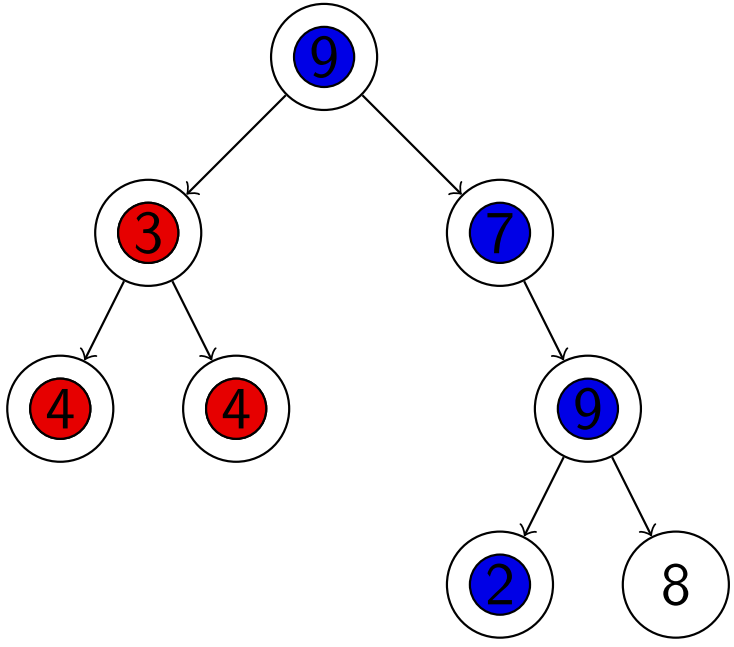

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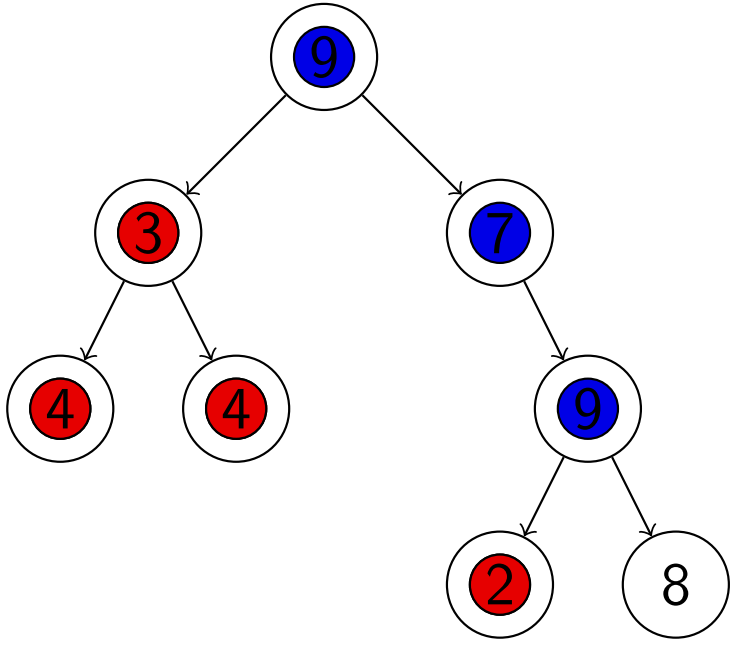

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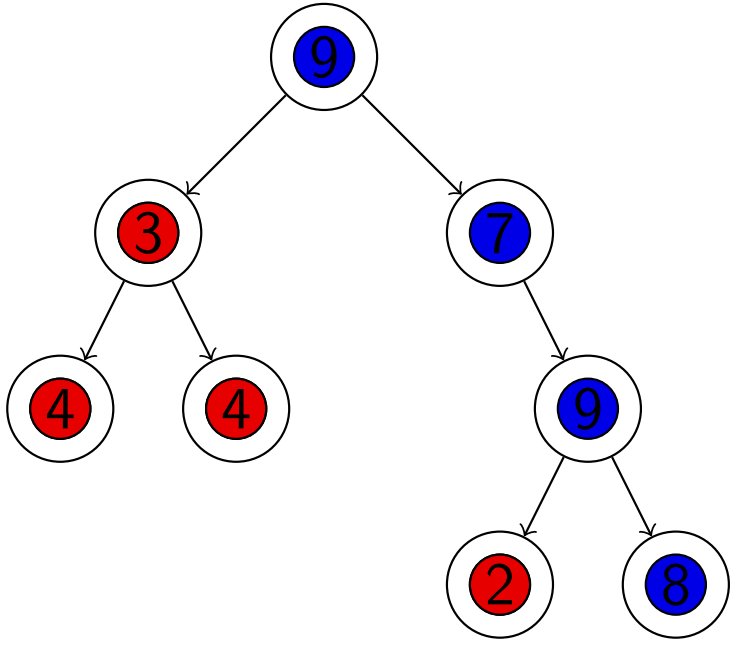

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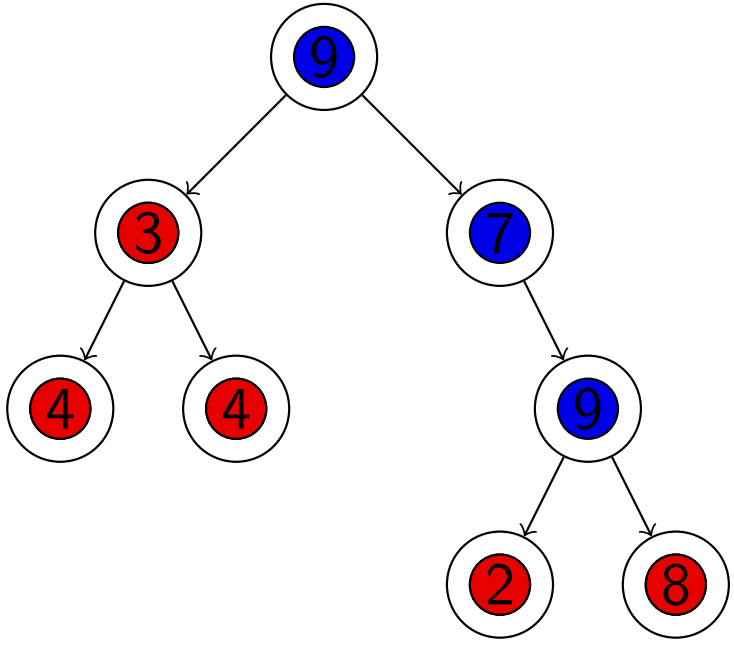

