Robotics 2

Midterm test in classroom – April 29, 2019

Exercise 1

Determine which of the following four 2×2 matrices can be, under the stated conditions, the inertia matrix associated to a real 2-dof robot with coordinates q_1 and q_2, and which not (and why):

DH COORDINATES

MUST NOT BE q_1 DEPEND.

$$M_A = \begin{pmatrix} a_1 + a_2 q_1^2 & a_2 \\ a_2 & a_2 \end{pmatrix}, \qquad a_1 > a_2 > 0;$$

$\det = a_2(a_2 q_1^2 + a_1 - a_2) > 0$ ALWAYS IF $a_1 > a_2 > 0$

ABS. COORDINATES

$$M_B = \begin{pmatrix} a_1 & a_3 \cos(q_2 - q_1) \\ a_3 \cos(\boxed{q_2} - \boxed{q_1}) & a_2 \end{pmatrix}, \qquad a_1 > 0, \quad a_1 a_2 > a_3^2 > 0;$$

$\rightarrow$ THIS IS CORRECT

$\det = a_1 a_2 - a_3^2 c_{q_2}^2 \geq a_1 a_2 - a_3^2 > 0$

since $a_1 a_2 > a_3^2$

$$M_C = \begin{pmatrix} a_1 + 2a_2 \cos q_2 & a_1 + a_2 \cos q_2 \\ a_1 + a_2 \cos q_2 & a_1 \end{pmatrix}, \qquad a_1 > 2a_2 > 0;$$

>0

BUT $\det \leq 0$ ALWAYS

$$M_D = \begin{pmatrix} m_1 + m_2 & -0.5 m_2 \\ -0.5 m_2 & m_2 \end{pmatrix}, \qquad m_1 > 0, \quad m_2 > 0.$$

$m_1 + m_2 > 0$ $\det > 0$ $\rightarrow$ THIS IS CORRECT

For each case that is feasible, sketch the possible structure of the associated robot.

Exercise 2

Consider the 3-dof robot in Fig. 1, moving on a horizontal plane.

- Using the generalized coordinates $\boldsymbol{q} \in \mathbb{R}^3$ and the dynamic parameters defined in Fig. 1, determine the dynamic model of this robot using a Lagrangian formulation.
- Assume that the kinematic parameters are known. Provide a linear parameterization of the dynamic model in the form

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{u},$$

being $\boldsymbol{u} \in \mathbb{R}^3$ the generalized force provided by the motors at the joints, such that the vector of dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^p$ has the minimum possible dimension p.

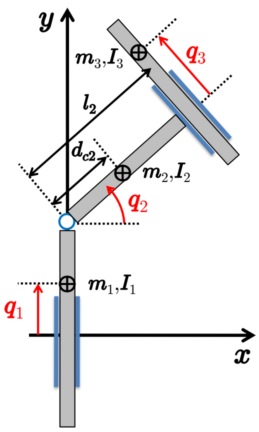

Figure 1: A 3-dof planar PRP robot, with its associated coordinates $\boldsymbol{q}$ and dynamic data.

Exercise 3

$\vec{g_0} = \begin{pmatrix} g_0 \\ 0 \end{pmatrix}$

The nR planar robot in Fig. 2 moves in the vertical plane under gravity. Each link has length $l_i > 0$ and has its center of mass on the kinematic axis at a distance $d_{ci} \gtreqless 0$ from the driving joint. Using the absolute coordinates $\boldsymbol{q}$ shown in Fig. 2, determine: *i)* the generic expressions of the components of the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ in the robot dynamic model; *ii)* all equilibrium configurations of the robot (i.e., all $\boldsymbol{q}_e \in \mathbb{R}^n$ such that $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$); and, *iii)* the generic conditions on the center of mass of each link such that the gravity vector vanishes identically (i.e., $\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}, \forall \boldsymbol{q}$).

$U = \sum_{i=1}^{n} -m_i g_0 P_{ci,x}$

$P_{ci,x} = \sum_{j=1}^{i-1} l_j \cdot \cos(q_j) + d_{ci} \cos(q_i)$

$\frac{\partial U}{\partial q} = \frac{\partial}{\partial q} \sum_{i=1}^{n} -m_i g_0 P_{ci,x}$

$= \sum_{i=1}^{n} -m_i g_0 \frac{\partial}{\partial q} P_{ci,x}$

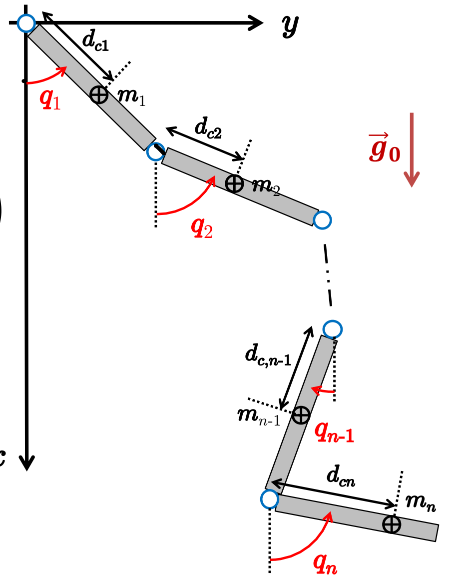

Figure 2: A nR planar robot, with its associated absolute coordinates $\boldsymbol{q}$

