Robotics II

October 28, 2016

Exercise 1

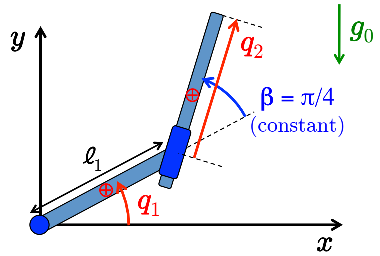

Figure 1: A planar RP robot with skewed prismatic joint.

Derive the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ and the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ in the dynamic model of the planar RP robot in Fig. 1, using the Lagrangian coordinates $\boldsymbol{q} = (q_1, q_2)$ defined therein and assuming uniform mass distribution for the two links. Determine all equilibrium configurations $\boldsymbol{q}_0$ under no external or dissipative forces/torques nor actuation inputs.

Exercise 2

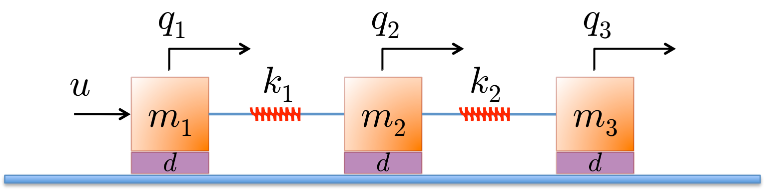

Figure 2: A mechanical system consisting of three masses connected by linear springs.

In the three-body mechanical system shown in Fig. 2, the first body (with mass $m_1 > 0$ and position q_1) is actuated by a force u and is connected to a second body (with mass $m_2 > 0$ and position q_2) through a spring of constant stiffness $k_1 > 0$. The second body is in turn connected to a third one (having mass $m_3 > 0$ and position q_3) through another spring of constant stiffness $k_2 > 0$. Each mass is subject to a dissipative force when moving on the ground, in the form of a viscous friction with coefficient $d > 0$ (equal for all three masses).

- Derive the dynamic model of this system by following a Lagrangian approach and using the set of generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$.
- Determine all equilibrium states, if any, of the system when $u = 0$, as well as all steady-state conditions when a constant force $u = \bar{u} > 0$ is being applied.
- Prove that a control law of the form

$$u = k_p(q_d - q_1), \qquad k_p > 0 \tag{1}$$

will globally asymptotically stabilize the closed-loop system to a unique equilibrium state (which one?). *Hint: Use either a energy-based Lyapunov approach or exploit the linearity of the system dynamics.*

[150 minutes; open books]

Solution

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Exercise 1

For $i = 1, 2$, let m_i be the mass of link i and I_i its inertia around an axis normal to the plane of motion, passing through the center of mass (which is at the midpoint of the link, because of the assumption on uniform mass distribution). Moreover, let $d > 0$ be the (constant) distance of the center of mass of link 2 (which is of unspecified length) from its tip, namely from the point characterized by the coordinate q_2. With this in mind, we follow a Lagrangian approach and compute the kinetic and the potential energy of the robot system in order to obtain, respectively, $\boldsymbol{B}(\boldsymbol{q})$ and $\boldsymbol{g}(\boldsymbol{q})$.

The kinetic energy of the first link is

$$T_1 = \frac{1}{2}\left(I_1 + m_1\left(\frac{\ell_1}{2}\right)^2\right)\dot{q}_1^2.$$

For the second link, the position of the center of mass is

$$\boldsymbol{p}_{c2} = \begin{pmatrix} \ell_1 \cos q_1 + (q_2 - d)\cos(q_1 + \pi/4) \\ \ell_1 \sin q_1 + (q_2 - d)\sin(q_1 + \pi/4) \end{pmatrix}.$$

Thus, its velocity is

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -\left(\ell_1 \sin q_1 + (q_2 - d)\sin(q_1 + \pi/4)\right)\dot{q}_1 + \cos(q_1 + \pi/4)\,\dot{q}_2 \\ \left(\ell_1 \cos q_1 + (q_2 - d)\cos(q_1 + \pi/4)\right)\dot{q}_1 + \sin(q_1 + \pi/4)\,\dot{q}_2 \end{pmatrix}.$$

