Robotics 2

February 3, 2022

Exercise #1

The RPR robot in Fig. 1 moves in a vertical plane and is controlled by the joint torque $\boldsymbol{\tau} \in \mathbb{R}^3$.

a) Provide a linear parametrization of the gravity term $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{a}_G$ in the robot dynamic model, where the matrix $\boldsymbol{G}(\boldsymbol{q})$ contains only known kinematic quantities (including the gravity acceleration g_0). Introduce kinematic and dynamic parameters as needed.

b) Design a control law $\boldsymbol{\tau} = \boldsymbol{\tau}_r(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{e}_p)$, driven by the Cartesian error $\boldsymbol{e}_p = \boldsymbol{p}_d - \boldsymbol{p}$, that achieves regulation of the end-effector position to a desired constant value $\boldsymbol{p}_d \in \mathbb{R}^2$, up to singularities. Give the explicit expression of all terms in this control law.

c) Find a robot configuration $\boldsymbol{q}_s$ and an associated desired position $\boldsymbol{p}_d$, with $\boldsymbol{e}_p = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}_s) \neq \boldsymbol{0}$, such that the robot will *not* move when at rest in $\boldsymbol{q}_s$ under the action of the previous control law $\boldsymbol{\tau} = \boldsymbol{\tau}_r(\boldsymbol{q}_s, \boldsymbol{0}, \boldsymbol{e}_p)$.

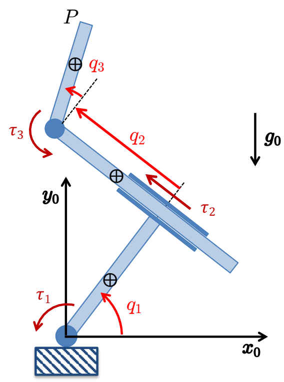

Figure 1: A planar RPR robot.

Exercise #2

Consider the planar 4R robot in Fig. 2, having equal links of unitary length. The robot is commanded by the joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^4$.

a) The end effector of the robot should follow a desired smooth position trajectory $\boldsymbol{p}_d(t) \in \mathbb{R}^2$. Provide the general form of the command $\ddot{\boldsymbol{q}}_a$ that executes the task in nominal conditions, while minimizing instantaneously the objective function

$$H = \frac{1}{2} \left\| \ddot{\boldsymbol{q}} + \boldsymbol{K}_d \, \dot{\boldsymbol{q}} \right\|^2, \qquad \boldsymbol{K}_d > 0. \tag{1}$$

Moreover, study the singularities that may be encountered during the execution of this task.

b) Consider again the problem in item a), but now with the desired task augmented in order to keep the end-effector angular speed at some constant value $\omega_{z,d} \in \mathbb{R}$. Provide the general form of the command $\ddot{\boldsymbol{q}}_b$ that executes the extended task.

c) Compute the numerical value of $\ddot{\boldsymbol{q}}_a$ when the robot is in the nominal state $\boldsymbol{x}_d = (\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d) \in \mathbb{R}^8$ and for a desired $\ddot{\boldsymbol{p}}_d \in \mathbb{R}^2$ given by

$$\boldsymbol{q}_d = \begin{pmatrix} \pi/4 \\ \pi/3 \\ -\pi/2 \\ 0 \end{pmatrix} \text{ [rad]}, \qquad \dot{\boldsymbol{q}}_d = \begin{pmatrix} -0.8 \\ 1 \\ 0.2 \\ 0 \end{pmatrix} \text{ [rad/s]}, \qquad \ddot{\boldsymbol{p}}_d = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ [m/s}^2\text{]},$$

having set $\boldsymbol{K}_d = \boldsymbol{I}_{4\times 4}$ in (1).

d) Compute the numerical value of $\ddot{\boldsymbol{q}}_b$ in the same conditions of item c). What are the values of $\boldsymbol{p}_d$, $\dot{\boldsymbol{p}}_d$, and $\omega_{z,d}$ in this case?

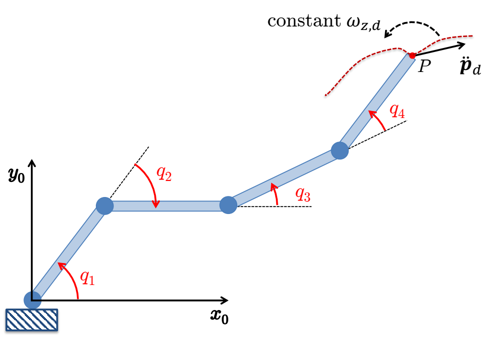

Figure 2: A planar 4R robot, with a sketch of its Cartesian tasks.

