Robotics 2

Regulation in the Joint Space

(with an introduction to stability)

Prof. Alessandro De Luca

Dipartimento di Ingegneria Informatica
Automatica e Gestionale Antonio Ruberti

Sapienza
Università di Roma

Equilibrium states of a robot

$$M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$$

$$\Rightarrow \quad \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -M^{-1}(x_1)[c(x_1,x_2) + g(x_1)] \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}(x_1) \end{pmatrix} u$$

$$\uparrow = 0 \qquad = f(x) + G(x_1)u$$

x_e unforced equilibrium $(u = 0)$ $\Rightarrow$ $f(x_e) = 0$ $\Rightarrow$ $\begin{cases} x_{e2} = 0 \\ g(x_{e1}) = 0 \end{cases}$

x_e forced equilibrium $(u = u(x))$ $\Rightarrow$ $f(x_e) + G(x_{e1})u(x_e) = 0$ $\Rightarrow$ $\begin{cases} x_{e2} = 0 \\ u(x_e) = g(x_{e1}) \end{cases}$

all equilibrium states of mechanical systems have zero velocity!

joint torques must balance gravity at the equilibrium!

Stability of dynamical systems

definitions - 1

$\dot{x} = f(x)$ e.g., a closed-loop system (under feedback control)

x_e equilibrium: $f(x_e) = 0$

(sometimes we consider as equilibrium state $x_e = 0$, e.g., when using errors as variables)

stability of x_e

$$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0: \; \|x(t_0) - x_e\| < \delta_\varepsilon \implies \|x(t) - x_e\| < \varepsilon, \forall t \geq t_0$$

asymptotic stability of x_e stability +

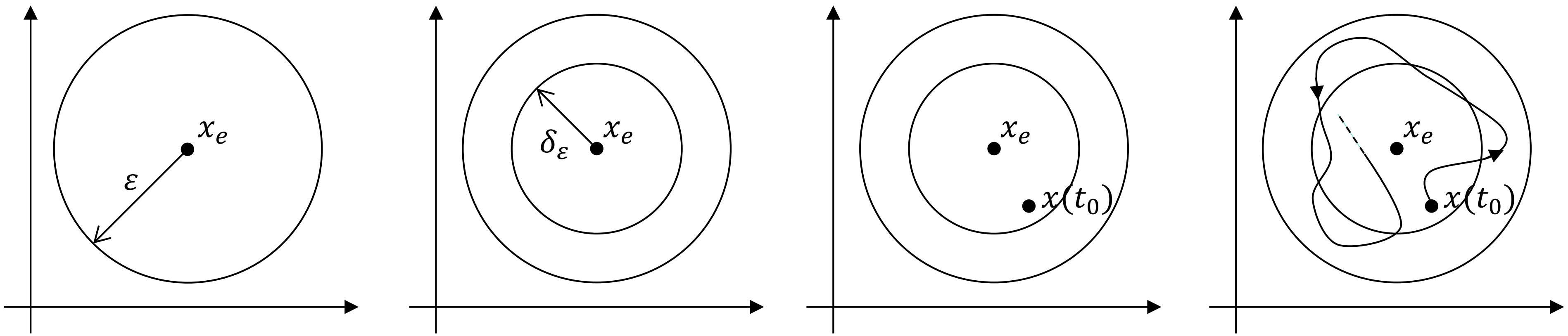
$$\exists \delta > 0: \; \|x(t_0) - x_e\| < \delta \implies \|x(t) - x_e\| \to 0, \text{ for } t \to \infty$$

asymptotic stability may become global ($\forall \delta > 0$, finite)

note: these are definitions of stability "in the sense of Lyapunov"

Stability vs. asymptotic stability

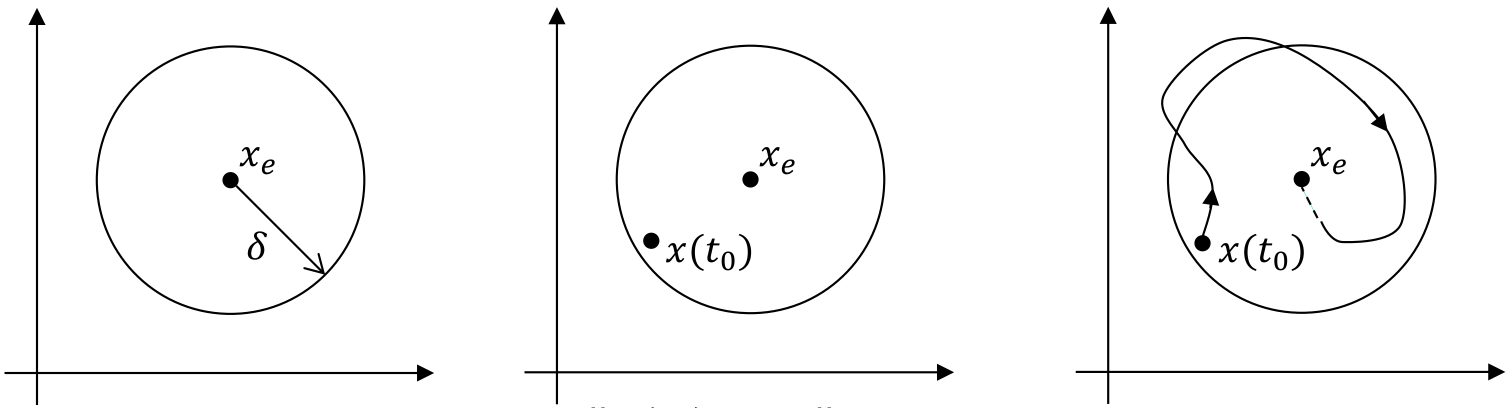
whiteboard…



$\forall \varepsilon > 0$ $\quad \exists \delta_\varepsilon > 0$ $\quad \|x(t_0) - x_e\| < \delta_\varepsilon$ $\quad \Longrightarrow \|x(t) - x_e\| < \varepsilon, \forall t \geq t_0$

equilibrium state x_e is stable

+



$\exists \delta > 0$ $\quad \|x(t_0) - x_e\| < \delta$ $\quad \Longrightarrow \|x(t) - x_e\| \to 0,$ for $t \to \infty$

equilibrium state x_e is asymptotically stable

Stability of dynamical systems

definitions - 2

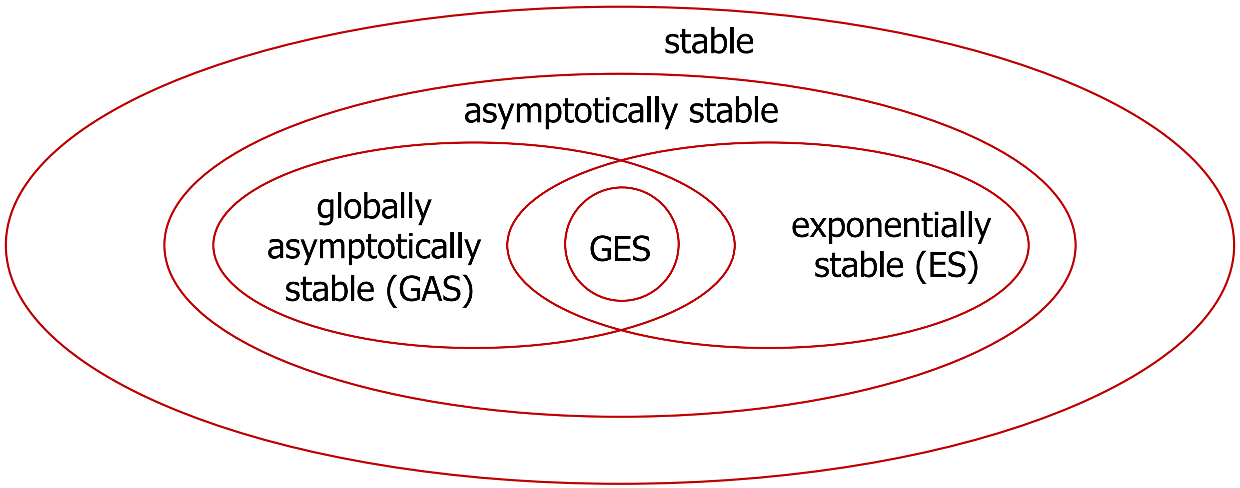
exponential stability of x_e

exponential rate λ

$$\exists \delta, c, \lambda > 0: \|x(t_0) - x_e\| < \delta \Longrightarrow \|x(t) - x_e\| \leq c e^{-\lambda(t-t_0)} \|x(t_0) - x_e\|$$

- allows to estimate the time needed to "approximately" converge: for $c = 1$, in $t - t_0 = 3 \times$ the time constant $\tau = 1/\lambda$, the initial error is reduced to 5%
- typically, this is a local property only (within some maximum finite radius δ) $\Rightarrow$ such "domain of attraction" is hard to be estimated accurately

taxonomy of stability definitions



a **necessary** condition for x_e to be GAS is that it is the **only** equilibrium state of the system

