Robotics 2

Remote Exam – June 5, 2020

Exercise #1

A 3R planar robot is subject to hard joint velocity limits $|\dot{q}_i| \leq V_i$, for $i = 1, 2, 3$, with $V_1 = 1$, $V_2 = 1.5$, and $V_2 = 2$ [rad/s]. At the current configuration, its task Jacobian is given by

$$\boldsymbol{J} = \begin{pmatrix} -1 & -1 & -0.5 \\ -0.366 & -0.866 & -0.866 \end{pmatrix}$$

and the task requires a Cartesian velocity $\boldsymbol{v} = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$ [m/s]. Apply the SNS algorithm to find a feasible solution $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ with the least possible norm, including task scaling if needed.

Exercise #2

Determine all the conditions on the constant parameters a, b, c and d, under which the following (linear) equations can be considered the dynamic model of an actual 2-dof robot:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{u}, \qquad \text{with} \quad \boldsymbol{q} \in \mathbb{R}^2,\ \boldsymbol{u} \in \mathbb{R}^2,$$

where

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad \boldsymbol{g} = \begin{pmatrix} 0 \\ d \end{pmatrix}.$$

Any guess about which type of robot this could be? Suppose now that only one generalized coordinate is actuated by a command $u_a \in \mathbb{R}$, while the other is passive. Which component of $\boldsymbol{u}$ should be actuated in order to guarantee the existence of an equilibrium? If we choose a value $u_a > 0$, what would be the instantaneous acceleration of the other (passive) coordinate?

Exercise #3

Consider the RP robot in Fig. 1, moving in a vertical plane. The prismatic joint has a limited range $d \leq q_2 \leq L$. Derive the gravity term $\boldsymbol{g}(\boldsymbol{q})$ in the dynamic model and find the expression (in symbolic form) of a constant $\alpha > 0$ that bounds $\|\partial \boldsymbol{g}(\boldsymbol{q})/\partial \boldsymbol{q}\|$ for all $\boldsymbol{q}$ within the robot workspace.

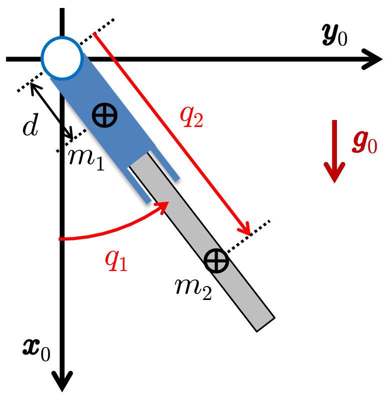

Figure 1: A RP robot, with associated coordinates $\boldsymbol{q}$ and relevant dynamic data.

Exercise #4

The Cartesian robot in Fig. 2 has the two links respectively of mass m_1 and m_2, and carries a payload m_p. It moves under gravity and has relevant viscous friction at both joints. In the absence of a priori information on the dynamic parameters, design an adaptive control law yielding global asymptotic stabilization of the tracking error for a desired trajectory $\boldsymbol{q}_d(t)$.

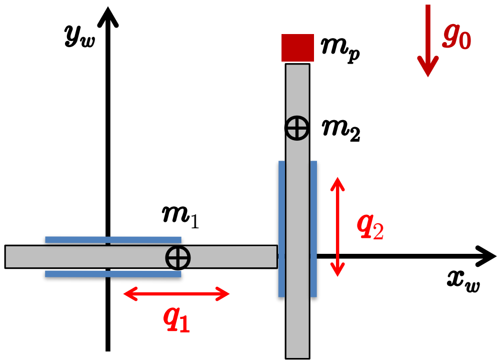

Figure 2: A Cartesian robot moving under gravity.

Exercise #5

A robot should close a door by firmly grasping its handle, pushing the door to the final position, and then turning the handle to lock the door. This interaction task is sketched in Fig. 3. Write down the natural and artificial constraints, defining a suitable task frame. Propose a time behavior for the hybrid references so as to cover the pushing phase and the final locking of the door. As usual in planning of hybrid force-velocity tasks, neglect any friction or environment compliance.

