Robotics 2

July 8, 2024

Exercise 1

Consider the robot in Fig. 1 with $n = 3$ joints, the first one prismatic and the other two revolute. Each link has uniformly distributed mass, center of mass on its major physical axis, and a diagonal barycentric inertia matrix. Assume that friction at the joints can be neglected. The robot is commanded at the joint level by a generalized vector of forces/torques $\boldsymbol{\tau} \in \mathbb{R}^3$.

a) Derive the dynamic model of the robot in the Lagrangian form $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}.\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$.

b) Find a linear parametrization $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{\tau}$ of the robot dynamics in terms of a vector $\boldsymbol{a} \in \mathbb{R}^p$ of dynamic coefficients and of a $3 \times p$ regressor matrix $\boldsymbol{Y}$. Discuss the minimality of p.

c) Determine which of the $10n = 30$ standard dynamic parameters of the links are irrelevant for the describing the motion of the robot.

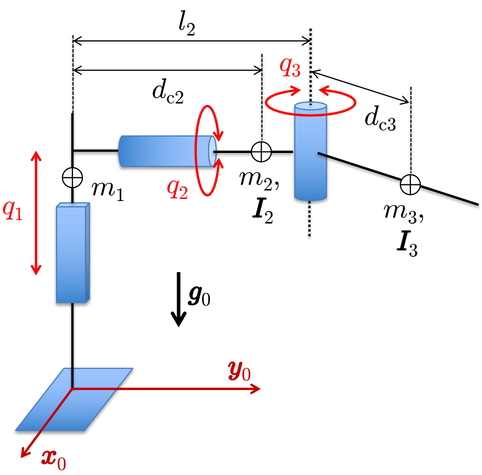


Figure 1: A PRR robot with its coordinates $\boldsymbol{q} = (q_1\ q_2\ q_3)^T$ and kinematic/dynamic parameters.

Exercise 2

The 2R planar robot in Fig. 2 has equal links of length l and is commanded by the joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^2$. The robot end effector has to perform a one-dimensional trajectory task $r_d(t) \in \mathbb{R}$ specified *only* along the x-direction. In a given robot state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, a desired task acceleration $\ddot{r}_d$ is assigned.

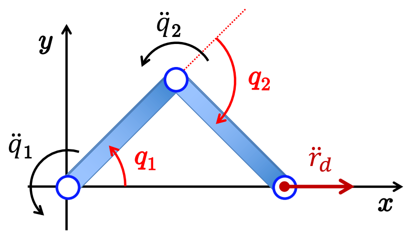


Figure 2: A 2R planar robot in a one-dimensional task.

Provide in symbolic form the command $\ddot{\boldsymbol{q}}$ that executes the task while minimizing the cost

$$H = \frac{1}{2} \left(\ddot{\boldsymbol{q}} - \ddot{\boldsymbol{q}}_0 \right)^T \left(\ddot{\boldsymbol{q}} - \ddot{\boldsymbol{q}}_0 \right), \qquad \ddot{\boldsymbol{q}}_0 = -\boldsymbol{K}_v \, \dot{\boldsymbol{q}},$$

for a diagonal matrix $\boldsymbol{K}_v > 0$. Evaluate then numerically the solution when the robot is in the configuration $\boldsymbol{q} = (\pi/4, -\pi/2)$ [rad] with joint velocity $\dot{\boldsymbol{q}} = (1, -1)$ [rad/s], the link length is $l = 1$ m, the task acceleration is $\ddot{r}_d = 1$ m/s^2, and $\boldsymbol{K}_v = \text{diag}\{2, 2\}$ [s^{-1}]. How would you modify the acceleration command if there was a task error in position and/or velocity?

Exercise 3

Figure 3 shows a mechanical system made of two masses B and M connected by a pulley and a damped elastic spring, viscous friction on the motion of the individual masses, an input force τ acting on the first mass, and gravity acting on the second mass only. The zero of the two position variables θ and q is associated to an undeformed spring. The spring has stiffness $K > 0$ and its elastic potential energy is quadratic in the deformation $q - \theta$.

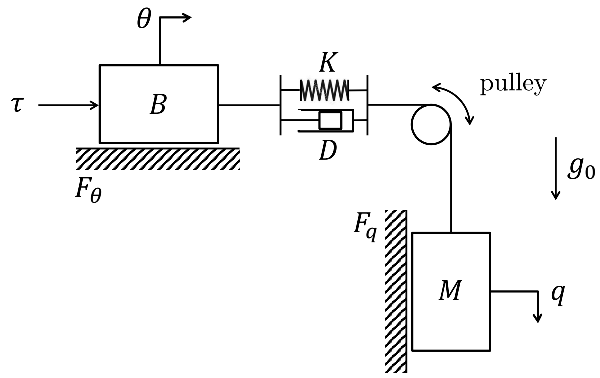


Figure 3: Two masses connected by a pulley and a damped elastic spring under gravity.

a) Derive the dynamic model of this system, including all non-conservative terms due to viscous friction (with coefficients $F_\theta > 0$ and $F_q > 0$, respectively) affecting the motion of the two masses and damping (with coefficient $D \geq 0$) on the time-varying deformation of the spring.
b) Provide the simplest feedback law that is able to asymptotically stabilize the position of the mass M to a constant desired height q_d. Prove the result using any preferred analysis method (linearization by Taylor expansion, Lyapunov/LaSalle, etc.).
c) Set now $D = 0$. Solve the inverse dynamics problem for a desired, sufficiently smooth trajectory $q_d(t)$. Provide the explicit expression of $\tau_d(t)$ as a function of $q_d(t)$ and its (higher order) time derivatives only.

