Robotics II

June 11, 2018

Exercise 1

The dynamic model of the planar RP robot in Fig. 1 moving in a vertical plane can be written in the usual form as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}. \tag{1}$$

- Define two different matrices $\boldsymbol{S}_1(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\boldsymbol{S}_2(\boldsymbol{q}, \dot{\boldsymbol{q}})$ that factorize the Coriolis and centrifugal terms (i.e., yielding $\boldsymbol{S}_i(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, for $i = 1, 2$), such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}_1$ is skew-symmetric, while $\dot{\boldsymbol{M}} - 2\boldsymbol{S}_2$ is not.
- Give the explicit symbolic expressions of the terms appearing in the definition of the momentum-based residual vector $\boldsymbol{r} \in \mathbb{R}^2$ that allows detection and isolation of collisions.
- Are there situations in which collision forces $\boldsymbol{F}_K \in \mathbb{R}^2$ in the plane of motion lead to poor or no detection, or to incorrect isolation of the involved link? Discuss the issue.

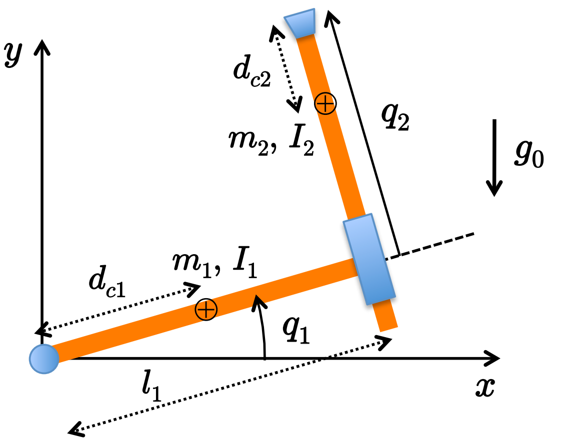

Figure 1: A planar RP robot, with the definition of joint variables and parameters.

Exercise 2

NOT TO BE ESTIMATED !

Consider an actuated pendulum with a link of $\boxed{\text{(known)}}$ length l that moves without friction in the vertical plane. The pendulum is driven by a DC motor at the base and carries a heavy and *unknown* payload. Assume that the motor inertia and the mass and inertia of the link are negligible with respect to the payload, which should be seen as a concentrated mass m at the tip of the pendulum. The drive gain k_i of the current-to-torque relation $\tau = k_i\, i_m$ of the motor is *unknown*, and only the motor current i_m can be commanded.

Design an adaptive control law for i_m that achieves global asymptotic tracking of a smooth desired trajectory $\theta_d(t)$ for the joint angle θ of the pendulum. Provide a (sketch of) proof of your result.

Exercise 3

With reference to Fig. 2, a mass $m > 0$ moves under the action of a control force F, in the presence of viscous friction with coefficient $d > 0$, and interacts with a stiff environment. We would like to regulate the contact force F_c to a constant desired value $F_d > 0$. The contact force is measured by a load cell of stiffness $k_s > 0$, i.e., $F_c = k_s x$ where $x = 0$ corresponds to the initial contact position with the environment. Consider a class of control laws of the form

$$F = \alpha k_f (F_d - F_c) + \beta F_d, \tag{2}$$

where $k_f > 0$, and with:

1. $(\alpha, \beta) = (0, 1)$ [pure feedforward];
2. $(\alpha, \beta) = (1, 0)$ [pure proportional feedback];
3. $(\alpha, \beta) = (1, 1)$ [combined feedback/feedforward].

- For each of the three above control cases, check the system equilibrium and verify its stability properties, giving a proof of your statements (e.g., via Lyapunov/LaSalle, or using Laplace analysis in view of the linearity of the system) and briefly discussing the benefits and limitations of each law.
- How robust are these results with respect to uncertainty in the knowledge of the physical parameters m. d, and k_s?
- Explain what happens under the action of the above control laws when there is no environment present ($F_c \equiv 0$). Would the system reach some form of steady state?

