# Robotics 2 - Midterm Test

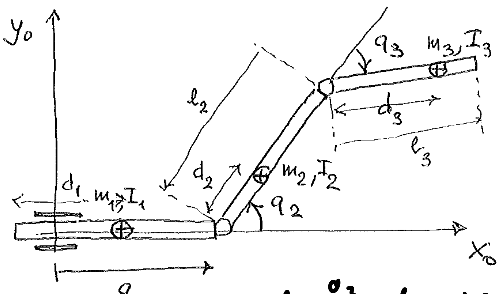
**April 13, 2016**

**Exercise 1**

For the PRR planar robot in Fig. 1, determine the symbolic expression of the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ and of the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$. Use the generalized coordinates and the scalar parameters shown in the figure.

$$P_{c2} = \begin{pmatrix} q_1 + d_2 C_2 \\ d_2 S_2 \end{pmatrix}$$

$$\dot{P}_{c2} = \begin{pmatrix} \dot{q}_1 - S_2 d_2 \dot{q}_2 \\ d_2 C_2 \dot{q}_2 \end{pmatrix}$$




$$\|v_{c2}\|^2 = \dot{q}_1^2 + d_2^2 \dot{q}_2^2 - 2 S_2 d_2 \dot{q}_1 \dot{q}_2$$

$$T_2 = \frac{1}{2} I_2 \dot{q}_2^2 + \frac{1}{2} m_2 \|v_{c2}\|^2$$

Figure 1: A planar PRR robot

**Exercise 2**

The 4R planar robot in Fig. 2 moves under gravity. For each link, the center of mass lies on its longitudinal axis of symmetry, at a generic distance from the driving joint. Determine: *i)* the expression of the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ in the robot dynamic model; *ii)* all equilibrium configurations of the robot (i.e., all $\boldsymbol{q}_e$ such that $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$; *iii)* a linear parametrization of the gravity vector in the form $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}_G(\boldsymbol{q})\boldsymbol{a}_G$; the particular location of the center of masses of the links such that the gravity vector vanishes (i.e., $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{0}$, for all $\boldsymbol{q}$).

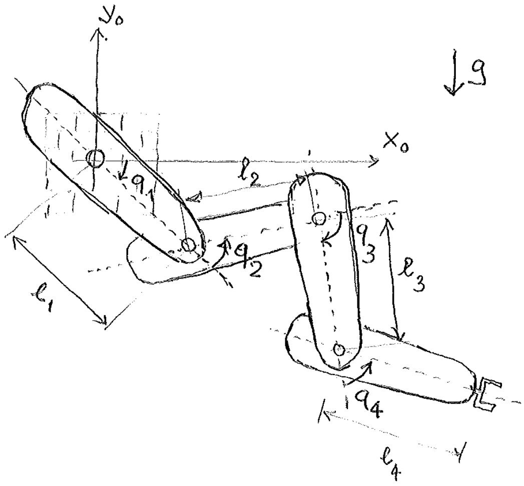



Figure 2: A 4R planar robot under gravity

**Exercise 3**

The 4R planar robot with all links of equal length $\ell$ in Fig. 3 needs to realize a motion task defined by a desired linear velocity $\boldsymbol{v}_d$ for its end-effector position $\boldsymbol{p}_e$ and by a desired angular velocity $\dot{\phi}_d$ for the orientation $\phi$ of its end-effector frame. Characterize first all the singular configurations of the robot for this specific task.

Assume then $\ell = 0.5$ [m], $\boldsymbol{q} = (0\ 0\ \pi/2\ 0)$, $\boldsymbol{v}_d = (1\ 0)$ [m/s], and $\dot{\phi}_d = 0.5$ [rad/s]. Moreover, the joints have limited motion range, i.e., $q_i \in [-2, 2]$ [rad], for $i = 1, \ldots, 4$. Determine the joint velocity $\dot{\boldsymbol{q}}$ that realizes the desired task while decreasing instantaneously the objective function that measures the distance from the midpoint of the joint ranges, i.e., in the form

$$H_{range}(\boldsymbol{q}) = \frac{1}{2N} \sum_{i=1}^{N} \left( \frac{q_i - \bar{q}_i}{q_{M,i} - q_{m,i}} \right)^2 .$$

