Robotics 2

June 10, 2022

Exercise #1

The PR robot in Fig. 1 moves on a *horizontal* plane. Its inertia matrix has the form

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} A & B\cos q_2 \\ B\cos q_2 & C \end{pmatrix} > 0. \tag{1}$$

Using only the symbolic coefficients A, B and C in (1), provide the expression of the regressor matrix $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)$ and of the complete *adaptive* control law that guarantees global asymptotic tracking of a smooth joint trajectory $\boldsymbol{q}_d(t)$, without a priori information on the values of the dynamic coefficients. Neglect all dissipative effects.
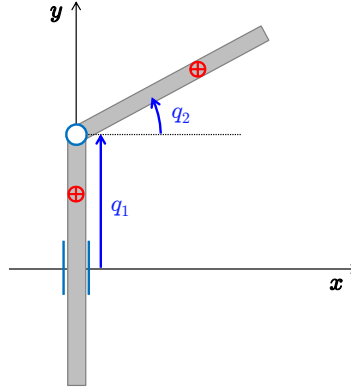

Figure 1: A planar PR robot with the definition of the joint variables q_1 and q_2.

Exercise #2

A macro-micro planar 4R robot is commanded by kinematic control at the joint velocity level. The first two links have equal length, $L_1 = L_2 = L$, and the last two links are also equal in length but four times shorter, $L_3 = L_4 = L/4$ (micro-manipulator). A trajectory $\boldsymbol{p}_{d,1}(t) \in \mathbb{R}^2$ is assigned to the robot end-effector, which is the highest priority task. A second trajectory $\boldsymbol{p}_{d,2}(t) \in \mathbb{R}^2$ is assigned to the tip of the second link, which is the end of the supporting macro part of the robot. Determine the arm configurations $\boldsymbol{q}_s$ at which *algorithmic* singularities occur for the extended Jacobian of the two simultaneous motion tasks. Specify the additional conditions needed in such algorithmic singularities under which a task priority approach would enable perfect execution of the primary task (with some least-squares error on the secondary task).

Exercise #3

The dynamic model of a robot with n elastic joints interacting with the environment can be expressed by two second-order differential equations, each of dimension n,

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}_J + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} \tag{2}$$

$$\boldsymbol{B}\ddot{\boldsymbol{\theta}} + \boldsymbol{\tau}_J = \boldsymbol{\tau}, \tag{3}$$

named respectively, the *link dynamics* and the *motor dynamics* of the elastic joint robot. In these equations, $\boldsymbol{\theta} \in \mathbb{R}^n$ are the motor variables (before joint elasticity), $\boldsymbol{q} \in \mathbb{R}^n$ are the link variables (after joint elasticity), and $\boldsymbol{\tau}_J = \boldsymbol{K}_J (\boldsymbol{\theta} - \boldsymbol{q}) \in \mathbb{R}^n$ is the elastic joint torque measured by the joint torque sensors, with a diagonal joint stiffness matrix $\boldsymbol{K}_J > 0$. Moreover, the robot is equipped with two encoders per joint, measuring both $\boldsymbol{\theta}$ and $\boldsymbol{q}$. The dynamic terms on the left-hand side of the top equation (2) are the same as in a rigid robot; on the right-hand side, there is also the end-effector robot Jacobian $\boldsymbol{J}(\boldsymbol{q}) = (\partial \boldsymbol{f}(\boldsymbol{q})/\partial \boldsymbol{q})$ associated to the Cartesian task vector $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{q}) \in \mathbb{R}^m$ and the related generalized Cartesian forces $\boldsymbol{F} \in \mathbb{R}^m$. On the other hand, in the bottom equation (3), the diagonal matrix $\boldsymbol{B} > 0$ collects the reflected motor inertias, while $\boldsymbol{\tau} \in \mathbb{R}^n$ are the input torques available for control. Suppose also that $m = n$.

