Robotics II

June 17, 2019

Exercise 1

Consider the Kawasaki S030 robot with six revolute joints and a spherical wrist shown in Fig. 1. For every link mass m_i, $i = 1, \ldots, 6$, the location of the center of mass is shown graphically in the different views (see also the distributed extra sheet in larger size; please, disregard any numerical value therein). Note that the position of the center of mass of the fifth link (m_5) as well as that of the sixth link (m_6) coincide with the wrist center —a simplifying assumption. The robot has been calibrated and all kinematic quantities are thus known.

Determine the symbolic expression of the gravity vector $\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^6$ and a possible linear parametrization in terms of the unknown dynamic coefficients $\boldsymbol{a}_g \in \mathbb{R}^p$, with the smallest value of p. Find all equilibrium configurations $\boldsymbol{q}_e$ of the robot (i.e., such that $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$).

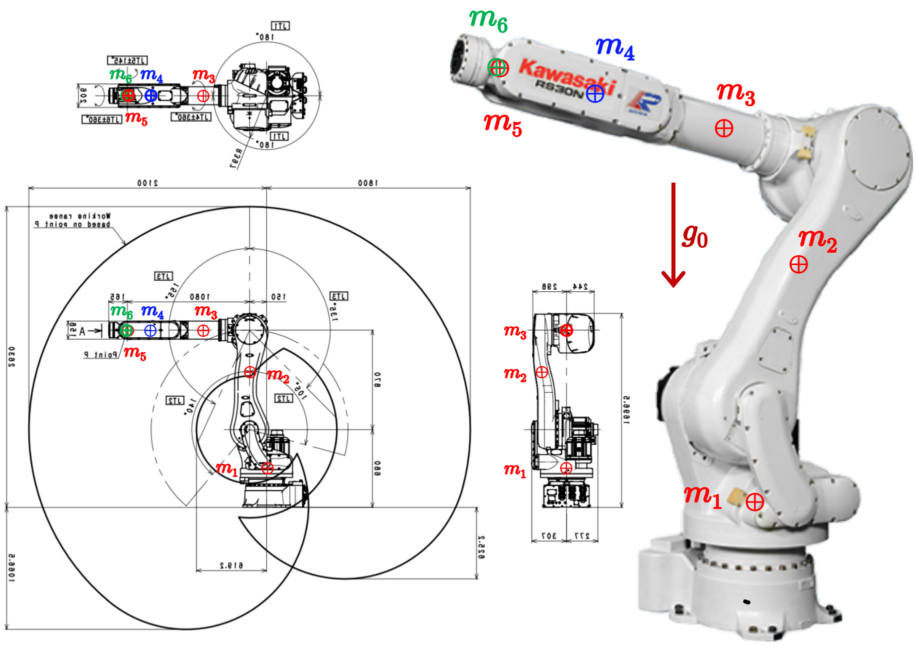

Figure 1: 6R Kawasaki S030 robot: Localization of the centers of mass of the six links.

Exercise 2

Consider the 3×3 matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + (a_2 \cos q_2 + a_3 \cos(q_2 + q_3))^2 & 0 & 0 \\ 0 & a_4 + a_5 + 2a_6 \cos q_3 & a_5 + a_6 \cos q_3 \\ 0 & a_5 + a_6 \cos q_3 & a_5 \end{pmatrix}.$$

Check whether this can be the inertia matrix of a 3-dof serial robot manipulator and, if so, under which conditions this holds true for the dynamic coefficients a_i ($i = 1, \ldots, 6$) appearing in $\boldsymbol{M}(\boldsymbol{q})$.

Exercise 3

A robot with $n > 3$ joints, parametrized by $\boldsymbol{q}$, is redundant w.r.t. a positional task $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{q})$ of dimension $m = 2$ or $m = 3$ (so, always with $m < n$), having task Jacobian $\boldsymbol{J}(\boldsymbol{q}) = \partial \boldsymbol{f}(\boldsymbol{q})/\partial \boldsymbol{q}$. At a configuration $\bar{\boldsymbol{q}}$ where rank $\boldsymbol{J}(\bar{\boldsymbol{q}}) = r < m$, consider the following two cases of joint velocity commands associated to a desired task velocity $\dot{\boldsymbol{x}}_d$:

$$\dot{\boldsymbol{q}}_A = \boldsymbol{J}^{\#}(\bar{\boldsymbol{q}})\, \dot{\boldsymbol{x}}_d, \qquad \dot{\boldsymbol{q}}_B = \boldsymbol{J}^T(\bar{\boldsymbol{q}})\, \dot{\boldsymbol{x}}_d.$$

