Robotics II

June 6, 2017

Exercise 1

Consider a planar 3R robot with unitary link lengths as in Fig. 1, where the generalized coordinates $\boldsymbol{q}$ are defined as the *absolute* angles of the links w.r.t. the $\boldsymbol{x}$-axis. The position of the robot end-effector $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q})$, as obtained through the direct kinematics, should follow the desired trajectory

$$\dot{p}_d = \begin{pmatrix} 2\cos(3t)\cdot 3 \\ -\sin(3t+\frac{\pi}{2})\cdot 3 \end{pmatrix}$$

$$\boldsymbol{p}_d(t) = \begin{pmatrix} 1 + 2\sin 3t \\ 2 + \cos\left(3t + \frac{\pi}{2}\right) \end{pmatrix}, \qquad \text{for } t \geq 0. \tag{1}$$

$t=0$ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$t=0$ $\begin{pmatrix} 6 \\ -3 \end{pmatrix}$

The robot is kinematically redundant for this task.

- Define a differential inversion scheme at the level of joint jerk commands $\dddot{\boldsymbol{q}}$ such that the squared norm $\|\dddot{\boldsymbol{q}}\|^2$ is locally minimized and the trajectory can be executed exactly right from the initial time $t = 0$.
- Provide numerical values for the initial joint position $\boldsymbol{q}(0)$, joint velocity $\dot{\boldsymbol{q}}(0)$, and joint acceleration $\ddot{\boldsymbol{q}}(0)$ such that there is a perfect initial matching with the desired trajectory. Provide also the numerical value of the initial locally optimal command $\dddot{\boldsymbol{q}}(0)$.
- Suppose that there is no perfect matching between the initial kinematic conditions of the robot and the trajectory at time $t = 0$. How can we modify the command law for $\dddot{\boldsymbol{q}}$ such that the error $\boldsymbol{e}(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t)$ and all its time derivatives will exponentially converge to zero?

$$\ddot{p}_d = \begin{pmatrix} -6\sin(3t)\cdot 3 \\ -\cos(3t+\frac{\pi}{2})\cdot 9 \end{pmatrix}$$

$t=0$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\dddot{p} = \begin{pmatrix} -18\cdot 3\cos(3t) \\ 9\cdot 3\sin(3t+\frac{\pi}{2}) \end{pmatrix}$$

$t=0$ $\begin{pmatrix} -54 \\ 27 \end{pmatrix}$

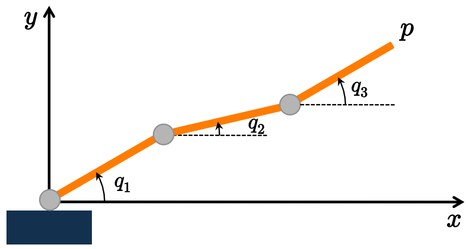

Figure 1: A planar 3R robot with absolute angles as generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$.

Exercise 2

For the same robot in Fig. 1, and using the same coordinates defined therein, assume that the three links have equal, uniformly distributed mass $m_i = m = 10$ kg, for $i = 1, 2, 3$. Each torque τ_i delivered by the motors and performing work on the absolute coordinate q_i is bounded as $|\tau_i| \leq T_{max} = 300$ Nm, for $i = 1, 2, 3$. Consider the Cartesian regulation control law

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\, \boldsymbol{K}_P \Big(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q})\Big) - \boldsymbol{K}_D \dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}), \qquad \text{with } \boldsymbol{p}_d = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{2}$$

$\dot{q} = 0$

where the gain matrices $\boldsymbol{K}_P$ and $\boldsymbol{K}_D$ are diagonal and positive definite. Let the robot starts at rest at $t = 0$ in the configuration $\boldsymbol{q}(0) = \begin{pmatrix} \pi/2 & 0 & 0 \end{pmatrix}^T$.

- If the gain matrices are of the form $\boldsymbol{K}_P = k_P \cdot \boldsymbol{I}_{2\times 2}$ and $\boldsymbol{K}_D = k_D \cdot \boldsymbol{I}_{2\times 2}$, provide the largest values for the scalars k_P and k_D such that $\boldsymbol{\tau}(0)$ in (2) does not violate its bounds.
- Let now the positional gain matrix be $\boldsymbol{K}_P = \text{diag}\{k_{Px}, k_{Py}\}$, while $\boldsymbol{K}_D$ is as before. Provide the largest values for the scalars k_{Px}, k_{Py}, and k_d such that $\boldsymbol{\tau}(0)$ in (2) does not violate its bounds.
- How would things change if the bounds were set as $|\tau_{\theta,i}| \leq T_{max} = 300$ Nm, where $\tau_{\theta,i}$ is the torque delivered by the motors and performing work on the *relative* (Denavit-Hartenberg) coordinate θ_i, for $i = 1, 2, 3$?

