Robotics II

March 27, 2018

Exercise 1

An automated crane can be seen as a mechanical system with two degrees of freedom that moves along a horizontal rail subject to the actuation force F, and that transports a swinging link connected with a passive and frictionless revolute joint, as sketched in Fig. 1. With reference to the kinematic variables and dynamic parameters defined therein:

- derive the dynamic model of this system using a Lagrangian formalism;
- provide a linear parameterization of the obtained model in terms of a minimal number of dynamic coefficients;
- provide a linear approximation of the nonlinear model for small variations around the unforced equilibrium state $\boldsymbol{x}_0 = \begin{pmatrix} q_1 & q_2 & \dot{q}_1 & \dot{q}_2 \end{pmatrix}^T = \mathbf{0}$;
- find the nonlinear state feedback law for the force $F = F(\boldsymbol{x}, a)$ that linearizes exactly the dynamics of the first coordinate as $\ddot{q}_1 = a$.

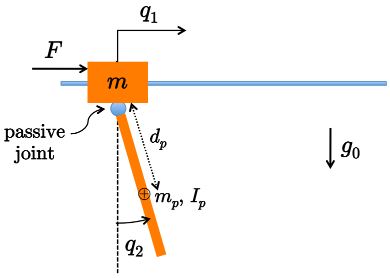
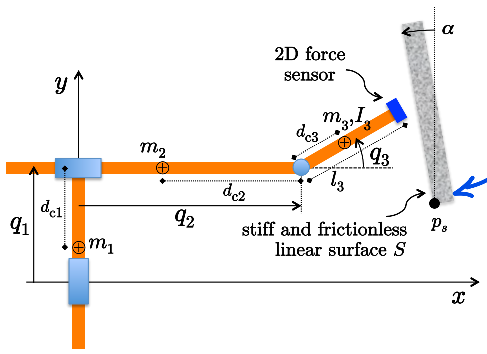


Figure 1: An automated crane with the relevant kinematic and dynamic definitions.

Exercise 2

The end-effector of a PPR robot moving on a horizontal plane and equipped with a 2D force sensor should follow a stiff and frictionless linear surface tilted by $\alpha > 0$ w.r.t. the absolute $\boldsymbol{y}$ axis, starting at time $t = t_0$ in the position $\boldsymbol{p}_s = \begin{pmatrix} x_s & y_s \end{pmatrix}^T$, with a tangential speed $V_t = V_t(t_0) + A_t(t - t_0)$ (with $V_t(t_0) > 0$ and a constant $A_t > 0$), and applying a constant normal force $F_n > 0$ (see Fig. 2). Assuming full knowledge of geometric, kinematic, and dynamic parameters, provide the symbolic expressions of the initial robot state and explicitly of all terms in the force/torque commands at the joints that will guarantee perfect execution of the desired task in nominal conditions. Is the solution unique? If not, provide the simplest one.



FROM HERE WE go UP (y ↑↑)
BUT (x GOES ↓↓)

Figure 2: A PPR robot should move in contact with a stiff and frictionless linear surface.

[150 minutes; open books]

Solution

March 27, 2018

Exercise 1

The crane is an underactuated mechanical system with $n = 2$ degrees of freedom, but with only a single control command ($p = 1$). To derive its dynamic model, we can follow a Lagrangian approach. For this, the position and velocity of the center of mass of the swinging link are[1]

$$\boldsymbol{p}_c = \begin{pmatrix} q_1 + d_p \sin q_2 \\ -\, d_p \cos q_2 \end{pmatrix}, \qquad \boldsymbol{v}_c = \dot{\boldsymbol{p}}_c = \begin{pmatrix} \dot q_1 + d_p \cos q_2 \, \dot q_2 \\ d_p \sin q_2 \, \dot q_2 \end{pmatrix}.$$