Taking advantage of the Singular Value Decomposition of the matrix $\bar{\boldsymbol{J}} = \boldsymbol{J}(\bar{\boldsymbol{q}})$, show that:

i. In both cases, the actual $\dot{\boldsymbol{x}}$ can be different from the desired $\dot{\boldsymbol{x}}_d$, but the vectors $\dot{\boldsymbol{x}}_d$ and $\dot{\boldsymbol{x}}$ make always a relative angle that is smaller than $\pi/2$;

ii. When $\dot{\boldsymbol{x}}_d \in \mathcal{R}\left(\bar{\boldsymbol{J}}\right)$, $\dot{\boldsymbol{q}}_A$ gives no task velocity error, while $\dot{\boldsymbol{q}}_B$ leads in general to an error $\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}}_d - \dot{\boldsymbol{x}} \neq \boldsymbol{0}$.

Exercise 4

For a 2R robot moving on a horizontal plane, determine (in symbolic or numerical form) all terms in the expression of a control law producing a joint torque $\boldsymbol{\tau} \in \mathbb{R}^2$ that is able to regulate the robot end-effector position $\boldsymbol{p} \in \mathbb{R}^2$ to a desired constant value $\boldsymbol{p}_d$, with a transient error $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p}$ which globally satisfies, up to kinematic singularities, the differential equations

$$\ddot{\boldsymbol{e}} + \begin{pmatrix} 20 & 0 \\ 0 & 10 \end{pmatrix} \dot{\boldsymbol{e}} + \begin{pmatrix} 100 & 0 \\ 0 & 50 \end{pmatrix} \boldsymbol{e} = \boldsymbol{0}.$$

When starting at rest from an initial configuration $\boldsymbol{q}_0$ that is not associated to $\boldsymbol{p}_d$, will the Cartesian behavior of the robot end-effector be oscillatory during the transient?

Exercise 5

A robot should slide a cube, firmly held by its end-effector gripper, on a flat surface, following an arbitrary path (see Fig. 2). For modeling purposes, assume that the surface is infinitely stiff and frictionless.

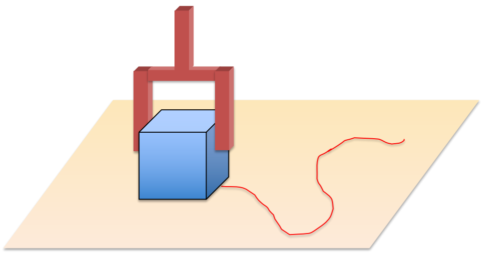

Figure 2: A cube sliding along a path on a flat surface.

Provide the set of natural constraints and a suitable set of artificial constraints for this interaction task with the environment. Consider both cases of a constant or arbitrary time-varying orientation of the cube while moving along the path. How many control loops of the generalized force or motion type are needed to achieve a perfectly linear and decoupled behavior in the task space? How many degrees of freedom are necessary for the robot in order to fulfil all control specifications? Can at least some of the desired control tasks be performed by a Scara robot? And by a 3R planar robot? If so, under which conditions?

[open books, 240 minutes]

Solution

June 17, 2019

Exercise 1

We need to define first a set of generalized coordinates $\boldsymbol{q}$, either by following a DH assignment of frames or by direct inspection of Fig. 1. The latter is more convenient here. In fact, it is rather easy to see that the definition of the link variables q_1 and (q_4, q_5, q_6) is irrelevant for the computation of the gravity term $\boldsymbol{g}(\boldsymbol{q})$ in the robot dynamics because:

- the rotation q_1 of joint 1 will not affect the height of the center of mass of any link;
- the position of the center of mass of link 4 will not depend on q_4, since m_4 lies exactly on the axis of joint 4;
- the position of the two center of masses m_5 and m_6 (which are assumed to be coincident[1] with the center of the spherical wrist) is independent from the joint variables of the wrist.

As a matter of fact, the potential energy of the system due to gravity, and thus the dynamic term $\boldsymbol{g}(\boldsymbol{q})$, will be a function of q_2 and q_3 only. Moreover, we will have $g_1 = g_4 = g_5 = g_6 \equiv 0$ for the components of $\boldsymbol{g}(q_2, q_3)$.