[Turn sheet for the next exercise]

Exercise 3

Consider the planar PRP robot in Fig. 2.

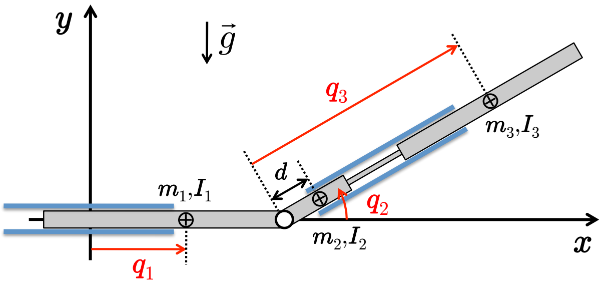

Figure 2: A planar PRP robot moving in a vertical plane, with definition of the generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ to be used.

- Determine the expressions of the inertial, Coriolis and centrifugal, and gravity terms in the dynamic model expressed in the usual Lagrangian form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}.$$

- Find a factorization $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}.\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ such that $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is a skew-symmetric matrix.
- Find all equilibrium configurations $\boldsymbol{q}_e$ (i.e., such that $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$), if any.
- Provide symbolic expressions for the scalar coefficients $\alpha > 0$ and $\beta > 0$ such that the following global linear bound holds for the Hessian of the gravitational potential energy $U_g(\boldsymbol{q})$:

$$\left\| \frac{\partial^2 U_g(\boldsymbol{q})}{\partial \boldsymbol{q}^2} \right\| = \left\| \frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} \right\| \le \alpha + \beta\, |q_3|, \qquad \forall \boldsymbol{q} \in \mathbb{R}^3.$$

[240 minutes; open books but no computer or smartphone]

Solution

June 6, 2017

Exercise 1

The direct kinematics of the planar 3R robot with unitary link lengths using absolute coordinates (i.e., the link angles w.r.t. the $\boldsymbol{x}$-axis) is

$$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} c_1 + c_2 + c_3 \\ s_1 + s_2 + s_3 \end{pmatrix}.$$

The associated first-order differential kinematics, with the Jacobian matrix, is

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \begin{pmatrix} -s_1 & -s_2 & -s_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\dot{\boldsymbol{q}}.$$

The second-order differential kinematics, with the first time-derivative $\dot{\boldsymbol{J}}$ of the Jacobian, is

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \begin{pmatrix} -c_1\dot{q}_1 & -c_2\dot{q}_2 & -c_3\dot{q}_3 \\ -s_1\dot{q}_1 & -s_2\dot{q}_2 & -s_3\dot{q}_3 \end{pmatrix}\dot{\boldsymbol{q}}.$$

The third-order differential kinematics, including the second time-derivative $\ddot{\boldsymbol{J}}$ of the Jacobian, is

$$\dddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\,\dddot{\boldsymbol{q}} + 2\dot{\boldsymbol{J}}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \ddot{\boldsymbol{J}}(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\,\dddot{\boldsymbol{q}} + 2\dot{\boldsymbol{J}}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \begin{pmatrix} s_1\dot{q}_1^2 - c_1\ddot{q}_1 & s_2\dot{q}_2^2 - c_2\ddot{q}_2 & s_3\dot{q}_3^2 - c_3\ddot{q}_3 \\ -c_1\dot{q}_1^2 - s_1\ddot{q}_1 & -c_2\dot{q}_2^2 - s_2\ddot{q}_2 & -c_3\dot{q}_3^2 - s_3\ddot{q}_3 \end{pmatrix}\dot{\boldsymbol{q}}.$$

