Assignment 2 in Signals and Transforms

Report

11/11-2022

1 Task 1

1.a

From the graph we can deduce that $x(t) = t \ for \ -1 \leq t \leq 1$
The following integral yields the result for n = 0

$$c_0 = \frac{1}{T_0}\int_{-1}^{1} x(t)dt = \frac{1}{T_0}\int_{-1}^{1} t\ dt = \frac{1}{T_0}\left[1\right]_{-1}^{1} = 0$$

For $n \neq 0$, integration by parts is used ($\int u(t)v'(t)dt = u(t)v(t) - \int u'(t)v(t)dt$)
Where $u(t) = t$ and $v'(t) = e^{-j\omega_0 t}dt$

$$c_n = \frac{1}{T_0}\int_{(T_0)} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T_0}\int_{-1}^{1} te^{-jn\omega_0 t}dt = \frac{1}{T_0}\left[-\frac{1}{jn\omega_0}\left[te^{-jn\omega_0 t}\right]_{-1}^{1} + \frac{1}{jn\omega_0}\int_{-1}^{1} e^{-jn\omega_0 t}dt\right]$$

The first part of the square bracket

$$-\frac{1}{jn\omega_0}\left[te^{-jn\omega_0 t}\right]_{-1}^{1} = -\frac{1}{jn\omega_0}(e^{-jn\omega_0} + e^{jn\omega_0}) = \frac{2j\cos(n\omega_0)}{n\omega_0}$$

The second part of the square bracket

$$\frac{1}{jn\omega_0}\int_{-1}^{1} e^{-jn\omega_0 t}dt = \frac{1}{(jn\omega_0)^2}\left[e^{-jn\omega_0 t}\right]_{-1}^{1} = -\frac{1}{(jn\omega_0)^2}(-e^{-jn\omega_0} + e^{jn\omega_0}) = -\frac{2j\sin(n\omega_0)}{(n\omega_0)^2}$$

Since a sine function of an integer multiplied by pi is always 0 and n is always going to be an integer $-\frac{2j\sin(n\omega_0)}{(n\omega_0)^2} = 0$
Thus

$$c_n = \frac{1}{T_0}\int_{(T_0)} x(t)e^{-jn\omega_0 t}dt = \frac{1}{T_0}\left[\frac{2j\cos(n\omega_0)}{n\omega_0}\right] = \frac{2}{T_0}\left[\frac{j\cos(n\omega_0)}{n\omega_0}\right]$$

From the graph it is clear that $T_0 = 2$ and therefore $\omega_0 = \frac{2\pi}{2} = \pi$

$$\frac{2}{T_0}\left[\frac{j\cos(n\omega_0)}{n\omega_0}\right] = \frac{2}{2}\left[\frac{j\cos(n\pi)}{n\pi}\right] = \frac{j\cos(n\pi)}{n\pi}$$

Using 3.5b from the lecture notes, it follows that

$$a_0 = c_0 = 0,\ a_n = 2Re\{c_n\} = 0,\ b_n = -2Im\{c_n\} = -2\frac{\cos(n\pi)}{n\pi} = -2\frac{(-1)^n}{n\pi}$$

Given by $\cos(n\pi) = (-1)^n$
The sawtooth signal $x(t)$ can therefore be expressed

$$x(t) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{\pi n}\sin(\pi n t)$$

Which gives $x_e(t) = 0$ and $x_o(t) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{\pi n}\sin(\pi n t)$

1.b

As shown in *Figure 1*, the odd component of the signal seems to be identical to the signal itself. The even component is 0 for all t. This corresponds well to the expression describing the signal in *1.a*, as the signal is described using only a sine function. It follows then that the even component will be zero.