Exercise #3

Figure 3 shows a 2R robot with unitary link lengths whose end effector is mechanically constrained to move only along the vertical segment between points $A = (0, 1)$ and $B = (0, \sqrt{2})$, under the action of the single available motor torque τ at the first joint. The second joint is passive and all dissipative effects can be neglected.

a) Derive a (one-dimensional) *reduced* dynamic model of the constrained robot.

b) If the robot is in an equilibrium state with its end effector in A, what is the applied torque τ_0?

c) Suppose that the end effector should execute a rest-to-rest motion from A and B with a sinusoidal acceleration profile at an angular frequency $\omega = 0.5$ [rad/s]. What is the explicit expression of the needed torque command $\tau_d(t)$?

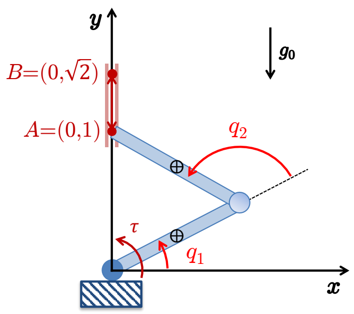

Figure 3: A 2R robot in constrained motion, with one actuator only at the first joint.

[210 minutes (3.5 hours); open books]

Solution

February 3, 2022

Exercise #1

We compute first the gravity term $\boldsymbol{g}(\boldsymbol{q})$ needed in the regulation law and provide a linear parametrization for it. The potential energy of each link is computed from the general expression (with vectors in $\mathbb{R}^2$, being the problem planar)

$$U_i = -m_i\, \boldsymbol{g}_0^T \boldsymbol{r}_{i,c_i}, \qquad \boldsymbol{g}_0 = \begin{pmatrix} 0 \\ -g_0 \end{pmatrix} \qquad \Rightarrow \qquad U_i = m_i g_0\, r_{i,c_{i_y}}$$

with $g_0 = 9.81$ [m/s^2]. Thus,

$$\begin{aligned} U_1 &= m_1 g_0\, d_1 \sin q_1, \\ U_2 &= m_2 g_0 \left(l_1 \sin q_1 + (q_2 - d_2)\cos q_1\right), \\ U_3 &= m_3 g_0 \left(l_1 \sin q_1 + q_2 \cos q_1 + d_3 \sin(q_1+q_3)\right), \end{aligned}$$

where l_1 is the length of link 1, d_1 and d_3 are the positions of the center mass of link 1 and link 3, respectively, computed from their proximal base, and d_2 is the position of the center mass of link 2 as computed from its distal base (i.e., the axis of joint 3). As a result, from $U = U_1 + U_2 + U_3$, we obtain

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = g_0 \begin{pmatrix} (m_1 d_1 + (m_2+m_3) l_1)\cos q_1 - (m_2(q_2-d_2) + m_3 q_2)\sin q_1 \\ + m_3 d_3 \cos(q_1+q_3) \\ (m_2+m_3)\cos q_1 \\ m_3 d_3 \cos(q_1+q_3) \end{pmatrix}. \tag{2}$$

This can be linearly parametrized by a (3×4) regressor matrix $\boldsymbol{G}(\boldsymbol{q})$ and a vector of dynamic coefficients $\boldsymbol{a}_G \in \mathbb{R}^4$ as follows:

$$\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{a}_G = \begin{pmatrix} g_0\,(l_1\cos q_1 - q_2 \sin q_1) & g_0\,\cos q_1 & g_0\,\sin q_1 & g_0\,\cos(q_1+q_3) \\ g_0\,\cos q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_0\,\cos(q_1+q_3) \end{pmatrix} \begin{pmatrix} m_2+m_3 \\ m_1 d_1 \\ m_2 d_2 \\ m_3 d_3 \end{pmatrix}.$$

Indeed, also other parametrizations that are still minimal (i.e., of dimension 4) can be found.