Exercise 4

Suppose that a 2R planar robot that is moving in a vertical plane has only one actuator at the first joint providing a torque τ. Find the expression of all forced equilibria $(\bar{\boldsymbol{q}}, \boldsymbol{0})$ associated to a generic constant input torque $\bar{\tau}$. Are you able to find a state feedback control law $\tau = f(\bar{\boldsymbol{q}}, \boldsymbol{q}, \dot{\boldsymbol{q}})$ that asymptotically stabilizes one of these equilibria, at least locally?

[240 minutes (4 hours); open books]

Solution

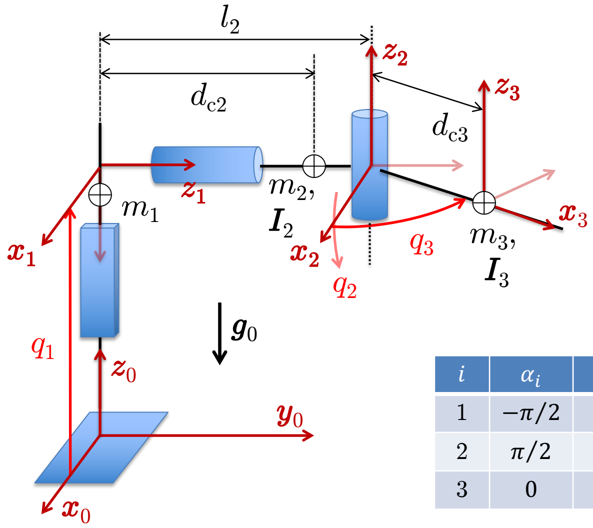
July 8, 2024

Exercise 1

The kinetic energies of the first two links are easy to compute:

$$T_1 = \frac{1}{2}\, m_1 \dot{q}_1^2 \qquad T_2 = \frac{1}{2}\, m_2 \dot{q}_1^2 + \frac{1}{2}\, I_{c2,yy}\, \dot{q}_1^2.$$

For the third link, one can use the moving frame recursive algorithm for obtaining ${}^3\boldsymbol{v}_{c3}$ and ${}^3\boldsymbol{\omega}_3$, or use the direct kinematics of the robot for computing the position $\boldsymbol{p}_{c3}(\boldsymbol{q})$ of the center of mass, and then differentiating it, as well as the orientation $\boldsymbol{R}_3(\boldsymbol{q})$ of the last frame, extracting then the angular velocity from its derivative. In any event, one should attach frames to the robot arm according to the Denavit-Hartenberg (DH) convention, as done in Fig. 4. Note that the last frame has been placed conveniently at the center of mass of the third link: this is reflected in the D-H parameter $a_3 = d_{c3}$.



i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	q_1	0
2	$\pi/2$	0	l_2	q_2
3	0	d_{c3}	0	q_3

Figure 4: DH frames and associated table of parameters for the PRR robot of Fig. 1.

Following the second approach, we compute via the DH direct kinematics

$$\boldsymbol{p}_{c3} = \begin{pmatrix} d_{c3} c_2 c_3 \\ l_2 + d_{c3} s_3 \\ q_1 - d_{c3} s_2 c_3 \end{pmatrix},$$

so that its velocity in the absolute (zero) frame and in the local (third) DH frame are

$$\boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} -d_{c3}\,(s_2 c_3\, \dot{q}_2 + c_3 s_3\, \dot{q}_3) \\ d_{c3} c_3\, \dot{q}_3 \\ \dot{q}_1 + d_{c3}\,(s_2 s_3\, \dot{q}_3 - c_2 c_3\, \dot{q}_2) \end{pmatrix} \quad \Rightarrow \quad {}^3\boldsymbol{v}_{c3} = {}^0\boldsymbol{R}_3^T(\boldsymbol{q})\, \boldsymbol{v}_{c3} = \begin{pmatrix} -s_2 c_3\, \dot{q}_1 \\ s_2 s_3\, \dot{q}_1 + d_{c3}\, \dot{q}_3 \\ c_2\, \dot{q}_1 - d_{c3} c_3\, \dot{q}_2 \end{pmatrix}.$$

Similarly, using the relationship

$$\boldsymbol{S}(\boldsymbol{\omega}_3) = {}^0\dot{\boldsymbol{R}}_3(\boldsymbol{q})\, {}^0\boldsymbol{R}_3^T(\boldsymbol{q}),$$

the angular velocity of the third link is

$$\boldsymbol{\omega}_3 = \begin{pmatrix} -S_{23}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ S_{13}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ -S_{12}(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{pmatrix} = \begin{pmatrix} s_2\,\dot{q}_3 \\ \dot{q}_2 \\ c_2\,\dot{q}_3 \end{pmatrix} \qquad \Rightarrow \qquad {}^3\boldsymbol{\omega}_3 = {}^0\boldsymbol{R}_3^T(\boldsymbol{q})\,\boldsymbol{\omega}_3 = \begin{pmatrix} s_3\,\dot{q}_2 \\ c_3\,\dot{q}_2 \\ \dot{q}_3 \end{pmatrix}.$$

Thus

$$\begin{aligned} T_3 &= \frac{1}{2}\,m_3\left\|{}^3\boldsymbol{v}_{c3}\right\|^2 + \frac{1}{2}\,{}^3\boldsymbol{\omega}_3^T\,{}^3\boldsymbol{I}_3\,{}^3\boldsymbol{\omega}_3 \\ &= \frac{1}{2}\,m_3\left((c_2\,\dot{q}_1 - d_{c3}c_3\,\dot{q}_2)^2 + (s_2s_3\,\dot{q}_1 - d_{c3}\,\dot{q}_3)^2 + s_2^2c_3^2\,\dot{q}_1^2\right) + \frac{1}{2}\left((I_{c3,xx}s_3^2 + I_{c3,yy}c_3^2)\,\dot{q}_2^2 + I_{c3,zz}\,\dot{q}_3^2\right). \end{aligned}$$

