WIP: The Optimization of Alcoholism under a Hypothetical Bartering System

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August 4, 2019

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1 Introduction to the hypothetical bartering system

You have \$10, and a beer is 2\$. Very quickly you can see that if you spend all 10\$, you will get 5 beers. Once you've drunken the 5 beers, you are left with 5 beer bottles and 5 caps. The store owner strikes you a deal. If you give him two empty bottles or four bottle caps, he'll give you a new bottle of beer. He is also kind enough to let you drink before you pay, but you are not allowed to be in bottle or cap-debt. How many drinks you can get?

2 Naive solution

3 Modeling amount of caps and bottles

3.1 Vector representation of caps and bottles

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

Will be the vector that represents the bottles and caps such that a is the amount of bottles, and b is the amount of caps.

3.2 Representation of the bartering system

If we can spend two empty bottles and receive a full drink, that is equivalent to spending two bottles and getting one bottle and one cap. We will represent this operation as the addition of the following vector;

$$\begin{bmatrix} -2+1 \\ 1 \end{bmatrix}$$

And since we can drink before we pay, having only one empty bottle is enough to drink.

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

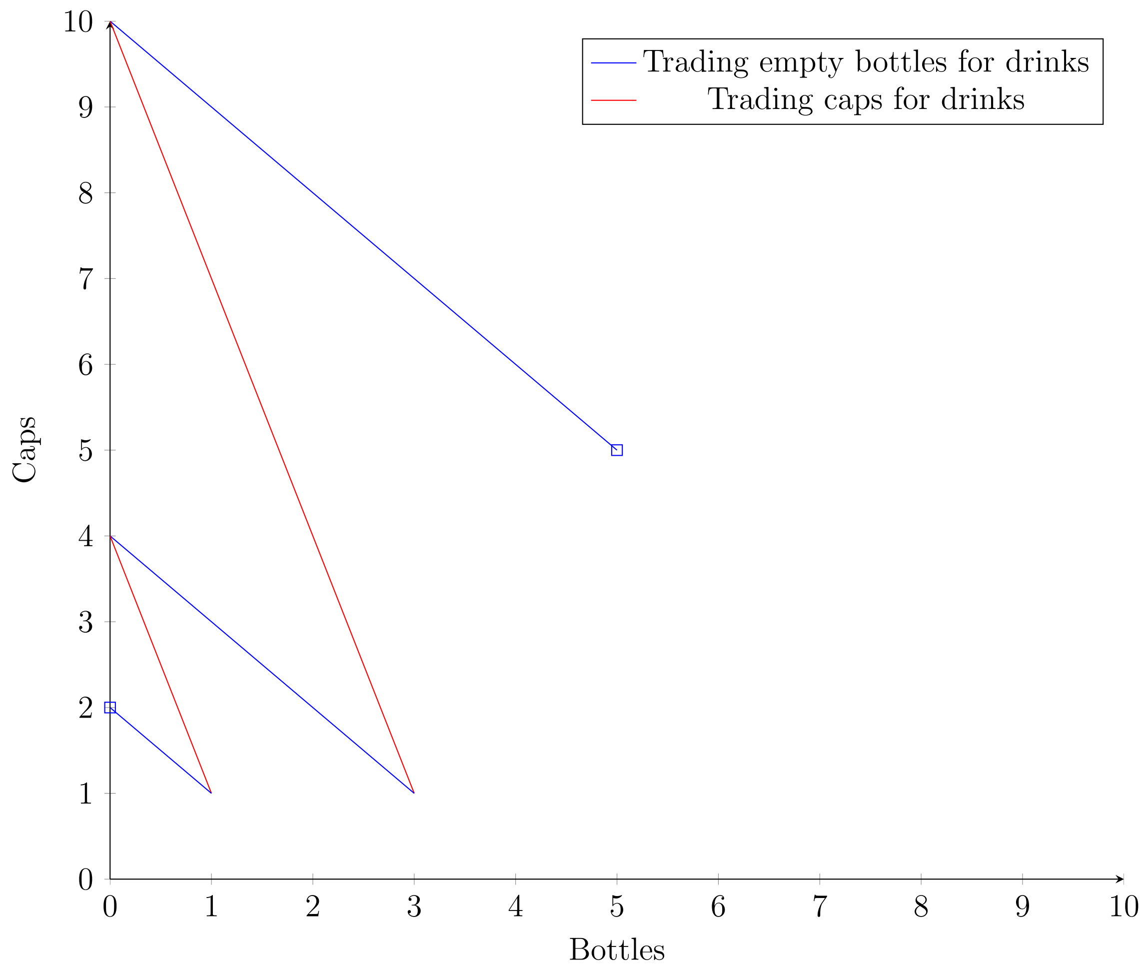
Using 4 caps can be represented simliarly

$$\begin{bmatrix} 1 \\ -4+1 \end{bmatrix}$$

Which can simply be evaluated to.