Exercise 4

a) A 2-dof robot has the following inertia matrix:

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 & a_2 \sin q_2 \\ a_2 \sin q_2 & a_3 \end{pmatrix}. \tag{1}$$

Find two matrices $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ that factorize the centrifugal/Coriolis terms in $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ (i.e., with $\boldsymbol{S}_i(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, for $i = 1, 2$), such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}_1$ is skew-symmetric and $\dot{\boldsymbol{M}} - 2\boldsymbol{S}_2$ is not.

b) Consider a robot with $n = 3$ joints, inertia matrix $\boldsymbol{M}(\boldsymbol{q}) > 0$, and centrifugal/Coriolis terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$, where the 3×3 factorizing matrix $\boldsymbol{S}$ is obtained through the Christoffel symbols. Show that one can always find another factorizing matrix $\boldsymbol{S}' \neq \boldsymbol{S}$ that satisfies $\boldsymbol{S}'(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and leads to a skew-symmetric matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}'$.

Exercise 5

For a 2R planar robot with links of length l_1 and l_2 and inertia matrix $J^{\#}_{M} = M^{-1} J^{T} (J M^{-1} J^{T})^{-1}$

$$M(q) = \begin{pmatrix} a_1 + 2a_2 \cos q_2 & a_3 + a_2 \cos q_2 \\ a_3 + a_2 \cos q_2 & a_3 \end{pmatrix}, \tag{2}$$

the task requires the end-effector to be moved so as to change its distance $\rho = \rho(q)$ from the base according to a desired smooth timing law $\rho_d(t)$. Define the instantaneous joint velocity $\dot{q}$ that realizes the task while minimizing the robot kinetic energy, and provide the symbolic expression of all the terms needed for its evaluation. Compute then the numerical value of $\dot{q}$ with the data:

$$q_1 = 0, \quad q_2 = \frac{\pi}{2}, \quad l_1 = l_2 = 1 \text{ [m]}, \qquad a_1 = 10, \quad a_2 = 2.5, \quad a_3 = \frac{5}{3}, \qquad \dot{\rho}_d = 0.5 \text{ [m/s]}. \tag{3}$$

Exercise 6

A 3R planar robot with links of unitary length is at rest in the Denavit-Hartenberg configuration $q = \begin{pmatrix} \pi/6 & \pi/6 & \pi/6 \end{pmatrix}$ [rad] and should instantaneously accelerate its end effector at $\ddot{p} = \begin{pmatrix} 4 & 2 \end{pmatrix}$ [m/s^2]. Find, if possible, a joint acceleration $\ddot{q} \in \mathbb{R}^3$ with the least possible norm that perfectly realizes the task under the bounds

$$|\ddot{q}_1| \leq 2.8 \text{ [rad/s}^2\text{]}, \qquad |\ddot{q}_2| \leq 3.6 \text{ [rad/s}^2\text{]}, \qquad |\ddot{q}_3| \leq 4 \text{ [rad/s}^2\text{]}. \tag{4}$$

[210 minutes (3.5 hours); open books, computer, but no internet and no smartphone]

Solution

April 29, 2019

Exercise 1

A) Matrix $\boldsymbol{M}_A$ is not a robot inertia matrix because it is a function of the first coordinate q_1 at the robot base. This can never be the case: the definition of q_1 is in fact arbitrary, as is the choice of a base frame for the robot. The inertia matrix is instead an intrinsic property of the manipulator structure.

B) $\boldsymbol{M}_B$ is the inertia matrix of a 2-dof robot with a parallelogram structure (see left of Fig. 3, which is taken from the lecture slides), in which we have $a_1 = I_{c1,zz} + m_1 l_{c1}^2 + I_{c3,zz} + m_3 l_{c3}^2 + m_4 l_1^2$, $a_2 = I_{c2,zz} + m_2 l_{c2}^2 + I_{c4,zz} + m_4 l_{c3}^2 + m_3 l_2^2$, and $a_3 = m_3 l_2 l_{c3} - m_4 l_1 l_{c2}$. Absolute coordinates have been used therein. The dependence of $\boldsymbol{M}_B$ only on the difference $q_2 - q_1$ confirms that a robot inertia matrix can only be a function of the internal configuration (thus, it does not depend on the choice of the base reference frame). This can be seen also analytically, by applying the linear change of coordinates

$$\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \boldsymbol{J}\boldsymbol{q}.$$

In the new coordinates $\boldsymbol{p} \in \mathbb{R}^2$, the transformed inertia matrix becomes

$$\widetilde{\boldsymbol{M}}_B = \boldsymbol{J}^{-T} \boldsymbol{M}_B \, \boldsymbol{J}^{-1} = \begin{pmatrix} a_1 + 2a_3 \cos p_2 & a_2 + a_3 \cos p_2 \\ a_2 + a_3 \cos p_2 & a_3 \end{pmatrix},$$

which is a function of the second coordinate p_2 only.