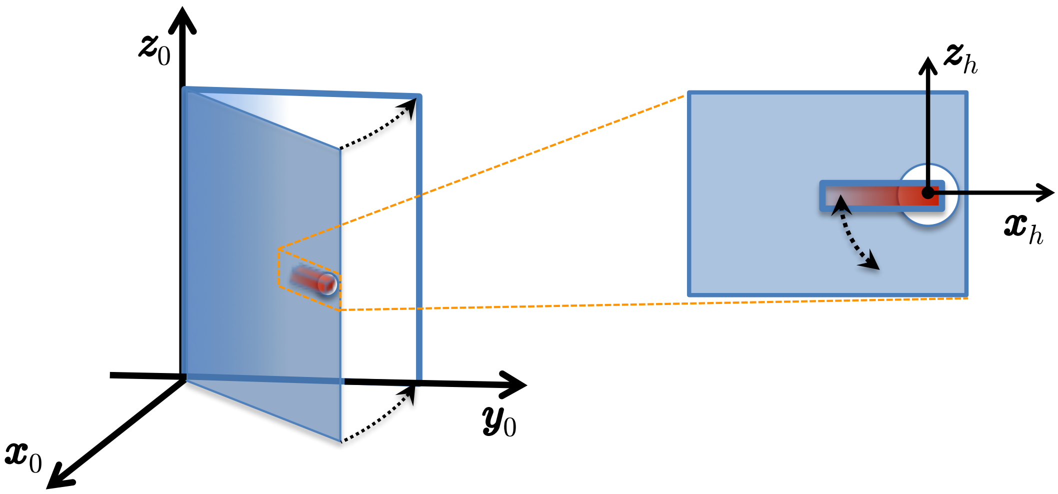

Figure 3: The task of closing a door [left], with handle motion for the final door locking [right].

Exercise #6

With reference to Fig. 4, a planar 2R robot with links of unitary length (in [m]) is in the configuration $\boldsymbol{q} = \begin{pmatrix} \pi/2 & -\pi/2 \end{pmatrix}^T$ [rad] and with a velocity $\dot{\boldsymbol{q}} = \begin{pmatrix} 0 & \pi/4 \end{pmatrix}^T$ [rad/s] when a collision occurs. Detail and comment the detection and isolation properties of the energy-based and momentum-based methods for the three cases of collision with the shown impact forces: *i)* $\boldsymbol{F}_{c1} = \begin{pmatrix} -1 & 0 \end{pmatrix}^T$ [N] at the end-effector; *ii)* $\boldsymbol{F}_{c2} = \begin{pmatrix} 0 & -1 \end{pmatrix}^T$ [N] at the elbow; *iii)* $\boldsymbol{F}_{c3} = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ [N] at the midpoint of the second link.

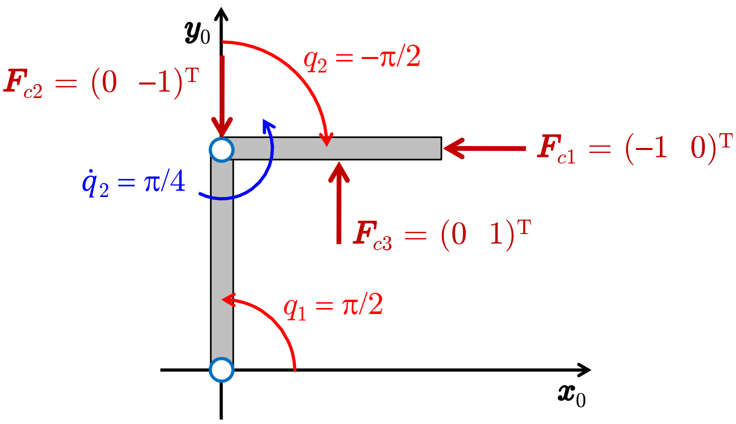

Figure 4: Three cases of collision for a 2R planar robot.