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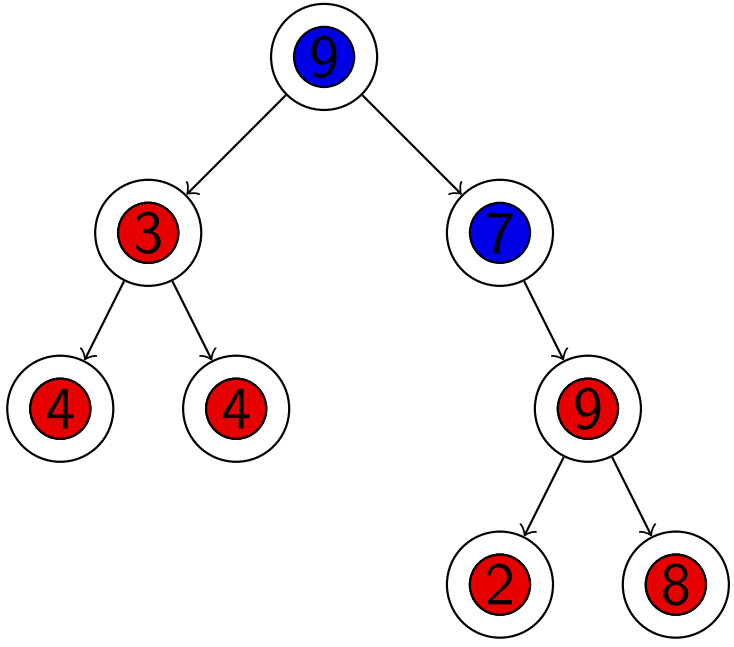

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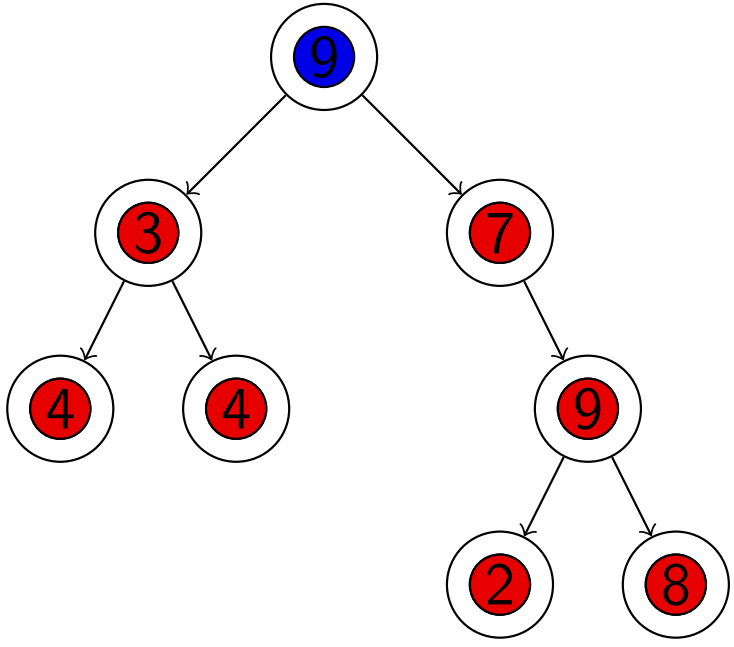

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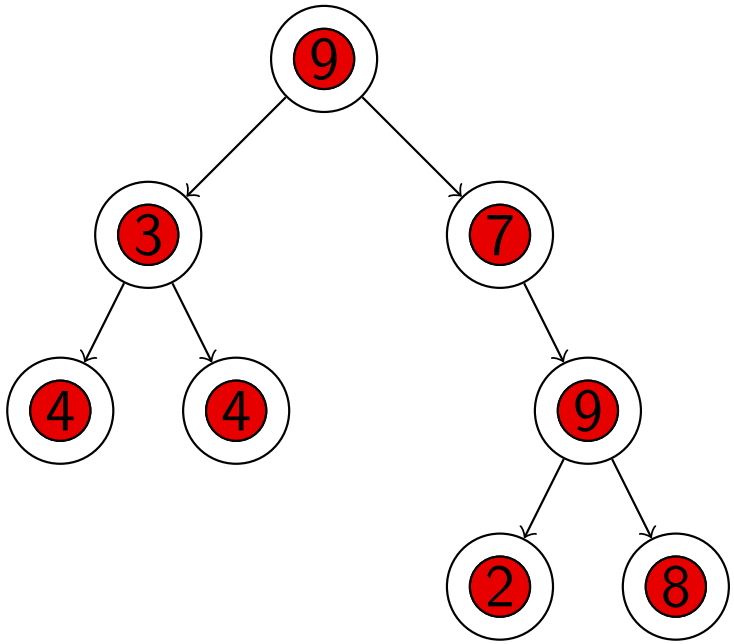
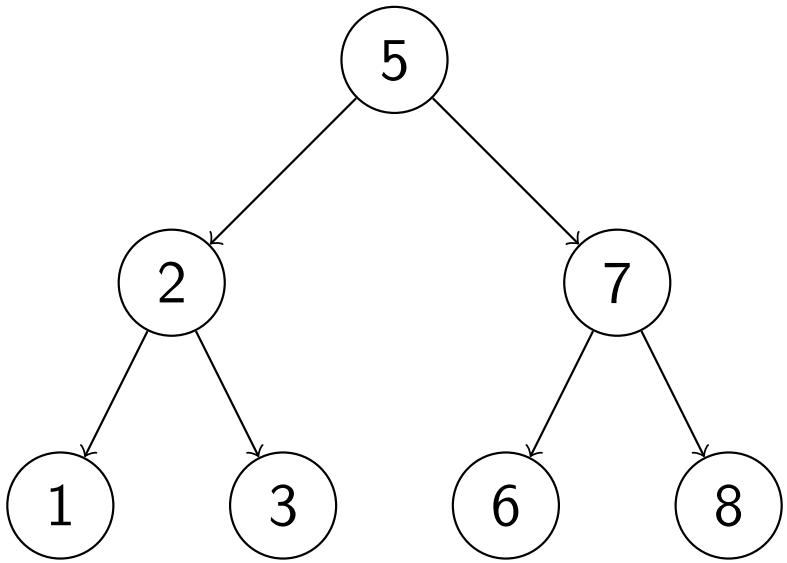