The required Cartesian regulator is given by[1]

$$\boldsymbol{\tau}_r = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{K}_P\left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})\right) - \boldsymbol{K}_D\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}), \qquad \text{with } \boldsymbol{K}_P > 0,\ \ \boldsymbol{K}_D > 0. \tag{3}$$

Having already given the gravity term in (2), the missing expressions in (3) are the direct kinematics

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} l_1\cos q_1 - q_2 \sin q_1 + l_3\cos(q_1+q_3) \\ l_1 \sin q_1 + q_2\cos q_1 + l_3 \sin(q_1+q_3) \end{pmatrix}$$

and its associated Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(l_1\sin q_1 + q_2\cos q_1 + l_3\sin(q_1+q_3)\right) & -\sin q_1 & -l_3\sin(q_1+q_3) \\ l_1\cos q_1 - q_2\sin q_1 + l_3\cos(q_1+q_3) & \cos q_1 & l_3\cos(q_1+q_3) \end{pmatrix}. \tag{4}$$

[1] Alternatively, one can replace the joint damping term $-\boldsymbol{K}_D\dot{\boldsymbol{q}}$ by a Cartesian damping $-\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{K}_{D,c}\,\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$, with $\boldsymbol{K}_{D,c} > 0$. However, there is no actual advantage in doing so.

To answer to item c), we need to find the singular configurations $\boldsymbol{q}_s$ of $\boldsymbol{J}(\boldsymbol{q})$. In fact, the controller (3) gets stuck with the robot at rest in an end-effector position $\boldsymbol{p}_s = \boldsymbol{f}(\boldsymbol{q}_s)$ different from the desired $\boldsymbol{p}_d$ if and only if the (gain scaled) error $\boldsymbol{K}_P\boldsymbol{e}_p$, with $\boldsymbol{e}_p = \boldsymbol{p}_d - \boldsymbol{p}_s \neq \boldsymbol{0}$, lies in the null space of the transpose of the Jacobian $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s)$. And this can happen only when the Jacobian in (4) loses rank. Analyzing the three minors (obtained by deleting one column of $\boldsymbol{J}$ at a time), we have

$$\det \boldsymbol{J}_{-1} = l_3 \sin q_3, \qquad \det \boldsymbol{J}_{-2} = l_3 \left(l_1 \sin q_3 - q_2 \cos q_3\right), \qquad \det \boldsymbol{J}_{-3} = -(q_2 + l_3 \sin q_3).$$

These are simultaneously zero if and only if $q_2 = 0$ and $q_3 = 0$ or π (with arbitrary q_1). Take for example $\boldsymbol{q}_s = (q_1, 0, 0)$. We have

$$\boldsymbol{p}_s = (l_1 + l_3) \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \qquad \boldsymbol{J}_s = \begin{pmatrix} -(l_1 + l_3)\sin q_1 & -\sin q_1 & -l_3 \sin q_1 \\ (l_1 + l_3)\cos q_1 & \cos q_1 & l_3 \cos q_1 \end{pmatrix}$$

and so

$$\mathcal{N}\left\{\boldsymbol{J}_s^T\right\} = \alpha \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}.$$

Consider a gain $\boldsymbol{K}_P = k_P \boldsymbol{I}_{2\times 2}$, $k_P > 0$. By choosing

$$\boldsymbol{p}_d = \Delta \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \quad \Delta < l_1 + l_3 \qquad \Rightarrow \qquad \boldsymbol{K}_P \boldsymbol{e}_p = k_P\left(\Delta - (l_1 + l_3)\right) \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} \neq \boldsymbol{0},$$

with the robot at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$), the control law (3) becomes simply $\boldsymbol{\tau}_r = \boldsymbol{\tau}_r(\boldsymbol{q}_s, \boldsymbol{0}, \boldsymbol{e}_p) = \boldsymbol{g}(\boldsymbol{q}_s)$. In the closed-loop system, with the robot dynamics evaluated at the state $\boldsymbol{x}_s = (\boldsymbol{q}_s, \boldsymbol{0})$, it is

$$\boldsymbol{M}(\boldsymbol{q}_s)\,\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}_s, \boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{q}_s) = \boldsymbol{g}(\boldsymbol{q}_s) \qquad \Rightarrow \qquad \boldsymbol{M}(\boldsymbol{q}_s)\,\ddot{\boldsymbol{q}} = \boldsymbol{0} \qquad \Leftrightarrow \qquad \ddot{\boldsymbol{q}} = \boldsymbol{0},$$

so that the state $\boldsymbol{x}_s$ is an equilibrium and the robot will not move under the action of (3), despite of the residual Cartesian position error.