When the initial conditions of the robot are perfectly matched with the desired end-effector trajectory,

$$\boldsymbol{p}(\boldsymbol{q}(0)) = \boldsymbol{p}_d(0), \qquad \boldsymbol{J}(\boldsymbol{q}(0))\,\dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{p}}_d(0), \qquad \boldsymbol{J}(\boldsymbol{q}(0))\,\ddot{\boldsymbol{q}}(0) + \dot{\boldsymbol{J}}(\boldsymbol{q}(0))\,\dot{\boldsymbol{q}}(0) = \ddot{\boldsymbol{p}}_d(0), \tag{3}$$

the nominal solution for executing $\boldsymbol{p}_d(t)$ with minimum norm of the joint jerk is (dropping dependencies)

$$\dddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\left(\dddot{\boldsymbol{p}}_d - 2\dot{\boldsymbol{J}}\,\ddot{\boldsymbol{q}} - \ddot{\boldsymbol{J}}\,\dot{\boldsymbol{q}}\right). \tag{4}$$

From (1), we have

$$\dot{\boldsymbol{p}}_d = \begin{pmatrix} 6\cos 3t \\ -3\sin\left(3t + \frac{\pi}{2}\right) \end{pmatrix}, \qquad \ddot{\boldsymbol{p}}_d = \begin{pmatrix} -18\sin 3t \\ -9\cos\left(3t + \frac{\pi}{2}\right) \end{pmatrix}, \quad \dddot{\boldsymbol{p}}_d = \begin{pmatrix} -54\cos 3t \\ 27\sin\left(3t + \frac{\pi}{2}\right) \end{pmatrix}.$$

Thus

$$\boldsymbol{p}_d(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \dot{\boldsymbol{p}}_d(0) = \begin{pmatrix} 6 \\ -3 \end{pmatrix}, \qquad \ddot{\boldsymbol{p}}_d(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \dddot{\boldsymbol{p}}_d(0) = \begin{pmatrix} -54 \\ 27 \end{pmatrix}.$$

It is easy to find an initial configuration $\boldsymbol{q}_0 = \boldsymbol{q}(0)$ that is matched with the initial position of the trajectory:

$$\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & \pi/2 \end{pmatrix}^T \text{ [rad]} \qquad \Rightarrow \qquad \boldsymbol{p}(\boldsymbol{q}_0) = \boldsymbol{p}_d(0).$$

In this configuration, the Jacobian is full rank and its pseudoinverse is easily computed as

$$\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{J}_0^{\#} = \boldsymbol{J}_0^T\left(\boldsymbol{J}_0\boldsymbol{J}_0^T\right)^{-1} = \begin{pmatrix} 0 & 1 \\ -0.5 & 0 \\ -0.5 & 0 \end{pmatrix}$$

The associated initial joint velocity $\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{q}}(0)$ and acceleration $\ddot{\boldsymbol{q}}_0 = \ddot{\boldsymbol{q}}(0)$ can be computed as minimum norm solutions at their differential level. We have

$$\dot{\boldsymbol{q}}_0 = \boldsymbol{J}_0^{\#}\dot{\boldsymbol{p}}_d(0) = \begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix} \text{ [rad/s]}.$$

From this, evaluating

$$\dot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 3 \end{pmatrix}\begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix} = \begin{pmatrix} -9 \\ -18 \end{pmatrix},$$

we obtain also

$$\ddot{\boldsymbol{q}}_0 = \boldsymbol{J}_0^{\#}\left(\ddot{\boldsymbol{p}}_d(0) - \dot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0\right) = -\boldsymbol{J}_0^{\#}\dot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 18 \\ -4.5 \\ -4.5 \end{pmatrix} \text{ [rad/s}^2\text{]}.$$

Evaluating now

$$\dot{\boldsymbol{J}}_0\,\ddot{\boldsymbol{q}}_0 = \begin{pmatrix} 54 \\ -27 \end{pmatrix}, \qquad \ddot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0 = \begin{pmatrix} -18 & 9 & 9 \\ -9 & 4.5 & 4.5 \end{pmatrix}\begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \boldsymbol{0},$$

from eq. (4) we finally obtain the jerk command at time $t = 0$:

$$\dddot{\boldsymbol{q}}(0) = \boldsymbol{J}_0^{\#}\left(\dddot{\boldsymbol{p}}_d(0) - 2\dot{\boldsymbol{J}}_0\,\ddot{\boldsymbol{q}}_0 - \ddot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0\right) = \begin{pmatrix} 0 & 1 \\ -0.5 & 0 \\ -0.5 & 0 \end{pmatrix}\left(\begin{pmatrix} -54 \\ 27 \end{pmatrix} - 2\begin{pmatrix} 54 \\ -27 \end{pmatrix}\right) = \begin{pmatrix} 81 \\ 81 \\ 81 \end{pmatrix} \text{ [rad/s}^3\text{]}.$$