The need for analysis and criteria

whiteboard…

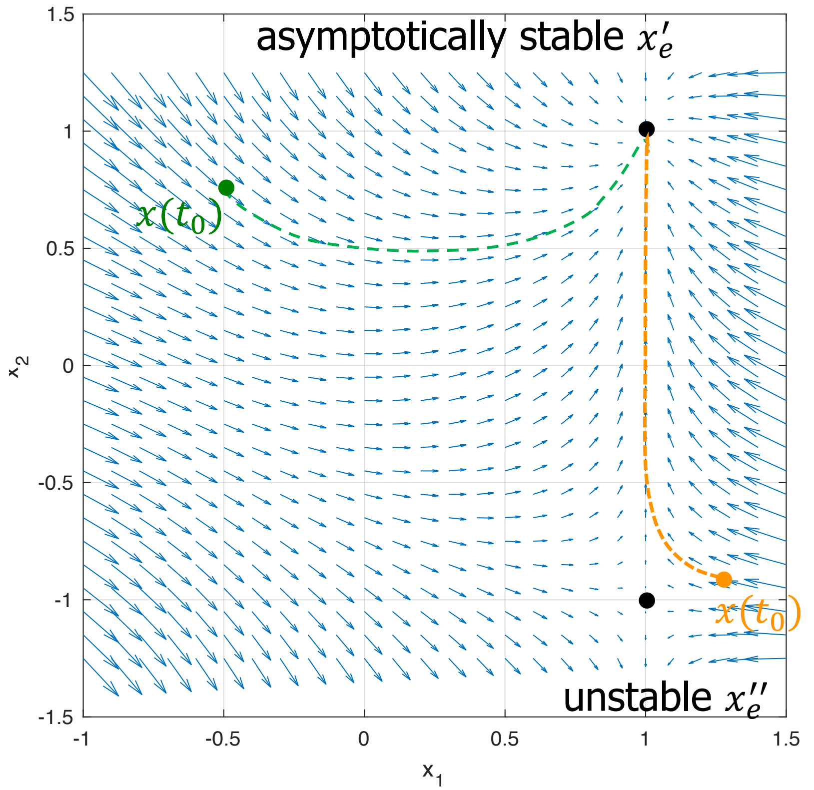
a nonlinear system $\dot{x} = f(x)$ in $\mathbb{R}^2$

$$\begin{cases} \dot{x}_1 = 1 - x_1^3 \\ \dot{x}_2 = x_1 - x_2^2 \end{cases}$$

two equilibria $f(x_e) = 0$

$$x_e' = (1,1), \quad x_e'' = (1,-1)$$

to assess (asymptotic) stability [or not] of equilibria, do we need to compute all system trajectories, starting from all possible initial states $x(t_0)$?



rather, we may be able to just look at the time evolution of a scalar function V, evaluated analytically along the state trajectories of the system (even in $\mathbb{R}^n$!)

Stability of dynamical systems

results - 1

Lyapunov candidate $V(x)\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$V(x_e) = 0, V(x) > 0, \forall x \neq x_e$$

positive definite function

typically, quadratic (e.g., $\frac{1}{2}(x - x_e)^T P(x - x_e)$) with level surfaces = ellipsoids)

may also be a local candidate only ($\forall x \neq x_e$: $\|x - x_e\| < \delta$)

sufficient condition of stability

$\exists V$ candidate: $\dot{V}(x) \leq 0$, along the trajectories of $\dot{x} = f(x)$

negative semi-definite function

sufficient condition of asymptotic stability

$\exists V$ candidate: $\dot{V}(x) < 0$, along the trajectories of $\dot{x} = f(x)$

negative definite function

sufficient condition of instability

$\exists V$ candidate: $\dot{V}(x) > 0$, along the trajectories of $\dot{x} = f(x)$

Stability of dynamical systems

results - 2

LaSalle Theorem

if $\exists V$ candidate: $\dot{V}(x) \leq 0$ along the trajectories of $\dot{x} = f(x)$

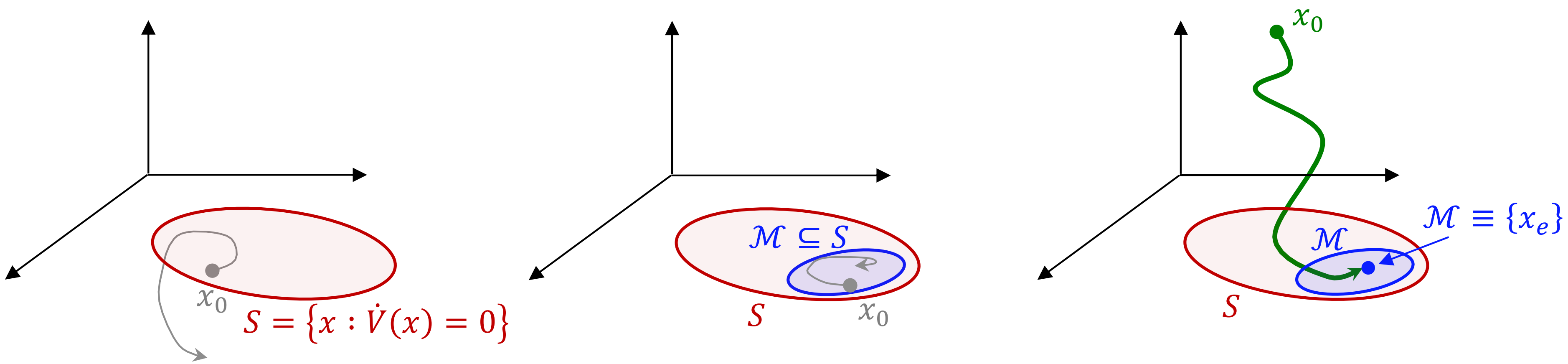
then system trajectories asymptotically converge to the

largest invariant set $\mathcal{M} \subseteq S = \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$

$\mathcal{M}$ is invariant if $x(t_0) \in \mathcal{M} \Longrightarrow x(t) \in \mathcal{M}, \forall t \geq t_0$

Corollary

$\mathcal{M} \equiv \{x_e\}$ ➡ asymptotic stability

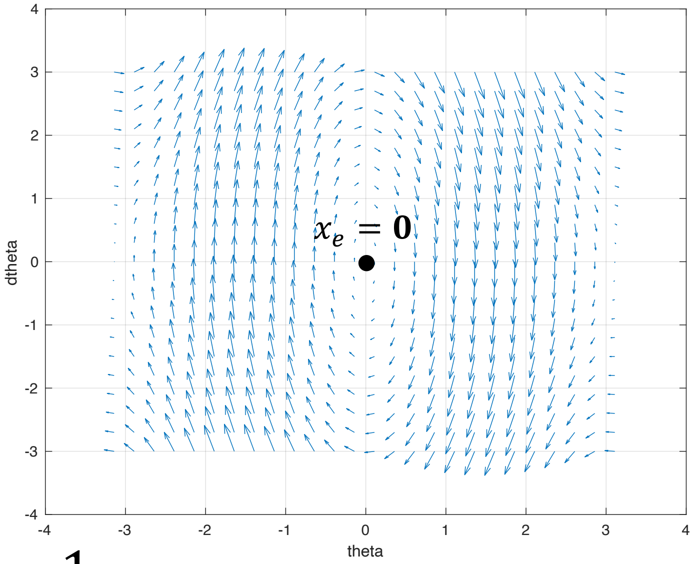


a mass m at the end of an unforced (passive) pendulum of length l

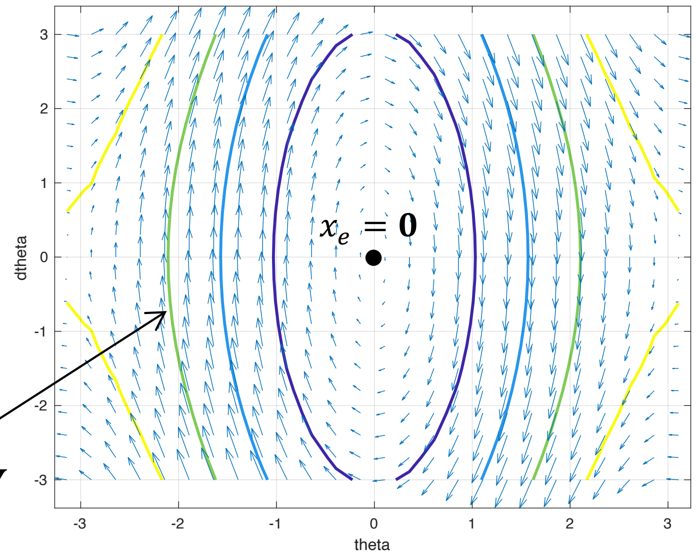
$$ml^2\ddot{\theta} + b\dot{\theta} + mlg_0 \sin\theta = 0 \quad \Rightarrow \quad x = (x_1, x_2) = (\theta, \dot{\theta}) \in \mathbb{R}^2 \quad \Rightarrow \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\left(\dfrac{g_0}{l}\right)\sin x_1 - \left(\dfrac{b}{ml^2}\right) x_2 \end{cases}$$

lower equilibrium at $\theta_e = 0$

phase plane



level sets of V



$$V = E = \frac{1}{2} ml^2\dot{\theta}^2 + mlg_0(1 - \cos\theta) \geq 0 \qquad V = 0 \iff x_e = (\theta_e, \dot{\theta}_e) = (0,0)$$

$$\dot{V} = \dot{\theta}\left(ml^2\ddot{\theta} + mlg_0 \sin\theta\right) = -b\dot{\theta}^2 \leq 0 \quad \Rightarrow$$

stability of equilibrium $x_e = 0$ (… at least!)

$\Rightarrow$ use LaSalle: $\dot{V} = 0 \iff \dot{\theta} = 0 \Rightarrow \ddot{\theta} = -\left(\frac{g_0}{l}\right)\sin\theta \neq 0$ unless $\theta = \theta_e = 0$ (or π!)

