Robotics 2

Dynamic model of robots:

Analysis, properties, extensions, uses

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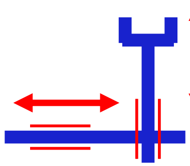
Analysis of inertial couplings

- Cartesian robot

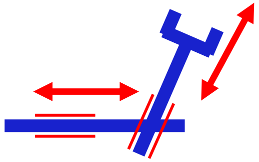


m11 and m22 are constant

$$M = \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix}$$

the fact that the matrix is diagonal shows no inertia coupling between the two joints: if we give a force to the first joint only the first joint accelerates and the same is valid for the second joint.

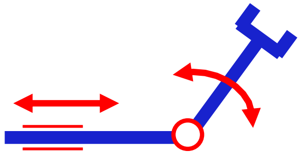
- Cartesian “skew” robot



$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}$$

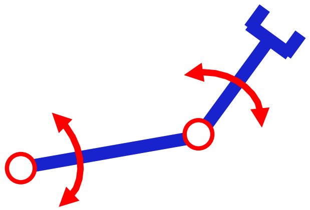
there is a diagonal coupling. However the matrix is still constant, so no coriolis effect

- PR robot



$$M = \begin{pmatrix} m_{11} & m_{12}(q_2) \\ m_{12}(q_2) & m_{22} \end{pmatrix}$$

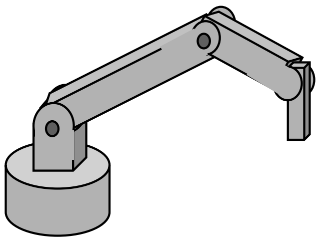
- 2R robot



$$M = \begin{pmatrix} m_{11}(q_2) & m_{12}(q_2) \\ m_{12}(q_2) & m_{22} \end{pmatrix}$$

- 3R articulated robot

(under simplifying assumptions on the CoMs)



if you apply force on the first joint only the first joint will accelerate
if you apply force on the second and third joint only the second and third joint will accelerate and the first not

$$M = \begin{pmatrix} m_{11}(q_2, q_3) & 0 & 0 \\ 0 & m_{22}(q_3) & m_{23}(q_3) \\ 0 & m_{23}(q_3) & m_{33} \end{pmatrix}$$

bottom-right part is the same of a 2R robot

Analysis of gravity term

- absence of gravity
  - constant U_g (motion on horizontal plane)
  - applications in remote space
- static balancing
  - distribution of masses (including motors)
- mechanical compensation
  - articulated system of springs
  - closed kinematic chains

$$\Rightarrow \quad g(q) \approx 0$$

Bounds on dynamic terms

- for an open-chain (serial) manipulator, there always exist positive real constants k_0 to k_7 such that, for **any** value of q and $\dot{q}$

$$k_0 \leq \|M(q)\| \leq k_1 + k_2\|q\| + k_3\|q\|^2$$ inertia matrix

$$\|S(q,\dot{q})\| \leq (k_4 + k_5\|q\|)\,\|\dot{q}\|$$ factorization matrix of Coriolis/centrifugal terms

$$\|g(q)\| \leq k_6 + k_7\|q\|$$ gravity vector

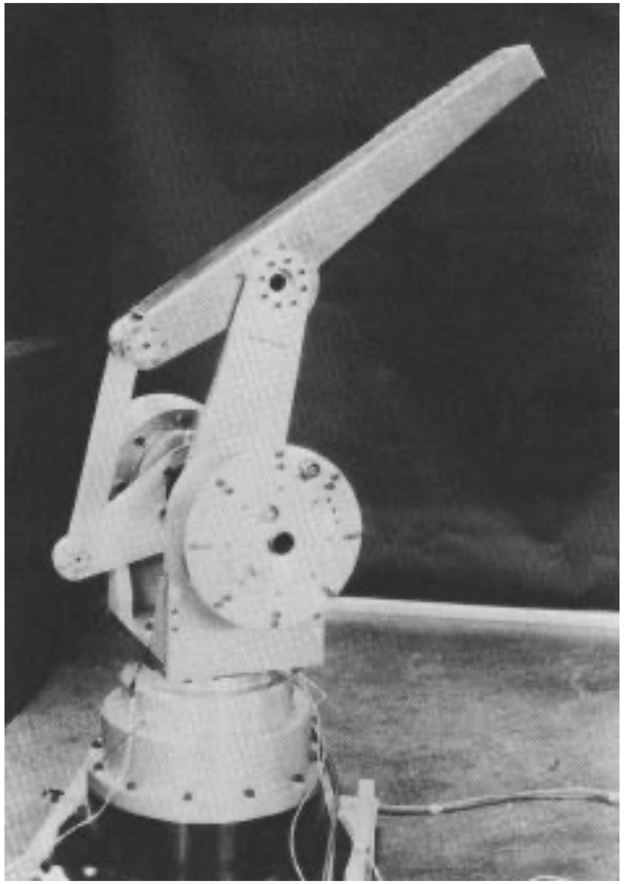
- if the robot has only **revolute** joints, these simplify to

$$\boxed{k_0 \leq \|M(q)\| \leq k_1 \quad \|S(q,\dot{q})\| \leq k_4\|\dot{q}\| \quad \|g(q)\| \leq k_6}$$

(the same holds true with bounds $q_{i,min} \leq q_i \leq q_{i,max}$ on prismatic joints)

NOTE: norms are either for vectors or for matrices (induced norms)