Binary Search Tree: Structure

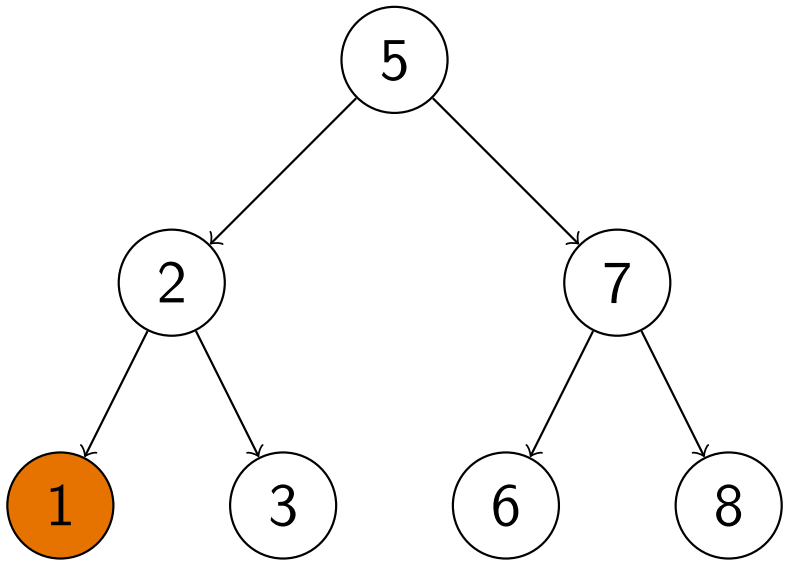
Binary Search Tree: A tree where:

1. each node has *at most two children*.
2. the value of a given node is greater than the value of every node in it's left subtree.
3. the value of a given node is less than the value of every node in it's right subtree.



Binary Tree: Terms and Basic Operations

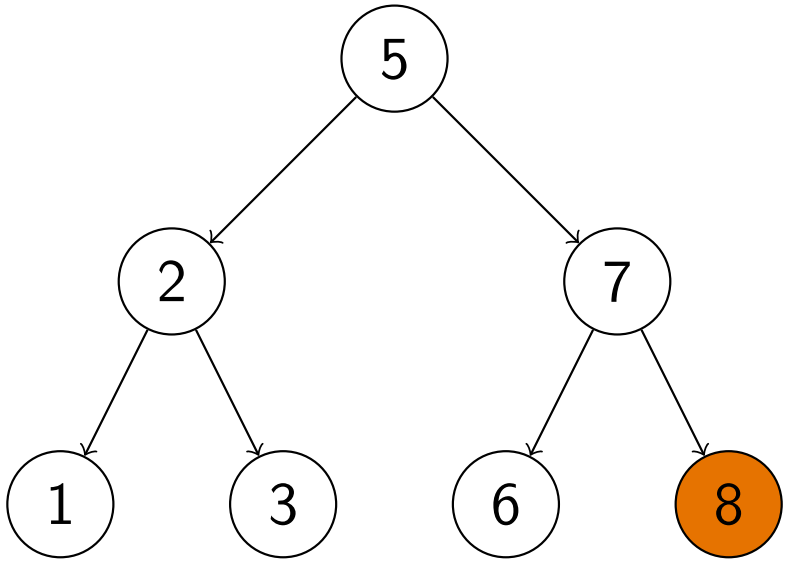
Function `Minimum(`*curr*`)`
  while *curr.left* $\neq$ *null* **do**
    curr = curr.left
  end
return curr



Minimum Node: Starting at a given node *curr*, the node with the least value under *curr* will be the leftmost node in it's left subtree.

Binary Search Tree: Maximum Node

Function `Maximum(`*curr*`)`
  while *curr.right* $\neq$ *null* **do**
    curr = curr.right
  end
return curr



Maximum Node: Starting at a given node *curr*, the node with the greatest value under *curr* will be the rightmost node in it's right subtree.

Binary Search Tree: Finding a Node

```
Function RecursiveSearch(Curr, Val)
    if Curr = NULL OR Val = Curr.Val then
        return curr
    end
    if Val < Curr.Val then
        return RecursiveSearch(Curr.Left, Val)
    end
    else
        return RecursiveSearch(Curr.Right, Val)
    end
return
```

- Though this can be done iteratively or recursively, both methods follow the pattern of finding and returning a reference to a node with a given value.

Binary Search Tree: Finding a Node

```
Function RecursiveSearch(Curr, Val)
    if Curr = NULL OR Val = Curr.Val then
        return curr
    end
    if Val < Curr.Val then
        return RecursiveSearch(Curr.Left, Val)
    end
    else
        return RecursiveSearch(Curr.Right, Val)
    end
return
```

- Though this can be done iteratively or recursively, both methods follow the pattern of finding and returning a reference to a node with a given value.
- Searching will also form the basis for adding and removing nodes.

Adding a Node

```
Function Add(Data)
    Root = Add(Root, Data);
return

Function Add(Curr)
    if Curr = NULL then
        return TreeNode(Data)
    end
    else if Data > Curr.Data then
        Curr.Right ← Add(Curr.Right, Data)
    end
    else
        Curr.Left ← Add(Curr.Left, Data)
    end
    return Curr
return
```

- **Case 1 (red):** Curr is null so we create and return the new node.
- **Case 2 (blue):** Curr isn't null so we need to keep traversing according to BST rules.