$\Rightarrow$ local asymptotic stability

Stability of dynamical systems

results - 3

- previous results are also valid for periodic time-varying systems

$$\dot{x} = f(x,t) = f\left(x, t + T_p\right) \Rightarrow V(x,t) = V\left(x, t + T_p\right)$$

- for general time-varying systems (e.g., in robot trajectory tracking control)

$$\dot{x} = f(x,t)$$

Barbalat Lemma

if i) a function $V(x,t)$ is lower bounded

ii) $\dot{V}(x,t) \leq 0$

then $\Rightarrow \exists \lim_{t\to\infty} V(x,t)$ (but this does not imply that $\lim_{t\to\infty} \dot{V}(x,t) = 0$)

if in addition iii) $\ddot{V}(x,t)$ is bounded

then $\Rightarrow \lim_{t\to\infty} \dot{V}(x,t) = 0$

Corollary

if a Lyapunov candidate $V(x,t)$ satisfies Barbalat Lemma along the trajectories of $\dot{x} = f(x,t)$, then the conclusions of LaSalle Theorem hold

Stability of dynamical systems

additional definition and result (for robust control)

"practical" stability of a set S

$$\exists T(x(t_0), S) \in \mathbb{R}: \; x(t) \in S, \forall t \geq t_0 + T(x(t_0), S)$$

a finite time

also known as u.u.b. stability

$\Rightarrow$ trajectories $x(t)$ are "uniformly ultimately bounded" (use in robust control)

sufficient condition of u.u.b. stability of a set S

$\exists V$ candidate: i) S is a level set of V for a given c_0

$$S = S(c_0) = \{x \in \mathbb{R}^n : V(x) \leq c_0\}$$

ii) $\dot{V}(x) < 0$ along trajectories of $\dot{x} = f(x), x \notin S$

Stability of linear systems

time-invariant case

$\dot{x} = Ax$ $\quad$ $x_e = 0$ is always an equilibrium state

I. asymptotic stability

II. global asymptotic stability

III. exponential stability

IV. $\sigma(A) \subset \mathbb{C}^-$ (all eigenvalues of A have negative real part)

V. $\forall Q > 0$ (positive definite), $\exists !\, P > 0$: $A^T P + PA = -Q$

Lyapunov equation ➡ $\frac{1}{2} x^T P x$ is a Lyapunov candidate

ALL EQUIVALENT !!

if $x_e = 0$ is an asymptotically stable equilibrium,
then it is necessarily the unique equilibrium

Stability of the linear approximation

Let $\Delta x = x - x_e$ and let $\dot{\Delta x} = \frac{df}{dx}|_{x=x_e}(x - x_e) = A\,\Delta x$ be the linear approximation of $\dot{x} = f(x)$ around the equilibrium x_e

A asymptotically stable ($\sigma(A) \subset \mathbb{C}^-$)

$\Updownarrow$

the original nonlinear system is exponentially stable at the origin

this is only a local result
(used also to estimate the domain of attraction)

PD control
(proportional + derivative action on the error)

robot $\boxed{M(q)\ddot{q} + c(q,\dot{q}) + g(q) = u}$

goal: asymptotic stabilization (= regulation) of the closed-loop equilibrium state

$$\boxed{q = q_d, \dot{q} = 0}$$

possibly obtained from kinematic inversion: $q_d = f^{-1}(r_d)$

control law $u = K_P(q_d - q) - K_D\dot{q}$

$K_P > 0, K_D > 0$ (positive definite), symmetric

Asymptotic stability with PD control

Theorem 1

In the absence of gravity ($g(q) \equiv 0$), the robot state $(q_d, 0)$ under the given PD joint control law is globally asymptotically stable

Proof

let $\boxed{e = q_d - q}$ (q_d constant)

Lyapunov candidate $\boxed{V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}e^T K_P e \geq 0}$ $V = 0 \Leftrightarrow e = \dot{e} = 0$

$$\dot{V} = \dot{q}^T M \ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}\dot{q} - e^T K_P \dot{q} = \dot{q}^T \left(u - \underbrace{S\dot{q} + \frac{1}{2}\dot{M}\dot{q}}_{=0,\ \text{due to energy conservation}}\right) - e^T K_P \dot{q}$$

$$= \cancel{\dot{q}^T K_P e} - \dot{q}^T K_D \dot{q} - \cancel{e^T K_P \dot{q}} = -\dot{q}^T K_D \dot{q} \leq 0$$ ($K_D > 0$, symmetric)

up to here, we proved stability only

but $\dot{V} = 0 \Leftrightarrow \dot{q} = 0$

continues ...

Asymptotic stability with PD control

$\dot{V} = 0 \Leftrightarrow \dot{q} = 0$ $\xrightarrow{\text{LaSalle}}$ system trajectories converge to the largest invariant set of states $\mathcal{M}$ where $\dot{q} \equiv 0$ (that is $\dot{q} = \ddot{q} = 0$)

$$\dot{q} = 0 \Rightarrow \underbrace{M(q)\ddot{q} = K_P e}_{\text{closed-loop dynamics}} \Rightarrow \ddot{q} = \underbrace{M^{-1}(q)K_P}_{\text{invertible}} e$$

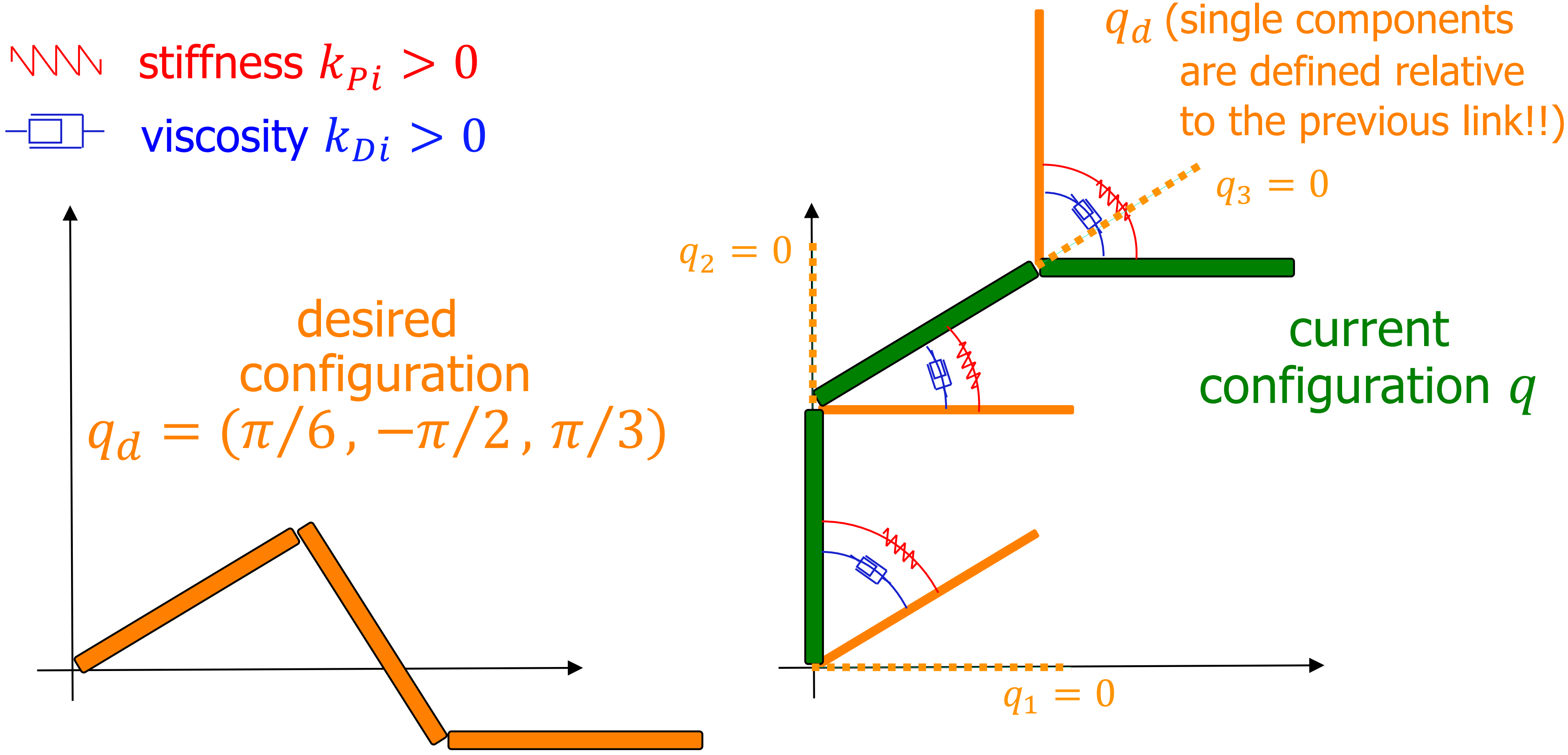
$$\dot{q} = 0, \ddot{q} = 0 \Leftrightarrow e = 0$$

$\Rightarrow$ $\boxed{\text{the only invariant state in } \dot{V} = 0 \text{ is given by } q = q_d, \dot{q} = 0}$

note: typically, $K_P = \text{diag}\{k_{Pi}\}$, $K_D = \text{diag}\{k_{Di}\}$,
$\Rightarrow$ decentralized linear control (local to each joint)

