Robotics 2

Robots with kinematic redundancy

Part 1: Fundamentals

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Redundant robots

- direct kinematics of the task $r = f(q)$

$$f : \mathrm{Q} \rightarrow \mathrm{R}$$

However we should match the task to the kinematic type of the robot: if we have a spatial task we can't complete it with a planar robot even if it has the necessary DoF (or more). So this redundancy analysis makes sense assuming we are using the proper type of robot for the task (e.g. spatial robot for spatial tasks)

joint space (dim Q = N)

task space (dim R = M)

- a robot is (kinematically) redundant for the task if $N > M$
  (more degrees of freedom than strictly needed for executing the task)
- r may contain the position and/or the orientation of the end-effector or, more in general, be any parameterization of the task (even not in the Cartesian workspace)
- "redundancy" of a robot is thus a relative concept, i.e., it holds with respect to a given task

Some E-E tasks and their dimensions

TASKS [for the robot end-effector (E-E)]	dimension M
position in the plane	2
position in 3D space	3
orientation in the plane	1
pointing in 3D space (two angles: rotation around vertical axis and elevation)	2
position and orientation in 3D space	6

a planar robot with $N = 3$ joints is redundant for the task of positioning its E-E in the plane ($M = 2$), but NOT for the task of positioning AND orienting the E-E in the plane ($M = 3$)

Typical cases of redundant robots

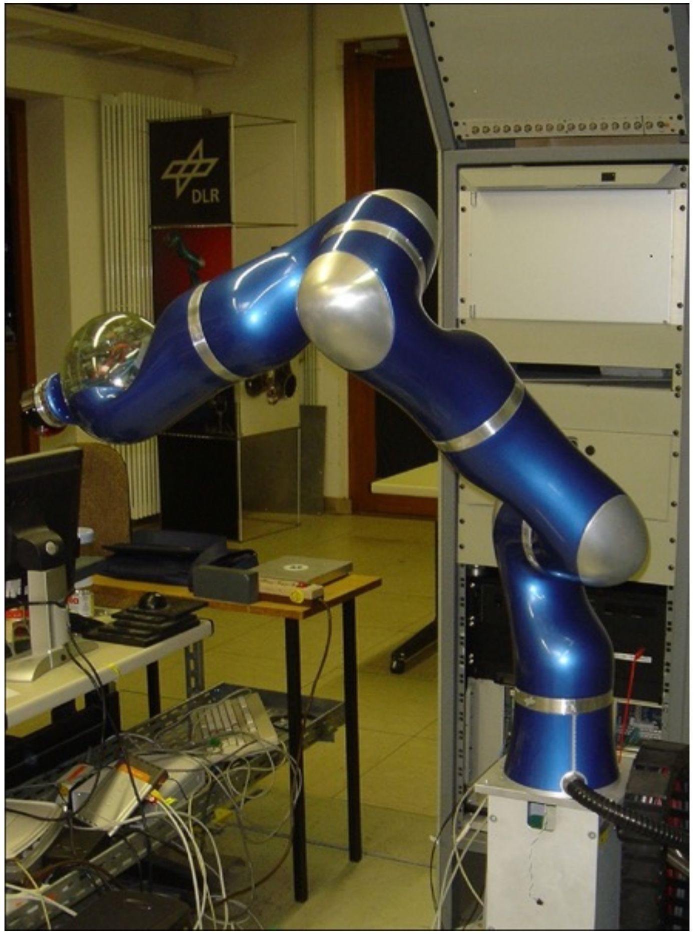
- 6R robot mounted on a linear track/rail
  - 7 dofs for positioning and orienting its end-effector in 3D space
- 6-dof robot used for arc welding tasks welding is a 5 dimensional task
  - task does not prescribe the final roll angle of the welding gun
- dexterous robotic hands
- multiple cooperating manipulators
- manipulator on a mobile base
- humanoid robots, team of mobile robots ...
- “kinematic” redundancy is not the only type…
  - redundancy of components (actuators, sensors)
  - redundancy in the control/supervision architecture

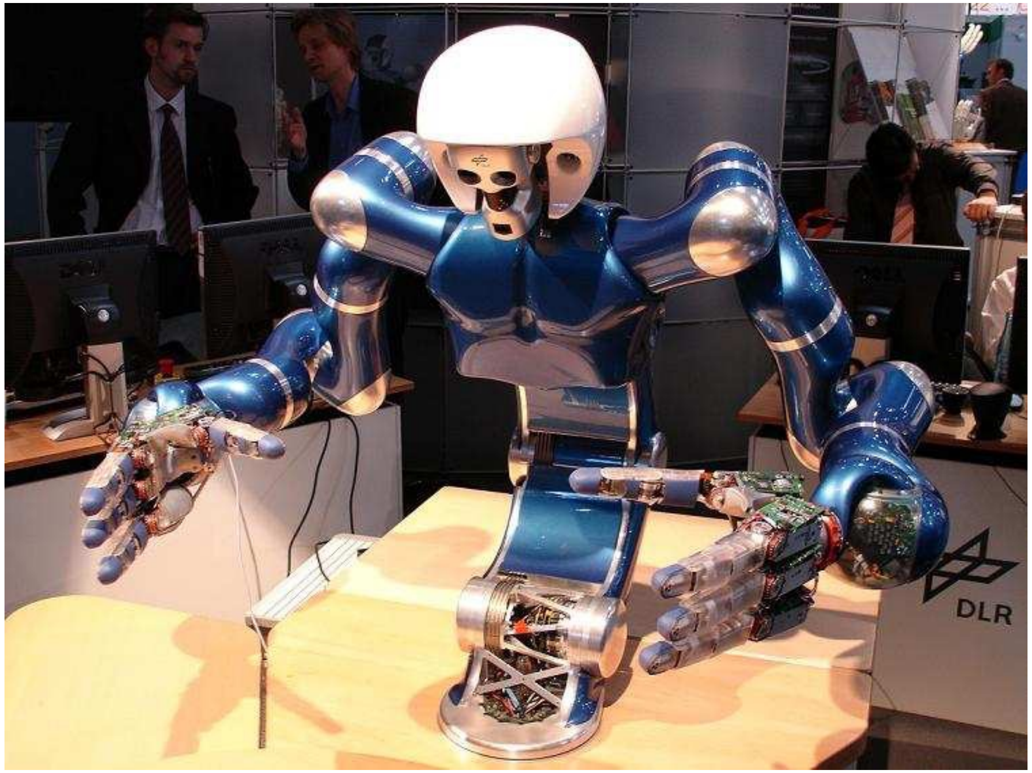
Uses of robot redundancy

- avoid collision with obstacles (in Cartesian space) …
- … or kinematic singularities (in joint space)
- stay within the admissible joint ranges
- increase manipulability in specified directions
- uniformly distribute/limit joint velocities and/or accelerations
- minimize energy consumption or needed motion torques
- optimize execution time
- increase dependability with respect to faults
- …

all objectives should be quantitatively "measurable"

DLR robots: LWR-III and Justin





7R LWR-III lightweight manipulator:
elastic joints (HD), joint torque sensing,
13.5 kg weight = payload

Justin two-arm upper-body humanoid:
43R actuated =
two arms (2×7) + torso (3*)
+ head (2) + two hands (2×12),
45 kg weight

* = one joint is dependent on the motion of the other two

Justin carrying a trailer

video

motion planning for DLR Justin robot in the configuration space, avoiding Cartesian obstacles and using robot redundancy

Dual-arm redundancy

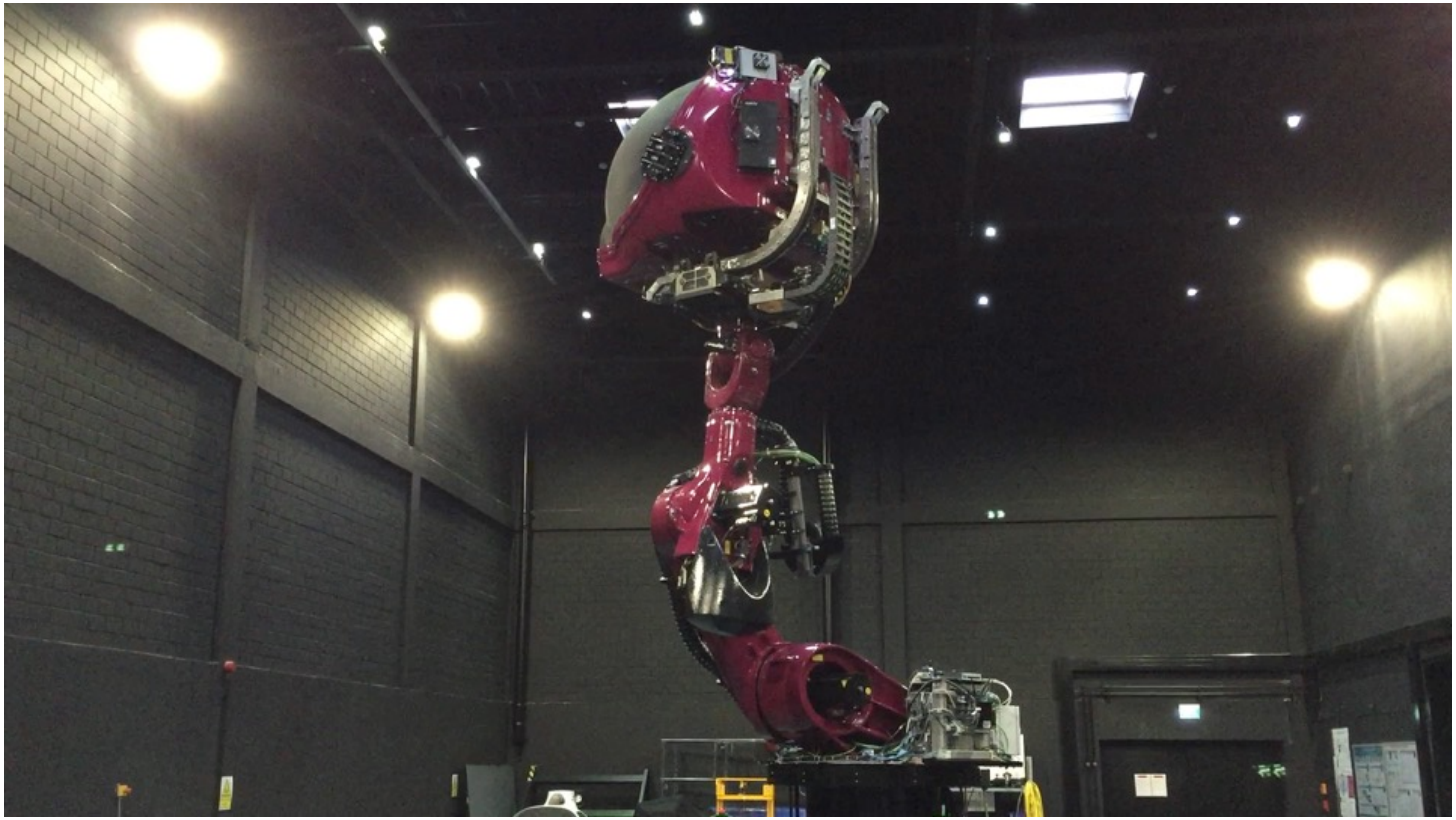
video

DIS, Uni Napoli

two 6R Comau robots, one mounted on a linear track (+1P)
coordinated 6D motion using the null-space of the right-side robot ($N - M = 1$)

Motion cueing from redundancy

video



Max Planck Institute for Biological Cybernetics, Tübingen

a 6R KUKA KR500 mounted on a linear track (+1P) with a sliding cabin (+1R), used as a dynamic emulation platform for human perception ($N - M = 2$)