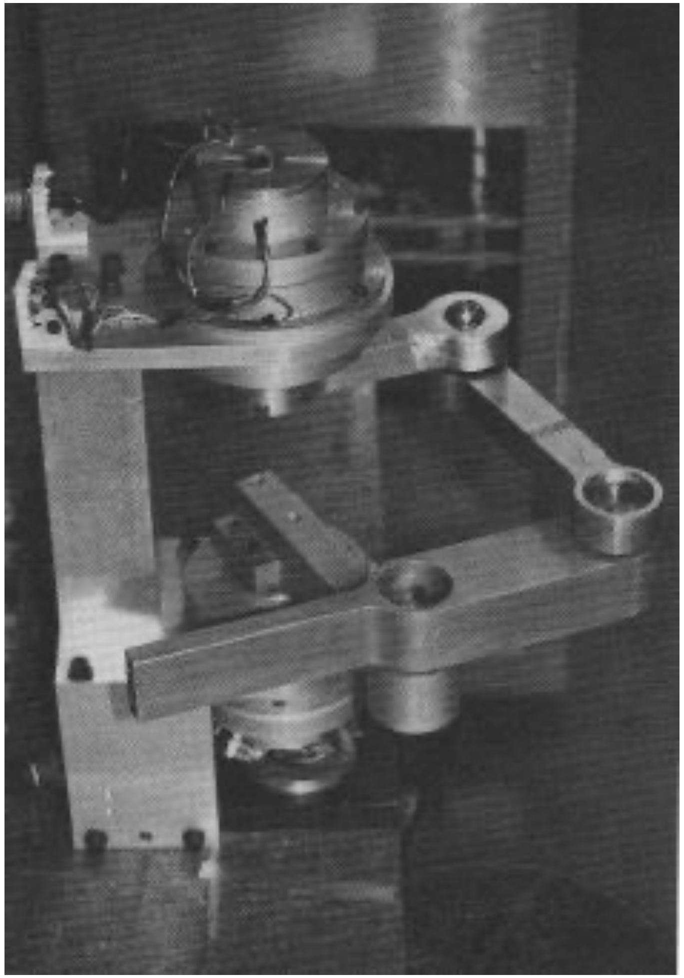
Robots with closed kinematic chains - 1



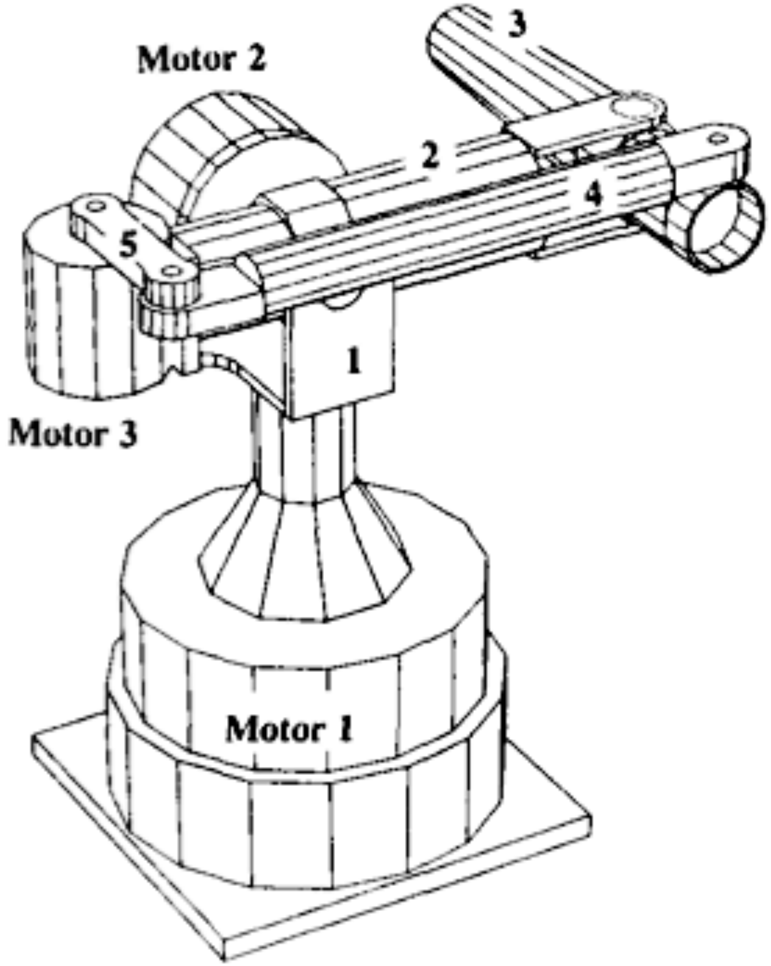
Comau Smart NJ130

MIT Direct Drive Mark II and Mark III

Robots with closed kinematic chains - 2



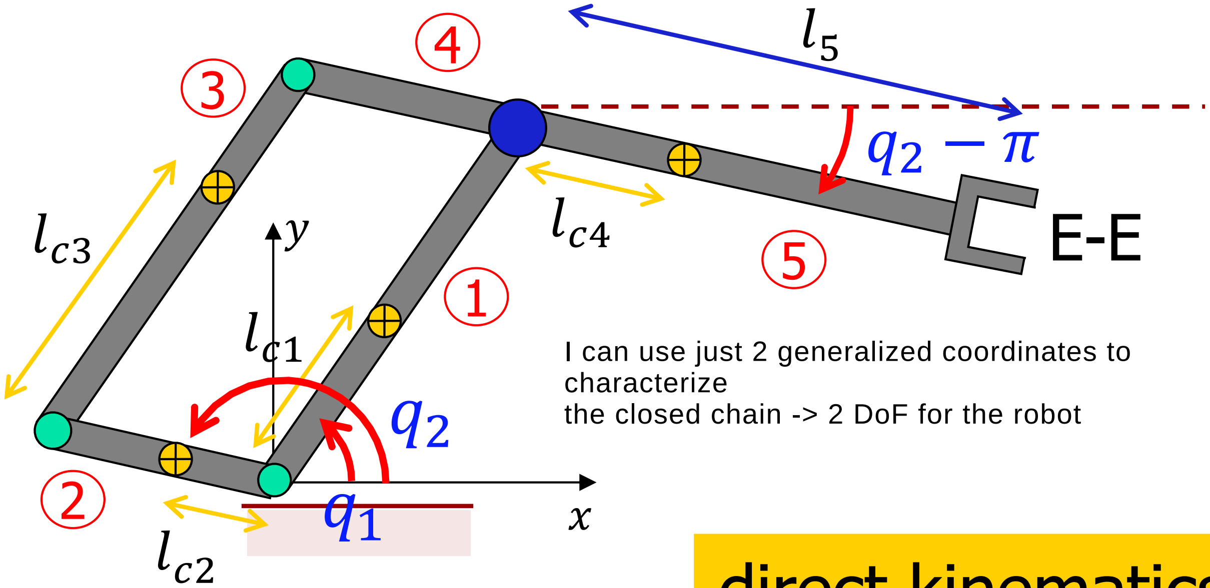
MIT Direct Drive Mark IV
(planar five-bar linkage)



UMinnesota Direct Drive Arm
(spatial five-bar linkage)

Robot with parallelogram structure

(planar) kinematics and dynamics



I can use just 2 generalized coordinates to characterize
the closed chain -> 2 DoF for the robot

⊕ center of mass: arbitrary l_{ci}

parallelogram:

$$l_1 = l_3$$
$$l_2 = l_4$$

direct kinematics

$$p_{EE} = \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} + \begin{pmatrix} l_5 \cos(q_2 - \pi) \\ l_5 \sin(q_2 - \pi) \end{pmatrix} = \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} - \begin{pmatrix} l_5 c_2 \\ l_5 s_2 \end{pmatrix}$$

position of center of masses

$$p_{c1} = \begin{pmatrix} l_{c1} c_1 \\ l_{c1} s_1 \end{pmatrix} \quad p_{c2} = \begin{pmatrix} l_{c2} c_2 \\ l_{c2} s_2 \end{pmatrix} \quad p_{c3} = \begin{pmatrix} l_2 c_2 \\ l_2 s_2 \end{pmatrix} + \begin{pmatrix} l_{c3} c_1 \\ l_{c3} s_1 \end{pmatrix} \quad p_{c4} = \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} - \begin{pmatrix} l_{c4} c_2 \\ l_{c4} s_2 \end{pmatrix}$$

Kinetic energy

linear/angular velocities

$$v_{c1} = \begin{pmatrix} -l_{c1}s_1 \\ l_{c1}c_1 \end{pmatrix}\dot{q}_1 \quad v_{c3} = \begin{pmatrix} -l_{c3}s_1 \\ l_{c3}c_1 \end{pmatrix}\dot{q}_1 + \begin{pmatrix} -l_2 s_2 \\ l_2 c_2 \end{pmatrix}\dot{q}_2 \quad \omega_1 = \omega_3 = \dot{q}_1$$

$$v_{c2} = \begin{pmatrix} -l_{c2}s_2 \\ l_{c2}c_2 \end{pmatrix}\dot{q}_2 \quad v_{c4} = \begin{pmatrix} -l_1 s_1 \\ l_1 c_1 \end{pmatrix}\dot{q}_1 + \begin{pmatrix} l_{c4}s_2 \\ -l_{c4}c_2 \end{pmatrix}\dot{q}_2 \quad \omega_2 = \omega_4 = \dot{q}_2$$

Note: a (planar) 2D notation is used here!

T_i

$$T_1 = \frac{1}{2}m_1 l_{c1}^2\dot{q}_1^2 + \frac{1}{2}I_{c1,zz}\dot{q}_1^2 \qquad T_2 = \frac{1}{2}m_2 l_{c2}^2\dot{q}_2^2 + \frac{1}{2}I_{c2,zz}\dot{q}_2^2$$

$$T_3 = \frac{1}{2}m_3\left(l_2^2\dot{q}_2^2 + l_{c3}^2\dot{q}_1^2 + 2l_2 l_{c3}c_{2-1}\dot{q}_1\dot{q}_2\right) + \frac{1}{2}I_{c3,zz}\dot{q}_1^2$$

$$T_4 = \frac{1}{2}m_4\left(l_1^2\dot{q}_1^2 + l_{c4}^2\dot{q}_2^2 - 2l_1 l_{c4}c_{2-1}\dot{q}_1\dot{q}_2\right) + \frac{1}{2}I_{c4,zz}\dot{q}_2^2$$

Robot inertia matrix

$$T = \sum_{i=1}^{4} T_i = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

$$M(q) = \begin{pmatrix} I_{c1,zz} + m_1 l_{c1}^2 + I_{c3,zz} + m_3 l_{c3}^2 + m_4 l_1^2 & \text{symm} \\ (m_3 l_2 l_{c3} - m_4 l_1 l_{c4}) c_{2-1} & I_{c2,zz} + m_2 l_{c2}^2 + I_{c4,zz} + m_4 l_{c4}^2 + m_3 l_2^2 \end{pmatrix}$$

structural condition
in mechanical design

$$m_3 l_2 l_{c3} = m_4 l_1 l_{c4}$$

(*)

this condition doesn't occurs in serial manipulator. If you set this condition the inertia matrix becomes diagonal and constant.

$M(q)$ diagonal and constant $\Rightarrow$ centrifugal and Coriolis terms $\equiv 0$

mechanically DECOUPLED and LINEAR dynamic model (up to the gravity term $g(q)$)

$$\begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

big advantage for the design of motion control laws!

Potential energy and gravity terms

from the y-components of vectors p_{ci}

U_i

$$U_1 = m_1 g_0 l_{c1} s_1 \qquad U_2 = m_2 g_0 l_{c2} s_2$$

$$U_3 = m_3 g_0 (l_2 s_2 + l_{c3} s_1) \qquad U_4 = m_4 g_0 (l_1 s_1 - l_{c4} s_2)$$

$$U = \sum_{i=1}^{4} U_i$$

$$g(q) = \left(\frac{\partial U}{\partial q}\right)^T = \begin{pmatrix} g_0(m_1 l_{c1} + m_3 l_{c3} + m_4 l_1) c_1 \\ g_0(m_2 l_{c2} + m_3 l_2 - m_4 l_{c4}) c_2 \end{pmatrix} = \begin{pmatrix} g_1(q_1) \\ g_2(q_2) \end{pmatrix}$$

gravity components are always "decoupled"

in addition, when (*) holds

$$\begin{aligned} m_{11}\ddot{q}_1 + g_1(q_1) &= u_1 \\ m_{22}\ddot{q}_2 + g_2(q_2) &= u_2 \end{aligned}$$

u_i are (non-conservative) torques performing work on q_i

further structural conditions in the mechanical design lead to $g(q) \equiv 0$!!

Adding dynamic terms ...

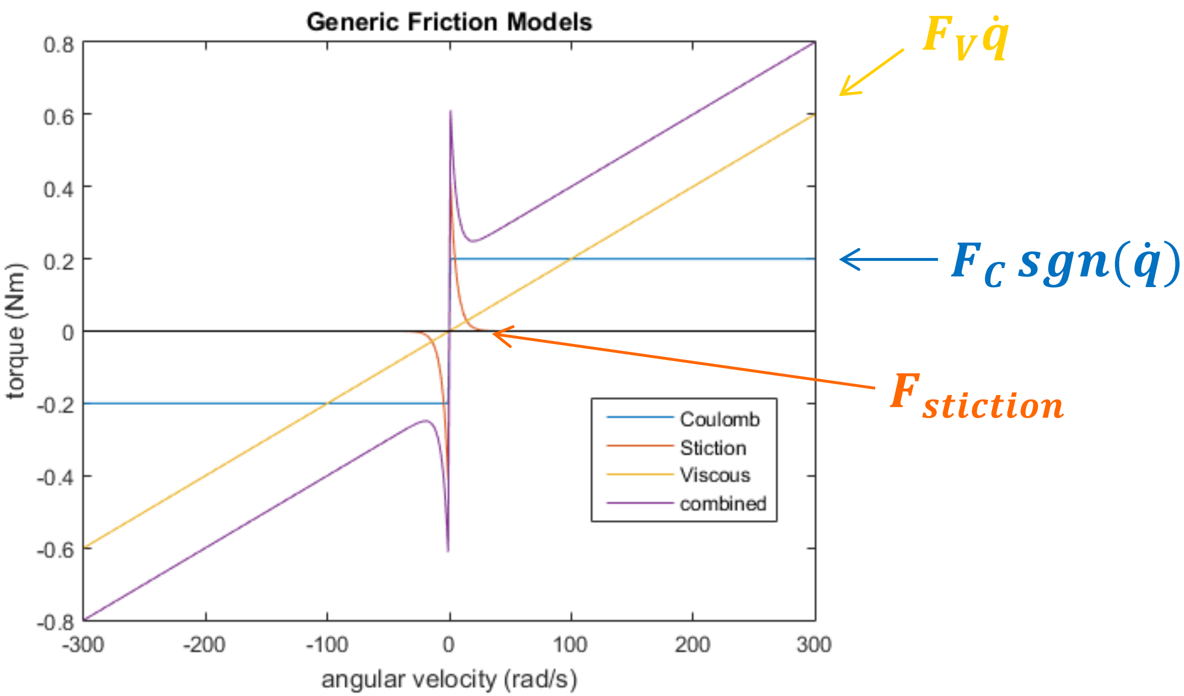
1) dissipative phenomena due to friction at the joints/transmissions
   - viscous, Coulomb, stiction, Stribeck, LuGre (dynamic)...
   - local effects at the joints
   - difficult to model in general, except for:

$$\boxed{u_{V,i} = -F_{V,i}\,\dot{q}_i} \qquad \boxed{u_{C,i} = -F_{C,i}\,\mathrm{sgn}(\dot{q}_i)}$$

in general:

$$u_{diss}^T\,\dot{q} < 0$$

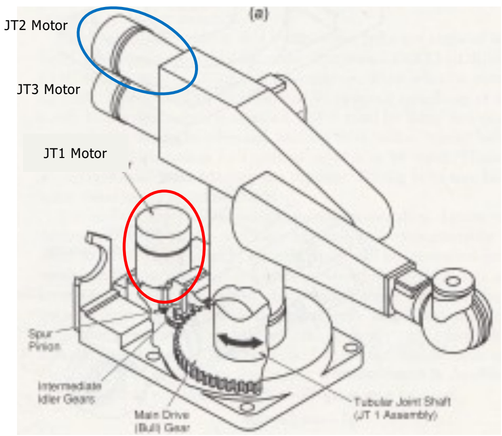
(component-wise too)