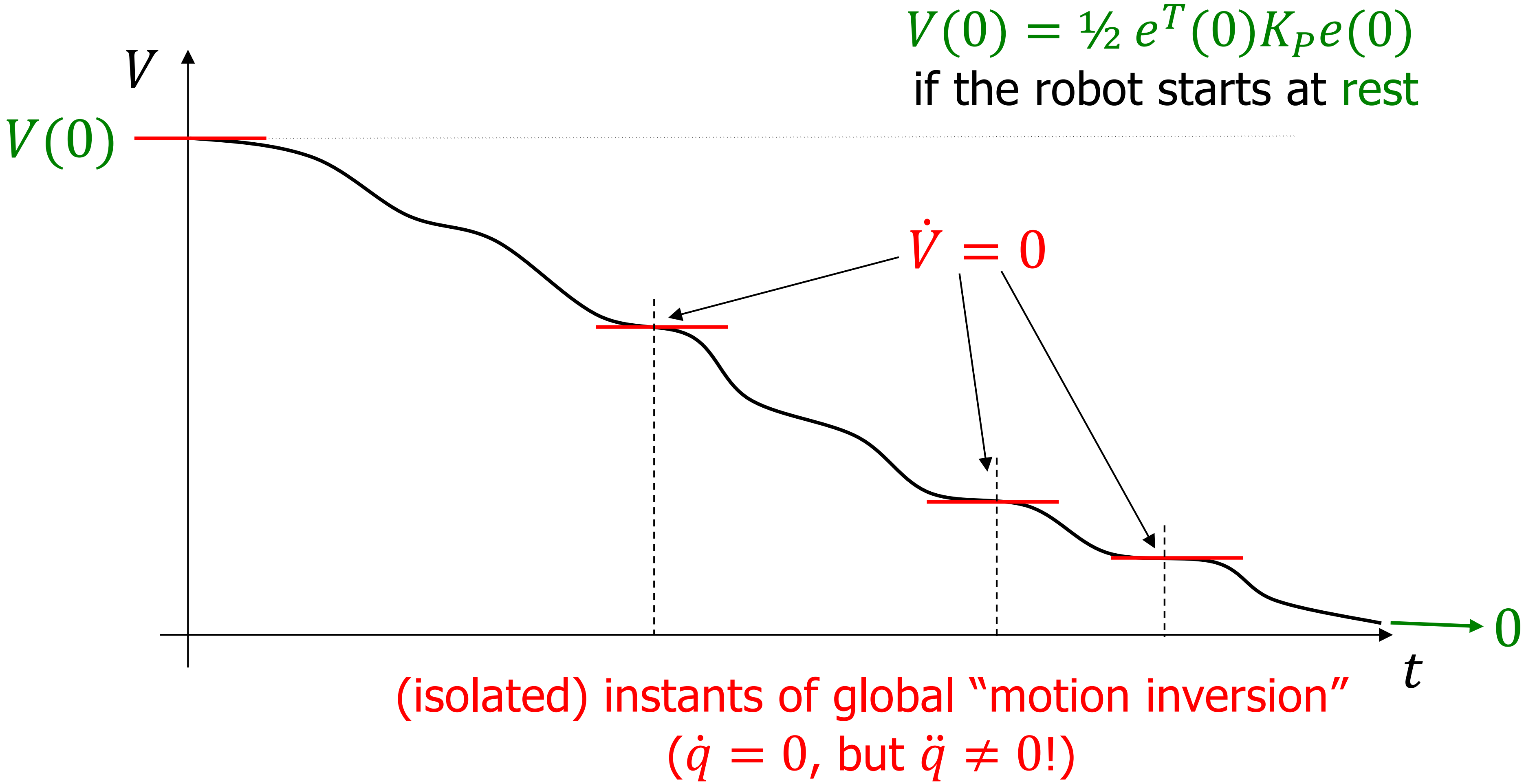
Mechanical interpretation

- for diagonal positive definite gain matrices K_P and K_D (thus, with positive diagonal elements), such values correspond to stiffness of "virtual" springs and viscosity of "virtual" dampers placed at the joints



Plot of the Lyapunov function V

- time evolution of the Lyapunov candidate



Comments on PD control - 1

- choice of control gains affects robot evolution during transients and practical settling times
  - hard to define values that are "optimal" in the whole workspace
  - "full" K_P and K_D gain matrices allow to assign desired eigenvalues to the linear approximation of the robot dynamics around the final desired state $(q_d, 0)$
- when (joint) viscous friction is present, the derivative term in the control law is not strictly necessary
  - $-F_V \dot{q}$ in the robot model acts similarly to $-K_D \dot{q}$ in the control law, but the latter can be modulated at will
- in the absence of tachometers, the actual realization of the derivative term in the feedback law requires some processing of joint position data measured by digital encoders (or analog resolvers/potentiometers)

Comments on PD control - 2

- analog or digital implementation of derivative action in the control law when only position is measured at the joints (e.g., through encoders)

continuous-time control law (design)

$$u(t) = K_P e(t) + K_D \dot{e}(t) \qquad e = q_d - q, \dot{e} = -\dot{q}$$

representation in the Laplace domain

$$u(s) = (K_P + K_D s)\, e(s)$$

not realizable as such (non-proper transfer function)

$$\boxed{u(s) = \left(K_P + \frac{K_D s}{1 + \tau s}\right) e(s)}$$

derivative action limited in bandwidth (up to $\omega \leq 1/\tau$)

transformation in the Zeta-domain (e.g., via backward differentiation rule on samples, every T_c sec)

$$u(z) = \left(K_P + K_D \frac{1 - z^{-1}}{T_c}\right) e(z) \qquad u(z) = \left(K_P + K_D \frac{\frac{1 - z^{-1}}{T_c}}{1 + \tau \frac{1 - z^{-1}}{T_c}}\right) e(z)$$

discrete-time implementations

$$\boxed{u_k = K_P e_k + K_D \frac{e_k - e_{k-1}}{T_c}}$$

$$\boxed{\begin{aligned} u_k = K_P e_k &+ \frac{K_D}{\tau + T_c}(e_k - e_{k-1}) \\ &+ \frac{\tau}{\tau + T_c}(u_{k-1} - K_P e_{k-1}) \end{aligned}}$$

both realizable

Inclusion of gravity

- in the presence of gravity, the same previous arguments (and proof) show that the control law

$$u = K_P(q_d - q) - K_D\dot{q} + g(q) \qquad \boxed{K_P > 0, K_D > 0}$$

will make the equilibrium state $(q_d, 0)$ globally asymptotically stable (nonlinear cancellation of gravity)

- if gravity is not cancelled or only approximately cancelled

$$u = K_P(q_d - q) - K_D\dot{q} + \hat{g}(q) \qquad \hat{g}(q) \neq g(q)$$

it is $q \to q^* \neq q_d, \dot{q} \to 0$, with a steady-state position error

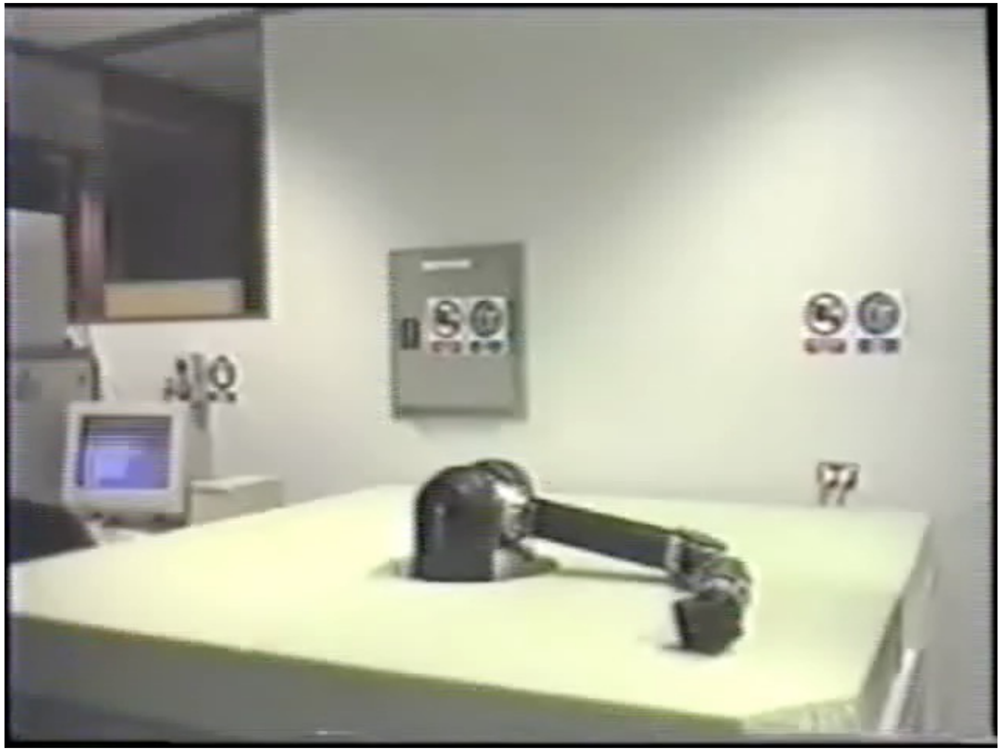
  - q^* is not unique in general, except when K_P is chosen large enough
  - explanation in terms of linear systems: there is no integral action before the point of access of the constant "disturbance" acting on the system

Approximate cancellation of gravity

WAM Barrett
(with some viscous friction)

$$u = \hat{g}(q)$$

$\hat{g}(q) = g(q)$



$\hat{g}(q) \neq g(q)$

two-part video

http://handbookofrobotics.org/view-chapter/69/videodetails/611

PD control + constant gravity compensation

if the robot potential energy $U(q)$ is bounded for all q, then its partial derivative $g(q)$ is also **bounded** everywhere and the following structural property holds

finite

$$\boxed{\exists \alpha > 0 \colon \left\| \frac{\partial^2 U}{\partial q^2} \right\| = \left\| \frac{\partial g}{\partial q} \right\| \leq \alpha, \forall q}$$

consequence ➡ $\|g(q) - g(q_d)\| \leq \alpha \|q - q_d\|$

note: induced norm of a matrix

$$\boxed{\|A\| = \sqrt{\lambda_{\max}(A^T A)} \triangleq A_M \geq A_m \triangleq \sqrt{\lambda_{\min}(A^T A)}}$$

LINEAR CONTROL law

$$\boxed{u = K_P(q_d - q) - K_D \dot{q} + g(q_d)}$$

$K_P, K_D > 0$ symmetric

linear feedback + **constant** feedforward

More on the basic assumption …

(in PD control + gravity compensation)

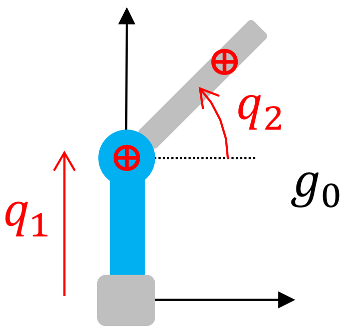
when is the (non-zero) gravity term $g(q)$ bounded for all q?