- Using feedback from joint torque sensors and motor velocities, design first a control law for $\boldsymbol{\tau}$ such that the motor dynamics becomes

$$\boldsymbol{B}_0 \, \ddot{\boldsymbol{\theta}} + \boldsymbol{D}_0 \dot{\boldsymbol{\theta}} + \boldsymbol{\tau}_J = \boldsymbol{u}, \qquad \text{for a diagonal, } \textit{arbitrary small } \boldsymbol{B}_0 > 0 \text{ and a suitable } \boldsymbol{D}_0 > 0.$$

- Thanks to this inertial reduction, the motor dynamics is made arbitrarily fast so that we can assume $\boldsymbol{\tau}_J \simeq \boldsymbol{u}$ (this fast dynamics can be seen as if it were always at steady-state). Complete then the control design on the robot link dynamics by imposing to the Cartesian task vector $\boldsymbol{x}$, the following impedance model

$$\boldsymbol{M}_x(\boldsymbol{q}) \, \ddot{\boldsymbol{x}} + (\boldsymbol{S}_x(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{D}_m) \, \dot{\boldsymbol{x}} + \boldsymbol{K}_m(\boldsymbol{x} - \boldsymbol{x}_d) = \boldsymbol{F} \tag{4}$$

where $\boldsymbol{x}_d$ is constant, $\boldsymbol{K}_m > 0$ and $\boldsymbol{D}_m > 0$ are desired, diagonal stiffness and damping matrices, and, assuming to work out of kinematic singularities,

$$\boldsymbol{M}_x(\boldsymbol{q}) = \left(\boldsymbol{J}(\boldsymbol{q}) \boldsymbol{M}^{-1}(\boldsymbol{q}) \boldsymbol{J}^T(\boldsymbol{q}) \right)^{-1}, \quad \boldsymbol{S}_x(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{J}^{-T}(\boldsymbol{q}) \boldsymbol{S}(\boldsymbol{q}, \boldsymbol{q}) \boldsymbol{J}^{-1}(\boldsymbol{q}) - \boldsymbol{M}_x(\boldsymbol{q}) \dot{\boldsymbol{J}}(\boldsymbol{q}) \boldsymbol{J}^{-1}(\boldsymbol{q}).$$

Write the final control law for the input torque $\boldsymbol{\tau} \in \mathbb{R}^n$ in explicit form only in terms of the original state variables $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\boldsymbol{\theta}$, and $\dot{\boldsymbol{\theta}}$. Moreover, if an external constant force $\boldsymbol{F} = \bar{\boldsymbol{F}}$ is being applied from the environment to the robot, which will the equilibrium $\boldsymbol{x} = \boldsymbol{x}_E$ at steady state and what will be the value $\boldsymbol{\tau}_E$ of the control torque $\boldsymbol{\tau}$ at this equilibrium?
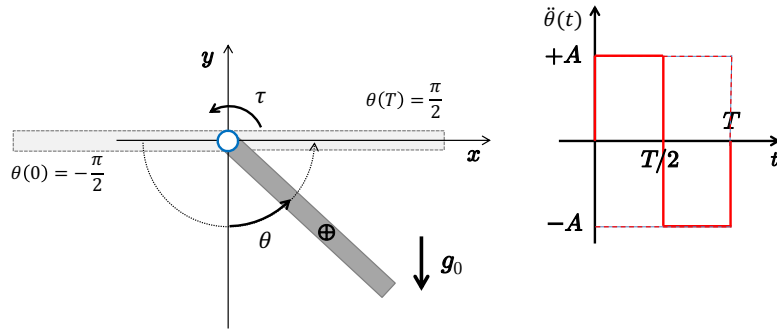
Exercise #4

Figure 2: A one-link arm moving under gravity and the desired joint acceleration profile.

Consider the actuated link under gravity in Fig. 2, with the input torque bounded as $|\tau| \leq \tau_{max}$. The link should perform a rest-to-rest motion from $\theta(0) = -\pi/2$ to $\theta(T) = \pi/2$ with the bang-bang acceleration profile $\ddot{\theta}(t)$ shown in the same figure. Determine analytically the minimum feasible time T to execute the desired motion with this type of trajectory.

