



## ***Robotics 2***

# **Dynamic model of robots: Algorithm for computing kinetic energy**

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DIPARTIMENTO DI INGEGNERIA INFORMATICA  
AUTOMATICA E GESTIONALE ANTONIO RUBERTI



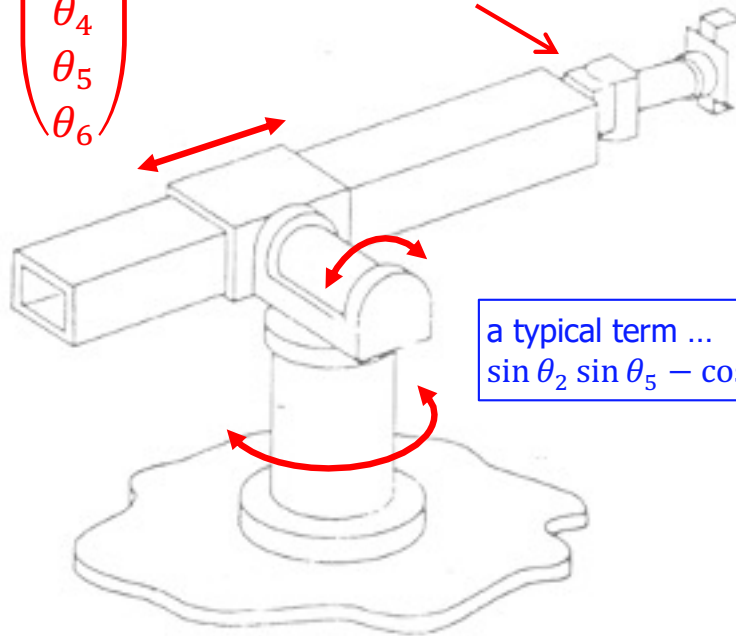
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# Complexity of robot inertia terms

## element $m_{11}(q)$ of Stanford arm

$$q = \begin{pmatrix} \theta_1 \\ \theta_2 \\ d_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{pmatrix}$$

3R spherical wrist



a typical term ...  
 $\sin \theta_2 \sin \theta_5 - \cos \theta_2 \sin \theta_4 \cos \theta_5$