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

These two operations can be represented geometrically as a translation of a point on a 2 dimensional Cartesian plane. For example, if we start with 5 empty bottles and 5 empty caps, we can trace the motion of the point as following.



4 Analysis of the graph

Since having either 1 bottle, or 3 caps is enough to do another transaction, we know from the context of the problem, that the final state must be one of the following points:

$$(0, 0)$$

$$(0, 1)$$

$$(0, 2)$$

Visually, you can already tell that it's impossible to get to the point $(0, 0)$ unless if you have already started there (sorry mate, you gotta buy some beer to play the game).

You can also tell that you cannot get to $(0, 1)$, because that means you came from $(1, 0)$ through a blue line, but that means you came from $(0, 2)$ from a red line, which is impossible because $(0, 2)$ is not enough to continue a transaction.

So already from this graph, you can tell that you will always end up with 2 caps left over (if we ignore the trivial case that you do not buy beer in the first place).

5 Algebraic Solution

Let $\vec{i}$ be the initial state, $\vec{f}$ be the final state, c be the number of cap-based transactions, and b be the number of bottle-based transactions.

$$\begin{aligned}
\vec{i} + c\begin{bmatrix}-3\\1\end{bmatrix} + b\begin{bmatrix}1\\-1\end{bmatrix} &= \vec{f} \\
c\begin{bmatrix}-3\\1\end{bmatrix} + b\begin{bmatrix}1\\-1\end{bmatrix} &= \vec{f} - \vec{i} \\
\begin{bmatrix}-3 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}c\\b\end{bmatrix} &= \vec{f} - \vec{i} \\
\begin{bmatrix}c\\b\end{bmatrix} &= \frac{1}{2}\begin{bmatrix}-1 & -1\\-1 & -3\end{bmatrix}(\vec{f} - \vec{i})
\end{aligned}$$

Initially we get the same amount of caps and bottles according to how many drinks i that we start with.

$$\begin{aligned}
\begin{bmatrix}c\\b\end{bmatrix} &= \frac{1}{2}\begin{bmatrix}-1 & -1\\-1 & -3\end{bmatrix}(\vec{f} - \vec{i}) \\
\begin{bmatrix}c\\b\end{bmatrix} &= \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}(\vec{i} - \vec{f}) \\
\begin{bmatrix}c\\b\end{bmatrix} &= \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix} i \begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}\vec{f} \\
\begin{bmatrix}c\\b\end{bmatrix} &= i\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}\vec{f}
\end{aligned}$$

If we simply make $\vec{f} = \vec{0}$, then we don't reach a contradiction even though we know from looking at the graph that it's not possible. This is because there's always a way to get to $(0, 0)$ if we choose to fall into debt, but since that's against the rules, we have to make a check that there exists a way to reach $\vec{f}$.

We can describe the validity v of transaction c and b using the following recursive function:

$$\begin{aligned}
v(c, b) &= (v(c-1, b) + v(c, b-1)) \times \underbrace{(\begin{bmatrix}-3 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}c\\b\end{bmatrix} + \vec{i} > \vec{f})}_{\text{1 if this is true, 0 if this is false}} \\
v(0, 0) &= 1
\end{aligned}$$

If we brute force using a script:

```
#!/usr/bin/python3
import numpy as np
import matplotlib.pyplot as plt

t = np.array([[-3, 1],
```