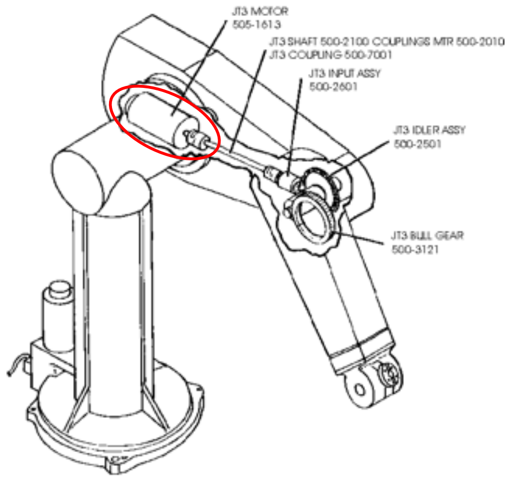


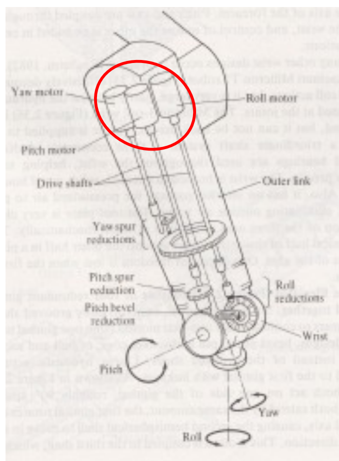
Adding dynamic terms ...

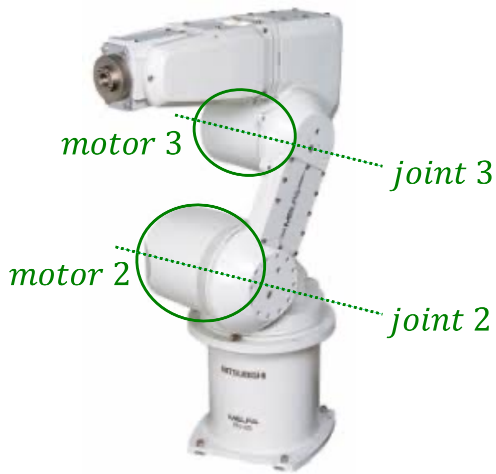
2) inclusion of electrical actuators (as additional rigid bodies)
   - motor i mounted on link $i-1$ (or before), with very few exceptions
   - often with its spinning axis aligned with joint axis i
   - (balanced) mass of motor included in total mass of carrying link
   - (rotor) inertia is to be added to robot kinetic energy
   - transmissions with reduction gears (often, large reduction ratios)
   - in some cases, multiple motors cooperate in moving multiple links: use a transmission coupling matrix Γ (with off-diagonal elements)

Unimation PUMA family



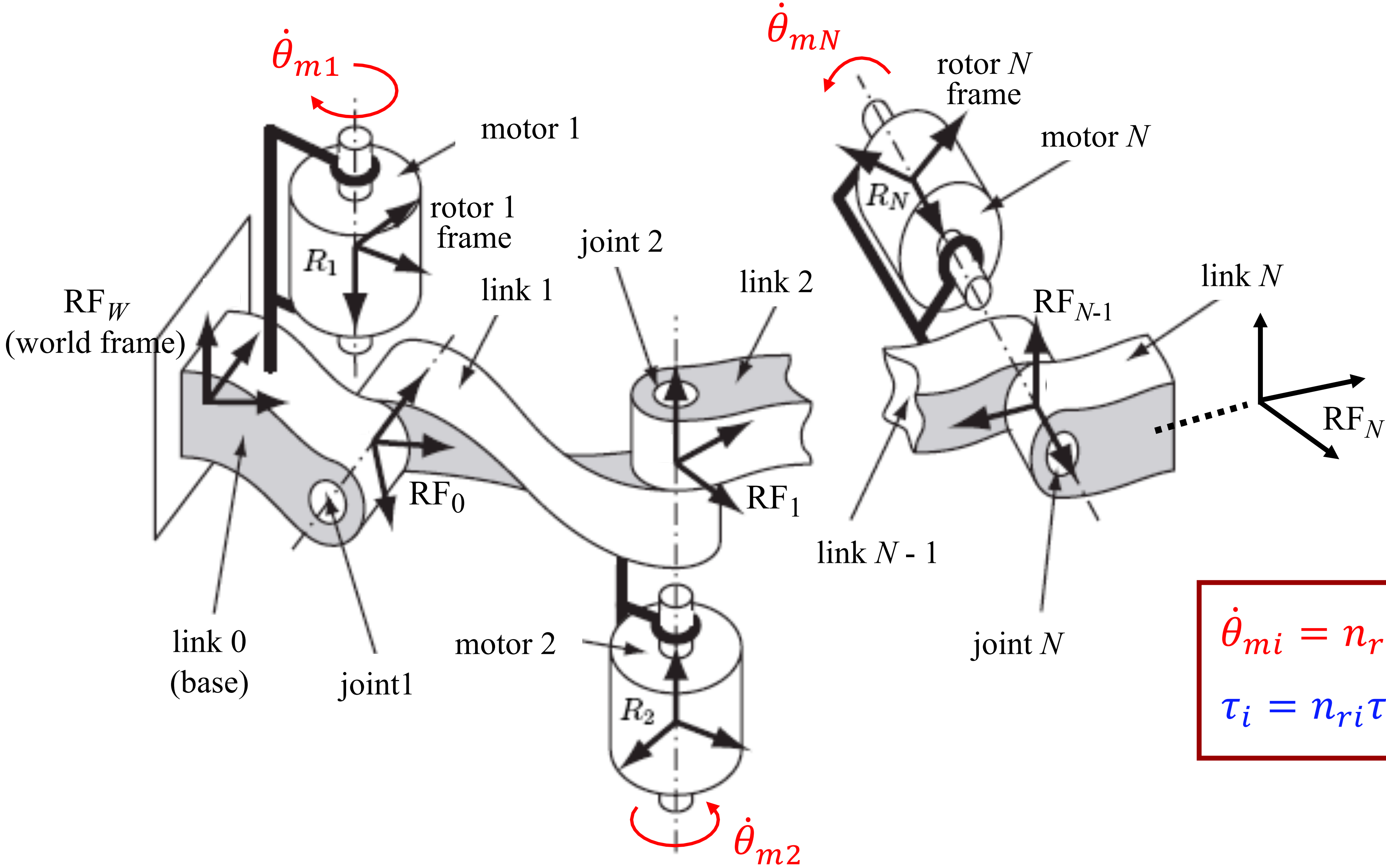






Mitsubishi RV-3S

Placement of motors along the chain



$$\dot{\theta}_{mi} = n_{ri}\dot{\theta}_i$$

$$\tau_i = n_{ri}\tau_{mi}$$

Resulting dynamic model

- simplifying assumption: in the rotational part of the kinetic energy, only the "spinning" rotor velocity is considered

the fact that the rotor, mounted on joint i, has an angular velocity because the previous link is rotating is neglected

$$T_{mi} = \frac{1}{2} I_{mi} \dot{\theta}_{mi}^2 = \frac{1}{2} I_{mi} n_{ri}^2 \dot{q}_i^2 = \frac{1}{2} B_{mi} \dot{q}_i^2 \qquad T_m = \sum_{i=1}^{N} T_{mi} = \frac{1}{2} \dot{q}^T \underbrace{B_m}_{\text{diagonal, } > 0} \dot{q}$$

- including all added terms, the robot dynamics becomes

$$(M(q) + \underbrace{B_m}_{\text{constant}})\ddot{q} + c(q, \dot{q}) + g(q) + \underbrace{F_V \dot{q} + F_C \, \mathrm{sgn}(\dot{q})}_{F_V > 0,\, F_C > 0 \text{ diagonal}} = \tau$$

moved to the left ...

constant → does NOT contribute to c

motor torques (**after** reduction gears)

- scaling by the reduction gears, looking from the motor side

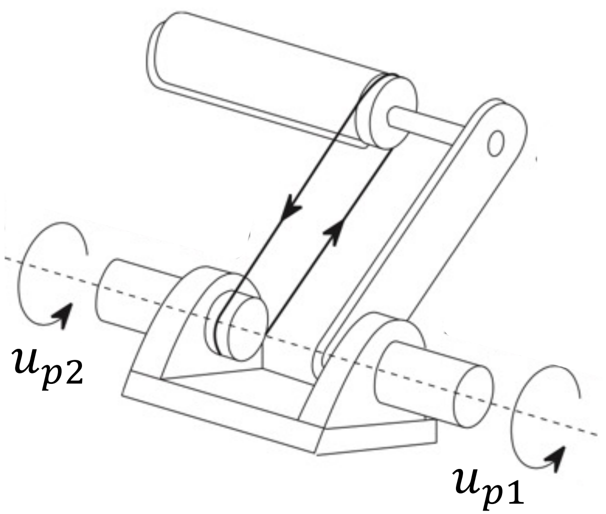
diagonal

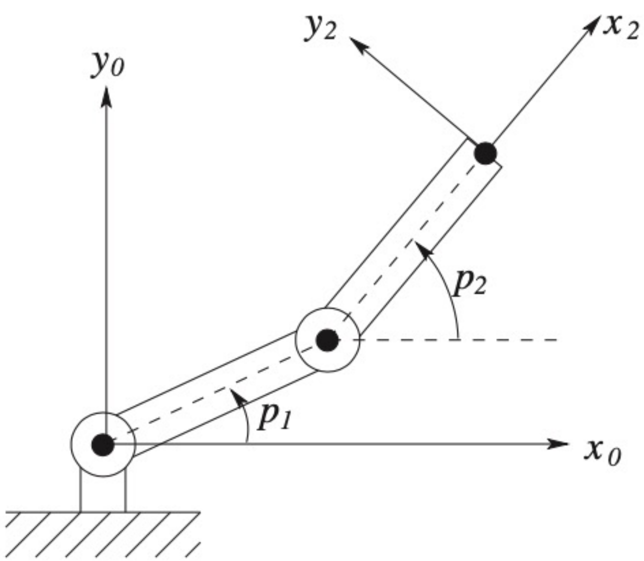
$$\left(I_m + \mathrm{diag}\left\{\frac{m_{ii}(q)}{n_{ri}^2}\right\}\right)\ddot{\theta}_m + \mathrm{diag}\left\{\frac{1}{n_{ri}}\right\}\left(\sum_{j=1}^{N} \bar{M}_j(q)\ddot{q}_j + f(q, \dot{q})\right) = \tau_m$$

except the terms m_{jj}

motor torques (**before** reduction gears)