Instead, when the initial conditions of the robot are not matched with the desired end-effector trajectory (i.e., if one or more of the identities in (3) is violated), in order to obtain exponential tracking of $\boldsymbol{p}_d(t)$, the solution with minimum norm of the joint jerk can be modified as (dropping dependencies)

$$\dddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\left(\dddot{\boldsymbol{p}}_d + k_2\left(\ddot{\boldsymbol{p}}_d - \boldsymbol{J}\ddot{\boldsymbol{q}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}\right) + k_1\left(\dot{\boldsymbol{p}}_d - \boldsymbol{J}\dot{\boldsymbol{q}}\right) + k_0\left(\boldsymbol{p}_d - \boldsymbol{p}\right) - 2\dot{\boldsymbol{J}}\,\ddot{\boldsymbol{q}} - \ddot{\boldsymbol{J}}\,\dot{\boldsymbol{q}}\right), \tag{5}$$

where the scalars k_0, k_1, and k_2 are such that

$$k(s) = s^3 + k_2 s^2 + k_1 s + k_0$$

is a Hurwitz polynomial, namely it has all roots in the left-hand side of the complex plane. From Routh criterion, this happens if and only if

$$k_0 > 0, \qquad k_1 > \frac{k_0}{k_2} > 0, \qquad k_2 > 0. \tag{6}$$

To show the transient properties of the control law (5), let the Cartesian position error be defined as $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p} \in \mathbb{R}^2$. From

$$\dddot{\boldsymbol{e}} = \dddot{\boldsymbol{p}}_d - \dddot{\boldsymbol{p}} = \dddot{\boldsymbol{p}}_d - \left(\boldsymbol{J}\,\dddot{\boldsymbol{q}} + 2\dot{\boldsymbol{J}}\,\ddot{\boldsymbol{q}} + \ddot{\boldsymbol{J}}\,\dot{\boldsymbol{q}}\right)$$

using (5) and being $\boldsymbol{J}\boldsymbol{J}^{\#} = \boldsymbol{I}_{2\times 2}$, it is easy to see that the following *linear* differential equation holds:

$$\dddot{\boldsymbol{e}} + k_2\,\ddot{\boldsymbol{e}} + k_1\,\dot{\boldsymbol{e}} + k_0\,\boldsymbol{e} = \boldsymbol{0}.$$

Under the conditions (6), the evolution of $\boldsymbol{e}(t)$ and of its time derivatives is that of the modes of an asymptotically stable linear system, namely exponentially or pseudo-exponentially converging to zero.

Exercise 2

We compute first the gravitational potential energy $U_g(\boldsymbol{q}) = U_1 + U_2 + U_3$. We have

$$U_1 = m_1 g_0 d_1 \sin q_1, \qquad U_2 = m_2 g_0 \left(\ell_1 \sin q_1 + d_2 \sin q_2\right),$$
$$U_3 = m_3 g_0 \left(\ell_1 \sin q_1 + \ell_2 \sin q_2 + d_3 \sin q_3\right).$$

Since $d_i = \ell_i/2 = 0.5$, for $i = 1,2,3$, it is

$$U_g(\boldsymbol{q}) = g_0 \left(\frac{m_1}{2} + m_2 + m_3\right) \sin q_1 + g_0 \left(\frac{m_2}{2} + m_3\right) \sin q_2 + g_0 \frac{m_3}{2} \sin q_3$$

and

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0\left((m_1/2) + m_2 + m_3\right) \cos q_1 \\ g_0\left((m_2/2) + m_3\right) \cos q_2 \\ g_0 (m_3/2) \cos q_3 \end{pmatrix}.$$