Exercise #2

We compute the direct and the differential kinematics up to the second order of the planar 4R robot for the positional task of its end effector. Being all links of unitary length, we have

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} c_1 + c_{12} + c_{123} + c_{1234} \\ s_1 + s_{12} + s_{123} + s_{1234} \end{pmatrix},$$

with the shorthand notation for trigonometric quantities (e.g., $c_{123} = \cos(q_1 + q_2 + q_3)$). Differentiating once w.r.t. time, we obtain

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with the (2×4) Jacobian matrix

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(s_1 + s_{12} + s_{123} + s_{1234}) & -(s_{12} + s_{123} + s_{1234}) & -(s_{123} + s_{1234}) & -s_{1234} \\ c_1 + c_{12} + c_{123} + c_{1234} & c_{12} + c_{123} + c_{1234} & c_{123} + c_{1234} & c_{1234} \end{pmatrix}. \tag{5}$$

Differentiating a second time, we get

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{6}$$

with the quadratic term in the joint velocities

$$\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = -\begin{pmatrix} c_1\,\dot{q}_1^2 + c_{12}\,(\dot{q}_1+\dot{q}_2)^2 + c_{123}\,(\dot{q}_1+\dot{q}_2+\dot{q}_3)^2 + c_{1234}\,(\dot{q}_1+\dot{q}_2+\dot{q}_3+\dot{q}_4)^2 \\ s_1\,\dot{q}_1^2 + s_{12}\,(\dot{q}_1+\dot{q}_2)^2 + s_{123}\,(\dot{q}_1+\dot{q}_2+\dot{q}_3)^2 + s_{1234}\,(\dot{q}_1+\dot{q}_2+\dot{q}_3+\dot{q}_4)^2 \end{pmatrix}.$$

At the current robot state $(\boldsymbol{q},\dot{\boldsymbol{q}})$ and for a given $\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_d$, finding a solution $\ddot{\boldsymbol{q}}$ to (6) that minimizes instantaneously the objective function (1) is a standard LQ problem with the unique solution

$$\begin{aligned} \ddot{\boldsymbol{q}}_a &= \boldsymbol{J}^{\#}(\boldsymbol{q})\,(\ddot{\boldsymbol{p}}_d - \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})) - \Big(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\Big)\,\boldsymbol{K}_d\,\dot{\boldsymbol{q}} \\ &= -\boldsymbol{K}_d\,\dot{\boldsymbol{q}} + \boldsymbol{J}^{\#}(\boldsymbol{q})\Big(\,\ddot{\boldsymbol{p}}_d - \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{K}_d\,\dot{\boldsymbol{q}}\Big). \end{aligned} \tag{7}$$

In fact, the *preferred* acceleration in the objective function (1), i.e., the one that would minimize H in the unconstrained case, is $\ddot{\boldsymbol{q}}_0 = -\boldsymbol{K}_d\dot{\boldsymbol{q}}$. This achieves damping of joint velocities in the null space of the task, as apparent from the first expression in (7). The second equivalent expression is however more efficient to evaluate.

The singularities of $\boldsymbol{J}(\boldsymbol{q})$ may affect the execution of the task and need to be analyzed in advance. Even if not strictly necessary, it is convenient for this purpose to simplify the Jacobian by the following factorization[2]

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1 & -s_{12} & -s_{123} & -s_{1234} \\ c_1 & c_{12} & c_{123} & c_{1234} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \boldsymbol{J}'(\boldsymbol{q})\,\boldsymbol{T}, \tag{8}$$

where $\boldsymbol{T}$ is a nonsingular matrix (with $\det \boldsymbol{T} = 1$). Thus, the configurations at which $\boldsymbol{J}'(\boldsymbol{q})$ loses rank coincide with those of $\boldsymbol{J}(\boldsymbol{q})$. By inspecting the six (2×2) minors of $\boldsymbol{J}'$, it is easy to see that this happens in the singular configurations

$$\boldsymbol{q}_s : \{\, q_1 \text{ is arbitrary}, \quad q_2 = 0 \ \text{ or } \ \pi, \quad q_3 = 0 \ \text{ or } \ \pi, \quad q_4 = 0 \ \text{ or } \ \pi \,\}.$$

In all these 8 types of singular configurations, the links of the robot are either stretched or folded along the single direction specified by q_1.