C) Matrix $\boldsymbol{M}_C$ resembles the inertia matrix of the usual 2R planar robot, but is not. A dynamic coefficient $a_3 > 0$ is missing in $M(1,1)$, and this destroys the positive definiteness for every $\boldsymbol{q}$, as it should be instead for an inertia matrix. In fact, the determinant

$$\det \boldsymbol{M}_C = -a_2^2 \cos^2 q_2$$

is never positive.

D) $\boldsymbol{M}_D$ is the inertia matrix of a 2P robot, with the second prismatic joint having a twist angle of $\alpha = \pm 120°$ w.r.t. the first prismatic joint. The first link has mass m_1 and the second link has mass m_2. A sketch of this robot is shown on the right of Fig. 3.

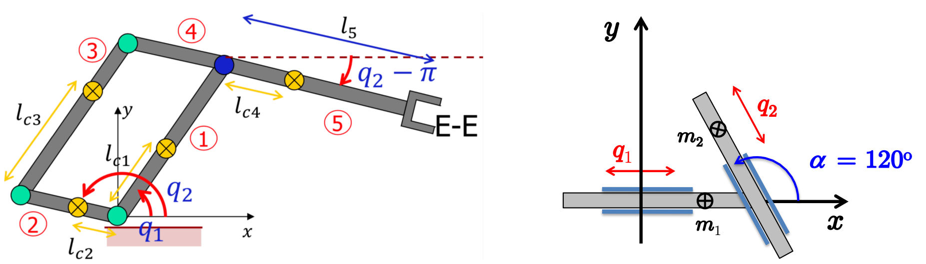

Figure 3: [left] 2-dof parallelogram robot; [right] PP robot with second axis twisted by $\alpha = 120°$.

Exercise 2

Following a Lagrangian approach, we derive the kinetic energy of the robot, $T(\boldsymbol{q},\dot{\boldsymbol{q}}) = T_1+T_2+T_3$. Using König theorem[1],

$$T_i = \frac{1}{2}\, m_i \|\boldsymbol{v}_{ci}\|^2 + \frac{1}{2}\, \boldsymbol{\omega}_i^T \boldsymbol{I}_i \boldsymbol{\omega}_i,$$

and performing computations for each link, one has

$$\begin{aligned}
&T_1 = \tfrac{1}{2}\, m_1 \dot{q}_1^2,\\
&\boldsymbol{p}_{c2} = \begin{pmatrix} d_{c2}\cos q_2 \\ q_1 + k_1 + d_{c2}\sin q_2 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -d_{c2}\sin q_2\, \dot{q}_2 \\ \dot{q}_1 + d_{c2}\cos q_2\, \dot{q}_2 \end{pmatrix},\\
&T_2 = \tfrac{1}{2}\, m_2 \left(\dot{q}_1^2 + d_{c2}^2\, \dot{q}_2^2 + 2\, d_{c2}\cos q_2\, \dot{q}_1\dot{q}_2\right) + \tfrac{1}{2}\, I_2\, \dot{q}_2^2,
\end{aligned}$$

where k_1 is the distance from the center of mass of link 1 to the axis of joint 2 (an irrelevant constant), and

$$\boldsymbol{p}_{c3} = \begin{pmatrix} l_2\cos q_2 - q_3 \sin q_2 \\ q_1 + k_1 + l_2 \sin q_2 + q_3 \cos q_2 \end{pmatrix} \Rightarrow \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} -\left(l_2\sin q_2 + q_3\cos q_2\right)\dot{q}_2 - \sin q_2\, \dot{q}_3 \\ \dot{q}_1 + \left(l_2\cos q_2 - q_3 \sin q_2\right)\dot{q}_2 + \cos q_2\, \dot{q}_3 \end{pmatrix},$$

$$T_3 = \tfrac{1}{2}\, m_3 \left(\dot{q}_1^2 + \left(l_2^2 + q_3^2\right)\dot{q}_2^2 + \dot{q}_3^2 + 2\left(l_2\cos q_2 - q_3\sin q_2\right)\dot{q}_1\dot{q}_2 + 2\cos q_2\, \dot{q}_1\dot{q}_3 + 2 l_2\, \dot{q}_2\dot{q}_3\right) + \tfrac{1}{2}\, I_3\, \dot{q}_2^2.$$

Thus

$$T = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \begin{pmatrix} m_1+m_2+m_3 & m_3\left(l_2\cos q_2 - q_3\sin q_2\right) + m_2 d_{c2}\cos q_2 & m_3\cos q_2 \\ & I_2 + m_2 d_{c2}^2 + I_3 + m_3\left(l_2^2+q_3^2\right) & m_3 l_2 \\ \mathit{symm} & & m_3 \end{pmatrix} \dot{\boldsymbol{q}}.$$