[180 minutes (3 hours); open books]

Solution

June 5, 2020

Exercise #1

The SNS algorithm at the velocity level starts with checking whether the minimum norm solution is feasible with respect to the joint velocity bounds. Using the pseudoinverse of the Jacobian $\boldsymbol{J}$ and the Cartesian velocity $\boldsymbol{v}$, we compute

$$\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^{\#}\boldsymbol{v} = \begin{pmatrix} -1.1333 & 0.9309 \\ -0.2124 & -0.3136 \\ 0.6914 & -1.2345 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1.3358 \\ -0.7383 \\ 0.1482 \end{pmatrix} \Leftarrow \begin{array}{l} < -V_1 = -1\,!! \\ \in [-V_2, V_2] = [-1.5, 1.5] \\ \in [-V_3, V_3] = [-2, 2] \end{array}$$

Therefore, we saturate the first joint velocity at its overdriven limit, $\dot{q}_1 = -V_1 = -1$ [rad/s], and recompute the task velocity $\boldsymbol{v}_1$ that needs to be executed with the remaining two joints,

$$\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{J}_1 \dot{q}_{0,1} = \boldsymbol{v} - \boldsymbol{J}_1(-V_1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -0.366 \end{pmatrix}(-1) = \begin{pmatrix} 1 \\ 0.634 \end{pmatrix},$$

where by $\boldsymbol{J}_i$, $i = 1, 2, 3$, we denote the columns of the Jacobian $\boldsymbol{J}$. We rewrite then the reduced problem as

$$\boldsymbol{J}_{(-1)} := \begin{pmatrix} \boldsymbol{J}_2 & \boldsymbol{J}_3 \end{pmatrix} = \begin{pmatrix} -1 & -0.5 \\ -0.8660 & -0.8660 \end{pmatrix}, \quad \dot{\boldsymbol{q}}_{(-1)} := \begin{pmatrix} \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{J}_{(-1)}\dot{\boldsymbol{q}}_{(-1)} = \boldsymbol{v}_1,$$

At this stage, the reduced solution is uniquely determined as

$$\dot{\boldsymbol{q}}_1 = \begin{pmatrix} \dot{q}_{1,1} \\ \dot{q}_{1,2} \end{pmatrix} = \boldsymbol{J}_{(-1)}^{-1}\boldsymbol{v}_1 = \begin{pmatrix} -2 & 1.1547 \\ 2 & -2.3094 \end{pmatrix} \begin{pmatrix} 1 \\ 0.634 \end{pmatrix} = \begin{pmatrix} -1.2679 \\ 0.5359 \end{pmatrix} \Leftarrow \text{ both feasible!}$$

Recombining the joint velocity vector, we have the (minimum norm) feasible solution

$$\dot{\boldsymbol{q}}^* = \begin{pmatrix} \dot{q}_{0,1} \\ \dot{q}_{1,1} \\ \dot{q}_{1,2} \end{pmatrix} = \begin{pmatrix} -1 \\ -1.2679 \\ 0.5359 \end{pmatrix} \text{ [rad/s]},$$

and we can check that is satisfies indeed $\boldsymbol{J}\dot{\boldsymbol{q}}^* = \boldsymbol{v}$. Thus, task scaling is not needed here.

Exercise #2

In order to be the dynamic model of an actual 2-dof robot, the only condition is on the parameters a and c, which need to guarantee the positive definiteness of the inertia matrix $\boldsymbol{M}$. Thus

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} > 0 \qquad \Longleftrightarrow \qquad a > 0, \quad \det \boldsymbol{M} = ac - b^2 > 0 \quad \Rightarrow \quad c > 0.$$

Being the inertia constant, there are no Coriolis and centrifugal effects. Also, friction is neglected. The 2-dof robot could be a planar arm with two prismatic joints (2P), whose axes are twisted by an angle $\alpha_1 \neq 0$. Moreover, in order to include a constant gravity term only on the second joint, the plane of motion is vertical and the first prismatic joint horizontal. We would have then for the kinetic and potential energy

$$T = T_1 + T_2 = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1^2 + \dot{q}_2^2 + 2\cos\alpha_1\,\dot{q}_1\dot{q}_2), \qquad U = U_2 = m_2 g_0\, q_2 \sin\alpha_1,$$

where $m_1 > 0$ and $m_2 > 0$ are the masses of the two links, so that the inertia matrix and the gravity vector for this 2P robot are