We extend now the task with the angular component, concerning the orientation of the end-effector. We proceed incrementally, using the results obtained so far for the two-dimensional task. We have

$$\boldsymbol{p}_e = \begin{pmatrix} \boldsymbol{f}(\boldsymbol{q}) \\ q_1+q_2+q_3+q_4 \end{pmatrix},$$

and thus

$$\dot{\boldsymbol{p}}_e = \boldsymbol{J}_e(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \begin{pmatrix} & \boldsymbol{J}(\boldsymbol{q}) & & \\ 1 & 1 & 1 & 1 \end{pmatrix} \dot{\boldsymbol{q}}$$

and

$$\ddot{\boldsymbol{p}}_e = \boldsymbol{J}_e(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_e(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \boldsymbol{J}_e(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{n}_e(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{J}_e(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \begin{pmatrix} \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ 0 \end{pmatrix}.$$

[2]This is a common trick for planar nR robot structures.

The solution of the LQ problem takes exactly the same form as in (7), once the augmented quantities are used, including the augmented desired acceleration $\ddot{\boldsymbol{p}}_{e,d}$, which is given by $\ddot{\boldsymbol{p}}_d \in \mathbb{R}^2$ with an extra third component equal to 0, or

$$\ddot{\boldsymbol{p}}_{e,d} = \begin{pmatrix} \ddot{\boldsymbol{p}}_d \\ \dot{\omega}_{z,d} \end{pmatrix} = \begin{pmatrix} \ddot{\boldsymbol{p}}_d \\ 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned} \ddot{\boldsymbol{q}}_b &= \boldsymbol{J}_e^{\#}(\boldsymbol{q}) \left(\ddot{\boldsymbol{p}}_{e,d} - \boldsymbol{n}_e(\boldsymbol{q}, \dot{\boldsymbol{q}})\right) - \left(\boldsymbol{I} - \boldsymbol{J}_e^{\#}(\boldsymbol{q})\boldsymbol{J}_e(\boldsymbol{q})\right) \boldsymbol{K}_d \, \dot{\boldsymbol{q}} \\ &= -\boldsymbol{K}_d \, \dot{\boldsymbol{q}} + \boldsymbol{J}_e^{\#}(\boldsymbol{q}) \Big(\ddot{\boldsymbol{p}}_{e,d} - \boldsymbol{n}_e(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{J}_e(\boldsymbol{q}) \boldsymbol{K}_d \, \dot{\boldsymbol{q}} \Big). \end{aligned} \tag{9}$$

The only difference will be in the analysis of the singularities of the (3×4) Jacobian $\boldsymbol{J}_e(\boldsymbol{q})$. Proceeding as before with the same transformation matrix $\boldsymbol{T}$, we have

$$\boldsymbol{J}_e(\boldsymbol{q}) = \boldsymbol{J}'_e(\boldsymbol{q})\, \boldsymbol{T}, \qquad \text{with} \quad \boldsymbol{J}'_e(\boldsymbol{q}) = \begin{pmatrix} -s_1 & -s_{12} & -s_{123} & -s_{1234} \\ c_1 & c_{12} & c_{123} & c_{1234} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

Accordingly, the singular configurations of $\boldsymbol{J}_e(\boldsymbol{q})$ are namely

$$\boldsymbol{q}_s : \{\, q_1 \text{ is arbitrary}, \;\; q_2 = 0 \text{ or } \pi, \;\; q_3 = 0 \text{ or } \pi, \;\; q_4 \text{ is arbitrary} \,\},$$

namely the same as those of the (2×3) Jacobian of a planar 3R robot in a positional task. In these 4 types of singular configurations, the first three links of the robot are either stretched or folded along the single direction specified again by q_1.