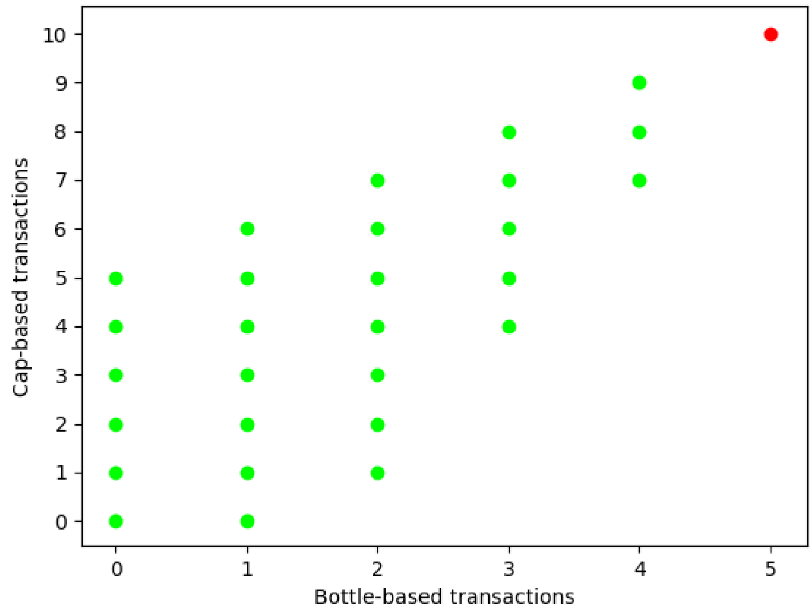
```
              [1, -1]])

def transaction(a, b):
    return np.dot(t, np.array([[a],[b]])) + [[5], [5]]

ok = [] # ok points that have already been checked
check = [(0, 0)] # ok points, but may lead to more
tmp = []
while not len(check) == 0:
    for i in check:
        if not i in ok:
            ok.append(i)
        tran = transaction(i[0], i[1] + 1)
        if tran[0] >=0 and tran[1] >=0:
            tmp.append((i[0], i[1] + 1))
        tran = transaction(i[0] + 1, i[1])
        if tran[0] >=0 and tran[1] >=0:
            tmp.append((i[0] + 1, i[1]))
    check = tmp
    tmp = []

plt.scatter([i[0] for i in ok], [i[1] for i in ok], c=(0, 1, 0))
plt.scatter([5], [10], c=(1, 0, 0))
plt.xlabel('Bottle-based transactions')
plt.ylabel('Cap-based transactions')
plt.xticks(range(6))
plt.yticks(range(11))
plt.savefig('debt.png')
```

Then we get the following result where the green dots represents the possible transactions:



The goal is to get as drunk as possible, which is reaching as many transactions as possible, thus we want to maximize $b+c$ with respect to the elements of f. The above matrix equation can be represented as the following set of equations.

$$c = \frac{1}{2}(-(f_c - i_c) - (f_b - i_b))$$
$$b = \frac{1}{2}(-(f_c - i_c) - 3(f_b - i_b))$$
$$c + b = \frac{1}{2}(-2(f_c - i_c) - 4(f_b - i_b))$$

Which can be simplified to:

$$c = \frac{1}{2}(-f_c - f_b + i_c + i_b))$$
$$b = \frac{1}{2}(-f_c - 3f_b + i_c + i_b))$$
$$c + b = -f_c - 2f_b + i_c + 2i_b$$

Using the typical optimization with derivatives isn't going to help us here, because the function is linear with respect to both f_c and f_b. Instead we can use the constraint that c, b, and $c+b$ must be natural numbers (in my definition, the natural numbers include 0).

$$c \in \mathbb{N}$$
$$-f_c - f_b + i_b + i_c \in 2\mathbb{N}$$

Since $i_b = i_c$, $i_b + i_c \in 2\mathbb{N}$

$$-f_c - f_b \in 2\mathbb{Z}$$
$$f_c + f_b \in 2\mathbb{N}$$

Thus, we can say that f_c and f_b have the same parity.

6 Using the final state of the vector to deduce the amount of drinks drunk

Since two empty bottles can get you a drink and a drink is worth \$2, then that means one bottle is worth \$1. Similarly since four bottle caps can get you a drink and a drink is worth \$2, then that means bottle cap is worth \$0.5.

Since a full drink is consisted of one cap, one bottle, and some drink, we can use simple algebra to deduce that:

$$\$2 = d + \$1 + \$0.5 \tag{1}$$

$$d = \$0.5 \tag{2}$$

the worth of the drink is \$0.5. If we started with \$10 dollars, and we are left with a bottles and b caps, then:

$$\$10 = \$0.5x + \$a + \$0.5b \tag{3}$$

$$x = \frac{\$10 - \$a - \$0.5b}{\$0.5} \tag{4}$$