- the robot has **all revolute** joints
  - all terms in $U(q)$ and thus in $g(q)$ are trigonometric (bounded)
- the robot has both types of joints, but **no prismatic variables in** $g(q)$
  - potential energy $U(q)$ may still be unbounded!
- all **prismatic** joints of the robot have a **limited range**
  - … ok, but one should take these limits into account in the control analysis

$$U = g_0\big(m_1 q_1 + m_2(q_1 + d_{c2}\sin q_2)\big) + U_0$$

$$g(q) = g_0 \begin{pmatrix} (m_1 + m_2) \\ m_2 d_{c2}\cos q_2 \end{pmatrix}$$

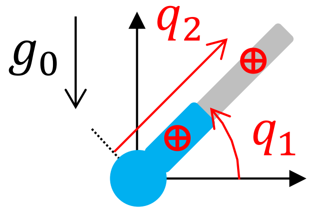
$$\left\|\frac{\partial g}{\partial q}\right\| \le \alpha = m_2 d_{c2} g_0$$



PR robot

$$U = g_0(m_1 d_{c1} + m_2 q_2)\sin q_1 + U_0$$

$$g(q) = g_0 \begin{pmatrix} (m_1 d_{c1} + m_2 q_2)\cos q_1 \\ m_2 \sin q_1 \end{pmatrix}$$



$$\left\|\frac{\partial g}{\partial q}\right\| \le\ ??$$

RP robot

PD control + constant gravity compensation

stability analysis

Theorem 2

K_P,m is the minimum eigenvalue of the K_P

If $\boxed{K_{P,m} > \alpha}$, the state $(q_d, 0)$ of the robot under joint-space PD control + constant gravity compensation at q_d is globally asymptotically stable

Proof

1. $(q_d, 0)$ is the unique closed-loop equilibrium state

in fact, for $\dot{q} = 0$ and $\ddot{q} = 0$, it is $K_P e = g(q) - g(q_d)$

which can hold only for $q = q_d$, because when $q \neq q_d$

$$\|K_P e\| \geq K_{P,m}\|e\| > \alpha\|e\| \geq \|g(q) - g(q_d)\|$$

$$\cancel{M(q)\ddot{q}} + \cancel{S(q,\dot{q})\dot{q}} + g(q) = K_P e - \cancel{K_D \dot{q}} + g(q_d)$$

PD control + constant gravity compensation

stability analysis

with $e = q_d - q$, $g(q) = \left(\frac{\partial U}{\partial q}\right)^T$, consider as Lyapunov candidate

$$\boxed{V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}e^T K_P e + U(q) - U(q_d) + e^T g(q_d)}$$

2. V is convex in $\dot{q}$ and e, and zero only for $e = \dot{q} = 0$

$$\left(\frac{\partial V}{\partial \dot{q}}\right)^T = M(q)\dot{q} \boxed{= 0 \text{ only for } \dot{q} = 0}$$

$$\frac{\partial^2 V}{\partial \dot{q}^2} = M(q) > 0$$

$(q_d, 0)$ is a global minimum of $V \geq 0$

$$\left(\frac{\partial V_{|\dot{q}=0}}{\partial e}\right)^T = K_P e - \left(\frac{\partial U}{\partial q}\right)^T + g(q_d) = K_P e + g(q_d) - g(q) \boxed{= 0}$$

$\partial e / \partial q = -I$

only for $q = q_d$

$$\frac{\partial^2 V_{|\dot{q}=0}}{\partial e^2} = K_P + \frac{\partial^2 U}{\partial q^2} > 0, \text{ since } \|K_P\| = K_{P,M} \geq K_{P,m} > \alpha$$

PD control + constant gravity compensation

stability analysis

differentiating $\boxed{V = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}e^T K_P e + U(q) - U(q_d) + e^T g(q_d)}$

$$
\begin{aligned}
\dot{V} &= \dot{q}^T\left(M(q)\ddot{q} + \frac{1}{2}\dot{M}(q)\dot{q}\right) - e^T K_P \dot{q} + \frac{\partial U(q)}{\partial q}\dot{q} - \dot{q}^T g(q_d) \\
&= \dot{q}^T\left(u \underbrace{- S(q,\dot{q})\dot{q} + \frac{1}{2}\dot{M}(q)\dot{q}}_{=0} - g(q)\right) - e^T K_P \dot{q} + \dot{q}^T\big(g(q) - g(q_d)\big) \\
&= \cancel{\dot{q}^T K_P e} - \dot{q}^T K_D \dot{q} + \cancel{\dot{q}^T\big(g(q_d) - g(q)\big)} - \cancel{e^T K_P \dot{q}} + \cancel{\dot{q}^T\big(g(q) - g(q_d)\big)} \\
&= -\dot{q}^T K_D \dot{q} \le 0
\end{aligned}
$$

for $\dot{V} = 0$ ($\Leftrightarrow \dot{q} = 0$), we have in the closed-loop system

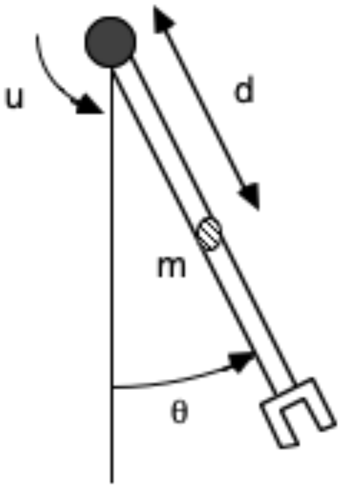
$$M(q)\ddot{q} + g(q) = K_P e + g(q_d)$$

$$\Rightarrow \quad \ddot{q} = M^{-1}(q)\big(K_P e + g(q_d) - g(q)\big) = 0 \Leftrightarrow e = 0$$

by LaSalle Theorem, the thesis follows

Example of a single-link robot

stability analysis



$$I\ddot{\theta} + mg_0 d \sin\theta = u$$

task: regulate the link position to the upward equilibrium

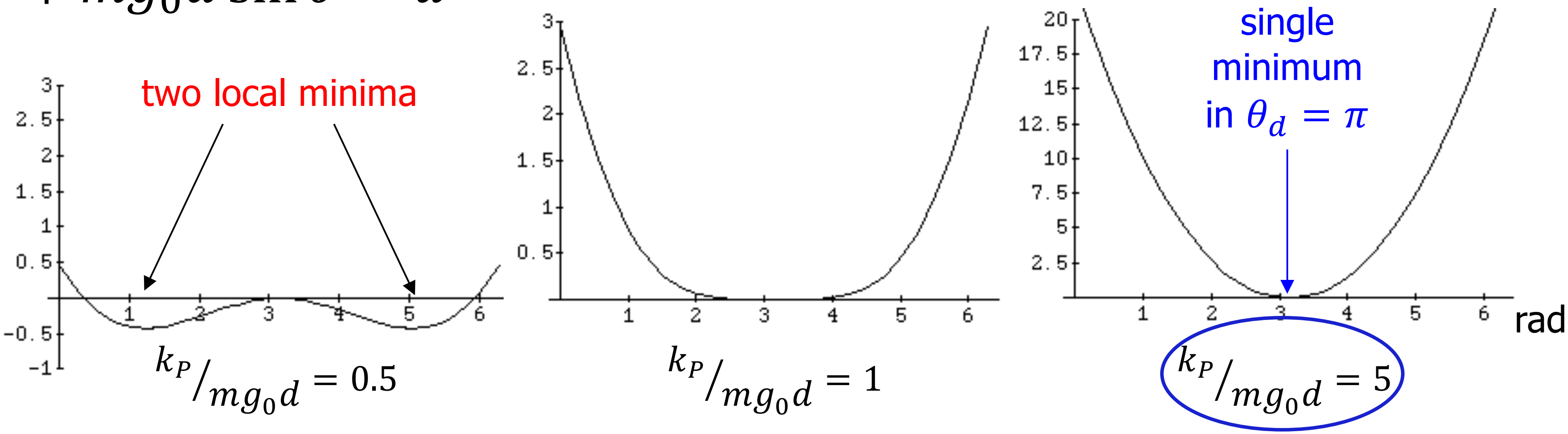
$$\theta_d = \pi \;\;\rightarrow\;\; g(\theta_d) = 0$$