Special actuation and associated coordinates
planar 2R robot with remotely driven forearm





- motor 1 moves link 1 by p_1
- motor 2 at the base moves the absolute angle p_2 of link 2
- derive the dynamic model from scratch using the $\boldsymbol{p}$ coordinates

$$M(p)\ddot{p} + c(p,\dot{p}) + g(p) = u_p$$

$$M(p) = \begin{pmatrix} a_1 - a_3 & a_2 c_{2-1} \\ a_2 c_{2-1} & a_3 \end{pmatrix}$$

$$c(p,\dot{p}) = \begin{pmatrix} -a_2 s_{2-1}\, \dot{p}_2^2 \\ a_2 s_{2-1}\, \dot{p}_1^2 \end{pmatrix}$$

no more Coriolis forces!

$$g(p) = \begin{pmatrix} a_4 c_1 \\ a_5 c_2 \end{pmatrix}$$

$c_1 = \cos p_1 \quad c_2 = \cos p_2$

$c_{2-1} = \cos(p_2 - p_1) \quad s_{2-1} = \sin(p_2 - p_1)$

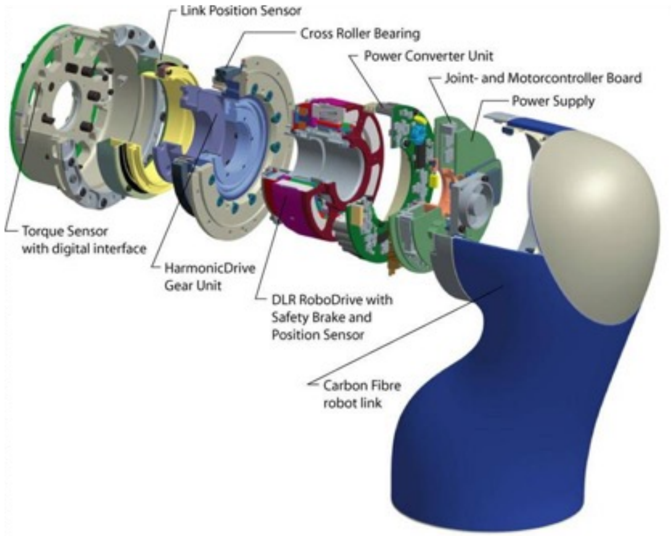
Including joint elasticity

- in industrial robots, use of motion transmissions based on
  - belts
  - harmonic drives
  - long shafts

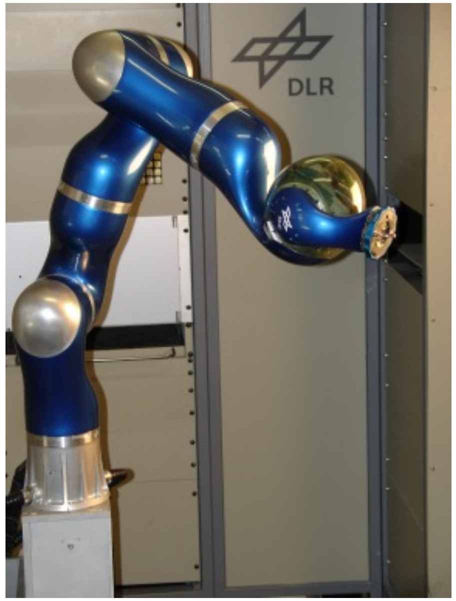
  introduces flexibility between actuating motors (input) and driven links (output)
- in research robots, compliance in transmissions is introduced on purpose for safety (human collaboration) and/or energy efficiency
  - actuator relocation by means of (compliant) cables and pulleys
  - harmonic drives and lightweight (but rigid) link design
  - redundant (macro-mini or parallel) actuation, with elastic couplings
- in both cases, flexibility is modeled as concentrated at the joints
- in most cases, assuming small joint deformation (elastic domain)

Robots with joint elasticity

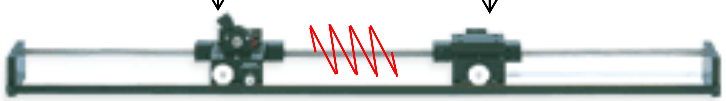
Dexter
with cable transmissions



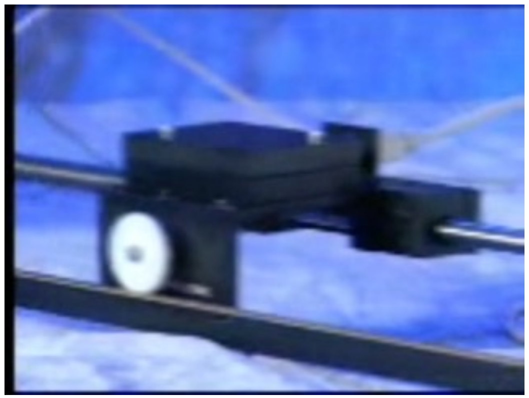
DLR LWR-III
with harmonic drives



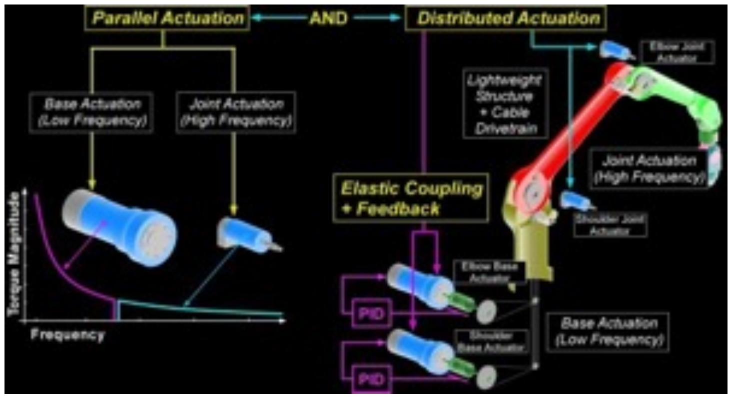
motor | elastic spring (stiffness K) | load/link



Quanser Flexible Joint
(1-dof linear, educational)



video



Stanford DECMMA
with micro-macro actuation

Dynamic model of robots with elastic joints

- introduce $2N$ generalized coordinates
  - $q = N$ link positions
  - $\theta = N$ motor positions (after reduction, $\theta_i = \theta_{mi}/n_{ri}$)
- add motor kinetic energy T_m to that of the links $T_q = \frac{1}{2}\dot{q}^T M(q)\dot{q}$

$$T_{mi} = \frac{1}{2} I_{mi}\dot{\theta}_{mi}^2 = \frac{1}{2} I_{mi} n_{ri}^2 \dot{\theta}_i^2 = \frac{1}{2} B_{mi}\dot{\theta}_i^2 \qquad T_m = \sum_{i=1}^{N} T_{mi} = \frac{1}{2}\dot{\theta}^T B_m \dot{\theta}$$

diagonal, > 0

- add elastic potential energy U_e to that due to gravity $U_g(q)$
  - $K =$ matrix of joint stiffness (diagonal, > 0)

$$U_{ei} = \frac{1}{2} K_i \left(q_i - \left(\frac{\theta_{mi}}{n_{ri}} \right) \right)^2 = \frac{1}{2} K_i (q_i - \theta_i)^2 \quad U_e = \sum_{i=1}^{N} U_{ei} = \frac{1}{2}(q-\theta)^T K (q-\theta)$$

- apply Euler-Lagrange equations w.r.t. (q, θ)

$2N$ 2nd-order differential equations

$$\begin{cases} M(q)\ddot{q} + c(q,\dot{q}) + g(q) + K(q-\theta) = 0 \\ B_m\ddot{\theta} + K(\theta - q) = \tau \end{cases}$$

no external torques performing work on q

if theta goes to q and K to infinity
inf*0 but somehow it disappear, and summing up the two equations we obtain the previous model, the one without elasticity

Use of the dynamic model
inverse dynamics

- given a desired trajectory $q_d(t)$
  - twice differentiable ($\exists\ \ddot{q}_d(t)$)
  - possibly obtained from a task/Cartesian trajectory $r_d(t)$, by (differential) kinematic inversion

the input torque needed to execute this motion (in free space) is

$$\tau_d = (M(q_d) + B_m)\ddot{q}_d + c(q_d, \dot{q}_d) + g(q_d) + F_V \dot{q}_d + F_C \operatorname{sgn}(\dot{q}_d)$$

(in contact, with an external wrench) $\ldots - J_{ext}^T(q_d) F_{ext,d}$

- useful also for control (e.g., nominal feedforward)
- however, this way of performing the algebraic computation ($\forall t$) is not efficient when using the Lagrangian modeling approach
  - symbolic terms grow much longer, quite rapidly for larger N N is DoF
  - in real time, numerical computation is based on Newton-Euler method

State equations
direct dynamics

Lagrangian dynamic model

$$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u$$

N differential 2^{nd} order equations

defining the vector of state variables as $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix} \in \mathbb{R}^{2N}$

state equations

$$\boxed{\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)[c(x_1,x_2) + g(x_1)] \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}(x_1) \end{pmatrix} u}$$

$$= \underbrace{f(x)}_{2N \times 1} + \underbrace{G(x)}_{2N \times N} u$$

The input appears in a linear fashion: outside the non-linear function of x. Is very convenient.