After substituting $s_3^2 = 1 - c_3^2$, the inertia matrix of the robot is found from the total kinetic energy

$$T = T_1 + T_2 + T_3 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1+m_2+m_3 & -m_3d_{c3}\,c_2c_3 & m_3d_{c3}\,s_2s_3 \\ -m_3d_{c3}\,c_2c_3 & I_{c2,yy} + I_{c3,xx} + \left(I_{c3,yy} - I_{c3,xx} + m_3d_{c3}^2\right)c_3^2 & 0 \\ m_3d_{c3}\,s_2s_3 & 0 & I_{c3,zz} + m_3d_{c3}^2 \end{pmatrix}.$$

Introducing the following $p = 5$ dynamic coefficients

$$\begin{aligned} \boldsymbol{a} &= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix}^T \\ a_1 &= m_1 + m_2 + m_3 \\ a_2 &= I_{c2,yy} + I_{c3,xx} \\ a_3 &= I_{c3,yy} - I_{c3,xx} + m_3d_{c3}^2 \\ a_4 &= m_3d_{c3} \\ a_5 &= I_{c3,zz} + m_3d_{c3}^2, \end{aligned} \tag{1}$$

the inertia matrix is rewritten more compactly as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 & -a_4\,c_2c_3 & a_4\,s_2s_3 \\ -a_4\,c_2c_3 & a_2 + a_3\,c_3^2 & 0 \\ a_4\,s_2s_3 & 0 & a_5 \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}_1(\boldsymbol{q}) & \boldsymbol{M}_2(\boldsymbol{q}) & \boldsymbol{M}_3(\boldsymbol{q}) \end{pmatrix}.$$

Using Christoffel's symbols, the Coriolis and centrifugal terms are computed as

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} a_4\left(s_2c_3\,(\dot{q}_2^2 + \dot{q}_3^2) + 2\,c_2s_3\,\dot{q}_2\dot{q}_3\right) \\ -2a_3\,s_3c_3\,\dot{q}_2\dot{q}_3 \\ a_3\,s_3c_3\,\dot{q}_2^2 \end{pmatrix}.$$

Finally, the potential energy of the three links due to gravity is $U = U_1 + U_2 + U_3$, with

$$U_1 = m_1g_0\,(q_1 - d_{c1}) \qquad U_2 = m_2g_0\,q_1 \qquad U_3 = m_3g_0\,(q_1 - d_{c3}s_2c_3)\,.$$

Thus

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0\,(m_1+m_2+m_3) \\ -m_3d_{c3}g_0\,c_2c_3 \\ m_3d_{c3}g_0\,s_2s_3 \end{pmatrix} = \begin{pmatrix} a_1g_0 \\ -a_4g_0\,c_2c_3 \\ a_4g_0\,s_2s_3 \end{pmatrix}, \tag{2}$$

where $g_0 = 9.81$ m/s^2 is assumed to be known (this allows to use in (2) the previously introduced coefficients a_1 and a_4).

The complete dynamic model is

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q}.\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{\tau},$$

with the 3×5 regressor matrix $\boldsymbol{Y}$ of the linear parametrization given by

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 + g_0 & 0 & 0 & s_2c_3\,(\dot{q}_2^2 + \dot{q}_3^2) + 2c_2s_3\,\dot{q}_2\dot{q}_3 - c_2c_3\,\ddot{q}_2 + s_2s_3\,\ddot{q}_3 & 0 \\ 0 & \ddot{q}_2 & c_3\,\ddot{q}_2 - 2s_3c_3\,\dot{q}_2\dot{q}_3 & -(\ddot{q}_1 + g_0)\,c_2c_3 & 0 \\ 0 & 0 & s_3c_3\,\dot{q}_2^2 & (\ddot{q}_1 + g_0)\,s_2s_3 & \ddot{q}_3 \end{pmatrix} \tag{3}$$

and the dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^5$ defined in (1).

Summarizing, out of the $3 \times 10 = 30$ standard dynamic parameters of the three robot links, half of them (15) has been removed from the beginning because of the simplifying assumptions made (center of mass on the kinematic/physical link axis, diagonal barycentric inertia matrix); other 7 parameters never appear, and play thus no role in the robot dynamics; the remaining 8 dynamic parameters appear in suitable combinations, generating the 5 dynamic coefficients in (1) — note that $a_4 = m_3 d_{c3}$ is both a standard dynamic parameter and a dynamic coefficient.

Exercise 2

The one-dimensional task kinematics of the 2R planar robot is

$$r = p_x(\boldsymbol{q}) = l_1 \cos q_1 + l_2 \cos(q_1 + q_2).$$

Therefore, we have

$$\dot{r} = \frac{\partial p_x}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}_r(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} -(l_1 \sin q_1 + l_2 \sin(q_1+q_2)) & -l_2 \sin(q_1+q_2) \end{pmatrix} \dot{\boldsymbol{q}},$$

with the 1×2 task Jacobian matrix $\boldsymbol{J}_r$, and

$$\ddot{r} = \boldsymbol{J}_r(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_r(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}_r(\boldsymbol{q})\ddot{\boldsymbol{q}} + h(\boldsymbol{q},\dot{\boldsymbol{q}})$$

where

$$\begin{aligned} h(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \begin{pmatrix} -(l_1 \cos q_1 \dot{q}_1 + l_2 \cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)) & -l_2 \cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) \end{pmatrix} \dot{\boldsymbol{q}} \\ &= -\left(l_1 \cos q_1\, \dot{q}_1^2 + l_2 \cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)^2\right). \end{aligned}$$

Note that the y-component of the end-effector position is not assigned by the task and its acceleration can take any value (this is why the 2R planar robot is redundant with respect to the given one-dimensional task).