Using the expressions of $\boldsymbol{p}(\boldsymbol{q})$ and $\boldsymbol{J}(\boldsymbol{q})$ from Exercise 1 and the mass data, we evaluate the control law (2) with $\boldsymbol{K}_P = k_P \cdot \boldsymbol{I}_{2\times 2}$, at the initial time $t = 0$, when $\boldsymbol{q}(0) = \begin{pmatrix} \pi/2 & 0 & 0 \end{pmatrix}^T$ and $\dot{\boldsymbol{q}}(0) = \boldsymbol{0}$:

$$\begin{aligned} \boldsymbol{\tau}(0) &= k_p \boldsymbol{J}^T(\boldsymbol{q}(0)) \left(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q}(0))\right) + \boldsymbol{g}(\boldsymbol{q}(0)) \\ &= k_p \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 15 g_0 \\ 5 g_0 \end{pmatrix} = \begin{pmatrix} k_p \\ k_p + 15 g_0 \\ k_p + 5 g_0 \end{pmatrix} = \begin{pmatrix} 300 \\ 300 \\ 300 \end{pmatrix}, \quad k_p > 0,\ g_0 = 9.81 > 0. \end{aligned} \tag{7}$$

Therefore, the largest value $k_p > 0$ that satisfies the bounds on the joint torques, $|\tau_i| \le T_{max} = 300$ Nm, for $i = 1,2,3$, is the one that saturates the second torque component, i.e.,

$$\tau_2(0) = k_p + 15 g_0 = 300 \text{ [Nm]} \qquad \Rightarrow \qquad k_p = 300 - 15 g_0 \simeq 152.85.$$

If $\boldsymbol{K}_P = \text{diag}\{k_{Px}, k_{Py}\}$ and all the rest is as before, the control law (2) is evaluated again as

$$\boldsymbol{\tau}(0) = \boldsymbol{J}^T(\boldsymbol{q}(0)) \,\text{diag}\{k_{Px}, k_{Py}\} \left(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q}(0))\right) + \boldsymbol{g}(\boldsymbol{q}(0)) = \begin{pmatrix} k_{Px} \\ k_{Py} + 15 g_0 \\ k_{Py} + 5 g_0 \end{pmatrix} = \begin{pmatrix} 300 \\ 300 \\ 300 \end{pmatrix}, \quad k_{Px} > 0,\ k_{Py} > 0. \tag{8}$$

Therefore, we can take as the largest gain values those that saturate the first two components of the torque $\boldsymbol{\tau}$, i.e.,

$$k_{Px} = \tau_1(0) = 300 \text{ [Nm]}, \qquad k_{Py} = 300 - 15 g_0 \simeq 152.85 \text{ [Nm]}.$$

In both cases, the value of $\boldsymbol{K}_D = k_D \cdot \boldsymbol{I}_{2\times 2}$ does not play any role (as long as $\dot{\boldsymbol{q}} = \boldsymbol{0}$).

Finally, consider the case of torque bounds in the form $|\tau_{\theta,i}| \le T_{max} = 300$ Nm, for $i = 1,2,3$, where $\boldsymbol{\tau_\theta}$ are the torques producing work on the relative coordinates $\boldsymbol{\theta}$ (of the Denavit-Hartenberg convention). Since

$$\boldsymbol{q} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \boldsymbol{\theta} = \boldsymbol{T}\boldsymbol{\theta} \quad \Rightarrow \quad \dot{\boldsymbol{q}} = \boldsymbol{T}\dot{\boldsymbol{\theta}},$$

$$\tau^T (T\dot{\theta}) = \tau_\theta^T \dot{\theta} \Rightarrow T^T \tau = \tau_\theta$$

from the principle of virtual work ($\boldsymbol{\tau}^T \dot{\boldsymbol{q}} = \boldsymbol{\tau}_{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}}$) we have

$$\boldsymbol{\tau_\theta} = \boldsymbol{T}^T \boldsymbol{\tau} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{\tau} \qquad \Rightarrow \qquad \boldsymbol{\tau} = \boldsymbol{T}^{-T} \boldsymbol{\tau_\theta} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{\tau_\theta}. \tag{9}$$

Therefore, taking for example the gain structure in (7), it follows that

$$\begin{pmatrix} -300 \\ -300 \\ -300 \end{pmatrix} \le \boldsymbol{\tau}_{\boldsymbol{\theta}}(0) = \boldsymbol{T}^T \boldsymbol{\tau}(0) = \boldsymbol{T}^T \begin{pmatrix} k_p \\ k_p + 15 g_0 \\ k_p + 5 g_0 \end{pmatrix} = \begin{pmatrix} 3k_p + 20 g_0 \\ 2k_p + 20 g_0 \\ k_p + 5 g_0 \end{pmatrix} \le \begin{pmatrix} 300 \\ 300 \\ 300 \end{pmatrix}.$$

The largest value $k_p > 0$ that satisfies all the above bounds is obtained then from the first component:

$$k_p = \frac{300 - 20 g_0}{3} \simeq 34.6 \text{ [Nm]}.$$

Note also that, from the linear transformations (9), a feasible cube of side $2T_{max} = 600$ Nm centered in the origin of the $\boldsymbol{\tau}_{\boldsymbol{\theta}}$-space becomes a skewed parallelepiped in the $\boldsymbol{\tau}$-space (and vice versa).