The (scalar) angular velocity of link 2 is simply $\dot{q}_1$. As a result, the kinetic energy of the second link is

$$T_2 = \frac{1}{2}\left(\left(I_2 + m_2\left(\ell_1^2 + (q_2 - d)^2 + 2\ell_1(q_2 - d)\cos(\pi/4)\right)\right)\dot{q}_1^2 + m_2\,\dot{q}_2^2 + 2m_2\ell_1\sin(\pi/4)\,\dot{q}_1\dot{q}_2\right).$$

Therefore,

$$T = T_1 + T_2 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\begin{pmatrix} b_{11}(q_2) & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\dot{\boldsymbol{q}} = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

from which the elements b_{ij} of the 2×2, symmetric, positive definite inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ are obtained as

$$\begin{aligned} b_{11}(q_2) &= I_1 + m_1\left(\frac{\ell_1}{2}\right)^2 + I_2 + m_2\left(\ell_1^2 + (q_2 - d)^2 + \ell_1(q_2 - d)\sqrt{2}\right) \\ b_{12} &= b_{21} = m_2\ell_1\frac{\sqrt{2}}{2} \\ b_{22} &= m_2, \end{aligned}$$

where $\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$ has been used.

For the potential energy $U = U_1 + U_2$ of the two robot links due to gravity, we use the expressions of the y-component of their center of mass:

$$
\begin{aligned}
U_1(q_1) &= m_1 g_0 \left(\frac{\ell_1}{2}\right) \sin q_1, \\
U_2(q_1, q_2) &= m_2 g_0 \left(\ell_1 \sin q_1 + (q_2 - d) \sin\left(q_1 + \pi/4\right)\right).
\end{aligned}
$$

Therefore, the gravity vector is

$$
\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = g_0 \begin{pmatrix} \left(\frac{m_1}{2} + m_2\right) \ell_1 \cos q_1 + m_2 (q_2 - d) \cos\left(q_1 + \pi/4\right) \\ m_2 \sin\left(q_1 + \pi/4\right) \end{pmatrix}.
$$

Solving for $\boldsymbol{g}(\boldsymbol{q}_0) = \boldsymbol{0}$ provides all equilibrium configurations $\boldsymbol{q}_0$. For this RP robot, there are two unforced equilibria:

$$
\boldsymbol{q}_0 = \begin{pmatrix} -\frac{\pi}{4} & d - \frac{\ell_1 \sqrt{2}}{2}\left(1 + \frac{m_1}{2 m_2}\right) \end{pmatrix}^T
$$

and

$$
\boldsymbol{q}_0 = \begin{pmatrix} \frac{3\pi}{4} & d - \frac{\ell_1 \sqrt{2}}{2}\left(1 + \frac{m_1}{2 m_2}\right) \end{pmatrix}^T.
$$

Exercise 2

By following a Lagrangian approach, we compute first the kinetic and the potential (elastic) energy of the mechanical system in Fig. 2:

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$$
T = \frac{1}{2}\left(m_1 \dot{q}_1^2 + m_2 \dot{q}_2^2 + m_3 \dot{q}_3^2\right), \qquad U = \frac{1}{2}\left(k_1 (q_1 - q_2)^2 + k_2 (q_2 - q_3)^2\right).
$$

Applying the Euler-Lagrange equations to $L = T - U$ and keeping into account the dissipative forces due to viscous friction, yields the three second-order differential equations

$$
\begin{aligned}
m_1 \ddot{q}_1 + k_1 (q_1 - q_2) &= u - d \dot{q}_1 \\
m_2 \ddot{q}_2 + k_1 (q_2 - q_1) + k_2 (q_2 - q_3) &= -d \dot{q}_2 \qquad (2) \\
m_3 \ddot{q}_3 + k_2 (q_3 - q_2) &= -d \dot{q}_3.
\end{aligned}
$$

Indeed, the same result would have been obtained by Newton's law (balance of forces) applied to the three masses.

