Robotics 2

January 11, 2022

Exercise #1

The RPR robot in Fig. 1 moves on a horizontal plane, carrying at its end effector a payload of mass m_p and inertia I_p. The links of the robot can be considered as uniform rods of length l_i and mass m_i, $i = 1, 2, 3$. The robot control architecture receives as reference input a high-level joint velocity command $\dot{\boldsymbol{q}}_r \in \mathbb{R}^3$.

[i] The first task requires to move the end-effector point P along a desired trajectory $\boldsymbol{p}_d(t) \in \mathbb{R}^2$, while locally minimizing the robot kinetic energy $T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$. Design a high-level control law $\dot{\boldsymbol{q}}_r^{[i]}$ realizing this task and provide the detailed symbolic expression of all its terms.

[ii] Consider next the presence of a circular obstacle $\mathcal{O}_{obs}$ of radius r that is placed in a known position P_{obs} in the robot workspace. Modify the previous control law into a $\dot{\boldsymbol{q}}_r^{[ii]}$ so as to try also avoiding collisions between the full robot body and the obstacle. Provide the symbolic expression of the additional terms in this law.

[iii] Using the following kinematic and dynamic data

$$l_1 = 0.45,\ l_2 = 0.7,\ l_3 = 0.35\ [\text{m}], \quad m_1 = m_2 = 10,\ m_3 = 4,\ m_p = 2\ [\text{kg}], \quad I_p = 0.01\ [\text{kgm}^2],$$

compute the numerical values of the command $\dot{\boldsymbol{q}}_r^{[i]}$ when the robot is in the configuration $\boldsymbol{q} = \bar{\boldsymbol{q}} = (\pi/4, 0.25, -\pi/4)$ [rad,m,rad] and the desired end-effector velocity is $\dot{\boldsymbol{p}}_d = (1, -1)$. For case [ii], repeat the same evaluation for $\dot{\boldsymbol{q}}_r^{[ii]}$, when the circular obstacle is placed at the point $P_{obs} = (0.1 + l_1\sqrt{2}, -0.1) \simeq (0.736, -0.1)$ [m] and has radius $r = 0.05$ [m].

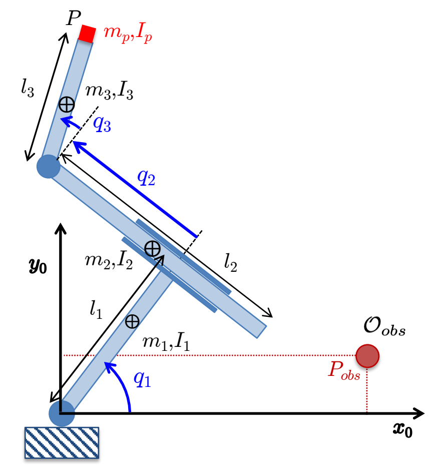

Figure 1: A planar RPR robot with a payload on the end effector. A circular obstacle $\mathcal{O}_{obs}$ may also be present in a generic position P_{obs} in the robot workspace.

Exercise #2

Consider a robot manipulator with n revolute joints and with dynamic model given by

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}. \tag{1}$$

At time $t = 0$, the robot is in a state $\boldsymbol{x}(0) = \begin{pmatrix} \boldsymbol{q}^T(0) & \dot{\boldsymbol{q}}^T(0) \end{pmatrix}^T = \begin{pmatrix} \boldsymbol{q}_0^T & \dot{\boldsymbol{q}}_0^T \end{pmatrix}^T \in \mathbb{R}^{2n}$, with $\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$.

[i] Define a joint torque law $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{x}, t)$ that is continuous w.r.t. time and that will bring the robot with a coordinated joint motion in *exactly* T seconds to an equilibrium state, i.e., to an arbitrary configuration $\boldsymbol{q}(T)$ with $\dot{\boldsymbol{q}}(0) = \mathbf{0}$, where the robot will remain for all $t \geq T$.

[ii] According to your choice, you should be able to provide in closed form both the reached joint configuration $\boldsymbol{q}_f = \boldsymbol{q}(T)$ and the resulting initial acceleration $\ddot{\boldsymbol{q}}_0 = \ddot{\boldsymbol{q}}(0)$.

[iii] Consider the case of a robot with $\boldsymbol{g}(\boldsymbol{q}) \equiv \mathbf{0}$. Assume that acceleration bounds $|\ddot{q}_i| \leq A_{max,i}$, $i = 1, \ldots, n$, are imposed on the already defined robot motion and that at least one of the joints exceeds its bound at some time instant $\bar{t} \in [0, T]$. Provide the expression of the minimum factor $k > 0$ such that the robot trajectory resulting from an uniform scaling of the motion time to $T' = kT$ will satisfy all the given bounds.

Exercise #3

The end effector of the PR robot in Fig. 2 is constrained to move along a line segment, between points A and B. Assume that all dissipative effects are negligible and that the robot dynamic model in free space has the form (1).

[i] Derive the (one-dimensional) *reduced* dynamic model of the constrained robot and the explicit expression of the force multiplier $\lambda \in \mathbb{R}$.

[ii] If the end-effector has to execute a rest-to-rest motion from A to B with a cubic profile in a total time T without generating any constraint force at the contact during the motion, what would be the explicit expression of the control law?

[iii] Using the following data

$$L = 1,\ d_2 = 0.6 \text{ [m]}, \quad m_1 = 15,\ m_2 = 8 \text{ [kg]}, \quad I_2 = 1.2 \text{ [kgm}^2\text{]},$$
$$A = (0.7, 2), \quad B = (0.5, 1) \text{ [m]}, \quad T = 2 \text{ [s]},$$

compute at the initial time $t = 0$ the numerical value $\boldsymbol{\tau}(0) \in \mathbb{R}^2$ of the force/torque commands at the joints that execute the task specified in [ii].