$\ddot{x} = 0$

$m\ddot{x} - k_s x - d\dot{x} = F$

case 1)

$m\ddot{x} + k_s x + d\dot{x} = F_d$

$x_1 = x$

$\dot{x}_1 = x_2$

$\ddot{x} = \dot{x}_2 = \frac{F - K_s x - d\dot{x}}{m} = \frac{F_d - K_s x_1 - d x_2}{m}$

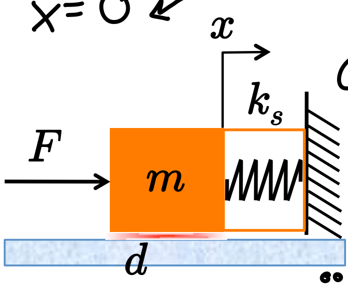

Figure 2: A mass in contact through a load cell with a stiff environment.

$A = \begin{pmatrix} 0 & 1 \\ -k_s/m & -d/m \end{pmatrix}$

EQ. when ACC $\ddot{x}$ AND VEL $\dot{x} = 0$

[240 minutes; open books, but no computer or smartphone]

$b = \begin{pmatrix} 0 \\ +F_d/m \end{pmatrix}$

$\dot{x}_1 = 0 = x_2$

$\dot{x}_2 = \frac{F_d - k_s x_1 - d\phi}{m} = \frac{F_d - k_s x_1}{m} = 0$

$x_1 = \frac{F_d}{k_s}$ eq. point

Solution

June 11, 2018

Exercise 1

For later use, we derive the terms in the dynamic model (1) following a Lagrangian approach. The kinetic energy of the first link is

$$T_1 = \frac{1}{2}(I_1 + m_1 d_{c1}^2)\dot{q}_1^2.$$

The position and velocity of the center of mass of the second link are, respectively,

$$\boldsymbol{p}_{c2} = \begin{pmatrix} l_1 \cos q_1 - (q_2 - d_{c2}) \sin q_1 \\ l_1 \sin q_1 + (q_2 - d_{c2}) \cos q_1 \end{pmatrix}, \qquad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -\sin q_1 (l_1 \dot{q}_1 + \dot{q}_2) - (q_2 - d_{c2}) \cos q_1 \, \dot{q}_1 \\ \cos q_1 (l_1 \dot{q}_1 + \dot{q}_2) - (q_2 - d_{c2}) \sin q_1 \, \dot{q}_1 \end{pmatrix}.$$

Therefore, the kinetic energy of the second link is

$$T_2 = \frac{1}{2} I_2 \, \dot{q}_1^2 + \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 = \frac{1}{2}(I_2 + m_2 (q_2 - d_{c2})^2)\dot{q}_1^2 + \frac{1}{2} m_2 (l_1 \dot{q}_1 + \dot{q}_2)^2.$$

The kinetic energy of the system is

$$T = T_1 + T_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with inertia matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_1 + m_1 d_{c1}^2 + I_2 + m_2 l_1^2 + m_2 (q_2 - d_{c2})^2 & m_2 l_1 \\ m_2 l_1 & m_2 \end{pmatrix}.$$

The generallzed momentum of the robot is then

$$\boldsymbol{p} = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} \left(I_1 + m_1 d_{c1}^2 + I_2 + m_2 (q_2 - d_{c2})^2\right) \dot{q}_1 + m_2 l_1 (l_1 \dot{q}_1 + \dot{q}_2) \\ m_2 (l_1 \dot{q}_1 + \dot{q}_2) \end{pmatrix}. \tag{3}$$

The Coriolis and centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are derived using the Christoffel symbols, i.e., for each component

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial \boldsymbol{q}_i}\right), \qquad i = 1, 2,$$

being $\boldsymbol{m}_i$ the ith column of the inertia matrix $\boldsymbol{M}$. We obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & m_2 (q_2 - d_{c2}) \\ m_2 (q_2 - d_{c2}) & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 2 m_2 (q_2 - d_{c2}) \dot{q}_1 \dot{q}_2$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} -m_2 (q_2 - d_{c2}) & 0 \\ 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_2 (q_2 - d_{c2}) \dot{q}_1^2,$$

and thus

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2 m_2 (q_2 - d_{c2}) \dot{q}_1 \dot{q}_2 \\ -m_2 (q_2 - d_{c2}) \dot{q}_1^2 \end{pmatrix}.$$