Binary Search Tree: Removing a Node

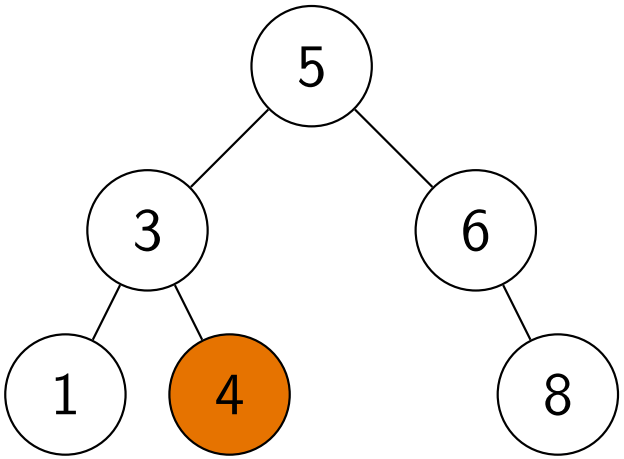


Figure 1: Leaf

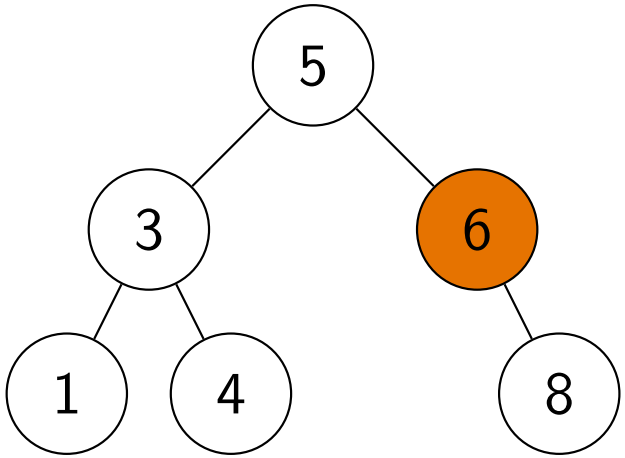


Figure 2: One Subtree

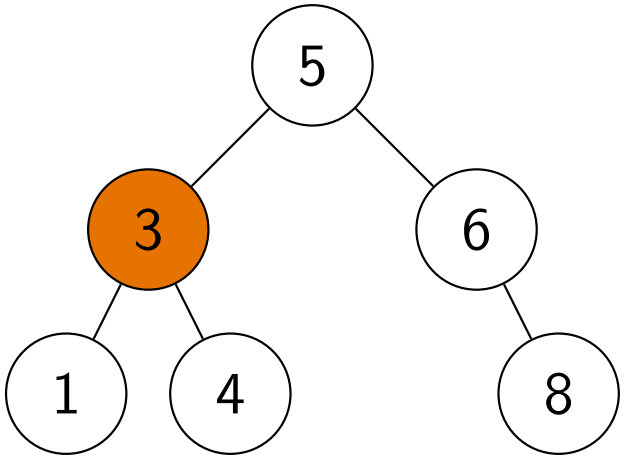
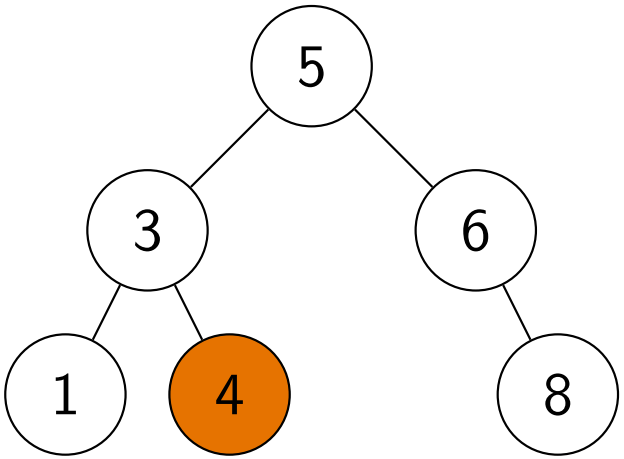


Figure 3: Two Subtrees

There are three cases for removing a node:

Binary Search Tree: Removing a Node



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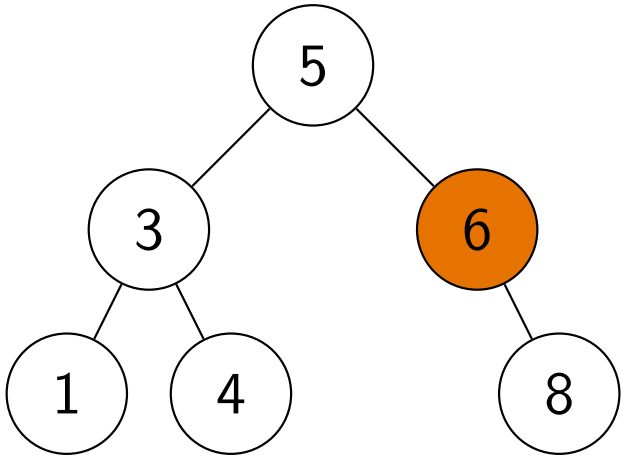


Figure 2: One Subtree

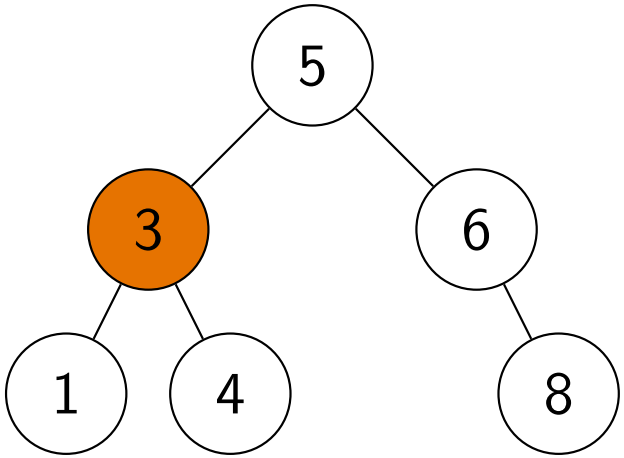
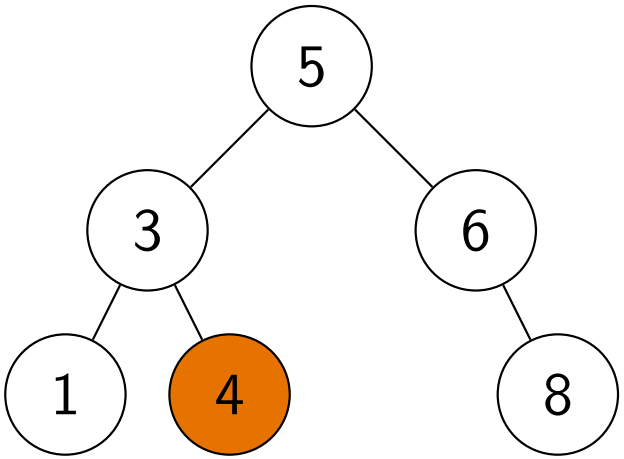


Figure 3: Two Subtrees

There are three cases for removing a node:

1. The node we are removing is a leaf (Figure 1).

Binary Search Tree: Removing a Node



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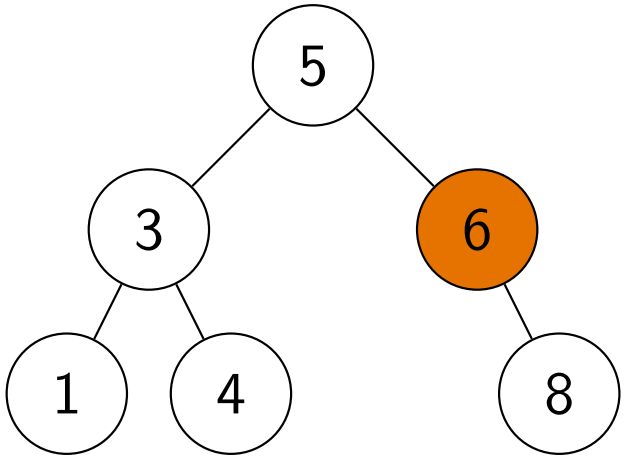