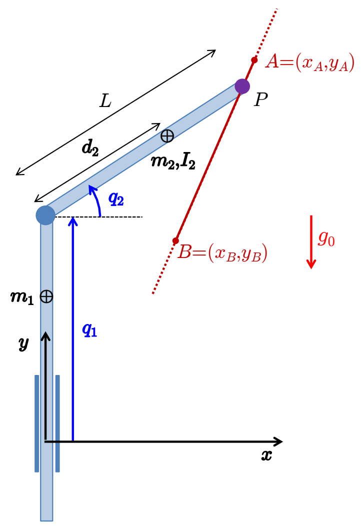

Figure 2: A planar PR robot with its end effector constrained on a segment AB of a line.

[210 minutes (3.5 hours); open books]

Solution

January 11, 2022

Exercise #1

There is a unique solution to the problem of finding a joint velocity command $\dot{\boldsymbol{q}}_r \in \mathbb{R}^N$ (here, $N = 3$) that realizes a given task velocity $\dot{\boldsymbol{p}}_d \in \mathbb{R}^M$, with $M < N$ (here, $M = 2$), by minimizing the kinetic energy of the robot, i.e., for problem [i],

$$\min_{\dot{\boldsymbol{q}}} \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \text{s.t.} \quad \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}_d,$$

where $\boldsymbol{M}(\boldsymbol{q}) > 0$ is the robot inertia matrix and $\boldsymbol{J}(\boldsymbol{q})$ is the task Jacobian, both evaluated at the current configuration $\boldsymbol{q}$. The solution is obtained by the weighted pseudoinverse

$$\dot{\boldsymbol{q}}_r^{[i]} = \boldsymbol{J}^{\#}_{\boldsymbol{M}}(\boldsymbol{q})\, \dot{\boldsymbol{p}}_d = \boldsymbol{M}^{-1}(\boldsymbol{q}) \boldsymbol{J}^T(\boldsymbol{q}) \left(\boldsymbol{J}(\boldsymbol{q}) \boldsymbol{M}^{-1}(\boldsymbol{q}) \boldsymbol{J}^T(\boldsymbol{q}) \right)^{-1} \dot{\boldsymbol{p}}_d. \tag{2}$$

For the planar RPR robot in Fig. 1, the Jacobian matrix is computed from the direct kinematics[1]

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} l_1 c_1 - q_2 s_1 + l_3 c_{13} \\ l_1 s_1 + q_2 c_1 + l_3 s_{13} \end{pmatrix}$$

as

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -(l_1 s_1 + q_2 c_1 + l_3 s_{13}) & -s_2 & -l_3 s_{13} \\ l_1 c_1 - q_2 s_1 + l_3 c_{13} & c_1 & l_3 c_{13} \end{pmatrix}. \tag{3}$$

Next, we compute the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ by extracting its elements from the total robot kinetic energy T. In doing so, we also use the fact that the links are uniform thin rods of length l_i, $i = 1, 2, 3$. The center of mass is then located at the link midpoint (at a distance $d_i = l_i/2$ to the end of the rod), while the barycentric inertia (around an axis normal to the plane) is $I_i = (1/12)\, m_i l_i^2$. The payload will be included in the kinetic energy of the robot.

Kinetic energy and inertia matrix

Link 1

$$T_1 = \frac{1}{2} \left(I_1 + m_1 d_1^2 \right) \dot{q}_1^2 = \frac{1}{2} \frac{m_1 l_1^2}{3} \dot{q}_1^2$$

Link 2

$$T_2 = \frac{1}{2} m_2 \left\| \boldsymbol{v}_{c2} \right\|^2 + \frac{1}{2} I_2 \dot{q}_1^2 \quad \Rightarrow$$

$$\boldsymbol{p}_{c2} = \begin{pmatrix} l_1 c_1 - (q_2 - d_2)\, s_1 \\ l_1 s_1 + (q_2 - d_2)\, c_1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -\left(l_1 s_1 + (q_2 - d_2)\, c_1\right) \dot{q}_1 - s_1 \dot{q}_2 \\ \left(l_1 c_1 - (q_2 - d_2)\, s_1\right) \dot{q}_1 + c_1 \dot{q}_2 \end{pmatrix}$$

$$\Rightarrow \quad \left\| \boldsymbol{v}_{c2} \right\|^2 = \left(l_1^2 + (q_2 - d_2)^2 \right) \dot{q}_1^2 + \dot{q}_2^2 + 2 l_1 \dot{q}_1 \dot{q}_2 \quad \Rightarrow$$

$$\begin{aligned} T_2 &= \frac{1}{2} \left(\left(I_2 + m_2 \left(l_1^2 + (q_2 - d_2)^2 \right) \right) \dot{q}_1^2 + m_2 \dot{q}_2^2 + 2 m_2 l_1 \dot{q}_1 \dot{q}_2 \right) \\ &= \frac{1}{2} \left(m_2 \left(l_1^2 + \frac{l_2^2}{3} + q_2^2 - l_2 q_2 \right) \dot{q}_1^2 + m_2 \dot{q}_2^2 + 2 m_2 l_1 \dot{q}_1 \dot{q}_2 \right) \end{aligned}$$

[1] In the following, the shorthand notation for trigonometric quantities is used (e.g., $s_{13} = \sin(q_1 + q_3)$).