PROJECTED GRADIENT

$\dot{q} = J^{\#}\dot{r} + (I - J^{\#}J)\dot{q}_0$

with $\dot{q}_0 = -\nabla_q H(q)$

while executing

$\dot{r} = \begin{pmatrix} v_d \\ \dot{\phi}_d \end{pmatrix}$ we increase

$-H(q)$

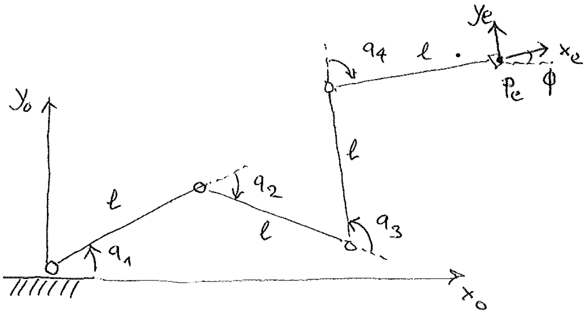



Figure 3: The kinematic skeleton of a planar 4R robot

**[150 minutes; open books]**

# Solution

April 13, 2016

**Exercise 1**

Since the motion is planar, we will use two-dimensional position and velocity vectors (in the $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ plane) and just the $z$-component of angular velocities. Also, the usual shorthand notation is adopted for trigonometric quantities, e.g., $s_2 = \sin q_2$, $c_{23} = \cos(q_2 + q_3)$.

*Kinetic energy*

For link 1, we have (the position of the center of mass on link 1, i.e., $d_1$, is irrelevant)

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2.$$

For link 2, we compute first the position of the center of mass and its velocity,

$$\boldsymbol{p}_{c2} = \begin{pmatrix} q_1 + d_2 c_2 \\ d_2 s_2 \end{pmatrix} \quad \rightarrow \quad \boldsymbol{v}_{c2} = \begin{pmatrix} \dot{q}_1 - d_2 s_2 \dot{q}_2 \\ d_2 c_2 \dot{q}_2 \end{pmatrix},$$

and then

$$\|\boldsymbol{v}_{c2}\|^2 = \dot{q}_1^2 + d_2^2 \dot{q}_2^2 - 2 d_2 s_2 \dot{q}_1 \dot{q}_2.$$

Since $\omega_{2z} = \dot{q}_2$, we obtain

$$T_2 = \frac{1}{2} m_2 \left(\dot{q}_1^2 + d_2^2 \dot{q}_2^2 - 2 d_2 s_2 \dot{q}_1 \dot{q}_2\right) + \frac{1}{2} I_2 \dot{q}_2^2.$$

Similarly, for link 3

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_1 + \ell_2 c_2 + d_3 c_{23} \\ \ell_2 s_2 + d_3 s_{23} \end{pmatrix} \quad \rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} \dot{q}_1 - \ell_2 s_2 \dot{q}_2 - d_3 s_{23} (\dot{q}_2 + \dot{q}_3) \\ \ell_2 c_2 \dot{q}_2 + d_3 c_{23} (\dot{q}_2 + \dot{q}_3) \end{pmatrix},$$

and then

$$\|\boldsymbol{v}_{c3}\|^2 = \dot{q}_1^2 + \ell_2^2 \dot{q}_2^2 + d_3 \left(\dot{q}_2 + \dot{q}_3\right)^2 - 2\ell_2 s_2 \dot{q}_1 \dot{q}_2 - 2 d_3 s_{23} \dot{q}_1 (\dot{q}_2 + \dot{q}_3) + 2 \ell_2 d_3 \left(s_2 s_{23} + c_2 c_{23}\right) \dot{q}_2 (\dot{q}_2 + \dot{q}_3).$$