Figure 2: One Subtree

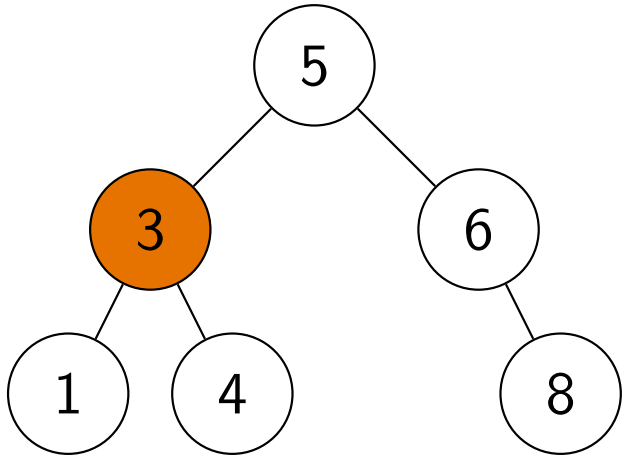
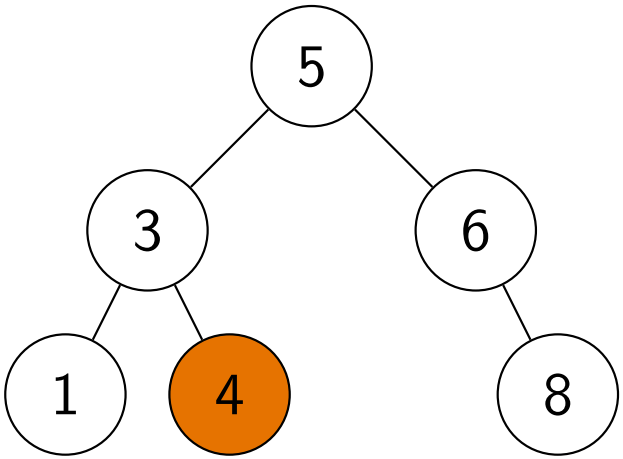


Figure 3: Two Subtrees

There are three cases for removing a node:

1. The node we are removing is a leaf (Figure 1).
2. The node we are removing has one subtree (Figure 2).

Binary Search Tree: Removing a Node



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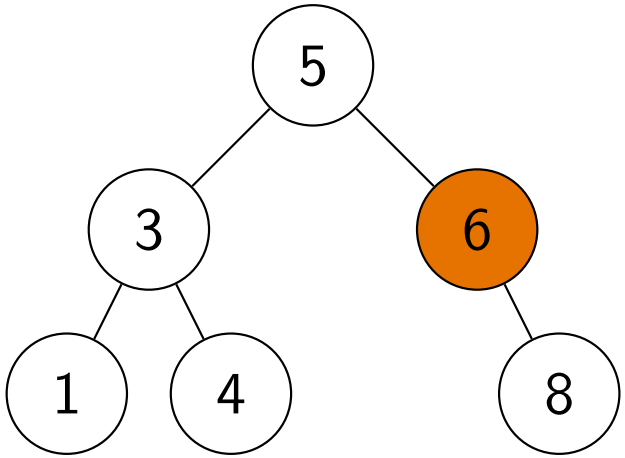

Figure 2: One Subtree

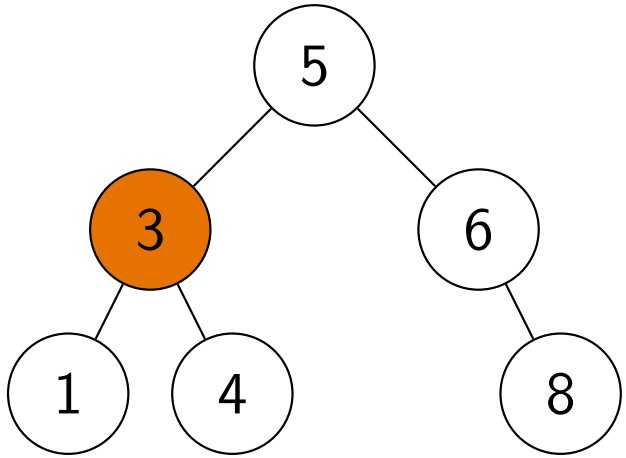

Figure 3: Two Subtrees

There are three cases for removing a node:

1. The node we are removing is a leaf (Figure 1).
2. The node we are removing has one subtree (Figure 2).
3. The node we are removing has two subtrees (Figure 3).

Case 1: Removing a Leaf

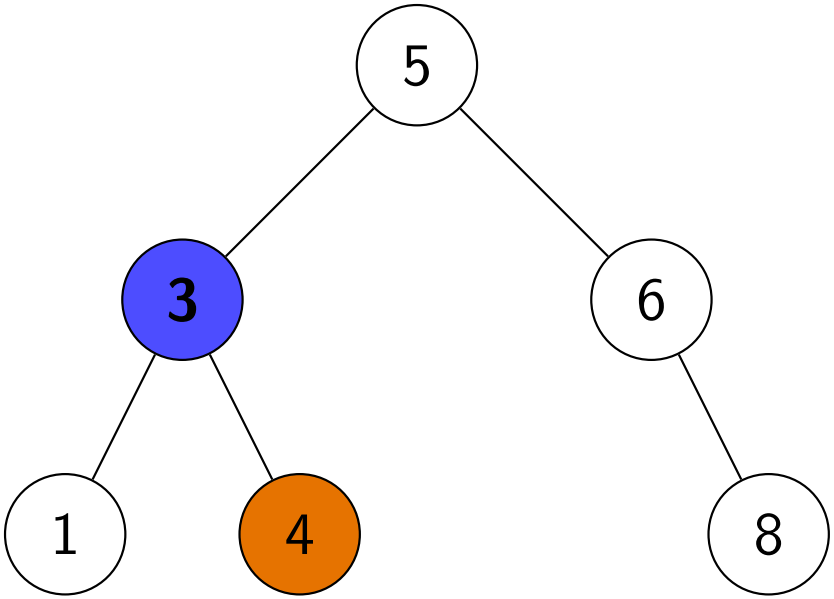

Figure 4: Find the node you want to remove (orange), in this case the one with value 4, and that node's parent (blue)