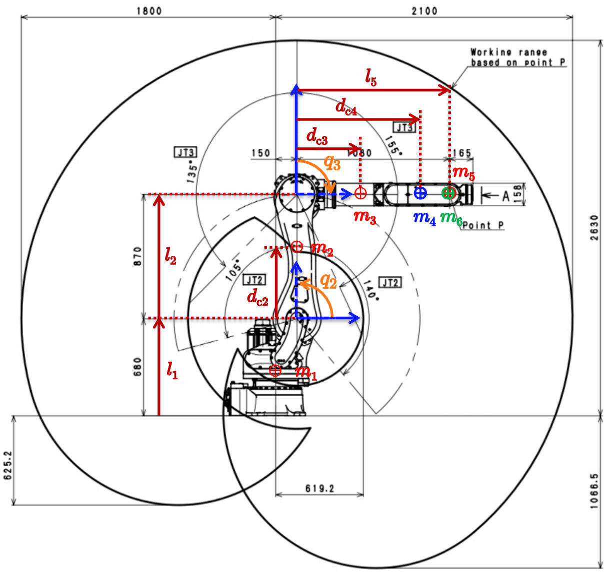

Figure 3: One possible choice of variables q_2 and q_3 for the second and third links of the Kawasaki robot, together with the definition kinematic/dynamic parameters.

With reference to Fig. 3, where the two joint variables q_2 and q_3 are defined (mimicking the classical assignment for the planar 2R case), together with the kinematic/dynamic parameters l_1, d_{c2}, l_2, d_{c3}, d_{c4} and l_5, we obtain

$$U_1 = \text{constant}, \quad U_2 = m_2 g_0 (l_1 + d_{c2} \sin q_2), \quad U_3 = m_3 g_0 (l_1 + l_2 \sin q_2 + d_{c3} \sin(q_2 + q_3)),$$

[1]This simplifying assumption is a very strong one, in particular concerning the location of m_6.

$U_4 = m_4 g_0 (l_1 + l_2 \sin q_2 + d_{c4} \sin(q_2+q_3))\,, \quad U_5+U_6 = (m_5{+}m_6) g_0 (l_1 + l_2 \sin q_2 + l_5 \sin(q_2+q_3))\,.$
As a result,

$$\begin{aligned} U = \textstyle\sum_{i=1}^{6} U_i &= g_0\,(m_2 d_{c2} + (m_3+m_4+m_5+m_6) l_2) \sin q_2 \\ &\quad + g_0\,(m_3 d_{c3} + m_4 d_{c4} + (m_5+m_6) l_5) \sin(q_2+q_3) + \text{constants} \\ &= a_1 \sin q_2 + a_2 \sin(q_2+q_3) + \text{constants}. \end{aligned}$$

Thus,

$$\begin{aligned} \boldsymbol{g}(q_2,q_3) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T &= \begin{pmatrix} 0 \\ a_1 \cos q_2 + a_2 \cos(q_2+q_3) \\ a_2 \cos(q_2+q_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \cos q_2 & \cos(q_2+q_3) \\ 0 & cos(q_2+q_3) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \boldsymbol{Y_g}(q_2,q_3)\,\boldsymbol{a}, \end{aligned} \tag{1}$$

with a linear parametrization expressed in terms of only $p = 2$ dynamic coefficients.

All equilibrium configurations $\boldsymbol{q}_e$ are found by setting $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$ in (1). We obtain

$$\boldsymbol{q}_e = \begin{pmatrix} \text{any} & \pm\pi/2 & 0 \text{ or } \pi & \text{any} & \text{any} & \text{any} \end{pmatrix}^T.$$

Exercise 2

The given matrix $\boldsymbol{M}(\boldsymbol{q})$ is symmetric and does not depend on q_1 —both are necessary conditions for being the inertia matrix of a serial manipulator. In order to be a positive definite matrix, it is necessary that all diagonal elements are strictly positive for all $\boldsymbol{q}$. This implies

$$a_1 > 0, \qquad a_4 + a_5 > 2|a_6| > 0, \qquad a_5 > 0. \tag{2}$$

The necessary and sufficient condition for positive definiteness of a symmetric matrix (Sylvester criterion) is that the leading minors are strictly positive (for all $\boldsymbol{q}$). Under (2), this boils down in checking that

$$\det \boldsymbol{M}_{[2:3]} = \det \begin{pmatrix} a_4 + a_5 + 2a_6 \cos q_3 & a_5 + a_6 \cos q_3 \\ a_5 + a_6 \cos q_3 & a_5 \end{pmatrix} > 0, \qquad \forall \boldsymbol{q}.$$