The potential energy of the robot is

$$U = U_1 + U_2 = m_1 g_0\, p_{c1,y} + m_2 g_0\, p_{c2,y} = m_1 g_0\, d_{c1} \sin q_1 + m_2 g_0 \left(l_1 \sin q_1 + (q_2 - d_{c2}) \cos q_1\right),$$

with $g_0 = 9.81$ [m/s^2]. The associated gravity vector is

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0\left((m_1 d_{c1} + m_2 l_1)\cos q_1 - m_2 (q_2 - d_{c2}) \sin q_1\right) \\ m_2 g_0 \cos q_1 \end{pmatrix}, \tag{4}$$

As for the factorizations of the Coriolis and centrifugal terms, using again the definition of Christoffel symbols, we compute

$$\boldsymbol{S}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} m_2(q_2 - d_{c2})\dot{q}_2 & m_2(q_2 - d_{c2})\dot{q}_1 \\ -m_2(q_2 - d_{c2})\dot{q}_1 & 0 \end{pmatrix}, \tag{5}$$

which guarantees the skew-symmetry of the matrix

$$\begin{aligned} \dot{\boldsymbol{M}} - 2\boldsymbol{S}_1 &= \begin{pmatrix} 2m_2(q_2 - d_{c2})\dot{q}_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2m_2(q_2 - d_{c2})\dot{q}_2 & 2m_2(q_2 - d_{c2})\dot{q}_1 \\ -2m_2(q_2 - d_{c2})\dot{q}_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2m_2(q_2 - d_{c2})\dot{q}_1 \\ 2m_2(q_2 - d_{c2})\dot{q}_1 & 0 \end{pmatrix}. \end{aligned}$$

On the other hand, the alternative choice

$$\boldsymbol{S}_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 0 & 2m_2(q_2 - d_{c2})\dot{q}_1 \\ -m_2(q_2 - d_{c2})\dot{q}_1 & 0 \end{pmatrix},$$

which produces another feasible factorization $\boldsymbol{S}_2(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, leads to the matrix

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}_2 = \begin{pmatrix} 2m_2(q_2 - d_{c2})\dot{q}_2 & -4m_2(q_2 - d_{c2})\dot{q}_1 \\ 2m_2(q_2 - d_{c2})\dot{q}_1 & 0 \end{pmatrix}$$

that is clearly *not* skew symmetric. This implies, e.g., that matrix $\boldsymbol{S}_2$ *cannot* be used in the definition of the residual vector $\boldsymbol{r}$ for collision detection and isolation.
The definition of the residual is

$$\boldsymbol{r}(t) = \boldsymbol{K}_I \left(\boldsymbol{p}(t) - \int_0^t \left(\boldsymbol{\tau} + \boldsymbol{S}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{r}\right) ds - \boldsymbol{p}(0)\right), \qquad \boldsymbol{K}_I > 0, \tag{6}$$

where $\boldsymbol{p} = \boldsymbol{M}\dot{\boldsymbol{q}}$ is given by (3), $\boldsymbol{p}(0) = \boldsymbol{0}$ iff $\dot{\boldsymbol{q}}(0) = \boldsymbol{0}$ (the robot starts at rest), $\boldsymbol{g}$ is given by (4), and matrix $\boldsymbol{S}$ should factorize the Coriolis and centrifugal terms so that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is skew symmetric. Choosing in particular $\boldsymbol{S} = \boldsymbol{S}_1$ in (5), we have in this case

$$\boldsymbol{S}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{pmatrix} m_2(q_2 - d_{c2})\dot{q}_2 & -m_2(q_2 - d_{c2})\dot{q}_1 \\ m_2(q_2 - d_{c2})\dot{q}_1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ m_2(q_2 - d_{c2})\dot{q}_1^2 \end{pmatrix}.$$

