# Robotics II

**September 15, 2010**

Consider the planar robot with three degrees of freedom (RPR) shown in Fig. 1.

$p_{c2} = \begin{pmatrix} q_2 c_1 \\ q_2 s_1 \\ 0 \end{pmatrix}$ $v_{c2} = \begin{pmatrix} \dot q_2 c_1 - s_1 q_2 \dot q_1 \\ \dot q_2 s_1 + c_1 q_2 \dot q_1 \\ 0 \end{pmatrix}$

$\|v_{c2}\|^2 = \dot q_2^2 + q_2^2 \dot q_1^2$

$T_1 = \frac{1}{2}(I_1 + m_1 d_{c1}^2)\dot q_1^2$

$T_2 = \frac{1}{2} I_2 \dot q_1^2 + \frac{1}{2} m_2 (\dot q_2^2 + q_2^2 \dot q_1^2)$

$T_3 = \frac{1}{2} I_3 \dot q_3 + \frac{1}{2} m_3 \|v_{c3}\|^2$

$p_{c3} = \begin{pmatrix} (q_2 + d_{c2}) c_1 + d_{c3} c_{12} \\ (q_2 + d_{c2}) s_1 + d_{c3} s_{12} \end{pmatrix}$

$\dot p_{c3} = \begin{pmatrix} \dot q_2 c_1 - (q_2 + d_{c2}) s_1 \dot q_1 - d_{c3} s_{12}(\dot q_1 + \dot q_3) \\ \dot q_2 s_1 + (q_2 + d_{c2}) c_1 \dot q_1 + d_{c3} c_{12}(\dot q_1 + \dot q_2) \end{pmatrix}$

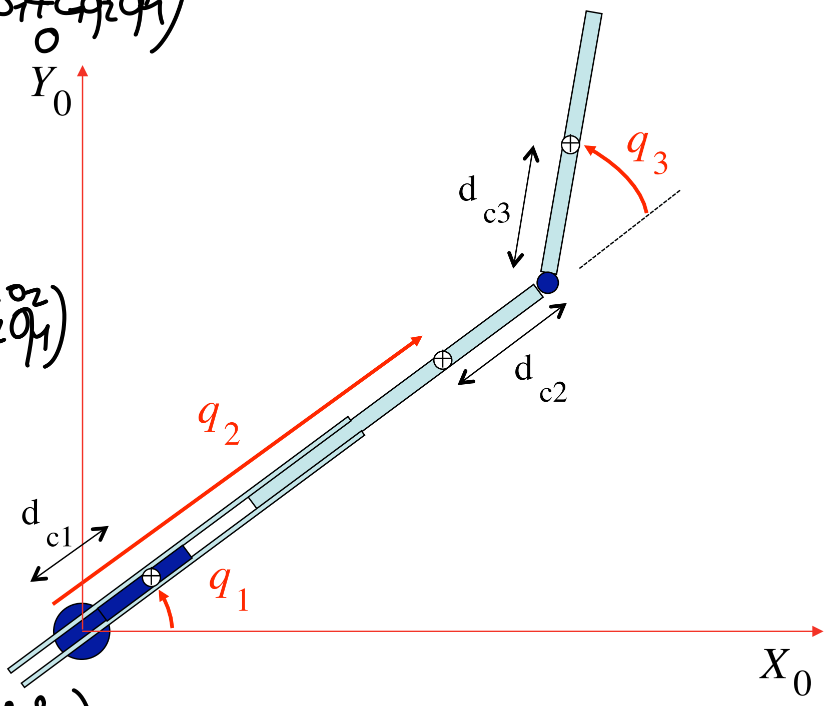


Figure 1: A planar RPR robot

1. Determine the symbolic expression of the robot inertia matrix $\boldsymbol{B}(\boldsymbol{q})$. Explicit all assumptions that are made.

2. Find a set of dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^p$, with a possibly minimal $p$, that provides a linear parameterization of the inertial term in the dynamic model, i.e., $\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{Y}(\boldsymbol{q}, \ddot{\boldsymbol{q}})\boldsymbol{a}$.

**[90 minutes; open books]**

# Solution

September 15, 2010

We compute the robot kinetic energy taking into account that the motion is planar: linear velocities are vectors in the plane $(x, y)$, angular velocities are scalars (in the $z$-direction). With standard notations, for the first link it is:

$$T_1 = \frac{1}{2}\boxed{\left(I_1 + m_1 d_{c1}^2\right)}\dot{q}_1^2.$$

PARALLEL AXIS THEOREM

For the second link, from

$$\boldsymbol{p}_{c2} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \begin{pmatrix} \dot{q}_2 \cos q_1 - q_2 \sin q_1 \dot{q}_1 \\ \dot{q}_2 \sin q_1 + q_2 \cos q_1 \dot{q}_1 \end{pmatrix},$$