Link 3

$$T_3 = \frac{1}{2}\, m_3 \left\|\boldsymbol{v}_{c3}\right\|^2 + \frac{1}{2}\, I_3 \left(\dot{q}_1 + \dot{q}_3\right)^2 \quad \Rightarrow$$

$$\boldsymbol{p}_{c3} = \begin{pmatrix} l_1 c_1 - q_2 s_1 + d_3 c_{13} \\ l_1 s_1 + q_2 c_1 + d_3 s_{13} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} -\left(l_1 s_1 + q_2 c_1 + d_3 s_{13}\right) \dot{q}_1 - s_1\, \dot{q}_2 - d_3 s_{13}\, \dot{q}_3 \\ \left(l_1 c_1 - q_2 s_1 + d_3 c_{13}\right) \dot{q}_1 + c_1\, \dot{q}_2 + d_3 c_{13}\, \dot{q}_3 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \quad \left\|\boldsymbol{v}_{c3}\right\|^2 &= \left(l_1^2 + q_2^2 + d_3^2 + 2 l_1 d_3 c_3 + 2 q_2 d_3 s_3\right) \dot{q}_1^2 + \dot{q}_2^2 + d_3^2\, \dot{q}_3^2 \\ &\quad + 2\left(l_1 + d_3 c_3\right) \dot{q}_1 \dot{q}_2 + 2 d_3 \left(d_3 + l_1 c_3 + q_2 s_3\right) \dot{q}_1 \dot{q}_3 + 2 d_3 c_3\, \dot{q}_2 \dot{q}_3 \quad \Rightarrow \end{aligned}$$

$$\begin{aligned} T_3 &= \frac{1}{2} \Big(\left(I_3 + m_3 \left(l_1^2 + q_2^2 + d_3^2 + 2 l_1 d_3 c_3 + 2 q_2 d_3 s_3\right)\right) \dot{q}_1^2 + m_3 \dot{q}_2^2 + \left(I_3 + m_3 d_3^2\right) \dot{q}_3^2 \\ &\quad + 2 m_3 \left(l_1 + d_3 c_3\right) \dot{q}_1 \dot{q}_2 + 2\left(I_3 + m_3 d_3 \left(d_3 + l_1 c_3 + q_2 s_3\right)\right) \dot{q}_1 \dot{q}_3 + 2 m_3 d_3 c_3\, \dot{q}_2 \dot{q}_3 \Big) \\ &= \frac{1}{2} \left(m_3 \left(l_1^2 + \frac{l_3^2}{3} + q_2^2 + l_1 l_3 c_3 + q_2 l_3 s_3 \right) \dot{q}_1^2 + m_3\, \dot{q}_2^2 + m_3 \frac{l_3^2}{3}\, \dot{q}_3^2 \right. \\ &\quad \left. + 2 m_3 \left(l_1 + \frac{l_3}{2} \right) \dot{q}_1 \dot{q}_2 + 2 m_3 \left(\frac{l_3^2}{3} + l_1 \frac{l_3}{2} c_3 + q_2 \frac{l_3}{2} s_3 \right) \dot{q}_1 \dot{q}_3 + 2 m_3 \frac{l_3}{2}\, \dot{q}_2 \dot{q}_3 \right) \end{aligned}$$

Payload

$$T_p = \frac{1}{2}\, m_p \left\|\boldsymbol{v}_p\right\|^2 + \frac{1}{2}\, I_p \left(\dot{q}_1 + \dot{q}_3\right)^2 \quad \Rightarrow$$

$$\boldsymbol{p}_p = \begin{pmatrix} l_1 c_1 - q_2 s_1 + l_3 c_{13} \\ l_1 s_1 + q_2 c_1 + l_3 s_{13} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_p = \begin{pmatrix} -\left(l_1 s_1 + q_2 c_1 + l_3 s_{13}\right) \dot{q}_1 - s_1\, \dot{q}_2 - l_3 s_{13}\, \dot{q}_3 \\ \left(l_1 c_1 - q_2 s_1 + l_3 c_{13}\right) \dot{q}_1 + c_1\, \dot{q}_2 + l_3 c_{13}\, \dot{q}_3 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \quad \left\|\boldsymbol{v}_p\right\|^2 &= \left(l_1^2 + q_2^2 + l_3^2 + 2 l_1 l_3 c_3 + 2 q_2 l_3 s_3\right) \dot{q}_1^2 + \dot{q}_2^2 + l_3^2\, \dot{q}_3^2 \\ &\quad + 2\left(l_1 + l_3 c_3\right) \dot{q}_1 \dot{q}_2 + 2 l_3 \left(l_3 + l_2 c_3 + q_3 s_3\right) \dot{q}_1 \dot{q}_3 + 2 l_3 c_3\, \dot{q}_2 \dot{q}_3 \quad \Rightarrow \end{aligned}$$

$$\begin{aligned} T_p &= \frac{1}{2} \Big(\left(I_p + m_p \left(l_1^2 + l_3^2 + q_2^2 + 2 l_1 l_3 c_3 + 2 q_2 l_3 s_3\right)\right) \dot{q}_1^2 + m_p \dot{q}_2^2 + \left(I_p + m_p l_3^2\right) \dot{q}_3^2 \\ &\quad + 2 m_p \left(l_1 + l_3 c_3\right) \dot{q}_1 \dot{q}_2 + 2\left(I_p + m_p \left(l_3^2 + l_1 l_3 c_3 + q_2 l_3 s_3\right)\right) \dot{q}_1 \dot{q}_3 + 2 m_p l_3 c_3\, \dot{q}_2 \dot{q}_3 \Big) \end{aligned}$$

Therefore, from

$$T = T_1 + T_2 + T_3 + T_p = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \frac{1}{2} \sum_{i,j=1}^{3} m_{ij}(\boldsymbol{q})\, \dot{q}_i \dot{q}_j$$

we obtain by extraction the coefficients of $\dot{q}_i \dot{q}_j$ the elements $m_{ij} = m_{ji}$ of the symmetric inertia matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix}, \tag{4}$$

with

$$
\begin{aligned}
m_{11} &= \frac{m_1 l_1^2}{3} + m_2\left(l_1^2 + \frac{l_2^2}{3} + q_2^2 - l_2 q_2\right) + m_3\left(l_1^2 + \frac{l_3^2}{3} + q_2^2 + l_1 l_3 c_3 + q_2 l_3 s_3\right) \\
&\quad + I_p + m_p\left(l_1^2 + l_3^2 + q_2^2 + 2 l_1 l_3 c_3 + 2 q_2 l_3 s_3\right) \\
m_{12} &= \left(m_2 + m_3 + m_p\right) l_1 + \left(m_3 \frac{l_3}{2} + m_p l_3\right) c_3 \\
m_{13} &= I_p + m_p l_3^2 + m_3 \frac{l_3^2}{3} + \left(m_3 \frac{l_3}{2} + m_p l_3\right)\left(l_1 c_3 + q_2 s_3\right) \\
m_{22} &= m_2 + m_3 + m_p \\
m_{23} &= \left(m_3 \frac{l_3}{2} + m_p l_3\right) c_3 \\
m_{33} &= m_3 \frac{l_3^2}{3} + I_p + m_p l_3^2.
\end{aligned}
$$