[210 minutes (3.5 hours); open books]

Solution

Exercise #1

We compute the Coriolis and centrifugal terms using the Christoffel symbols:

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i} \right), \qquad i = 1, 2,$$

where $\boldsymbol{m}_i(\boldsymbol{q})$ is the i-th column of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. This leads to

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 \\ 0 & -B \sin q_2 \end{pmatrix} \qquad \Rightarrow \qquad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -B \sin q_2 \, \dot{q}_2^2,$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \boldsymbol{O} \qquad \Rightarrow \qquad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) \equiv 0.$$

Accordingly, the factorization of these terms with the skew-symmetric property is trivial (being only a centrifugal term present):

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}} = \begin{pmatrix} 0 & -B \sin q_2 \dot{q}_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} -B \sin q_2 \, \dot{q}_2^2 \\ 0 \end{pmatrix}.$$

Having defined $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}(\boldsymbol{q}_d - \boldsymbol{q}) = \dot{\boldsymbol{q}}_d + \boldsymbol{K}_D^{-1} \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q})$, the adaptive control law is

$$\boldsymbol{\tau} = \hat{\boldsymbol{M}}(\boldsymbol{q}) \ddot{\boldsymbol{q}}_r + \hat{\boldsymbol{S}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}}_r + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}})$$

$$= \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r) \hat{\boldsymbol{a}} + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}), \qquad \text{(diagonal) } \boldsymbol{K}_P > 0, \boldsymbol{K}_D > 0),$$

with

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r) = \begin{pmatrix} \ddot{q}_{r1} & \ddot{q}_{r2} \cos q_2 - \dot{q}_2 \dot{q}_{2r} \sin q_2 & 0 \\ 0 & \ddot{q}_{r1} \cos q_2 & \ddot{q}_{r2} \end{pmatrix}, \qquad \hat{\boldsymbol{a}} = \begin{pmatrix} \hat{A} \\ \hat{B} \\ \hat{C} \end{pmatrix},$$

and adaptation law

$$\dot{\hat{\boldsymbol{a}}} = \boldsymbol{\Gamma} \boldsymbol{Y}^T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r) \left(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}} \right), \qquad \boldsymbol{\Gamma} > 0.$$

Exercise #2

The two tasks of dimension $m_1 = m_2 = 2$ are defined by the following kinematics:

$$\boldsymbol{p}_1 = \boldsymbol{f}_1(\boldsymbol{q}) = L \begin{pmatrix} c_1 + c_{12} \\ s_1 + s_{12} \end{pmatrix} + \frac{L}{4} \begin{pmatrix} c_{123} + c_{1234} \\ s_{123} + s_{1234} \end{pmatrix} = \boldsymbol{p}_2 + \frac{L}{4} \begin{pmatrix} c_{123} + c_{1234} \\ s_{123} + s_{1234} \end{pmatrix},$$

$$\boldsymbol{p}_2 = \boldsymbol{f}_2(\boldsymbol{q}) = L \begin{pmatrix} c_1 + c_{12} \\ s_1 + s_{12} \end{pmatrix}.$$

The associated Jacobians $\boldsymbol{J}_i(\boldsymbol{q}) = \partial \boldsymbol{f}_i(\boldsymbol{q}) / \partial \boldsymbol{q}$, for $i = 1, 2$, are

$$\boldsymbol{J}_1(\boldsymbol{q}) = L \begin{pmatrix} -\left(s_1 + s_{12} + \dfrac{s_{123} + s_{1234}}{4} \right) & -\left(s_{12} + \dfrac{s_{123} + s_{1234}}{4} \right) & -\dfrac{s_{123} + s_{1234}}{4} & -\dfrac{s_{1234}}{4} \\ c_1 + c_{12} + \dfrac{c_{123} + c_{1234}}{4} & c_{12} + \dfrac{c_{123} + c_{1234}}{4} & \dfrac{c_{123} + c_{1234}}{4} & \dfrac{c_{1234}}{4} \end{pmatrix}$$

$$\boldsymbol{J}_2(\boldsymbol{q}) = L \begin{pmatrix} -(s_1 + s_{12}) & -s_{12} & 0 & 0 \\ c_1 + c_{12} & c_{12} & 0 & 0 \end{pmatrix}.$$

Since $n = m_1 + m_2 = 4$, the simultaneous execution of both task leads to the square *extended Jacobian*

$$\boldsymbol{J}_E(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_1(\boldsymbol{q}) \\ \boldsymbol{J}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{11}(\boldsymbol{q}_M, \boldsymbol{q}_m) & \boldsymbol{J}_{12}(\boldsymbol{q}_M, \boldsymbol{q}_m) \\ \boldsymbol{J}_{21}(\boldsymbol{q}_M) & \boldsymbol{O} \end{pmatrix}$$

with $\boldsymbol{q}_M = (q_1, q_2)$ and $\boldsymbol{q}_m = (q_3, q_4)$ being respectively the variables of the macro- and micro-manipulator.