Introduce now a (minimal) parametrization of the robot inertia matrix, collecting the 4 dynamic coefficients that appear in $\boldsymbol{M}(\boldsymbol{q})$ and defined as follows[2]:

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} m_1+m_2+m_3 \\ I_2 + m_2 d_{c2}^2 + I_3 \\ m_3 \\ m_2 d_{c2} \end{pmatrix}. \tag{5}$$

As a result, the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ takes the linearly parametrized form

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 & a_3\left(l_2\cos q_2 - q_3\sin q_2\right) + a_4\cos q_2 & a_3\cos q_2 \\ a_3\left(l_2\cos q_2 - q_3\sin q_2\right) + a_4\cos q_2 & a_2 + a_3\left(l_2^2+q_3^2\right) & a_3 l_2 \\ a_3\cos q_2 & a_3 l_2 & a_3 \end{pmatrix}. \tag{6}$$

The components of the Coriolis and centrifugal vector are computed from (6) using the Christoffel's symbols

$$c_i(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{M}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{M}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i}\right),$$

[1] Being the motion planar, we restrict all linear vectors (e.g., $\boldsymbol{p}_{c2}$ or $\boldsymbol{v}_{c2}$) to be two-dimensional (living in the plane $(\boldsymbol{x},\boldsymbol{y})$), while angular velocities are just scalars (the component of $\boldsymbol{\omega}_i$ along the $\boldsymbol{z}$-axis normal to the plane).

[2] As indicated in the text, we consider the kinematic parameter l_2 to be known.

being $\boldsymbol{M}_i$ the ith column of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. We have

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & -a_3\sin q_2 \\ 0 & -a_3\sin q_2 & 0 \end{pmatrix}\right. \\
&\qquad \left. + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & -a_3\sin q_2 \\ 0 & -a_3\sin q_2 & 0 \end{pmatrix} - \mathbf{0}\right) \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & -a_3\sin q_2 \\ 0 & -a_3\sin q_2 & 0 \end{pmatrix}
\end{aligned}$$

$$\Rightarrow \qquad c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) = -\left(a_3(l_2\sin q_3 + q_3\cos q_2) + a_4\sin q_2\right)\dot{q}_2^2 - 2\,a_3\sin q_2\,\dot{q}_2\dot{q}_3.$$

Similarly,

$$\begin{aligned}
\boldsymbol{C}_2(\boldsymbol{q}) &= \frac{1}{2}\left(\begin{pmatrix} 0 & -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & -a_3\sin q_2 \\ 0 & 0 & 2a_3q_3 \\ 0 & 0 & 0 \end{pmatrix}\right. \\
&\qquad + \begin{pmatrix} 0 & 0 & 0 \\ -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & 0 & 0 \\ -a_3\sin q_2 & 2a_3q_3 & 0 \end{pmatrix} \\
&\quad \left. - \begin{pmatrix} 0 & -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & -a_3\sin q_2 \\ -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & 0 & 0 \\ -a_3\sin q_2 & 0 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_3q_3 \\ 0 & a_3q_3 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_2(\boldsymbol{q},\dot{\boldsymbol{q}}) = 2\,a_3q_3\,\dot{q}_2\dot{q}_3
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{C}_3(\boldsymbol{q}) &= \frac{1}{2}\left(\begin{pmatrix} 0 & -a_3\sin q_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -a_3\sin q_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -a_3\sin q_2 & 0 \\ -a_3\sin q_2 & 2a_3q_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_3q_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_3(\boldsymbol{q},\dot{\boldsymbol{q}}) = -a_3q_3\,\dot{q}_2^2.
\end{aligned}$$

Summarizing, we have the final expression:

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} -\left(a_3(l_2\sin q_2 + q_3\cos q_2) + a_4\sin q_2\right)\dot{q}_2^2 - 2\,a_3\sin q_2\,\dot{q}_2\dot{q}_3 \\ 2\,a_3q_3\,\dot{q}_2\dot{q}_3 \\ -a_3q_3\,\dot{q}_2^2 \end{pmatrix}. \tag{7}$$

Using the inertia matrix in (6) and the quadratic velocity vector in (7), the complete robot dynamic model can be expressed in the linearly parametrized form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}}.\ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{u},$$

where $\boldsymbol{a} \in \mathbb{R}^4$ is defined in (5) and the 3×4 matrix $\boldsymbol{Y}$ is given by

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 & 0 & & \cos q_2\,\ddot{q}_2 - \sin q_2\,\dot{q}_2^2 \\ 0 & \ddot{q}_2 & \boldsymbol{Y}_3 & \cos q_2\,\ddot{q}_1 \\ 0 & 0 & & 0 \end{pmatrix}, \tag{8}$$

being the third column

$$\boldsymbol{Y}_3 = \begin{pmatrix} (l_2\cos q_2 - q_3\sin q_2)\,\ddot{q}_2 + \cos q_2\,\ddot{q}_3 - (l_2\sin q_2 + q_3\cos q_2)\,\dot{q}_2^2 - 2\sin q_2\,\dot{q}_2\dot{q}_3 \\ (l_2\cos q_2 - q_3\sin q_2)\,\ddot{q}_1 + \left(l_2^2 + q_3^2\right)\ddot{q}_2 + l_2\,\ddot{q}_3 + 2q_3\,\dot{q}_2\dot{q}_3 \\ \cos q_2\,\ddot{q}_1 + l_2\,\ddot{q}_2 + \ddot{q}_3 - q_3\,\dot{q}_2^2 \end{pmatrix}.$$