$$m_{11} = m_1 k_{122}^2$$

... (derived by hand) in JPL  
 Tech. Memo. 33-669, 1974

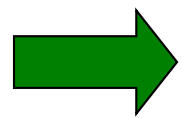
$$\begin{aligned} &+ m_2 \left[ k_{211}^2 s^2 \theta_2 + k_{233}^2 c^2 \theta_2 + r_2 (2\bar{y}_2 + r_2) \right] \\ &+ m_3 \left[ k_{322}^2 s^2 \theta_2 + k_{333}^2 c^2 \theta_2 + r_3 (2\bar{z}_3 + r_3) s^2 \theta_2 + r_2^2 \right] \\ &+ m_4 \left\{ \frac{1}{2} k_{411}^2 \left[ s^2 \theta_2 (2s^2 \theta_4 - 1) + s^2 \theta_4 \right] + \frac{1}{2} k_{422}^2 (1 + c^2 \theta_2 + s^2 \theta_4) \right. \\ &\quad \left. + \frac{1}{2} k_{433}^2 \left[ s^2 \theta_2 (1 - 2s^2 \theta_4) - s^2 \theta_4 \right] + r_3^2 s^2 \theta_2 + r_2^2 - 2\bar{y}_4 r_3 s^2 \theta_2 + 2\bar{z}_4 (r_2 s \theta_4 + r_3 s \theta_2 c \theta_2 c \theta_4) \right\} \\ &+ m_5 \left\{ \frac{1}{2} (-k_{511}^2 + k_{522}^2 + k_{533}^2) \left[ (s \theta_2 s \theta_5 - c \theta_2 s \theta_4 c \theta_5)^2 + c^2 \theta_4 c^2 \theta_5 \right] \right. \\ &\quad \left. + \frac{1}{2} (k_{511}^2 - k_{522}^2 + k_{533}^2) (s^2 \theta_4 + c^2 \theta_2 c^2 \theta_4) \right. \\ &\quad \left. + \frac{1}{2} (k_{511}^2 + k_{522}^2 - k_{533}^2) \left[ (s \theta_2 c \theta_5 + c \theta_2 s \theta_4 s \theta_5)^2 + c^2 \theta_4 s^2 \theta_5 \right] + r_3^2 s^2 \theta_2 + r_2^2 \right. \\ &\quad \left. + 2\bar{z}_5 \left[ r_3 (s^2 \theta_2 c \theta_5 + s \theta_2 s \theta_4 c \theta_4 s \theta_5) - r_2 c \theta_4 s \theta_5 \right] \right\} \\ &+ m_6 \left\{ \frac{1}{2} (-k_{611}^2 + k_{622}^2 + k_{633}^2) \left[ (s \theta_2 s \theta_5 c \theta_6 - c \theta_2 s \theta_4 c \theta_5 c \theta_6 - c \theta_2 c \theta_4 s \theta_6)^2 + (c \theta_4 c \theta_5 c \theta_6 - s \theta_4 s \theta_6)^2 \right] \right. \\ &\quad \left. + \frac{1}{2} (k_{611}^2 - k_{622}^2 + k_{633}^2) \left[ (c \theta_2 s \theta_4 c \theta_5 s \theta_6 - s \theta_2 s \theta_5 s \theta_6 - c \theta_2 c \theta_4 c \theta_6)^2 + (c \theta_4 c \theta_5 s \theta_6 + s \theta_4 c \theta_6)^2 \right] \right. \\ &\quad \left. + \frac{1}{2} (k_{611}^2 + k_{622}^2 - k_{633}^2) \left[ (c \theta_2 s \theta_4 s \theta_5 + s \theta_2 c \theta_5)^2 + c^2 \theta_4 s^2 \theta_5 \right] \right. \\ &\quad \left. + \left[ r_6 c \theta_2 s \theta_4 s \theta_5 + (r_6 c \theta_5 + r_3) s^2 \theta_2 \right]^2 + (r_6 c \theta_4 s \theta_5 - r_2)^2 \right. \\ &\quad \left. + 2\bar{z}_6 \left[ r_6 (s^2 \theta_2 c^2 \theta_5 + c^2 \theta_4 s^2 \theta_5 + c^2 \theta_2 s^2 \theta_4 s^2 \theta_5 + 2s \theta_2 c \theta_2 s \theta_4 s \theta_5 c \theta_5) \right. \right. \\ &\quad \left. \left. + r_3 (s \theta_2 c \theta_2 s \theta_4 s \theta_5 + s^2 \theta_2 c \theta_5) - r_2 c \theta_4 s \theta_5 \right] \right\} \end{aligned}$$

radius of gyration factors  $k_{ijj}^2$  are being used here  
 for a body of mass  $m_i$  and moment of inertia  $I_j$  with respect  
 to an axis  $z_j$ , the radius of gyration  $k_{ijj}$  is the distance of  
 the mass  $m_i$  from the same axis, such that  $I_j = m_i k_{ijj}^2$