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} m_1 + m_2 & m_2 \cos\alpha_1 \\ m_2 \cos\alpha_1 & m_2 \end{pmatrix}, \qquad \boldsymbol{g} = \begin{pmatrix} 0 \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ m_2 g_0 \sin\alpha_1 \end{pmatrix},$$

with $\det \boldsymbol{M} = m_1 m_2 + m_2^2(1 - \cos\alpha_1) > 0$. Note that the coefficient $b = m_2 \cos\alpha_1$ could be positive or negative (depending on the twist angle $|\alpha_1| < \pi/2$ or, respectively, $> \pi/2$). Same for $d = m_2 g_0 \sin\alpha_1$ (depending on the sign of α_1).

Looking now at the equilibrium condition (i.e., $\ddot{\boldsymbol{q}} = \boldsymbol{0}$) in the underactuated case, it is evident that the second joint has to be the actuated one,

$$\boldsymbol{g} = \boldsymbol{u} \qquad \Rightarrow \qquad \begin{pmatrix} 0 \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ u_a \end{pmatrix}.$$

The equilibrium force at the second joint is $u_a = d$ $(= m_2 g_0 \sin\alpha_1)$, the same for all equilibrium configurations $\boldsymbol{q}_e \in \mathbb{R}^2$. When choosing a value $u_a > 0$, the instantaneous acceleration $\ddot{q}_1$ of the first (passive) joint is obtained from

$$\ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \boldsymbol{M}^{-1}(\boldsymbol{u} - \boldsymbol{g}) = \frac{1}{\det \boldsymbol{M}} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} 0 \\ u_a - d \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} -b \\ a \end{pmatrix} (u_a - d),$$

and thus $\ddot{q}_1 = b(d - u_a)/(ac - b^2)$. Being the determinant of $\boldsymbol{M}$ positive, the sign of $\ddot{q}_1$ will depend on the sign of the product $b(d - u_a)$. For instance, when $b > 0$ and for a large $u_a > |d|$, the acceleration of the first (passive) joint will be negative. On the other hand, the acceleration of the second (actuated) joint will always be positive, as soon as the control force overcomes the gravity term ($u_a > |d|$).

Exercise #3

From Fig. 1, we have for the potential energy due to gravity

$$U(\boldsymbol{q}) = U_1(q_1) + U_2(\boldsymbol{q}) = -m_1 g_0 d \cos q_1 - m_2 g_0 q_2 \cos q_1.$$

Therefore, using the compact notation for trigonometric quantities, we have

$$\boldsymbol{g}(\boldsymbol{q}) = \frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} (m_1 d + m_2 q_2) g_0 s_1 \\ -m_2 g_0 c_1 \end{pmatrix},$$

and then the symmetric matrix (representing the Hessian of $U(\boldsymbol{q})$)

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} (m_1 d + m_2 q_2) g_0 c_1 & m_2 g_0 s_1 \\ m_2 g_0 s_1 & 0 \end{pmatrix}.$$

This matrix is not definite in sign. Therefore, in order to evaluate its norm, we have to use the general form

$$\|\boldsymbol{A}(\boldsymbol{q})\| = \sqrt{\lambda_{\max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right)},$$

and compute the real eigenvalues of the positive semi-definite, symmetric matrix[1]

$$\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q}) = \begin{pmatrix} \left((m_1 d + m_2 q_2) g_0 c_1\right)^2 + (m_2 g_0 s_1)^2 & m_2 (m_1 d + m_2 q_2) g_0^2 s_1 c_1 \\ m_2 (m_1 d + m_2 q_2) g_0^2 s_1 c_1 & (m_2 g_0 s_1)^2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}.$$

[1] We have $a_1 > 0$, $a_3 \geq 0$, and thus $a_1 + a_3 > 0$. Also, $a_1 a_3 - a_2^2 \geq 0$.