The joint acceleration command $\ddot{\boldsymbol{q}}$ that realizes the desired task $\ddot{r}_d$ (out of singularities, i.e., where $\boldsymbol{J}_r(\boldsymbol{q})$ does not vanish) while minimizing instantaneously the complete quadratic objective H, where $\ddot{\boldsymbol{q}}_0 = -\boldsymbol{K}_v\dot{\boldsymbol{q}}$ is the preferred joint acceleration, is given by

$$\begin{aligned} \ddot{\boldsymbol{q}} &= \boldsymbol{J}_r^{\#}(\boldsymbol{q})\,(\ddot{r}_d - h(\boldsymbol{q},\dot{\boldsymbol{q}})) - \left(\boldsymbol{I} - \boldsymbol{J}_r^{\#}(\boldsymbol{q})\boldsymbol{J}_r(\boldsymbol{q})\right)\boldsymbol{K}_v\dot{\boldsymbol{q}} \\ &= -\boldsymbol{K}_v\dot{\boldsymbol{q}} + \boldsymbol{J}_r^{\#}(\boldsymbol{q})\,(\ddot{r}_d - h(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{J}_r(\boldsymbol{q})\boldsymbol{K}_v\dot{\boldsymbol{q}}), \end{aligned} \tag{4}$$

with $\boldsymbol{J}_r^{\#}(\boldsymbol{q}) = \boldsymbol{J}_r^T(\boldsymbol{q})(\boldsymbol{J}_r(\boldsymbol{q})\boldsymbol{J}_r^T(\boldsymbol{q}))^{-1}$ (if $\boldsymbol{J}_r(\boldsymbol{q}) \neq \boldsymbol{0}^T$). Substituting the numerical values of the problem in (4) gives

$$\ddot{\boldsymbol{q}} = \begin{pmatrix} -2 \\ 2.4142 \end{pmatrix} \text{[rad/s}^2\text{]}. \tag{5}$$

Note that the two components of the joint acceleration both oppose their respective velocity (i.e., $\ddot{q}_i\dot{q}_i < 0$, for $i = 1, 2$), which confirms that the damping action specified by $\ddot{\boldsymbol{q}}_0$ is being pursued. On the other hand, with the acceleration (5) produced in the given state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, the resulting acceleration of the end-effector is

$$\ddot{\boldsymbol{p}} = \begin{pmatrix} \ddot{p}_x \\ \ddot{p}_y \end{pmatrix} = \begin{pmatrix} 1 \\ -1.8284 \end{pmatrix} \; [\mathrm{m/s^2}]$$

showing that $\ddot{p}_x = 1 = \ddot{r}_d$ has been correctly realized (while $\ddot{p}_y \neq 0$ is just a result of the chosen redundancy resolution scheme).

In the presence of a task error, the command (4) is modified by adding a PD action to the desired task acceleration,

$$\ddot{\boldsymbol{q}} = -\boldsymbol{K}_v\dot{\boldsymbol{q}} + \boldsymbol{J}_r^{\#}(\boldsymbol{q})\left(\ddot{r}_d + K_D(\dot{r}_d - \boldsymbol{J}_r(\boldsymbol{q})\dot{\boldsymbol{q}}) + K_P(r_d - p_x(\boldsymbol{q})) - h(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{J}_r(\boldsymbol{q})\boldsymbol{K}_v\dot{\boldsymbol{q}}\right), \tag{6}$$

so as to bring the task error back to zero at an exponential rate governed by the choice of the two gains $K_P > 0$ and $K_D > 0$.

Exercise 3

The dynamic model of the system in Fig. 3 is given by the two second-order differential equations (one for each mass)

$$B\,\ddot{\theta} + K\,(\theta - q) + D\,(\dot{\theta} - \dot{q}) + F_\theta\,\dot{\theta} = \tau \tag{7}$$

$$M\,\ddot{q} + K\,(q - \theta) - Mg_0 + D\,(\dot{q} - \dot{\theta}) + F_q\,\dot{q} = 0, \tag{8}$$

which can be easily derived either from an energy-based Lagrangian approach with

I think this should be theta and q without the dot

$$\mathcal{T}_B = \frac{1}{2}\,B\dot{\theta}^2 \qquad \mathcal{T}_M = \frac{1}{2}\,M\dot{q}^2 \qquad \mathcal{U}_g = -Mg_0q \qquad \mathcal{U}_e = \frac{1}{2}\,K\,(\dot{\theta} - \dot{q})^2,$$

including all non-conservative terms, or simply by the balance of forces acting on the two masses in a Newton approach. Note that the dynamics (7) is *linear*, whereas eq. (8) is *affine* since it contains the offset constant term $-Mg_0$ due to gravity.

Under the action of a constant force $\bar{\tau}$, any forced equilibrium configuration $(\bar{\theta}, \bar{q})$ for eqs. (7), (8) should satisfy

$$K\,(\bar{\theta} - \bar{q}) = \bar{\tau} \qquad K\,(\bar{\theta} - \bar{q}) - Mg_0 = 0,$$

and thus

$$\bar{\tau} = -Mg_0 \qquad \bar{\theta} = \bar{q} - K^{-1}Mg_0. \tag{9}$$

With the above in mind, for the regulation problem consider the simplest linear feedback law with constant feedforward

$$\tau = \tau_d + K_P(\theta_d - \theta) \qquad K_P > 0, \tag{10}$$

where

$$\tau_d = -Mg_0 \qquad \theta_d = q_d - K^{-1}Mg_0 \tag{11}$$

satisfy the conditions (9) for achieving an equilibrium; θ_d is the required position of mass B when mass M is in equilibrium at q_d. Note that no derivative term is present in (10), as customary instead in a PD control law: the presence of various sources of dissipation in the system (viscous friction with coefficients F_θ and F_q, damping D on the elastic spring) makes this additional control action no longer needed for stabilization purposes.