Self-motion

video

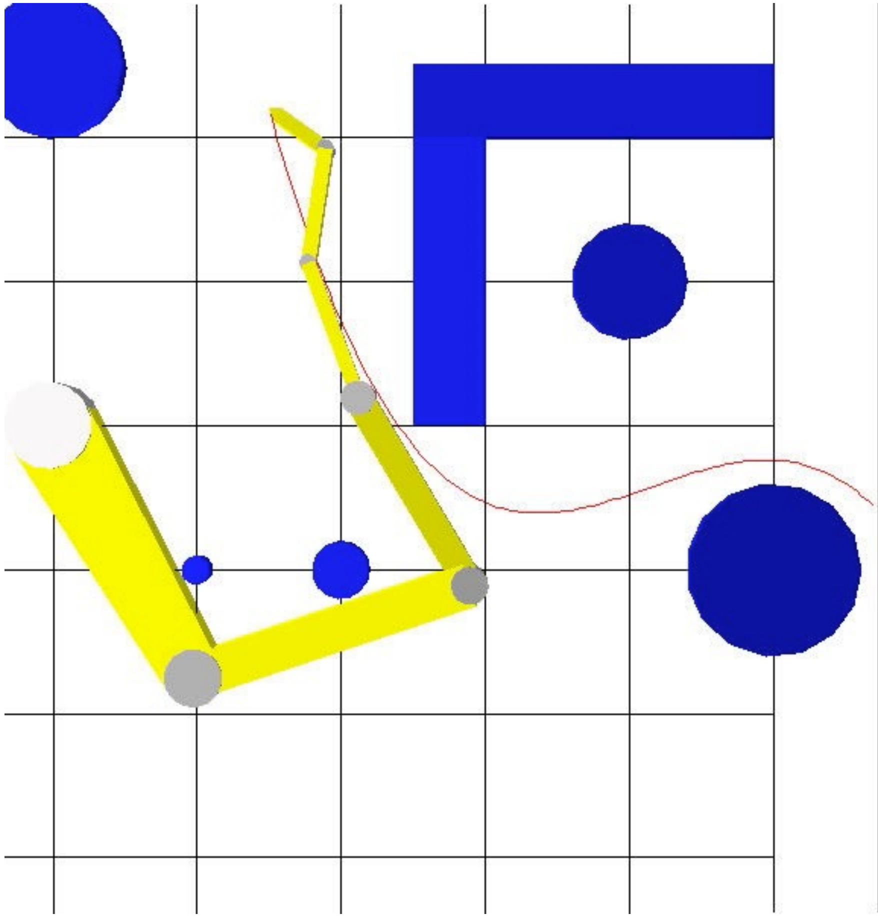
video

Nakamura's Lab, Uni Tokyo

8R Dexter: self-motion with constant 6D pose of E-E ($N - M = 2$)

6R robot with spherical shoulder in compliant tasks for the Cartesian E-E position ($N - M = 3$)

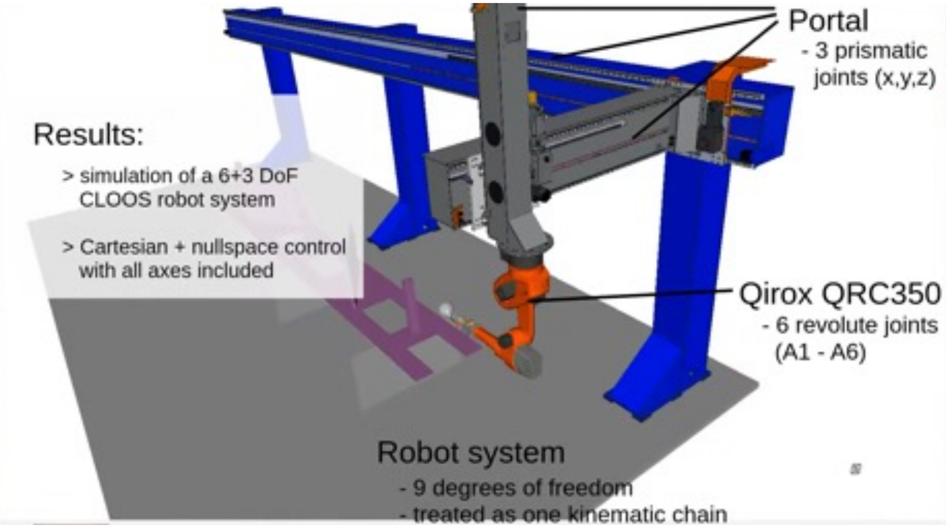
Obstacle avoidance

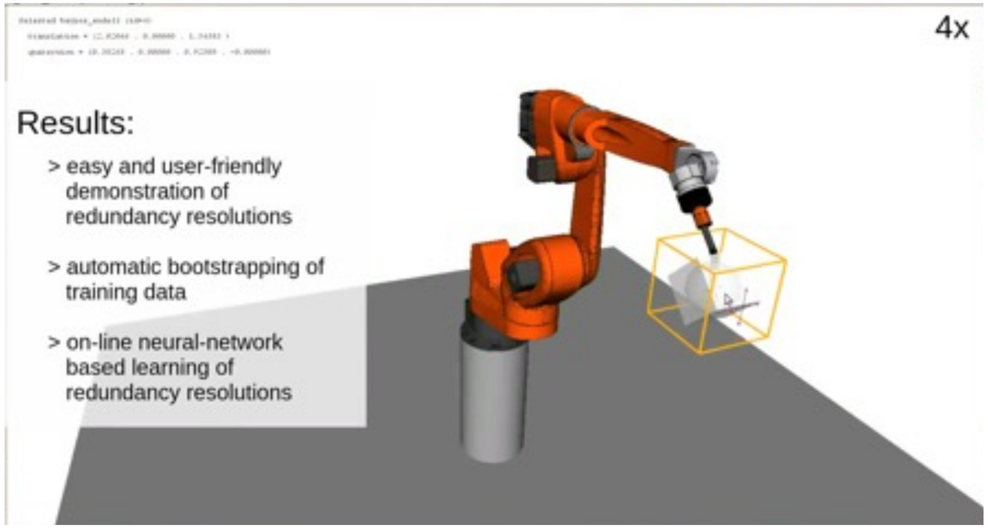


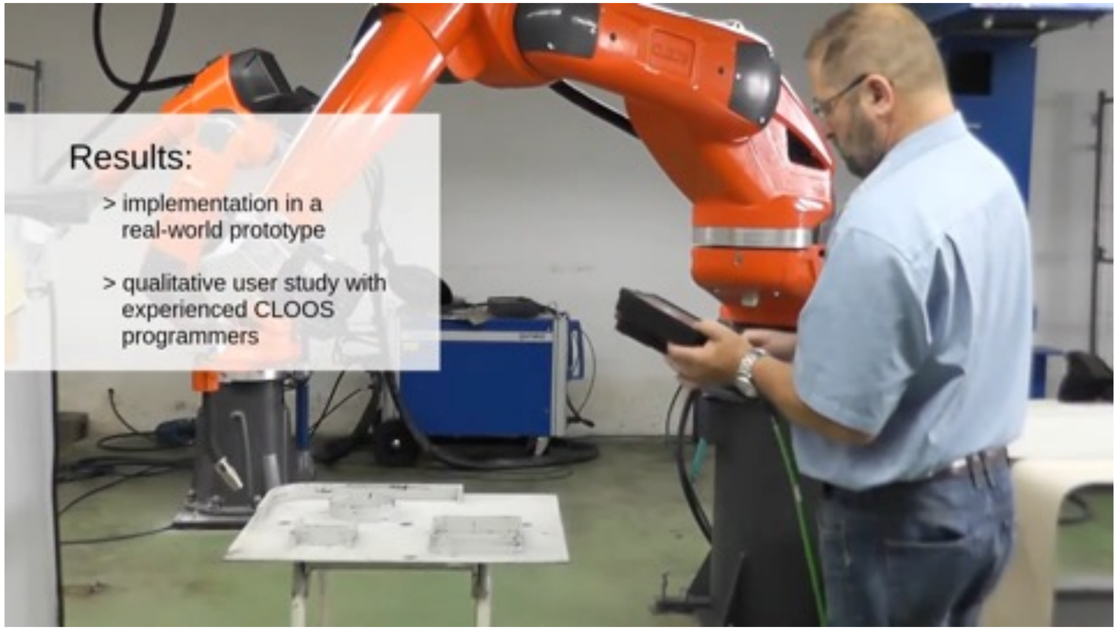
video

6R planar arm moving on a given geometric path for the E-E ($N - M = 4$)

An Echord++ industrial experiment





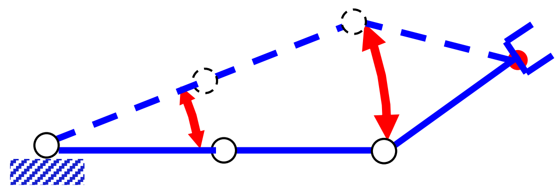


3 videos

Inverse kinematics problem

- find $q(t)$ that realizes the task: $f\big(q(t)\big) = r(t)$ (at all times t)
- infinite solutions exist when the robot is redundant
  (even for $r(t) = r$ = constant)

$N = 3 > 2 = M$



r = constant
E-E position

- the robot arm may have "internal displacements" that are unobservable at the task level (e.g., not contributing to E-E motion)
  - these joint displacements can be chosen so as to improve/optimize in some way the behavior of the robotic system
- self-motion: an arm reconfiguration in the joint space that does not change/affect the value of the task variables r
- solutions are mainly sought at differential level (e.g., velocity)

Redundancy resolution

via optimization of an objective function

Local methods

given $\dot{r}(t)$ and $q(t)$, $t = kT_s$

optimization of $H(q,\dot{q})$

$\dot{q}(kT_s)$ ← ON-LINE

$$q\big((k+1)T_s\big) = q(kT_s) + T_s\,\dot{q}(kT_s)$$

discrete-time form

Global methods

given $r(t)$, $t \in [t_0, t_0+T]$, $q(t_0)$

optimization of $\int_{t_0}^{t_0+T} H(q,\dot{q})dt$

$$q(t),\ t \in [t_0, t_0+T]$$

OFF-LINE

Linear Quadratic (linear constraint) -> closed form solution

relatively EASY (a LQ problem) ⟷ quite DIFFICULT (nonlinear TPBV problems arise)

Local resolution methods

three classes of methods for solving $\dot{r} = J(q)\dot{q}$

1. Jacobian-based methods (here, analytic Jacobian in general!)

   among the infinite solutions, one is chosen, e.g., that minimizes a suitable (possibly weighted) norm For instance using pseudo-inverse

2. null-space methods Jacobian-based method to execute the task + null space velocities which generates r_dot = 0. Although this terms doesn't affect the task, it is additional velocity that can be helpful for some reason

   a term is added to the previous solution so as not to affect execution of the task trajectory, i.e., belonging to the null-space $\mathcal{N}(J(q))$

3. task augmentation methods

   redundancy is reduced/eliminated by adding $S \leq N - M$ further auxiliary tasks (when $S = N - M$, the problem has been "squared")

$$r = f(q) \Rightarrow \dot{r} = J(q)\dot{q}$$

1 Jacobian-based methods

we look for a solution to $\dot{r} = J(q)\dot{q}$ in the form

$$J = \underbrace{\boxed{}}_{N}\Big\}M \qquad \dot{q} = K(q)\dot{r} \qquad K = \underbrace{\boxed{}}_{M}\Big\}N$$

minimum requirement for K: $J(q)K(q)J(q) = J(q)$

(➡ K = generalized inverse of J)

r_dot is a linear combination of the columns of J

using q_dot = K*r_dot

here we write r_dot as a combination of the column of J with q_dot

$$\forall \dot{r} \in \mathcal{R}(J(q)) \Rightarrow J(q)[K(q)\dot{r}] = J(q)K(q)J(q)\dot{q} = J(q)\dot{q} = \dot{r}$$

example:

if $J = [J_a \ \ J_b]$, $\det(J_a) \neq 0$, one such generalized inverse of J is $K_r = \begin{pmatrix} J_a^{-1} \\ 0 \end{pmatrix}$

(actually, this is a stronger right-inverse)

the zero means that we are zeroing the motion of the redundant DoF, mantaining only the motion of the joints corresponding to the non-singular minor of the jacobian