Using the given data, we have from (3)

$$
\boldsymbol{J}(\bar{\boldsymbol{q}}) = \begin{pmatrix} -0.495 & -0.7071 & 0 \\ 0.4914 & 0.7071 & 0.35 \end{pmatrix}
$$

and from (4)

$$
\boldsymbol{M}(\bar{\boldsymbol{q}}) = \begin{pmatrix} 5.6126 & 8.1899 & 0.6163 \\ 8.1899 & 16 & 0.9899 \\ 0.6163 & 0.9899 & 0.4183 \end{pmatrix}.
$$

Thus, we compute numerically the solution (2) obtaining

$$
\dot{\boldsymbol{q}}_r^{[i]} = \boldsymbol{J}_{\boldsymbol{M}}^{\#}(\bar{\boldsymbol{q}})\, \dot{\boldsymbol{p}}_d = \begin{pmatrix} -2.0329 & 0.1042 \\ 0.0088 & -0.0729 \\ 2.8365 & 2.8582 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2.1371 \\ 0.0818 \\ -0.0217 \end{pmatrix} \quad \text{[rad/s,m/s,rad/s]}. \tag{5}
$$

For case [ii], the solution includes a null space term with the gradient of the distance function $H_{obs}(\boldsymbol{q})$ between the fixed obstacle $\mathcal{O}_{obs}$ and the entire robot body $\mathcal{R} = \mathcal{R}(\boldsymbol{q})$ —the robot occupies a region in the Cartesian space that depends indeed on the current configuration $\boldsymbol{q}$. This results in

$$
\begin{aligned}
\dot{\boldsymbol{q}}_r^{[ii]} &= \boldsymbol{J}_{\boldsymbol{M}}^{\#}(\boldsymbol{q})\, \dot{\boldsymbol{p}}_d + \alpha\left(\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{M}}^{\#}(\boldsymbol{q}) \boldsymbol{J}(\boldsymbol{q})\right) \nabla_q H_{obs}(\boldsymbol{q}) \\
&= \alpha \nabla_q H_{obs}(\boldsymbol{q}) + \boldsymbol{J}_{\boldsymbol{M}}^{\#}(\boldsymbol{q})\big(\dot{\boldsymbol{p}}_d - \alpha \boldsymbol{J}(\boldsymbol{q}) \nabla_q H_{obs}(\boldsymbol{q})\big),
\end{aligned} \tag{6}
$$

for a sufficiently small step size $\alpha > 0$. The distance function from the obstacle (also known as the *clearance* of the robot) is defined as

$$
H_{obs}(\boldsymbol{q}) = \min_{\substack{\boldsymbol{a}(\boldsymbol{q}) \in \mathcal{R} \\ \boldsymbol{b} \in \mathcal{O}_{obs}}} \|\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{b}\|, \tag{7}
$$

with the Euclidean norm

$$
\|\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{b}\| = \sqrt{(\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{b})^T (\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{b})}.
$$

Accordingly, the gradient of $H_{obs}(\boldsymbol{q})$ is evaluated[2] as

$$\nabla_q H_{obs}(\boldsymbol{q}) = \left(\frac{\partial H_{obs}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \frac{1}{2}\,\frac{1}{\|\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{b}\|}\left(\frac{\partial \boldsymbol{a}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T (\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{b}). \tag{8}$$

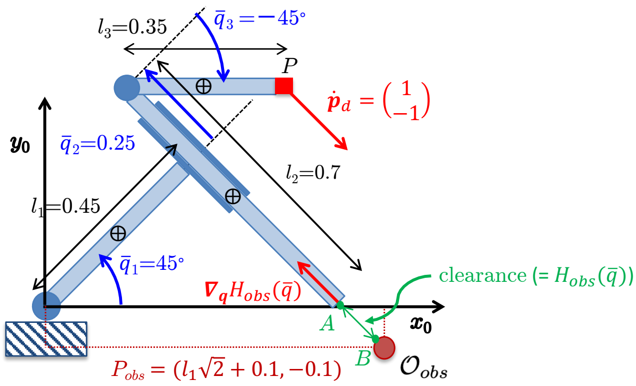

Figure 3: The clearance between the RPR robot and the circular obstacle $\mathcal{O}_{obs}$ when the robot is in the configuration $\bar{\boldsymbol{q}} = (\pi/4, 0.25, -\pi/4)$.

We use now the given data and illustrate the situation in Fig. 3. At $\boldsymbol{q} = \bar{\boldsymbol{q}}$, the two points on the robot $\mathcal{R}$ and on the obstacle $\mathcal{O}_{obs}$ that are giving the clearance of the robot are point A at the lower end of the second link and point B on the boundary of the circular obstacle (at a distance $r = 0.05$ [m] from its center $P_{obs} = (0.736, -0.1)$ [m]), along the line having the *same* orientation of the second link. For varying $\boldsymbol{q}$, the position of the point A on the robot and its Jacobian are

$$\boldsymbol{a}(\boldsymbol{q}) = \begin{pmatrix} l_1 c_1 - (q_2 - l_2)s_1 \\ l_1 s_1 + (q_2 - l_2)c_1 \end{pmatrix} \quad\Rightarrow\quad \frac{\partial \boldsymbol{a}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -l_1 s_1 - (q_2 - l_2)c_1 & -s_1 & 0 \\ l_1 c_1 - (q_2 - l_2)s_1 & c_1 & 0 \end{pmatrix}.$$