PD control + constant gravity compensation (here, zero!)

$$u = k_P(\pi - \theta) - k_D\dot{\theta}$$

by Theorem 2, it is sufficient (here, also necessary*) to choose

$$k_P > \alpha = mg_0 d, \quad k_D > 0$$



plots of $V(\theta)$ (for $\dot{\theta} = 0$)

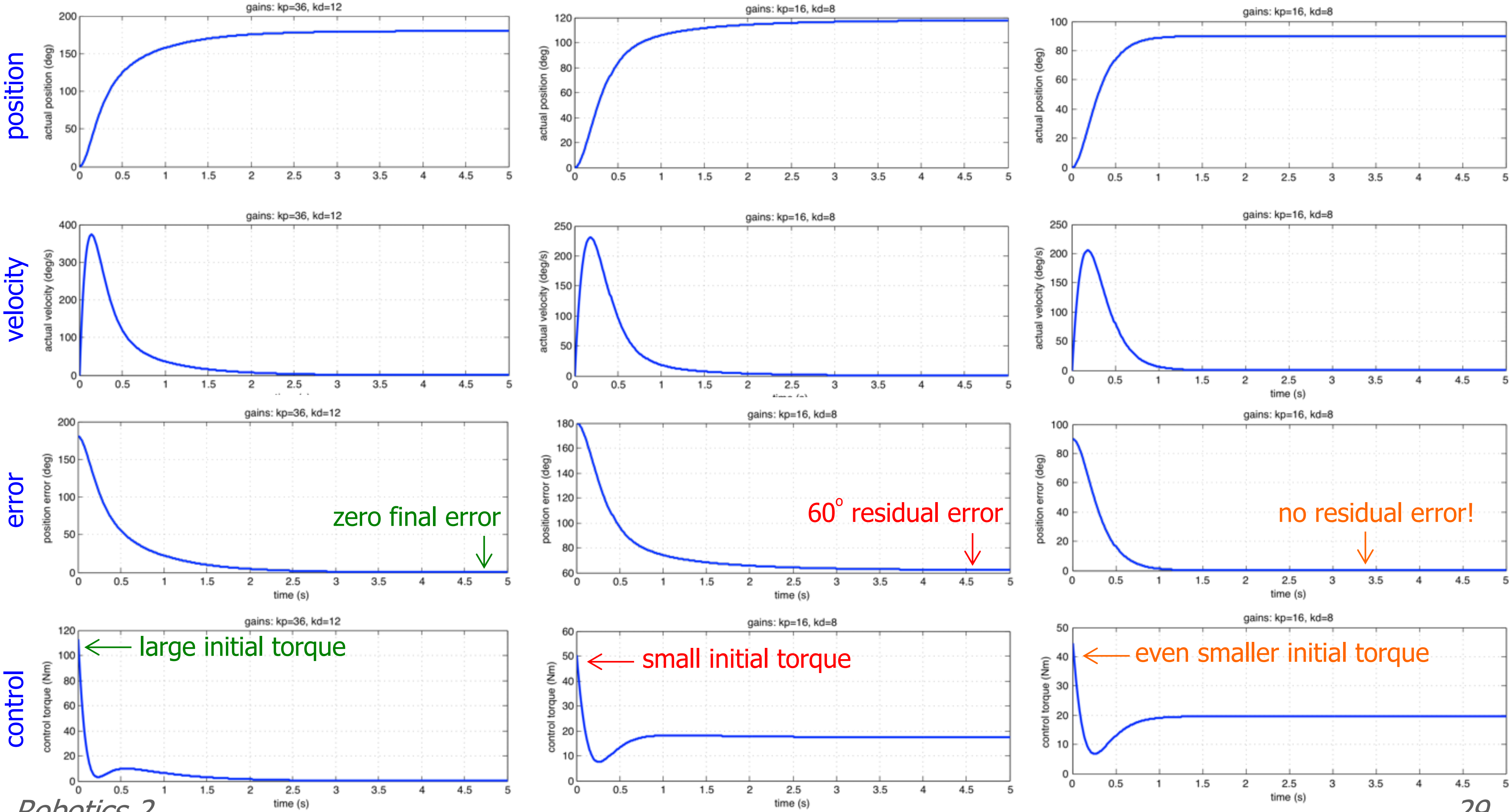
* by a local analysis of the linear approximation at π

Example of a single-link robot

simulations with data: $I = 0.9333,\ mg_0d = 19.62\ (= \alpha)$

$\theta_d = 180° \rightarrow g(\theta_d) = 0$ $\qquad$ $\theta_d = 90° \rightarrow g(\theta_d) = mg_0d$

sufficient P gain: $k_P = 36, k_D = 12$ $\qquad$ low P gain: $k_P = 16, k_D = 8$ $\qquad$ low P gain: $k_P = 16, k_D = 8$



Approximate gravity compensation

the approximate control law

$$u = K_P(q_d - q) - K_D\dot{q} + \hat{g}(q_d)$$

leads, under similar hypotheses, to a closed-loop equilibrium q^*

- its uniqueness is not guaranteed (unless K_P is large enough)
- for $K_P \to \infty$, one has $q^* \to q_d$

conclusion: in the presence of gravity, the previous regulation control laws require an accurate knowledge of the gravity term in the dynamic model to guarantee the zeroing of the position error (since we can only use "finite" control gains ⇒ in practice, not too large)

PID control

- in linear systems, the addition of an integral control action is used to eliminate a constant error in the step response at steady state
- in robots, a PID may be used to recover such a position error due to an incomplete (or absent) gravity compensation/cancellation

➡ the control law $$u(t) = K_P(q_d - q(t)) + K_I \int_0^t (q_d - q(\tau))\, d\tau - K_D\dot{q}(t)$$

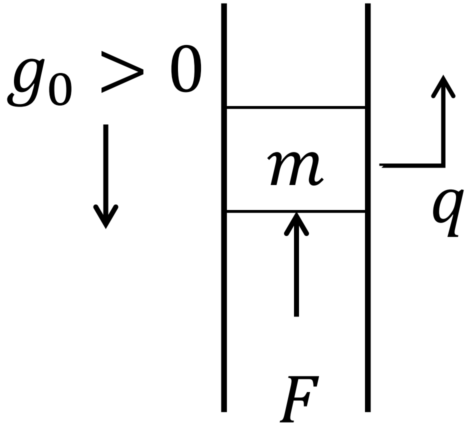
- is independent from any robot dynamic model term
- **if** the desired closed-loop equilibrium is asymptotically stable under PID control, the integral term is "loaded" at steady state to the value

$$\boxed{K_I \int_0^\infty (q_d - q(\tau))\, d\tau = g(q_d)}$$

- however, one can show only local asymptotic stability of this law, i.e., for $q(0) \in \Delta(q_d)$, under complex conditions on K_P, K_I, K_D and $e(0)$

Linear example with PID control

whiteboard…



$$\boxed{m\ddot{q} + m g_0 = F}$$ (no friction)

$$e(t) = q_d - q(t)$$
$$\dot{e}(t) = -\dot{q}(t)$$

$$F = k_P(q_d - q) - k_D\dot{q}$$

(PD $\Rightarrow$ steady-state error $e = q_d - \bar{q}$, with $\bar{q} = q_d - \frac{m g_0}{k_P}$)

$$F = k_P(q_d - q) - k_D\dot{q} + m g_0$$

(PD + gravity cancellation $\Rightarrow$ regulation $\forall k_P > 0, k_D > 0$)

$$F = k_P(q_d - q) - k_D\dot{q} + k_I\int_0^t (q_d - q(\tau))d\tau$$

(PID $\Rightarrow$ regulation $\forall k_I > 0, k_D > 0, k_P > \frac{m k_I}{k_D} > 0$)

with global exponential stability!

Laplace domain analysis: $e(s) = \mathcal{L}[e(t)]$, $d(s) = \mathcal{L}[m g_0]$ + Routh criterion

$$\frac{e(s)}{d(s)} = W_d(s) = \frac{s}{m s^3 + k_D s^2 + k_P s + k_I}$$

3	m	k_P
2	k_D	k_I
1	$(k_D k_P - m k_I)/k_D$	
0	k_I	

Saturated PID control

- more in general, one can prove global asymptotic stability of $(q_d, 0)$, under lower bound limitations for K_P, K_I, K_D (depending on suitable "bounds" on the terms in the dynamic model), for a nonlinear PID law

$$u(t) = K_P(q_d - q(t)) + K_I \int_0^t \Phi(q_d - q(\tau))\, d\tau - K_D \dot{q}$$

where $\Phi(q_d - q)$ is a saturation-type function, such as

$$\Phi(x) = \begin{cases} \sin x, & |x| \le \pi/2 \\ 1, & x > \pi/2 \\ -1, & x < -\pi/2 \end{cases} \quad \text{or} \quad \Phi(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

(see paper by R. Kelly, IEEE TAC, 1998; available as extra material on the course web)

Limits of robot regulation controllers

- response times needed for reaching the desired steady state are not easily predictable in advance
  - depend heavily on robot dynamics, on PD/PID gains, on the required total displacement, and on the interested area of the robot workspace
  - integral term (when present) needs some time to "unload" itself from the error history accumulated during transients
    - large initial errors are stored in the integral term
    - anti-windup schemes stop the integration when commands saturate
    - … an intuitive explanation for the success of "saturated" PID law
- control efforts in the few first instants of motion typically exceed by far those required at steady state
  - especially for high positional gains
  - may lead to saturation (hard nonlinearity) of robot actuators