$2N$ differential 1^{st} order equations

another choice... $\boxed{\tilde{x} = \begin{pmatrix} q \\ M(q)\dot{q} \end{pmatrix}}$ ← generalized momentum

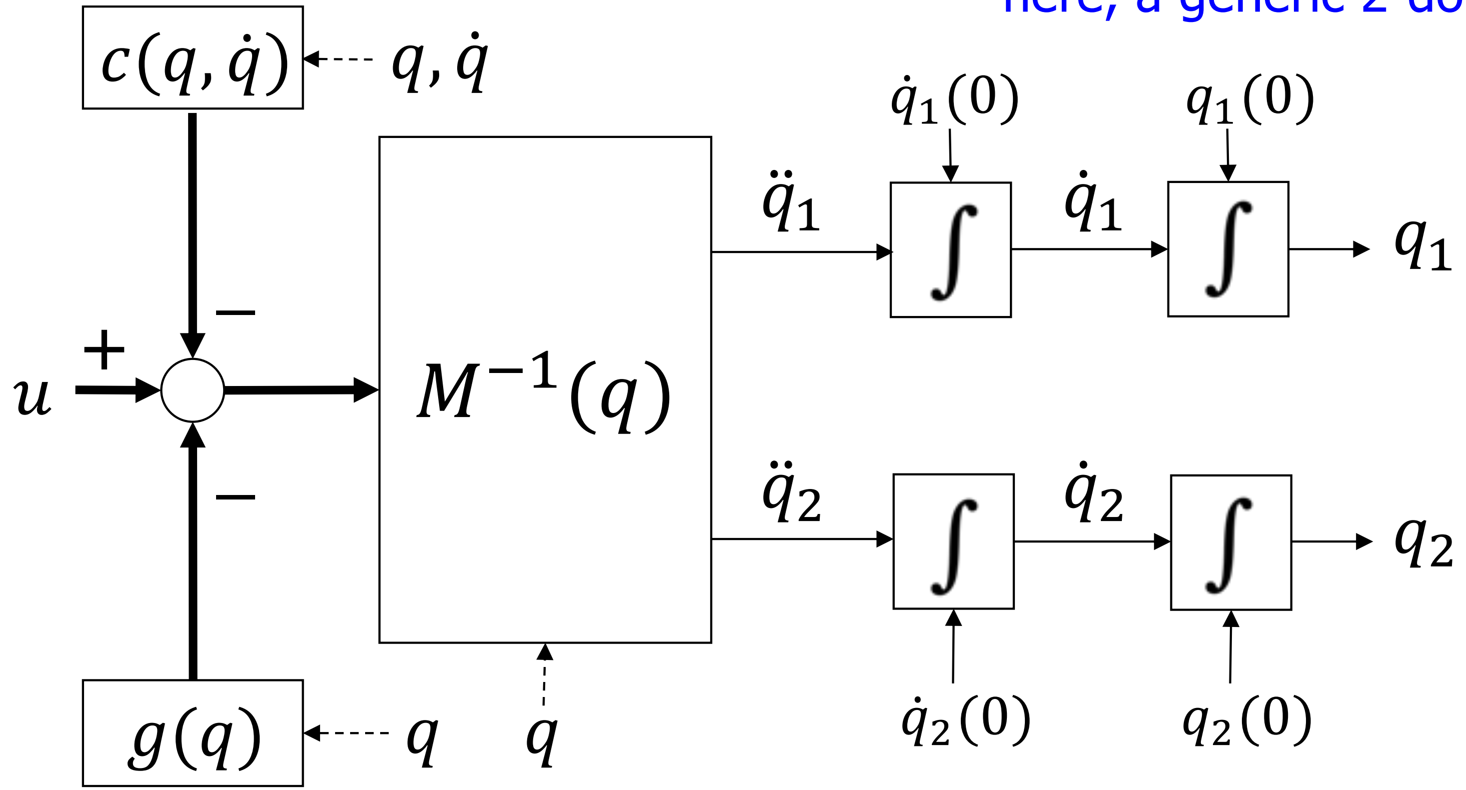
$\dot{\tilde{x}} = \ldots$ (do it as exercise)

Dynamic simulation

Simulink block scheme

here, a generic 2-dof robot

input torque command (open-loop or in feedback)



including "inv(M)"

- initialization (dynamic coefficients and initial state)
- calls to (user-defined) Matlab functions for the evaluation of model terms
- choice of a numerical integration method (and of its parameters)

e.g., 4th-order Runge-Kutta (ode45)

The model is non linear but when you are approaching a final destination (e.g. point to point destination, while approaching the final destination when you want to be there at zero velocity) then locally your dynamics can be approximated as a linear model

Approximate linearization

- we can derive a linear dynamic model of the robot, which is valid locally around a given operative condition
  - useful for analysis, design, and gain tuning of linear (or of the linear part of) control laws
  - approximation by Taylor series expansion, up to the first order
  - linearization around a (constant) equilibrium state or along a (nominal, time-varying) equilibrium trajectory
  - usually, we work with (nonlinear) state equations; for mechanical systems, it is more convenient to directly use the 2nd order model
    - same result, but easier derivation

velocity zero since you don't want to move away

equilibrium torque that balances gravity

equilibrium state $(q,\dot{q}) = (q_e, 0)\ [\ \ddot{q} = 0\ ] \quad \Rightarrow \quad g(q_e) = u_e$

equilibrium trajectory $(q,\dot{q}) = (q_d(t), \dot{q}_d(t))\ \ [\ \ddot{q} = \ddot{q}_d(t)\ ]$

$$\Rightarrow \quad M(q_d)\ddot{q}_d + c(q_d, \dot{q}_d) + g(q_d) = u_d$$

Linearization at an equilibrium state

- variations around an equilibrium state

$$q = q_e + \Delta q \quad \dot{q} = \cancel{\dot{q}_e} + \Delta\dot{q} = \Delta\dot{q} \quad \ddot{q} = \cancel{\ddot{q}_e} + \Delta\ddot{q} = \Delta\ddot{q} \quad u = u_e + \Delta u$$

- keeping into account the quadratic dependence of c terms on velocity (thus, neglected around the zero velocity)

delta_q * delta_q = delta_q^2 is an infinitesimal of higher order than delta_q so is neglectable. Since c terms have delta_q^2 we neglect the c terms

$$M(q_e)\Delta\ddot{q} + \cancel{g(q_e)} + \underbrace{\left.\frac{\partial g}{\partial q}\right|_{q=q_e}}_{G(q_e)} \Delta q + \underbrace{\cancel{\mathrm{o}\left(\|\Delta q\|, \|\Delta\dot{q}\|\right)}}_{\text{infinitesimal terms of second or higher order}} = \cancel{u_e} + \Delta u$$

- in state-space format, with $\Delta x = \begin{pmatrix} \Delta q \\ \Delta\dot{q} \end{pmatrix}$

$$\Delta\dot{x} = \begin{pmatrix} 0 & I \\ -M^{-1}(q_e)G(q_e) & 0 \end{pmatrix} \Delta x + \begin{pmatrix} 0 \\ M^{-1}(q_e) \end{pmatrix} \Delta u = A\,\Delta x + B\,\Delta u$$

Linearization along a trajectory

- variations around an equilibrium trajectory

$$q = q_d + \Delta q \qquad \dot{q} = \dot{q}_d + \Delta\dot{q} \qquad \ddot{q} = \ddot{q}_d + \Delta\ddot{q} \qquad u = u_d + \Delta u$$

- developing to 1st order the terms in the dynamic model ...

$$M(q_d + \Delta q)\left(\ddot{q}_d + \Delta\ddot{q}\right) + c(q_d + \Delta q, \dot{q}_d + \Delta\dot{q}) + g(q_d + \Delta q) = u_d + \Delta u$$

$$M(q_d + \Delta q) \cong M(q_d) + \sum_{i=1}^{N} \left.\frac{\partial M_i}{\partial q}\right|_{q=q_d} e_i^T \Delta q$$

i-th row of the identity matrix

$$g(q_d + \Delta q) \cong g(q_d) + G(q_d)\Delta q$$

$$c\left(q_d + \Delta q, \dot{q}_d + \Delta\dot{q}\right) \cong c(q_d, \dot{q}_d) + \overbrace{\left.\frac{\partial c}{\partial q}\right|_{\substack{q=q_d \\ \dot{q}=\dot{q}_d}}}^{C_1(q_d, \dot{q}_d)} \Delta q + \underbrace{\left.\frac{\partial c}{\partial \dot{q}}\right|_{\substack{q=q_d \\ \dot{q}=\dot{q}_d}}}_{C_2(q_d, \dot{q}_d)} \Delta\dot{q}$$

Linearization along a trajectory (cont)

- after simplifications …

$$M(q_d)\ddot{\Delta q} + C_2(q_d,\dot{q}_d)\dot{\Delta q} + D(q_d,\dot{q}_d,\ddot{q}_d)\Delta q = \Delta u$$

with

$$D(q_d,\dot{q}_d,\ddot{q}_d) = G(q_d) + C_1(q_d,\dot{q}_d) + \sum_{i=1}^{N} \left.\frac{\partial M_i}{\partial q}\right|_{q=q_d} \ddot{q}_d e_i^T$$

- in state-space format

$$\dot{\Delta x} = \begin{pmatrix} 0 & I \\ -M^{-1}(q_d)D(q_d,\dot{q}_d,\ddot{q}_d) & -M^{-1}(q_d)C_2(q_d,\dot{q}_d) \end{pmatrix}\Delta x + \begin{pmatrix} 0 \\ M^{-1}(q_d) \end{pmatrix}\Delta u = A(t)\,\Delta x + B(t)\,\Delta u$$

a linear, but time-varying system!!

Coordinate transformation

$q \in \mathbb{R}^N$ $\quad \boxed{M(q)\ddot{q} + c(q,\dot{q}) + g(q) = M(q)\ddot{q} + n(q,\dot{q}) = u_q}$ 1

if we wish/need to use a new set of generalized coordinates p

$p \in \mathbb{R}^N$ $\quad \boxed{p = f(q)} \rightarrow \boxed{q = f^{-1}(p)}$

by duality
(principle of virtual work)

$\boxed{\dot{p} = \frac{\partial f}{\partial q}\dot{q} = J(q)\dot{q}} \rightarrow \boxed{\dot{q} = J^{-1}(q)\dot{p}}$ $\quad \boxed{u_q = J^T(q)u_p}$

$\boxed{\ddot{p} = J(q)\ddot{q} + \dot{J}(q)\dot{q}} \rightarrow \boxed{\ddot{q} = J^{-1}(q)\left(\ddot{p} - \dot{J}(q)J^{-1}(q)\dot{p}\right)}$

$\rightarrow$ 1

$$\boxed{M(q)J^{-1}(q)\ddot{p} - M(q)J^{-1}(q)\dot{J}(q)J^{-1}(q)\dot{p} + n(q,\dot{q}) = J^T(q)u_p}$$

$J^{-T}(q)\cdot$ pre-multiplying the whole equation...