Using the values $\boldsymbol{q}_d$, $\dot{\boldsymbol{q}}_d$, and $\ddot{\boldsymbol{p}}_d$ specified in item c) of the text of this Exercise, we evaluate (7) and (9) as

$$\ddot{\boldsymbol{q}}_a = \begin{pmatrix} -0.3074 \\ -1.6378 \\ 2.0496 \\ 1.1248 \end{pmatrix} \text{ [rad/s}^2\text{]}, \qquad \ddot{\boldsymbol{q}}_b = \begin{pmatrix} 0.3163 \\ -2.5735 \\ 3.0503 \\ -0.7932 \end{pmatrix} \text{ [rad/s}^2\text{]}.$$

The extended position and orientation task vector and the end-effector linear and angular velocity at the current state $\boldsymbol{x}_d = (\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d)$ are

$$\boldsymbol{p}_e = \begin{pmatrix} \boldsymbol{p}_d \\ \phi_d \end{pmatrix} = \begin{pmatrix} 2.3801 \\ 2.1907 \\ 0.2618 \end{pmatrix} \text{ [m, m, rad]}, \qquad \dot{\boldsymbol{p}}_e = \begin{pmatrix} \dot{\boldsymbol{p}}_d \\ \omega_{z,d} \end{pmatrix} = \begin{pmatrix} 0.1654 \\ 0.1553 \\ 0.4 \end{pmatrix} \text{ [m/s, m/s, rad/s]}.$$

Exercise #3

Apart from the knowledge of unitary link lengths, no dynamic information is provided about the planar 2R robot in Fig. 3, Therefore, in its dynamic model,

$$\boldsymbol{M}(\boldsymbol{q})\, \ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u} + \boldsymbol{A}^T(\boldsymbol{q})\lambda = \begin{pmatrix} \tau \\ 0 \end{pmatrix} + \boldsymbol{A}^T(\boldsymbol{q})\lambda, \tag{11}$$

we will use a parametric form for the inertia, Coriolis/centrifugal, and gravity terms

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2c_2 & a_3 + a_2c_2 \\ a_3 + a_2c_2 & a_3 \end{pmatrix}, \;\; \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -a_2 s_2 \left(\dot{q}_2 + 2\dot{q}_1\right) \dot{q}_2 \\ a_2 s_2 \dot{q}_1^2 \end{pmatrix}, \;\; \boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} a_4 c_1 + a_5 c_{12} \\ a_5 c_{12} \end{pmatrix},$$

with dynamic coefficients a_i, $i = 1\ldots,5$, and using the shorthand notation for trigonometric quantities. On the right-hand side of (11), we have taken into account that there is no motor at the second joint (the joint is passive) and that the robot is subject to a holonomic constraint, providing a reaction force $\lambda \in \mathbb{R}$. The scalar constraint on the robot end effector is written as

$$k(\boldsymbol{p}) = p_x = 0 \qquad \Rightarrow \qquad h(\boldsymbol{q}) = k\left(\boldsymbol{f}(\boldsymbol{q})\right) = c_1 + c_{12} = 0.$$

The Jacobian of this constraint is

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(s_1 + s_{12}\right) & -s_{12} \end{pmatrix}. \tag{12}$$

In the assumed hypothesis on the location of the two points A and B, the matrix $\boldsymbol{A}(\boldsymbol{q})$ is always well defined and with full rank ($= 1$).

In order to obtain the reduced dynamic model of the constrained 2R robot, the basic step is to define a 1×2 (row) vector $\boldsymbol{D}(\boldsymbol{q})$ that is linearly independent from $\boldsymbol{A}(\boldsymbol{q})$, possibly everywhere in the region of interest. A useful choice is given by the following matrix (also with rank one in the constrained space)

$$\boldsymbol{D}(\boldsymbol{q}) = \begin{pmatrix} c_1 + c_{12} & c_{12} \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q}), \tag{13}$$

generating in this way the (2×2) robot Jacobian. This matrix is always nonsingular in the constrained region of operation of the 2R robot. Thus,

$$\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix}^{-1} = \boldsymbol{J}^{-1}(\boldsymbol{q}) = \frac{\operatorname{adj}\left\{\boldsymbol{J}(\boldsymbol{q})\right\}}{\det \boldsymbol{J}(\boldsymbol{q})} = \frac{1}{s_2} \begin{pmatrix} c_{12} & s_{12} \\ -\left(c_1 + c_{12}\right) & -\left(s_1 + s_{12}\right) \end{pmatrix} \triangleq \begin{pmatrix} \boldsymbol{E}(\boldsymbol{q}) & \boldsymbol{F}(\boldsymbol{q}) \end{pmatrix}.$$