The closed-loop equations (7), (8) with the control law (10) are

$$B\,\ddot{\theta} + K\,(\theta - q) + D\,(\dot{\theta} - \dot{q}) + F_\theta\,\dot{\theta} = -Mg_0 + K_P(\theta_d - \theta) \tag{12}$$

$$M\,\ddot{q} + K\,(q - \theta) - Mg_0 + D\,(\dot{q} - \dot{\theta}) + F_q\,\dot{q} = 0. \tag{13}$$

At steady state, $\dot{\theta} = \ddot{\theta} = \dot{q} = \ddot{q} = 0$, it is

$$K\,(\bar{\theta} - \bar{q}) = -Mg_0 + K_P(\theta_d - \bar{\theta}) \qquad K\,(\bar{q} - \bar{\theta}) - Mg_0 = 0 \tag{14}$$

which imply $K_P(\theta_d - \bar{\theta}) = 0$, so that $(\bar{\theta}, \bar{q}) = (\theta_d, q_d) = (q_d - K^{-1}Mg_0, q_d)$ is the unique equilibrium configuration.

To verify the asymptotic stability of the closed-loop equilibrium, we present next two alternative methods: the first is a global approach based on Lyapunov analysis, completed by the use of LaSalle theorem as done during the course; the second analyzes the linearized version of the system dynamics, obtained by a first-order Taylor expansion around the desired closed-loop equilibrium point, and has in general only a local validity.[1]

1. Lyapunov analysis. Define the following energy-based function

$$V = \frac{1}{2}\,B\,\dot{\theta}^2 + \frac{1}{2}\,M\,\dot{q}^2 + \frac{1}{2}\,K\,(\theta - q)^2 - Mg_0\,q + \frac{1}{2}\,K_P\,(\theta_d - \theta)^2 - Mg_0\,(\theta_d - \theta)\,, \tag{15}$$

which contains the kinetic energy of the two masses, the potential energy of the elastic spring, the potential energy due to gravity (acting only on mass M), together with a virtual potential energy introduced by control (in terms of the position error of the mass B, with $K_P > 0$), and an additional term that is linear in the position error. The addition of this last term guarantees that $V \geq 0$ for all states $\boldsymbol{x} = (\theta, q, \dot{\theta}, \dot{q}) \in \mathbb{R}^4$ and $V = 0$ if and only if $\boldsymbol{x} = \boldsymbol{x}_e = (\theta_d, q_d, 0, 0)$, namely the desired equilibrium specified in (14), so that (15) is a Lyapunov candidate. On one hand, V is quadratic and positive definite with respect to the velocities $\dot{\theta}$ and $\dot{q}$. On the other hand, consider the remaining terms of V collected in the configuration-dependent function

$$P(\theta, q) = V|_{\dot{\theta}=\dot{q}=0} = \frac{1}{2}\,K\,(\theta - q)^2 + \frac{1}{2}\,K_P\,(\theta_d - \theta)^2 - Mg_0\,(\theta_d + q - \theta)\,.$$

It is easy to see that P is a convex function with global minimum at (θ_d, q_d): in fact, its gradient is

$$\nabla P = \begin{pmatrix} \nabla_\theta P \\ \nabla_q P \end{pmatrix} = \begin{pmatrix} K\,(\theta - q) - K_P\,(\theta_d - \theta) + Mg_0 \\ K\,(q - \theta) - Mg_0 \end{pmatrix}$$

and the stationarity condition $\nabla P = \mathbf{0}$ holds if and only if $(\theta, q) = (\theta_d, q_d)$ in agreement with (14); moreover, being the Hessian of P

$$\nabla^2 P = \begin{pmatrix} \nabla^2_{\theta\theta} P & \nabla^2_{\theta q} P \\ \nabla^2_{q\theta} P & \nabla^2_{qq} P \end{pmatrix} = \begin{pmatrix} K + K_P & -K \\ -K & K \end{pmatrix} > 0,$$

the desired configuration is a minimum for P. As a result, V is a valid Lyapunov candidate.

The time derivative of (15) evaluated along the trajectories of the closed-loop system is

$$\begin{aligned}
\dot{V} &= B\ddot{\theta}\,\dot{\theta} + M\ddot{q}\,\dot{q} + K\,(q - \theta)\,(\dot{q} - \dot{\theta}) - Mg_0\,(\dot{q} - \dot{\theta}) - K_P\,(\theta_d - \theta)\,\dot{\theta} \\
&= \Big(K\,(q - \theta) + D\,(\dot{q} - \dot{\theta}) - F_\theta\,\dot{\theta} - Mg_0 + K_P(\theta_d - \theta)\Big)\,\dot{\theta} \\
&\quad + \Big(K\,(\theta - q) + Mg_0 + D\,(\dot{\theta} - \dot{q}) - F_q\,\dot{q}\Big)\,\dot{q} \\
&\quad + K\,(q - \theta)\,(\dot{q} - \dot{\theta}) - Mg_0\,(\dot{q} - \dot{\theta}) - K_P\,(\theta_d - \theta)\,\dot{\theta} \\
&= -D\,(\dot{\theta} - \dot{q})^2 - F_\theta\,\dot{\theta}^2 - F_q\,\dot{q}^2 \leq 0.
\end{aligned}$$

[1] This method should be part of the background knowledge of any student exposed to linear dynamical systems.