The residual $\boldsymbol{r}$ in (6) will be affected by a non-zero contact force $\boldsymbol{F}_K \in \mathbb{R}^2$ lying in the plane $(\boldsymbol{x}, \boldsymbol{y})$ and acting on one of the robot links through the joint torque $\boldsymbol{\tau}_K = \boldsymbol{J}_K^T(\boldsymbol{q})\boldsymbol{F}_K$, except for some singular cases. Essentially, these are directions along which the contact point cannot be given by means of $\dot{\boldsymbol{q}}$ a linear velocity in the plane of motion. In the following, we distinguish between collisions on the first or on the second link (see Fig. 3).

- **Collision on link 1**. The position of the contact point along the first link and the associated contact Jacobian are

$$\boldsymbol{p}_{K1} = \rho_1 \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \quad \text{with } \rho_1 \in [0, l_1] \quad \Rightarrow \quad \boldsymbol{J}_{K1} = \begin{pmatrix} -\sin q_1 & 0 \\ \cos q_1 & 0 \end{pmatrix},$$

and thus collision is not detected when

$$\boldsymbol{F}_{K1} = \alpha \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \quad \text{for arbitrary } \alpha = \|\boldsymbol{F}_{K1}\| \quad \Rightarrow \quad \boldsymbol{J}_{K1}^T \boldsymbol{F}_{K1} = \boldsymbol{0},$$

namely, when the contact force is aligned with the first link (Fig. 3a). The closer is the alignment of $\boldsymbol{F}_{K1}$ with the axis of link 1, the poorer will be the detection.

- **Collision on link 2**. The position of the contact point along the second link is[1]

$$\boldsymbol{p}_{K2} = \begin{pmatrix} l_1 \cos q_1 - (q_2 - \rho_2) \sin q_1 \\ l_1 \sin q_1 + (q_2 - \rho_2) \cos q_1 \end{pmatrix} = \mathrm{Rot}_{2\times 2}(q_1) \begin{pmatrix} l_1 \\ q_2 - \rho_2 \end{pmatrix}, \quad \text{with } \rho_2 \in [0, l_{2,max}]$$

and the associated contact Jacobian is

$$\Rightarrow \quad \boldsymbol{J}_{K2} = \begin{pmatrix} -\left(l_1 \sin q_1 + (q_2 - \rho_2) \cos q_1\right) & -\sin q_1 \\ l_1 \cos q_1 - (q_2 - \rho_2) \sin q_1 & \cos q_1 \end{pmatrix}, \qquad \det \boldsymbol{J}_{K2} = \rho_2 - q_2.$$

Thus, collision is not detected when $\rho_2 = q_2$ and

$$\boldsymbol{F}_{K2} = \alpha \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \quad \text{for arbitrary } \alpha = \pm \|\boldsymbol{F}_{K2}\| \quad \Rightarrow \quad \boldsymbol{J}_{K2|_{q_2=\rho_2}}^T \boldsymbol{F}_{K2} = \boldsymbol{0},$$

namely, when the contact occurs at the second joint location and the force is orthogonal to the second link (Fig. 3b). On the other hand, we obtain still detection but wrong isolation when the contact force is orthogonal to the second link, as before, and the contact point is not along the first link axis ($\rho_2 \neq q_2$). In this case (see Fig. 3c), we would have

$$\boldsymbol{r} \simeq \boldsymbol{\tau}_{K2} = \boldsymbol{J}_{K2}^T \boldsymbol{F}_{K2} = \alpha \cdot \boldsymbol{J}_{K2}^T \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} = \begin{pmatrix} \alpha \left(q_2 - \rho_2\right) \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix},$$

indicating incorrectly that a collision occurred on link 1, rather than on link 2. Finally, if

$$\boldsymbol{F}'_{K2} = \beta \begin{pmatrix} l_1 \cos q_1 - (q_2 - \rho_2) \sin q_1 \\ l_1 \sin q_1 + (q_2 - \rho_2) \cos q_1 \end{pmatrix}, \quad \text{for arbitrary } \beta \neq 0,$$

namely, when the line of action of the contact force passes through the axis of joint 1, we obtain (see Fig. 3d)

$$\boldsymbol{r} \simeq \boldsymbol{\tau}'_{K2} = \boldsymbol{J}_{K2}^T \boldsymbol{F}'_{K2} = \begin{pmatrix} 0 \\ \beta(q_2 - \rho_2) \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix},$$

indicating correctly that the collision is on link 2 (the largest index with a non-zero component in $\boldsymbol{r}$), although the first component of the residual is vanishing ($r_1 = 0$).