We have

$$\det \boldsymbol{M}_{[2:3]} = a_4 a_5 - a_6^2 \cos^2 q_3 > 0 \quad \Rightarrow \quad a_4 a_5 > a_6^2 \geq 0 \quad \Rightarrow \quad a_4 > 0, \tag{3}$$

the latter being implied by the previous condition $a_5 > 0$. Joining conditions (2) and (3) leads to the necessary and sufficient conditions[2]

$$a_1 > 0, \qquad a_4 > 0, \qquad a_5 > 0, \qquad a_4 + a_5 > 2|a_6|, \qquad a_4 a_5 > a_6^2. \tag{4}$$

[2] Matrix $\boldsymbol{M}(\boldsymbol{q})$ is in fact the inertia matrix of the robot considered in the midterm test of Robotics 2 during the academic year 2016/17, with some additional simplifying assumptions. As such, the conditions (4) are automatically satisfied in that case by the explicit expressions of the dynamic coefficients.

Exercise 3

We use the Singular Value Decomposition of the $m \times n$ matrix $\bar{\boldsymbol{J}} = \boldsymbol{J}(\bar{\boldsymbol{q}})$ (with $m < n$). From

$$\bar{\boldsymbol{J}} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{U}\begin{pmatrix} \operatorname{diag}\{\sigma_1, \ldots, \sigma_r\} & \boldsymbol{O}_{r\times(m-r)} & \boldsymbol{O}_{m\times(n-m)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \ldots, 0\} & \end{pmatrix}\boldsymbol{V}^T,$$

where $\boldsymbol{U} = \begin{pmatrix} \boldsymbol{u}_1 \ \boldsymbol{u}_2 \ \ldots \ \boldsymbol{u}_r \ \boldsymbol{u}_{r+1} \ \ldots \ \boldsymbol{u}_m \end{pmatrix}$ and $\boldsymbol{V}$ are two orthonormal matrices, respectively of dimension m and n, and the singular values of $\bar{\boldsymbol{J}}$ have been ordered as $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, $\sigma_{r+1} = \cdots = \sigma_m = 0$, we have

$$\bar{\boldsymbol{J}}^{\#} = \boldsymbol{V}\boldsymbol{\Sigma}^{\#}\boldsymbol{U}^T = \boldsymbol{V}\begin{pmatrix} \operatorname{diag}\left\{\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_r}\right\} & \boldsymbol{O}_{r\times(m-r)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \ldots, 0\} \\ \multicolumn{2}{c}{\boldsymbol{O}_{(n-m)\times n}} \end{pmatrix}\boldsymbol{U}^T$$

and

$$\bar{\boldsymbol{J}}^T = \boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{U}^T = \boldsymbol{V}\begin{pmatrix} \operatorname{diag}\{\sigma_1, \ldots, \sigma_r\} & \boldsymbol{O}_{r\times(m-r)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \ldots, 0\} \\ \multicolumn{2}{c}{\boldsymbol{O}_{(n-m)\times n}} \end{pmatrix}\boldsymbol{U}^T.$$

Thus, the result of the two command choices $\dot{\boldsymbol{q}}_A = \bar{\boldsymbol{J}}^{\#}\dot{\boldsymbol{x}}_d$ and $\dot{\boldsymbol{q}}_B = \bar{\boldsymbol{J}}^T\dot{\boldsymbol{x}}_d$ is

$$\begin{aligned}\dot{\boldsymbol{x}}_A = \bar{\boldsymbol{J}}\dot{\boldsymbol{q}}_A &= \bar{\boldsymbol{J}}\bar{\boldsymbol{J}}^{\#}\dot{\boldsymbol{x}}_d = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{\Sigma}^{\#}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\#}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d \\ &= \boldsymbol{U}\begin{pmatrix} \boldsymbol{I}_{r\times r} & \boldsymbol{O}_{r\times(m-r)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \ldots, 0\} \end{pmatrix}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d = \begin{pmatrix} \boldsymbol{u}_1 \ \boldsymbol{u}_2 \ \ldots \ \boldsymbol{u}_r \ \boldsymbol{0} \ \ldots \ \boldsymbol{0} \end{pmatrix}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d\end{aligned}$$