Exercise 3

The use of absolute coordinates makes it simpler to derive the solution for the gravity term in the general case of a planar (serial) robot arm with n revolute joints. Since

$$\boldsymbol{g}_0 = \begin{pmatrix} g_0 \\ 0 \\ 0 \end{pmatrix}, \qquad g_0 = 9.81\ [\mathrm{m/s^2}],$$

the potential energy due to gravity of link j of the robot, for $j = 1,\ldots,n$, is

$$U_j(\boldsymbol{q}) = -m_j \boldsymbol{g}_0^T \boldsymbol{r}_{0,cj} = -m_j g_0 \left(\sum_{k=1}^{j-1} l_k\cos q_k + d_{cj}\cos q_j\right). \tag{9}$$

The total potential energy due to gravity is then

$$\begin{aligned} U(\boldsymbol{q}) = \sum_{j=1}^{n} U_j(\boldsymbol{q}) &= -g_0\sum_{j=1}^{n} m_j \left(\sum_{k=1}^{j-1} l_k\cos q_k + d_{cj}\cos q_j\right) \\ &= -g_0 \sum_{i=1}^{n}\left(\left(\sum_{j=i+1}^{n} m_j\right) l_i + m_i d_{ci}\right)\cos q_i\,. \end{aligned} \tag{10}$$

For instance, setting $n = 5$, we have from eq. (9)

$$\begin{aligned} U_1 &= -m_1 g_0\, d_{c1}\cos q_1 \\ U_2 &= -m_2 g_0\,(l_1\cos q_1 + d_{c2}\cos q_2) \\ U_3 &= -m_3 g_0\,(l_1\cos q_1 + l_2\cos q_2 + d_{c3}\cos q_3) \\ U_4 &= -m_4 g_0\,(l_1\cos q_1 + l_2\cos q_2 + l_3\cos q_3 + d_{c4}\cos q_4) \\ U_5 &= -m_5 g_0\,(l_1\cos q_1 + l_2\cos q_2 + l_3\cos q_3 + l_4\cos q_4 + d_{c5}\cos q_5) \\ U(\boldsymbol{q}) &= \sum_{j=1}^{5} U_j, \end{aligned}$$

where the structure of the last expression of $U(\boldsymbol{q})$ in eq. (10) can be easily recognized.

According to (10), the components $g_i(\boldsymbol{q})$ of the gravity vector $\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^n$ are

$$g_i(\boldsymbol{q}) = \frac{\partial U(\boldsymbol{q})}{\partial q_i} = g_0 \left(\left(\sum_{j=i+1}^{n} m_j \right) l_i + m_i d_{ci} \right) \sin q_i, \qquad i = 1, \dots, n. \tag{11}$$

The unforced equilibrium configurations of the robot (independently of the values of kinematic and dynamic parameters) are then characterized by

$$q_i = q_{e,i} = \{0, \pi\}, \qquad \forall i \in \{1, \dots, n\} \qquad \Rightarrow \qquad \boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}.$$

All these configurations correspond to the robot links being stretched or folded along the vertical axis $\boldsymbol{x}$. The total number of such equilibria is $N_e = 2^n$.

In order to have a robot always balanced with respect to gravity, i.e., $\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}$, the following conditions should hold from (11):

$$\left(\sum_{j=i+1}^{n} m_j \right) l_i + m_i d_{ci} = 0 \qquad \Longleftrightarrow \qquad d_{ci} = -\frac{\left(\sum_{j=i+1}^{n} m_j \right) l_i}{m_i} < 0, \qquad i = 1, \dots, n-1,$$

and

$$d_{cn} = 0.$$

In words, the mass and location of the center of mass of each link should balance (at the associated joint axis) the total mass of the following links in the chain, as if it were concentrated at the end of the link. Any configuration $\boldsymbol{q} \in \mathbb{R}^n$ would then be an equilibrium.

