Robotics II

January 9, 2013

Exercise 1

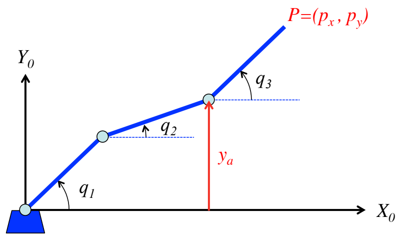

Figure 1: A $3R$ planar robot with unitary link lengths and two sets of task variables

Consider the $3R$ planar robot of Fig. 1, having links of unitary length and with the generalized coordinates defined therein. This robot is redundant for the task of positioning its end-effector at $\boldsymbol{p} = (p_x, p_y)$, as well as for the task of imposing a value to the second link end-point height y_a.

a) For each *separate* task, define the associated task Jacobian and its singularities.

b) Characterize the so-called *algorithmic* singularities (configurations where each task can be executed separately, but not both tasks simultaneously).

c) For the simultaneous execution of both tasks, provide the expression of an inverse differential kinematic solution at the velocity level, based on a *task-priority* strategy that assigns higher priority to the end-effector position task.

Exercise 2

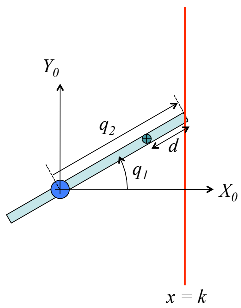

Figure 2: A RP robot moving on a horizontal plane with its end-effector constrained on a line

The end-effector of the RP robot in Fig. 2 is constrained to move on the Cartesian line $x = k$, with $k > 0$. For this operative condition, derive the expression of the *constrained* robot dynamics (in this case, two second-order differential equations, with a dynamically consistent projection matrix acting on forces/torques so as to automatically satisfy the motion constraint in any admissible robot state).

[210 minutes; open books]

Solutions

January 9, 2013

Exercise 1

Being the generalized coordinates q_i $(i = 1, 2, 3)$ the absolute angles of the links w.r.t. the $\boldsymbol{x}_0$ axis, the end-effector position is expressed as

$$\boldsymbol{p} = \begin{pmatrix} \cos q_1 + \cos q_2 + \cos q_3 \\ \sin q_1 + \sin q_2 + \sin q_3 \end{pmatrix} = \boldsymbol{f}_1(\boldsymbol{q})$$

The associated task Jacobian is

$$\boldsymbol{J}_1(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_1}{\partial \boldsymbol{q}} = \begin{pmatrix} -\sin q_1 & -\sin q_2 & -\sin q_3 \\ \cos q_1 & \cos q_2 & \cos q_3 \end{pmatrix}$$

and is singular if and only if

$$\sin(q_2 - q_1) = \sin(q_3 - q_2) = 0, \qquad (\Rightarrow\ \sin(q_3 - q_1) = 0) \tag{1}$$

or, in terms of Denavit-Hartenberg relative link angles $\theta_i = q_i - q_{i-1}$ (for $i = 2.3$), when $\sin\theta_2 = \sin\theta_3 = 0$. This occurs only when all three links are folded or stretched along a common radial line originating at the robot base.

The height y_a of the end-point of the second link and its associated task Jacobian are given by

$$y_a = \sin q_1 + \sin q_2 = f_2(\boldsymbol{q}) \quad \Rightarrow \quad \boldsymbol{J}_2(\boldsymbol{q}) = \frac{\partial f_2}{\partial \boldsymbol{q}} = \begin{pmatrix} \cos q_1 & \cos q_2 & 0 \end{pmatrix}.$$

This Jacobian is singular if and only if

$$\cos q_1 = \cos q_2 = 0, \tag{2}$$

namely when the first two links are either folded or stretched *and* the end-point of the second link is on the $\boldsymbol{y}_0$ axis.