Thus, the closed-loop system is certainly stable. To conclude about asymptotic stability, we use LaSalle theorem. Since

$$\dot{V} = 0 \quad \Longleftrightarrow \quad \dot{\theta} = \dot{q} = 0,$$

we analyze the closed-loop eqs. (12) and (13) under this condition:

$$\begin{aligned} B\,\ddot{\theta} + K\,(\theta - q) &= -Mg_0 + K_P(\theta_d - \theta) \\ M\,\ddot{q} + K\,(q - \theta) - Mg_0 &= 0. \end{aligned}$$

Since for any set of states contained in $\mathcal{S} = \{\boldsymbol{x} : \dot{V} = 0\}$ to be invariant both $\ddot{\theta}$ and $\ddot{q}$ must also vanish, we have

$$\begin{aligned} K\,(\theta - q) &= -Mg_0 + K_P(\theta_d - \theta) \\ K\,(q - \theta) - Mg_0 &= 0. \end{aligned}$$

As shown in (14), these equations have $(\theta, q) = (\theta_d, q_d)$ as the only solution. Thus, the maximal set of invariant states contained in $\mathcal{S}$ reduces to the singleton $\boldsymbol{x}_e = (\theta_d, q_d, 0, 0)$, which is then a global, asymptotically stable equilibrium. This concludes the proof.

It should be noted that a control law of the form

$$\tau = -Mg_0 + K_P(q_d - q) \qquad K_P > 0, \tag{16}$$

similar to eq. (10) and maybe more natural at first sight, produces the same desired equilibrium point. However, showing asymptotic stability with a Lyapunov argument is quite difficult — either when trying to define a correct candidate function $V \geq 0$, with $V = 0$ only at the desired equilibrium, or in proving that $\dot{V} \leq 0$ is obtained. Moreover, a restriction to the maximum gain in the control law (16) would apply, while a global result would be hard if not impossible to obtain. The reason for this behavior will become clearer when pursuing the alternative approach.

2. Analysis by local approximate linearization. A simpler and systematic approach consists in linearizing the dynamic equations (7), (8) around the desired equilibrium point $\boldsymbol{x}_d = (\theta_d, q_d, 0, 0)$. In this case, the approximate linearization procedure by Taylor expansion boils down to simply removing the constant offset term $-Mg_0$ due to gravity. Furthermore, one can apply Laplace transforms to the linearized equations and then conveniently use a SISO transfer function for describing the process to be controlled.

Let $\Delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_d = (\theta - \theta_d, q - q_d, \dot{\theta}, \dot{q}) = (\Delta\theta, \Delta q, \Delta\dot{\theta}, \Delta\dot{q})$ and $\Delta\tau = \tau - \tau_d = \tau + Mg_0$. Then, replacing in eqs. (7), (8)

$$\theta = \theta_d + \Delta\theta \qquad q = q_d + \Delta q \qquad \dot{\theta} = \Delta\dot{\theta} \qquad \dot{q} = \Delta\dot{q} \qquad \tau = \tau_d + \Delta\tau = -Mg_0 + \Delta\tau,$$

as well as $\ddot{\theta} = \Delta\ddot{\theta}$ and $\ddot{q} = \Delta\ddot{q}$, yields

$$\begin{aligned} B\,\Delta\ddot{\theta} + K\,(\Delta\theta - \Delta q) + D\,(\Delta\dot{\theta} - \Delta\dot{q}) + F_\theta\,\Delta\dot{\theta} &= \Delta\tau \\ M\,\Delta\ddot{q} + K\,(\Delta q - \Delta\theta) - Mg_0 + D\,(\Delta\dot{q} - \Delta\dot{\theta}) + F_q\,\Delta\dot{q} &= 0, \end{aligned}$$

where the identity $K(\theta_d - q_d) + Mg_0 = 0$ coming from (11) has been used. In the Laplace domain, we obtain

$$\begin{aligned} \big(Bs^2 + (D + F_\theta)s + K\big)\,\Delta\theta(s) - (Ds + K)\,\Delta q(s) &= \Delta\tau(s) \\ \big(Ms^2 + (D + F_q)s + K\big)\,\Delta q(s) - (Ds + K)\,\Delta\theta(s) &= 0. \end{aligned}$$

Thus, after some algebraic manipulation, the transfer function from $\Delta\tau$ to Δq is

$$P_q(s) = \frac{\Delta q(s)}{\Delta \tau(s)} = \frac{Ds + K}{s\,\mathrm{den}_3(s)}, \tag{17}$$

with the third-order polynomial in the denominator

$$\mathrm{den}_3(s) = BMs^3 + ((D{+}F_\theta)M + (D{+}F_q)B)\,s^2 + ((B{+}M)K + (F_\theta{+}F_q)D + F_\theta F_q)\,s + (F_\theta{+}F_q)K.$$

On the other hand, the transfer function from $\Delta\tau$ to $\Delta\theta$ is

$$P_\theta(s) = \frac{\Delta\theta(s)}{\Delta\tau(s)} = \frac{\Delta\theta(s)}{\Delta q(s)}\,P_q(s) = \frac{Ms^2 + (D + F_q)s + K}{s\,\mathrm{den}_3(s)}. \tag{18}$$

The transfer function $P_\theta(s)$ has a pole-zero excess (also called relative degree) equal to two. Being all physical coefficients positive, the two zeros of its numerator have negative real part and the three poles from $\mathrm{den}_3(s)$ (one certainly real) have all negative real parts —as can be shown from the Routh table this polynomial; finally, the fourth pole is at the origin. According to elementary feedback theory and using the properties of the root locus method, if one considers a proportional feedback of the form

$$\Delta\tau = -K_P \Delta\theta \qquad K_P > 0, \tag{19}$$

the four closed-loop poles will remain in the left-hand side of the complex plane *for all* positive values of the gain K_P. In particular, when increasing this gain, two poles converge to the open-loop zeros, while the other two approach the vertical asymptotes (whose center is located at a value $s_0 < 0$). A numerical example of such behavior is shown in Fig. 5.