Exercise 4

a) From the inertia matrix (1) of this robot, which is a PR planar arm already treated in the lecture slides, the matrices of Christoffel symbols and the velocity vector (containing only a centrifugal term in the present case) are computed as

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 \\ 0 & a_2 \cos q_2 \end{pmatrix}, \qquad \boldsymbol{C}_2(\boldsymbol{q}) = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} a_2 \cos q_2 \, \dot{q}_2^2 \\ 0 \end{pmatrix} \tag{12}$$

with $a_2 = -m_2 d_{c2}$. The time derivative of the inertia matrix (1) is

$$\dot{\boldsymbol{M}} = \begin{pmatrix} 0 & a_2 \cos q_2 \, \dot{q}_2 \\ a_2 \cos q_2 \, \dot{q}_2 & 0 \end{pmatrix}.$$

A factorization $\boldsymbol{S}_1 \dot{\boldsymbol{q}}$ of vector $\boldsymbol{c}$ in (12) leading to the skew symmetry of $\dot{\boldsymbol{M}} - 2\boldsymbol{S}_1$ is found by using the matrices of Christoffel symbols (or just by trivial inspection). It is easy to check that the matrix

$$\boldsymbol{S}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} 0 & a_2 \cos q_2 \, \dot{q}_2 \\ 0 & 0 \end{pmatrix} \tag{13}$$

provides

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}_1 = \begin{pmatrix} 0 & -a_2 \cos q_2 \, \dot{q}_2 \\ a_2 \cos q_2 \, \dot{q}_2 & 0 \end{pmatrix},$$

which satisfies the desired skew-symmetric property. On the other hand, a different feasible factorization that uses the matrix

$$
\begin{aligned}
\boldsymbol{S}_2(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \begin{pmatrix} \dot{q}_2 & a_2\cos q_2\,\dot{q}_2 - \dot{q}_1 \\ 0 & 0 \end{pmatrix} \\
&= \boldsymbol{S}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) + \begin{pmatrix} \dot{q}_2 & -\dot{q}_1 \\ 0 & 0 \end{pmatrix} = \boldsymbol{S}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{S}_0(\dot{\boldsymbol{q}}), \quad \text{with } \boldsymbol{S}_0(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{0},
\end{aligned}
\tag{14}
$$

satisfies $\boldsymbol{S}_2\dot{\boldsymbol{q}} = \boldsymbol{c}$ and provides

$$
\dot{\boldsymbol{M}} - 2\boldsymbol{S}_2 = \begin{pmatrix} -2\dot{q}_2 & -a_2\sin q_2\,\dot{q}_2 + 2\dot{q}_1 \\ a_2\sin q_2\,\dot{q}_2 & 0 \end{pmatrix},
$$

which is clearly not a skew-symmetric matrix.

b) We use here the simple fact that, when $\dot{\boldsymbol{q}} \in \mathbb{R}^3$, one has

$$
\dot{\boldsymbol{q}} \times \dot{\boldsymbol{q}} = \boldsymbol{S}_0(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{pmatrix} 0 & -\dot{q}_3 & \dot{q}_2 \\ \dot{q}_3 & 0 & -\dot{q}_1 \\ -\dot{q}_2 & \dot{q}_1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \boldsymbol{0},
$$

where $\boldsymbol{S}_0$ is by construction a skew-symmetric matrix. If a factorization $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ leading to a skew symmetric matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is available, then matrix $\boldsymbol{S}' = \boldsymbol{S} + \boldsymbol{S}_0 \neq \boldsymbol{S}$ will be another feasible factorization, since

$$
\boldsymbol{S}'(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \left(\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{S}_0(\dot{\boldsymbol{q}})\right)\dot{\boldsymbol{q}} = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}).
$$

Moreover, the matrix

$$
\dot{\boldsymbol{M}} - 2\boldsymbol{S}' = \left(\dot{\boldsymbol{M}} - 2\boldsymbol{S}\right) + (-2\boldsymbol{S}_0)
$$

is also skew symmetric, being the sum of two skew-symmetric matrices. Indeed, matrix $\boldsymbol{S}'$ is not obtained using the Christoffel symbols only, and it is also clear that an infinite number of such feasible factorizations exists, all leading to the skew-symmetric property.

This construction can be easily generalized to the case of arbitrary $n \geq 3$, by considering for instance a velocity vector $\dot{\boldsymbol{q}}_0 \in \mathbb{R}^n$ with all zero components but three consecutive ones, and an associated $n \times n$ skew-symmetric matrix $\boldsymbol{S}_0(\dot{\boldsymbol{q}}_0)$ with a single non-zero 3×3 block at the proper place on the diagonal, constructed as in the three-dimensional case.

Exercise 5

The direct kinematics of the 2R planar robot is

$$
\boldsymbol{p} = \begin{pmatrix} l_1\cos q_1 + l_2\cos(q_1+q_2) \\ l_1\sin q_1 + l_2\sin(q_1+q_2) \end{pmatrix}.
$$

Therefore, the distance of the robot end effector from the base is

$$
\rho(\boldsymbol{q}) = \|\boldsymbol{p}\| = \sqrt{l_1^2 + l_2^2 + 2\,l_1 l_2\cos q_2}.
$$

The task Jacobian is then a 1×2 matrix

$$
\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial\rho(\boldsymbol{q})}{\partial\boldsymbol{q}} = \begin{pmatrix} 0 & -\dfrac{l_1 l_2\sin q_2}{\|\boldsymbol{p}\|} \end{pmatrix},
\tag{15}
$$

which, for $l_1 \neq l_2$, is full rank except when $q_2 = \{0, \pi\}$. For $l_1 = l_2$, the rank of $\boldsymbol{J}$ drops only at $q_2 = 0$, whereas the element $J_{12}(q_2)$ has a discontinuity at $q_2 = \pm\pi$ (the two limits for $q_2 \to +\pi$ and $q_2 \to -\pi$ are non-zero and different).

