Robotics II

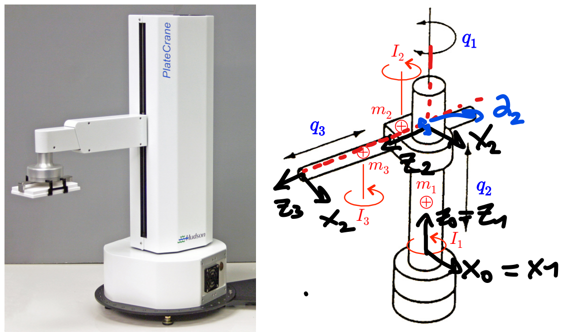
July 11, 2017

Exercise 1

For the RPP cylindrical robot in Fig. 1, using the generalized coordinates defined therein, provide the symbolic expression of each term of the dynamic model that appears in the control law

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}_d + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_d + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}), \qquad \boldsymbol{K}_P > 0, \quad \boldsymbol{K}_D > 0, \tag{1}$$

so that global asymptotic tracking of a desired joint trajectory $\boldsymbol{q}_d(t) \in C^2$ is guaranteed. Joint axis 3 has a DH offset $a_2 \neq 0$ from joint axis 2. Moreover, the center of mass of links 1 and 3 is placed on the joint axis having the same index, while the center of mass of link 2 is at a distance $r_2 > 0$ from joint axis 2.



α	a	d	q
0	0	0	q_1
$\frac{\pi}{2}$	0	q_2	0
0	$-a_2$	q_3	0

$${}^1\omega_1 = \begin{pmatrix} 0 \\ 0 \\ \dot q_1 \end{pmatrix}$$

$${}^0p_{c2} = \begin{pmatrix} r_2 S_1 \\ r_2 C_1 \\ q_2 \end{pmatrix}$$

$${}^0v_{c2} = \begin{pmatrix} r_2 C_1 \dot q_1 \\ -r_2 S_1 \dot q_1 \\ \dot q_2 \end{pmatrix}$$

$$\|v_{c2}\|^2 = r_2^2 \dot q_1^2 + \dot q_2^2$$

Figure 1: A RPP cylindrical robot with its generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$.

Exercise 2

Consider a Cartesian robot moving in a vertical place and having its end effector constrained to an ideal (rigid, frictionless) linear surface tilted by an acute angle $\alpha > 0$ from the $\boldsymbol{x}$-axis, as in Fig. 2. For this robot in constrained motion, provide the explicit symbolic expressions of

- the *reduced dynamics*, i.e., the differential equation relating the pseudo-acceleration $\dot{v} \in \mathbb{R}$ to the pseudo-velocity $v \in \mathbb{R}$, the configuration $\boldsymbol{q} \in \mathbb{R}^2$, and the input forces at the joints $\boldsymbol{u} \in \mathbb{R}^2$;
- the *multiplier* $\lambda \in \mathbb{R}$, i.e., the scalar reaction force that would act on the robot end effector when attempting to violate the geometric constraint.

Is it possible, by a suitable choice of $\boldsymbol{u}$, to realize a uniform motion with constant velocity $v = V$ on the surface, while inducing no reaction force, i.e., with $\lambda = 0$? If so, when and how?

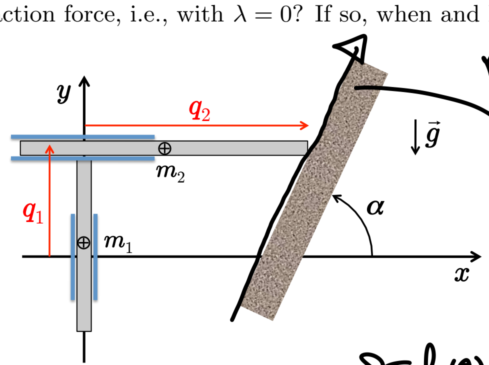
$$T_1 = \frac{1}{2} m_1 \dot q_1^2$$

$$T_2 = \frac{1}{2} m_2 (\dot q_2^2 + \dot q_1^2)$$

$$T = \frac{1}{2}\left(\dot q_1^2 (m_1 + m_2) + m_2 \dot q_2^2\right)$$

$$M = \begin{pmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$g = \begin{pmatrix} g_0 (m_1 + m_2) \\ 0 \end{pmatrix}$$



$$r = \begin{pmatrix} q_2 \\ q_1 \end{pmatrix}$$

r_x

$r_y = m x + q$, $m \to \tan(\alpha)$, $q \to \beta$

$\beta \cos(\alpha)$

$$q_1 = \tan(\alpha) q_2 + \beta$$

$$q_1 \cos(\alpha) = q_2 \sin(\alpha) + h_0$$

so $h(q) = -q_1 C_\alpha + q_2 S_\alpha + h_0 = 0$

Figure 2: A PP robot with the end effector in constrained motion on a linear surface.