For the unforced ($u = 0$) equilibrium configurations, we set $\dot{\boldsymbol{q}} = \ddot{\boldsymbol{q}} = \boldsymbol{0}$ in (2) and obtain an infinity of equilibria $\boldsymbol{q}_0$, all with equal $q_1 = q_2 = q_3$ at a common arbitrary value. The two springs will be undeformed in this rest condition.

For the forced case with $u = \bar{u} > 0$, it is rather tedious but straightforward exercise (using, e.g., Laplace transforms) to verify that the system response to a constant force input will asymptotically reach a steady-state condition, where all masses will be moving at the same constant speed $\dot{q}_1 = \dot{q}_2 = \dot{q}_3 = \bar{u}/3d$. This specific value can be found by setting $\ddot{\boldsymbol{q}} = \boldsymbol{0}$ in (2) and summing up the three equations (the elastic forces cancel each other). In this steady-state condition, the deformation of the two springs can be computed again from the model as $\rightarrow \ddot{q} = 0, \dot{q}_i = \frac{\bar{u}}{3d}\ \forall i, \bar{u} = \text{cost.}$

$$
\delta_{12} = q_1 - q_2 = \frac{1}{k_1} \frac{2\bar{u}}{3}, \qquad \delta_{23} = q_2 - q_3 = \frac{1}{k_2} \frac{\bar{u}}{3},
$$

being in general $\delta_{12} \neq \delta_{23}$. Note also that the steady-state deformations are independent from the actual values of the masses (these will influence instead the transient behavior).

We can prove that the control law (1) asymptotically stabilizes the *unique* closed-loop equilibrium state $q_1 = q_2 = q_3 = q_d$, $\dot{\boldsymbol{q}} = \mathbf{0}$, by using a simple Lyapunov/LaSalle argument. Define the Lyapunov candidate as the total system energy $E = T + U$, plus the control energy in the form of a virtual spring of stiffness $k_p > 0$, with rest condition in $q_1 = q_d$:

$$V = \frac{1}{2}\left(m_1\dot{q}_1^2 + m_2\dot{q}_2^2 + m_3\dot{q}_3^2\right) + \frac{1}{2}\left(k_1(q_1 - q_2)^2 + k_2(q_2 - q_3)^2 + k_p(q_d - q_1)^2\right).$$

Indeed $V \geq 0$, whereas $V = 0$ if and only if the closed-loop system is at the desired equilibrium state. Taking the time derivative of V, using (2), and simplifying terms leads to

$$\dot{V} = -d\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) \leq 0.$$

The non-positivity of $\dot{V}$ is thus guaranteed by the presence of viscous friction in the mass-ground contact (otherwise, we could inject dissipation by just adding a damping term $-k_d\dot{q}_1$, $k_d > 0$, in the control law). To complete the proof, we invoke LaSalle's theorem. From $\dot{V} = 0 \iff \dot{\boldsymbol{q}} = \mathbf{0}$, we determine the largest invariant set of states contained in $\dot{V} = 0$. When $\dot{\boldsymbol{q}} \equiv \mathbf{0}$, from the third equation in (2), $\ddot{q}_3 = 0$ necessarily implies $q_3 = q_2$. Substituting backward in the second equation under the same operative conditions yields necessarily $q_2 = q_1$. Finally, from the first equation

$$0 = m_1\ddot{q}_1 = k_p(q_d - q_1) - k_1(q_1 - q_2) - d\dot{q}_1 = k_p(q_d - q_1) \qquad \Rightarrow \qquad q_1 = q_d,$$

and thus the configuration $\boldsymbol{q}_e$ having $q_1 = q_2 = q_3 = q_d$ with velocity $\dot{\boldsymbol{q}} = \mathbf{0}$ is the only invariant state for the closed-loop system. The result is thus proven. Since the system is linear, asymptotic stability will be equivalent to exponential stability. For the same reason, one could have analyzed the characteristic polynomial of the closed-loop linear system, using the Routh criterion to establish stability. However, this would require longer computations in order to prove the same result.

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