From

$$\det\left(\lambda \boldsymbol{I} - \boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right) = \det\begin{pmatrix} \lambda - a_1 & -a_2 \\ -a_2 & \lambda - a_3 \end{pmatrix} = \lambda^2 - (a_1+a_3)\lambda + (a_1a_3 - a_2^2),$$

we obtain the maximum (real and positive) eigenvalue as

$$\lambda_{\max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right) = \frac{a_1+a_3}{2} + \frac{\sqrt{(a_1+a_3)^2 - 4(a_1a_3 - a_2^2)}}{2} = \frac{a_1+a_3+\sqrt{(a_1-a_3)^2+4a_2^2}}{2} > 0.$$

Substituting the expressions of the a_i's, we get

$$\begin{aligned}
&\lambda_{\max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right)\\
&= \frac{1}{2}\left(\left((m_1d+m_2q_2)g_0c_1\right)^2 + 2\left(m_2g_0s_1\right)^2 + \sqrt{\left((m_1d+m_2q_2)g_0c_1\right)^4 + 4\left(m_2(m_1d+m_2q_2)g_0^2s_1c_1\right)^2}\right)\\
&= \frac{1}{2}\left(\left((m_1d+m_2q_2)g_0c_1\right)^2 + 2\left(m_2g_0s_1\right)^2 + (m_1d+m_2q_2)\,g_0^2c_1\sqrt{(m_1d+m_2q_2)^2c_1^2 + 4\left(m_2s_1\right)^2}\right).
\end{aligned}$$

This expression can be upper bounded in different ways, using also the upper limit for the prismatic joint $q_2 \le L$. Replacing for instance $c_1 \to 1$, $s_1 \to 1$, and $q_2 = L$, we finally obtain the upper bound

$$\|\boldsymbol{A}(\boldsymbol{q})\| = \sqrt{\lambda_{\max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right)} \le \alpha$$

with[2]

$$\alpha = \frac{g_0}{\sqrt{2}}\sqrt{\left(m_1d+m_2L\right)^2 + 2\,m_2^2 + (m_1d+m_2L)\sqrt{(m_1d+m_2L)^2 + 4\,m_2^2}} > 0.$$

This constant is used, e.g., in the proof of the global asymptotic stability of a PD control law with gravity compensation $\boldsymbol{g}(\boldsymbol{q}_d)$, in which the minimum (positive) value $\boldsymbol{K}_{P,m}$ of the diagonal matrix of proportional gains $\boldsymbol{K}_P$ should satisfy $\boldsymbol{K}_{P,m} > \alpha$.

Exercise #4

The dynamic model of the Cartesian robot in Fig. 2 is very simple (in fact, this robot has a linear and decoupled dynamics). We have

$$T = T_1 + T_2 + T_p = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}(m_2+m_p)\left(\dot{q}_1^2 + \dot{q}_2^2\right), \qquad U = U_2 + U_p = (m_2+m_p)g_0q_2,$$

and thus, from the Euler-Lagrange equations, considering also the presence of viscous friction

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} + \boldsymbol{F}_v\dot{\boldsymbol{q}} = \boldsymbol{u}, \tag{1}$$

with

$$\boldsymbol{M} = \begin{pmatrix} m_1+m_2+m_p & 0 \\ 0 & m_2+m_p \end{pmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} 0 \\ (m_2+m_p)g_0 \end{pmatrix}, \quad \boldsymbol{F}_v = \begin{pmatrix} f_{v1} & 0 \\ 0 & f_{v2} \end{pmatrix}.$$

[2] One may notice that there is a unit inconsistency among the terms in the expression of α. In fact, we are taking the norm of a matrix $\boldsymbol{A}(\boldsymbol{q})$ that has elements expressed in different units. This happens because the robot has joints of different nature (revolute and prismatic): the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ has the first component expressed in [Nm] (a torque) and the second in [N] (a force).