Robot dynamic model
after coordinate transformation

$$\boxed{J^{-T}(q)M(q)J^{-1}(q)\ddot{p} + J^{-T}(q)\left(n(q,\dot{q}) - M(q)J^{-1}(q)\dot{J}(q)J^{-1}(q)\dot{p}\right) = u_p}$$

$\boxed{q \to p}$

for actual computation, these inner substitutions are not strictly necessary

$\boxed{(q,\dot{q}) \to (p,\dot{p})}$

non-conservative generalized forces performing work on p

$$\boxed{M_p(p)\ddot{p} + c_p(p,\dot{p}) + g_p(p) = u_p}$$

$$M_p = J^{-T}MJ^{-1}$$ symmetric, positive definite (out of singularities)

$$g_p = J^{-T}g$$

j_dot is a function of p and p_dot so quadratic dependence of p_dot

$$c_p = J^{-T}\left(c - MJ^{-1}\dot{J}J^{-1}\dot{p}\right) = J^{-T}c - M_p\dot{J}J^{-1}\dot{p}$$ quadratic dependence on $\dot{p}$

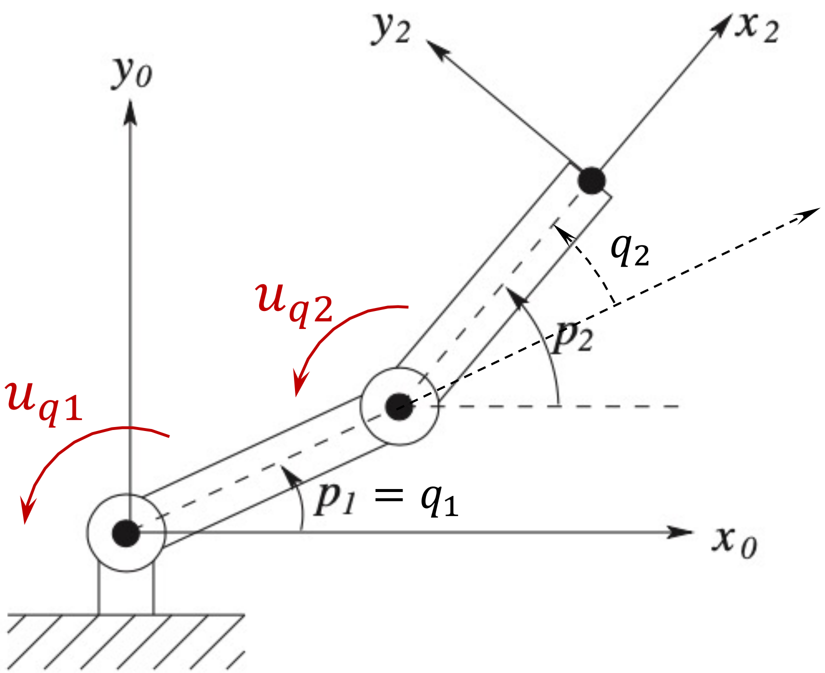
$$c_p(p,\dot{p}) = S_p(p,\dot{p})\,\dot{p} \qquad \dot{M}_p - 2S_p$$ skew-symmetric

when p = E-E pose, this is the robot dynamic model in Cartesian coordinates

NOTE: in this case, we have implicitly assumed than $M = N$ (no redundancy!)

Example of coordinate transformation

planar 2R robot using absolute coordinates



- motor 1 at joint 1, motor 2 at joint 2
- in place of DH angles $\boldsymbol{q}$, use the absolute angles $p_1 = q_1$ and $p_2 = q_1 + q_2$

$$p = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} q = J\, q$$

a linear transformation

$$J^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad J^{-T} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

- from $M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u_q$ obtained with DH relative coordinates

blue terms are the same found in a direct way in slide #15

$$M_p(p) = J^{-T} M\, J^{-1} = \begin{pmatrix} a_1 - a_3 & a_2 c_{2-1} \\ a_2 c_{2-1} & a_3 \end{pmatrix} \qquad g_p(p) = J^{-T} g = \begin{pmatrix} a_4 c_1 \\ a_5 c_2 \end{pmatrix}$$

$$c_p(p,\dot{p}) = J^{-T} c = \begin{pmatrix} -a_2 s_{2-1}\, \dot{p}_2^2 \\ a_2 s_{2-1}\, \dot{p}_1^2 \end{pmatrix} \qquad u_p = J^{-T} u_q = \begin{pmatrix} u_{q1} - u_{q2} \\ u_{q2} \end{pmatrix}$$

Robot dynamic model
in the task/Cartesian space, with redundancy

dynamic model in the joint space

$$M(q)\ddot{q} + n(q,\dot{q}) = \tau$$

$$\boxed{\begin{gathered} q \in \mathbb{R}^N \\ r = f(q) \in \mathbb{R}^M \\ M < N \end{gathered}}$$

second-order task kinematics

$$\ddot{r} = J(q)\ddot{q} + \dot{J}(q)\dot{q}$$

J is full rank = M

1) isolate the joint acceleration from the dynamics ➡ $\ddot{q} = M^{-1}(q)\left(\tau - n(q,\dot{q})\right)$

2) decompose the joint torques in two complementary spaces

$$\tau = \underbrace{J^T(q)F}_{\in \mathcal{R}(J^T)} + \underbrace{(I - J^T(q)H(q))\tau_0}_{\in \mathcal{N}(J^T H)}$$

H is a generalized inverse of J^T

$$J^T H J^T = J^T$$

torques coming from generalized forces F in the task space …

… and joint torques $\tau_0 \notin \mathcal{R}(J^T)$

➡ $\tau_0 = J^T(q)F, \forall F \in \mathbb{R}^M \Rightarrow \left(I - J^T(q)H(q)\right)J^T(q)F = 0$

3) substitute 1) and 2) in the differential task kinematics

➡ $$\ddot{r} = \underbrace{J(q)M^{-1}(q)\left(J^T(q)\right.}F + (I - J^T(q)H(q))\tau_0 - n(q,\dot{q})\Big) + \dot{J}(q)\dot{q}$$

4) isolate on the right-hand side the generalized forces F in the task space …

Robot dynamic model
in the task/Cartesian space, with redundancy

$$\Rightarrow \left(J(q)M^{-1}(q)J^T(q)\right)^{-1}\ddot{r} = F + \left(J(q)M^{-1}(q)J^T(q)\right)^{-1}\left(J(q)M^{-1}(q)\left(\left(I - J^T(q)H(q)\right)\tau_0 - n(q,\dot{q})\right) + \dot{J}(q)\dot{q}\right)$$

5) choose as generalized inverse $H = (JM^{-1}J^T)^{-1}JM^{-1} = \left(J_M^{\#}\right)^T$, i.e., the transpose of the inertia-weighted pseudoinverse of the task Jacobian (see block of slides #2)

$\Rightarrow$ in this way, the joint torque component τ_0 will **NOT** affect the task acceleration $\ddot{r}$

$$\left(J(q)M^{-1}(q)J^T(q)\right)^{-1}\ddot{r} = F + \left(J(q)M^{-1}(q)J^T(q)\right)^{-1}\left(\dot{J}(q)\dot{q} - J(q)M^{-1}(q)\, n(q,\dot{q})\right)$$

6) the resulting (M −dimensional) task dynamics is then

$$\boxed{M_r(q)\ddot{r} + n_r(q,\dot{q}) = F} \ldots + F_{ext}$$

external forces can be added on the rhs of the equations in a dynamically consistent way!

with

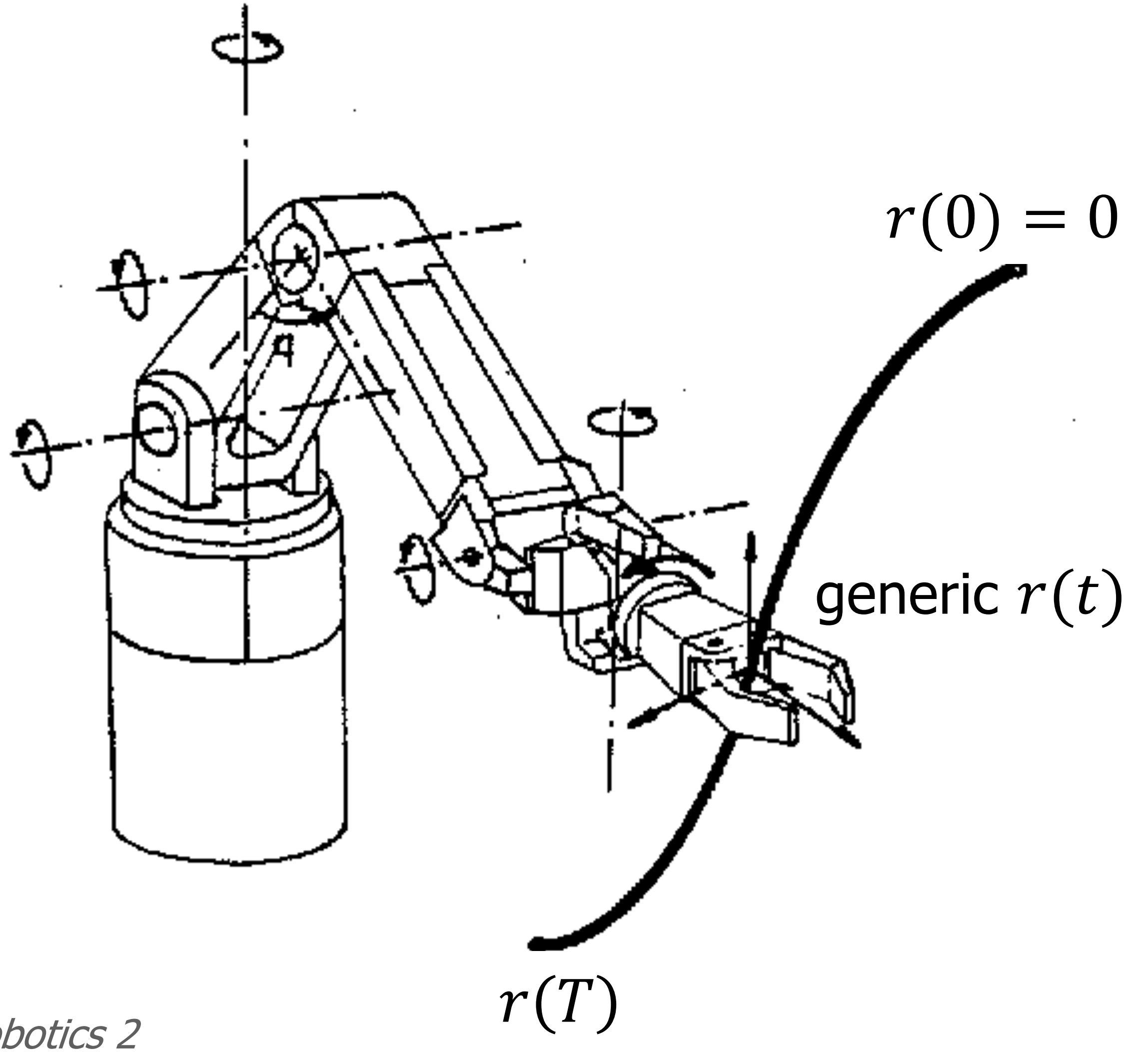
$$\left.\begin{aligned} M_r(q) &= \left(J(q)M^{-1}(q)J^T(q)\right)^{-1} \quad \text{task inertia matrix} \\ n_r(q,\dot{q}) &= M_r(q)\left(J(q)M^{-1}(q)\, n(q,\dot{q}) - \dot{J}(q)\dot{q}\right) \end{aligned}\right\} \text{for } M = N \text{, these terms are identical to slide \#27}$$