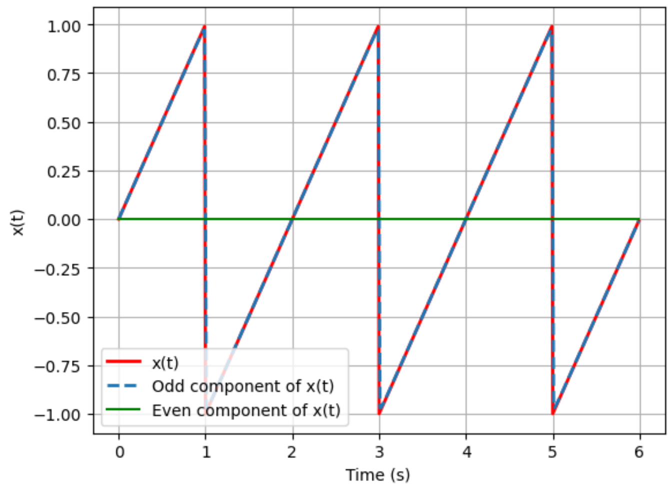
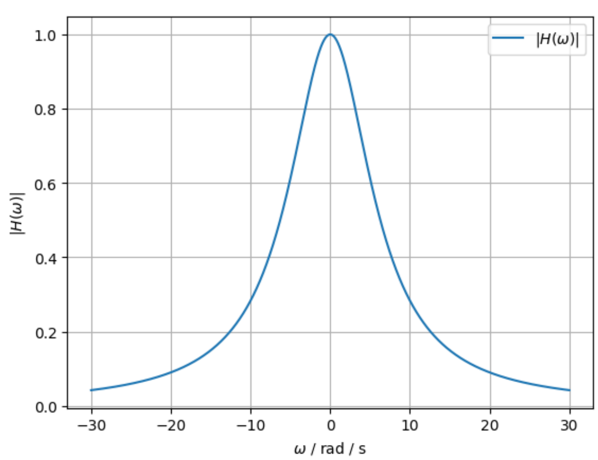


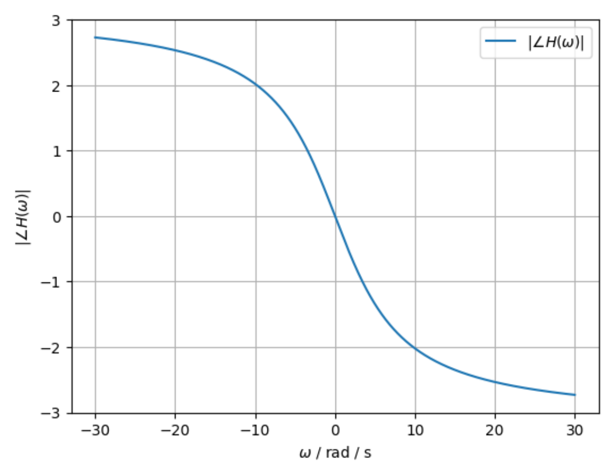
Figure 1: Signal x(t), and its odd and even components.

2 Task 2

2.a



(a) Figure 2



(b) Figure 3

As the frequency response of a system is the Fourier transform of the impulse response, the frequency response is given by the following expression.

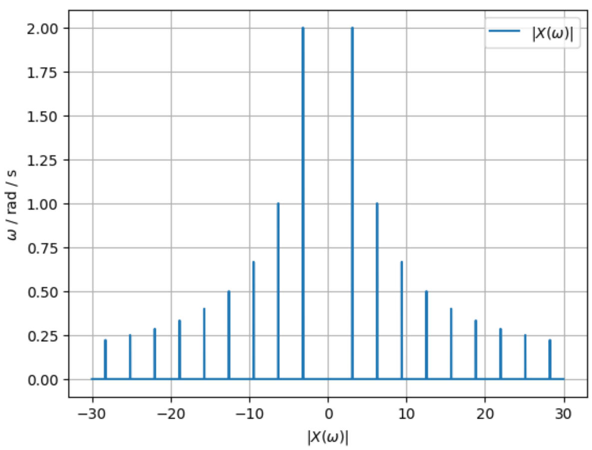
$$H(\omega) = \int_{-\infty}^{\infty} \alpha^2 t e^{-\alpha t} u(t) e^{-j\omega t} dt = \alpha^2 \int_0^{\infty} t e^{-\alpha t} e^{-j\omega t} dt = \alpha^2 \int_0^{\infty} t e^{-(\alpha t + j\omega)t} dt$$

$$= \alpha^2 \left[\frac{e^{-(\alpha t + j\omega)t}}{-(\alpha t j\omega)} \right]_0^\infty = \frac{\alpha^2}{(\alpha + j\omega)^2} = \frac{4\pi^2}{(2\pi + j\omega)^2}$$

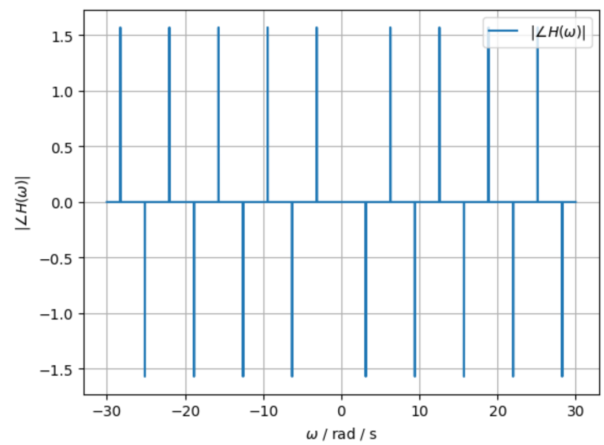
Low frequencies yield a higher magnitude in this system, which can be seen in *Figure 2*. *Figure 3* shows that higher (positive) frequencies grant a negative phase, while low (negative) frequencies show a positive phase. The phase also increases in the respective direction as the frequency increases or decreases.