The kinetic energy of the system is

$$T = T_m + T_p = \frac{1}{2} m \, \dot q_1^2 + \frac{1}{2} m_p \|\boldsymbol{v}_c\|^2 + \frac{1}{2} I_p \dot q_2^2 = \frac{1}{2}\left((m + m_p)\dot q_1^2 + (I_p + m_p d_p^2)\dot q_2^2 + 2 m_p d_p \cos q_2 \dot q_1 \dot q_2\right),$$

and thus

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad \Rightarrow \qquad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m + m_p & m_p d_p \cos q_2 \\ m_p d_p \cos q_2 & I_p + m_p d_p^2 \end{pmatrix}.$$

Using the Christoffel's symbols, we found only a centrifugal term

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -m_p d_p \sin q_2 \, \dot q_2^2 \\ 0 \end{pmatrix}.$$

The potential energy and the associated gravity vector are

$$U = U_0 - m_p g_0 d_p \cos q_2 \qquad \Rightarrow \qquad \boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ m_p g_0 d_p \sin q_2 \end{pmatrix},$$

with $g_0 = 9.81$ [m/s^2]. Assuming the possible presence of a viscous friction term (with a viscous coefficient $f_v \geq 0$) on the movement along the rail, the dynamic equations take the scalar form

$$\begin{aligned} (m + m_p)\ddot q_1 + m_p d_p \cos q_2 \, \ddot q_2 - m_p d_p \sin q_2 \, \dot q_2^2 + f_v \dot q_1 &= F \\ m_p d_p \cos q_2 \, \ddot q_1 + (I_p + m_p d_p^2)\ddot q_2 + m_p g_0 d_p \sin q_2 &= 0 \end{aligned} \tag{1}$$

Equations (1) can be rewritten in the linearly parametrized form

$$\begin{pmatrix} \ddot q_1 & 0 & \cos q_2 \, \ddot q_2 - \sin q_2 \, \dot q_2^2 & \dot q_1 \\ 0 & \ddot q_2 & \cos q_2 \, \ddot q_1 + g_0 \sin q_2 & 0 \end{pmatrix} \begin{pmatrix} m + m_p \\ I_p + m_p d_p^2 \\ m_p d_p \\ f_v \end{pmatrix} = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\boldsymbol{\pi} = \begin{pmatrix} F \\ 0 \end{pmatrix}.$$

The linear approximation of the dynamic equations of the crane around the (stable) equilibrium state $\boldsymbol{x}_0 = \begin{pmatrix} \boldsymbol{q}_0^T & \dot{\boldsymbol{q}}_0^T \end{pmatrix}^T = \begin{pmatrix} q_1 & q_2 & \dot q_1 & \dot q_2 \end{pmatrix}^T = \boldsymbol{0}$, which satisfies (1) with $F = F_0 = 0$ (unforced), is obtained by setting in (1)

$$\boldsymbol{q} = \boldsymbol{q}_0 + \boldsymbol{\Delta q} = \boldsymbol{\Delta q}, \quad \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_0 + \boldsymbol{\Delta} \dot{\boldsymbol{q}} = \boldsymbol{\Delta} \dot{\boldsymbol{q}}, \quad \ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}_0 + \boldsymbol{\Delta} \ddot{\boldsymbol{q}} = \boldsymbol{\Delta} \ddot{\boldsymbol{q}}, \quad F = F_0 + \Delta F = \Delta F,$$

[1] We have taken the x-axis along the rail, and the y-axis in the vertical upward direction.

and neglecting second- and higher-order increments (e.g., setting $\sin \Delta q_2 \simeq \Delta q_2$ and $\cos \Delta q_2 \simeq 1$):

$$\begin{pmatrix} m + m_p & m_p d_p \\ m_p d_p & I_p + m_p d_p^2 \end{pmatrix} \begin{pmatrix} \Delta \ddot{q}_1 \\ \Delta \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} f_v & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta \dot{q}_1 \\ \Delta \dot{q}_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & m_p g_0 d_p \end{pmatrix} \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix} = \begin{pmatrix} \Delta F \\ 0 \end{pmatrix}.$$

