Robotics II

June 11, 2012

Exercise 1

In an image-based visual servoing scheme, a pin-hole camera with constant focal length $\lambda > 0$ is mounted on the end-effector of a robot manipulator. A fixed Cartesian point $P = (X\ Y\ Z)^T$, whose coordinates are expressed in the camera frame and having $Z > 0$, is associated to a point feature $\boldsymbol{f} = (u\ v)^T$ in the image plane. The 2×6 interaction matrix $\boldsymbol{J}_p$ in

$$\dot{\boldsymbol{f}} = \boldsymbol{J}_p(u, v, Z) \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix}$$

provides the velocity of the point feature $\boldsymbol{f}$ when the robot imposes to the camera a linear velocity $\boldsymbol{v} \in \mathbb{R}^3$ and an angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$, both expressed in the camera frame. Determine a basis for all possible motions of the camera that *do not move* the point feature in the image plane, i.e., yielding $\dot{\boldsymbol{f}} = \boldsymbol{0}$. How many independent *pure translation* motions (i.e., with $\boldsymbol{\omega} = \boldsymbol{0}$) of this kind exist for the camera? And how many independent *pure rotation* motions (i.e., with $\boldsymbol{v} = \boldsymbol{0}$)?

Exercise 2

A planar 3R robot with equal uniform thin rod links of unitary length moves in the vertical plane, actuated by the joint torques $\boldsymbol{\tau} \in \mathbb{R}^3$ and subject to a constant contact force $\boldsymbol{F}_c$ applied to the midpoint of the second link and pointing always upward (see Fig. 1).

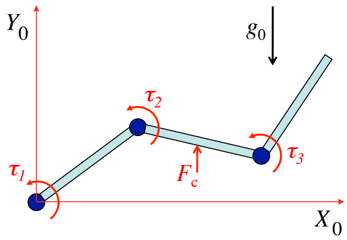

Figure 1: A planar 3R robot with a contact force

- Define a control law for $\boldsymbol{\tau}$ that regulates the robot to a desired generic equilibrium state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$ and give the explicit expression of the kinematic and dynamic terms needed for its implementation. The contact force and its location are assumed to be known, and thus there is no need of a force sensor. Provide a Lyapunov candidate that allows to prove the global asymptotic stability of the closed-loop system using Lyapunov/LaSalle arguments.
- Suppose that the robot is in its desired steady state $(\boldsymbol{q}_d, \boldsymbol{0})$ under the action of the previously defined control law. The contact force is then suddenly doubled in intensity, but the control law is left unchanged. Does the third joint accelerate instantaneously? Assuming that the system remains stable, find a characterization of the new final configuration reached at the end of the transient in terms of the parameters of the controller.
- If the contact force could be measured, would it be possible to preserve the global asymptotic stability of the original state $(\boldsymbol{q}_d, \boldsymbol{0})$ by suitably modifying the control law? If so, how?

[180 minutes; open books]

Solution

June 11, 2012

Exercise 1

The interaction matrix $\boldsymbol{J}_p$ for a point feature takes the form (see lecture slides)

$$\boldsymbol{J}_p = \begin{pmatrix} -\dfrac{\lambda}{Z} & 0 & \dfrac{u}{Z} & \dfrac{uv}{\lambda} & -\dfrac{u^2}{\lambda} - \lambda & v \\ 0 & \dfrac{\lambda}{Z} & \dfrac{v}{Z} & \dfrac{v^2}{\lambda} + \lambda & -\dfrac{uv}{\lambda} & -u \end{pmatrix} = \begin{pmatrix} \boldsymbol{J_v} & \boldsymbol{J_\omega} \end{pmatrix}. \tag{1}$$

Since $\lambda > 0$ and Z is limited (by the range of the camera), the interaction matrix as well as its two (2×3) blocks $\boldsymbol{J_v}$ and $\boldsymbol{J_\omega}$ will all have rank 2, independently from the values of u and v.

The problem requires to define a basis for the null space of $\boldsymbol{J}_p$ in the six-dimensional space of linear and angular camera velocity $(\boldsymbol{v}, \boldsymbol{\omega})$. From the analysis of the structure of $\boldsymbol{J}_p$ in (1), we have that

$$\dim \mathcal{N}\{\boldsymbol{J}_p\} = 4, \qquad \dim \mathcal{N}\{\boldsymbol{J_v}\} = 1, \qquad \dim \mathcal{N}\{\boldsymbol{J_\omega}\} = 1.$$