Pseudoinverse

Is a generalized inverse
with more properties

$$\dot{q} = J^{\#}(q)\dot{r}$$

... a very common choice: $K = J^{\#}$

- $J^{\#}$ always exists, and is the unique matrix satisfying

$$J\,J^{\#}J = J \qquad J^{\#}J\,J^{\#} = J^{\#}$$
$$(J\,J^{\#})^T = J\,J^{\#} \qquad (J^{\#}J)^T = J^{\#}J$$

symmetry

- if J is full (row) rank, $J^{\#} = J^T(J\,J^T)^{-1}$; else, it is computed numerically using the SVD (Singular Value Decomposition) of J (`pinv` of Matlab)
- the pseudo-inverse joint velocity is the only that minimizes the norm $\|\dot{q}\|^2 = \dot{q}^T\dot{q}$ among all joint velocities that minimize the task error norm $\|\dot{r} - J(q)\dot{q}\|^2$

if we have linear and angular quantity in the same vector q, minimizing the norm makes no sense, so we need to deal with this inconsistency

- if the task is feasible ($\dot{r} \in \mathcal{R}(J(q))$), there will be no task error

Weighted pseudoinverse

$$\dot{q} = J_W^{\#}(q)\dot{r}$$

another choice: $K = J_W^{\#}$

- the solution $\dot{q}$ minimizes the weighted norm

$$\|\dot{q}\|_W^2 = \dot{q}^T W \, \dot{q}$$

$W > 0$, symmetric (often diagonal)

- if J is full (row) rank, $J_W^{\#} = W^{-1}J^T(JW^{-1}J^T)^{-1}$
- large weight $W_i \Rightarrow$ small $\dot{q}_i$
  - larger weights for proximity joints (carrying/moving more "mass")
  - weights chosen proportionally to the inverse of the joint ranges W = 1 / (q_max - q_min)
- it is NOT a "pseudoinverse" (4th relation does not hold), but it shares similar properties

Singular Value Decomposition (SVD)

- the SVD routine of Matlab applied to J provides two orthonormal matrices $U_{M\times M}$ and $V_{N\times N}$, and a matrix $\Sigma_{M\times N}$ of the form

If you had a tall matrix
the sigmas are on top and the zeros are on the bottom

$$\Sigma = \left(\begin{array}{cccc|c} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & 0_{M\times(N-M)} \\ & & & \sigma_M & \end{array}\right)$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_\rho > 0$$
$$\sigma_{\rho+1} = \cdots = \sigma_M = 0$$

singular values of J

where $\rho = \mathrm{rank}(J) \leq M$, so that their product is

so the number of non zero sigmas
is the rank of J

$$\boxed{J = U\Sigma V^T}$$

- the columns of U are eigenvectors of $J\,J^T$ (associated to its non-negative eigenvalues ${\sigma_i}^2$), the columns of V are eigenvectors of J^TJ

Proof of the first statement is on xournal++

- the last $N-\rho$ columns of V are a basis for the null space of J

$$Jv_i = \sigma_i u_i \quad (\mathrm{i} = 1, \cdots, \rho) \qquad \boxed{Jv_i = 0 \quad (\mathrm{i} = \rho+1, \cdots, N)}$$

Computation of pseudoinverses

- show that the pseudoinverse of J is equal to

$$J = U\Sigma V^T \quad \Rightarrow \quad J^\# = V\Sigma^\# U^T \qquad \Sigma^\# = \left(\begin{array}{cccc} \frac{1}{\sigma_1} & & & \\ & \ddots & & \\ & & \frac{1}{\sigma_\rho} & \\ & & & 0_{(M-\rho)\times(M-\rho)} \\ \hline & & 0_{(N-M)\times M} & \end{array}\right)$$

for any rank ρ of J

- show that matrix $J_W^\#$ appears when solving the constrained linear-quadratic (LQ) optimization problem (with $W > 0$, symmetric, and assuming J of full rank)

$$\min \frac{1}{2} \|\dot{q}\|_W^2 \quad \text{s.t.} \quad J(q)\dot{q} - \dot{r} = 0$$

and that the pseudoinverse is a particular case for $W = I$

- show that a weighted pseudoinverse of J can be computed by $\mathrm{SVD/pinv}$ as

$$J_{aux} = JW^{-1/2} \qquad J_W^\# = W^{-1/2}\,\mathrm{pinv}(J_{aux})$$

applies equally to square and non-square matrices

Singularity robustness
Damped Least Squares (DLS)

you want to have it closer to a singularity but you don't want mu when not closer to a singularity since it introduces an error

unconstrained minimization of a **suitable** objective function

$$\min_{\dot q} H(\dot q) = \frac{\mu^2}{2}\|\dot q\|^2 + \frac{1}{2}\|\dot r - J\dot q\|^2$$

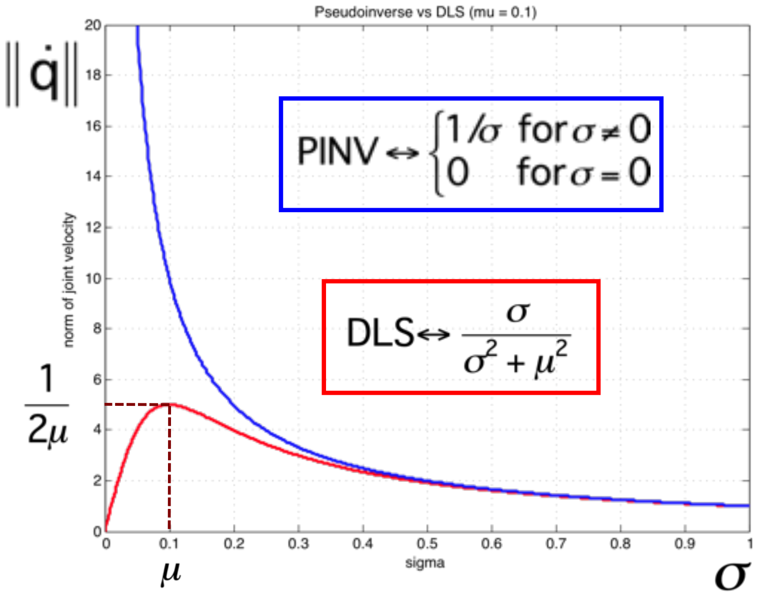
compromise between large joint velocity and task accuracy

the result is always a positive definite matrix and therefore invertible

SOLUTION $\dot q = J_{DLS}(q)\dot r = J^T\left(J\,J^T + \mu^2 I_M\right)^{-1}\dot r$

- induces a robust behavior when crossing singularities, but in its basic version gives always a task error $\dot e = \mu^2\left(J\,J^T + \mu^2 I_M\right)^{-1}\dot r$ (as for $N = M$)
  - J_{DLS} is not a generalized inverse K
  - using SVD: $J = U\,\Sigma\,V^T \Rightarrow J_{DLS} = V\Sigma_{DLS}U^T$, $\Sigma_{DLS} = \left(\begin{array}{c|c} \underset{\rho\times\rho}{\boxed{diag\left\{\frac{\sigma_i}{\sigma_i^2+\mu^2}\right\}}} & 0_{(M-\rho)\times(M-\rho)} \\ \hline 0_{(N-M)\times\rho} & 0_{(N-M)\times(M-\rho)} \end{array}\right)$
  - choice of a variable damping factor $\mu^2(q) \geq 0$, function of the minimum singular value $\sigma_\rho(q) > 0$ of $J \cong$ a measure of distance from a singularity (if $\rho = M$) or of further loss of rank (when $\rho < M$) when the singularity has gone you turn mu off putting it to zero, so to not have error
  - numerical filtering: introduces damping only/mostly in non-feasible directions for the task (see Maciejewski and Klein, *J of Rob Syst*, 1988)

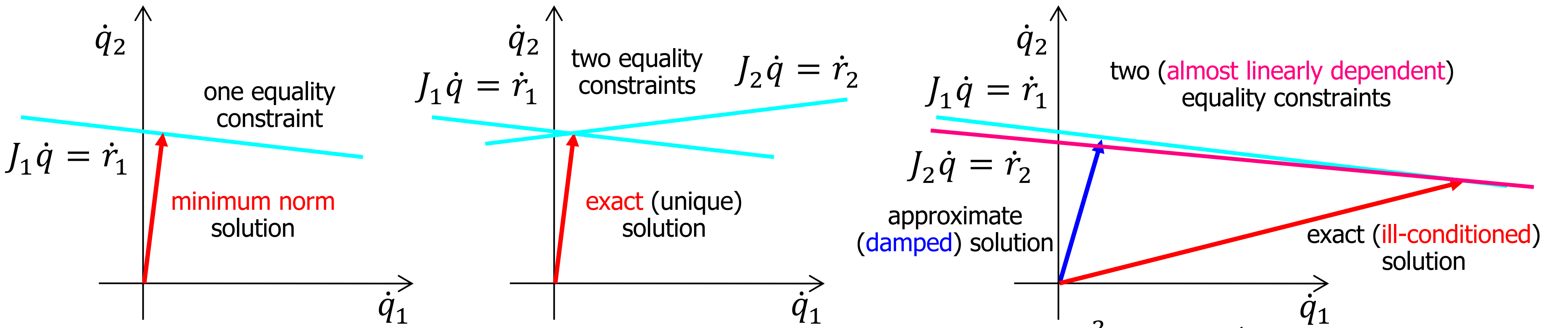
Behavior of DLS solution



a. comparison of joint velocity norm with PINV (pseudoinverse) or DLS solutions

- in a task direction along a vector u of U, when the associated singular value $\sigma \longrightarrow 0$
- PINV goes to infinity (and then is 0 at $\sigma = 0$)
- DLS peaks a value of $1/2\mu$ at $\sigma = \mu$ (and then smoothly goes to 0...)

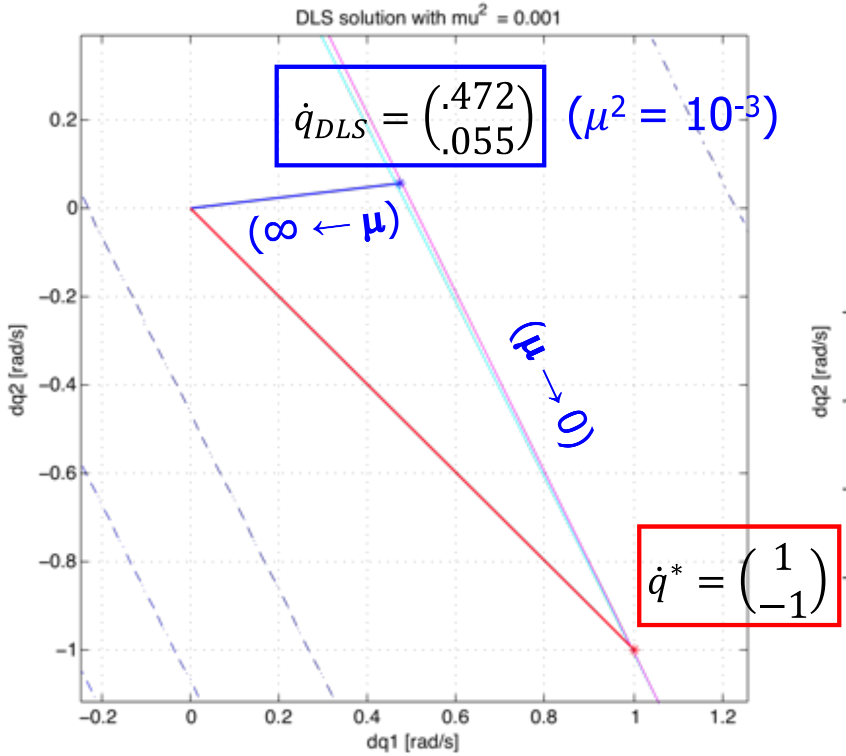
b. graphical interpretation of “damping” effect (here $M = N = 2$, for simplicity)