[1] We assume that the prismatic joint has a maximum excursion of $q_2 \in [0, l_{2,max}]$.

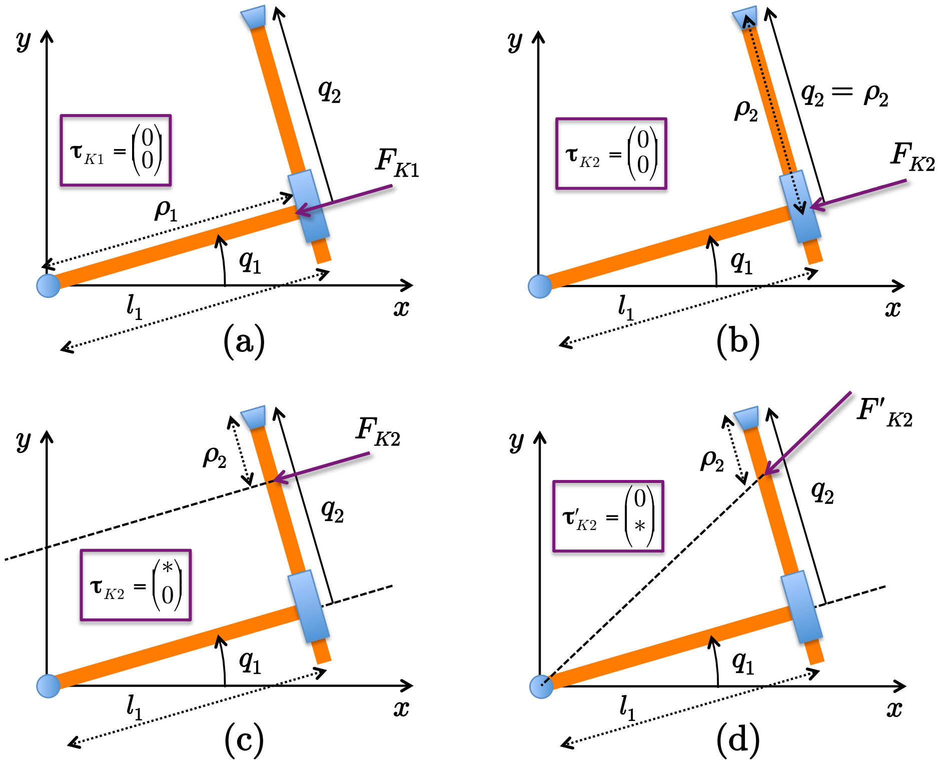

Figure 3: Possible collision situations leading to no detection or wrong isolation: (a) no detection of a contact force on link 1; (b) no detection of a contact force on link 2; (c) wrong isolation of a contact force on link 2; (d) isolation of a contact force on link 2, despite the residual at joint 1 is not being affected.

Exercise 2

Define as coordinate of the pendulum link its angle θ from the downward vertical (positive if counterclockwise). Under the simplifying assumptions made, the Lagrangian dynamics of the actuated pendulum is computed from the kinetic and potential energy

$$T = \frac{1}{2} m l^2 \dot{\theta}^2, \qquad U = -m g_0 \, l \cos\theta,$$

as

$$m l^2 \ddot{\theta} + m g_0 \, l \sin\theta = \tau = k_i i_m, \tag{7}$$

where the drive gain of the DC motor has been included. The unknowns in (7) are the payload (concentrated) mass m and the drive gain k_i. Dividing by $k_i > 0$, this equation can be rewritten in the standard linearly parametrized form (and without unknown parameters affecting the input)

wow!