This means that there will be four independent velocity vector of the camera that will not move the feature (u, v) in the image plane, and that one such vector can be given the form of a pure translation $(\boldsymbol{v}, \boldsymbol{0})$ and another one of a pure rotation $(\boldsymbol{0}, \boldsymbol{\omega})$. The other two vectors will be associated to roto-translations. With this in mind, we can solve the problem by inspection or by resorting to the following symbolic Matlab code.

```
% null space of point feature interaction matrix

clear all
clc
syms lam u v Z real

J =
[ -lam/Z, 0, u/Z, (u*v)/lam, - u^2/lam - lam, v]
[ 0, -lam/Z, v/Z, v^2/lam + lam, -(u*v)/lam, -u];

N=null(J)

% check special case for u=v=0 (point at the center of the image plane)

Norigin=subs(N,[u,v],[0,0])

% check alternative solution (a pure rotation) that Matlab does not provide as such

nrot=simplify(u*N(:,2)+v*N(:,3)+lam*N(:,4))
verifynrot=simplify(J*nrot)

% OUTPUT

N =
[ u/lam, (Z*u*v)/lam^2, -(Z*(lam^2 + u^2))/lam^2, (Z*v)/lam]
[ v/lam, (Z*(lam^2 + v^2))/lam^2, -(Z*u*v)/lam^2, -(Z*u)/lam]
[ 1, 0, 0, 0]
[ 0, 1, 0, 0]
[ 0, 0, 1, 0]
[ 0, 0, 0, 1]
```

```
Norigin =
[ 0, 0, -Z, 0]
[ 0, Z, 0, 0]
[ 1, 0, 0, 0]
[ 0, 1, 0, 0]
[ 0, 0, 1, 0]
[ 0, 0, 0, 1]
nrot =
0
0
0
u
v
lam
verifyrot =
0
0
```

The columns of matrix `N` provide the solution. It is apparent that the first vector (conveniently scaled by λ) is a pure translation

$$\lambda\ \mathtt{N}(:,1) = \begin{pmatrix} u \\ v \\ \lambda \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{v} \\ \mathbf{0} \end{pmatrix}.$$

This represents a linear motion of the camera along the ray connecting the point (u, v) on the image plane to the Cartesian point $(X, Y.Z)$. On the other hand, the remaining three basis vectors for the null space of $\boldsymbol{J}_p$ found by Matlab have all the form of roto-translations. However, it is easy to see that their linear combination

$$u\ \mathtt{N}(:,2) + v\ \mathtt{N}(:,3) + \lambda\ \mathtt{N}(:,4) = \begin{pmatrix} \mathbf{0} \\ u \\ v \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\omega} \end{pmatrix}$$

is a pure rotation along the previously defined ray. Indeed, if not properly guided, Matlab will find the null space of a matrix by suitable orthogonalizations and normalizations (see the resulting (4×4) identity matrix in the lower part of `N`), without providing automatically a solution with a desired physical meaning.

Finally, the two remaining null space vectors can be given the form of roto-translations, either both with linear and angular components in the image plane

$$\lambda^2\ \mathtt{N}(:,2) = \begin{pmatrix} uvZ \\ (v^2+\lambda^2)Z \\ 0 \\ \lambda^2 \\ 0 \\ 0 \end{pmatrix}, \qquad \lambda^2\ \mathtt{N}(:,3) = \begin{pmatrix} -(u^2+\lambda^2)Z \\ -uvZ \\ 0 \\ 0 \\ \lambda^2 \\ 0 \end{pmatrix},$$

or one still of this type and the other with linear component in the image plane and angular component along the camera main axis

$$\lambda^2 \;\; (\mathtt{N}(:,2) + \mathtt{N}(:,3)) = \begin{pmatrix} -\left(u^2+\lambda^2-uv\right)Z \\ \left(v^2+\lambda^2-uv\right)Z \\ 0 \\ \lambda^2 \\ \lambda^2 \\ 0 \end{pmatrix}, \qquad \lambda\, \mathtt{N}(:,4) = \begin{pmatrix} vZ \\ -uZ \\ 0 \\ 0 \\ 0 \\ \lambda \end{pmatrix}.$$

Exercise 2

The robot dynamics, under the action of the contact force $\boldsymbol{F}_c$ and the motor control torque $\boldsymbol{\tau}$, is described by

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c,$$

AT EQUILIBRIUM. $q = q_d$, $\dot{q} = 0$
$c = 0$, $\tau = g - J_c^T F_c + K_P 0 + K_D 0$
$= g - J_c^T F_c$

where the Jacobian $\boldsymbol{J}_c(\boldsymbol{q})$ is associated to the linear velocity (in the plane) of the mid point of the second robot link, where the contact force is applied in the upward direction. For a regulation task in the presence of the constant and *known* force $\boldsymbol{F}_c$, we can use the control law

$$\boldsymbol{\tau} = \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\dot{\boldsymbol{q}}. \tag{2}$$

PD + GRAVITY COMP.
+ F_c COMP.

In fact, by canceling gravity and the effect of the contact force at the joint level, the task becomes in this way one of free regulation in the absence of gravity. This is solvable by a PD joint error action with $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$, yielding global asymptotic stability of the desired equilibrium state $(\boldsymbol{q}_d, \boldsymbol{0})$. The proof uses the Lyapunov candidate

$$V = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\left(\boldsymbol{q}_d - \boldsymbol{q}\right)^T\boldsymbol{K}_P\left(\boldsymbol{q}_d - \boldsymbol{q}\right)$$

and LaSalle analysis.