We define then the pseudo-velocity $v_y \in \mathbb{R}$ on the Cartesian line and the inverse mapping to $\dot{\boldsymbol{q}}$ as

$$v_y = \boldsymbol{D}(\boldsymbol{q})\,\dot{\boldsymbol{q}} = c_1\,\dot{q}_1 + c_{12}\left(\dot{q}_1 + \dot{q}_2\right), \qquad \dot{\boldsymbol{q}} = \boldsymbol{F}(\boldsymbol{q})\,v_y = \begin{pmatrix} s_{12} \\ -\left(s_1 + s_{12}\right) \end{pmatrix} v_y. \tag{14}$$

Note that a subscript y has been added to v since this is exactly the y-component of the end-effector velocity, the only allowed by the constraint. Since the elements of the matrix $\boldsymbol{F}(\boldsymbol{q})$ are available analytically, we can obtain its time derivative in closed form[3]

$$\dot{\boldsymbol{F}}(\boldsymbol{q}) = \frac{1}{s_2} \begin{pmatrix} c_{12}\left(\dot{q}_1 + \dot{q}_2\right) \\ -c_1\dot{q}_1 - c_{12}\left(\dot{q}_1 + \dot{q}_2\right) \end{pmatrix} - \frac{c_2\,\dot{q}_2}{s_2^2} \begin{pmatrix} s_{12} \\ -\left(s_1 + s_{12}\right) \end{pmatrix}$$

Thus, the reduced dynamic model of the constrained 2R robot is given by the single differential equation

$$\left(\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\right)\dot{v}_y = \boldsymbol{F}^T(\boldsymbol{q})\left(\boldsymbol{u} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{F}}(\boldsymbol{q})\,v_y\right), \tag{15}$$

with the (always positive) reduced inertia given by the scalar

$$\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = \frac{1}{s_2^2}\left(a_3 s_1^2 + (a_1 - a_3)s_{12}^2 - 2a_2 s_1 c_2 s_{12}\right).$$

[3]This means that we don't need to replace the term $\dot{\boldsymbol{F}}(\boldsymbol{q})\dot{\boldsymbol{q}}$ by the longer expression $-\left(\dot{\boldsymbol{E}}(\boldsymbol{q})\dot{\boldsymbol{A}}(\boldsymbol{q}) + \boldsymbol{F}(\boldsymbol{q})\dot{\boldsymbol{D}}(\boldsymbol{q})\right)\dot{\boldsymbol{q}}$ within the derivations of the constrained dynamics. This is reflected in the form of the resulting equation (15).

Being only the single torque τ available as input, we have

$$\boldsymbol{F}^T(\boldsymbol{q})\,\boldsymbol{u} = \frac{1}{s_2}\begin{pmatrix} s_{12} & -(s_1+s_{12}) \end{pmatrix}\begin{pmatrix} \tau \\ 0 \end{pmatrix} = \frac{s_{12}}{s_2}\,\tau, \tag{16}$$

with $s_2 \neq 0$ in the domain of interest. For the following developments, the derivation of the expression of the force multiplier λ is not needed.

For item b), we need to find the robot configuration associated with point A. The inverse kinematic solution for this point can be found by geometric inspection: $\boldsymbol{q}_A = (\pi/6,\, 2\pi/3)$. Setting the robot at rest, we have that

$$\dot{\boldsymbol{q}} = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{c}(\boldsymbol{q}_A, \boldsymbol{0}) = \boldsymbol{0}, \quad v_y = \boldsymbol{D}(\boldsymbol{q}_A)\dot{\boldsymbol{q}} = 0,$$

and from (15) it follows

$$\left(\boldsymbol{F}^T(\boldsymbol{q}_A)\boldsymbol{M}(\boldsymbol{q}_A)\boldsymbol{F}(\boldsymbol{q}_A)\right)\dot{v}_y = \boldsymbol{F}^T(\boldsymbol{q}_A)\,\boldsymbol{u} - \boldsymbol{F}^T(\boldsymbol{q}_A)\,\boldsymbol{g}(\boldsymbol{q}_A) = \left.\frac{s_{12}}{s_2}\right|_{\boldsymbol{q}=\boldsymbol{q}_A}\tau - \left.\frac{s_{12}a_4c_1 - s_1a_5c_{12}}{s_2}\right|_{\boldsymbol{q}=\boldsymbol{q}_A}.$$