and

$$\begin{aligned}\dot{\boldsymbol{x}}_B = \bar{\boldsymbol{J}}\dot{\boldsymbol{q}}_B &= \bar{\boldsymbol{J}}\bar{\boldsymbol{J}}^T\dot{\boldsymbol{x}}_d = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{U}^T\dot{\boldsymbol{x}}_d = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\boldsymbol{U}^T\dot{\boldsymbol{x}}_d \\ &= \boldsymbol{U}\begin{pmatrix} \operatorname{diag}\{\sigma_1^2, \ldots, \sigma_r^2\} & \boldsymbol{O}_{r\times(m-r)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \ldots, 0\} \end{pmatrix}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d = \begin{pmatrix} \sigma_1^2\boldsymbol{u}_1 \ \sigma_2^2\boldsymbol{u}_2 \ \ldots \ \sigma_r^2\boldsymbol{u}_r \ \boldsymbol{0} \ \ldots \ \boldsymbol{0} \end{pmatrix}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d.\end{aligned}$$

Based on these expressions, we can immediately see that[3]

$$\dot{\boldsymbol{x}}_d^T\dot{\boldsymbol{x}}_A = \dot{\boldsymbol{x}}_d^T\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\#}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d = \dot{\boldsymbol{x}}_d^T\boldsymbol{U}\begin{pmatrix} \boldsymbol{I}_{r\times r} & \boldsymbol{O}_{r\times(m-r)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \ldots, 0\} \end{pmatrix}^2\boldsymbol{U}^T\dot{\boldsymbol{x}}_d = \boldsymbol{w}^T\boldsymbol{w} = \|\boldsymbol{w}\|^2 \geq 0,$$

having set

$$\boldsymbol{w} = \begin{pmatrix} \boldsymbol{I}_{r\times r} & \boldsymbol{O}_{r\times(m-r)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \ldots, 0\} \end{pmatrix}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d.$$

Similarly, one can show that $\dot{\boldsymbol{x}}_B^T\dot{\boldsymbol{x}}_d \geq 0$. From the definition of the scalar products it follows that

$$\dot{\boldsymbol{x}}_d^T\dot{\boldsymbol{x}}_A = \|\dot{\boldsymbol{x}}_d\| \cdot \|\dot{\boldsymbol{x}}_A\| \cos\alpha_A \geq 0, \qquad \dot{\boldsymbol{x}}_d^T\dot{\boldsymbol{x}}_B = \|\dot{\boldsymbol{x}}_d\| \cdot \|\dot{\boldsymbol{x}}_B\| \cos\alpha_B \geq 0.$$

Therefore, each of the obtained Cartesian velocities $\dot{\boldsymbol{x}}_A$ and $\dot{\boldsymbol{x}}_B$ will form an angle $\alpha_i \leq \pi/2$, $i = A, B$, with the desired $\dot{\boldsymbol{x}}_d$.

[3] Matrix $\boldsymbol{A} := \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\#}$ is diagonal and idempotent. Thus we can write $\boldsymbol{A} = \boldsymbol{A}^2 = \boldsymbol{A}^T\boldsymbol{A}$.

Furthermore, when the desired Cartesian velocity $\dot{\boldsymbol{x}}_d$ is in the image of the Jacobian $\bar{\boldsymbol{J}}$, it is always spanned by the first r colums of the orthonormal matrix $\boldsymbol{U}$, namely

$$\dot{\boldsymbol{x}}_d \in \mathcal{R}\left(\bar{\boldsymbol{J}}\right) \qquad \Longrightarrow \qquad \dot{\boldsymbol{x}}_d = \sum_{i=1}^{r} \lambda_i \boldsymbol{u}_i = \left(\boldsymbol{u}_1 \ \boldsymbol{u}_2 \ \dots \ \boldsymbol{u}_r \right) \boldsymbol{\lambda},$$

for a generic vector $\boldsymbol{\lambda} = \left(\lambda_1 \ \lambda_2 \ \dots \ \lambda_r \right)^T \in \mathbb{R}^m$. In this case, the choice $\dot{\boldsymbol{q}}_A$ yields