Being $\omega_{3z} = \dot{q}_2 + \dot{q}_3$, we obtain (after trigonometric simplification)

$$T_3 = \frac{1}{2} m_3 \left(\dot{q}_1^2 + \ell_2^2 \dot{q}_2^2 + d_3 \left(\dot{q}_2 + \dot{q}_3\right)^2 - 2\ell_2 s_2 \dot{q}_1 \dot{q}_2 - 2 d_3 s_{23} \dot{q}_1 (\dot{q}_2 + \dot{q}_3) + 2 \ell_2 d_3 c_3 \dot{q}_2 (\dot{q}_2 + \dot{q}_3)\right) + \frac{1}{2} I_3 \left(\dot{q}_2 + \dot{q}_3\right)^2.$$

*Robot inertia matrix*

From

$$T = \sum_{i=1}^{3} T_i = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q}) \dot{\boldsymbol{q}},$$

we obtain the (symmetric) elements $b_{ij} = b_{ji}$ of the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ as

$$\begin{aligned}
b_{11} &= m_1 + m_2 + m_3 =: a_1 \\
b_{22} &= I_2 + m_2 d_2^2 + I_3 + m_3 d_3^2 + m_3 \ell_2^2 + 2 m_3 \ell_2 d_3 c_3 =: a_2 + 2 a_3 c_3 \\
b_{33} &= I_3 + m_3 d_3^2 =: a_4 \\
b_{12} &= -\left(m_2 d_2 + m_3 \ell_2\right) s_2 - m_3 d_3 s_{23} =: -a_5 s_2 - a_6 s_{23} \\
b_{13} &= -m_3 d_3 s_{23} = -a_6 s_{23} \\
b_{23} &= I_3 + m_3 d_3^2 + m_3 \ell_2 d_3 c_3 = a_4 + a_3 c_3.
\end{aligned}$$

where we have introduced the dynamic coefficients $a_i$ ($i = 1, \ldots, 6$) for the constant factors, in order to have more compact expressions. Thus, the positive definite, symmetric robot inertia matrix can be rewritten as

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 & -(a_5 s_2 + a_6 s_{23}) & -a_6 s_{23} \\ -(a_5 s_2 + a_6 s_{23}) & a_2 + 2a_3 c_3 & a_4 + a_3 c_3 \\ -a_6 s_{23} & a_4 + a_3 c_3 & a_4 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1(\boldsymbol{q}) & \boldsymbol{b}_2(\boldsymbol{q}) & \boldsymbol{b}_3(\boldsymbol{q}) \end{pmatrix}. \tag{1}$$

*Coriolis and centrifugal vector*

From (1) and

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{pmatrix}, \quad c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \quad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left\{ \frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{B}(\boldsymbol{q})}{\partial q_i} \right\} \ (i = 1, 2, 3),$$

we compute

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \frac{1}{2}\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(a_5 c_2 + a_6 c_{23}) & -a_6 c_{23} \\ 0 & -a_6 c_{23} & -a_6 c_{23} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(a_5 c_2 + a_6 c_{23}) & -a_6 c_{23} \\ 0 & -a_6 c_{23} & -a_6 c_{23} \end{pmatrix}^T - \boldsymbol{0} \right\} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(a_5 c_2 + a_6 c_{23}) & -a_6 c_{23} \\ 0 & -a_6 c_{23} & -a_6 c_{23} \end{pmatrix} \\
\boldsymbol{C}_2(\boldsymbol{q}) &= \frac{1}{2}\left\{ \begin{pmatrix} 0 & -(a_5 c_2 + a_6 c_{23}) & -a_6 c_{23} \\ 0 & 0 & -2a_3 s_3 \\ 0 & 0 & -a_3 s_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -(a_5 c_2 + a_6 c_{23}) & 0 & 0 \\ -a_6 c_{23} & -2a_3 s_3 & -a_3 s_3 \end{pmatrix} \right. \\
&\qquad \left. - \begin{pmatrix} 0 & -(a_5 c_2 + a_6 c_{23}) & -a_6 c_{23} \\ -(a_5 c_2 + a_6 c_{23}) & 0 & 0 \\ -a_6 c_{23} & 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a_3 s_3 \\ 0 & -a_3 s_3 & -a_3 s_3 \end{pmatrix} \\
\boldsymbol{C}_3(\boldsymbol{q}) &= \frac{1}{2}\left\{ \begin{pmatrix} 0 & -a_6 c_{23} & -a_6 c_{23} \\ 0 & 0 & -a_3 s_3 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -a_6 c_{23} & 0 & 0 \\ -a_6 c_{23} & -a_3 s_3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -a_6 c_{23} & -a_6 c_{23} \\ -a_6 c_{23} & -2a_3 s_3 & -a_3 s_3 \\ -a_6 c_{23} & -a_3 s_3 & 0 \end{pmatrix} \right\} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_3 s_3 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