Finally, partial feedback linearization of the crane dynamics as concerns the motion of q_1 is obtained as follows. Solve (globally!) for the revolute joint acceleration $\ddot{q}_2$ from the second equation in (1),

$$\ddot{q}_2 = -\frac{1}{I_p + m_p d_p^2} \left(m_p d_p \cos q_2 \, \ddot{q}_1 + m_p d_p g_0 \sin q_2 \right)$$

and substitute it in the first one, yielding

$$\left((m + m_p) - \frac{m_p^2 d_p^2 \cos^2 q_2}{I_p + m_p d_p^2} \right) \ddot{q}_1 - \frac{m_p^2 d_p^2 \, g_0 \sin q_2 \cos q_2}{I_p + m_p d_p^2} - m_p d_p \sin q_2 \, \dot{q}_2^2 + f_v \dot{q}_1 = F.$$

From this, it is immediate to see that the nonlinear state feedback law

$$F = \frac{(m + m_p) I_p + m \, m_p d_p^2 + m_p^2 d_p^2 \sin^2 q_2}{I_p + m_p d_p^2} \, a - \frac{m_p^2 d_p^2 \, g_0 \sin q_2 \cos q_2}{I_p + m_p d_p^2} - m_p d_p \sin q_2 \, \dot{q}_2^2 + f_v \dot{q}_1$$

yields (again, globally) $\ddot{q}_1 = a$. Accordingly, the second equation in (1) becomes

$$(I_p + m_p d_p^2) \ddot{q}_2 + m_p g_0 d_p \sin q_2 = -m_p d_p \cos q_2 \, a$$

Exercise 2

Noting that q_1 affects the y-coordinate and q_2 the x-coordinate, the direct/differential kinematics of the end-effector position, velocity, and acceleration are given respectively by

$$\boldsymbol{p} = \begin{pmatrix} q_2 + l_3 \cos q_3 \\ q_1 + l_3 \sin q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}),$$

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} \, \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \begin{pmatrix} 0 & 1 & -l_3 \sin q_3 \\ 1 & 0 & l_3 \cos q_3 \end{pmatrix} \dot{\boldsymbol{q}} \tag{2}$$

and

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q}) \ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q}) \ddot{\boldsymbol{q}} + \begin{pmatrix} 0 & 0 & -l_3 \cos q_3 \, \dot{q}_3 \\ 0 & 0 & -l_3 \sin q_3 \, \dot{q}_3 \end{pmatrix} \dot{\boldsymbol{q}}, \tag{3}$$

where the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ has been introduced.

The robot has $n = 3$ joints and the (hybrid) planar task has dimension $m = 2$ (one in force, the other in motion/velocity). In the presence of $n - m = 1$ degree of redundancy, the task can be executed in an infinite number of ways, beginning right from the different initial choices of an inverse kinematic configuration $\boldsymbol{q}(t_0)$ at time t_0, among those associated to the initial Cartesian point $\boldsymbol{p}(t_0) = \boldsymbol{p}_s$, and of the initial joint velocity $\dot{\boldsymbol{q}}(t_0)$, among those associated to the initial end-effector velocity

$$\dot{\boldsymbol{p}}(t_0) = V_t(t_0) \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}.$$

GOING NEGATIVE WITH X

It is then possible to parametrize the joint-space motion in terms of one variable. In this case, the easy choice is to pick the third joint variable q_3 as the parametrizing one. We set an arbitrary (but sufficiently smooth) time profile for it

$$q_3(t) = \beta(t), \qquad \dot{q}_3(t) = \dot{\beta}(t), \qquad \forall t \geq t_0,$$

and thus

$$q_3(t_0) = \beta(t_0), \qquad \dot{q}_3(t_0) = \dot{\beta}(t_0).$$

As a result, the two prismatic joints will be initialized at

$$\begin{pmatrix} q_1(t_0) \\ q_2(t_0) \end{pmatrix} = \begin{pmatrix} y_s - l_3 \sin\beta(t_0) \\ x_s - l_3 \cos\beta(t_0) \end{pmatrix},$$

with initial velocity

$$\begin{pmatrix} \dot{q}_1(t_0) \\ \dot{q}_2(t_0) \end{pmatrix} = \begin{pmatrix} V_t(t_0)\cos\alpha - l_3\,\dot{\beta}(t_0)\cos\beta(t_0) \\ -V_t(t_0)\sin\alpha - l_3\,\dot{\beta}(t_0)\sin\beta(t_0) \end{pmatrix}.$$