$$\begin{pmatrix} A \\ D \end{pmatrix} = \begin{pmatrix} -C_\alpha & S_\alpha \\ S_\alpha & C_\alpha \end{pmatrix}$$

Exercise 3

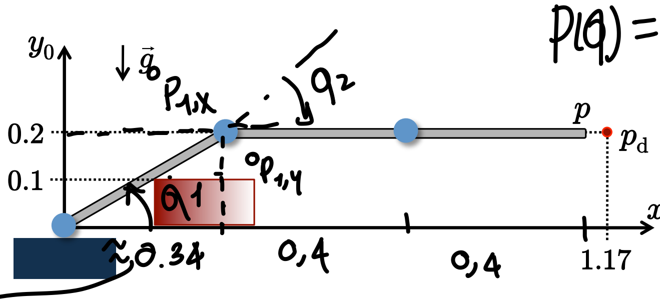
Consider a planar 3R robot, with equal link lengths $\ell = 0.4$ m and equal, uniformly distributed link masses $m = 2.5$ kg, that moves in a vertical plane. The generalized coordinates $\boldsymbol{q} \in \mathbb{R}^3$ are defined by a standard Denavit-Hartenberg convention. Under the action of the Cartesian control law (with gravity cancelation)

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\,\boldsymbol{K}_P\Big(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q})\Big) - \boldsymbol{K}_D\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}), \tag{2}$$

where $\boldsymbol{p}_d = \begin{pmatrix} 1.17 & 0.2 \end{pmatrix}^T$, $\boldsymbol{K}_P = 400 \cdot \boldsymbol{I}_{2\times 2}$, $\boldsymbol{K}_D = 40 \cdot \boldsymbol{I}_{2\times 2}$, and $\boldsymbol{p}(\boldsymbol{q})$ is the direct kinematics of the end-effector position, the robot has reached the equilibrium condition shown in Fig. 3, in which the first link is in contact with a rigid obstacle. In this steady state, determine the numerical values of

- the control torque $\boldsymbol{\tau} \in \mathbb{R}^3$ at the joints;
- the joint torque $\boldsymbol{\tau}_c \in \mathbb{R}^3$ at the joints due to the contact force $\boldsymbol{F}_c \in \mathbb{R}^2$ acting on the first link;
- the momentum-based residual $\boldsymbol{r} \in \mathbb{R}^3$ for collision detection/isolation, when $\boldsymbol{K}_I = 10 \cdot \boldsymbol{I}_{3\times 3}$;
- if possible, the components of the contact force $\boldsymbol{F}_c$ acting on the first link (expressed in frame RF_0 or in frame RF_1).

$$p(q) = \begin{pmatrix} \ell(c_1 + c_{12} + c_{123}) \\ \ell(s_1 + s_{12} + s_{123}) \end{pmatrix}$$



PITHAGORAS THEOREM

$$^0P_1 = \begin{pmatrix} 0.34 \\ 0.2 \end{pmatrix} = \begin{pmatrix} \ell c_1 \\ \ell s_1 \end{pmatrix} \Longrightarrow$$

$$q_{1e} = \mathrm{ATAN2}(0.34, 0.2) = \pi/6$$

Figure 3: A planar 3R in contact with a rigid obstacle.

$$^0P_2 = \begin{pmatrix} 0.74 \\ 0.2 \end{pmatrix} = \begin{pmatrix} \ell(c_1 + c_{12}) \\ \ell(s_1 + s_{12}) \end{pmatrix}$$