and thus

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -a_5 c_2\, \dot{q}_2^2 - a_6 c_{23}\, (\dot{q}_2 + \dot{q}_3)^2 \\ -a_3 s_3\, (2\dot{q}_2 + \dot{q}_3)\, \dot{q}_3 \\ a_3 s_3\, \dot{q}_2^2 \end{pmatrix} = \begin{pmatrix} -(m_2 d_2 + m_3 \ell_2)\, c_2\, \dot{q}_2^2 - m_3 d_3\, c_{23}\, (\dot{q}_2 + \dot{q}_3)^2 \\ -m_3 \ell_2 d_3\, s_3\, (2\dot{q}_2 + \dot{q}_3)\, \dot{q}_3 \\ m_3 \ell_2 d_3\, s_3\, \dot{q}_2^2 \end{pmatrix}. \tag{2}$$

**Exercise 2**

Again, the robot motion occurs in a (vertical) plane and we will use for simplicity two-dimensional position vectors in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$. The total potential energy is

$$U = \sum_{i=1}^{4} U_i, \qquad U_i = -m_i \boldsymbol{g}^T \boldsymbol{r}_{0,c_i}, \quad i = 1, \ldots, 4.$$

Since

$$\boldsymbol{g}^T = \begin{pmatrix} 0 & -g_0 & 0 \end{pmatrix}, \qquad g_0 = 9.81\,[\mathrm{m/s^2}],$$

we need to compute only the $y$-component of the position vector $\boldsymbol{r}_{0,c_i}$ of the center of mass of the link $i$, for $i = 1, \dots, 4$. We have

$$\begin{aligned} r_{0,c_{1,y}} &= d_1 s_1 \\ r_{0,c_{2,y}} &= \ell_1 s_1 + d_2 s_{12} \\ r_{0,c_{3,y}} &= \ell_1 s_1 + \ell_2 s_{12} + d_3 s_{123} \\ r_{0,c_{4,y}} &= \ell_1 s_1 + \ell_2 s_{12} + \ell_3 s_{123} + d_4 s_{1234}, \end{aligned}$$

where $d_i$ is the (signed) distance of the center of mass of link $i$ from the axis of joint $i$ ($i = 1, \dots, 4$). Thus

$$\begin{aligned} U &= g_0 m_1 d_1 s_1 + g_0 m_2 (\ell_1 s_1 + d_2 s_{12}) + g_0 m_3 (\ell_1 s_1 + \ell_2 s_{12} + d_3 s_{123}) + g_0 m_4 (\ell_1 s_1 + \ell_2 s_{12} + \ell_3 s_{123} + d_4 s_{1234}) \\ &= g_0 \Big\{ [m_1 d_1 + (m_2 + m_3 + m_4)\ell_1]\, s_1 + [m_2 d_2 + (m_3 + m_4)\ell_2]\, s_{12} + [m_3 d_3 + m_4 \ell_3]\, s_{123} + m_4 d_4 s_{1234} \Big\} \\ &=: a_{G1} s_1 + a_{G2} s_{12} + a_{G3} s_{123} + a_{G4} s_{1234}, \end{aligned}$$

where we have introduced the dynamic coefficients $a_{Gi}$ ($i = 1, \dots, 4$) for the constant factors related to gravity.