The determinant of $\boldsymbol{J}_E(\boldsymbol{q})$ is easily computed as

$$\det \boldsymbol{J}_E(\boldsymbol{q}) = \det \boldsymbol{J}_{12}(\boldsymbol{q}_M, \boldsymbol{q}_m) \cdot \det \boldsymbol{J}_{21}(\boldsymbol{q}_M) = \frac{L^4}{16} \sin q_2 \sin q_4.$$

In particular, $\sin q_2 = 0$ corresponds certainly to a rank loss of the secondary task Jacobian $\boldsymbol{J}_2$ (namely, a singularity of the square $\boldsymbol{J}_{21}$ matrix), whereas $\sin q_4 = 0$ corresponds to a singularity of the square block $\boldsymbol{J}_{12}$. However, it could be that the 2×4 primary task Jacobian $\boldsymbol{J}_2$ has still full rank even if $\sin q_4 = 0$. In this case, the primary task may still be generically realized by a task priority strategy.

The analysis of the rank deficiencies of the extended Jacobian $\boldsymbol{J}_E$ and of its sub-parts can be simplified by using the following two invertible transformations (acting on the columns and on the rows of the matrix).

- Redefining the joint velocity as

$$\dot{\boldsymbol{q}} = \boldsymbol{T}\,\dot{\boldsymbol{q}}' \qquad \text{with } \boldsymbol{T} = \frac{1}{L} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -4 & 4 \end{pmatrix},$$

leads to

$$\boldsymbol{J}_E(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}_E(\boldsymbol{q})\boldsymbol{T}\,\dot{\boldsymbol{q}}' = \boldsymbol{J}'_E(\boldsymbol{q})\dot{\boldsymbol{q}}'$$

with the simpler form

$$\boldsymbol{J}'_E(\boldsymbol{q}) = \begin{pmatrix} -s_1 & -s_{12} & -s_{123} & -s_{1234} \\ c_1 & c_{12} & c_{123} & c_{1234} \\ -s_1 & -s_{12} & 0 & 0 \\ c_1 & c_{12} & 0 & 0 \end{pmatrix}.$$

- Expressing the task velocities in the frame rotated with joint 1,

$$\begin{pmatrix} {}^1\dot{\boldsymbol{p}}_1 \\ {}^1\dot{\boldsymbol{p}}_2 \end{pmatrix} = \begin{pmatrix} {}^0\boldsymbol{R}_1^T(\boldsymbol{q}_1) & \boldsymbol{O} \\ \boldsymbol{O} & {}^0\boldsymbol{R}_1^T(\boldsymbol{q}_1) \end{pmatrix} \begin{pmatrix} {}^0\dot{\boldsymbol{p}}_1 \\ {}^0\dot{\boldsymbol{p}}_2 \end{pmatrix}$$

$$= \begin{pmatrix} {}^0\boldsymbol{R}_1^T(\boldsymbol{q}_1) & \boldsymbol{O} \\ \boldsymbol{O} & {}^0\boldsymbol{R}_1^T(\boldsymbol{q}_1) \end{pmatrix} \boldsymbol{J}'_E(\boldsymbol{q})\dot{\boldsymbol{q}}' = {}^1\boldsymbol{J}'_E(\boldsymbol{q})\dot{\boldsymbol{q}}',$$

further simplifies the extended Jacobian to

$${}^1\boldsymbol{J}'_E(\boldsymbol{q}) = \begin{pmatrix} 0 & -s_2 & -s_{23} & -s_{234} \\ 1 & c_2 & c_{23} & c_{234} \\ 0 & -s_2 & 0 & 0 \\ 1 & c_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} {}^1\boldsymbol{J}'_1(\boldsymbol{q}) \\ {}^1\boldsymbol{J}'_2(\boldsymbol{q}) \end{pmatrix}.$$