# Expression of $v_{ci}$ and $\omega_i$

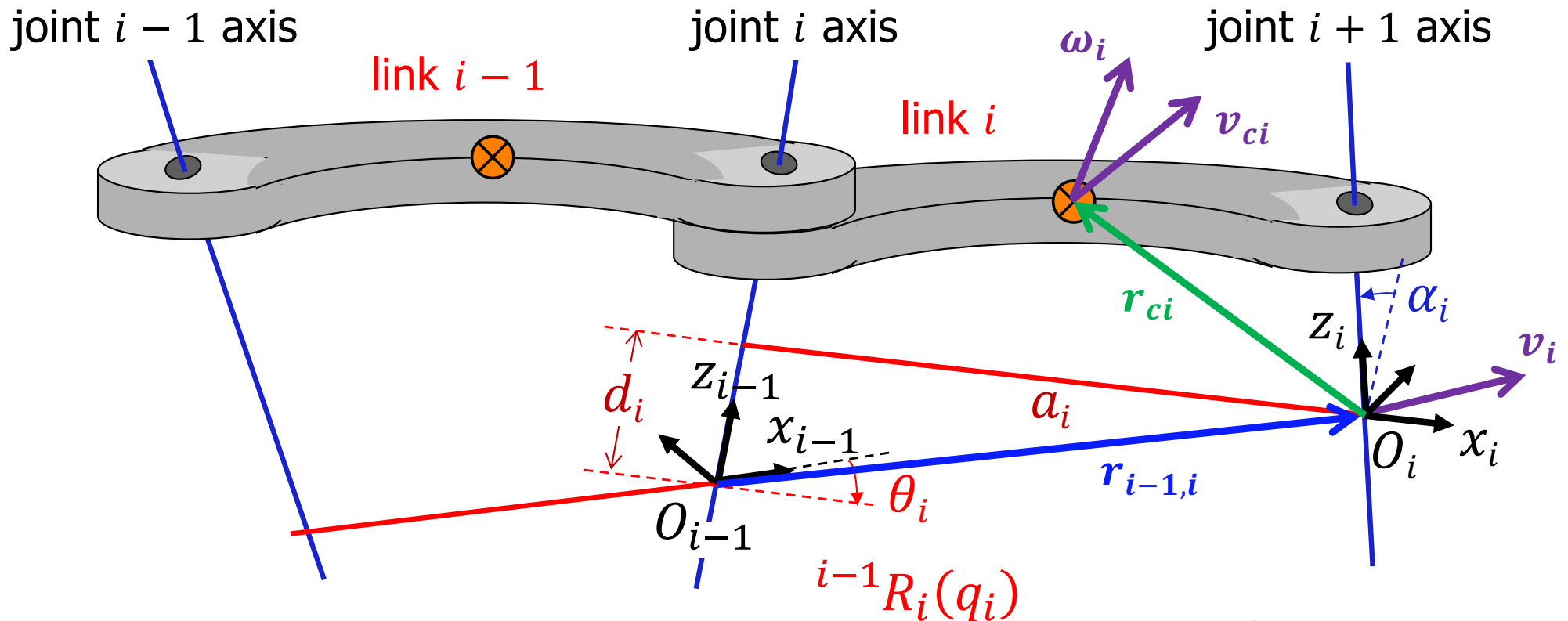
- $v_{ci}$  and  $\omega_i$  can be written using the relations of the robot differential kinematics (partial Jacobians)
- it is useful however to operate in a **recursive** way, expressing each vector quantity related to link  $i$  in the “**moving**” frame  $RF_i$  attached to link  $i$  (with the notation  **${}^i\text{vector}_i$** )
  - particularly convenient when using algebraic/symbolic manipulation languages (Matlab Symbolic Toolbox, Maple, Mathematica, ...) for computing the kinetic energy of a (open chain) robot arm, when the number of joints increases (e.g., for  $N \geq 4$ )



Moving Frames



# Recall: D-H frames



$${}^{i-1}A_i(q_i) = \begin{pmatrix} \cos \theta_i & -\cos \alpha_i \sin \theta_i & \sin \alpha_i \sin \theta_i & a_i \cos \theta_i \\ \sin \theta_i & \cos \alpha_i \cos \theta_i & -\sin \alpha_i \cos \theta_i & a_i \sin \theta_i \\ 0 & \sin \alpha_i & \cos \alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



# Moving Frames algorithm

velocity of  
center of mass  
of link  $i$

$$\boxed{v_{ci} = v_i + \omega_i \times r_{ci}}$$

velocity of  $O_i$   
(origin of  $RF_i$ )

angular velocity  
of link  $i$

position of  
center of mass  
of link  $i$  w.r.t.  $O_i$

$$\text{set } \sigma_i = \begin{cases} 0 & \text{revolute joint} \\ 1 & \text{prismatic joint} \end{cases}$$

$${}^i\omega_i = {}^{i-1}R_i^T(q_i) \left[ {}^{i-1}\omega_{i-1} + (1 - \sigma_i) \dot{q}_i {}^{i-1}z_{i-1} \right] = {}^{i-1}R_i^T(q_i) {}^{i-1}\omega_i$$

$z$ -axis of  $RF_{i-1}$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$${}^i v_i = {}^{i-1}R_i^T(q_i) \left[ {}^{i-1}v_{i-1} + \sigma_i \dot{q}_i {}^{i-1}z_{i-1} + {}^{i-1}\omega_i \times {}^{i-1}r_{i-1,i} \right]$$