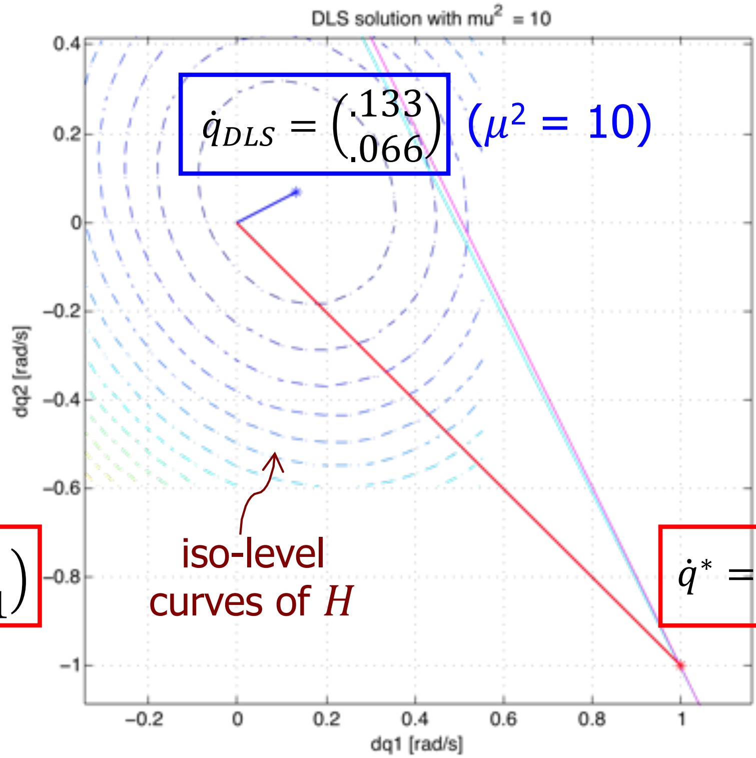


$$H(\dot{q}) = \frac{\mu^2}{2}\|\dot{q}\|^2 + \frac{1}{2}\|\dot{r} - J\dot{q}\|^2$$

Numerical example of DLS solution

planar 2R arm, unit links, close to (stretched) singular configuration $q_1 = 45°$, $q_2 = 1.5°$)





$$\dot r = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$\in \mathcal{R}(J)$ even @singularity!

exact solution (μ=0)

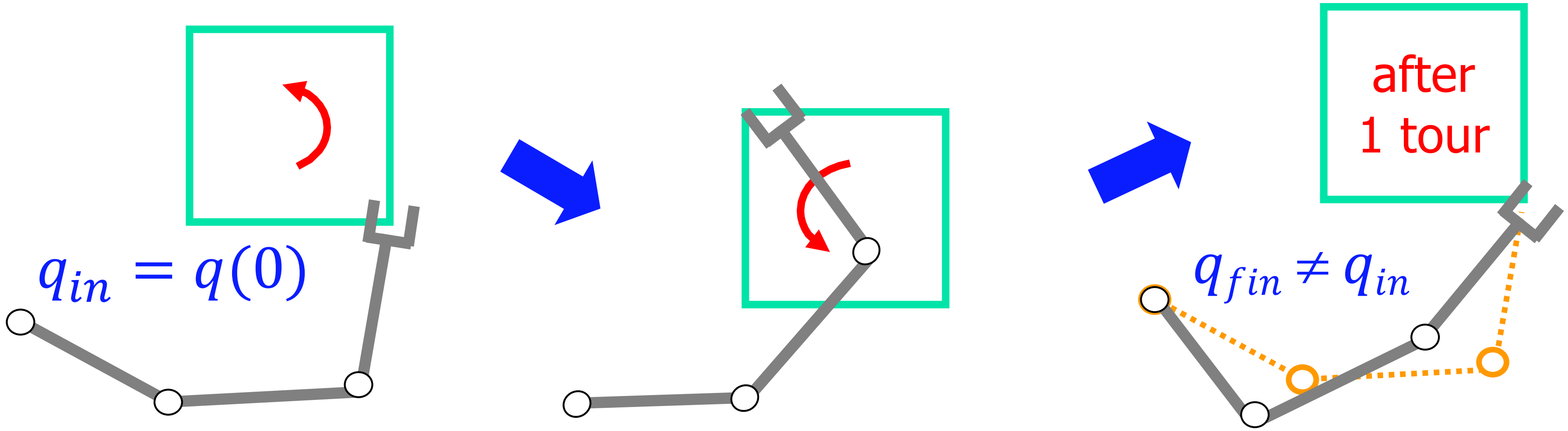
$$H = \frac{\mu^2}{2} \|\dot q\|^2 + \frac{1}{2} \|\dot r - J \dot q\|^2$$

μ^2	0	10^{-4}	10^{-3}	10^{-2}	10
$\|\dot q\|$	√2	.8954	.4755	.4467	.1490
$\|\dot e\|$	0	$6.6 \cdot 10^{-3}$	$1.4 \cdot 10^{-2}$	$1.6 \cdot 10^{-2}$	.6668
H_{min}	0	$7.7 \cdot 10^{-5}$	$2.2 \cdot 10^{-4}$	$1.2 \cdot 10^{-3}$	$3.4 \cdot 10^{-1}$

Limits of Jacobian-based methods

- no guarantee that singularities are globally avoided during task execution
  - despite joint velocities are kept to a minimum, this is only a local property and "avalanche" phenomena may occur
- typically lead to non-repeatable motion in the joint space
  - cyclic motions in task space do not map to cyclic motions in joint space

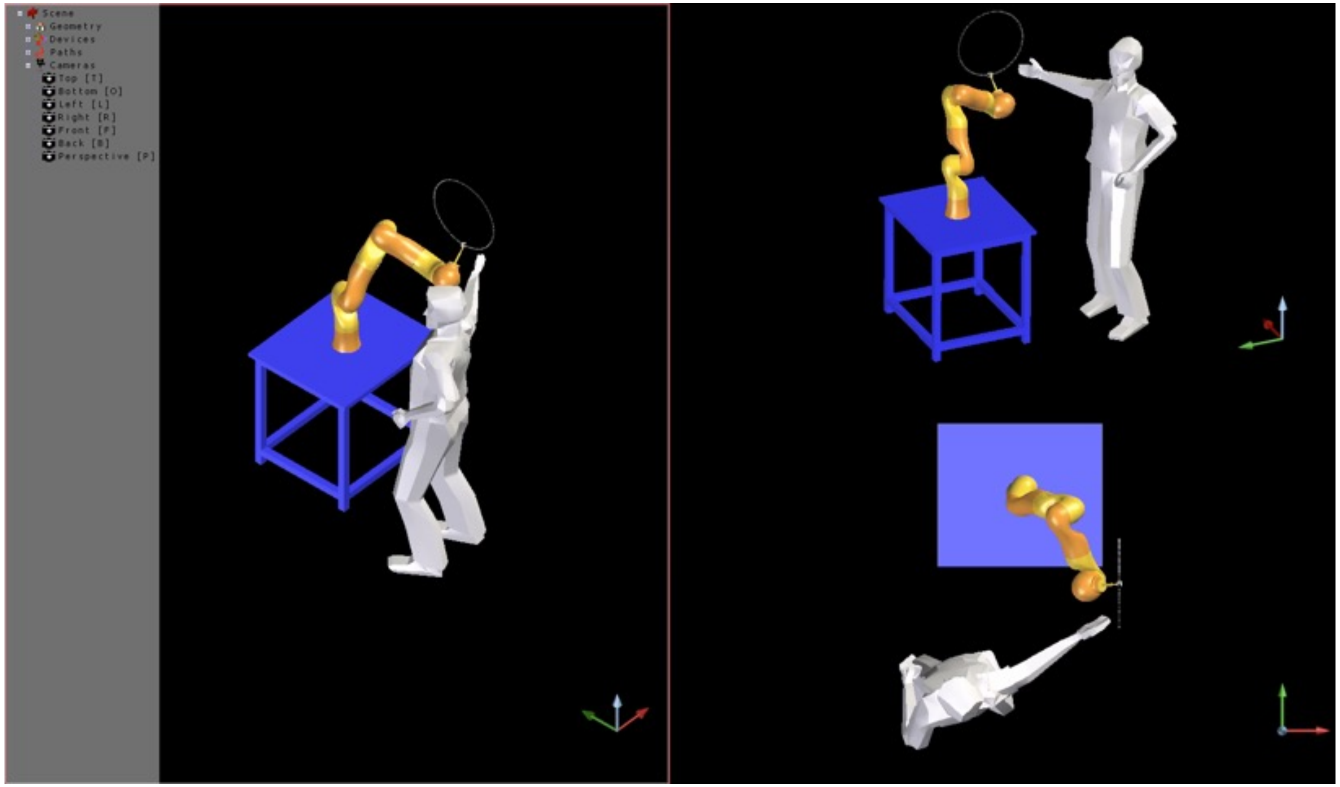
Infinite solutions since the robot is 1 degree redundant. If you have a constrained workspace, changing configuration you may collide with an object



$$q(t) = q(0) + \int_0^t K\big(q(\tau)\big)\dot{r}(\tau)\, d\tau$$

Drift with Jacobian pseudoinverse

- a 7R KUKA LWR4 robot moves in the vicinity of a human operator
- we command a cyclic Cartesian path (only in position, $M = 3$), to be repeated several times using the pseudoinverse solution
- (unexpected) collision of a link occurs during the third cycle ...



video

2 Null-space methods

general solution of $J\dot{q} = \dot{r}$

$$\dot{q} = \underbrace{J^{\#}\dot{r}}_{\text{a particular solution}} + \underbrace{\left(I - J^{\#}J\right)}_{\text{"orthogonal" projection}}\dot{q}_0$$

all solutions of the associated homogeneous equation $J\dot{q} = 0$ (self-motions)

a particular solution (here, the pseudoinverse) in $\mathcal{R}(J^T)$

"orthogonal" projection of $\dot{q}_0$ in $\mathcal{N}(J)$

properties of projector $[I - J^{\#}J]$

- symmetric
- idempotent: $[I - J^{\#}J]^2 = [I - J^{\#}J]$
- $[I - J^{\#}J]^{\#} = [I - J^{\#}J]$
- $J^{\#}\dot{r}$ is orthogonal to $[I - J^{\#}J]\dot{q}_0$

even more in general...