A joint velocity $\dot{\boldsymbol{q}}$ that realizes the task while minimizing the robot kinetic energy uses the inertia-weighted pseudoinverse of the task Jacobian,

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_{WPs} = \boldsymbol{J}^{\#}_{\boldsymbol{M}}(\boldsymbol{q})\,\dot{\rho}_d = \boldsymbol{M}^{-1}(\boldsymbol{q})\,\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{M}^{-1}(\boldsymbol{q})\,\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\dot{\rho}_d, \tag{16}$$

where the last equality holds provided that $\boldsymbol{J}$ is full rank. In this case, using in (16) the symbolic expressions (2) and (15), we obtain

$$\dot{\boldsymbol{q}}_{WPs} = -\frac{l_1 l_2 \sin q_2}{\|\boldsymbol{p}\| \cdot \det \boldsymbol{M}(\boldsymbol{q})}\begin{pmatrix} -(a_3 + a_2\cos q_2) \\ a_1 + 2a_2\cos q_2 \end{pmatrix} \cdot \left(\frac{(l_1 l_2 \sin q_2)^2 (a_1 + 2a_2 \cos q_2)}{\|\boldsymbol{p}\|^2 \cdot \det \boldsymbol{M}(\boldsymbol{q})}\right)^{-1}\dot{\rho}_d,$$

where $\det \boldsymbol{M}(\boldsymbol{q}) = a_3(a_1 - a_3) - a_2^2\cos^2 q_2 > 0$. Simplifying, this yields

$$\dot{\boldsymbol{q}}_{WPs} = \frac{\|\boldsymbol{p}\|}{l_1 l_2 \sin q_2}\begin{pmatrix} \dfrac{a_3 + a_2\cos q_2}{a_1 + 2a_2\cos q_2} \\ -1 \end{pmatrix}\dot{\rho}_d. \tag{17}$$

Substituting in (17) the numerical data of the problem, we obtain finally

$$\dot{\boldsymbol{q}}_{WPs} = \begin{pmatrix} 0.1179 \\ -0.7071 \end{pmatrix}. \tag{18}$$

An alternative (and simpler) solution would have been to seek for the minimum norm joint velocity, i.e., using the pseudoinverse of $\boldsymbol{J}$,

$$\dot{\boldsymbol{q}}_{Ps} = \boldsymbol{J}^{\#}(\boldsymbol{q})\,\dot{\rho}_d = \begin{pmatrix} 0 \\ -\dfrac{\|\boldsymbol{p}\|}{l_1 l_2 \sin q_2} \end{pmatrix}\dot{\rho}_d,$$

leading to the numerical value

$$\dot{\boldsymbol{q}}_{Ps} = \begin{pmatrix} 0 \\ -0.7071 \end{pmatrix}. \tag{19}$$

Indeed, a different target is reached by the two solutions $\dot{\boldsymbol{q}}_{WPs}$ in (18) and $\dot{\boldsymbol{q}}_{Ps}$ in (19) in terms of the objective function that is being minimized. We have in fact

$$\frac{1}{2}\,\dot{\boldsymbol{q}}_{WPs}^T\,\boldsymbol{M}\,\dot{\boldsymbol{q}}_{WPs} = \min_{\dot{\boldsymbol{q}}:\,\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\rho}_d} \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}\,\dot{\boldsymbol{q}} = 0.3472 < 0.4167 = \frac{1}{2}\,\dot{\boldsymbol{q}}_{Ps}^T\,\boldsymbol{M}\,\dot{\boldsymbol{q}}_{Ps}$$

and, viceversa,

$$\frac{1}{2}\,\dot{\boldsymbol{q}}_{Ps}^T\,\dot{\boldsymbol{q}}_{Ps} = \min_{\dot{\boldsymbol{q}}:\,\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\rho}_d} \frac{1}{2}\,\dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}} = 0.25 < 0.2569 = \frac{1}{2}\,\dot{\boldsymbol{q}}_{WPs}^T\,\dot{\boldsymbol{q}}_{WPs}.$$

Exercise 6

Using Denavit-Hartenberg coordinates, the Jacobian of a 3R planar robot with unitary links is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix},$$

with the usual compact notation. At the configuration $\boldsymbol{q} = \begin{pmatrix} \pi/6 & \pi/6 & \pi/6 \end{pmatrix}^T$ [rad], it becomes

$$\boldsymbol{J} = \begin{pmatrix} -2.3660 & -1.8660 & -1 \\ 1.3660 & 0.5 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_2 & \boldsymbol{J}_3 \end{pmatrix}. \tag{20}$$

Note that rank $\boldsymbol{J} = 2$. Moreover, its three 2×2 minors are all different from zero in this case. Since the robot is at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$), the second-order differential mapping to be inverted is simply

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}\ddot{\boldsymbol{q}}, \qquad \text{for} \quad \ddot{\boldsymbol{p}} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \text{ [m/s}^2\text{]}.$$

In order to get a solution that is feasible with respect to the hard bounds (4) and has the least possible norm, we can apply the SNS (Saturation in the Null Space) method on joint accelerations[3].