The gravity vector of this robot is then

$$\boldsymbol{g}(\boldsymbol{q}) = \left( \frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} a_{G1} c_1 + a_{G2} c_{12} + a_{G3} c_{123} + a_{G4} c_{1234} \\ a_{G2} c_{12} + a_{G3} c_{123} + a_{G4} c_{1234} \\ a_{G3} c_{123} + a_{G4} c_{1234} \\ a_{G4} c_{1234} \end{pmatrix}, \tag{3}$$

and its linear parametrization is

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} c_1 & c_{12} & c_{123} & c_{1234} \\ 0 & c_{12} & c_{123} & c_{1234} \\ 0 & 0 & c_{123} & c_{1234} \\ 0 & 0 & 0 & c_{1234} \end{pmatrix} \begin{pmatrix} a_{G1} \\ a_{G2} \\ a_{G3} \\ a_{G4} \end{pmatrix} = \boldsymbol{Y}_G(\boldsymbol{q}) \boldsymbol{a}_G. \tag{4}$$

All equilibrium configurations $\boldsymbol{q}_e$ are found by analyzing recursively the vector equation $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$ from the last component backwards:

$$\begin{aligned} & g_4(\boldsymbol{q}_e) = 0 \quad \rightarrow \quad c_{1234} = 0 \\ & g_3(\boldsymbol{q}_e) = 0 \quad \rightarrow \quad \text{being already } c_{1234} = 0 \quad \rightarrow \quad c_{123} = 0 \\ & g_2(\boldsymbol{q}_e) = 0 \quad \rightarrow \quad \text{being already } c_{1234} = 0, c_{123} = 0 \quad \rightarrow \quad c_{12} = 0 \\ & g_1(\boldsymbol{q}_e) = 0 \quad \rightarrow \quad \text{being already } c_{1234} = 0, c_{123} = 0, c_{12} = 0 \quad \rightarrow \quad c_1 = 0. \end{aligned}$$

Thus, the unforced equilibria of the robot (assuming a generic mass distribution) are characterized by

$$q_{e1} = \pm \frac{\pi}{2} \, \cap \, q_{e2} = \{0, \pi\} \, \cap \, q_{e3} = \{0, \pi\} \, \cap \, q_{e4} = \{0, \pi\},$$

namely with the robot being stretched or folded along the vertical direction only.

Finally, perfect balancing in all configurations (i.e., $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{0}$) is obtained for when the mass distribution zeroes the vector of dynamic coefficients, namely $\boldsymbol{a}_G = \boldsymbol{0}$. Starting again from the last component and proceeding backwards, we obtain

$$\begin{aligned} & a_{G4} = 0 \quad \rightarrow \quad d_4 = 0 \\ & a_{G3} = 0 \quad \rightarrow \quad m_3 d_3 + m_4 \ell_3 = 0 \quad \rightarrow \quad d_3 = -\frac{m_4}{m_3}\, \ell_3 \\ & a_{G2} = 0 \quad \rightarrow \quad m_2 d_2 + (m_3 + m_4)\ell_2 = 0 \quad \rightarrow \quad d_2 = -\frac{m_3 + m_4}{m_2}\, \ell_2 \\ & a_{G1} = 0 \quad \rightarrow \quad m_1 d_1 + (m_2 + m_3 + m_4)\ell_1 = 0 \quad \rightarrow \quad d_1 = -\frac{m_2 + m_3 + m_4}{m_1}\, \ell_1. \end{aligned}$$

**Exercise 3**

The task vector for this 4R planar robot is defined as

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{p}_e \\ \phi \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ \phi \end{pmatrix} = \begin{pmatrix} \ell\,(c_1 + c_{12} + c_{123} + c_{1234}) \\ \ell\,(s_1 + s_{12} + s_{123} + s_{1234}) \\ q_1 + q_2 + q_3 + q_4 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}).$$

Differentiating $\boldsymbol{r}$ w.r.t. to time yields

$$\dot{\boldsymbol{r}} = \begin{pmatrix} \boldsymbol{v} \\ \dot{\phi} \end{pmatrix} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with the task Jacobian given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell\,(s_1 + s_{12} + s_{123} + s_{1234}) & -\ell\,(s_{12} + s_{123} + s_{1234}) & -\ell\,(s_{123} + s_{1234}) & -\ell\,s_{1234} \\ \ell\,(c_1 + c_{12} + c_{123} + c_{1234}) & \ell\,(c_{12} + c_{123} + c_{1234}) & \ell\,(c_{123} + c_{1234}) & \ell\,c_{1234} \\ 1 & 1 & 1 & 1 \end{pmatrix}. \qquad (5)$$