Moreover the position of the point B on the obstacle is given by

$$\boldsymbol{b} = \boldsymbol{p}_{obs} + r\,\frac{\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{p}_{obs}}{\|\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{p}_{obs}\|}.$$

Thus, we have from (7)

$$\boldsymbol{a}(\bar{\boldsymbol{q}}) = \begin{pmatrix} 0.6364 \\ 0 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0.7007 \\ -0.0646 \end{pmatrix} \quad\Rightarrow\quad H_{obs}(\bar{\boldsymbol{q}}) = \|\boldsymbol{a}(\bar{\boldsymbol{q}}) - \boldsymbol{b}\| = 0.0911 \text{ [m]},$$

with the gradient in (8) being

$$\nabla_q H_{obs}(\bar{\boldsymbol{q}}) = \begin{pmatrix} 0.4509 \\ 1 \\ 0 \end{pmatrix}.$$

[2]The expression in (8) holds in situations when the function $H_{obs}(\boldsymbol{q})$ is differentiable. This occurs when the two points on the robot $\mathcal{R}$ and on the obstacle $\mathcal{O}_{obs}$ that determine the minimum distance at the current configuration $\boldsymbol{q}$ do not jump form one location to another. In practice, the gradient $\nabla_q H_{obs}(\boldsymbol{q})$ is often computed numerically by finite differences of $H_{obs}(\boldsymbol{q})$ between two successive configuration $\boldsymbol{q}(t_k)$ and $\boldsymbol{q}(t_{k-1})$ attained during motion (with some safeguarding rule to obtain bounded variations).

Setting for instance $\alpha = 1$, the evaluation of (6) yields the solution

$$\dot{\boldsymbol{q}}_r^{[ii]} = \begin{pmatrix} -3.6742 \\ 1.1577 \\ -0.0373 \end{pmatrix} \text{[rad/s,m/s,rad/s]}. \tag{9}$$

Compare now the two joint velocity commands in (5) and (9). It is evident that the presence of the obstacle will modify the inertia-weighted minimum norm solution by pushing away the second link through the sliding of the second (prismatic) joint in the positive direction. In order to still achieve the desired end-effector velocity $\dot{\boldsymbol{p}}_d$ while compensating for this extra joint motion, the first robot joint in $\dot{\boldsymbol{q}}_r^{[ii]}$ will rotate in the clockwise direction by a larger amount than in $\dot{\boldsymbol{q}}_r^{[i]}$.

Exercise #2

To address the general problem of a robot motion that has to be completed in a prescribed *finite* time, it is convenient to apply first to (1) the nonlinear feedback law

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{u}(t) + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}), \tag{10}$$

obtaining the linear and decoupled system $\ddot{\boldsymbol{q}} = \boldsymbol{u}$. Then, we use the acceleration command $\boldsymbol{u}(t)$, which has to be continuous w.r.t. time for all $t > 0$, to plan a state-to-rest trajectory $\boldsymbol{q}_d(t)$ in exactly T seconds. Since the final configuration is not specified a priori, but the state to be reached at time $t = T$ should be an equilibrium, we will impose the following minimal set of (asymmetric) boundary conditions to the interpolating trajectory:

$$\boldsymbol{q}_d(0) = \boldsymbol{q}_0, \qquad \dot{\boldsymbol{q}}_d(0) = \dot{\boldsymbol{q}}_0 \neq 0, \qquad \dot{\boldsymbol{q}}_d(T) = \boldsymbol{0}, \qquad \ddot{\boldsymbol{q}}_d(T) = \boldsymbol{0}. \tag{11}$$

One can satisfy these conditions by choosing a cubic trajectory in normalized time $\sigma = t/T \in [0, 1]$,

$$\boldsymbol{q}_d(\sigma) = \boldsymbol{a}\,\sigma^3 + \boldsymbol{b}\,\sigma^2 + \boldsymbol{c}\,\sigma + \boldsymbol{d}, \tag{12}$$

with $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^n$, $\boldsymbol{c} \in \mathbb{R}^n$, and $\boldsymbol{d} \in \mathbb{R}^n$. The joint motion will automatically be coordinated since the total time T is the same for all joints. Imposing the boundary conditions (11) on the vector function (12) leads to

$$\boldsymbol{q}_d(0) = \boldsymbol{q}_0 \quad \Rightarrow \quad \boldsymbol{d} = \boldsymbol{q}_0, \qquad\qquad \dot{\boldsymbol{q}}_d(0) = \dot{\boldsymbol{q}}_0 \quad \Rightarrow \quad \boldsymbol{c} = \dot{\boldsymbol{q}}_0,$$

and

$$\begin{cases} \dot{\boldsymbol{q}}_d(1) = \boldsymbol{0} \\ \ddot{\boldsymbol{q}}_d(1) = \boldsymbol{0} \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{1}{T}\,(3\boldsymbol{a} + 2\boldsymbol{b} + \dot{\boldsymbol{q}}_0) = \boldsymbol{0} \\ \dfrac{1}{T^2}\,(6\boldsymbol{a} + 2\boldsymbol{b}) = \boldsymbol{0} \end{cases} \quad \Rightarrow \quad \begin{cases} \boldsymbol{a} = \dfrac{\dot{\boldsymbol{q}}_0}{3} \\ \boldsymbol{b} = -\dot{\boldsymbol{q}}_0. \end{cases}$$

As a result, we obtain

$$\boldsymbol{q}_d(t) = \frac{\dot{\boldsymbol{q}}_0}{3}\left(\frac{t}{T}\right)^3 - \dot{\boldsymbol{q}}_0\left(\frac{t}{T}\right)^2 + \dot{\boldsymbol{q}}_0\left(\frac{t}{T}\right) + \boldsymbol{q}_0, \tag{13}$$

and thus the requested values

$$\boldsymbol{q}_f = \boldsymbol{q}_d(T) = \boldsymbol{q}_0 + \frac{\dot{\boldsymbol{q}}_0}{3}, \qquad\qquad \ddot{\boldsymbol{q}}_0 = \ddot{\boldsymbol{q}}_d(0) = -\frac{2\dot{\boldsymbol{q}}_0}{T^2}. \tag{14}$$