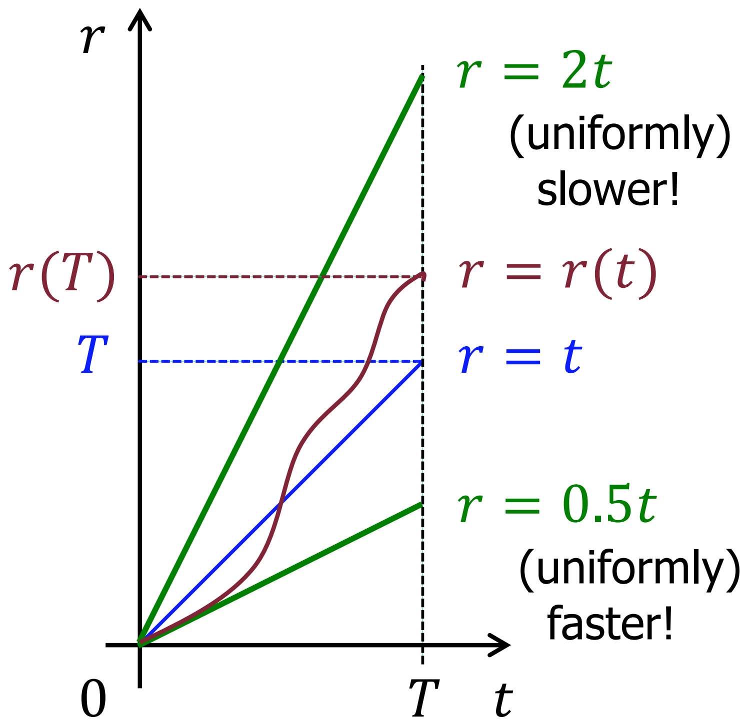
7) an additional $(N - M)$-dimensional second-order dynamics is needed to describe the full robot!

Dynamic scaling of trajectories

uniform time scaling of motion

- given a smooth original trajectory $q_d(t)$ of motion for $t \in [0, T]$
  - suppose to rescale time as $t \to r(t)$ (a strictly **increasing** function of t)





Dynamic scaling of trajectories

uniform time scaling of motion

- in the new time scale, the scaled trajectory $q_s(r)$ satisfies

$$q_d(t) = q_s(r(t)) \quad \Rightarrow \quad \dot{q}_d(t) = \frac{dq_d}{dt} = \frac{dq_s}{dr}\frac{dr}{dt} = q_s'(r)\,\dot{r}(t)$$

same path executed
(at different instants of time)

$$\Downarrow$$

$$\ddot{q}_d(t) = \frac{d\dot{q}_d}{dt} = \left(\frac{dq_s'}{dr}\frac{dr}{dt}\right)\dot{r} + q_s'\frac{d\dot{r}}{dt} = q_s''(r)\dot{r}^2(t) + q_s'(r)\ddot{r}(t)$$

- uniform scaling of the trajectory occurs when $r(t) = kt$

$$\dot{q}_d(t) = kq_s'(kt) \qquad \ddot{q}_d(t) = k^2 q_s''(kt)$$

Q: what is the new input torque needed to execute the scaled trajectory?
(suppose dissipative terms can be neglected)

Dynamic scaling of trajectories

inverse dynamics under uniform time scaling

- the new torque could be recomputed through the inverse dynamics, for every $r = kt \in [0, T_s] = [0, kT]$ along the scaled trajectory, as

$$\tau_s(kt) = M(q_s)q_s'' + c(q_s, q_s') + g(q_s)$$

- however, being the dynamic model linear in the acceleration and quadratic in the velocity, it is

$$\tau_d(t) = M(q_d)\ddot{q}_d + c(q_d, \dot{q}_d) + g(q_d) = M(q_s)k^2 q_s'' + c(q_s, kq_s') + g(q_s)$$
$$= k^2\big(M(q_s)q_s'' + c(q_s, q_s')\big) + g(q_s) = k^2\big(\tau_s(kt) - g(q_s)\big) + g(q_s)$$

- thus, saving separately the total torque $\tau_d(t)$ and gravity torque $g_d(t)$ in the inverse dynamics computation along the original trajectory, the new input torque is obtained directly as

$$\boxed{\tau_s(kt) = \frac{1}{k^2}\big(\tau_d(t) - g(q_d(t))\big) + g(q_d(t))}$$

$k > 1$: slow down $\Rightarrow$ reduce torque

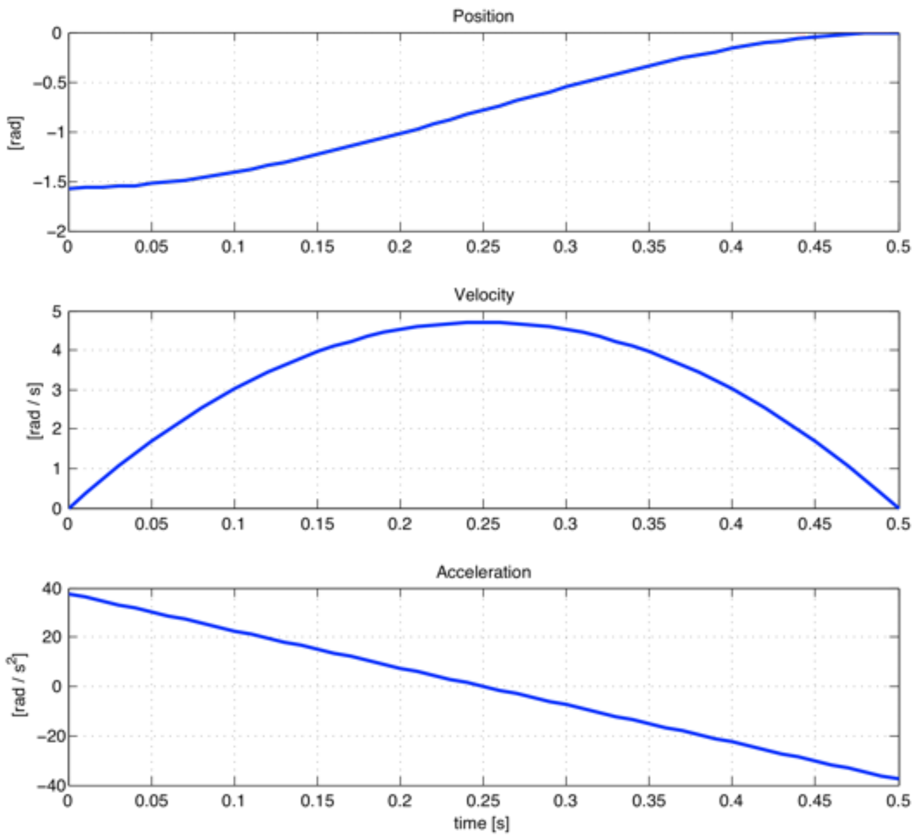
$k < 1$: speed up $\Rightarrow$ increase torque

gravity term (only position-dependent): does NOT scale!

Dynamic scaling of trajectories

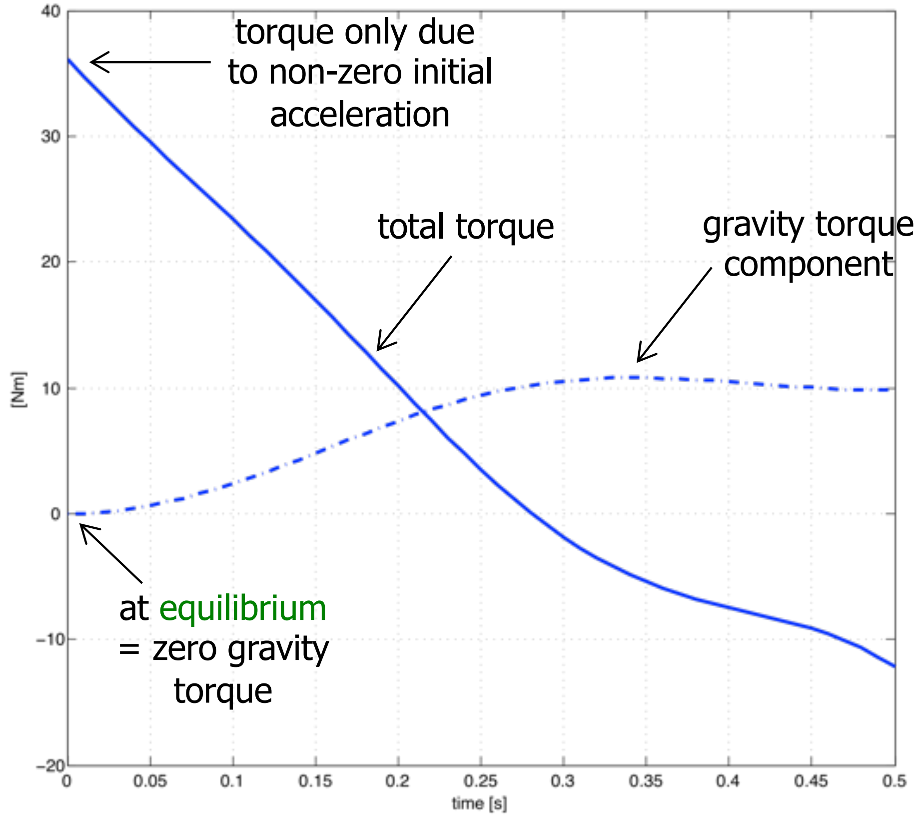
numerical example

- rest-to-rest motion with cubic polynomials for planar 2R robot under gravity (from downward equilibrium to horizontal link 1 & upward vertical link 2)
- original trajectory lasts $T = 0.5$ s (but say, it violates the torque limit at joint 1)