To implement the control law (2), we need to measure only $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ (no force sensing is needed) and compute $\boldsymbol{g}(\boldsymbol{q})$ and $\boldsymbol{J}(\boldsymbol{q})$. From the gravitational potential energy

$$U = U_1 + U_2 + U_3 = g_0\left(m_1\frac{\ell_1}{2}s_1 + m_2\left(\ell_1 s_1 + \frac{\ell_2}{2}s_{12}\right) + m_3\left(\ell_1 s_1 + \ell_2 s_{12} + \frac{\ell_3}{2}s_{123}\right)\right),$$

setting $\ell_1 = \ell_2 = \ell_3 = 1$, we obtain

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \frac{g_0}{2}\begin{pmatrix} m_1 c_1 + m_2\left(2c_1 + c_{12}\right) + m_3\left(2c_1 + 2c_{12} + c_{123}\right) \\ m_2 c_{12} + m_3\left(2c_{12} + c_{123}\right) \\ m_3 c_{123} \end{pmatrix}.$$

From the positional kinematics of the mid point of the second link

$$\boldsymbol{p}_c = \boldsymbol{f}_c(\boldsymbol{q}) = \begin{pmatrix} \ell_1 c_1 + 0.5\,\ell_2 c_{12} \\ \ell_1 s_1 + 0.5\,\ell_2 s_{12} \end{pmatrix},$$

setting $\ell_1 = \ell_2 = 1$, we obtain

$$\boldsymbol{J}_c(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \frac{1}{2}\begin{pmatrix} -\left(2s_1 + s_{12}\right) & -s_{12} & 0 \\ 2c_1 + c_{12} & c_{12} & 0 \end{pmatrix}.$$

Therefore

$$
\boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c = \boldsymbol{J}_c^T(\boldsymbol{q}) \begin{pmatrix} 0 \\ F_{c,y} \end{pmatrix} = \frac{F_{c,y}}{2} \begin{pmatrix} 2c_1 + c_{12} \\ c_{12} \\ 0 \end{pmatrix}.
$$

!

Note that no additional control torque is present at the third joint since the contact force is applied at the second link. However, this contact force, if not compensated, will in general accelerate also the third joint because of the inertial coupling.

With the robot at the desired equilibrium state $(\boldsymbol{q}_d, \boldsymbol{0})$ under the action of the control law (2), suppose that the contact force is doubled from $\boldsymbol{F}_c$ to $2\boldsymbol{F}_c$. The dynamic equations become in this state

τ IS STILL $g - J_c F_c$ WHILE F_c DOUBLED

$$
\boldsymbol{B}(\boldsymbol{q}_d)\ddot{\boldsymbol{q}} = -\boldsymbol{J}_c^T(\boldsymbol{q}_d)\boldsymbol{F}_c + 2\boldsymbol{J}_c^T(\boldsymbol{q}_d)\boldsymbol{F}_c = \boldsymbol{J}_c^T(\boldsymbol{q}_d)\boldsymbol{F}_c
$$

and thus the instantaneous acceleration is

$$
\ddot{\boldsymbol{q}} = \boldsymbol{B}^{-1}(\boldsymbol{q}_d)\boldsymbol{J}_c^T(\boldsymbol{q}_d)\boldsymbol{F}_c.
$$

Being in general the inertia matrix a full matrix (and so its inverse), all joints will have a non-zero acceleration $\ddot{q}_i$, including $\ddot{q}_3$.

Suppose now that a new steady state is reached, with $\boldsymbol{q} = \bar{\boldsymbol{q}}$ and $\dot{\boldsymbol{q}} = \ddot{\boldsymbol{q}} = \boldsymbol{0}$. In this equilibrium state, it is necessarily

$\underbrace{M \cdot \emptyset + \emptyset + g}_{\emptyset} = \tau + 2J_c^T F_c = J_c^T F_c + K_P(q_D - \bar{q})$

$= g - J_c^T F_c + K_P(q_d - \bar{q}) + K_D \cdot \emptyset$

$$
\boldsymbol{K}_P(\boldsymbol{q}_d - \bar{\boldsymbol{q}}) + \boldsymbol{J}_c^T(\bar{\boldsymbol{q}})\boldsymbol{F}_c = \boldsymbol{0}. \tag{3}
$$

We can determine the equilibrium configuration $\bar{\boldsymbol{q}}$ only by solving numerically the nonlinear equation (3). However, assuming that $\boldsymbol{K}_P$ is chosen diagonal as usual, since the last component of $\boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c$ is always zero, the third equation in (3) becomes

$$
K_{P,3}\,(q_{d,3} - \bar{q}_3) = 0 \qquad \Rightarrow \qquad \bar{q}_3 = q_{d,3}
$$

while $\bar{q}_i \neq q_{d,i}$ for $i = 1, 2$. This means that the third joint will resume its desired value after the perturbation due to the doubling of the contact force acting on the second link.

→ AFTER PERTURBATION = NEXT EQ. $\bar{q}$

Finally, if the contact force is measured (in some way) and used in feedback control, then the same law (2) —now with any value for the measured quantity $\boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c$— will guarantee that $(\boldsymbol{q}_d, \boldsymbol{0})$ will always remain the unique globally asymptotically stable equilibrium for the robot system.

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