Case 1: Removing a Leaf

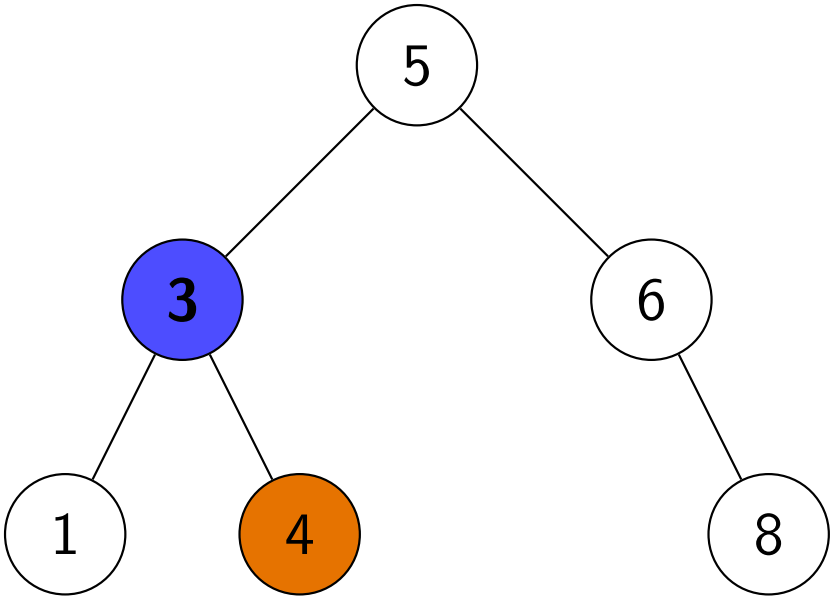

Figure 4: Find the node you want to remove (orange), in this case the one with value 4, and that node's parent (blue)

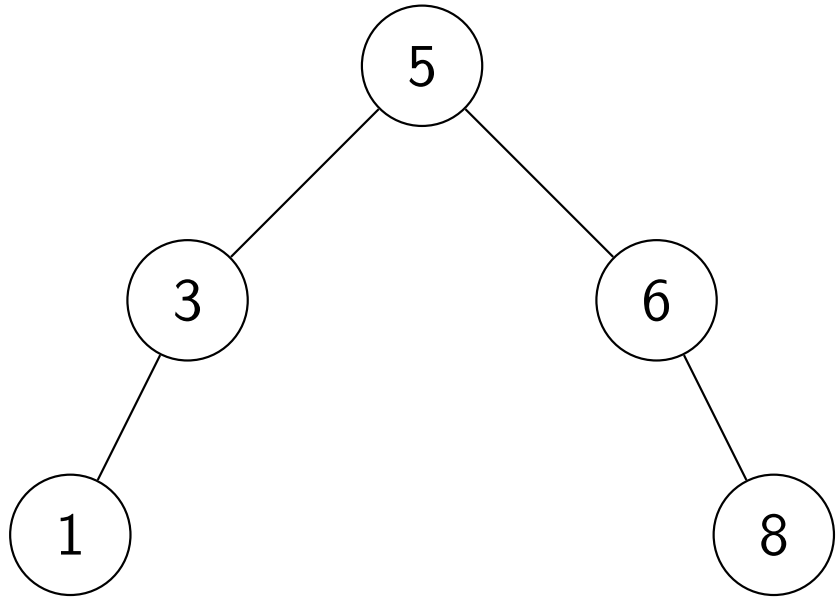

Figure 5: Set the parent's reference to that node, in this case parent.left equal to null, to remove the node

Case 2: Removing a Node with One Subtree

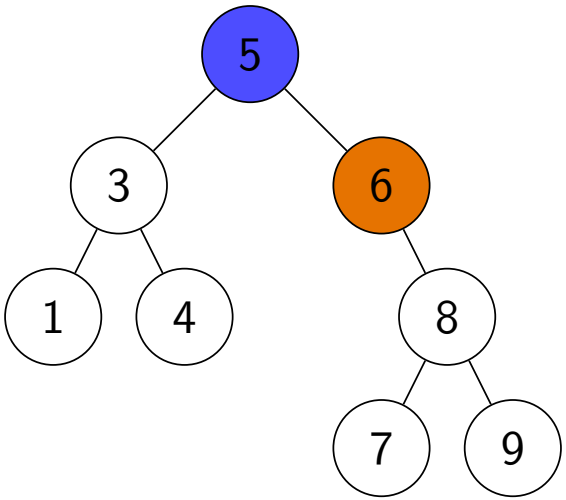

Figure 6: Find the node you want to remove (orange) and that node's parent (blue)

Case 2: Removing a Node with One Subtree

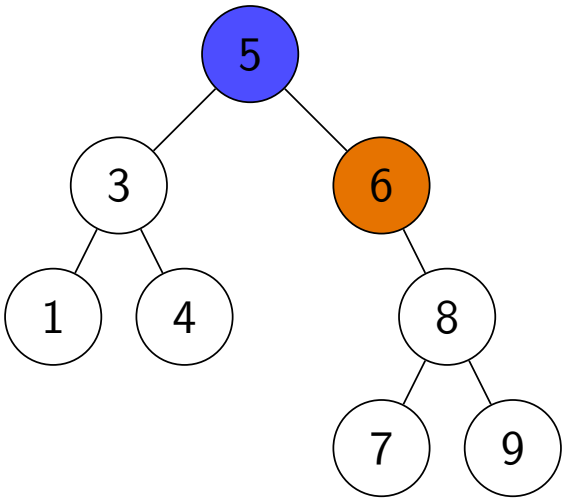

Figure 6: Find the node you want to remove (orange) and that node's parent (blue)

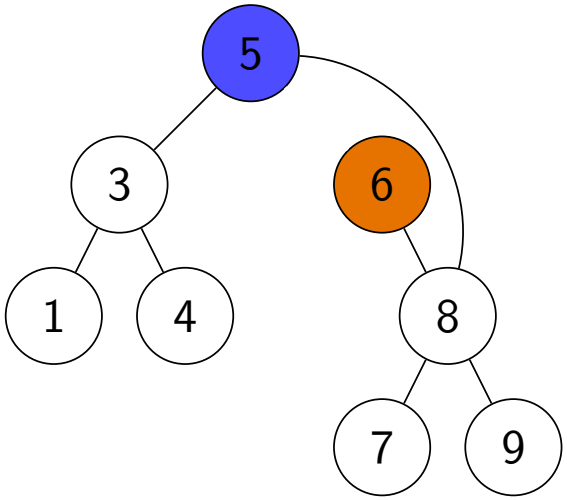

Figure 7: Set parent.right to the root of the node we want to remove's subtree.