$$\dot{q} = K_1\dot{r} + (I - K_2 J)\dot{q}_0$$

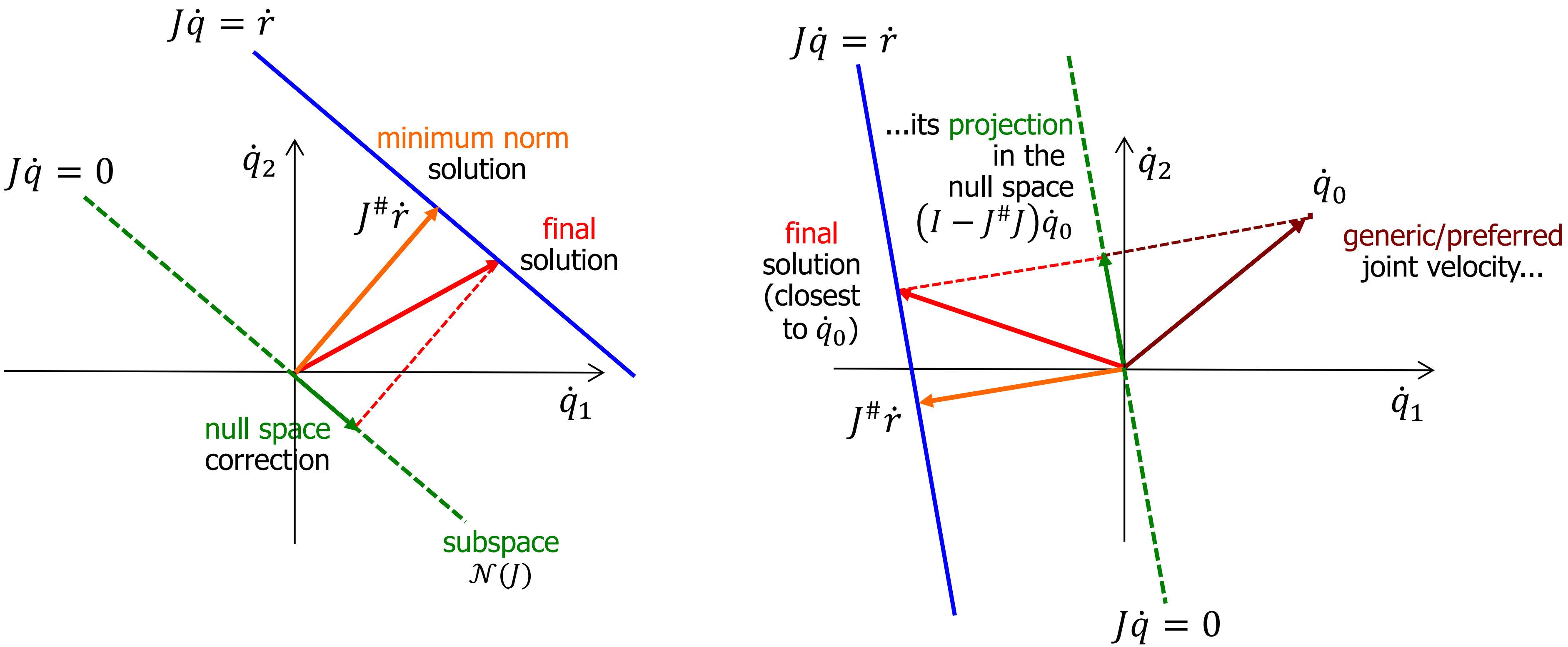
K_1, K_2 generalized inverses of J $(JK_iJ = J)$

... but with less nice properties!

$\boxed{\text{how do we choose } \dot{q}_0\text{?}}$

Geometric view on Jacobian null space

in the space of velocity commands



a correction is added to the original pseudoinverse (minimum norm) solution
i) which is in the null space of the Jacobian
ii) and possibly satisfies additional criteria or objectives

Linear-Quadratic Optimization

generalities

$$\min_x H(x) = \frac{1}{2}(x - x_0)^T W (x - x_0)$$

$$\text{s.t.} \quad \underbrace{J}_{M \times N} x = y$$

$x \in \mathbb{R}^N$

$W > 0$ (symmetric)

$y \in \mathbb{R}^M$

$\text{rank}(J) = \rho(J) = M$

$$L(x, \lambda) = H(x) + \lambda^T (Jx - y)$$ ← Lagrangian (with multipliers λ)

necessary conditions

$$\nabla_x L = \left(\frac{\partial L}{\partial x}\right)^T = W(x - x_0) + J^T \lambda = 0 \quad \Rightarrow \quad \boxed{x = x_0 - W^{-1} J^T \lambda}$$

$$\nabla_\lambda L = \left(\frac{\partial L}{\partial \lambda}\right)^T = Jx - y = 0 \quad \Rightarrow \quad \boxed{J x_0 - J W^{-1} J^T \lambda - y = 0}$$

\+ sufficient condition for a minimum

$$\nabla_x^2 L = W > 0$$

$$\boxed{\lambda = \underbrace{(J W^{-1} J^T)^{-1}}_{M \times M \text{ invertible}} (J x_0 - y)} \quad \Rightarrow \quad x = x_0 + W^{-1} J^T (J W^{-1} J^T)^{-1} (y - J x_0)$$

Linear-Quadratic Optimization

application to robot redundancy resolution

PROBLEM

$$\min_{\dot q} H(\dot q) = \frac{1}{2}(\dot q - \dot q_0)^T W (\dot q - \dot q_0)$$

$$\text{s.t. } J\dot q = \dot r$$

$\dot q_0$ is a "privileged" joint velocity

SOLUTION

$$\boxed{\dot q = \dot q_0 + \underbrace{W^{-1}J^T(JW^{-1}J^T)^{-1}}_{J_W^{\#}}(\dot r - J\dot q_0)}$$

$$\dot q = \underbrace{J_W^{\#}\dot r}_{\substack{\text{minimum weighted norm} \\ \text{solution (for } \dot q_0 = 0)}} + \underbrace{\left(I - J_W^{\#}J\right)}_{\substack{\text{"projection" matrix in} \\ \text{the null-space } \mathcal{N}(J)}}\dot q_0$$

Projected Gradient (PG)

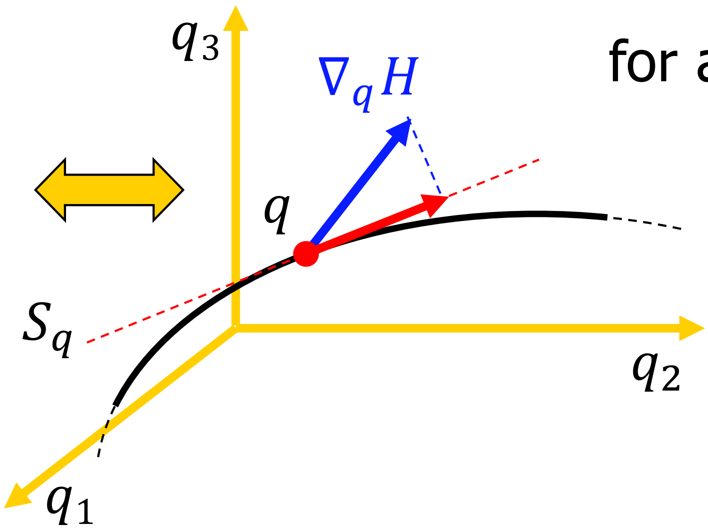
$$\dot{q} = J^{\#}\dot{r} + \left(I - J^{\#}J\right)\dot{q}_0$$

the choice $\dot{q}_0 = \nabla_q H(q) \rightarrow$ differentiable objective function

realizes one step of a constrained optimization algorithm

while executing the time-varying task $r(t)$
the robot tries to increase the value of $H(q)$

projected gradient



for a fixed $\bar{r}$: $S_q = \{q \in \mathbb{R}^N : f(q) = \bar{r}\}$

$$\Rightarrow \quad \dot{q} = \left(I - J^{\#}J\right)\nabla_q H$$

N-dimensional $\boxed{\left(I - J^{\#}J\right)\nabla_q H = 0}$

is a necessary condition
of constrained optimality

Typical objective functions $H(q)$

- manipulability (maximize the “distance” from singularities)

$$H_{\text{man}}(q) = \sqrt{\det[J(q)J^T(q)]}$$

- joint range (minimize the “distance” from the mid points of the joint ranges)

$$q_i \in [q_{m,i}, q_{M,i}] \qquad \bar{q}_i = \frac{q_{M,i} + q_{m,i}}{2}$$

$$H_{\text{range}}(q) = \frac{1}{2N} \sum_{i=1}^{N} \left(\frac{q_i - \bar{q}_i}{q_{M,i} - q_{m,i}} \right)^2$$

$$\dot{q}_0 = -\nabla_q H(q)$$

- obstacle avoidance (maximize the minimum distance to Cartesian obstacles)

also known as “clearance”

$$H_{\text{obs}}(q) = \min_{\substack{a \in \text{robot} \\ b \in \text{obstacles}}} \|a(q) - b\|^2$$

potential difficulties due to non-differentiability (this is a max-min problem)

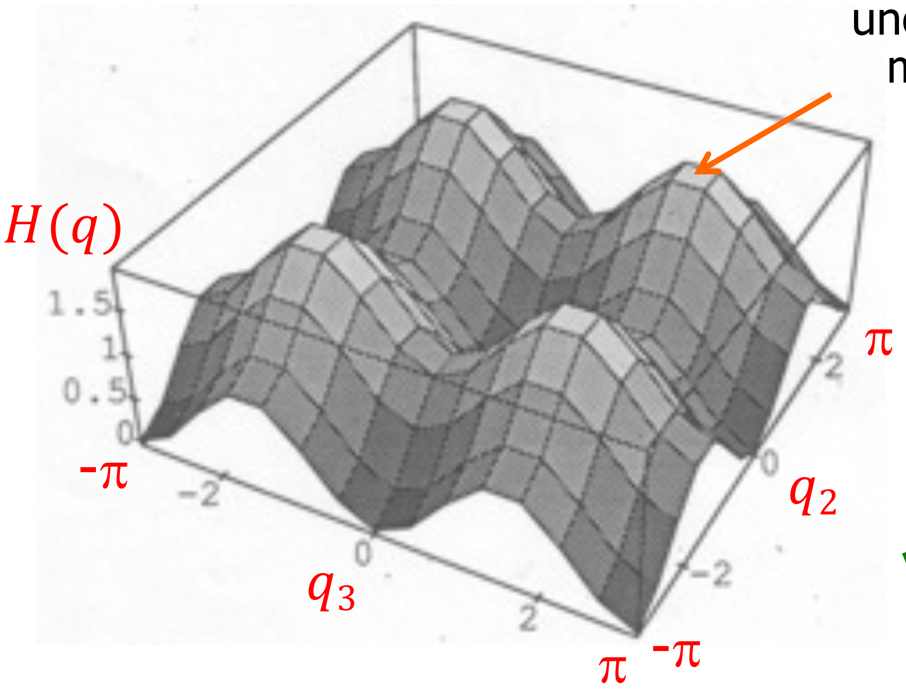
Singularities of planar 3R arm

the robot is redundant for a positioning task in the plane ($M = 2$)