Regulation in industrial robots

- in industrial robots, the planner generates a reference trajectory $q_r(t)$ even when the task requires only positioning/regulation of the robot
  - "smooth" enough, with a user-defined transfer time T
  - reference trajectory interpolates initial and final desired position

$$q_r(0) = q(0) \qquad q_r(t \geq T) = q_d$$

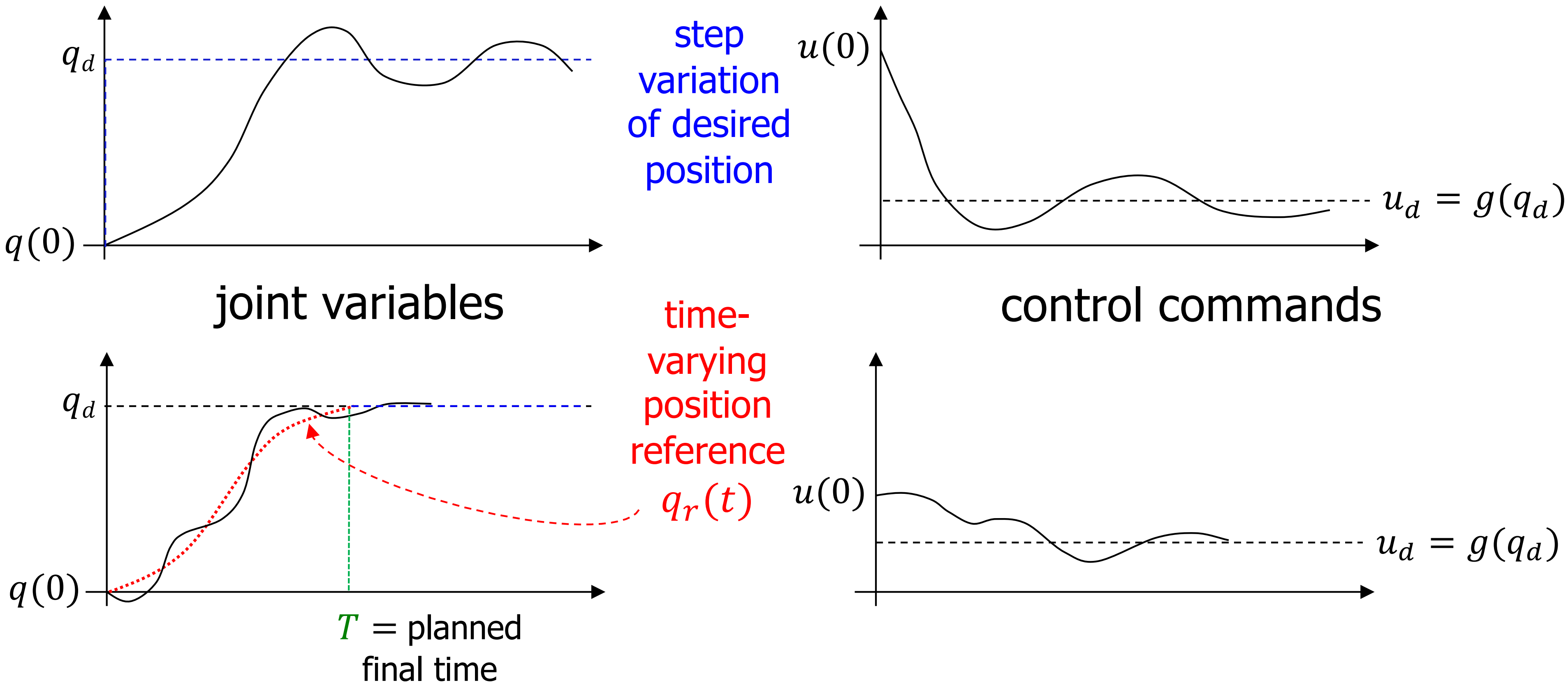
- $q_r(t)$ is used within a control law of the form

$$u = K_P(q_r(t) - q) + K_D(\underbrace{\dot{q}_r(t)}_{\text{often neglected}} - \dot{q}) + g(q)$$

e.g., PD with gravity cancellation

  - in this way, the position error is initially zero
  - robot motion stays only "in the vicinity" of the reference trajectory until $t = T$, typically with small position errors (gains can be larger!)
  - final regulation is only a "local" problem ($e(T) = q_d - q(T)$ is small)

Qualitative comparison



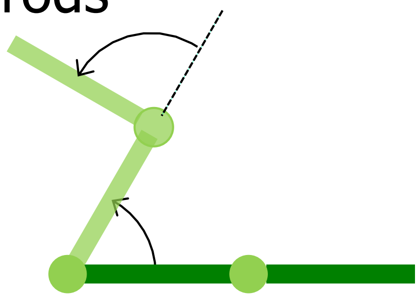
- no saturation of commands: in principle, much larger gains can be used
- better prediction of settling times: local exponential convergence (designed on the linear approximation of the robot dynamics around $(q_d, 0)$)
- "fine tuning" of control gains made easier, but still a tedious and delicate task

Quantitative comparison

planar 2R arm

m_1	10 [kg]
m_2	5 [kg]
l_1	0.5 [m]
l_2	0.5 [m]
d_1	0.25 [m]
d_2	0.25 [m]
I_1	$5/24$ [kg m²]
I_2	$5/48$ [kg m²]

robot data: links are uniform thin rods



no gravity (horizontal plane)

rest-to-rest motion task:

$q(0) = (0,0)$ (straight) $\rightarrow q_d = (\pi/3, \pi/2)$

interpolating trajectory: cubic polynomials

three case studies

a) low gains (overdamped) $K_P = \text{diag}\{80, 40\}, K_D = \text{diag}\{60, 30\}$
vs interpolating trajectory in $T = 2$ s

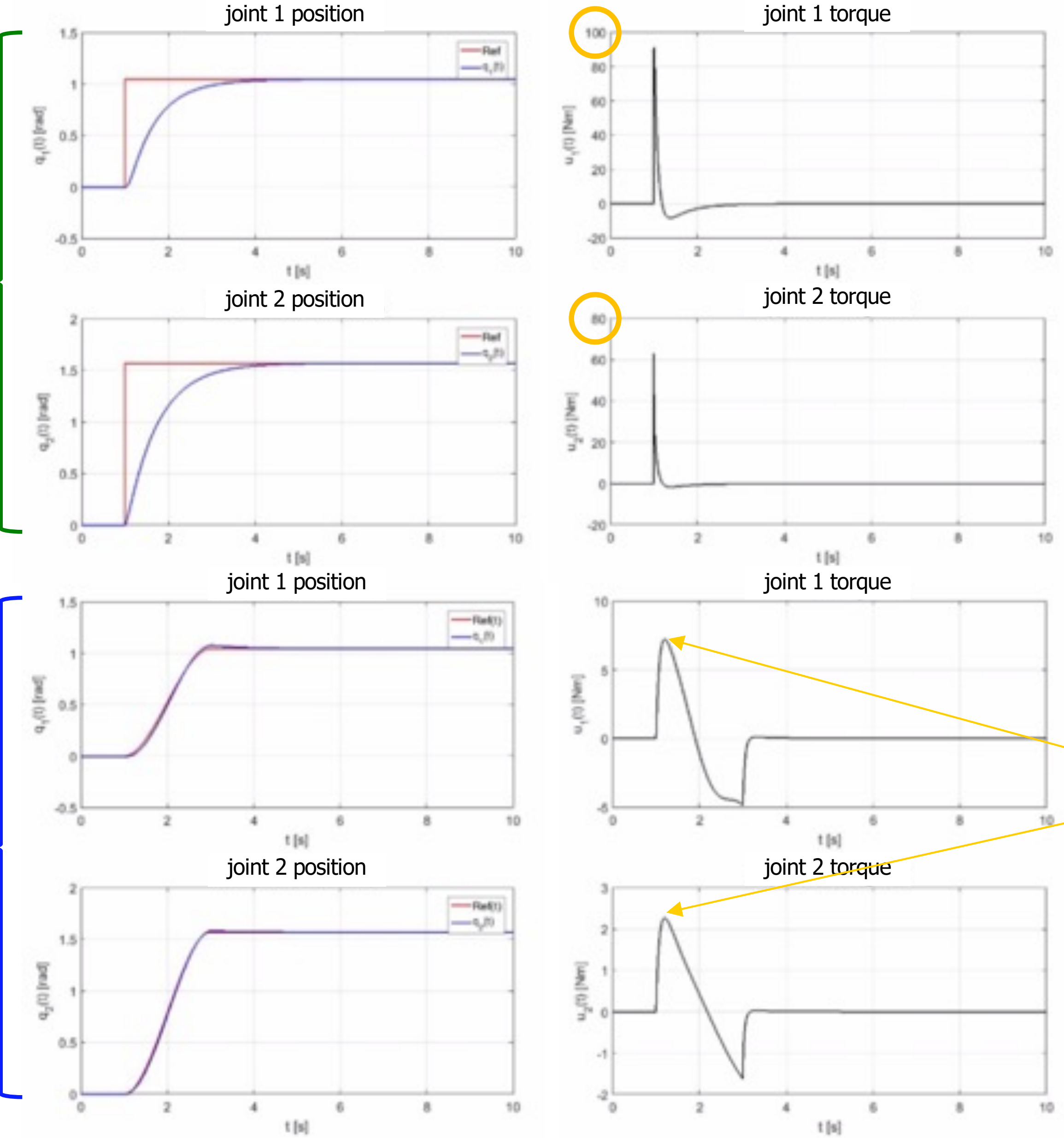
b) medium gains (very overdamped) $K_P = \text{diag}\{200, 100\}, K_D = \text{diag}\{200, 100\}$
vs interpolating trajectory in $T = 2$ s

c) high gains $K_P = \text{diag}\{1250, 180\}, K_D = \text{diag}\{200, 70\}$
vs interpolating trajectory in $T = 1$ s, with torque saturation $u_{1,\max} = 30$ Nm, $u_{2,\max} = 10$ Nm