Exercise 3

Following a Lagrangian approach, we compute first the kinetic energy $T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T_1 + T_2 + T_3$. We have

$$\begin{aligned} T_1 &= \tfrac{1}{2} m_1 \dot{q}_1^2, \qquad T_2 = \tfrac{1}{2} m_2 \left(\dot{q}_1^2 + d^2 \dot{q}_2^2 - 2d \sin q_2 \, \dot{q}_1 \dot{q}_2 \right) + \tfrac{1}{2} I_2 \, \dot{q}_2^2, \\ T_3 &= \tfrac{1}{2} m_3 \left(\dot{q}_1^2 + q_3^2 \, \dot{q}_2^2 + \dot{q}_3^2 - 2 q_3 \sin q_2 \, \dot{q}_1 \dot{q}_2 + 2 \cos q_2 \, \dot{q}_1 \dot{q}_3 \right) + \tfrac{1}{2} I_3 \, \dot{q}_2^2. \end{aligned}$$

Thus

$$T = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \frac{1}{2} \dot{\boldsymbol{q}}^T \begin{pmatrix} m_1 + m_2 + m_3 & -(m_2 d + m_3 q_3) \sin q_2 & m_3 \cos q_2 \\ & I_2 + m_2 d^2 + I_3 + m_3 q_3^2 & 0 \\ \text{symm} & & m_3 \end{pmatrix} \dot{\boldsymbol{q}}.$$

The components of the Coriolis and centrifugal vector are computed using the Christoffel's symbols

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial \boldsymbol{q}_i} \right),$$

being $\boldsymbol{m}_i$ the ith column of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. We have

$$\begin{aligned} \boldsymbol{C}_1(\boldsymbol{q}) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(m_2 d + m_3 q_3) \cos q_2 & -m_3 \sin q_2 \\ 0 & -m_3 \sin q_2 & 0 \end{pmatrix} \\ \Rightarrow \qquad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -(m_2 d + m_3 q_3) \cos q_2 \, \dot{q}_2^2 - 2\, m_3 \sin q_2 \, \dot{q}_2 \dot{q}_3. \end{aligned}$$

Similarly

$$\boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m_3 q_3 \\ 0 & m_3 q_3 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 2\, m_3 q_3 \, \dot{q}_2 \dot{q}_3 \,,$$

and

$$\boldsymbol{C}_3(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -m_3 q_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 q_3 \, \dot{q}_2^2 \,.$$

A factorization of the Coriolis and centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}}$ that satisfies the skew-symmetric property is given by

$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_3(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} 0 & -(m_2 d + m_3 q_3) \cos q_2 \, \dot{q}_2 - m_3 \sin q_2 \, \dot{q}_3 & -m_3 \sin q_2 \, \dot{q}_2 \\ 0 & m_3 q_3 \, \dot{q}_3 & m_3 q_3 \, \dot{q}_2 \\ 0 & -m_3 q_3 \, \dot{q}_2 & 0 \end{pmatrix}.$$

Being

$$\dot{\boldsymbol{M}}(\boldsymbol{q}) = \begin{pmatrix} 0 & -(m_2 d + m_3 q_3)\cos q_2 \dot{q}_2 - m_3 \sin q_2\, \dot{q}_3 & -m_3 \sin q_2\, \dot{q}_2 \\ -(m_2 d + m_3 q_3)\cos q_2\, \dot{q}_2 - m_3 \sin q_2\, \dot{q}_3 & 2\, m_3 q_3\, \dot{q}_3 & 0 \\ -m_3 \sin q_2\, \dot{q}_2 & 0 & 0 \end{pmatrix},$$

it is easy to check that the matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is skew-symmetric.