$$\frac{m}{k_i} l^2 \ddot{\theta} + \frac{m}{k_i} g_0 \, l \sin\theta = \left(l^2 \ddot{\theta} + g_0 \, l \sin\theta \right) \frac{m}{k_i} = Y(\theta, \ddot{\theta}) \, a = i_m, \tag{8}$$

being $a = m/k_i > 0$ the only dynamic coefficient that matters ($p = 1$). We note that a plays the role of a scaled mechanical inertia. Being the motor current i_m the input to the system, it is easy to derive from eq. (8) an adaptive control law for trajectory tracking that mimics the classical one derived when the input is directly the motor torque τ. Given a twice-differentiable desired trajectory $\theta_d(t)$ for the joint variable, define such adaptive law as

$$\begin{aligned} i_m &= Y(\theta,\ddot{\theta}_r)\,\hat{a} + k_p e + k_d \dot{e}, \qquad k_p > 0, k_d > 0, \qquad e = \theta_d - \theta, \qquad \dot{\theta}_r = \dot{\theta}_d + \lambda e = \dot{\theta}_d + \frac{k_p}{k_d} e, \\ \dot{\hat{a}} &= \gamma\, Y(\theta,\ddot{\theta}_r)\,(\dot{\theta}_r - \dot{\theta}), \qquad \gamma > 0, \qquad \dot{\theta}_r - \dot{\theta} = \dot{e} + \lambda e, \end{aligned} \tag{9}$$

with

$$Y(\theta,\ddot{\theta}_r) = l^2\ddot{\theta}_r + g_0\, l \sin\theta, \qquad \hat{a} = \widehat{\left(\frac{m}{k_i}\right)} \in \mathbb{R}.$$

The global asymptotic tracking of the smooth trajectory $\theta_d(t)$ can be proven by following the same arguments as in the classical case, i.e., via a Lyapunov candidate and the use of Barbalat lemma and LaSalle theorem. However, one should carefully define the candidate by considering the scaled mechanical inertia of the system. Therefore, noting the absence of dissipative terms, define the candidate function

$$V = \frac{1}{2}\,\frac{ml^2}{k_i}(\dot{\theta}_r - \dot{\theta})^2 + \frac{1}{2}\,R\,e^2 + \frac{1}{2\gamma}\,\tilde{a}^2 \geq 0,$$

with $R = 2k_p > 0$ and $\tilde{a} = a - \hat{a}$. We have that $V = 0$ iff $e = \dot{e} = \tilde{a} = 0$. For the closed-loop system (8–9), we can write

$$Y(\theta,\ddot{\theta})\,a = \left(l^2\ddot{\theta} + g_0\, l\sin\theta\right)\frac{m}{k_i} = \left(l^2\ddot{\theta}_r + g_0\, l\sin\theta\right)\widehat{\left(\frac{m}{k_i}\right)} + k_p e + k_d\dot{e} = Y(\theta,\ddot{\theta}_r)\,\hat{a} + k_p e + k_d \dot{e}.$$

Subtracting both sides of this equality from $Y(\theta,\ddot{\theta}_r)\,a$, one obtains

$$Y(\theta,\ddot{\theta}_r)\,a - Y(\theta,\ddot{\theta})\,a = \frac{ml^2}{k_i}(\ddot{\theta}_r - \ddot{\theta}) = \left(l^2\ddot{\theta}_r + g_0\, l\sin\theta\right)\tilde{a} - k_p e - k_d\dot{e} = Y(\theta,\ddot{\theta}_r)\,\tilde{a} - k_p e - k_d \dot{e}. \tag{10}$$