The actual torque law realizing the task will be given by (10) with

$$\boldsymbol{u}(t) = \begin{cases} \ddot{\boldsymbol{q}}_d(t) = \dfrac{2\dot{\boldsymbol{q}}_0}{T^2}\left(\dfrac{t}{T} - 1\right), & t \in [0, T] \\ \boldsymbol{0}, & t \geq T. \end{cases} \tag{15}$$

We note that, after stopping the robot at time $t = T$ in $\boldsymbol{q} = \boldsymbol{q}_f$, the applied torque (10) with (15) will become equal to $\boldsymbol{\tau}(t) = \boldsymbol{g}(\boldsymbol{q}_f)$, keeping the robot in equilibrium for all times $t \geq T$ as requested.

The choice of a cubic function (12) is also convenient for addressing the presence of bounds on the joint accelerations. In fact, the accelerations in (15) are linear functions of time and their maximum (absolute) value is always attained at $t = 0$, as given by the second equation in (14). We have

$$|\ddot{q}_i(t)| \leq A_{max,i}, \quad \forall t \in [0, +\infty) \qquad \Longleftrightarrow \qquad \max_{t \in [0,T]} |\ddot{q}_{d,i}(t)| = \frac{2|\dot{q}_{0,i}|}{T^2} \leq A_{max,i}, \qquad i = 1, \dots, n.$$

Assuming that at least one of the joints exceeds its bound during the planned motion implies that this will happen at $t = 0$. Define the index i^* of the maximum violating joint as

$$i^* = \arg\left\{\max_{i=1,\dots,n} \frac{|\dot{q}_{0,i}|}{A_{max,i}}\right\}, \qquad \text{with } |\ddot{q}_{d,i^*}(0)| = \frac{2|\dot{q}_{0,i^*}|}{T^2} > A_{max,i^*}.$$

Then, the uniform scaling of motion time with minimum $k > 0$ that guarantees satisfaction of all acceleration bounds is given by

$$\frac{2|\dot{q}_{0,i^*}|}{(kT)^2} = A_{max,i^*} \qquad \Rightarrow \qquad k = \frac{1}{T}\sqrt{\frac{2|\dot{q}_{0,i^*}|}{A_{max,i^*}}} > 1 \qquad \Rightarrow \qquad T' = kT. \tag{16}$$

The absence of the gravity term $\boldsymbol{g}(\boldsymbol{q})$ plays no role in the solution. In that case, when the robot comes to rest at the end of the motion, the control law will simply vanish ($\boldsymbol{\tau}(t) = \boldsymbol{0}$, for $t \geq T$).

Exercise #3

We derive first the dynamic model of the PR robot in Fig. 2 when moving in an unconstrained way in the vertical plane (under gravity).

Kinetic energy and inertia matrix

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} I_2 \dot{q}_2^2,$$

$$\boldsymbol{p}_{c2} = \begin{pmatrix} d_2 \cos q_2 \\ q_1 + d_2 \sin q_2 \end{pmatrix} \;\Rightarrow\; \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -d_2 \sin q_2 \, \dot{q}_2 \\ \dot{q}_1 + d_2 \cos q_2 \, \dot{q}_2 \end{pmatrix} \;\Rightarrow\; \|\boldsymbol{v}_{c2}\|^2 = \dot{q}_1^2 + d_2^2 \dot{q}_2^2 + 2 d_2 \cos q_2 \, \dot{q}_1 \dot{q}_2$$

$$\Rightarrow \quad T = T_1 + T_2 = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \text{with} \quad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1 + m_2 & m_2 d_2 \cos q_2 \\ m_2 d_2 \cos q_2 & I_2 + m_2 d_{c2}^2 \end{pmatrix}.$$

Potential energy and gravity vector

$$U_1 = m_1 g_0 \, q_1, \qquad U_2 = m_2 g_0 \, (q_1 + \sin q_2) \quad \Rightarrow \quad U = U_1 + U_2$$

$$\Rightarrow \quad \boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0 \, (m_1 + m_2) \\ g_0 m_2 d_2 \cos q_2 \end{pmatrix}.$$

Coriolis and centrifugal vector

$$
\begin{aligned}
& \boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{m}_1}{\partial \boldsymbol{q}}\right) + \left(\frac{\partial \boldsymbol{m}_1}{\partial \boldsymbol{q}}\right)^T - \left(\frac{\partial \boldsymbol{M}}{\partial q_1}\right)\right) = \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_2 \sin q_2 \end{pmatrix} \\
\Rightarrow \quad & c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \dot{\boldsymbol{q}} = -m_2 d_2 \sin q_2 \, \dot{q}_2^2 \\
& \boldsymbol{C}_2(\boldsymbol{q}) = \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{m}_2}{\partial \boldsymbol{q}}\right) + \left(\frac{\partial \boldsymbol{m}_2}{\partial \boldsymbol{q}}\right)^T - \left(\frac{\partial \boldsymbol{M}}{\partial q_2}\right)\right) = \boldsymbol{0} \quad \Rightarrow \quad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \\
\Rightarrow \quad & \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -m_2 d_2 \sin q_2 \, \dot{q}_2^2 \\ 0 \end{pmatrix}.
\end{aligned}
$$

Dynamic model

$$
\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \quad \Longleftrightarrow
$$

$$
\begin{cases} (m_1 + m_2)\,\ddot{q}_1 + m_2 d_2 \cos q_2 \,\ddot{q}_2 - m_2 d_2 \sin q_2 \,\dot{q}_2^2 + g_0\,(m_1 + m_2) = \tau_1 \\ m_2 d_2 \cos q_2 \,\ddot{q}_1 + \left(I_2 + m_2 d_2^2\right)\ddot{q}_2 + g_0 m_2 d_2 \cos q_2 = \tau_2. \end{cases} \tag{17}
$$