4

With the above in mind, a true *algorithmic* singularity occurs when $\sin q_2 \neq 0$, i.e., the second task Jacobian $\boldsymbol{J}_2$ is full rank, the first task Jacobian $\boldsymbol{J}_1$ is also of full rank, but the extended Jacobian $\boldsymbol{J}_E$ is singular, which implies then necessarily $\sin q_4 = 0$. To check if this situation is at all possible, consider the simplified Jacobian $^1\boldsymbol{J}_1'$ evaluated in particular for $q_4 = 0$ (but with $\sin q_2 \neq 0$). We have —this is labeled below as case a:

$$\left.^1\boldsymbol{J}_1'(\boldsymbol{q})\right|_{q_4=0} = \begin{pmatrix} 0 & -s_2 & -s_{23} & -s_{23} \\ 1 & c_2 & c_{23} & c_{23} \end{pmatrix} \qquad \Rightarrow \qquad \text{has rank} = 2 \text{ (just as } \boldsymbol{J}_1).$$

This means that adopting a task priority solution will allow the highest priority task to be exactly executed (in this case, together with the second one being also rank $\boldsymbol{J}_2 = 2$), whereas inversion of the square extended Jacobian $\boldsymbol{J}_E$ would be impossible.

A further interesting situation is when the secondary task is singular ($\sin q_2 = 0$), together with the sub-Jacobian $\boldsymbol{J}_{12}$ of the first task ($\sin q_4 = 0$). For instance, when $q_2 = q_4 = 0$ one has

$$\left.^1\boldsymbol{J}_1'(\boldsymbol{q})\right|_{q_2=q_4=0} = \begin{pmatrix} 0 & 0 & -s_3 & -s_3 \\ 1 & 1 & c_3 & c_3 \end{pmatrix} \qquad \Rightarrow \qquad \text{has rank} = 2 \text{ iff } \sin q_3 \neq 0.$$

Also in this situation, labeled below as case b, the task priority solution would allow a correct execution of the highest priority task, whereas the secondary task would report an error (minimized in a least square sense).
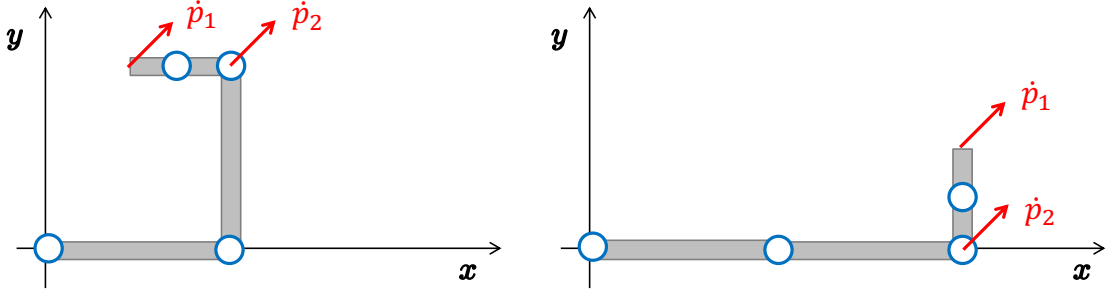

Figure 3: Cases a (left) and b (right) of singularities for the extended Jacobian $\boldsymbol{J}_E$ of the 4R macro-micro robot.

The above two situations are depicted in Fig. 3. The task priority solutions are computed as

$$\dot{\boldsymbol{q}}_{TP} = \boldsymbol{J}_1^{\#}(\boldsymbol{q})\dot{\boldsymbol{p}}_1 + (\boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{P}_1(\boldsymbol{q}))^{\#}\left(\dot{\boldsymbol{p}}_2 - \boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{J}_1^{\#}(\boldsymbol{q})\dot{\boldsymbol{p}}_1\right), \tag{5}$$

where $\boldsymbol{P}_1(\boldsymbol{q}) = \boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q})\boldsymbol{J}_1(\boldsymbol{q})$. In both cases, we have set the link length parameter and the task velocities to

$$L = 1 \text{ [m]}, \qquad \dot{\boldsymbol{p}}_1 = \dot{\boldsymbol{p}}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ [m/s]}.$$