Case 2: Removing a Node with One Subtree

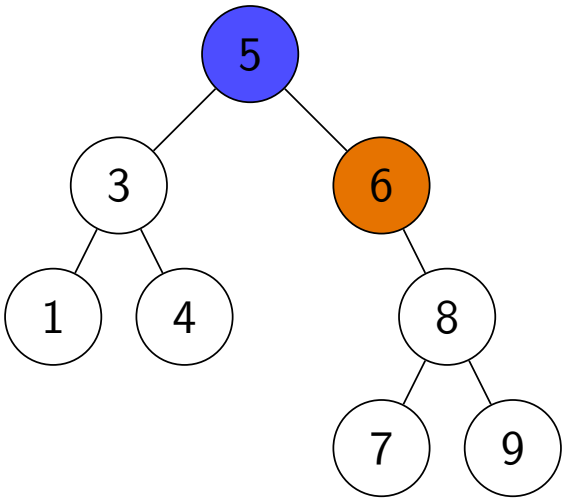

Figure 6: Find the node you want to remove (orange) and that node's parent (blue)

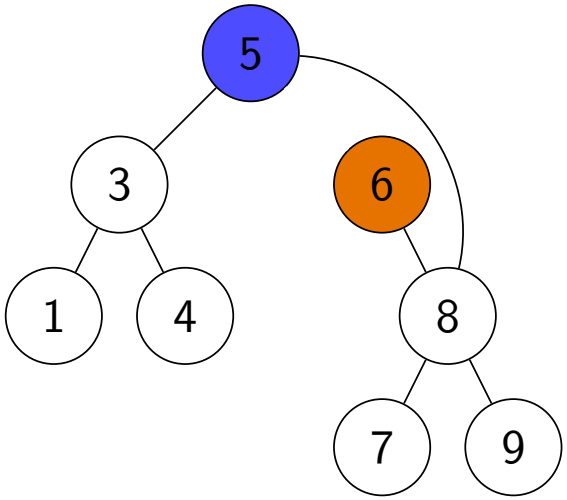

Figure 7: Set parent.right to the root of the node we want to remove's subtree.

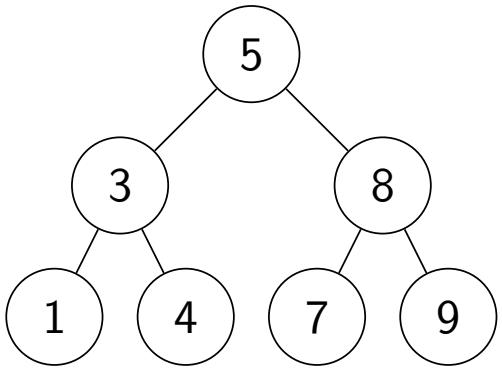

Figure 8: Set the node we want to remove's reference to it's subtree is null, thus removing it from the tree

Case 3: Removing a Node with Two Subtrees

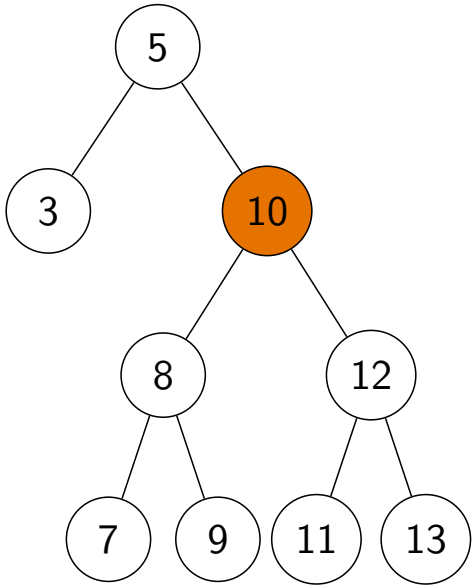

Figure 9: Find the node you want to remove (orange)

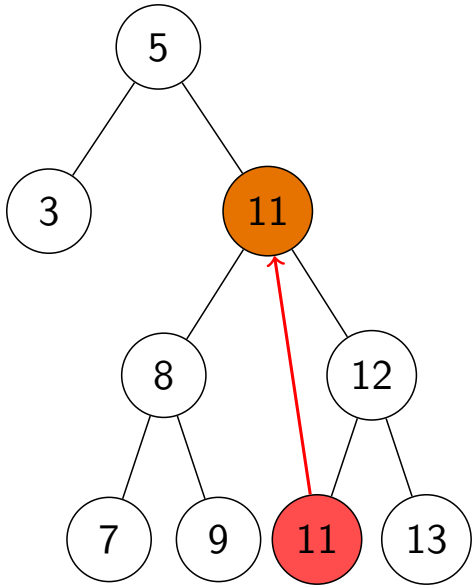

Figure 10: Find the successor node (red) and copy the successor's value to the node we want to remove.

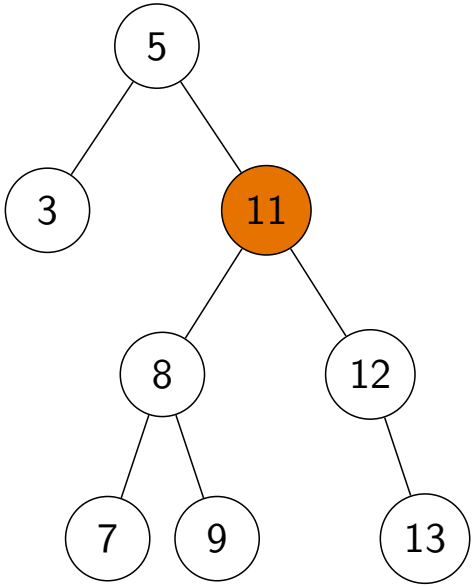

Figure 11: Call the removal method on the successor node.

Removing a Node

```
Function Remove(Val)
    root = Remove(Root, Val);

Function Remove(Curr, Val)
    if curr = null then
        return null
    end
    if Curr.val < Val then
        Curr.right ← Remove(Curr.right, Val)
    end
    else if Curr.val > Val then
        Curr.left ← Remove(Curr.left, Val)
    end
    else
        if Curr.Left is NULL AND Curr.Right is NULL then
            return NULL
        end
        else if Curr.left is null then
            return Curr.right
        end
        else if Curr.right is null then
            return Curr.left
        end
        else
            MinNode ← Minimum(Curr.right)
            Curr.data ← MinNode.data
            Curr.right ← Remove(Curr.right, MinNode.data)
            return Curr
        end
    end
```

- **Step 1 (red):** Here we are recurring down the tree to search for the node we want to remove.