[210 minutes; open books but no computer or smartphone]

$$\frac{0.74}{\ell} - 0.86 = \cos(q_2 + \tfrac{\pi}{6})$$

$$\frac{0.2}{\ell} - 0.5 = \sin(q_2 + \tfrac{\pi}{6})$$

$$q_{2e} = \mathrm{ATAN2}\left(\frac{0.2}{\ell} - 0.5, \frac{0.74}{\ell} - 0.86\right) - \frac{\pi}{6}$$

$$q_{3e} = 0$$

We compute $J(q_e)$, then we can compute:

$$\tau = J(q_e)^T \boxed{K_p}\,\underbrace{(P_d - P(q_e))}_{\text{we have}} - \boxed{K_D}\,\underbrace{\boxed{\dot{q}}}_{\dot{q}=0} + \underbrace{g(q_e)}_{\text{easy to compute}}$$

PROBLEM GAVE US THEM

Solution

July 11, 2017

Exercise 1

Following a Lagrangian approach, we compute first the kinetic energy $T = T_1 + T_2 + T_3$. We have

$$\begin{array}{c} T_1 = \frac{1}{2} I_1\, \dot{q}_1^2 \qquad T_2 = \frac{1}{2} I_2\, \dot{q}_1^2 + \frac{1}{2} m_2 \left(r_2^2\, \dot{q}_1^2 + \dot{q}_2^2 \right) \\ T_3 = \frac{1}{2} I_3\, \dot{q}_1^2 + \frac{1}{2} m_3 \left(\left(a_2^2 + q_3^2 \right) \dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2 a_2 \dot{q}_1 \dot{q}_3 \right) \end{array} \quad \Rightarrow \quad T = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}},$$

with

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_1 + I_2 + I_3 + m_2 r_2^2 + m_3 a_2^2 + m_3 q_3^2 & 0 & m_3 a_2 \\ 0 & m_2 + m_3 & 0 \\ m_3 a_2 & 0 & m_3 \end{pmatrix}. \tag{3}$$

In order to guarantee global asymptotic trajectory tracking for the control law (1), the factorization $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}}$ of the Coriolis and centrifugal terms should be such that $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is a skew-symmetric matrix. This is automatically guaranteed if the components of the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are computed using the Christoffel's symbols, i.e.,

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial \boldsymbol{q}_i} \right), \qquad i = 1, 2, 3, \tag{4}$$

being $\boldsymbol{m}_i$ the ith column of the inertia matrix $\boldsymbol{M}$. Define $I_0 = I_1 + I_2 + I_3 + m_2 r_2^2 + m_3 a_2^2$, so that $m_{11}(\boldsymbol{q}) = I_0 + m_3 q_3^2$. I_0 is one of the three dynamic coefficients in the complete robot model, the other two being m_3 (a_2 is a kinematic parameter supposed to be known) and $(m_2 + m_3)$. Using (4), we obtain

$$\begin{aligned} \boldsymbol{C}_1(\boldsymbol{q}) &= \begin{pmatrix} 0 & 0 & m_3 q_3 \\ 0 & 0 & 0 \\ m_3 q_3 & 0 & 0 \end{pmatrix} & \Rightarrow \quad & c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 2 m_3 q_3\, \dot{q}_1 \dot{q}_3, \\ \boldsymbol{C}_2(\boldsymbol{q}) &= \boldsymbol{0} & \Rightarrow \quad & c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \\ \boldsymbol{C}_3(\boldsymbol{q}) &= \begin{pmatrix} -m_3 q_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \Rightarrow \quad & c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 q_3\, \dot{q}_1^2. \end{aligned}$$

A factorization that satisfies the skew-symmetric property is then given by

$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_3(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} m_3 q_3\, \dot{q}_3 & 0 & m_3 q_3\, \dot{q}_1 \\ 0 & 0 & 0 \\ -m_3 q_3\, \dot{q}_1 & 0 & 0 \end{pmatrix}. \tag{5}$$

For the potential energy due to gravity, $U_g = U_1 + U_2 + U_3$, we have (up to a constant)

$$U_1 = 0, \qquad U_2 = m_2 g_0\, q_2, \qquad U_3 = m_3 g_0\, q_2,$$

and thus

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} 0 \\ (m_2 + m_3) g_0 \\ 0 \end{pmatrix}. \tag{6}$$

Summarizing, the terms from the robot dynamics used in the control law (1) are given by (3), (5), and (6).