When $\boldsymbol{q}_a = (0, \pi/2, \pi/2, 0)$ (algorithmic singularity, case a), the joint velocity (5) provides the correct solution for both tasks:

$$\dot{\boldsymbol{q}}_{TP,a} = \begin{pmatrix} 1 \\ -2 \\ 0.8 \\ 0.4 \end{pmatrix} \text{ [rad/s]} \qquad \Rightarrow \qquad \boldsymbol{J}_E(\boldsymbol{q}_a)\dot{\boldsymbol{q}}_{TP,a} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ [m/s]}.$$

When $\boldsymbol{q}_b = (0, 0, \pi/2, 0)$ (case b), the Jacobian $\boldsymbol{J}_2$ of the second task is not of full rank, and thus $\boldsymbol{J}_E$ is singular), the joint velocity (5) still provides the correct solution for the high-priority task

$$\dot{\boldsymbol{q}}_{TP,b} = \begin{pmatrix} 0.7586 \\ -0.5172 \\ -1.793 \\ -0.8966 \end{pmatrix} \text{[rad/s]} \quad \Rightarrow \quad \boldsymbol{J}_E(\boldsymbol{q}_b)\dot{\boldsymbol{q}}_{TP,b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{[m/s]}.$$

while the x-component of the second task velocity $\dot{\boldsymbol{p}}_2$ is clearly not realized.

Exercise #3

The first step in the control design is achieved simply by choosing $\quad$ this is just the inverse formula of theta_ddot in the first request, plugged in the motor equations

$$\boldsymbol{\tau} = \left(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{B}_0^{-1}\right)\boldsymbol{\tau}_J + \boldsymbol{B}\boldsymbol{B}_0^{-1}\left(\boldsymbol{u} - \boldsymbol{D}_0\dot{\boldsymbol{\theta}}\right). \tag{6}$$

Assuming that $\boldsymbol{\tau}_J = \boldsymbol{u}$, then the rest of the procedure follows a standard impedance control design for a task that involves a constant reference $\boldsymbol{x}_d$ and when the apparent inertia is chosen as the natural Cartesian inertia of the robot, with consistent Coriolis/centrifugal terms included in the model to be matched. Thus, no force feedback will be required. The Cartesian model of the link dynamics (viz., that of a rigid robot) is

$$\boldsymbol{M}_x(\boldsymbol{q})\ddot{\boldsymbol{x}} + \boldsymbol{S}_x(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{x}} + \boldsymbol{g}_x(\boldsymbol{q}) = \boldsymbol{F} + \boldsymbol{J}^{-T}(\boldsymbol{q})\boldsymbol{u}. \tag{7}$$

Comparing (7) with the target model (4), the equivalence is obtained by choosing $\quad$ this is obtained isolating u from the classical cartesian dynamic model (7) plugging then it in the desired impedance model given in the text

$$\boldsymbol{u} = \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{K}_m(\boldsymbol{x}_d - \boldsymbol{x}) - \boldsymbol{D}_m\dot{\boldsymbol{x}} + \boldsymbol{g}_x(\boldsymbol{q})\right) = \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{K}_m(\boldsymbol{x}_d - \boldsymbol{x}) - \boldsymbol{D}_m\dot{\boldsymbol{x}}\right) + \boldsymbol{g}(\boldsymbol{q}), \tag{8}$$

being $\boldsymbol{g}_x(\boldsymbol{q}) = \boldsymbol{J}^{-T}(\boldsymbol{q})\boldsymbol{g}(\boldsymbol{q})$. Putting together eqs. (6) and (8) gives finally

$$\boldsymbol{\tau} = \left(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{B}_0^{-1}\right)\boldsymbol{K}_J(\boldsymbol{\theta} - \boldsymbol{q}) + \boldsymbol{B}\boldsymbol{B}_0^{-1}\left(\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{K}_m(\boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{D}_m\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{D}_0\dot{\boldsymbol{\theta}}\right).$$

If an external constant force $\boldsymbol{F} = \bar{\boldsymbol{F}}$ is applied at the robot end effector, the system will find the steady-state equilibrium

$$\boldsymbol{x}_E = \boldsymbol{x}_d + \boldsymbol{K}_m^{-1}\bar{\boldsymbol{F}}.$$