$$\begin{aligned}
\dot{\boldsymbol{x}}_A = \bar{\boldsymbol{J}} \dot{\boldsymbol{q}}_A &= \left(\boldsymbol{u}_1 \ \boldsymbol{u}_2 \ \dots \ \boldsymbol{u}_r \ \boldsymbol{0} \ \dots \ \boldsymbol{0} \right) \boldsymbol{U}^T \dot{\boldsymbol{x}}_d \\
&= \left(\boldsymbol{u}_1 \ \boldsymbol{u}_2 \ \dots \ \boldsymbol{u}_r \ \boldsymbol{0} \ \dots \ \boldsymbol{0} \right) \begin{pmatrix} \boldsymbol{u}_1^T \\ \dots \\ \boldsymbol{u}_r^T \\ \boldsymbol{u}_{r+1}^T \\ \dots \\ \boldsymbol{u}_m^T \end{pmatrix} \left(\boldsymbol{u}_1 \ \boldsymbol{u}_2 \ \dots \ \boldsymbol{u}_r \right) \boldsymbol{\lambda} \\
&= \left(\boldsymbol{u}_1 \ \boldsymbol{u}_2 \ \dots \ \boldsymbol{u}_r \ \boldsymbol{0} \ \dots \ \boldsymbol{0} \right) \begin{pmatrix} \boldsymbol{I}_{r \times r} \\ \boldsymbol{O}_{(m-r) \times r} \end{pmatrix} \boldsymbol{\lambda} = \left(\boldsymbol{u}_1 \ \boldsymbol{u}_2 \ \dots \ \boldsymbol{u}_r \right) \boldsymbol{\lambda} = \dot{\boldsymbol{x}}_d
\end{aligned}$$

with no Cartesian velocity error. On the other hand, with the choice $\dot{\boldsymbol{q}}_B$ we do not generate the desired $\dot{\boldsymbol{x}}_d$:

$$\begin{aligned}
\dot{\boldsymbol{x}}_B = \bar{\boldsymbol{J}} \dot{\boldsymbol{q}}_B &= \left(\sigma_1^2 \boldsymbol{u}_1 \ \sigma_2^2 \boldsymbol{u}_2 \ \dots \ \sigma_r^2 \boldsymbol{u}_r \ \boldsymbol{0} \ \dots \ \boldsymbol{0} \right) \boldsymbol{U}^T \dot{\boldsymbol{x}}_d \\
&= \dots = \left(\sigma_1^2 \boldsymbol{u}_1 \ \ \sigma_2^2 \boldsymbol{u}_2 \ \ \dots \ \ \sigma_r^2 \boldsymbol{u}_r \right) \boldsymbol{\lambda} \neq \dot{\boldsymbol{x}}_d.
\end{aligned}$$

Exercise 4

In order to achieve the desired linear and decoupled dynamics for the Cartesian error $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p}$ at the end-effector level, the torque command $\boldsymbol{\tau}$ in

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau}$$

for the controlled 2R planar robot cannot be chosen as the simple Cartesian PD regulation law

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q}) \left(\boldsymbol{K}_P \boldsymbol{e} - \boldsymbol{K}_D \dot{\boldsymbol{p}} \right), \qquad \boldsymbol{K}_P, \boldsymbol{K}_D > 0,$$

where $\dot{\boldsymbol{e}} = -\dot{\boldsymbol{p}}$ being $\dot{\boldsymbol{p}}_d = \boldsymbol{0}$. Rather, we should resort to a feedback linearization control law in the Cartesian space. In the considered case of a square and (assumed) nonsingular robot Jacobian, this law can be designed in two equivalent ways. Either by

$$\left. \begin{array}{l} \boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \quad \Rightarrow \quad \ddot{\boldsymbol{q}} = \boldsymbol{a} \\ \ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{K}_P \boldsymbol{e} - \boldsymbol{K}_D \dot{\boldsymbol{p}} \end{array} \right\} \quad \Rightarrow \quad \boldsymbol{a} = \boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\boldsymbol{K}_P (\boldsymbol{p}_d - \boldsymbol{p}) - \boldsymbol{K}_D \dot{\boldsymbol{p}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \right),$$

and thus

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\boldsymbol{K}_P (\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q})) - \boldsymbol{K}_D \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \right) + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{5}$$

with the matrix control gains chosen as

$$\boldsymbol{K}_P = \begin{pmatrix} 100 & 0 \\ 0 & 50 \end{pmatrix} > 0, \qquad \boldsymbol{K}_D = \begin{pmatrix} 20 & 0 \\ 0 & 10 \end{pmatrix} > 0. \tag{6}$$