When considering the two tasks together, the *Extended* Jacobian is square

$$\boldsymbol{J}_E(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_1(\boldsymbol{q}) \\ \boldsymbol{J}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -\sin q_1 & -\sin q_2 & -\sin q_3 \\ \cos q_1 & \cos q_2 & \cos q_3 \\ \cos q_1 & \cos q_2 & 0 \end{pmatrix}.$$

Algorithmic singularities will occur when both $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are full (row) rank, but

$$\det \boldsymbol{J}_E = -\cos q_3 \cdot \sin(q_2 - q_1) = 0. \tag{3}$$

Comparing eqs. (1–2) with (3), this happens when

- the third link is vertical ($\cos q_3 = 0$), while the first two are not; or,
- the first two links are aligned ($\sin(q_2 - q_1) = 0$) but not vertical, and the third link is not aligned with the first two.

Indeed, the above are only particular conditions for singularity of the Extended Jacobian. In fact, $\boldsymbol{J}_E$ is not invertible as soon as the third link is vertical and/or the first two links are aligned, no matter what is the situation of the other links.

Let $\boldsymbol{v}_d \in \mathbb{R}^2$ be a desired velocity for the robot end-effector and $\dot{y}_{a,d}$ a desired height variation rate for the end-point of the second link. An inverse solution of the form

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_E^{-1}(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{v}_d \\ \dot{y}_{a,d} \end{pmatrix}$$

will blow out as soon as a singularity occurs for $\boldsymbol{J}_E$. A task-priority solution, with the first task (of dimension $m_1 = 2$) of higher priority than the second one (of dimension $m_2 = 1$), is given by

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_1^{\#}(\boldsymbol{q})\, \boldsymbol{v}_d + \left(\boldsymbol{J}_2(\boldsymbol{q}) \left(\boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q}) \boldsymbol{J}_1(\boldsymbol{q}) \right) \right)^{\#} \left(\dot{y}_{a,d} - \boldsymbol{J}_2(\boldsymbol{q}) \boldsymbol{J}_1^{\#}(\boldsymbol{q})\, \boldsymbol{v}_d \right). \tag{4}$$

This will guarantee perfect execution of the first task even when $\boldsymbol{J}_E$ is singular (i.e., eq. (3) holds), provided that eq. (1) is *not* satisfied (in particular, in algorithmic singularities, where eq. (2) is *not* satisfied too).

Using the properties of projection matrices (symmetry and idempotency), and being the matrix $\boldsymbol{J}_2(\boldsymbol{I} - \boldsymbol{J}_1^{\#}\boldsymbol{J}_1)$ a row vector in our case, the solution (4) can also be rewritten as

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_1^{\#}(\boldsymbol{q})\, \boldsymbol{v}_d + \alpha \left(\boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q}) \boldsymbol{J}_1(\boldsymbol{q}) \right) \boldsymbol{J}_2^T(\boldsymbol{q}),$$

with the scalar

$$\alpha = \alpha(\boldsymbol{q}, \boldsymbol{v}_d, \dot{y}_{a,d}) = \frac{\dot{y}_{a,d} - \boldsymbol{J}_2(\boldsymbol{q}) \boldsymbol{J}_1^{\#}(\boldsymbol{q})\, \boldsymbol{v}_d}{\boldsymbol{J}_2(\boldsymbol{q}) \left(\boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q}) \boldsymbol{J}_1(\boldsymbol{q}) \right) \boldsymbol{J}_2^T(\boldsymbol{q})}.$$

Exercise 2

Following the Lagrangian approach, with multipliers $\boldsymbol{\lambda}$ used to weigh the holonomic constraints $\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}$, the dynamic equations (in the absence of gravity) take the form

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u} + \boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{\lambda} \qquad s.t. \quad \boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0},$$

with $\boldsymbol{A}(\boldsymbol{q}) = \partial \boldsymbol{h}(\boldsymbol{q})/\partial \boldsymbol{q}$. By further elaboration, one can eliminate the multipliers (the forces that arise when attempting to violate the constraints) and obtain the so-called *constrained* robot dynamics in the form