Moreover, we can also invert the second-order differential kinematics (3) in a parametrized way as

$$\ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \begin{pmatrix} \ddot{p}_y + l_3\left(-\cos\beta\,\ddot{\beta} + \sin\beta\,\dot{\beta}^2\right) \\ \ddot{p}_x + l_3\left(\sin\beta\,\ddot{\beta} + \cos\beta\,\dot{\beta}^2\right) \\ \ddot{\beta} \end{pmatrix}. \tag{4}$$

With this in mind, the dynamic model of the planar PPR robot (in the absence of gravity and without dissipative effects), when in contact with a stiff environment environment takes the form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau} + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F}. \tag{5}$$

where $\boldsymbol{F} \in \mathbb{R}^2$ is the contact force applied by the environment on the robot end-effector (equal and opposite to the one applied by the robot on the environment) and $\boldsymbol{J}$ has been defined in (2). We provide next the explicit symbolic expressions of the dynamic terms appearing in (5). Note first that the position and velocity of the center of mass of the third link are

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_2 + d_{c3}\cos q_3 \\ q_1 + d_{c3}\sin q_3 \end{pmatrix}, \qquad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} \dot{q}_2 - d_{c3}\sin q_3\,\dot{q}_3 \\ \dot{q}_1 + d_{c3}\cos q_3\,\dot{q}_3 \end{pmatrix}.$$

Following a Lagrangian approach, we compute the total kinetic energy $T = T_1 + T_2 + T_3$ as:

$$\left.\begin{array}{c} T_1 = \frac{1}{2}m_1\dot{q}_1^2, \qquad T_2 = \frac{1}{2}m_2\left(\dot{q}_1^2 + \dot{q}_2^2\right), \\ T_3 = \frac{1}{2}m_3\left(\dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2\,\dot{q}_3^2 + 2d_{c3}\dot{q}_3\left(\cos q_3\dot{q}_1 - \sin q_3\dot{q}_2\right)\right) + \frac{1}{2}I_3\,\dot{q}_3^2 \end{array}\right. \quad \Rightarrow \quad T = \tfrac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1 + m_2 + m_3 & 0 & m_3 d_{c3}\cos q_3 \\ 0 & m_2 + m_3 & -m_3 d_{c3}\sin q_3 \\ m_3 d_{c3}\cos q_3 & -m_3 d_{c3}\sin q_3 & I_3 + m_3 d_{c3}^2 \end{pmatrix}.$$

The Coriolis and centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are derived using the Christoffel's symbols, i.e., for each component

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T\boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \quad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial \boldsymbol{q}_i}\right), \quad i = 1, 2, 3,$$

being $\boldsymbol{m}_i$ the ith column of the inertia matrix $\boldsymbol{M}$. We obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m_3 d_{c3} \sin q_3 \end{pmatrix} \qquad \Rightarrow \qquad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 d_{c3} \sin q_3 \, \dot{q}_3^2,$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m_3 d_{c3} \cos q_3 \end{pmatrix} \qquad \Rightarrow \qquad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 d_{c3} \cos q_3 \, \dot{q}_3^2,$$

$$\boldsymbol{C}_3(\boldsymbol{q}) = \boldsymbol{0} \qquad \Rightarrow \qquad c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0.$$

We note that there are only centrifugal terms and no Coriolis torques.
Applying now to (5) the feedback linearizing control law

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} \tag{6}$$

will transform the system into a set of decoupled input-output integrators

$$\ddot{\boldsymbol{q}} = \boldsymbol{a}.$$

For the specified hybrid task, the desired end-effector acceleration and contact force are respectively

$$\ddot{\boldsymbol{p}}_d = A_t \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}, \qquad \boldsymbol{F}_d = F_n \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}.$$

From the desired end-effector acceleration, using (4), we obtain also the desired joint acceleration in parametrized form

$$\ddot{\boldsymbol{q}}_d = \begin{pmatrix} A_t \cos\alpha + l_3 \left(-\cos\beta \, \ddot{\beta} + \sin\beta \, \dot{\beta}^2 \right) \\ -A_t \sin\alpha + l_3 \left(\sin\beta \, \ddot{\beta} + \cos\beta \, \dot{\beta}^2 \right) \\ \ddot{\beta} \end{pmatrix}$$