To start with, we look for a solution with minimum acceleration norm,

$$\ddot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^{\#}\ddot{\boldsymbol{p}} = \begin{pmatrix} 0.1715 & 0.9832 \\ -0.4686 & -0.6861 \\ -0.5314 & -1.0460 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.6524 \\ -3.2466 \\ -4.2175 \end{pmatrix}. \tag{21}$$

The third joint acceleration violates its maximum bound, $\ddot{q}_{PS,3} = -4.2175 < -4 = -A_3$, so this solution is not feasible. Next, we search for a feasible solution by saturating the (single) overdriven joint, i.e., setting $\ddot{q}_3 = -A_3 = -4$ [rad/s^2] (this is step 1 of the SNS algorithm). The original task is modified by removing the contribution of the saturated acceleration of the third joint (and by discarding the associated column of $\boldsymbol{J}$). We rewrite this as

$$\ddot{\boldsymbol{p}}_1 = \ddot{\boldsymbol{p}} - \boldsymbol{J}_3(-A_3) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot (-4) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2.3660 & -1.8660 \\ 1.3660 & 0.5 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \boldsymbol{J}_{-3}\, \ddot{\boldsymbol{q}}_{-3},$$

where $\boldsymbol{J}_{-i}$ is the matrix obtained by deleting the ith column from the Jacobian in (20) and, accordingly, $\ddot{\boldsymbol{q}}_{-i}$ is the vector of joint accelerations without the ith component. For the modified task, we compute the contribution of the two active joints by inverting matrix $\boldsymbol{J}_{-3}$, which is now square and nonsingular. Thus, we obtain the unique solution

$$\ddot{\boldsymbol{q}}_{-3} = \left(\boldsymbol{J}_{-3}\right)^{-1} \ddot{\boldsymbol{p}}_1 = \begin{pmatrix} 0.3660 & 1.3660 \\ -1 & -1.7321 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.7321 \\ -3.4641 \end{pmatrix} \text{ [rad/s}^2\text{]}.$$

All bounds are now satisfied and the resulting joint acceleration, with the third component fixed at $\ddot{q}_3 = -4 = -A_3$, is feasible. Thus, there is no need to scale down the original task acceleration $\ddot{\boldsymbol{p}}$. Recomposing the complete joint acceleration vector, we have the solution

$$\ddot{\boldsymbol{q}}^* = \begin{pmatrix} 2.7321 \\ -3.4641 \\ -4 \end{pmatrix} \text{ [rad/s}^2\text{]}, \qquad \text{with } \boldsymbol{J}\ddot{\boldsymbol{q}}^* = \ddot{\boldsymbol{p}} \quad \text{and} \quad \|\ddot{\boldsymbol{q}}^*\| = 5.9552. \tag{22}$$

This solution is not unique, but the underlying SNS method which has been followed guarantees that a feasible solution of minimum norm has been obtained.

For instance, another feasible solution can be obtained by setting the acceleration of the second joint to its lower bound ($\ddot{q}_2 = -A_2 = -3.6$ [rad/s^2]), which is the saturation level that is closer

[3] The solution is quite intuitive in this case and could also be obtained without any knowledge of the SNS method.

to the value of this component in the unconstrained minimum norm solution (21), and adjusting accordingly the other two joint accelerations. Since

$$\ddot{\boldsymbol{q}}_{-2} = \left(\boldsymbol{J}_{-2}\right)^{-1}\left(\ddot{\boldsymbol{p}} - \boldsymbol{J}_2 \cdot (-A_2)\right) = \begin{pmatrix} 0 & 0.7321 \\ -1 & -1.7321 \end{pmatrix} \begin{pmatrix} -2.7177 \\ 3.8 \end{pmatrix} = \begin{pmatrix} 2.7818 \\ -3.8641 \end{pmatrix} \text{ [rad/s}^2\text{]},$$

the complete solution $\ddot{\boldsymbol{q}}^{\diamond}$ is feasible but has slightly larger norm than the SNS solution $\ddot{\boldsymbol{q}}^{*}$ in (22):

$$\ddot{\boldsymbol{q}}^{\diamond} = \begin{pmatrix} 2.7818 \\ -3.6 \\ -3.8641 \end{pmatrix} \text{ [rad/s]}, \qquad \text{with } \boldsymbol{J}\ddot{\boldsymbol{q}}^{\diamond} = \ddot{\boldsymbol{p}} \quad \text{and} \quad \|\dot{\boldsymbol{q}}^{\diamond}\| = 5.9691.$$

Finally, it is easy to see that a solution having the acceleration of the first joint saturated at its upper bound, $\ddot{\boldsymbol{q}} = \begin{pmatrix} 2.8 & -3.6497 & -3.8144 \end{pmatrix}^T$, would instead be unfeasible (the second component is out of bounds).

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