Or, by using the Cartesian dynamics of the robot

$$\boldsymbol{M}_p(\boldsymbol{q})\ddot{\boldsymbol{p}} + \boldsymbol{c}_p(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{F},$$

with

$$\boldsymbol{M}_p(\boldsymbol{q}) = \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}, \quad \boldsymbol{c}_p(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{J}^{-T}(\boldsymbol{q})\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right),$$

and choosing

$$\left.\begin{array}{l} \boldsymbol{F} = \boldsymbol{M}_p(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{c}_p(\boldsymbol{q},\dot{\boldsymbol{q}}) \;\;\Rightarrow\;\; \ddot{\boldsymbol{p}} = \boldsymbol{a} \\ \ddot{\boldsymbol{p}} = \boldsymbol{K}_P\boldsymbol{e} - \boldsymbol{K}_D\dot{\boldsymbol{p}}, \quad \boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} \end{array}\right\} \Rightarrow \boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{M}_p(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{p}) - \boldsymbol{K}_D\dot{\boldsymbol{p}}\right) + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{c}_p(\boldsymbol{q},\dot{\boldsymbol{q}}).$$

By elaborating further the latter expression of $\boldsymbol{\tau}$,

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{p}) - \boldsymbol{K}_D\dot{\boldsymbol{p}}\right) + \left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) \\ &= \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{p}) - \boldsymbol{K}_D\dot{\boldsymbol{p}}\right) + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}, \end{aligned}$$

we recover (5) as expected.

The requested symbolic form of the terms in (5) are easily obtained for a 2R planar robot (see lecture slides). The kinematic terms are

$$\begin{aligned} \boldsymbol{p}(\boldsymbol{q}) &= \begin{pmatrix} l_1\cos q_1 + l_2\cos(q_1+q_2) \\ l_1\sin q_1 + l_2\sin(q_1+q_2) \end{pmatrix}, \\ \boldsymbol{J}(\boldsymbol{q}) &= \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -(l_1\sin q_1 + l_2\sin(q_1+q_2)) & -l_2\sin(q_1+q_2) \\ l_1\cos q_1 + l_2\cos(q_1+q_2) & l_2\cos(q_1+q_2) \end{pmatrix}, \\ \dot{\boldsymbol{J}}(\boldsymbol{q}) &= -\begin{pmatrix} l_1\cos q_1\dot{q}_1 + l_2\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) & l_2\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) \\ l_1\sin q_1\dot{q}_1 + l_2\sin(q_1+q_2)(\dot{q}_1+\dot{q}_2) & l_2\sin(q_1+q_2)(\dot{q}_1+\dot{q}_2) \end{pmatrix}. \end{aligned}$$

The dynamic terms are

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q}) &= \begin{pmatrix} a_1 + 2a_2\cos q_2 & a_3 + a_2\cos q_2 \\ a_3 + a_2\cos q_2 & a_3 \end{pmatrix}, \\ \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \begin{pmatrix} -a_2\sin q_2\,(2\dot{q}_1 + \dot{q}_2)\,\dot{q}_2 \\ a_2\sin q_2\,\dot{q}_1^2 \end{pmatrix}, \end{aligned}$$

with dynamic coefficients $a_1 = I_{c1,zz} + m_1 d_{c1}^2 + I_{c2,zz} + m_2 d_{c2}^2 + m_2 l_1^2 > 0$, $a_2 = m_2 l_1 d_{c2}$ and $a_3 = I_{c2,zz} + m_2 d_{c2}^2 > 0$. The numerical values used in (5) are those of the matrix gains given by (6).

In order to study the characteristics of the transient behavior of the error $\boldsymbol{e}(t) \to \boldsymbol{0}$, one may compute the roots of the following two algebraic equations in the Laplace domain:

$$\begin{aligned} \left(s^2 + 20s + 100\right) e_x(s) &= (s+10)^2\, e_x(s) = 0, \\ \left(s^2 + 10s + 50\right) e_y(s) &= (s + 5 - 5i)\,(s + 5 + 5i)\, e_y(s) = 0. \end{aligned}$$

As a result, when starting at rest with an initial $e_x(0) \neq 0$, $e_x(t)$ will converge to zero without overshoot (double negative real root), whereas $e_y(t)$ will converge to zero from an initial $e_y(0) \neq 0$