Removing a Node

```
Function Remove(Val)
    root = Remove(Root, Val);

Function Remove(Curr, Val)
    if curr = null then
        return null
    end
    if Curr.val < Val then
        Curr.right ← Remove(Curr.right, Val)
    end
    else if Curr.val > Val then
        Curr.left ← Remove(Curr.left, Val)
    end
    else
        if Curr.Left is NULL AND Curr.Right is NULL then
            return NULL
        end
        else if Curr.left is null then
            return Curr.right
        end
        else if Curr.right is null then
            return Curr.left
        end
        else
            MinNode ← Minimum(Curr.right)
            Curr.data ← MinNode.data
            Curr.right ← Remove(Curr.right, MinNode.data)
            return Curr
        end
    end
```

- **Step 1 (red):** Here we are recurring down the tree to search for the node we want to remove.
- **Step 2:** Remove using cases

Removing a Node

```
Function Remove(Val)
    root = Remove(Root, Val);

Function Remove(Curr, Val)
    if curr = null then
        return null
    end
    if Curr.val < Val then
        Curr.right ← Remove(Curr.right, Val)
    end
    else if Curr.val > Val then
        Curr.left ← Remove(Curr.left, Val)
    end
    else
        if Curr.Left is NULL AND Curr.Right is NULL then
            return NULL
        end
        else if Curr.left is null then
            return Curr.right
        end
        else if Curr.right is null then
            return Curr.left
        end
        else
            MinNode ← Minimum(Curr.right)
            Curr.data ← MinNode.data
            Curr.right ← Remove(Curr.right, MinNode.data)
            return Curr
        end
    end
```

- **Step 1 (red):** Here we are recurring down the tree to search for the node we want to remove.
- **Step 2:** Remove using cases
  - **Case 1 (green):** No subtree.

Removing a Node

```
Function Remove(Val)
    root = Remove(Root, Val);

Function Remove(Curr, Val)
    if curr = null then
        return null
    end
    if Curr.val < Val then
        Curr.right ← Remove(Curr.right, Val)
    end
    else if Curr.val > Val then
        Curr.left ← Remove(Curr.left, Val)
    end
    else
        if Curr.Left is NULL AND Curr.Right is NULL then
            return NULL
        end
        else if Curr.left is null then
            return Curr.right
        end
        else if Curr.right is null then
            return Curr.left
        end
        else
            MinNode ← Minimum(Curr.right)
            Curr.data ← MinNode.data
            Curr.right ← Remove(Curr.right, MinNode.data)
            return Curr
        end
    end
```

- **Step 1 (red):** Here we are recurring down the tree to search for the node we want to remove.
- **Step 2:** Remove using cases
  - **Case 1 (green):** No subtree.
  - **Case 2 (blue):** One subtree.

Removing a Node

```
Function Remove(Val)
    root = Remove(Root, Val);

Function Remove(Curr, Val)
    if curr = null then
        return null
    end
    if Curr.val < Val then
        Curr.right ← Remove(Curr.right, Val)
    end
    else if Curr.val > Val then
        Curr.left ← Remove(Curr.left, Val)
    end
    else
        if Curr.Left is NULL AND Curr.Right is NULL then
            return NULL
        end
        else if Curr.left is null then
            return Curr.right
        end
        else if Curr.right is null then
            return Curr.left
        end
        else
            MinNode ← Minimum(Curr.right)
            Curr.data ← MinNode.data
            Curr.right ← Remove(Curr.right, MinNode.data)
            return Curr
        end
    end
```

- **Step 1 (red):** Here we are recurring down the tree to search for the node we want to remove.
- **Step 2:** Remove using cases
  - **Case 1 (green):** No subtree.
  - **Case 2 (blue):** One subtree.
  - **Case 3 (purple):** Both subtrees.

Searching Unsorted Linear Data Structures vs Searching Binary Search Tree

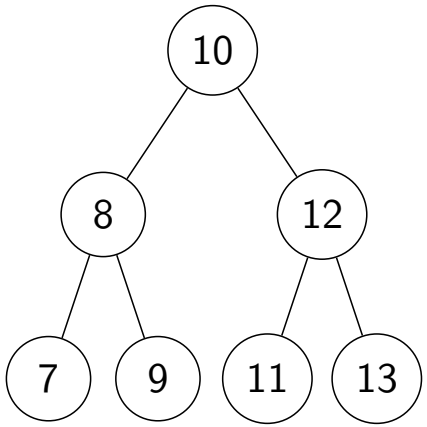

Figure 12: Balanced BST

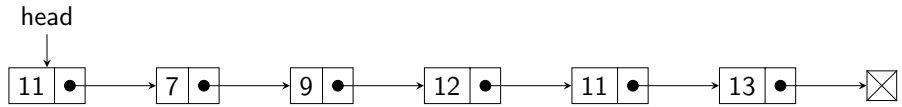

Figure 13: Linked List

- Searching a *balanced* BST, search is $O(log_2(n))$ where n is the number of nodes in the tree.
- Recall, searching a linked list is $(log_2(n))$ where n is the number of nodes in the list.

The Issue of Unbalanced Binary Search Trees

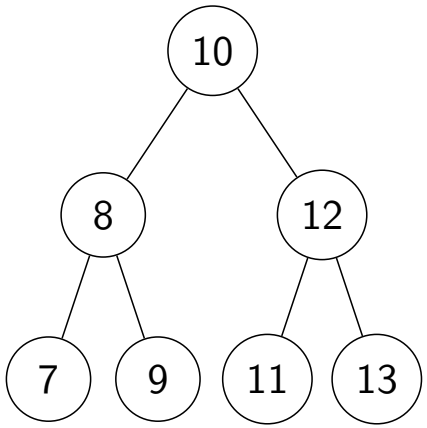

Figure 14: Balanced BST

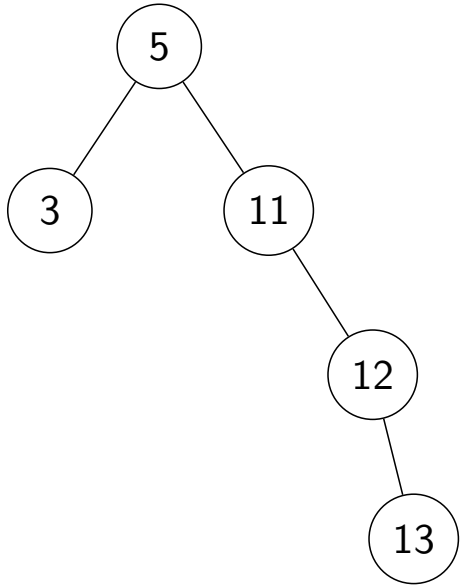

Figure 15: Unbalanced BST

- For *balanced* BST, search is $O(log_2(n))$ where n is the number of nodes in the tree.
- For *unbalanced* BST, search is $O(h)$ where h is the height of the tree.
- We want to make sure our tree remains balanced when we insert/delete but this can't be done with traditional BST.
- **Next week:** AVL self balancing trees