Using (10), the time derivative of V computed along the trajectories of the closed-loop system (8–9) is evaluated as

$$\begin{aligned} \dot{V} &= \frac{ml^2}{k_i}(\ddot{\theta}_r - \ddot{\theta})\,(\dot{\theta}_r - \dot{\theta}) + 2k_p\, e\,\dot{e} - \frac{1}{\gamma}\,\tilde{a}\,\dot{\hat{a}} \\ &= \left(Y(\theta,\ddot{\theta}_r)\,\tilde{a} - k_p e - k_d\dot{e}\right)(\dot{\theta}_r - \dot{\theta}) + 2k_p\, e\,\dot{e} - \frac{1}{\gamma}\,\gamma\, Y(\theta,\ddot{\theta}_r)\,\tilde{a}\,(\dot{\theta}_r - \dot{\theta}) \\ &= -\,(k_p e + k_d \dot{e})\,(\dot{e} + \frac{k_p}{k_d} e) + 2k_p\, e\,\dot{e} \\ &= -\,k_d\,\dot{e}^2 - \frac{k_p^2}{k_d}\,e^2 \leq 0. \end{aligned}$$

The rest of the proof is completed just like in the classical case.

Exercise 3

The dynamic model of the system in Fig. 2 is

$$m\ddot{x} = F - F_c - d\dot{x},$$

with all non-conservative forces performing work on x on the right-hand side. Since $F_c = k_s x$ (as the single compliant element in the contact), this equation can be rewritten as

$$m\ddot{x} + d\dot{x} + k_s x = F, \tag{11}$$

with all physical coefficients being positive. Applying the class of force control laws (2) yields

$$m\ddot{x} + d\dot{x} + k_s x = \alpha k_f (F_d - F_c) + \beta F_d = \alpha k_f (F_d - k_s x) + \beta F_d. \tag{12}$$

At the equilibrium, $\dot{x} = \ddot{x} = 0$, we have

$$k_s x = \alpha k_f (F_d - k_s x) + \beta F_d,$$

which is solved for a position $x = x_e$ and an associated contact force $F_c = F_e = k_s x_e$ as

$$x_e = \frac{\beta + \alpha k_f}{k_s(1 + \alpha k_f)} F_d, \qquad F_e = \frac{\beta + \alpha k_f}{1 + \alpha k_f} F_d \quad \Rightarrow \quad e_f = F_d - F_e = \frac{1 - \beta}{1 + \alpha k_f} F_d. \tag{13}$$

Therefore, the correct desired contact force F_d is obtained at the equilibrium if and only if $\beta = 1$ (presence of the constant feedforward), no matter if $\alpha = 0$ (no feedback) or $\alpha = 1$ (combined situation). In such case, it is in fact

$$x_e = \frac{1}{k_s} F_d, \qquad F_e = F_d \quad \Rightarrow \quad e_f = 0. \tag{14}$$

On the other hand, for $\beta = 0$ and $\alpha = 1$ (pure proportional feedback), we have at the equilibrium

$$x_e = \frac{k_f}{k_s(1 + k_f)} F_d, \qquad F_e = \frac{k_f}{1 + k_f} F_d \quad \Rightarrow \quad e_f = \frac{1}{1 + k_f} F_d, \tag{15}$$

which shows that only for $k_f \to \infty$ (or, for large proportional gains) we can drive the force error to zero (or, below a given tolerance).

Indeed, we need to show that the above equilibria are asymptotically stable. In view of the linearity of the system, whenever this property holds, the equilibrium will also be a global, exponentially stable one.

Consider first the case of a pure feedforward command $F = F_d$ ($\alpha = 0$, $\beta = 1$). The system dynamics is

$$m\ddot{x} + d\dot{x} + k_s x = F_d$$

In order to study the asymptotic stability of the equilibrium (14), we choose as Lyapunov candidate

$$V_1 = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k_s (x - x_e)^2 \geq 0 \tag{16}$$

Its time derivative is evaluated on the controlled system as

$$\dot{V}_1 = m\dot{x}\ddot{x} + k_s(x - x_e)\dot{x} = \dot{x}(-d\dot{x} - k_s x_e + F_d) = -d\dot{x}^2 \leq 0.$$