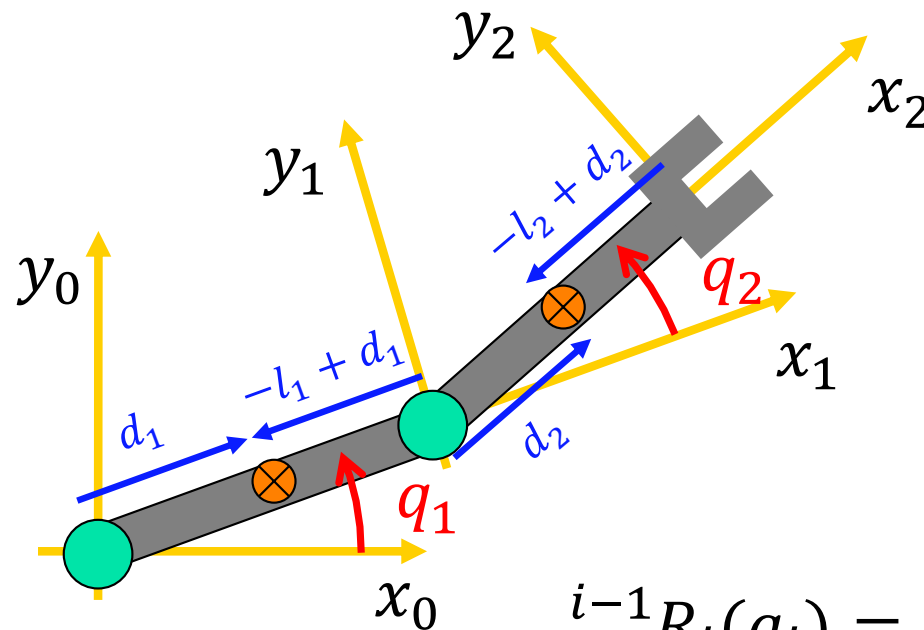
... =  ${}^i\omega_i$  already computed

... =  ${}^i r_{i-1,i}$  (constant,  
if joint  $i$  is revolute!)



# Dynamic model of a 2R robot

## application of the algorithm



$\sigma_i = 0, \quad i = 1, 2$   
(revolute joints)

moving in the  
vertical plane

$$g = \begin{pmatrix} 0 \\ -g_0 \\ 0 \end{pmatrix}$$
$$g_0 = 9.81$$

$${}^{i-1}R_i(q_i) = \begin{pmatrix} c_i & -s_i & 0 \\ s_i & c_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad {}^i r_{ci} = \begin{pmatrix} -l_i + d_i \\ 0 \\ 0 \end{pmatrix}$$

**assumption:** center of mass of  
each link is on its kinematic axis

initialization:  $i = 0$

$${}^0\omega_0 = 0$$

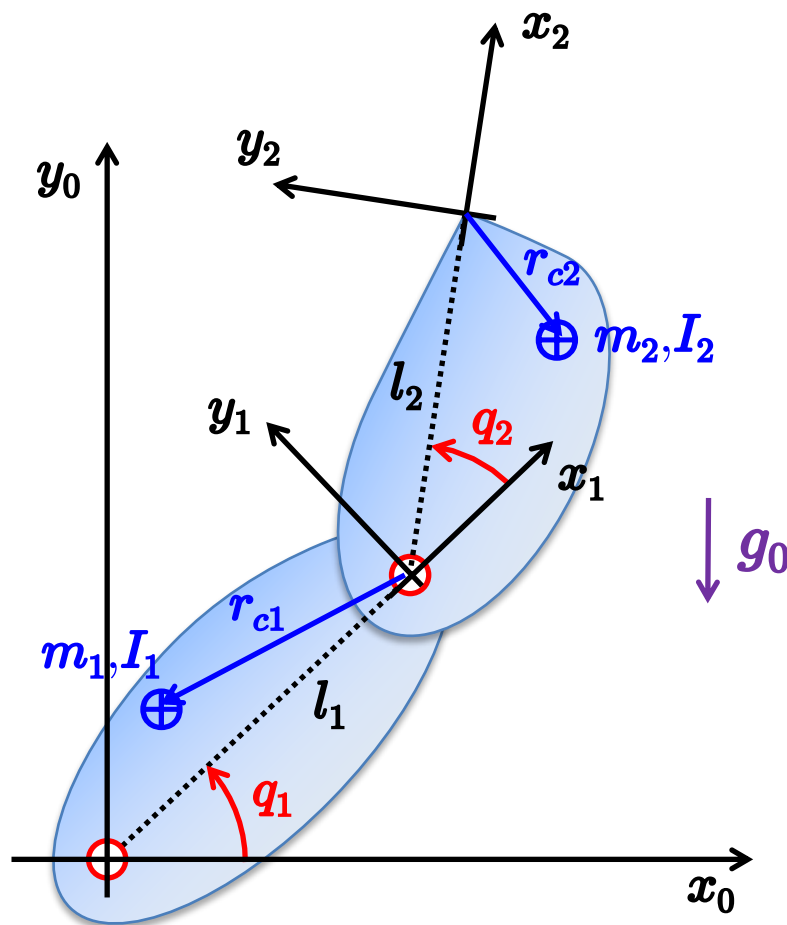
$${}^0v_0 = 0$$



# Dynamic model of a 2R robot

what if the CoM is not on the kinematic axis ...