2.b

As per the transform table, the continuous time Fourier transform for the signal is given by $X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0)$. Entering the expression for c_n yields the transform $X = 2\pi \sum_{n=-\infty}^{\infty} ((-1)^n j \frac{1}{n\pi}) \delta(\omega - n\omega_0)$. The magnitude is shown in *Figure 4*, and the phase in *Figure 5*.

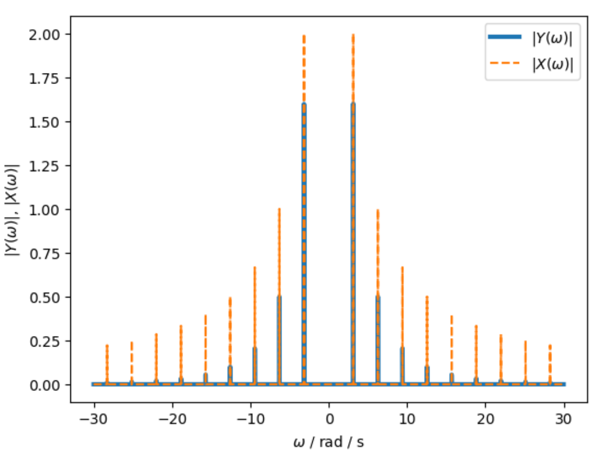


(a) Figure 4

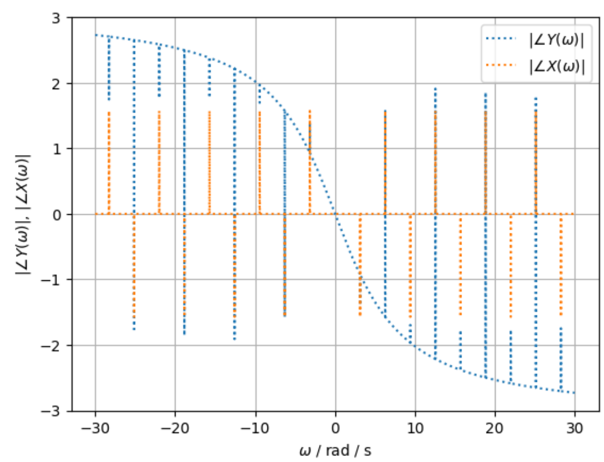


(b) Figure 5

2.c



(a) Figure 6



(b) Figure 7

The spectrum of the output signal Y(ω) is given by the product of the input signals Fourier transform and the frequency response. This means that

$$Y(\omega) = H(\omega)X(\omega) => Y(\omega) = \frac{8\pi^2 j(-1)^n}{n(2\pi + jw)^2}$$

The magnitude of Y(ω) is smaller than the magnitude of X(ω), as can be seen in *Figure 6*. The phase is also affected, seemingly acquiring a wave-like appearance in contrast to the input signals discrete phase. The effect on phase is due to the fact that the phases of Y(ω) and X(ω) are added together, which causes the wave-like look. The magnitude is due to scaling when multiplying the transform by a complex number (the frequency response).

2.d

The output signals are identical to each other.

The signals in *Task 3(b)* of assignment one are generated using a convolution of the input signal and system in the time domain, while the signals in this task are generated using multiplication in the frequency domain. These methods are so called transform pairs, and it is as such expected that the results will be the same.

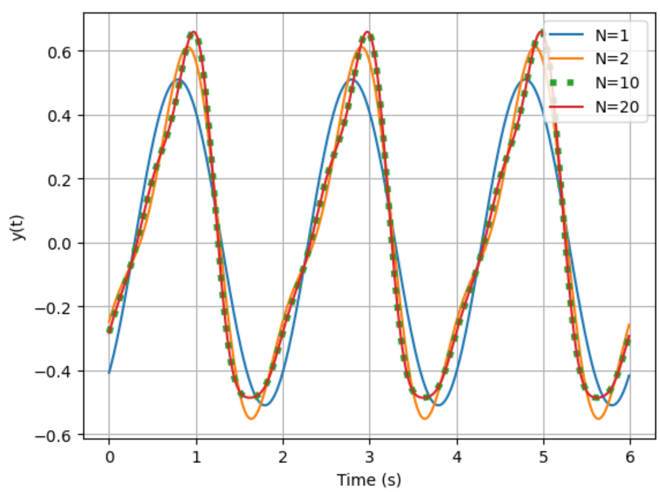


Figure 5: Signal y(t)