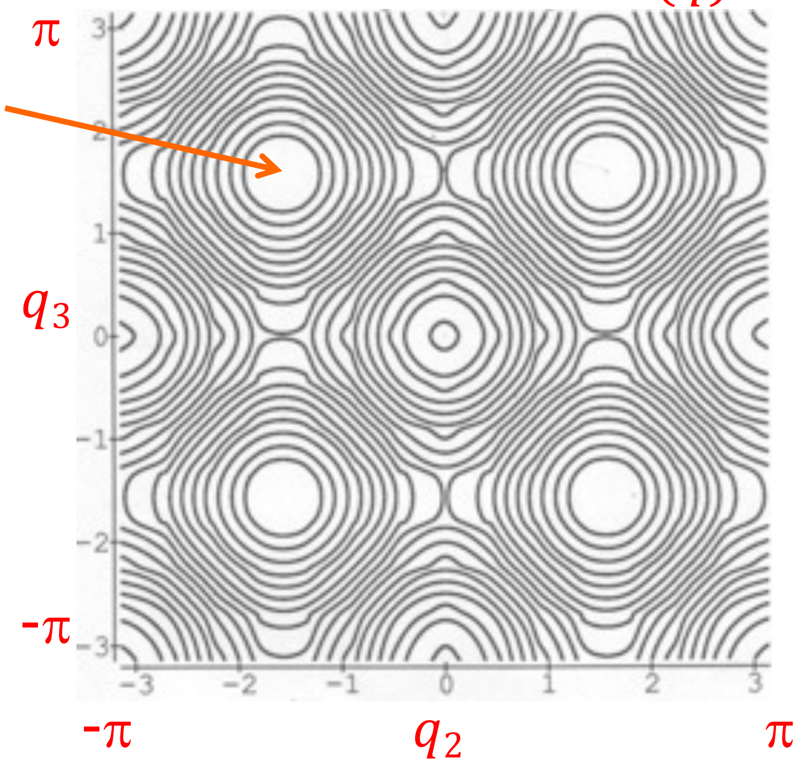
$$H(q) = \sin^2 q_2 + \sin^2 q_3$$

this H is **not** H_{man} but has the same minima

unconstrained maxima of $H(q)$

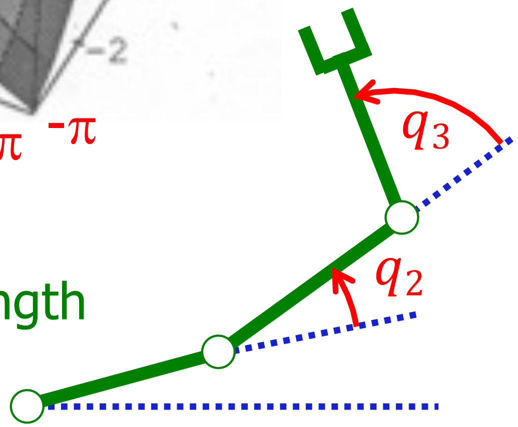


iso-level curves of $H(q)$

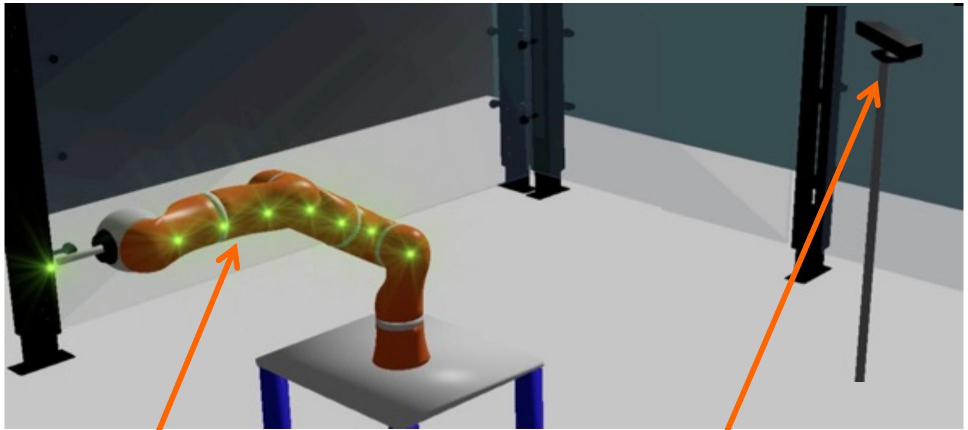


independent from q_1!

links of **equal** (unit) length



Minimum distance computation
in human-robot interaction

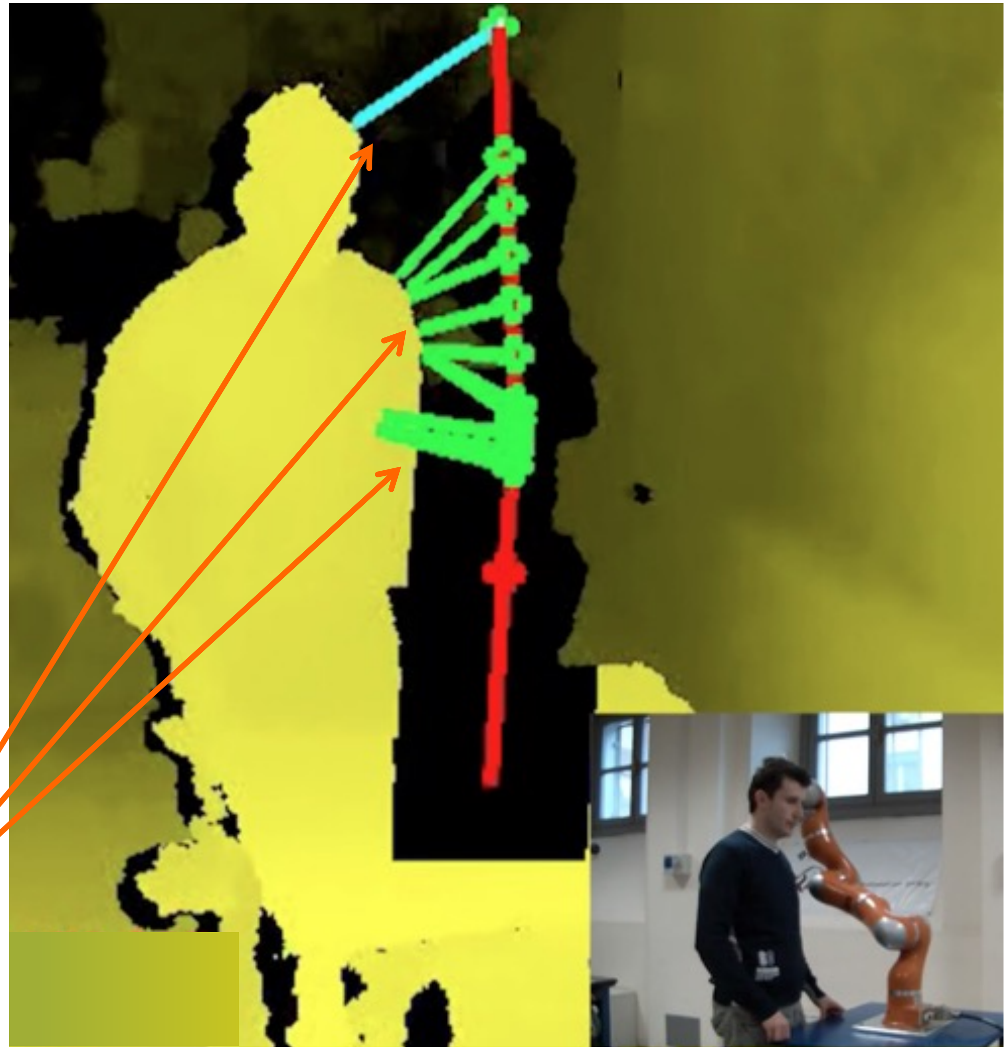


LWR4 robot with a finite number of control points $\boldsymbol{a}(\boldsymbol{q})$ (8, including the E-E)

a Kinect sensor monitors the workspace giving the 3D position of points $\boldsymbol{b}$ on obstacles that are fixed or moving (like humans)

distances in 3D (and then the clearance) are computed in this case as

$$\min_{\substack{a \,\in\, \{\text{control points}\} \\ b \,\in\, \text{human body}}} \|a(q) - b\|^2$$



Comments on null-space methods

- the projection matrix $(I - J^{\#}J)$ has dimension $N \times N$, but only rank $N - M$ (if J is full rank M), with some waste of information
- actual (efficient) evaluation of the solution

$$\dot{q} = J^{\#}\dot{r} + \left(I - J^{\#}J\right)\dot{q}_0 = \dot{q}_0 + J^{\#}(\dot{r} - J\dot{q}_0)$$

  but the pseudoinverse is needed anyway, and this is computationally intensive (SVD in the general case)
- in principle, the additional complexity of a redundancy resolution method should depend only on the redundancy degree $N - M$
- a constrained optimization method is available, which is known to be more efficient than the projected gradient (PG) —at least when the Jacobian has full rank …

Decomposition of joint space

- if $\rho\big(J(q)\big) = M$, there exists a decomposition of the set of joints (possibly, after a reordering)

$$q = \begin{pmatrix} q_a \\ q_b \end{pmatrix} \begin{matrix} \} \; M \\ \} \; N-M \end{matrix} \qquad \text{such that } \overbrace{J_a(q) = \frac{\partial f}{\partial q_a}}^{M \times M} \text{ is nonsingular}$$

- from the implicit function theorem, there exists an inverse function g

$$f(q_a, q_b) = r \quad \Rightarrow \quad q_a = g(r, q_b)$$

$$\text{with } \frac{\partial g}{\partial q_b} = -\left(\frac{\partial f}{\partial q_a}\right)^{-1} \frac{\partial f}{\partial q_b} = -J_a^{-1}(q) J_b(q)$$

- the $N-M$ variables q_b can be selected independently (e.g., they are used for optimizing an objective function $H(q)$, "reduced" via the use of g to a function of q_b only)
- $q_a = g(r, q_b)$ is then chosen so as to correctly execute the task

Reduced Gradient (RG)

- $H(q) = H(q_a, q_b) = H(g(r, q_b), q_b) = H'(q_b)$, with r at current value
- the Reduced Gradient (w.r.t. q_b only, but still keeping the effects of this choice into account) is

$$\nabla_{q_b} H' = \begin{bmatrix} -(J_a^{-1} J_b)^T & I_{N-M} \end{bmatrix} \nabla_q H$$

($\neq \nabla_{q_b} H$ only!!)

$$\boxed{\nabla_{q_b} H' = 0}$$

is a "compact" (i.e., $N - M$ dimensional) necessary condition of constrained optimality

- algorithm

$$\dot{q}_b = \nabla_{q_b} H'$$

step in the gradient direction of the reduced $(N - M)$-dim space

$$J_a \dot{q}_a + J_b \dot{q}_b = \dot{r}$$

satisfaction of the M-dim task constraints

$$\dot{q}_a = J_a^{-1}(\dot{r} - J_b \dot{q}_b)$$

Comparison between PG and RG

- Projected Gradient (PG)

$$\dot{q} = J^{\#}\dot{r} + \left(I - J^{\#}J\right)\nabla_q H$$

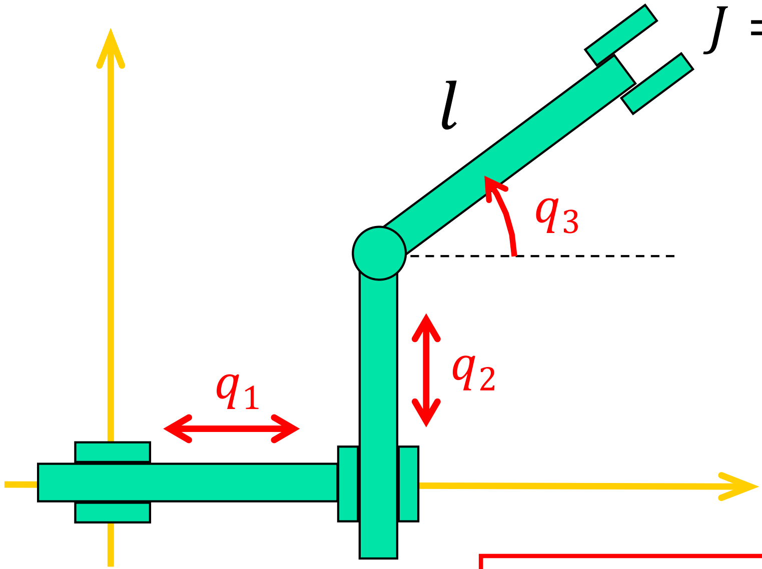
- Reduced Gradient (RG)

$$\dot{q} = \begin{pmatrix} \dot{q}_a \\ \dot{q}_b \end{pmatrix} = \begin{pmatrix} J_a^{-1} \\ 0 \end{pmatrix} \dot{r} + \begin{pmatrix} -J_a^{-1}J_b \\ I \end{pmatrix} \left(-(J_a^{-1}J_b)^T \quad I\right) \nabla_q H$$

- RG is analytically simpler and numerically faster than PG, but requires the search for a non-singular minor (J_a) of the robot Jacobian
- if r = cost & $N - M = 1$ ⇒ same (unique) direction for $\dot{q}$, but RG has automatically a larger optimization step size
- else ⇒ RG and PG methods provide always different evolutions

Analytic comparison

PPR robot



$$J = \begin{pmatrix} 1 & 0 & -ls_3 \\ 0 & 1 & lc_3 \end{pmatrix} = (\, J_a \mid J_b \,) \quad q_a = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \quad q_b = q_3$$

RG: $\dot{q} = \begin{pmatrix} J_a^{-1} \\ 0 \end{pmatrix} \dot{r} + \begin{pmatrix} -J_a^{-1} J_b \\ I \end{pmatrix} \left(-(J_a^{-1} J_b)^T \quad I \right) \nabla_q H$

$$\dot{q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \dot{r} + \begin{pmatrix} ls_3 \\ -lc_3 \\ 1 \end{pmatrix} \left(ls_3 \quad -lc_3 \quad 1 \right) \nabla_q H$$

PG: $\dot{q} = J^{\#} \dot{r} + \left(I - J^{\#} J \right) \nabla_q H$

$$J^{\#} = \frac{1}{1 + l^2} \begin{pmatrix} 1 + l^2 c_3^2 & l^2 s_3 c_3 \\ l^2 s_3 c_3 & 1 + l^2 s_3^2 \\ -ls_3 & lc_3 \end{pmatrix} \qquad I - J^{\#} J = \frac{1}{1 + l^2} \begin{pmatrix} l^2 s_3^2 & l^2 s_3 c_3 & ls_3 \\ l^2 s_3 c_3 & l^2 c_3^2 & -lc_3 \\ ls_3 & -lc_3 & 1 \end{pmatrix}$$

always < 1!!