Exercise 2

The dynamic model of the PP robot in free motion is

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{u} \qquad \Longleftrightarrow \qquad \begin{aligned}(m_1+m_2)\ddot{q}_1 + (m_1+m_2)g_0 &= u_1\\ m_2\ddot{q}_2 &= u_2.\end{aligned}$$

The linear constraint on the end-effector position is expressed by its implicit form[1] in terms of the angle α as (note the exchanged order of joint variables!)

$$\boldsymbol{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} q_2 \\ q_1 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \qquad k(\boldsymbol{r}) = r_x \sin\alpha - r_y \cos\alpha + k_0 = 0$$

$$\Rightarrow \quad h(\boldsymbol{q}) = k\left(\boldsymbol{f}(\boldsymbol{q})\right) = -q_1\cos\alpha + q_2 \sin\alpha + k_0 = 0.$$

Accordingly, the constraint Jacobian $\boldsymbol{A}$ is a constant, 1×2 matrix (always of full rank)

$$\boldsymbol{A} = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\cos\alpha & \sin\alpha \end{pmatrix}$$

The kinematic constraint is

$$\boldsymbol{A}\,\dot{\boldsymbol{q}} = 0 \qquad \Rightarrow \qquad -\dot{q}_1\cos\alpha + \dot{q}_2\sin\alpha = 0.$$

The dynamic model of the PP robot constrained to the surface becomes

$$\begin{aligned}&\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{u} + \boldsymbol{A}^T\lambda\\ &\text{s.t.} \quad h(\boldsymbol{q}) = 0\end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned}(m_1+m_2)\ddot{q}_1 + (m_1+m_2)g_0 &= u_1 - \lambda\cos\alpha\\ m_2\ddot{q}_2 &= u_2 + \lambda\sin\alpha\\ \text{s.t.} \ -q_1\cos\alpha + q_2\sin\alpha + k_0 &= 0.\end{aligned}$$

To proceed with the reduced dynamics approach, we can border $\boldsymbol{A}$ with a row matrix $\boldsymbol{D}$, so as to build a (globally) nonsingular matrix:

$$\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix} = \begin{pmatrix} -\cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix}^{-1} = \begin{pmatrix} -\cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{F} \end{pmatrix}.$$

The pseudo-velocity v that automatically satisfies the differential constraint is given by

$$v = \boldsymbol{D}\dot{\boldsymbol{q}} = \dot{q}_1\sin\alpha + \dot{q}_2\cos\alpha,$$

whereas the admissible joint velocities and accelerations are given by

$$\dot{\boldsymbol{q}} = \boldsymbol{F}v = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} v, \qquad \ddot{\boldsymbol{q}} = \boldsymbol{F}\dot{v} = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}\dot{v}.$$

Since

$$\begin{aligned}\boldsymbol{F}^T\boldsymbol{M}\boldsymbol{F} &= \begin{pmatrix} \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} = m_1\sin^2\alpha + m_2\\ \boldsymbol{F}^T\boldsymbol{g} &= \begin{pmatrix} \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} (m_1+m_2)g_0 \\ 0 \end{pmatrix} = (m_1+m_2)g_0\sin\alpha\end{aligned}$$

the reduced dynamic model is given by

$$\left(\boldsymbol{F}^T\boldsymbol{M}\boldsymbol{F}\right)\dot{v} + \boldsymbol{F}^T\boldsymbol{g} = \boldsymbol{F}^T\boldsymbol{u} \qquad \Longleftrightarrow \qquad \left(m_1\sin^2\alpha + m_2\right)\dot{v} + (m_1+m_2)g_0\sin\alpha = u_1\sin\alpha + u_2\cos\alpha, \quad (7)$$

[1] The geometric expression of a line in a plane (r_x, r_y) having an angular coefficient $m = \tan\alpha$, for $\alpha \in (0, \pi/2)$, is $r_y = (\tan\alpha)\, r_x + \beta$. Multiplying by $\cos\alpha \neq 0$, one easily obtains the expression $k(\boldsymbol{r}) = 0$, where $k_0 = \beta\cos\alpha$.

while, being $\boldsymbol{E}^T = \begin{pmatrix} -\cos\alpha & \sin\alpha \end{pmatrix}$, the force multiplier is given by

$$\lambda = \boldsymbol{E}^T \left(\boldsymbol{M}\boldsymbol{F}\,\dot{v} + \boldsymbol{g} - \boldsymbol{u}\right) \qquad \Longleftrightarrow \qquad \lambda = -m_1 \cos\alpha \sin\alpha\, \dot{v} - (m_1 + m_2) g_0 \cos\alpha + u_1 \cos\alpha - u_2 \sin\alpha. \quad (8)$$

Note that for $\alpha = \pi/2$, equations (7–8) collapse to

$$(m_1 + m_2)\,\dot{v} + (m_1 + m_2) g_0 = u_1, \qquad \lambda = -u_2,$$

showing in particular that each force applied by the second actuator will result in an equal reaction force in the opposite direction.