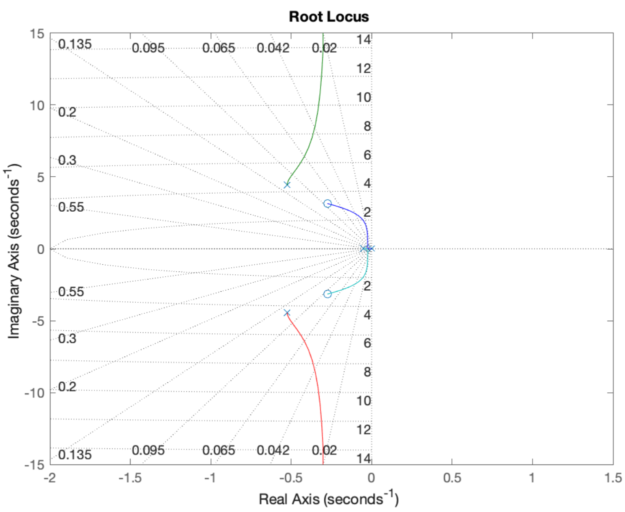


Figure 5: The root locus of the process (18) for varying $K_P > 0$.

As a result, the desired equilibrium is asymptotically (actually, exponentially) stabilized by control laws of the form

$$\tau = \tau_d + \Delta\tau = \tau_d - K_P\Delta\theta = -Mg_0 + K_P(\theta_d - \theta), \qquad \forall K_P > 0,$$

just like in eq. (10). Although an approach based on approximate linearization has usually only a local validity, in the present case the method was only used to remove a constant bias (and no other nonlinear terms). The analysis can then be considered of global validity, as already confirmed by the previous Lyapunov method.

The transfer function $P_q(s)$ in (17) also explains why it is preferable to close the proportional feedback loop (19) on the position variable θ of the first mass B (using the modified reference θ_d, as computed from q_d in (11), rather than closing a feedback loop $K_P(q_d - q)$ directly on the desired position of the second mass M. In fact, while the transfer function $P_q(s)$ shares most of the characteristics of $P_\theta(s)$, its pole-zero excess is instead equal to three (there is only one zero in the numerator). Again from elementary feedback theory and the properties of the root locus, such a feedback would lead to a stable closed-loop system only for a very limited positive range of K_P, going unstable for larger values. This is a consequence of the physical *non-colocation* between the control input τ and the output q to be controlled, because of the presence of elastic dynamics in the mechanical transmission between the two masses B and M.

Finally, assume that $D = 0$ in eqs. (7), (8) and that we would like to reproduce a trajectory $q_d(t)$ that is *four* times differentiable. Setting $q = q_d(t)$ in (8) and solving for θ gives

$$\theta_d(t) = q_d(t) + K^{-1}(M\,\ddot{q}_d(t) + F_q\,\dot{q}_d(t) - Mg_0).$$

Differentiating this once and twice provides

$$\dot{\theta}_d(t) = \dot{q}_d(t) + K^{-1}(M\,\dddot{q}_d(t) + F_q\,\ddot{q}_d(t)).$$

and

$$\ddot{\theta}_d(t) = \ddot{q}_d(t) + K^{-1}(M\,\ddddot{q}_d(t) + F_q\,\dddot{q}_d(t)).$$

By replacing these expressions for θ, $\dot{\theta}$ and $\ddot{\theta}$ in (7), we obtain the required inverse dynamics torque

$$\begin{aligned}
\tau_d(t) &= B\,\ddot{\theta}_d(t) + K(\theta_d(t) - q_d(t)) + F_\theta\,\dot{\theta}_d(t) \\
&= B\,\ddot{q}_d(t) + BK^{-1}(M\,\ddddot{q}_d(t) + F_q\,\dddot{q}_d(t)) + M\,\ddot{q}_d(t) + F_q\,\dot{q}_d(t) - Mg_0 \\
&\quad + F_\theta\,\dot{q}_d(t) + F_\theta K^{-1}(M\,\dddot{q}_d(t) + F_q\,\ddot{q}_d(t)) \\
&= BMK^{-1}\,\ddddot{q}_d(t) + (F_\theta M + F_q B)\,K^{-1}\,\dddot{q}_d(t) \\
&\quad + ((B+M) + F_\theta F_q)K^{-1})\,\ddot{q}_d(t) + (F_\theta + F_q)\,\dot{q}_d(t) - Mg_0,
\end{aligned}$$

which is eventually expressed only in terms of $q_d(t)$ and its first four time derivatives, as requested.

Exercise 4

The dynamic model of a 2R robot in the vertical plane is given by (see, e.g., the lecture slides)

$$\begin{aligned}
&\begin{pmatrix} a_1 + 2a_2\cos q_2 & a_3 + a_2\cos q_2 \\ a_3 + a_2\cos q_2 & a_3 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -a_2\sin q_2\left(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\right) \\ a_2\sin q_2\dot{q}_1^2 \end{pmatrix} \\
&\qquad + \begin{pmatrix} a_4\cos q_1 + a_5\cos(q_1+q_2) \\ a_5\cos(q_1+q_2) \end{pmatrix} = \begin{pmatrix} \tau \\ 0 \end{pmatrix},
\end{aligned} \tag{20}$$

for suitable dynamic coefficients a_i, $i = 1, \dots, 5$, and with a zero in the right-hand side of the second scalar equation due to the missing actuation (this underactuated robot is usually called Pendubot). The forced equilibrium conditions for this mechanical system are determined by setting $\tau = \bar{\tau}$ (constant) and $\dot{\boldsymbol{\theta}} = \ddot{\boldsymbol{\theta}} = \mathbf{0}$ in (20). This gives

$$\begin{aligned}
a_4\cos\bar{q}_1 + a_5\cos(\bar{q}_1 + \bar{q}_2) &= \bar{\tau} \\
a_5\cos(\bar{q}_1 + \bar{q}_2) &= 0,
\end{aligned} \tag{21}$$

implying

$$\bar{\tau} = a_4 \cos\bar{q}_1 = g_0(m_1 d_{c1} + m_2 l_1)\cos\bar{q}_1. \tag{22}$$

Therefore, we have two continuum families of equilibria, each parametrized by the value $\bar{q}_1$, with

$$\bar{q}_2 = -\bar{q}_1 \pm \frac{\pi}{2},$$

i.e., the second link is vertical upward or downward, and with the corresponding equilibrium torque $\bar{\tau}$ given by (22).