We write next the Cartesian constraint on the end-effector position $\boldsymbol{p} = (p_x, p_y)$: point P should to belong to the line $\mathcal{L}$ passing through the two points A and B. Assuming that both $x_A \neq x_B$ and $y_A \neq y_B$ hold true, we can use the parametric expression of the line

$$
\mathcal{L}: \quad \frac{x - x_B}{x_A - x_B} = \frac{y - y_B}{y_A - y_B} \quad \Rightarrow \quad P \in \mathcal{L}: \; k(\boldsymbol{p}) = \frac{p_y - y_B}{y_A - y_B} - \frac{p_x - x_B}{x_A - x_B} = 0. \tag{18}
$$

Substituting the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ for the point P in (18) yields

$$
\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} L\cos q_2 \\ q_1 + L\sin q_2 \end{pmatrix} \quad \Rightarrow \quad h(\boldsymbol{q}) = k\left(\boldsymbol{f}(\boldsymbol{q})\right) = \frac{q_1 + L\sin q_2 - y_B}{y_A - y_B} - \frac{L\cos q_2 - x_B}{x_A - x_B} = 0, \tag{19}
$$

with the Jacobian of the scalar constraint given by

$$
\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} \dfrac{1}{y_A - y_B} & \dfrac{L\cos q_2}{y_A - y_B} + \dfrac{L\sin q_2}{x_A - x_B} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}. \tag{20}
$$

In the assumed hypothesis on the relative location of the two points A and B, the matrix $\boldsymbol{A}(\boldsymbol{q})$ is always well defined and has full rank.

In order to obtain the reduced dynamic model of the constrained PR robot, the basic step is to define a 1×2 (row) vector $\boldsymbol{D}(\boldsymbol{q})$ that is linearly independent from $\boldsymbol{A}(\boldsymbol{q})$. A simple choice is the following constant matrix with rank one:

$$
\boldsymbol{D} = \begin{pmatrix} 0 & y_A - y_B \end{pmatrix} \quad \Rightarrow \quad \det \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D} \end{pmatrix} = 1. \tag{21}
$$

Thus,

$$
\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D} \end{pmatrix}^{-1} = \begin{pmatrix} y_A - y_B & -\left(\dfrac{L\cos q_2}{y_A - y_B} + \dfrac{L\sin q_2}{x_A - x_B}\right) \\ 0 & \dfrac{1}{y_A - y_B} \end{pmatrix} \triangleq \begin{pmatrix} \boldsymbol{E} & \boldsymbol{F}(\boldsymbol{q}) \end{pmatrix}.
$$

We define then the pseudo-velocity $v \in \mathbb{R}$ on the Cartesian line and the inverse mapping to $\dot{\boldsymbol{q}}$ as

$$v = \boldsymbol{D}\,\dot{\boldsymbol{q}} = (y_A - y_B)\,\dot{q}_2\,, \qquad \dot{\boldsymbol{q}} = \boldsymbol{F}(\boldsymbol{q})v = \begin{pmatrix} -\left(\dfrac{L\cos q_2}{y_A - y_B} + \dfrac{L\sin q_2}{x_A - x_B}\right) \\ \dfrac{1}{y_A - y_B} \end{pmatrix} v. \tag{22}$$

Being

$$\dot{\boldsymbol{D}} = \boldsymbol{0} \qquad \text{and} \qquad \dot{\boldsymbol{A}}(\boldsymbol{q}) = \begin{pmatrix} 0 & \left(\dfrac{L\cos q_2}{x_A - x_B} - \dfrac{L\sin q_2}{y_A - y_B}\right)\dot{q}_2 \end{pmatrix},$$

the reduced dynamic model of the constrained PR robot is given by the single differential equation

$$\left(\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\right)\dot{v} = \boldsymbol{F}^T(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right) + \boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{E}\,\dot{\boldsymbol{A}}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\,v, \tag{23}$$

with the scalars

$$\begin{aligned}\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) &= (m_1 + m_2)\left(\frac{L\cos q_2}{y_A - y_B} + \frac{L\sin q_2}{x_A - x_B}\right)^2 + \frac{I_2 + m_2 d_{c2}^2}{(y_A - y_B)^2} \\ &\quad - 2\,\frac{m_2 d_2 \cos q_2}{y_A - y_B}\left(\frac{L\cos q_2}{y_A - y_B} + \frac{L\sin q_2}{x_A - x_B}\right)\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{E}\,\dot{\boldsymbol{A}}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) &= -(m_1 + m_2)\left(\frac{L\cos q_2}{x_A - x_B} - \frac{L\sin q_2}{y_A - y_B}\right)^2 \dot{q}_2 \\ &\quad \frac{m_2 d_2 \cos q_2}{y_A - y_B}\left(\frac{L\cos q_2}{x_A - x_B} - \frac{L\sin q_2}{y_A - y_B}\right)\dot{q}_2\,.\end{aligned}$$

Similarly, the multiplier $\lambda \in \mathbb{R}$ that produces the normal force when attempting to violate the constraint is

$$\lambda = \left(\boldsymbol{E}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\right)\dot{v} - \left(\boldsymbol{E}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{E}\,\dot{\boldsymbol{A}}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\right)v + \boldsymbol{E}^T\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{\tau}\right), \tag{24}$$

with the scalars

$$\boldsymbol{E}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = (y_A - y_B)\left(\frac{m_2 d_2\cos q_2}{y_A - y_B} - (m_1 + m_2)\left(\frac{L\cos q_2}{y_A - y_B} + \frac{L\sin q_2}{x_A - x_B}\right)\right)$$

and

$$\boldsymbol{E}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{E}\,\dot{\boldsymbol{A}}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = (m_1 + m_2)(y_A - y_B)\left(\frac{L\cos q_2}{x_A - x_B} - \frac{L\sin q_2}{y_A - y_B}\right)\dot{q}_2\,.$$