Finally, it is possible to maintain a uniform motion with constant velocity $v = V$ on the linear surface without having a reaction force $\lambda = 0$ at all times, under following (necessary and sufficient) conditions:

- the robot is in an initial configuration $\boldsymbol{q}_0$ satisfying $h(\boldsymbol{q}_0) = 0$, i.e., the end effector is already on the linear surface;
- the robot has an initial velocity $\dot{\boldsymbol{q}}_0$ satisfying $\boldsymbol{A}\dot{\boldsymbol{q}}_0 = 0$ and such that $\|\dot{\boldsymbol{r}}_0\| = V$ or

$$\dot{\boldsymbol{q}}_0 = V \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{r}}_0 = V \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

  i.e, the end-effector velocity is tangential to the surface and has the right module;
- the input $\boldsymbol{u}$ is computed by solving eqs. (7–8) for $\dot{v} = 0$ and $\lambda = 0$, or

$$(m_1 + m_2) g_0 \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} = \begin{pmatrix} \sin\alpha & \cos\alpha \\ \cos\alpha & -\sin\alpha \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (m_1 + m_2) g_0 \\ 0 \end{pmatrix};$$

  namely, it is sufficient that the first actuator sustains the total gravity load of the robot.

Exercise 3

The direct kinematics for the end-effector position of the planar 3R robot with equal link lengths ℓ and standard DH angles is

$$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q}) = \ell \begin{pmatrix} \cos q_1 + \cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3) \\ \sin q_1 + \sin(q_1 + q_2) + \sin(q_1 + q_2 + q_3) \end{pmatrix} = \ell \begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{pmatrix},$$

where a shorthand notation has been introduced. The associated analytic Jacobian matrix is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \ell \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix}.$$

The equilibrium configuration of the robot in Fig. 2 is easily recognized to be $\boldsymbol{q} = \begin{pmatrix} \pi/6 & -\pi/6 & 0 \end{pmatrix}^T$. Using this configuration and the data of the problem, we obtain numerical values for the following terms in the control law (2):

$$\boldsymbol{p} = \begin{pmatrix} 1.1464 \\ 0.2 \end{pmatrix}, \qquad \boldsymbol{F}_e = \boldsymbol{K}_P\,(\boldsymbol{p}_d - \boldsymbol{p}) = \begin{pmatrix} 9.4359 \\ 0 \end{pmatrix}, \qquad \boldsymbol{J}^T = \begin{pmatrix} -0.2 & 1.1464 \\ 0 & 0.8 \\ 0 & 0.4 \end{pmatrix}.$$

To compute the gravity vector, we start from the potential energy $U_g(\boldsymbol{q}) = U_1 + U_2 + U_3$. Having each link the same uniformly distributed mass m, we have

$$U_1 = m\,g_0 \ell\, \frac{1}{2}\, s_1, \qquad U_2 = m\,g_0 \ell \left(s_1 + \frac{1}{2}\, s_{12}\right), \qquad U_3 = m\,g_0 \ell \left(s_1 + s_{12} + \frac{1}{2}\, s_{123}\right),$$

and thus

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = m\, g_0 \ell \begin{pmatrix} 2.5\, c_1 + 1.5\, c_{12} + 0.5\, c_{123} \\ 1.5\, c_{12} + 0.5\, c_{123} \\ 0.5\, c_{123} \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{g} = \begin{pmatrix} 40.8593 \\ 19.6200 \\ 4.9050 \end{pmatrix}.$$

Therefore, being $\dot{\boldsymbol{q}} = \boldsymbol{0}$, the control torque at steady state is

$$\boldsymbol{\tau} = \boldsymbol{J}^T \boldsymbol{F}_e + \boldsymbol{g} = \begin{pmatrix} 38.9721 \\ 19.6200 \\ 4.9050 \end{pmatrix} \text{ [Nm]}.$$