The dynamic model (1) can be linearly re-parametrized as

$$\boldsymbol{Y}(\dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{u} \tag{2}$$

with

$$\boldsymbol{Y}(\dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \begin{pmatrix} \ddot{q}_1 & 0 & \dot{q}_1 & 0 \\ 0 & \ddot{q}_2 + g_0 & 0 & \dot{q}_2 \end{pmatrix}, \qquad \boldsymbol{a} = \begin{pmatrix} m_1 + m_2 + m_p \\ m_2 + m_p \\ f_{v1} \\ f_{v2} \end{pmatrix}.$$

The adaptive control law for tracking a desired trajectory $\boldsymbol{q}_d(t)$ (at least twice differentiable w.r.t. time) is

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{Y}(\dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\,\hat{\boldsymbol{a}} + \boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_D \dot{\boldsymbol{e}}, \qquad \boldsymbol{K}_P, \boldsymbol{K}_D > 0, \\ \dot{\hat{\boldsymbol{a}}} &= \boldsymbol{\Gamma} \boldsymbol{Y}^T(\dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\,(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}})\,, \quad \boldsymbol{\Gamma} > 0, \end{aligned} \tag{3}$$

with $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$, $\dot{\boldsymbol{e}} = \dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}$, $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}\boldsymbol{e}$, and $\ddot{\boldsymbol{q}}_r = \ddot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}\dot{\boldsymbol{e}}$ ($\boldsymbol{\Lambda} = \boldsymbol{K}_P \boldsymbol{K}_D^{-1} > 0$), and where

$$\boldsymbol{Y}(\dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r) = \begin{pmatrix} \ddot{q}_{r1} & 0 & \dot{q}_{r1} & 0 \\ 0 & \ddot{q}_{r2} + g_0 & 0 & \dot{q}_{r2} \end{pmatrix}.$$

The gain matrices $\boldsymbol{K}_P$, $\boldsymbol{K}_D$, and $\boldsymbol{\Gamma}$ are taken diagonal. We additionally remark the following.

- The problem is fully decoupled into parallel subproblems for each of the two prismatic joints. Consider for instance the first joint. From eqs. (1), (2), and (3), we have for the closed-loop system

$$\begin{aligned} (m_1 + m_2 + m_p)\ddot{q}_1 + f_{v1}\dot{q}_1 &= u_1, \\ u_1 &= \hat{a}_1 \ddot{q}_{r1} + \hat{a}_3 \dot{q}_{r1} + k_{p1} e_1 + k_{d1} \dot{e}_1, \\ \dot{\hat{a}}_1 &= \gamma_1\, \ddot{q}_{r1}(\dot{q}_{r1} - \dot{q}_1), \\ \dot{\hat{a}}_3 &= \gamma_3\, \dot{q}_{r1}(\dot{q}_{r1} - \dot{q}_1), \end{aligned}$$

  with $e_1 = q_{d1} - q_1$, $\dot{e}_1 = \dot{q}_{d1} - \dot{q}_1$, $\dot{q}_{r1} = \dot{q}_{d1} + \frac{k_{P1}}{k_{D1}}\, e_1$, and $\ddot{q}_{r1} = \ddot{q}_{d1} + \frac{k_{P1}}{k_{D1}}\, \dot{e}_1$.

- Despite the linear dynamics of the considered robot, the closed-loop system is indeed still nonlinear because of the interplay between the robot state $\boldsymbol{x} = (\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathbb{R}^4$ and the state $\hat{\boldsymbol{a}} \in \mathbb{R}^4$ of the adaptive controller.

Exercise #5

With reference to Fig. 5, where a possible task frame is defined, the $\boldsymbol{x}_t$ and $\boldsymbol{z}_t$ axes of the task frame have been chosen to coincide with the homologous ones at the door handle. The natural constraints are the following:

$$v_x = 0, \qquad v_z = 0, \qquad \omega_x = 0, \qquad \omega_z = 0, \qquad f_y = 0, \qquad \mu_y = 0.$$

In these constraints, we neglect any friction effect and mass/inertia or compliance of the environment (which is thus assumed to be purely geometric). The complementary artificial constraints are chosen then as:

$$f_x = 0, \qquad f_z = 0, \qquad \mu_x = 0, \qquad \mu_z = 0, \qquad v_y = v_{d,y}(t), \qquad \omega_y = \omega_{d,y}(t).$$