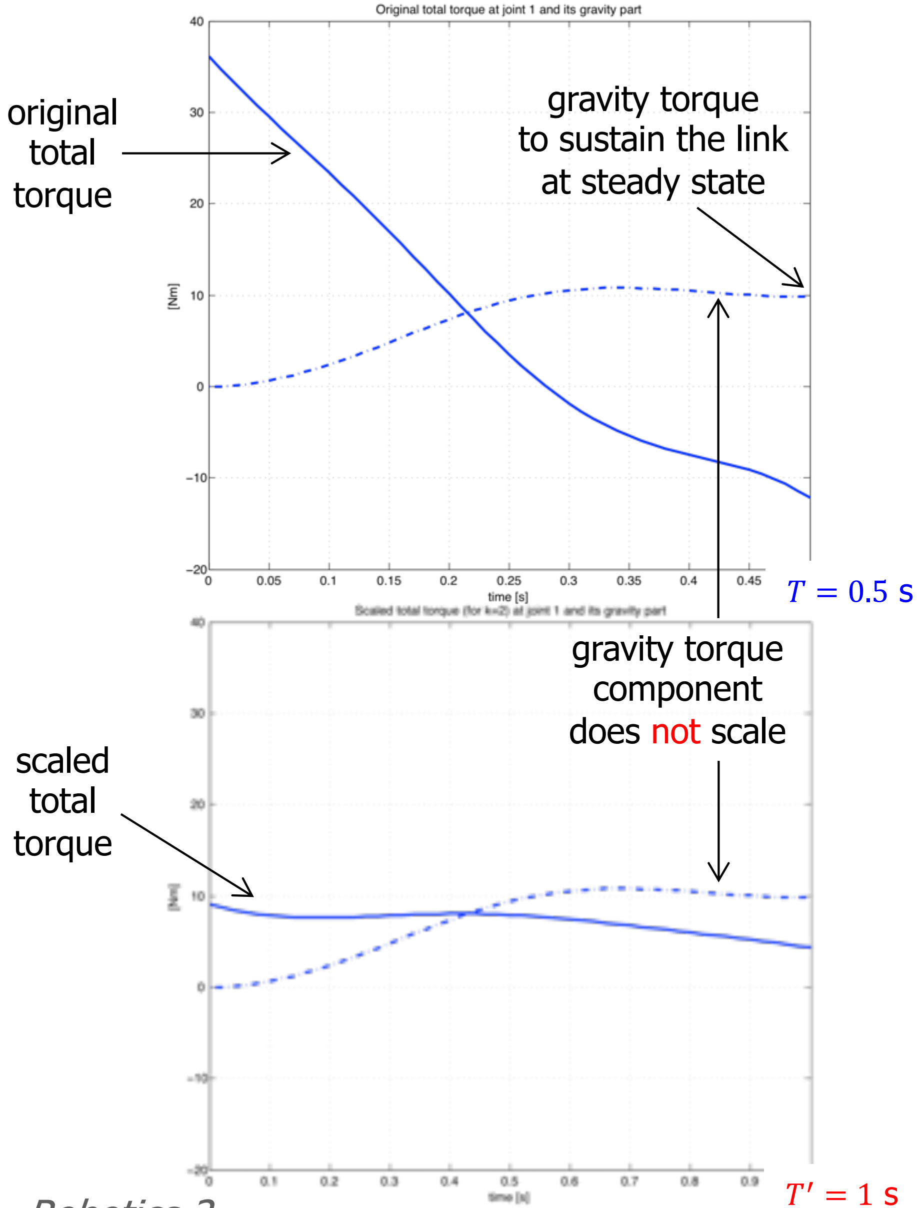


for both joints

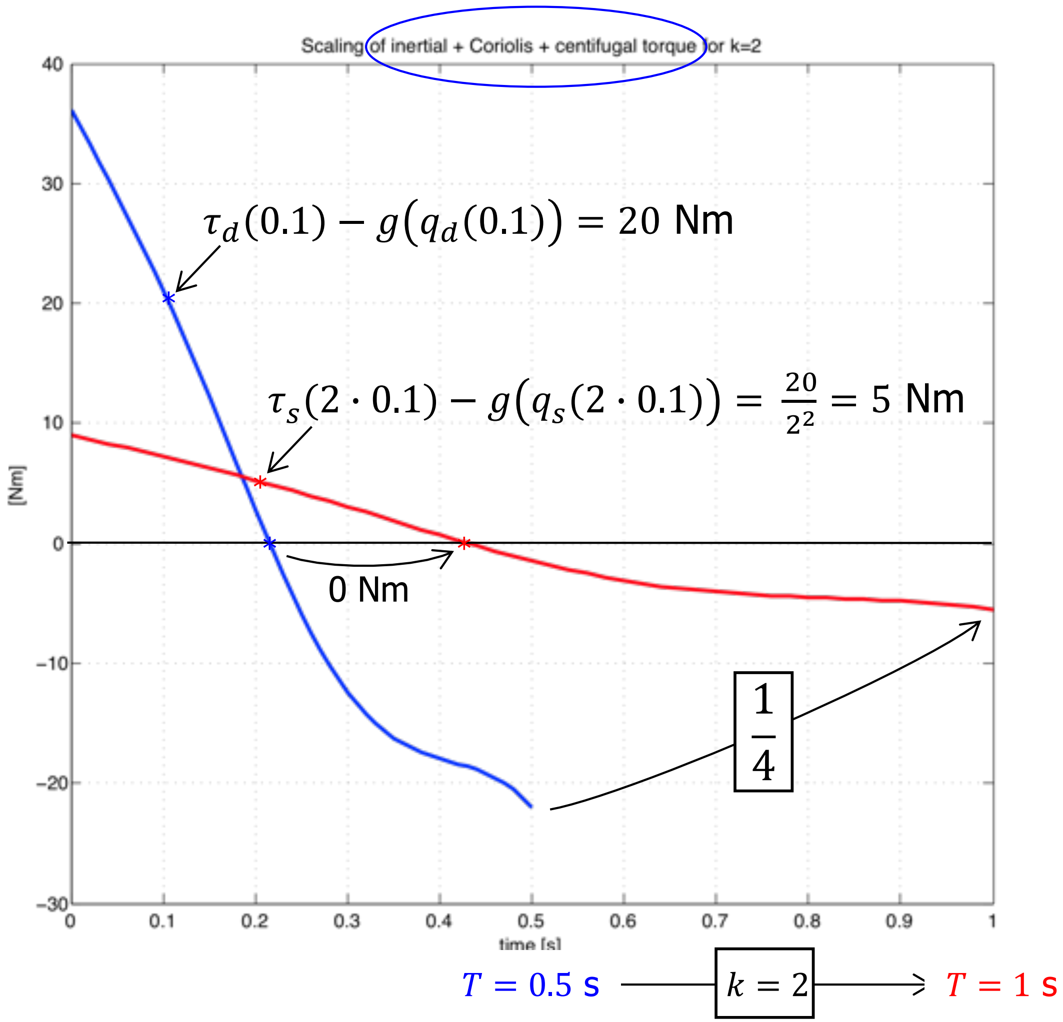
only joint 1 torque is shown



Dynamic scaling of trajectories
numerical example



- scaling with $k = 2$ (slower) $\rightarrow T' = 1$ s

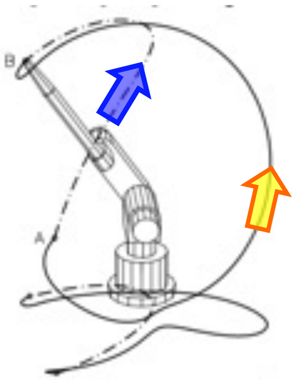


Optimal point-to-point robot motion

considering the dynamic model

- given the initial ($\Rightarrow$ A) and final ($\Rightarrow$ B) robot configurations (at rest) and the actuator torque bounds, find
  - the minimum-time T_{min} motion
  - the (global/integral) minimum-energy E_{min} motion

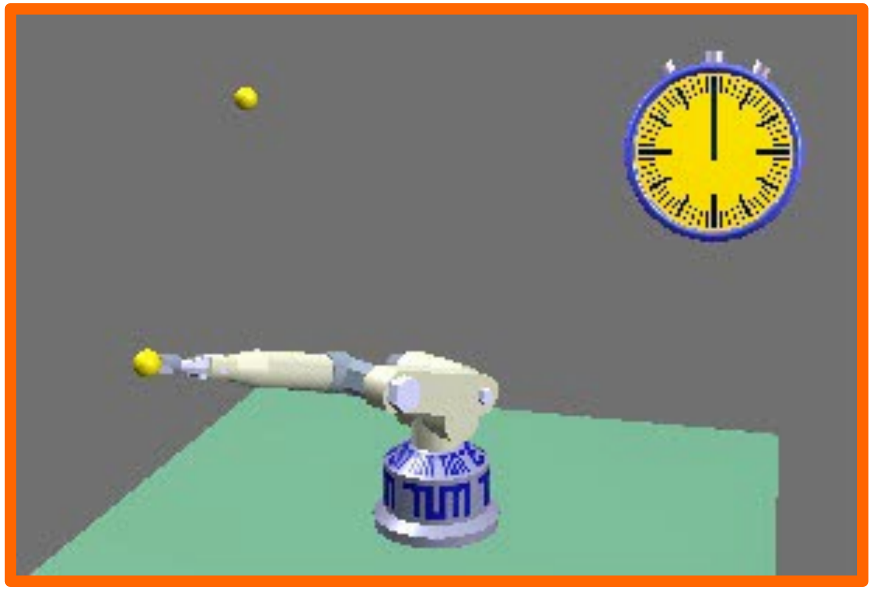
  and the associated command torques needed to execute them



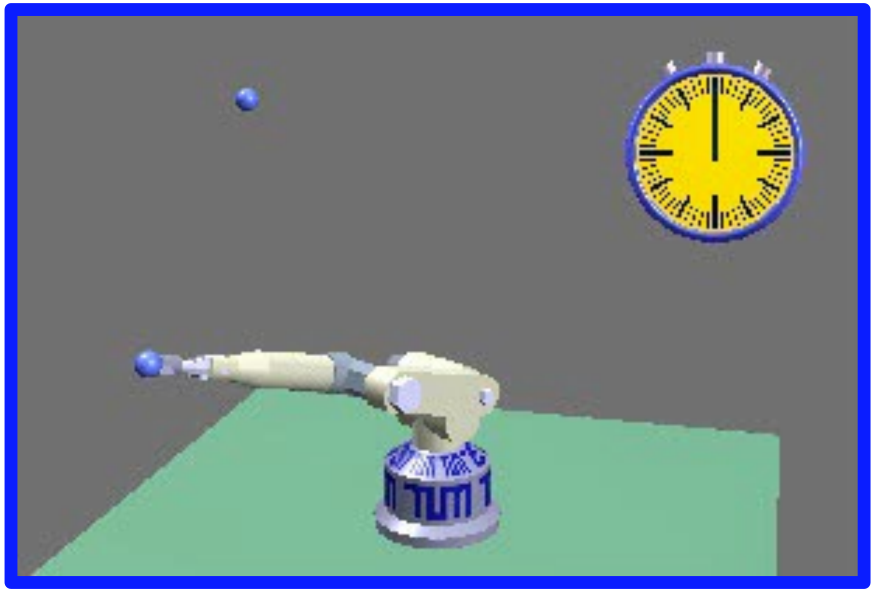
- a complex nonlinear optimization problem solved numerically

video

video



T_{min}= 1.32 s, E = 306



T = 1.60 s, E_{min} = 6.14