The desired rest-to-rest motion from A to B on the line $\mathcal{L}$ in time T is planned in a parametric way by defining the path as

$$\boldsymbol{p}_d(s) = \boldsymbol{p}_A + s\,\frac{\boldsymbol{p}_B - \boldsymbol{p}_A}{\Delta}, \qquad s \in [0, \Delta]\,, \qquad \Delta = \|\boldsymbol{p}_B - \boldsymbol{p}_A\| = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2},$$

where $\boldsymbol{p}_A \in \mathbb{R}^2$ and $\boldsymbol{p}_B \in \mathbb{R}^2$ are, respectively, the position vectors of point A and point B, and the timing law with a cubic profile as

$$s = s_d(t) = \Delta\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right), \qquad t \in [0, T]\,.$$

Note that the parameter s is here the actual length of the path traced during motion. The desired pseudo-velocity and pseudo-acceleration are then

$$v_d(t) = \dot{s}_d(t) = 6\Delta\,\frac{t}{T}\left(1-\frac{t}{T}\right), \qquad \dot{v}_d(t) = \frac{6\Delta}{T^2}\left(1-2\,\frac{t}{T}\right), \qquad t\in[0,T]\,.$$

At time $t=0$, the position $\boldsymbol{p}$ of the robot end effector should be matched with the position $\boldsymbol{p}_A$ of the point A (on the linear constraint $h(\boldsymbol{q})=0$). The initial configuration $\boldsymbol{q}(0)$ is obtained by solving the inverse kinematics problem for the PR robot

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} L\cos q_2 \\ q_1 + L\sin q_2 \end{pmatrix} = \begin{pmatrix} x_A \\ y_A \end{pmatrix} = \boldsymbol{p}_A.$$

This yields[3]

$$\boldsymbol{q}_0 = \boldsymbol{q}(0) = \boldsymbol{f}^{-1}(\boldsymbol{p}_A) = \begin{pmatrix} y_A - \sin\left(\arccos\left(\frac{x_A}{L}\right)\right) \\ \arccos\left(\frac{x_A}{L}\right) \end{pmatrix} = \begin{pmatrix} y_A - \sqrt{1-\left(\frac{x_A}{L}\right)^2} \\ \arccos\left(\frac{x_A}{L}\right) \end{pmatrix}.$$

Moreover, since the robot starts at rest, it is $\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{q}}(0) = \boldsymbol{0}$ (consistently with $v_d(0)=0$).

To execute the desired constrained motion with $\dot{v} = \dot{v}_d(t)$ and $\lambda = \lambda_d = 0$ (no forces are generated normal to the line $\mathcal{L}$), we apply the inverse constrained dynamics control law

q_ddot is from slide #14 n. 33

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\,\dot{v}_d + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{M}(\boldsymbol{q})\boldsymbol{E}\,\dot{\boldsymbol{A}}(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{25}$$

with

$$\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = \begin{pmatrix} -(m_1+m_2)\left(\dfrac{L\cos q_2}{x_A-x_B} - \dfrac{L\sin q_2}{y_A-y_B}\right) + \dfrac{m_2 d_2\cos q_2}{y_A-y_B} \\ -m_2 d_2\cos q_2\left(\dfrac{L\cos q_2}{x_A-x_B} - \dfrac{L\sin q_2}{y_A-y_B}\right) + \dfrac{I_2+m_2 d_2^2}{y_A-y_B} \end{pmatrix}$$

and

$$\boldsymbol{M}(\boldsymbol{q})\boldsymbol{E}\,\dot{\boldsymbol{A}}(\boldsymbol{q}) = \begin{pmatrix} 0 & (m_1+m_2)\,(y_A-y_B)\left(\dfrac{L\cos q_2}{x_A-x_B} - \dfrac{L\sin q_2}{y_A-y_B}\right)\dot{q}_2 \\ 0 & m_2 d_2\cos q_2\,(y_A-y_B)\left(\dfrac{L\cos q_2}{x_A-x_B} - \dfrac{L\sin q_2}{y_A-y_B}\right)\dot{q}_2 \end{pmatrix}.$$

At time $t=0$, we have

$$\boldsymbol{\tau}_0 = \boldsymbol{\tau}(0) = \boldsymbol{M}(\boldsymbol{q}_0)\boldsymbol{F}(\boldsymbol{q}_0)\dot{v}_d(0) + \boldsymbol{g}(\boldsymbol{q}_0) \tag{26}$$

With the given numerical data, we compute the following relevant quantities: the path length and the initial pseudo-acceleration

$$\Delta = 1.0198 \text{ [m]}, \qquad \dot{v}_d(0) = 1.5297 \text{ [m/s}^2\text{]};$$

the inverse kinematics solution at the point A

$$\boldsymbol{q}_0 = \begin{pmatrix} 1.2859 \\ 0.7954 \end{pmatrix} \text{ [m,rad]};$$

[3] We chose arbitrarily only one of the two inverse solutions. The other solution has a sign $-$ in front of arccos and a sign $+$ in front of the square root.

the inertia matrix, the gravity vector, and the $\boldsymbol{F}$ term evaluated at the initial configuration $\boldsymbol{q}_0$

$$\boldsymbol{M}(\boldsymbol{q}_0) = \begin{pmatrix} 23 & 3.36 \\ 3.36 & 4.08 \end{pmatrix}, \qquad \boldsymbol{g}(\boldsymbol{q}_0) = \begin{pmatrix} 225.63 \\ 32.96 \end{pmatrix}, \qquad \boldsymbol{F}(\boldsymbol{q}_0) = \begin{pmatrix} 4.27 \\ 1 \end{pmatrix}.$$

Finally, plugging in (26) the above values, we obtain

$$\boldsymbol{\tau}_0 = \begin{pmatrix} 381.02 \\ 61.15 \end{pmatrix} \text{ [N,Nm]}.$$

* * * * *