see **Robotics 2 Midterm 2021** (14 April)



$${}^{i-1}R_i(q_i) = \begin{pmatrix} c_i & -s_i & 0 \\ s_i & c_i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$${}^i r_{ci} = \begin{pmatrix} r_{ci,x} \\ r_{ci,y} \\ 0 \end{pmatrix}$$

may also work in 2D (planar case)  
 ${}^{i-1}\bar{R}_i(q_i), {}^i\bar{r}_{ci}$

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \|v_{c1}\|^2 + \frac{1}{2} \omega_1^T I_1 \omega_1 \\ &= \frac{1}{2} m_1 \left( (l_1 + r_{c1,x})^2 + r_{c1,y}^2 \right) \dot{q}_1^2 + \frac{1}{2} I_1 \dot{q}_1^2 \end{aligned}$$

$${}^0 p_{c2} = \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} + {}^0 \bar{R}_2(q_1, q_2) \begin{pmatrix} l_2 + r_{c2,x} \\ r_{c2,y} \end{pmatrix}$$

$${}^0 \bar{R}_2(q_1, q_2) = \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix} \text{ etc ...}$$



# First step (link 1)

$$i = 1$$

$${}^1\omega_1 = {}^0R_1^T(q_1) \left[ {}^0\omega_0 + \dot{q}_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix}$$

$${}^1v_1 = {}^0R_1^T(q_1) \left[ {}^0v_0 + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix} \times \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{pmatrix} \right] = {}^0R_1^T(q_1) \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix} \times {}^0R_1^T(q_1) \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix} \times \begin{pmatrix} l_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ l_1 \dot{q}_1 \\ 0 \end{pmatrix}$$

there is not dependence on  $q_1$

$${}^1v_{c1} = \begin{pmatrix} 0 \\ l_1 \dot{q}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix} \times \begin{pmatrix} -l_1 + d_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ d_1 \dot{q}_1 \\ 0 \end{pmatrix}$$





# Kinetic energy of link 1

$${}^1\omega_1 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix}$$

$${}^1v_{c1} = \begin{pmatrix} 0 \\ d_1 \dot{q}_1 \\ 0 \end{pmatrix}$$



$$T_1 = \frac{1}{2} m_1 d_1^2 \dot{q}_1^2 + \frac{1}{2} I_{c1,zz} \dot{q}_1^2 = \frac{1}{2} \underbrace{(I_{c1,zz} + m_1 d_1^2)}_{\substack{\text{the actual inertia around the rotation axis} \\ \text{of the first joint (parallel axis theorem)}}} \dot{q}_1^2$$

the actual inertia around the rotation axis  
of the first joint (parallel axis theorem)



## Second step (link 2)

$$i = 2$$

$${}^2\omega_2 = {}^1R_2^T(q_2) \left[ {}^1\omega_1 + \dot{q}_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{pmatrix}$$

$${}^2v_2 = {}^1R_2^T(q_2) \left[ {}^1v_1 + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} l_2 c_2 \\ l_2 s_2 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} l_1 s_2 \dot{q}_1 \\ l_1 c_2 \dot{q}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} l_2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} l_1 s_2 \dot{q}_1 \\ l_1 c_2 \dot{q}_1 + l_2 (\dot{q}_1 + \dot{q}_2) \\ 0 \end{pmatrix}$$



## Kinetic energy of link 2

$$i = 2$$

$${}^2v_{c2} = {}^2v_2 + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} -l_2 + d_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 s_2 \dot{q}_1 \\ l_1 c_2 \dot{q}_1 + d_2 (\dot{q}_1 + \dot{q}_2) \\ 0 \end{pmatrix}$$



$$T_2 = \frac{1}{2} m_2 \left( l_1^2 \dot{q}_1^2 + d_2^2 (\dot{q}_1 + \dot{q}_2)^2 + 2 l_1 d_2 c_2 \dot{q}_1 (\dot{q}_1 + \dot{q}_2) \right) + \frac{1}{2} I_{c2,zz} (\dot{q}_1 + \dot{q}_2)^2$$