The global stabilization of any such equilibrium by a PD control law with gravity compensation/cancellation is made very hard by the fact that the robot is underactuated: no torque command can be delivered at joint 2. Therefore, in order to define a state feedback control law that *locally* asymptotically stabilizes one of these equilibria, we use the approximate linearization of (20) around $(\bar{\boldsymbol{q}}, \boldsymbol{0})$, with $\bar{\boldsymbol{q}} = (\bar{q}_1, \bar{q}_2)$ satisfying (21) and with a (small) control action $\Delta\tau$ around the equilibrium torque $\bar{\tau}$ given by (22). Substituting in (20)

$$\boldsymbol{q} = \bar{\boldsymbol{q}} + \Delta\boldsymbol{q} \qquad \dot{\boldsymbol{q}} = \Delta\dot{\boldsymbol{q}} \qquad \ddot{\boldsymbol{q}} = \Delta\ddot{\boldsymbol{q}} \qquad \tau = \bar{\tau} + \Delta\tau,$$

and neglecting second- and higher-order terms in the variations $\Delta(\cdot)$, we obtain

$$\begin{aligned} &\begin{pmatrix} a_1 + 2a_2\cos\bar{q}_2 & a_3 + a_2\cos\bar{q}_2 \\ a_3 + a_2\cos\bar{q}_2 & a_3 \end{pmatrix} \begin{pmatrix} \Delta\ddot{q}_1 \\ \Delta\ddot{q}_2 \end{pmatrix} \\ -&\begin{pmatrix} a_4\sin\bar{q}_1 + a_5\sin(\bar{q}_1+\bar{q}_2) & a_5\sin(\bar{q}_1+\bar{q}_2) \\ a_5\sin(\bar{q}_1+\bar{q}_2) & a_5\sin(\bar{q}_1+\bar{q}_2) \end{pmatrix} \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix} = \begin{pmatrix} \Delta\tau \\ 0 \end{pmatrix}, \end{aligned} \tag{23}$$

or in compact form

$$\bar{\boldsymbol{M}}\,\Delta\ddot{\boldsymbol{q}} + \bar{\boldsymbol{G}}\,\Delta\boldsymbol{q} = \begin{pmatrix} \Delta\tau \\ 0 \end{pmatrix}, \qquad \text{with} \quad \bar{\boldsymbol{M}} = \boldsymbol{M}(\bar{\boldsymbol{q}}) \qquad \bar{\boldsymbol{G}} = \boldsymbol{G}(\bar{\boldsymbol{q}}) = \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right|_{\boldsymbol{q}=\bar{\boldsymbol{q}}}.$$

The system can be put in state-space format by choosing, e.g.,

$$\Delta\boldsymbol{x} = \begin{pmatrix} \Delta\boldsymbol{x}_1 \\ \Delta\boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} \Delta\boldsymbol{q} \\ \bar{\boldsymbol{M}}\Delta\dot{\boldsymbol{q}} \end{pmatrix},$$

leading to

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}\,\Delta\boldsymbol{x} + \boldsymbol{b}\,\Delta\tau, \qquad \text{with} \quad \boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \bar{\boldsymbol{M}}^{-1} \\ -\bar{\boldsymbol{G}} & \boldsymbol{O} \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} \boldsymbol{0} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}.$$

At this stage, provided that the 4×4 controllability matrix

$$\mathcal{C} = \begin{pmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \boldsymbol{A}^2\boldsymbol{b} & \boldsymbol{A}^3\boldsymbol{b} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} & \bar{\boldsymbol{M}}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \boldsymbol{0} & -\boldsymbol{M}^1\bar{\boldsymbol{G}}\bar{\boldsymbol{M}}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \boldsymbol{0} & -\bar{\boldsymbol{G}}\bar{\boldsymbol{M}}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \boldsymbol{0} \end{pmatrix}$$

is nonsingular,[2] we can obtain a local asymptotic stabilization of the system at the chosen equilibrium state $\boldsymbol{x}_e = (\bar{\boldsymbol{q}}, \boldsymbol{0})$ (and actually also assign all the closed-loop eigenvalues as desired), by means of the full state feedback law

$$\Delta\tau = -\boldsymbol{K}\Delta\boldsymbol{x} = -\boldsymbol{K}_1\Delta\boldsymbol{q} - \boldsymbol{K}_2\bar{\boldsymbol{M}}\Delta\dot{\boldsymbol{q}} = -K_1\Delta q_1 - K_2\Delta q_2 - K_3\begin{pmatrix} 1 & 0 \end{pmatrix}\bar{\boldsymbol{M}}\Delta\dot{\boldsymbol{q}} - K_4\begin{pmatrix} 0 & 1 \end{pmatrix}\bar{\boldsymbol{M}}\Delta\dot{\boldsymbol{q}},$$

[2]The controllability condition is generically satisfied.

with $\boldsymbol{K}$ such that $\sigma(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{K}) \in \mathbb{C}^-$. As a result, the required complete control torque will be

$$\tau = \bar{\tau} + \Delta\tau = a_4 \cos \bar{q}_1 + \begin{pmatrix} K_1 & K_2 \end{pmatrix} \begin{pmatrix} \bar{q}_1 - q_1 \\ \bar{q}_2 - q_2 \end{pmatrix} - \begin{pmatrix} K_3 & K_4 \end{pmatrix} \bar{\boldsymbol{M}} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix},$$

in the form of a PD-type feedback with constant feedforward.

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