The contact with the obstacle occurs at the midpoint of link 1, with

$$\boldsymbol{p}_c = \frac{\ell}{2}\begin{pmatrix} c_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 0.1732 \\ 0.1 \end{pmatrix}, \qquad \boldsymbol{J}_c^T = \frac{\ell}{2}\begin{pmatrix} -s_1 & c_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -0.1 & 0.1732 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The unknown contact force $\boldsymbol{F}_c$ generates a joint torque $\boldsymbol{\tau}_c = \boldsymbol{J}_c^T \boldsymbol{F}_c$, which will have a non-zero component τ_{c1} only at joint 1, the only joint preceding the contact point. Moreover, because gravity is being compensated by control (and thus removed from this analysis) and the robot is in a static equilibrium, from the balance of joint torques it follows that

$$\boldsymbol{\tau}_c + \boldsymbol{\tau}_e = \boldsymbol{0} \qquad \Longleftrightarrow \qquad \boldsymbol{J}_c^T \boldsymbol{F}_c + \boldsymbol{J}^T \boldsymbol{F}_e = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{\tau}_c = -\boldsymbol{J}^T \boldsymbol{F}_e = \begin{pmatrix} 1.8872 \\ 0 \\ 0 \end{pmatrix} \text{ [Nm]}.$$

This is also the value reached by the momentum-based residual at steady state, i.e.,

$$\dot{\boldsymbol{r}} = \boldsymbol{K}_I\left(\boldsymbol{\tau}_c - \boldsymbol{r}\right) \qquad \Rightarrow \qquad \boldsymbol{r}_\infty = \lim_{t\to\infty} \boldsymbol{r}(t) = \boldsymbol{\tau}_c = \begin{pmatrix} 1.8872 \\ 0 \\ 0 \end{pmatrix} \text{ [Nm]}.$$

Note that the internal structure of $\boldsymbol{r}$ supports its *isolation* property: in fact, the components of $\boldsymbol{r}$ associated to joints that are beyond the contacting link are always zero. This is true not only at steady state, but also in dynamic conditions, when the robot is moving and may hit an obstacle.

At this stage, knowing exactly the contact point on link 1, we can recover at least the component ${}^nF_c$ of the contact force $\boldsymbol{F}_c$ that is normal to the first link. Instead, the internal force component ${}^tF_c$ that is aligned with link 1 produces no torque at joint 1, and thus cannot be reconstructed from τ_{c1} only. Expressing the contact force in the frame RF_1 rotated with link 1, we have

$${}^1\boldsymbol{F}_c = \begin{pmatrix} {}^tF_c \\ {}^nF_c \end{pmatrix}, \qquad \tau_{c1} = {}^nF_c\,\frac{\ell}{2} \qquad \Rightarrow \qquad {}^nF_c = \frac{2\tau_{c1}}{\ell} = 9.4359 \text{ [N]}.$$

This partial result can be obtained also from pseudo-inversion of the static relation $\boldsymbol{\tau}_c = \boldsymbol{J}_c^T \boldsymbol{F}_c$. Note first that the contact Jacobian in frame 1 is

$${}^1\boldsymbol{J}_c(\boldsymbol{q}) = \mathbf{Rot}^T(q_1)\,\boldsymbol{J}_c(\boldsymbol{q}) = \begin{pmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{pmatrix}\frac{\ell}{2}\begin{pmatrix} -s_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \ell/2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0 \end{pmatrix}.$$

From

$$\boldsymbol{\tau}_c = \boldsymbol{J}_c^T \boldsymbol{F}_c = \left({}^1\boldsymbol{J}_c\right)^T \cdot {}^1\boldsymbol{F}_c$$

we obtain

$${}^1\boldsymbol{F}_c = \left({}^1\boldsymbol{J}_c^T\right)^\# \boldsymbol{\tau}_c = \begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1.8872 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 9.4359 \end{pmatrix} = \begin{pmatrix} {}^tF_c \\ {}^nF_c \end{pmatrix}.$$

Since pseudo-inversion yields always a minimum norm solution, the obtained tangential component of the contact force is ${}^tF_c = 0$, i.e., is automatically set to zero. Finally, the contact force expressed in the base reference frame RF_0 is computed as

$$\boldsymbol{F}_c = \mathbf{Rot}(q_1)\,{}^1\boldsymbol{F}_c = \begin{pmatrix} -4.7180 \\ 8.1718 \end{pmatrix} \text{ [N]}.$$

Indeed, $\|\boldsymbol{F}_c\| = \left\|{}^1\boldsymbol{F}_c\right\| = 9.4359$.

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