Substituting $\boldsymbol{a} = \ddot{\boldsymbol{q}}_d$ and $\boldsymbol{F} = -\boldsymbol{F}_d$ in the feedback linearizing law (6), yields the desired nominal control commands

$$\boldsymbol{\tau}_d = \boldsymbol{M}(\boldsymbol{q}) \begin{pmatrix} A_t \cos\alpha + l_3 \left(-\cos\beta \, \ddot{\beta} + \sin\beta \, \dot{\beta}^2 \right) \\ -A_t \sin\alpha + l_3 \left(\sin\beta \, \ddot{\beta} + \cos\beta \, \dot{\beta}^2 \right) \\ \ddot{\beta} \end{pmatrix} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{J}^T(\boldsymbol{q}) \begin{pmatrix} F_n \cos\alpha \\ F_n \sin\alpha \end{pmatrix}, \tag{7}$$

where the dependence of the inertia matrix $\boldsymbol{M}$ and of the Jacobian $\boldsymbol{J}$ is actually only on $q_3 = \beta$ and that of the centrifugal terms $\boldsymbol{c}$ is only on $q_3 = \beta$ and $\dot{q}_3 = \dot{\beta}$. This can be made more explicit by rewriting (7) in extended form as

$$\boldsymbol{\tau}_d = \begin{pmatrix} (m_1+m_2+m_3) \left(A_t \cos\alpha + l_3 \left(-\cos\beta \, \ddot{\beta} + \sin\beta \, \dot{\beta}^2 \right) \right) + m_3 d_{c3} \left(\cos\beta \, \ddot{\beta} - \sin\beta \, \dot{\beta}^2 \right) + F_n \sin\alpha \\ (m_1+m_3) \left(-A_t \sin\alpha + l_3 \left(\sin\beta \, \ddot{\beta} + \cos\beta \, \dot{\beta}^2 \right) \right) - m_3 d_{c3} \left(\sin\beta \, \ddot{\beta} + \cos\beta \, \dot{\beta}^2 \right) + F_n \cos\alpha \\ \left(I_3 + m_3 d_{c3}^2 \right) \ddot{\beta} + m_3 d_{c3} \left(A_t \cos(\alpha - \beta) - l_3 \ddot{\beta} \right) + F_n \, l_3 \sin(\alpha - \beta) \end{pmatrix}.$$

Note that the first two components of $\boldsymbol{\tau}_d$ are forces (the units of all terms are [N] = [kg·m/s^2]), while the last component is a torque (units in [Nm]). Moreover, thanks to the fact that the initial robot state is matched with the task at the initial time $t = t_0$, there will be no need of a feedback action on task errors in the nominal control commands (7) in order to execute the entire task in ideal conditions.

The above parametrized control law is one of the many realizing the desired task, depending on the choice of the time evolution $\beta(t)$ for the variable q_3 of the revolute joint. Indeed, simplifications arise for specific choices. The simplest one is choosing to keep q_3 at a constant value β, with $\dot{\beta} = \ddot{\beta} = 0$. We obtain

$$\boldsymbol{\tau}_d = \begin{pmatrix} \tau_{d1} \\ \tau_{d2} \\ \tau_{d3} \end{pmatrix} = \begin{pmatrix} A_t(m_1 + m_2 + m_3)\cos\alpha + F_n \sin\alpha \\ -A_t(m_1 + m_3)\sin\alpha + F_n \cos\alpha \\ A_t\, m_3 d_{c3} \cos(\alpha - \beta) + F_n\, l_3 \sin(\alpha - \beta) \end{pmatrix}.$$

Having chosen to keep the third joint at rest for the entire motion, the robot behaves kinematically as a 2P robot. In particular, when placing the third robot link normal to the frictionless surface, we have $\beta = \alpha$ and the third control component reduces to $\tau_{d3} = A_t\, m_3 d_{c3}$.

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