SINCE $\omega$ IS SCALAR (ONLY Z COMPONENT) $\Rightarrow$ $\omega^T I \omega = I\omega^2$

we have:

$$T_2 = \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} m_2 \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} = \frac{1}{2}\left(I_2 \dot{q}_1^2 + m_2 (q_2^2 \dot{q}_1^2 + \dot{q}_2^2)\right).$$

I THINK IT'S $\omega_1$

For the third link, from

$$\boldsymbol{p}_{c3} = \begin{pmatrix} (q_2 + d_{c2}) \cos q_1 + d_{c3} \cos(q_1 + q_3) \\ (q_2 + d_{c2}) \sin q_1 + d_{c3} \sin(q_1 + q_3) \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} \dot{q}_2 \cos q_1 - (q_2 + d_{c2}) \sin q_1 \dot{q}_1 - d_{c3} \sin(q_1 + q_3)(\dot{q}_1 + \dot{q}_3) \\ \dot{q}_2 \sin q_1 + (q_2 + d_{c2}) \cos q_1 \dot{q}_1 + d_{c3} \cos(q_1 + q_3)(\dot{q}_1 + \dot{q}_3) \end{pmatrix},$$

we have:

$$\begin{aligned} T_3 &= \frac{1}{2} I_3 \omega_3^2 + \frac{1}{2} m_3 \boldsymbol{v}_{c3}^T \boldsymbol{v}_{c3} = \frac{1}{2} I_3 (\dot{q}_1 + \dot{q}_3)^2 \\ &\quad + \frac{1}{2} m_3 \left((q_2 + d_{c2})^2 \dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2 (\dot{q}_1 + \dot{q}_3)^2 + 2 d_{c3} ((q_2 + d_{c2}) \cos q_3 \dot{q}_1 - \sin q_3 \dot{q}_2)(\dot{q}_1 + \dot{q}_3)\right). \end{aligned}$$

From

$$T = \sum_{i=1}^{3} T_i = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \text{with} \quad \boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix},$$

we obtain for the single elements of the symmetric inertia matrix:

$$\begin{aligned} b_{11} &= I_1 + m_1 d_{c1}^2 + I_2 + m_2 q_2^2 + I_3 + m_3 d_{c3}^2 + m_3 (q_2 + d_{c2})^2 + 2 m_3 d_{c3} (q_2 + d_{c2}) \cos q_3 \\ &= a_1 + a_2 q_2 + a_3 q_2^2 + 2 a_4 \cos q_3 + 2 a_5 q_2 \cos q_3 \end{aligned}$$

$$b_{12} = -m_3 d_{c3} \sin q_3 = -a_5 \sin q_3$$

$$b_{13} = I_3 + m_3 d_{c3}^2 + m_3 d_{c3} (q_2 + d_{c2}) \cos q_3 = a_6 + a_5 q_2 \cos q_3 + a_4 \cos q_3$$

$$b_{22} = m_2 + m_3 = a_3$$

$$b_{23} = -m_3 d_{c3} \sin q_3 = -a_5 \sin q_3$$

$$b_{33} = I_3 + m_3 d_{c3}^2 = a_6.$$

Therefore, the inertia matrix can be compactly rewritten as

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 + a_2 q_2 + a_3 q_2^2 + 2a_4 \cos q_3 + 2a_5 q_2 \cos q_3 & -a_5 \sin q_3 & a_6 + a_5 q_2 \cos q_3 + a_4 \cos q_3 \\ -a_5 \sin q_3 & a_3 & -a_5 \sin q_3 \\ a_6 + a_5 q_2 \cos q_3 + a_4 \cos q_3 & -a_5 \sin q_3 & a_6 \end{pmatrix}$$

and the (minimal) parametrization of $\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{Y}(\boldsymbol{q}, \ddot{\boldsymbol{q}})\boldsymbol{a}$ is thus of dimension $p = 6$, with

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} I_1 + m_1 d_{c1}^2 + I_2 + I_3 + m_3(d_{c2}^2 + d_{c3}^2) \\ 2m_3 d_{c2} \\ m_2 + m_3 \\ m_2 d_{c2} d_{c3} \\ m_3 d_{c3} \\ I_3 + m_3 d_{c3}^2 \end{pmatrix}$$

and

$$\boldsymbol{Y}(\boldsymbol{q}, \ddot{\boldsymbol{q}}) = \begin{pmatrix} \ddot{q}_1 & q_2\ddot{q}_1 & q_2^2\ddot{q}_1 & \cos q_3(2\ddot{q}_1 + \ddot{q}_3) & q_2 \cos q_3(2\ddot{q}_1 + \ddot{q}_3) - \sin q_3 \ddot{q}_2 & \ddot{q}_3 \\ 0 & 0 & \ddot{q}_2 & 0 & -\sin q_3(\ddot{q}_1 + \ddot{q}_3) & 0 \\ 0 & 0 & 0 & \cos q_3 \ddot{q}_1 & q_2 \cos q_3 \ddot{q}_1 - \sin q_3 \ddot{q}_2 & \ddot{q}_1 + \ddot{q}_3 \end{pmatrix}.$$

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