Balance Integrator

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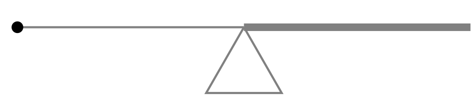
Abstract

A description of a balance/scale that can integrate using torque. The left side is a point mass at 1 meter away, and the right side is a bent rod. The mass of the point mass should be the solution to the definite integral, while the right side should be of uniform mass. Notice that the way that the rod is bent is not necessarily the curve of the function we are trying to integrate.

Contents

1 A prototype using varying density

The left side is a point mass 1 meter away from the center, thus the torque from it is $\tau = mgr = mg$. The right side is a straight bar, but with varying density.



Proof. Let L be the length of the bar, $f(x)$ the function we wish to integrate, and $\rho(x)$ the density of the rod.

$$\tau = \int_0^L x dF$$
$$\tau = \int_0^L x g dm$$
$$\tau = \int_0^L x g \rho(x) dx$$

The torque on the right side is same (in opposite magnitude) as the torque on the left.

$$mg = \int_0^L x g \rho(x) dx$$
$$mg = g \int_0^L x \rho(x) dx$$
$$m = \int_0^L x \rho(x) dx$$

And the mass should be the same value as our arbitrary integral ($\int_0^L f(x)dx$).

$$\int_0^L f(x)dx = \int_0^L x\rho(x)dx$$

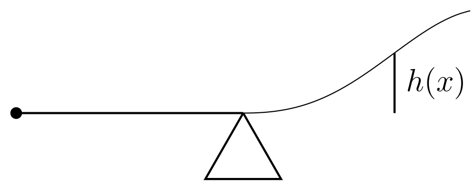
Although it is not true that the integral being the same implies the integrand is the same, the integrand being the same is *a* solution and we will use it.

$$f(x) = x\rho(x)$$
$$\rho(x) = \frac{f(x)}{x}$$

This implies that manufacturing a rod with varying density $\frac{f(x)}{x}$ will have the same torque as a point mass with mass $\int_0^L f(x)dx$. $\square$

2 Using bent rod instead of varying density

A varying density in the rod will have two drawbacks, the first being the (probably I'm not an material scientist) difficulty of manufacturing one, and the second being the difficulty of visualizing the integrator at work (the coolest part!).



Proof. Let $h(x)$ be the height of the bar with respect to the level of the scale.

$$\tau = \int_0^L \sqrt{x^2 + h^2(x)} \times dF$$

$$\tau = \int_0^L \sqrt{x^2 + h^2(x)} \sin\theta dF$$

$$\tau = \int_0^L \sqrt{x^2 + h^2(x)} \frac{x}{h(x)} dF$$

$$\tau = \int_0^L x \sqrt{\frac{x^2 + h^2(x)}{h^2(x)}} dF$$

$$\tau = \int_0^L x \sqrt{1 + \frac{x^2}{h^2(x)}} dF$$

$$\tau = \int_0^L x \sqrt{1 + \frac{x^2}{h^2(x)}} g\rho dx$$

In this case, ρ is a constant.

$$\tau = g\rho \int_0^L x \sqrt{1 + \frac{x^2}{h^2(x)}} dx$$

$$m = \rho \int_0^L x\sqrt{1 + \frac{x^2}{h^2(x)}}dx$$

Also like last time, m is the solution to the integral $\int_0^L f(x)dx$

$$\int_0^L f(x)dx = \rho \int_0^L x\sqrt{1 + \frac{x^2}{h^2(x)}}dx$$

Therefore, the integrands being the same is a solution to the above equation.

$$\begin{aligned}
f(x) &= \rho x\sqrt{1 + \frac{x^2}{h^2(x)}} \\
\frac{f(x)}{x\rho} &= \sqrt{1 + \frac{x^2}{h^2(x)}} \\
(\frac{f(x)}{x\rho})^2 &= 1 + \frac{x^2}{h^2(x)} \\
\frac{x^2}{h^2(x)} &= (\frac{f(x)}{x\rho})^2 - 1 \\
\frac{h^2(x)}{x^2} &= \frac{1}{(\frac{f(x)}{x\rho})^2 - 1} \\
h^2(x) &= \frac{x^2}{(\frac{f(x)}{x\rho})^2 - 1}
\end{aligned}$$

$$h(x) = \sqrt{\frac{x^2}{(\frac{f(x)}{x\rho})^2 - 1}}$$

$$h(x) = \sqrt{\frac{x^4}{(\frac{f(x)}{\rho})^2 - x^2}}$$

$$h(x) = x^2\sqrt{\frac{1}{(\frac{f(x)}{\rho})^2 - x^2}}$$

$$h(x) = x^2\sqrt{\frac{\rho^2}{f^2(x) - (\rho x)^2}}$$

$$h(x) = \rho x^2\sqrt{\frac{1}{f^2(x) - (\rho x)^2}}$$

$\square$

3 Use case for approximating π

A classic example of a use case for computation would be π. We will use the following definite integral.

$$\pi = \int_0^1 \frac{4}{1+x^2} dx$$

Thus, $f(x)$ and $h(x)$ would be:

$$f(x) = \frac{4}{1+x^2}$$

$$h(x) = \rho x^2\sqrt{\frac{1}{f^2(x) - (\rho x)^2}}$$

$$h(x) = \rho x^2\sqrt{\frac{1}{(\frac{4}{1+x^2})^2 - (\rho x)^2}}$$