Comparison on a planar 2R arm – case a

PD with low gains
$K_P = \mathrm{diag}\{80, 40\}$
$K_D = \mathrm{diag}\{60, 30\}$
(overdamped)

PD with same gains
on interpolating
trajectory of $T = 2$ s

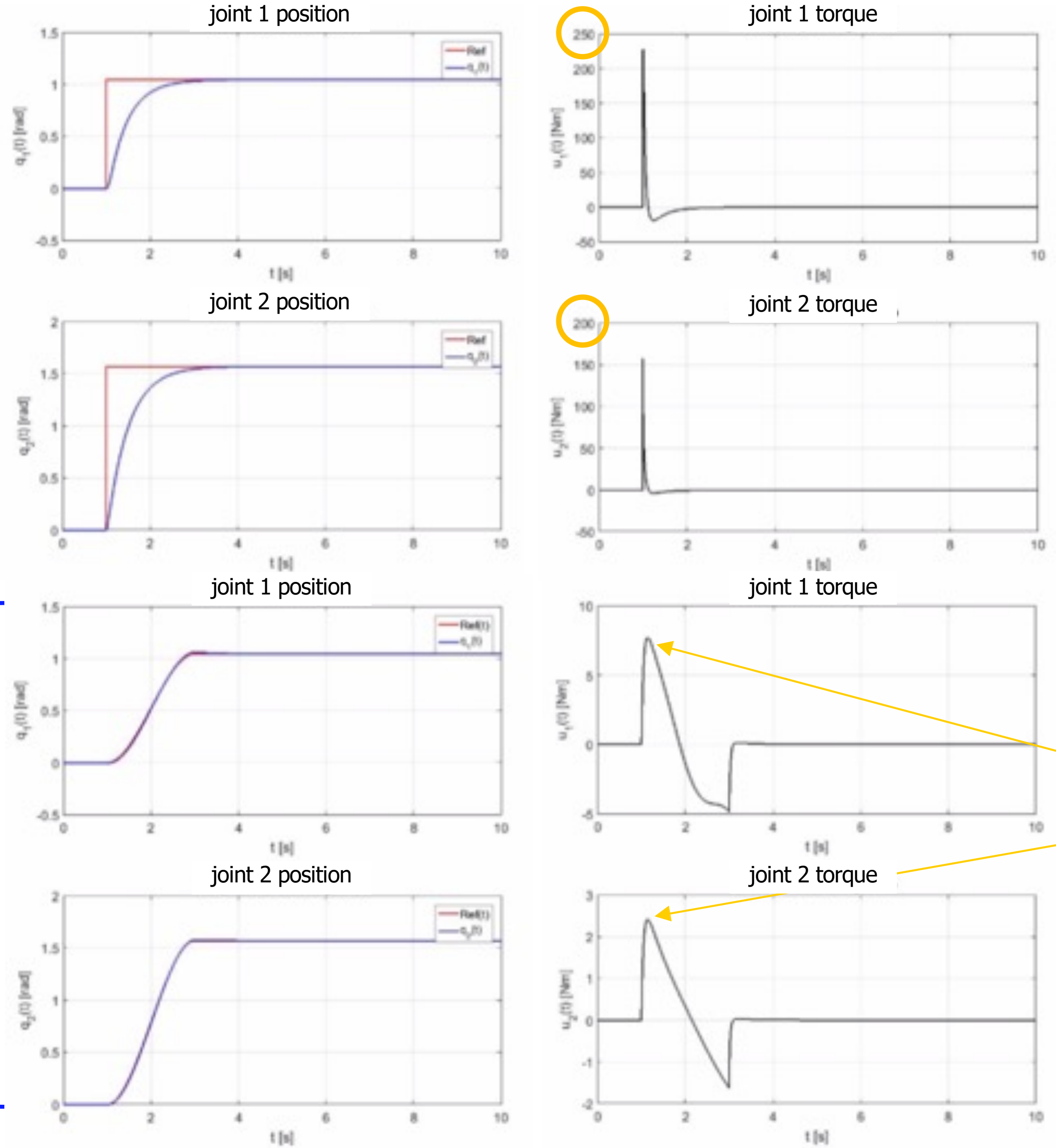


a reduction of the peak control effort by a factor of 100 on joint 1 & by a factor of 30 on joint 2

max torques of 7 and 2.3 Nm

Comparison on a planar 2R arm – case b

PD with medium gains
$K_P = \mathrm{diag}\{200, 100\}$
$K_D = \mathrm{diag}\{200, 100\}$
(very overdamped)



even stronger peak reduction, with similar total control effort, plus improved tracking of reference trajectory on both joints

max torques of 7.5 and 2.4 Nm

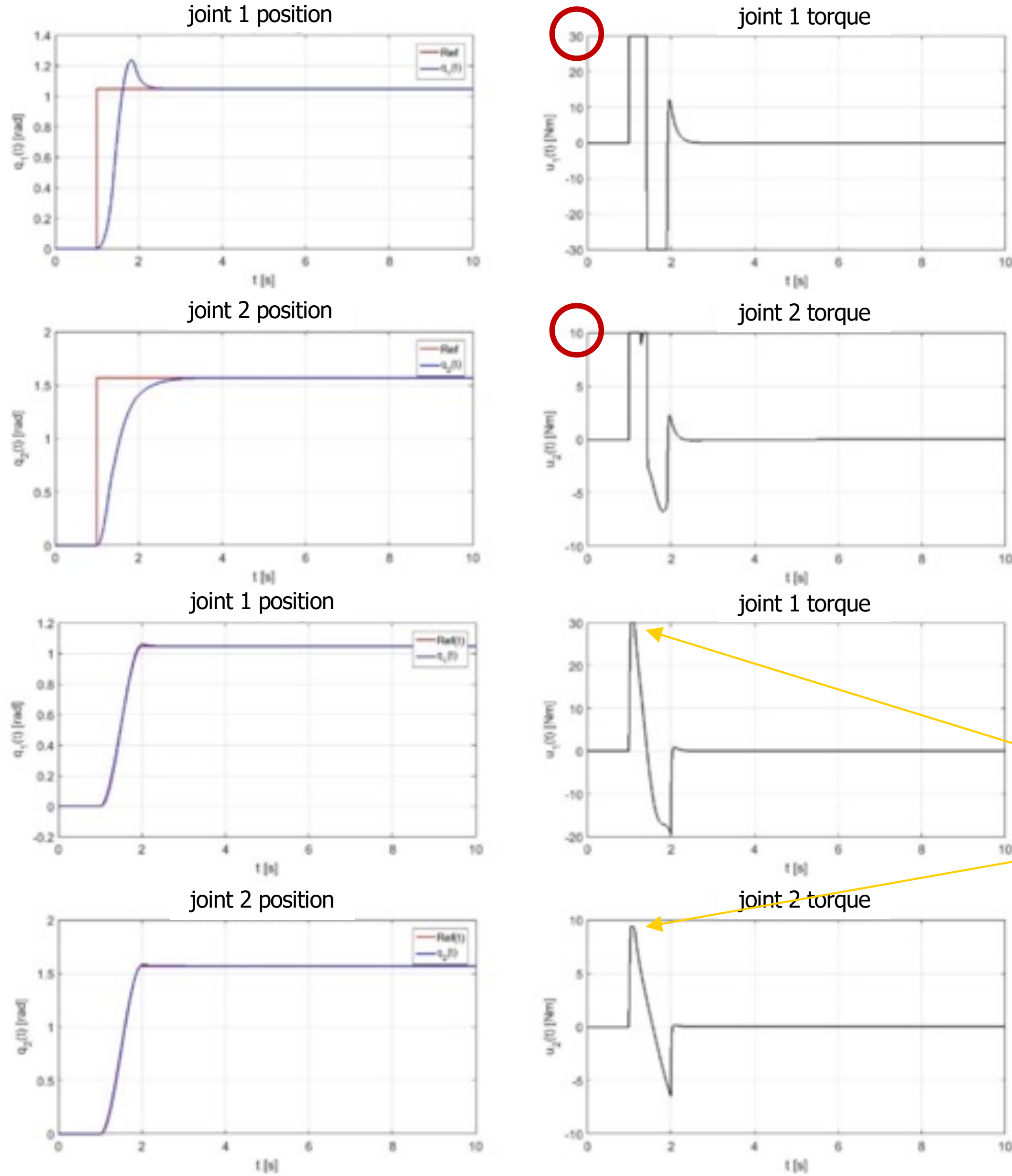
PD with same gains on interpolating trajectory of $T = 2$ s

Comparison on a planar 2R arm – case c

PD with high gains
$K_P = \text{diag}\{1250, 180\}$
$K_D = \text{diag}\{200, 70\}$

torque saturation
$u_{1,\max} = 30$ Nm
$u_{2,\max} = 10$ Nm

PD with same gains
on interpolating
trajectory of $T = 1$ s



position overshoot
and long saturations
are avoided,
with very good
tracking of the
faster reference
trajectory

max torques
of 30 and 9.5 Nm