# Robot inertia matrix $M(q)$

$$T = T_1 + T_2 = \frac{1}{2} (\dot{q}_1 \quad \dot{q}_2)^T \begin{pmatrix} m_{11}(q) & m_{12}(q) \\ m_{21}(q) & m_{22}(q) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

$$\begin{aligned} m_{11}(q) &= I_{c1,zz} + m_1 d_1^2 + I_{c2,zz} + m_2 d_2^2 + m_2 l_1^2 + 2m_2 l_1 d_2 c_2 \\ &= a_1 + 2a_2 \cos q_2 \end{aligned}$$

$$m_{12}(q) = m_{21}(q) = I_{c2,zz} + m_2 d_2^2 + m_2 l_1 d_2 c_2 = a_3 + a_2 \cos q_2$$

$$m_{22} = I_{c2,zz} + m_2 d_2^2 = a_3$$

**NOTE:** introduction of **dynamic coefficients**  $a_i$  is a convenient **regrouping** of the dynamic parameters (more on this later → **linear parametrization of dynamics**)



# Centrifugal and Coriolis terms

$$C_1(q) = \frac{1}{2} \left( \frac{\partial M_1}{\partial q} + \left( \frac{\partial M_1}{\partial q} \right)^T - \frac{\partial M}{\partial q_1} \right) = \frac{1}{2} \left( \begin{pmatrix} 0 & -2a_2s_2 \\ 0 & -a_2s_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -2a_2s_2 & -a_2s_2 \end{pmatrix} \right)$$
$$= \begin{pmatrix} 0 & -a_2s_2 \\ -a_2s_2 & -a_2s_2 \end{pmatrix} \quad \rightarrow \quad c_1(q, \dot{q}) = -a_2s_2(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2)$$

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$$C_2(q) = \frac{1}{2} \left( \frac{\partial M_2}{\partial q} + \left( \frac{\partial M_2}{\partial q} \right)^T - \frac{\partial M}{\partial q_2} \right) = \dots = \begin{pmatrix} a_2s_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$c_2(q, \dot{q}) = a_2s_2\dot{q}_1^2$$



# Gravity terms

$$U_1 = -m_1 g^T r_{0,c1} = -m_1 \begin{pmatrix} 0 & -g_0 & 0 \end{pmatrix} \begin{pmatrix} * \\ d_1 s_1 \\ * \end{pmatrix} = m_1 g_0 d_1 s_1$$

$$U_2 = -m_2 g^T r_{0,c2} = m_2 g_0 (l_1 s_1 + d_2 s_{12})$$

$$U = U_1 + U_2$$

$$g(q) = \left( \frac{\partial U}{\partial q} \right)^T = \begin{pmatrix} g_0(m_1 d_1 c_1 + m_2 l_1 c_1 + m_2 d_2 c_{12}) \\ g_0 m_2 d_2 c_{12} \end{pmatrix} = \begin{pmatrix} a_4 c_1 + a_5 c_{12} \\ a_5 c_{12} \end{pmatrix}$$

There are only 5 dynamics coefficients that matters: a1 to a5



# Dynamic model of a 2R robot

$$(a_1 + 2a_2c_2)\ddot{q}_1 + (a_2c_2 + a_3)\ddot{q}_2 - a_2s_2(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2) + a_4c_1 + a_5c_{12} = u_1$$

$$(a_2c_2 + a_3)\ddot{q}_1 + a_3\ddot{q}_2 + a_2s_2\dot{q}_1^2 + a_5c_{12} = u_2$$

$a_2 = m_2 \cdot l_1^2 \cdot d_2 \rightarrow d_2$  is the only one that can go to 0: the second link has the CoM on the joint axis 2  
if  $a_2 = 0$  it means that the inertia matrix is constant  $\rightarrow$  no coriolis centrifugal term.  
moreover if  $d_2 = 0 \rightarrow a_5 = 0$  so also the gravity will have no effect since the CoM is on the joint axis

**Q1:** is it  $a_2 = 0$  possible? ...physical interpretation? ...consequences?

$a_5 = 0$  if second link balanced ( $d_2 = 0$ ).  $a_4 = 0$  when  $d_1 = -l_1$ , so the CoM balances its own mass and the mass of the link that is beyond (GUARDA FINE VIDEO LECTURE 09)

**Q2:** is it  $a_4 = a_5 = 0$  possible as well? ...physical interpretation?

**Q3:** based on the expressions of the dynamic coefficients  $a_1, a_2, a_3$ , check that the robot inertia matrix is **always** positive definite, and in particular that the diagonal elements are **always** positive ( $\forall q$ )

**Q4:** provide two different matrices  $S'$  and  $S''$  for the factorization of the quadratic velocity terms, respectively **satisfying** and **not satisfying** the skew-symmetry of  $\dot{M} - 2S$