Joint range limits

$$q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \theta = T\theta$$

$$\boxed{-90° \leq \theta_i \leq 90°}$$

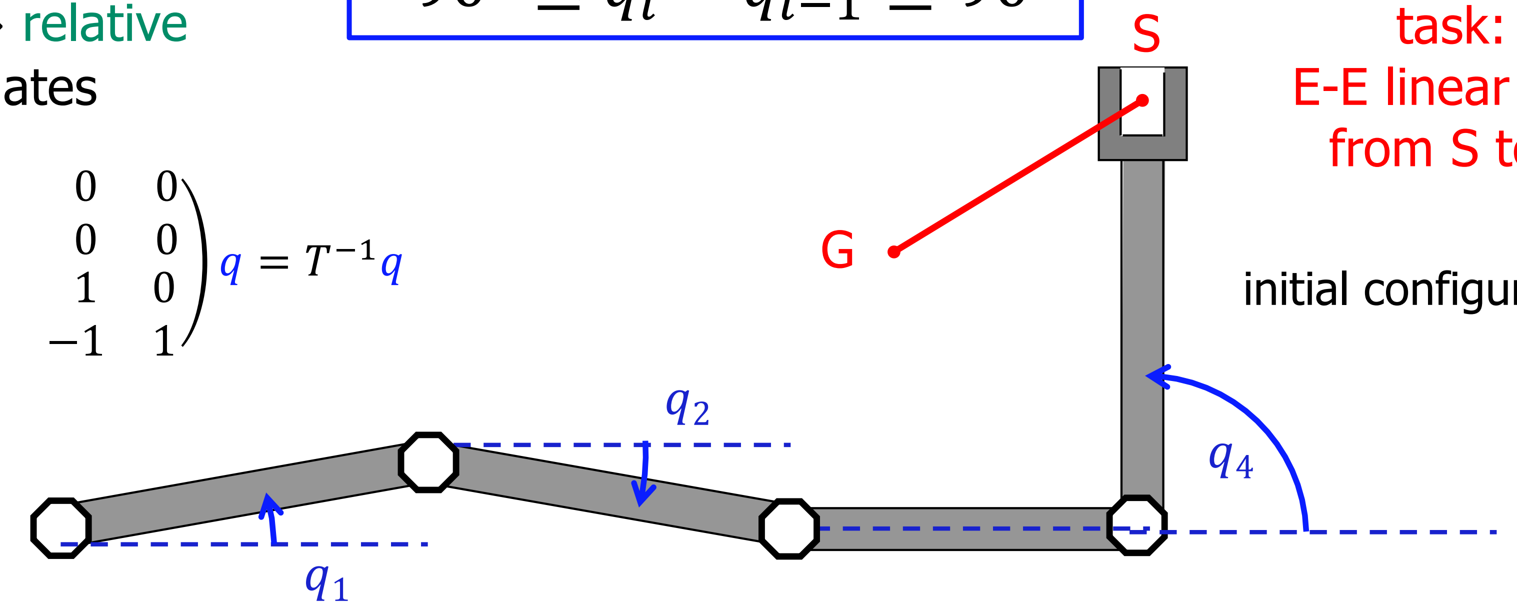
$$\Updownarrow$$

$$\boxed{-90° \leq q_i - q_{i-1} \leq 90°}$$

absolute ⇔ relative
coordinates

$$\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} q = T^{-1} q$$

task:
E-E linear path
from S to G

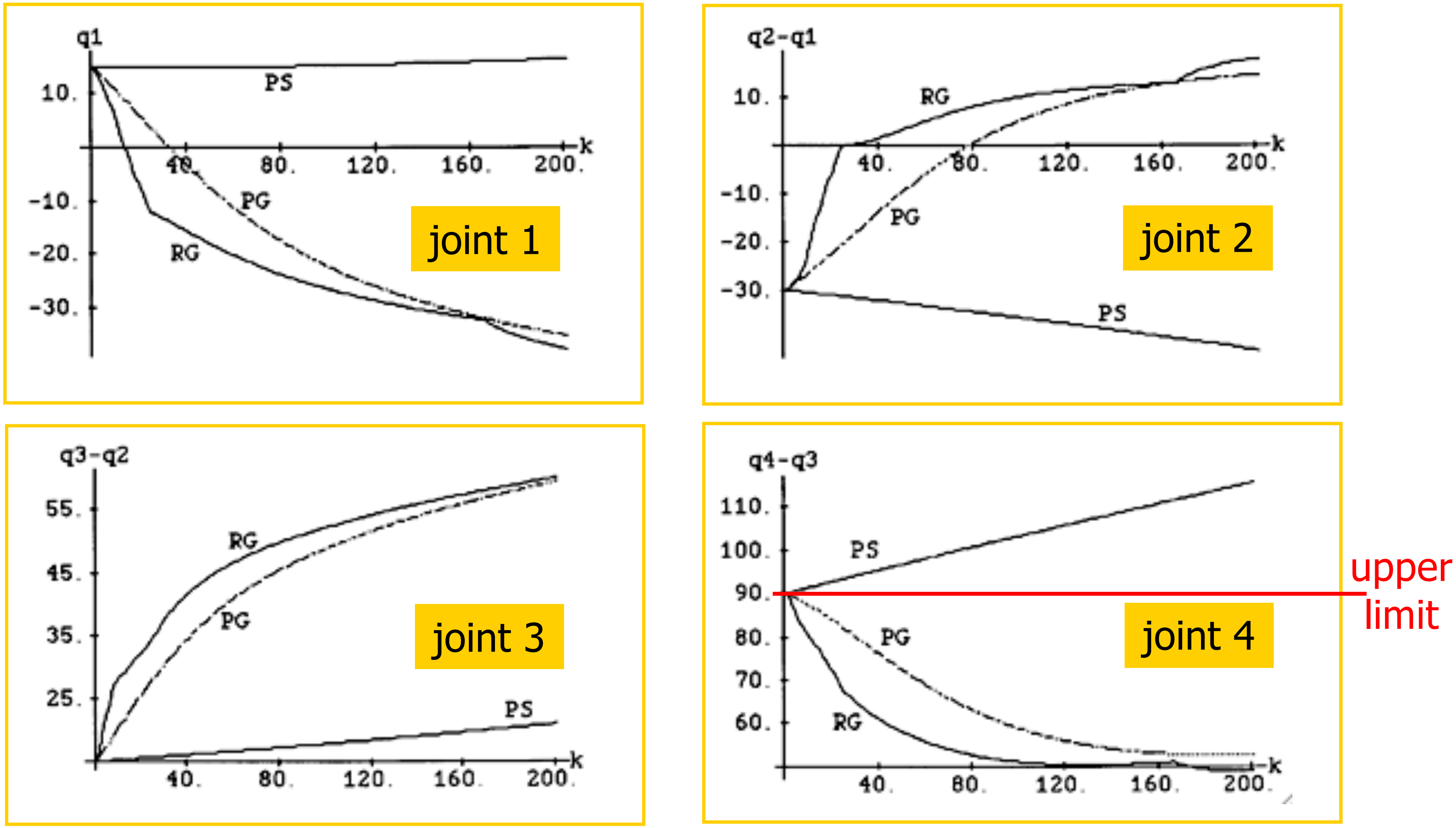


initial configuration

numerical comparison among pseudoinverse (PS), projected gradient (PG), and reduced gradient (RG) methods

Numerical results

minimizing distance from mid joint range



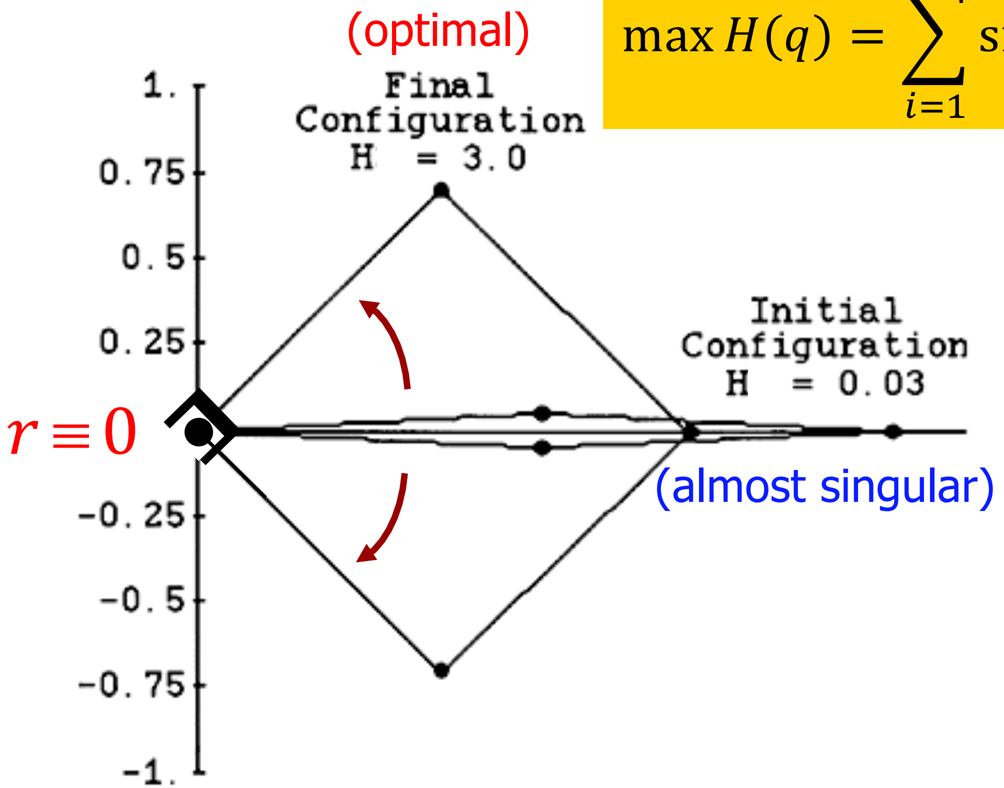
steps of numerical simulation

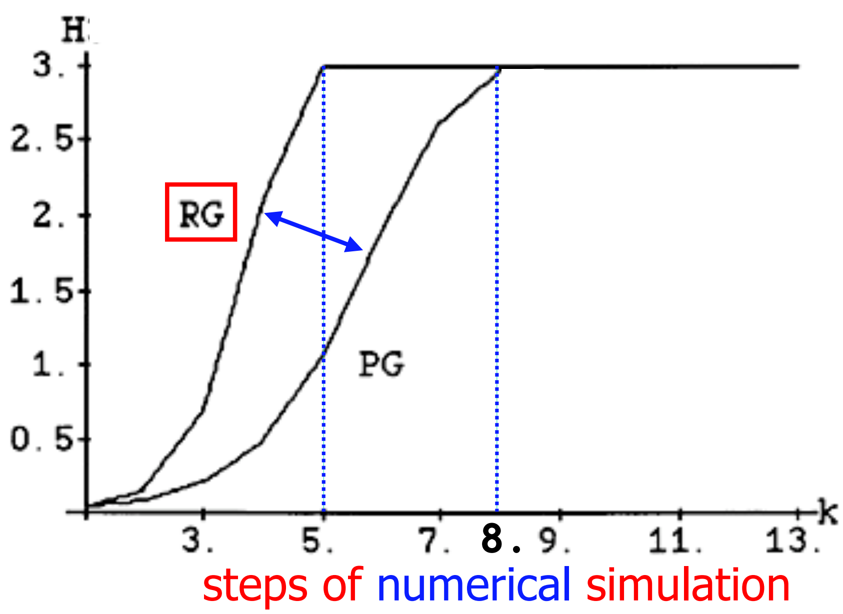
Numerical results

self-motion for escaping singularities

$$\max H(q) = \sum_{i=1}^{3} \sin^2(q_{i+1} - q_i)$$

this function is NOT the manipulability index, but has the same minima (= 0)