For the potential energy due to gravity, $U_g(\boldsymbol{q}) = U_1 + U_2 + U_3$, we have (up to a constant)

$$U_1 = 0, \qquad U_2 = m_2 g_0\, d \sin q_2, \qquad U_3 = m_3 g_0\, q_3 \sin q_2.$$

Thus

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ (m_2 d + m_3 q_3) g_0 \cos q_2 \\ m_3 g_0 \sin q_2 \end{pmatrix}.$$

The unforced equilibrium configurations are

$$\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0} \quad \Rightarrow \quad q_{e,1} = any, \quad q_{e,2} = \{0, \pi\}, \quad q_{e,3} = -\frac{m_2}{m_3}\, d.$$

Taking a further partial derivative of $\boldsymbol{g}$ w.r.t. $\boldsymbol{q}$, we obtain the matrix

$$\frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \frac{\partial^2 U_g(\boldsymbol{q})}{\partial \boldsymbol{q}^2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(m_2 d + m_3 q_3) g_0 \sin q_2 & m_3 g_0 \cos q_2 \\ 0 & m_3 g_0 \cos q_2 & 0 \end{pmatrix} = \boldsymbol{A}(\boldsymbol{q}).$$

Matrix $\boldsymbol{A}$ is symmetric, thus it has real eigenvalues. To have all non-negative eigenvalues (so that we can order them and find their maximum, as requested by the definition of norm of a matrix that we use), we compute the semi-positive definite matrix

$$\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_0^2\left((m_2 d + m_3 q_3)^2 \sin^2 q_2 + m_3^2 \cos^2 q_2\right) & -g_0^2 m_3\,(m_2 d + m_3 q_3) \sin q_2 \cos q_2 \\ 0 & -g_0^2 m_3\,(m_2 d + m_3 q_3) \sin q_2 \cos q_2 & g_0^2 m_3^2 \cos^2 q_2 \end{pmatrix},$$

which has clearly one zero eigenvalue. Denote by $\boldsymbol{B}$ the lower 2×2 block on the diagonal of this matrix. The characteristic polynomial of $\boldsymbol{A}^T\boldsymbol{A}$ is then

$$\det\left(\lambda \boldsymbol{I} - \boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right) = \lambda \cdot \det\left(\lambda \boldsymbol{I} - \boldsymbol{B}(\boldsymbol{q})\right) = \lambda\left(\lambda^2 - \text{trace}\{\boldsymbol{B}(\boldsymbol{q})\}\lambda + \det\{\boldsymbol{B}(\boldsymbol{q})\}\right)$$

with $\text{trace}\{\boldsymbol{B}(\boldsymbol{q})\} > 0$ and $\det\{\boldsymbol{B}(\boldsymbol{q})\} > 0$. Therefore, the maximum eigenvalue of $\boldsymbol{A}^T\boldsymbol{A}$ is

$$\lambda_{max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right) = \frac{1}{2}\,\text{trace}\{\boldsymbol{B}(\boldsymbol{q})\} + \frac{1}{2}\sqrt{(\text{trace}\{\boldsymbol{B}(\boldsymbol{q})\})^2 - 4\det\{\boldsymbol{B}(\boldsymbol{q})\}}$$

Since we are looking for a bound on the norm of $\boldsymbol{A}(\boldsymbol{q})$, we can write the chain of inequalities

$$\begin{aligned}\lambda_{max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right) &\leq \text{trace}\{\boldsymbol{B}(\boldsymbol{q})\} = g_0^2\left((m_2 d + m_3 q_3)^2 \sin^2 q_2 + 2\, m_3^2 \cos^2 q_2\right) \\ &< g_0^2\left((m_2 d + m_3 q_3)^2 + 2\, m_3^2\right) < g_0^2 (m_2 d + m_3 |q_3| + \sqrt{2}\, m_3)^2.\end{aligned}$$

Therefore, we finally obtain the requested bound

$$\left\|\frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right\| = \|\boldsymbol{A}(\boldsymbol{q})\| = \sqrt{\lambda_{max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right)} < g_0\,(m_2 d + m_3 |q_3| + m_3\sqrt{2}) = \alpha + \beta\,|q_3|, \quad \forall \boldsymbol{q} \in \mathbb{R}^3,$$

with

$$\alpha = g_0(m_2 d + m_3\sqrt{2}), \qquad \beta = g_0 m_3.$$

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