The value of the control signal $\boldsymbol{u}$ at this equilibrium is

$$\boldsymbol{u}_E = \boldsymbol{J}^T(\boldsymbol{q}_E)\bar{\boldsymbol{F}} + \boldsymbol{g}(\boldsymbol{q}_E),$$

where the equilibrium configuration $\boldsymbol{q}_E$ is such that $\boldsymbol{f}(\boldsymbol{q}_E) = \boldsymbol{x}_E$. Moreover, since the elastic torque at steady state will be $\boldsymbol{\tau}_{J,E} = \boldsymbol{u}_E$, it follows from (6) that the control torque at the equilibrium is $\boldsymbol{\tau}_E = \boldsymbol{u}_E$.

Exercise #4

The dynamics of the actuated pendulum under gravity is given by

$$I\ddot{\theta} + mg_0 d\sin\theta = \tau,$$

where $m > 0$ is the link mass, $d > 0$ is the distance of its center of mass from the joint, and $I = I_c + md^2 > 0$ is the link inertia around the joint axis at the base. With the bang-bang acceleration profile $\ddot{\theta}(t)$ given in Fig. 2, when starting at rest in $\theta(0) = \theta_0$ one has by integration

$$\dot{\theta}(t) = \begin{cases} At, & t \in [0, T/2] \\ A(T-t), & t \in [T/2, T] \end{cases} \qquad \theta(t) = \begin{cases} \theta_0 + \frac{1}{2}At^2, & t \in [0, T/2] \\ \theta_0 + \frac{1}{4}AT^2 - \frac{1}{2}A(T-t)^2, & t \in [T/2, T] \end{cases}$$

yielding $\theta(T) = \theta_0 + \frac{1}{4}AT^2$. Thus, for a rest-to-rest angular motion $\Delta = \theta(T) - \theta_0$ to be executed in T seconds, the bang-bang value A of the acceleration will have to be

$$A = \frac{4\Delta}{T^2}.$$

From the inverse dynamics for the desired motion, one should note that the maximum of the gravity torque $\tau_g = mg_0 d \sin\theta$ in *absolute value* occurs at $\theta = \pm\pi/2$. However, the gravity torque has always opposite sign of the inertial torque $I\ddot{\theta} = \pm IA$ (see the dotted blue line in the right of Fig. 4), subtracting from the total torque that the actuator needs to deliver for the desired motion. Stated differently, it helps in the acceleration phase ($\tau_g(t)$ has negative sign when $t \in [0, T/2)$) as well as in the deceleration phase ($\tau_g(t)$ has positive sign when $t \in (T/2, T]$). Therefore, we easily see that

$$\max_{t \in [0,T]} |\tau(t)| = |\tau(T/2)| = IA = I\frac{4\Delta}{T^2} \leq \tau_{max},$$

i.e., at $t = T/2$, where $\tau_g(T/2) = \tau_g(\theta = 0) = 0$. From this, the minimum motion time for the required $\Delta = \pi$ is

$$T_{min} = \sqrt{\frac{4\Delta I}{\tau_{max}}} = \sqrt{\frac{4\pi I}{\tau_{max}}}.$$
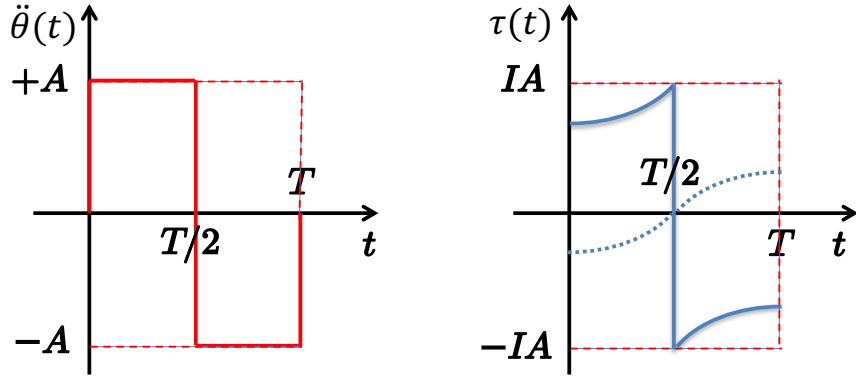

Figure 4: The joint acceleration profile and the associated joint torque profile.

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