RG is faster than PG
(keeping the same accuracy on r)

3 Task augmentation methods

- an auxiliary task is added (task augmentation)

$$S \updownarrow \quad \boxed{f_y(q) = y} \qquad S \leq N - M$$

corresponding to some desirable feature for the solution

$$r_A = \begin{pmatrix} r \\ y \end{pmatrix} = \begin{pmatrix} f(q) \\ f_y(q) \end{pmatrix} \Rightarrow \dot{r}_A = \begin{pmatrix} J(q) \\ J_y(q) \end{pmatrix} \dot{q} = J_A(q)\dot{q} \qquad \underbrace{\boxed{J_A}}_{N} \} M+S$$

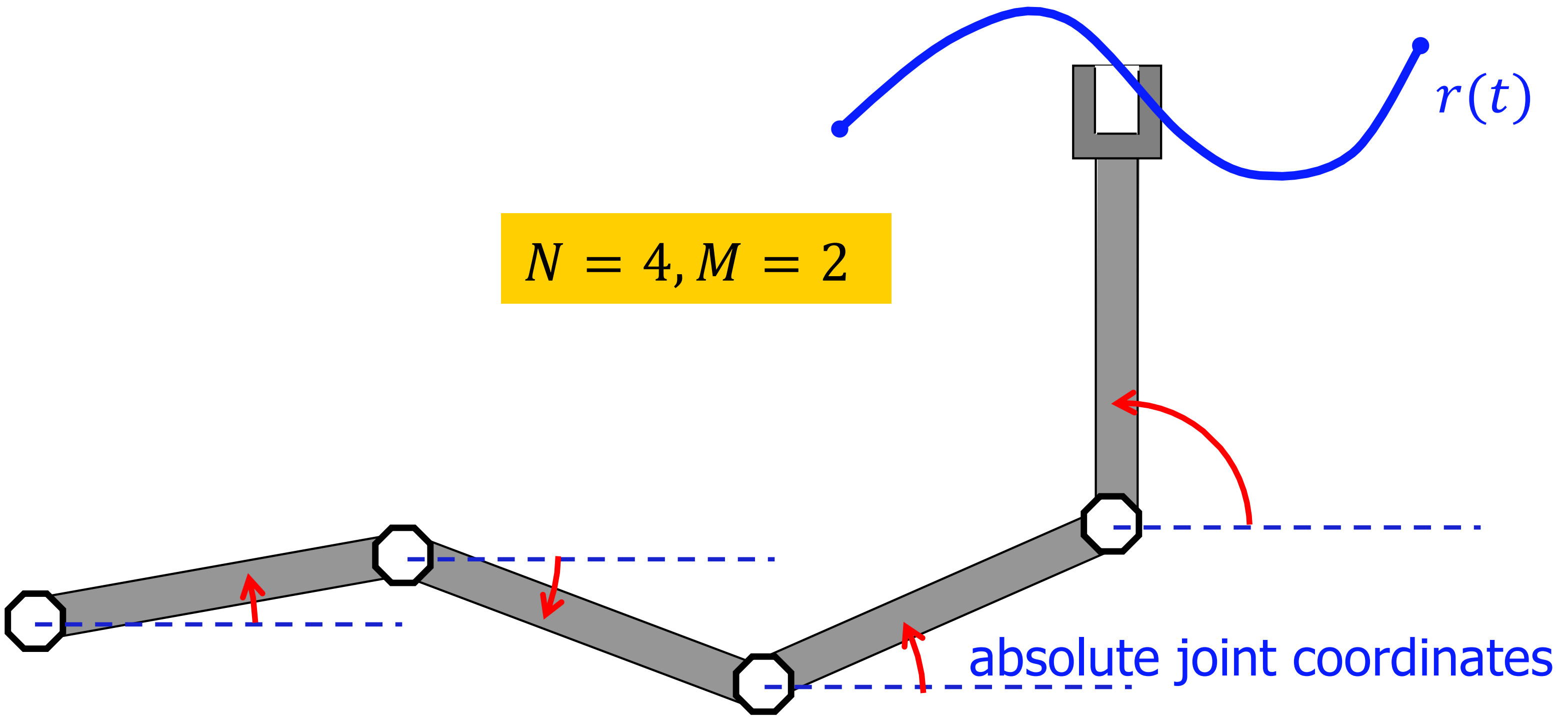
- a solution is chosen still in the form of a generalized inverse

$$\boxed{\dot{q} = K_A(q)\dot{r}_A}$$

or by adding a term in the null space of the augmented Jacobian matrix J_A

Augmented task

example



$$f_y(q) = q_4 = \pi/2 \qquad (S = 1)$$

last link is to be held vertical...

Augmenting the task …

- advantage: better shaping of the inverse kinematic solution

- disadvantage: algorithmic singularities are introduced when

$$\rho(J) = M \qquad \rho(J_y) = S \qquad \text{but} \qquad \rho(J_A) < M + S$$

to avoid this, it should be always $\boxed{\mathcal{R}(J^T) \cap \mathcal{R}(J_y^T) = \emptyset}$

difficult to be obtained globally!

$\Updownarrow$

rows of J AND rows of J_y
are all together linearly independent

Extended Jacobian ($S = N-M$)

- square J_A: in the absence of algorithmic singularities, we can choose

$$\dot{q} = J_A^{-1}(q)\dot{r}_A$$

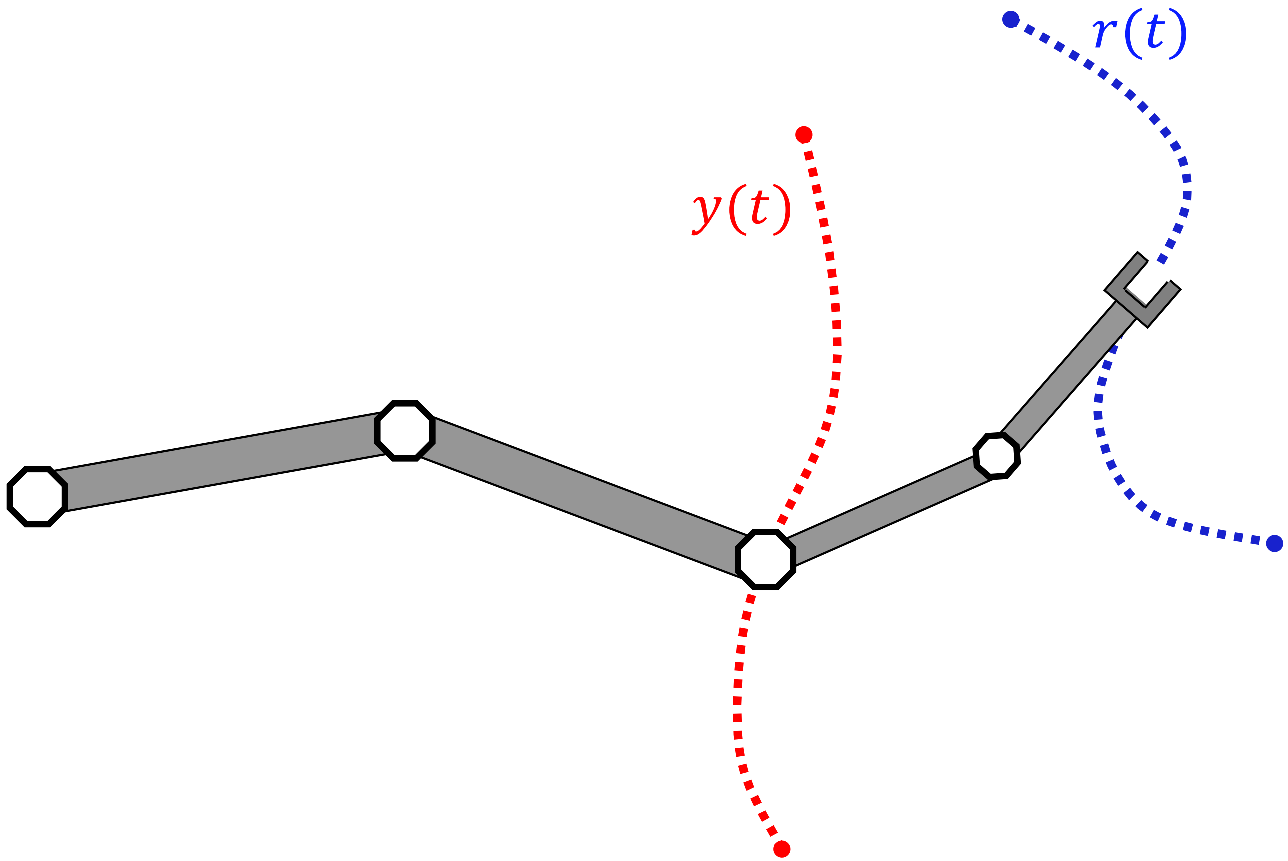
- the scheme is then repeatable
  - provided no singularities are encountered during a complete task cycle*
- when the $N - M$ conditions $f_y(q) = 0$ correspond to necessary (and sufficient) conditions for constrained optimality of a given objective function $H(q)$ (see RG method, slide #36), this scheme guarantees that constrained optimality is locally preserved during task execution
- in the vicinity of algorithmic singularities, for the simultaneous execution of the original task and the auxiliary task(s), one can use the DLS method; however, both tasks will be affected by errors

* there exists an unexpected phenomenon in some 3R manipulators having "generic" kinematics: the robot may sometimes perform a pose change after a full cycle, even if no singularity has been encountered during motion (see J. Burdick, *Mech. Mach. Theory*, 30(1), 1995)

Extended Jacobian

example

MACRO-MICRO manipulator



$N = 4, M = 2$

$$\dot{r} = J(q_1, \dots, q_4)\dot{q}$$

$$\dot{y} = J_y(q_1, q_2)\dot{q}$$

$$\Downarrow$$

$$J_A = \left(\begin{array}{c|c} * & * \\ * & 0 \end{array}\right) \quad 4\times 4$$

Task Priority

if the original (primary) task $\dot{r}_1 = J_1(q)\dot{q}$ has higher priority than the auxiliary (secondary) task $\dot{r}_2 = J_2(q)\dot{q}$

- we first address the task with highest priority

$$\dot{q} = J_1^{\#}\dot{r}_1 + \left(I - J_1^{\#}J_1\right)v_1$$

- and then choose v_1 so as to satisfy, if possible, also the secondary (lower priority) task

$$\dot{r}_2 = J_2\dot{q} = J_2\,J_1^{\#}\dot{r}_1 + J_2\left(I - J_1^{\#}J_1\right)v_1 = J_2\,J_1^{\#}\dot{r}_1 + J_2 P_1 v_1$$

the general solution for v_1 takes the usual form

this second term is an extra term

$$v_1 = (J_2P_1)^{\#}\left(\dot{r}_2 - J_2\,J_1^{\#}\dot{r}_1\right) + \left(I - (J_2P_1)^{\#}(J_2P_1)\right)v_2$$

v_2 is available for the execution of further tasks of lower (ordered) priorities

Task Priority (continue)

- substituting the expression of v_1 in $\dot{q}$

$$\dot{q} = J_1^{\#}\dot{r}_1 + P_1(J_2P_1)^{\#}(\dot{r}_2 - J_2J_1^{\#}\dot{r}_1) + P_1\left(I - (J_2P_1)^{\#}(J_2P_1)\right)v_2$$

$P(BP)^{\#} = (BP)^{\#}$
when matrix P is
idempotent and symmetric

$= (J_2P_1)^{\#}$

possibly $= 0$

- main advantage: highest priority task is ideally no longer affected by algorithmic singularities (error is restricted only to secondary task)