For the purpose of singularity analysis, the matrix $\boldsymbol{J}(\boldsymbol{q})$ can be rewritten as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell\,s_1 & -\ell\,s_{12} & -\ell\,s_{123} & -\ell\,s_{1234} \\ \ell\,c_1 & \ell\,c_{12} & \ell\,c_{123} & \ell\,c_{1234} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \boldsymbol{J}_a(\boldsymbol{q})\,\boldsymbol{T},$$

where the square matrix $\boldsymbol{T}$ is clearly nonsingular. Thus, $\boldsymbol{J}$ and $\boldsymbol{J}_a$ have always the same rank. In particular, the Jacobian $\boldsymbol{J}$ will be full (row) rank if and only if the $2 \times 3$ upper left block of matrix $\boldsymbol{J}_a$ will have rank equal to 2. This matrix block corresponds to the well-known Jacobian of a planar 3R robot (with equal links of length $\ell$) performing a positional task with its end-effector. The singularities of the 4R arm for the given task occur then if and only if

$$q_2 = \{0, \pi\} \;\cap\; q_3 = \{0, \pi\},$$

namely when its *first three* links are stretched or folded along a single direction.

Plugging the link length $\ell = 0.5$ [m] and the given configuration $\boldsymbol{q} = (0 \;\; 0 \;\; \pi/2 \;\; 0)$ in (5) provides

$$\boldsymbol{J} = \begin{pmatrix} -1 & -1 & -1 & -0.5 \\ 1 & 0.5 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

whose pseudoinverse is computed (by hand or using Matlab) as

$$\boldsymbol{J}^{\#} = \boldsymbol{J}^T \left(\boldsymbol{J}\boldsymbol{J}^T\right)^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0.5 & 1 \\ -1 & 0 & 1 \\ -0.5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3.25 & -1.5 & -3.5 \\ -1.5 & 1.25 & 1.5 \\ -3.5 & 1.5 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 1 & 1/6 \\ -2/3 & 0 & -1/3 \\ -5/3 & -1 & -5/6 \\ 2 & 0 & 2 \end{pmatrix}.$$

The desired velocity task is specified by

$$\dot{\boldsymbol{r}}_d = \begin{pmatrix} \boldsymbol{v}_d \\ \phi_d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0.5 \end{pmatrix}.$$

In view of the separability of the objective function $H_{range}(\boldsymbol{q}) = \sum_{i=1}^{N} H_{range,i}(q_i)$ that measures the distance from the midpoint of the joint ranges, its gradient takes the form

$$\nabla_{\boldsymbol{q}} H_{range}(\boldsymbol{q}) = \left(\frac{\partial H_{range}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T, \qquad \text{with} \quad \frac{\partial H_{range}(\boldsymbol{q})}{\partial q_i} = \frac{\partial H_{range,i}(q_i)}{\partial q_i} = \frac{1}{N}\frac{q_i - \bar{q}_i}{(q_{M,i} - q_{m,i})^2}.$$

With the data $N = 4$, $q_{M,i} = -q_{m,i} = 2$, and thus $\bar{q}_i = 0$, for $i = 1, \ldots, 4$, the gradient at the given configuration $\boldsymbol{q} = (0 \;\, 0 \;\, \pi/2 \;\, 0)$ is

$$\nabla_{\boldsymbol{q}} H_{range} = \frac{1}{64}\begin{pmatrix} 0 \\ 0 \\ \pi/2 \\ 0 \end{pmatrix}$$

The joint velocity solution that realizes the desired task while *decreasing* instantaneously the objective function $H_{range}$ is evaluated then as

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{r}}_d - \left(\boldsymbol{I} - \boldsymbol{J}^{\#}\boldsymbol{J}\right)\nabla_{\boldsymbol{q}} H_{range} = -\nabla_{\boldsymbol{q}} H_{range} + \boldsymbol{J}^{\#}\left(\dot{\boldsymbol{r}}_d + \boldsymbol{J}\,\nabla_{\boldsymbol{q}} H_{range}\right) = \begin{pmatrix} 0.4126 \\ -0.8252 \\ -2.0874 \\ 3 \end{pmatrix} \; [\text{rad/s}].$$

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