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \left(\boldsymbol{I} - \boldsymbol{A}^T(\boldsymbol{q}) \left(\boldsymbol{A}_B^{\#}(\boldsymbol{q}) \right)^T \right) (\boldsymbol{u} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})) - \boldsymbol{B}(\boldsymbol{q}) \boldsymbol{A}_B^{\#}(\boldsymbol{q}) \dot{\boldsymbol{A}}(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

where

$$\boldsymbol{A}_B^{\#}(\boldsymbol{q}) = \boldsymbol{B}^{-1}(\boldsymbol{q}) \boldsymbol{A}^T(\boldsymbol{q}) \left(\boldsymbol{A}(\boldsymbol{q}) \boldsymbol{B}^{-1}(\boldsymbol{q}) \boldsymbol{A}^T(\boldsymbol{q}) \right)^{-1}$$

is the (dynamically consistent) pseudoinverse of $\boldsymbol{A}$, weighted by the robot inertia matrix.

We need thus to provide the robot inertia matrix $\boldsymbol{B}$, the Coriolis and centrifugal vector $\boldsymbol{c}$, the matrix $\boldsymbol{A}$ and its time derivative $\dot{\boldsymbol{A}}$. The kinetic energy[1] is

$$T = T_1 + T_2 = \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2} \left(I_2 \dot{q}_1^2 + m_2 \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} \right).$$

[1] For simplicity, it is assumed that the first link has its center of mass on the axis of the first joint. Otherwise, if the center of mass is at a distance d_{c1}, simply replace I_1 by $I_1 + m_1 d_{c1}^2$ in the following.

Since

$$\boldsymbol{p}_{c2} = \begin{pmatrix} (q_2 - d)\cos q_1 \\ (q_2 - d)\sin q_1 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -(q_2 - d)\sin q_1 \dot{q}_1 + \dot{q}_2 \cos q_1 \\ (q_2 - d)\cos q_1 \dot{q}_1 + \dot{q}_2 \sin q_1 \end{pmatrix},$$

it follows

$$T = \frac{1}{2}\left(I_1 + I_2 + m_2(q_2 - d)^2\right)\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \begin{pmatrix} I_1 + I_2 + m_2(q_2 - d)^2 & 0 \\ 0 & m_2 \end{pmatrix} \dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

From the inertia matrix, using the Christoffel symbols, we obtain

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2m_2(q_2 - d)\dot{q}_1\dot{q}_2 \\ -m_2(q_2 - d)\dot{q}_1^2 \end{pmatrix}.$$

The (scalar) Cartesian constraint on the end-effector is

$$h(\boldsymbol{q}) = q_2 \cos q_1 - k = 0.$$

Thus,

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \end{pmatrix}$$

and

$$\dot{\boldsymbol{A}}(\boldsymbol{q}) = \begin{pmatrix} -\dot{q}_2 \sin q_1 - q_2 \cos q_1 \dot{q}_1 & -\sin q_1 \dot{q}_1 \end{pmatrix}.$$

Since q_2 is never allowed to go to zero (by the constraint $x = k > 0$ on the end-effector), matrix $\boldsymbol{A}$ has always full rank and all expressions in the constrained dynamics hold without singularities. For instance, the dynamically consistent weighted pseudoinverse takes the final expression

$$\boldsymbol{A}_B^{\#}(\boldsymbol{q}) = \frac{m_2(I_1 + I_2 + m_2(q_2 - d)^2)}{I_1 + I_2 + m_2 q_2^2 + m_2 d(d - 2q_2)\cos^2 q_1} \begin{pmatrix} -\dfrac{q_2 \sin q_1}{I_1 + I_2 + m_2(q_2 - d)^2} \\ \dfrac{\cos q_1}{m_2} \end{pmatrix}.$$

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