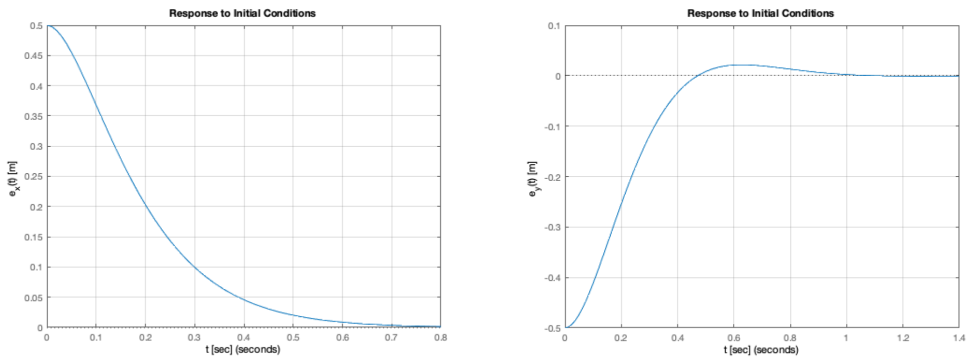

Figure 4: The evolution of $e_x(t)$ [left] and $e_y(t)$ [right] when the robot under control (5) starts at rest in a configuration $\boldsymbol{q}_0$ with initial Cartesian position error $\boldsymbol{e}(0) = (e_x(0), e_y(0)) = (0.5, -0.5) \neq \mathbf{0}$.

always changing its sign during the transient (a pair of complex conjugate roots, with negative real part). See the numerical example in Fig. 4.

Exercise 5

With reference to the task frame $(\boldsymbol{x}_t, \boldsymbol{y}_t, \boldsymbol{z}_t)$ shown in Fig. 5, the six natural constraints on the task are:

$$v_z = 0, \quad \omega_x = 0, \quad \omega_y = 0, \quad F_x = 0, \quad F_y = 0, \quad M_z = 0.$$

The six complementary artificial constraints specify the way in which the interaction task should be executed:

$$F_z = F_{z,d} < 0, \quad M_x = M_{x,d} = 0, \quad M_y = M_{y,d} = 0, \quad v_x = v_{x,d}, \quad v_y = v_{y,d}, \quad \omega_z = \omega_{z,d}\,.$$

Here, $|F_{z,d}| \neq 0$ is the intensity of the normal force to the plane that the robot should apply to the cube so as to keep one of its faces in permanent contact. The desired moments around the axes $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$ are both set to zero, so as to minimize the actual strain on the cube. The desired trajectory on the plane will be followed with a scalar speed $\dot{s} = \sqrt{v_{x,d}^2 + v_{y,d}^2} > 0$. Finally, the choice of either a constant or arbitrary time-varying orientation of the cube while moving along the path is made by setting either $\omega_{z,d} = 0$ or, respectively, $\omega_{z,d} = \omega_{z,d}(t)$.

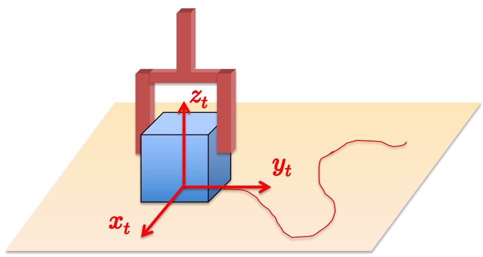

Figure 5: The instantaneous task frame associated to the cube moving on a flat surface.

As a result, in a hybrid force-velocity formulation, there will be three control loops on the generalized force components and three control loops on the planar motion components. Out of singularities, hybrid force-velocity control will achieve a perfectly linear and decoupled behavior of these six controlled outputs associated to the task space. In general, a robot with six degrees of freedom will be necessary in order to fulfil all control specifications.

With a Scara-type robot (four parallel joint axes, three revolute joints providing motion on a plane and a prismatic joint acting orthogonally), only four control specifications can be satisfied. If the joint axes of the robot are (perfectly) normal to the plane of motion of the cube, then the remaining two specifications $M_{x,d} = M_{y,d} = 0$ are automatically satisfied (although any desired value different from zero for these quantities could not be realized).

When using a 3R planar robot (with all revolute joint axes normal to the plane of motion of the cube), the task specification of a non-zero normal force along the axis $\boldsymbol{z}_t$ cannot be accomplished. The dofs of this robot are instead necessary and sufficient to execute a complete planar motion, with arbitrary values of $v_{x,d}$, $v_{y,d}$ and $\omega_{z,d}$.

* * * * *