In order to have $\dot{v}_y = 0$, i.e., an equilibrium, we need to apply the torque

$$\tau_0 = \left.\left(a_4c_1 - \frac{s_1}{s_{12}}\,a_5c_{12}\right)\right|_{\boldsymbol{q}=\boldsymbol{q}_A} = \frac{\sqrt{3}}{2}\,(a_4+a_5)\,. \tag{17}$$

The value in (17) implicitly takes into account the reaction force (i.e., λ) imposed by the constraint at the end effector, which helps sustaining the robot against gravity in the configuration $\boldsymbol{q}_A$, even in the absence of a motor torque at joint 2. In fact, note that this is *not* the first component of the gravity torque $\boldsymbol{g}(\boldsymbol{q}_A)$, which is

$$g_1(\boldsymbol{q}_A) = \left.(a_4c_1 + a_5c_{12})\right|_{\boldsymbol{q}=\boldsymbol{q}_A} = \frac{\sqrt{3}}{2}\,(a_4-a_5)\,.$$

For item c), we design first the required motion trajectory for the end-effector. We start from the desired acceleration profile

$$\dot{v}_{y,d}(t) = \Delta\,\sin\omega t, \qquad t\in[0,T], \quad T = \frac{2\pi}{\omega}, \tag{18}$$

which is zero at start (for $t=0$) and end (for $t=T$). The amplitude $\Delta > 0$ needs yet to be defined. Integrating this acceleration profile, and imposing zero speed at the motion start and end, we obtain

$$v_{y,d}(t) = \frac{\Delta}{\omega}\,(1-\cos\omega t)\,, \qquad t\in[0,T]. \tag{19}$$

The positional trajectory starting from $A=(0,1)$ at time $t=0$ is then

$$p_{y,d}(t) = y_A + \frac{\Delta}{\omega}\left(t - \frac{1}{\omega}\sin\omega t\right).$$

We impose that the motion ends in point $B=(0,\sqrt{2})$ at time $t=T=2\pi/\omega$, obtaining eventually the value of Δ

$$p_{y,d}(T) = y_A + \frac{\Delta}{\omega}\,\frac{2\pi}{\omega} = y_B \qquad \Rightarrow \qquad \Delta = \frac{\omega^2}{2\pi}\,(y_B - y_A) = \frac{\omega^2}{2\pi}\left(\sqrt{2}-1\right) > 0.$$

To compute the torque $\tau_d(t)$ that realizes the motion, we use (18), (19), and again (15), evaluated along the nominal state trajectory $\boldsymbol{x}_d(t) = (\boldsymbol{q}_d(t), \dot{\boldsymbol{q}}_d(t))$:

$$\begin{aligned}\tau_d(t) = \left.\frac{s_2}{s_{12}}\right|_{\boldsymbol{q}=\boldsymbol{q}_d(t)} \cdot \boldsymbol{F}^T(\boldsymbol{q}_d(t))\Big(&\boldsymbol{M}(\boldsymbol{q}_d(t))\boldsymbol{F}(\boldsymbol{q}_d(t))\,\dot{v}_{y,d}(t) + \boldsymbol{c}(\boldsymbol{q}_d(t), \dot{\boldsymbol{q}}_d(t)) + \boldsymbol{g}(\boldsymbol{q}_d(t)) \\ &+ \boldsymbol{M}(\boldsymbol{q}_d(t))\dot{\boldsymbol{F}}(\boldsymbol{q}_d(t))\,v_{y,d}(t)\Big)\,.\end{aligned} \tag{20}$$

To determine the state trajectory $\boldsymbol{x}_d(t)$, $t \in [0, T]$, to be used in (20), one starts with

$$\boldsymbol{q}_d(0) = \boldsymbol{q}_A, \qquad \dot{\boldsymbol{q}}_d(0) = \boldsymbol{0},$$

and integrates forward in time the joint acceleration

$$\ddot{\boldsymbol{q}}_d = \boldsymbol{F}(\boldsymbol{q}_d)\,\dot{v}_{y,d} + \dot{\boldsymbol{F}}(\boldsymbol{q}_d)\boldsymbol{F}(\boldsymbol{q}_d)\dot{\boldsymbol{q}}_d,$$

as driven by the desired (reduced) acceleration profile $\dot{v}_{y,d}(t)$ in (18).

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