We have $\dot{V}_1 = 0$ if and only if $\dot{x} = 0$. The system behaves then as $m\ddot{x} = F_d - k_s x$, showing that there will be an acceleration iff the contact force $F_c = k_s x$ is different from the desired one. By LaSalle theorem, we conclude the asymptotic stability of the equilibrium state $x = x_e$, $\dot{x} = 0$. Stated differently, by pushing constantly on the mass with the desired force $F_d > 0$, a steady state is reached with the desired contact force (thanks to the asymptotic stability of the original open-loop system). However, the transient behavior will be specified only by the actual physical mass m, sensor stiffness k_s, and viscous (damping) coefficient d. Moreover, the pure feedforward scheme is highly sensitive to unmodeled disturbance forces acting on the system.

Consider now the case of a pure proportional feedback of the force error, namely $F = k_f(F_d - F_c)$ ($\alpha = 1$, $\beta = 0$). The closed-loop system is then

$$m\ddot{x} + d\dot{x} + k_s x = k_f(F_d - F_c) = k_f(F_d - k_s x) \quad \Rightarrow \quad m\ddot{x} + d\dot{x} + k_s(1 + k_f)x = k_f F_d$$

In order to study the asymptotic stability of the (incorrect) equilibrium (15), we choose as Lyapunov candidate

$$V_2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_s(1+k_f)(x-x_e)^2 \geq 0. \tag{17}$$

Its time derivative, evaluated along the trajectories of the closed-loop system, is

$$\dot{V}_2 = m\dot{x}\ddot{x} + k_s(1+k_f)(x-x_e)\dot{x} = \dot{x}\left(-d\dot{x} - k_s(1+k_f)x + k_f F_d + k_s(1+k_f)(x-x_e)\right) = -d\dot{x}^2 \leq 0,$$

where we replaced the expression of x_e in (15) in order to simplify terms. The conclusion about asymptotic stability follows from a LaSalle analysis similar to the previous case. Indeed, the steady-state force error $e_f \neq 0$ can be decreased by increasing k_f, modifying accordingly the transient behavior. However, when increasing k_f the system response will become faster but soon underdamped. As a matter of fact, a useful additional damping action of the form $-d_c\dot{x}$, with $d_c > 0$, is actually missing in the considered control law.

On the other hand, when combining the feedforward and feedback actions in the control law ($\alpha = \beta = 1$), the steady-state error will vanish without the need of increasing the feedback gain k_f. As a result, this can be tuned so as to obtain the best transient behavior and possibly reduce the effects of extra disturbing forces. The analysis of the asymptotic stability of x_e in (14) can be conducted as before, using the same Lyapunov candidate V_2 in (17).

For those more acquainted with Laplace transformation methods in linear control systems, it is worth mentioning that the above stability analyses could have been conducted more easily (and quickly) by looking at the location of poles (with Routh criterion, or even with simpler methods) of suitable transfer functions in the Laplace domain s, both for the open-loop system

$$\frac{F_c(s)}{F(s)} = \frac{k_s}{ms^2 + ds + k_s},$$

and for the closed-loop system, e.g., under combined feedback/feedforward

$$\frac{F_c(s)}{F_d(s)} = \frac{k_s(1+k_f)}{ms^2 + ds + k_s(1+k_f)}.$$

We note also that all the obtained stability results are completely independent from the parameters m, d, and k_s (as long as they remain physically meaningful, i.e., positive). These quantities were invoked in the analysis, but are never used for force control design, which inherits therefore some intrinsic robustness. Nonetheless, the values of these parameters will affect the quality of the transient behavior in response to reference values F_d.

At last, when $F_c = 0$ (no interaction with the environment), all controllers will behave in a similar way. We would have in that case

$$m\ddot{x} + d\dot{x} = F = \begin{cases} F_d & \text{[pure feedforward]} \\ k_f F_d & \text{[pure proportional feedback]} \\ (1+k_f)F_d & \text{[combined feedback/feedforward]} \end{cases} \tag{18}$$

and the mass would always reach a constant steady-state velocity (with $\ddot{x} = 0$) equal to

$$\dot{x}_{ss} = \frac{F}{d}$$

with the constant F as specified in (18).

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