The first four (zero) values for forces and moments that are assigned as references to the hybrid task controller reflect the desire to limit the mechanical stress on the door handle grasped by the robot end-effector. The last two time-varying references are used instead to specify the way feasible motions are handled during the task of closing a door. While moving the door, we set $\omega_{d,y} = 0$ (no motion around the rotation axis $\boldsymbol{y}_h = \boldsymbol{y}_t$ of the door handle). On the other hand, the time law $v_{d,y}(t)$ will define the way the door closing should be performed, e.g., with a trapezoidal profile for a rest-to-rest motion from the initial position (door open) to the desired approach position (door nearly closed), using a fast or slow cruise speed. When the approach position is reached, the linear motion is ended ($v_{d,y} = 0$) and $\omega_{d,y}(t)$ is used to turn the handle, preparing it for the final phase of door locking. Note finally that when the door touches the door frame on the wall, the contact situation changes and, accordingly, also the definition of natural and artificial constraints.

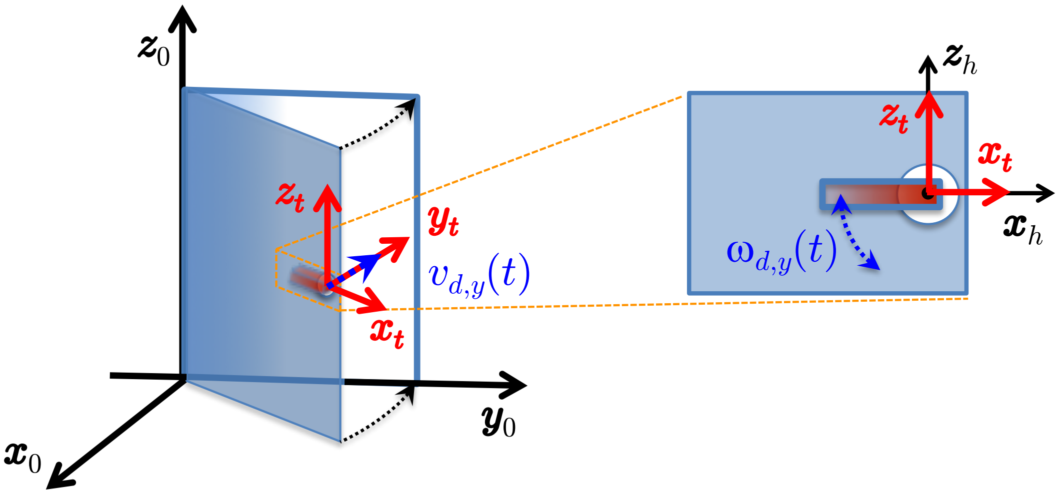

Figure 5: Two views of the task frame assignment for closing a door, with associated time-varying specification of two artificial constraints.

Exercise #6

The energy-based method fails to detect a collision when $\dot{\boldsymbol{q}} = \boldsymbol{0}$ or, more in general, when the colliding force $\boldsymbol{F}_c$ is orthogonal to the velocity $\boldsymbol{v}_c$ of the contact point. In fact, in this case we have $\boldsymbol{v}_c^T \boldsymbol{F}_c = \left(\boldsymbol{J}_c(\boldsymbol{q})\dot{\boldsymbol{q}}\right)^T \boldsymbol{F}_c = \dot{\boldsymbol{q}}^T \left(\boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c\right) = \dot{\boldsymbol{q}}^T \boldsymbol{\tau}_c = 0$, thus not exciting the scalar residual σ. In the given configuration $\boldsymbol{q} = \begin{pmatrix} \pi/2 & -\pi/2 \end{pmatrix}^T$ of the planar 2R robot, we verify this condition for the three considered cases:

$$\boldsymbol{v}_{c1} = \boldsymbol{J}_{c1}\dot{\boldsymbol{q}} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \pi/4 \end{pmatrix} = \begin{pmatrix} 0 \\ \pi/4 \end{pmatrix} \text{ [m/s]} \quad \Rightarrow \quad \boldsymbol{v}_{c1}^T \boldsymbol{F}_{c1} = \begin{pmatrix} 0 & \pi/4 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0;$$

$$\boldsymbol{v}_{c2} = \boldsymbol{J}_{c2}\dot{\boldsymbol{q}} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \pi/4 \end{pmatrix} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{v}_{c2}^T \boldsymbol{F}_{c2} = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0;$$

$$\boldsymbol{v}_{c3} = \boldsymbol{J}_{c3}\dot{\boldsymbol{q}} = \begin{pmatrix} -1 & 0 \\ 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0 \\ \pi/4 \end{pmatrix} = \begin{pmatrix} 0 \\ \pi/8 \end{pmatrix} \text{ [m/s]} \Rightarrow \boldsymbol{v}_{c3}^T \boldsymbol{F}_{c3} = \begin{pmatrix} 0 & \pi/8 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \pi/8 \text{ [Nm]}.$$

Therefore, only in the third case we are able to detect the occurrence of a collision with the energy-based method.

As for the momentum-based method, a collision is detected (and possibly isolated) provided $\boldsymbol{F}_c$ is not in the null space of the transpose of the contact Jacobian $\boldsymbol{J}_c$. In fact, if $\boldsymbol{F}_c \in \mathcal{N}\left\{\boldsymbol{J}_c^T(\boldsymbol{q})\right\}$ the contact force is balanced by the reaction of the rigid robot structure, yielding $\boldsymbol{\tau}_c = \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c = \mathbf{0}$ and thus not exciting the residual vector $\boldsymbol{r}$. We verify next if the detection condition holds for the three considered cases, drawing also conclusions on the isolation property. For the tip contact on link 2:

$$\mathcal{N}\left\{\boldsymbol{J}_{c1}^T\right\} = \mathcal{N}\left\{\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\right\} = \emptyset \;\Rightarrow\; \boldsymbol{F}_{c1} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \notin \mathcal{N}\left\{\boldsymbol{J}_{c1}^T\right\},\; \boldsymbol{\tau}_{c1} = \boldsymbol{J}_{c1}^T \boldsymbol{F}_{c1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ [Nm]}.$$

Therefore, this collision will be detected with the momentum-based method (contrary to what happens with the energy-based method). However, the second component r_2 of the residual vector will be unaffected (being $\tau_{c1,2} = 0$), leading to the wrong conclusion that the contact occurred on link 1. Indeed, this is a singular situation for the isolation property (the contact force vector $\boldsymbol{F}_{c1}$ passes through the axis of joint 2). For the contact at the robot elbow:

$$\mathcal{N}\left\{\boldsymbol{J}_{c2}^T\right\} = \mathcal{N}\left\{\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\} = \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad\Rightarrow\quad \boldsymbol{F}_{c2} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \mathcal{N}\left\{\boldsymbol{J}_{c2}^T\right\},\; \boldsymbol{\tau}_{c2} = \boldsymbol{J}_{c2}^T \boldsymbol{F}_{c2} = \mathbf{0}.$$

Therefore, also the momentum-based method is not able to detect this collision. In fact, the contact force vector $\boldsymbol{F}_{c2}$ passes through both joint axes and is balanced entirely by the internal reaction force of the robot structure. Finally, for the contact at the midpoint of link 2:

$$\mathcal{N}\left\{\boldsymbol{J}_{c3}^T\right\} = \mathcal{N}\left\{\begin{pmatrix} -1 & 0.5 \\ 0 & 0.5 \end{pmatrix}\right\} = \emptyset \;\Rightarrow\; \boldsymbol{F}_{c3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin \mathcal{N}\left\{\boldsymbol{J}_{c3}^T\right\},\; \boldsymbol{\tau}_{c3} = \boldsymbol{J}_{c3}^T \boldsymbol{F}_{c3} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \text{ [Nm]}.$$

Therefore, the momentum-based method detects the collision and also correctly isolates the contact as occurring on the second link.

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