Robotics II

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For the planar RP robot under gravity shown in Fig. 1, consider a class of one-dimensional tasks defined only in terms of the y-component of the end-effector Cartesian position

$$y = p_y(q_1, q_2).$$

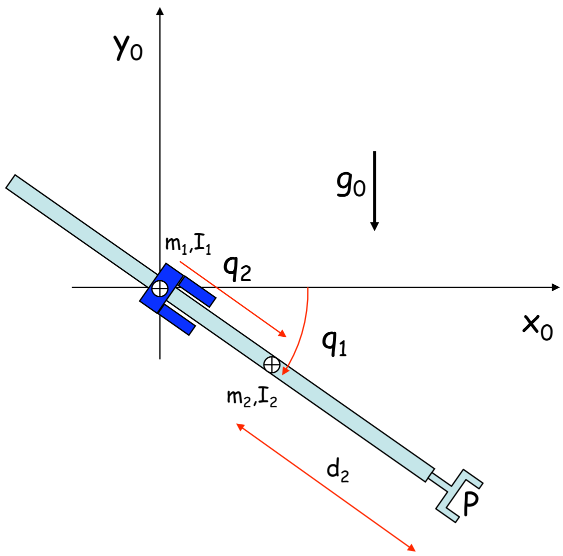

Figure 1: RP robot in the vertical plane, with definition of coordinates ($d_2 > 0$ is a constant)

Noting that the robot is redundant for this class of tasks, determine the explicit expression of the actuation input $\boldsymbol{\tau} = (\tau_1, \tau_2)$ that, at a generic robot state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, realizes a desired $\ddot{y}_d = A$ and has the *minimum norm* property.

[90 minutes; open books]

Solution

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The dynamic model of the RP robot

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{1}$$

should be obtained first.

With reference to Fig. 1, the robot kinetic energy T is given by

$$p_{c2} = \begin{pmatrix} q_2 c_1 \\ q_2 s_1 \\ 0 \end{pmatrix} \Rightarrow \|\dot{p}_{c2}\| = \dot{q}_2^2 + q_2^2 \dot{q}_1^2$$

$$\begin{aligned}
T_1 &= \frac{1}{2} I_1 \dot{q}_1^2 \\
T_2 &= \frac{1}{2} m_2 \|v_{c2}\|^2 + \frac{1}{2} I_2 \dot{q}_1^2 = \frac{1}{2}(I_2 + m_2 q_2^2)\dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 \\
T &= T_1 + T_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} \quad \Rightarrow \quad \boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} I_1 + I_2 + m_2 q_2^2 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} b_{11}(q_2) & 0 \\ 0 & b_{22} \end{pmatrix}.
\end{aligned}$$

Using the Christoffel's symbols for the components of the velocity vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$

$$\left(\frac{\partial b_1}{\partial q_1} \ \frac{\partial b_1}{\partial q_2} \right)$$

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right) + \left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \left(\frac{\partial \boldsymbol{B}(\boldsymbol{q})}{\partial q_i} \right) \right) \qquad i = 1, 2,$$

the Coriolis and centrifugal terms are determined as follows:

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \begin{pmatrix} 0 & m_2 q_2 \\ m_2 q_2 & 0 \end{pmatrix} \quad \Rightarrow \quad c_1(q_2, \dot{q}_1, \dot{q}_2) = 2\, m_2 q_2 \dot{q}_1 \dot{q}_2 \\
\boldsymbol{C}_2(\boldsymbol{q}) &= \begin{pmatrix} -2\, m_2 q_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad c_2(q_1, \dot{q}_1) = -m_2 q_2 \dot{q}_1^2.
\end{aligned}$$

The robot potential energy U is given by

$$\begin{aligned}
U_1 &= U_{10} \qquad U_2 = m_2 g_0 q_2 \sin q_1 + U_{20} \\
U &= U_1 + U_2 = m_2 g_0 q_2 \sin q_1 + U_{10} + U_{20} \\
&\Rightarrow \quad \boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} m_2 g_0 q_2 \cos q_1 \\ m_2 g_0 \sin q_1 \end{pmatrix} = \begin{pmatrix} g_1(q_1, q_2) \\ g_2(q_1) \end{pmatrix},
\end{aligned}$$

with $g_0 = 9.81 > 0$.

The direct kinematics associated to the end-effector position of the RP robot is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} (d_2 + q_2)\cos q_1 \\ (d_2 + q_2)\sin q_1 \end{pmatrix},$$

where $d_2 > 0$ is the constant length shown in Fig. 1. Being the task defined only in terms of the p_y component, it is

$$\dot{p}_y = \begin{pmatrix} (d_2 + q_2)\cos q_1 & \sin q_1 \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

and then

$$\ddot{p}_y = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \begin{pmatrix} \cos q_1 \dot{q}_2 - (d_2 + q_2)\sin q_1 \dot{q}_1 & \cos q_1 \dot{q}_1 \end{pmatrix} \dot{\boldsymbol{q}}. \tag{2}$$

Note that the task Jacobian $\boldsymbol{J}$ is singular if and only if $\sin q_1 = 0$ *and* $q_2 = -d_2$.

Replacing in (2) the accelerations $\ddot{\boldsymbol{q}}$ from (1) yields

$$\ddot{p}_y = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{B}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right) + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Setting then $\ddot{p}_y = A$ and reorganizing terms, we obtain

$$\boldsymbol{M}(\boldsymbol{q})\boldsymbol{\tau} = A - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{B}^{-1}(\boldsymbol{q})\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})\right) =: d(\boldsymbol{q},\dot{\boldsymbol{q}}),$$

having defined also

$$\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{B}^{-1}(\boldsymbol{q}) = \begin{pmatrix} \dfrac{(d_2+q_2)\cos q_1}{b_{11}(q_2)} & \dfrac{\sin q_1}{b_{22}} \end{pmatrix}.$$

At a generic robot state $(\boldsymbol{q},\dot{\boldsymbol{q}})$, the question at hand is then formulated as a linear-quadratic optimization problem in the standard form

$$\min \frac{1}{2}\|\boldsymbol{\tau}\|^2 = \frac{1}{2}\left(\tau_1^2 + \tau_2^2\right) \quad \text{s.t.} \quad \boldsymbol{M}\boldsymbol{\tau} = d.$$

The optimal solution is simply

$$\boldsymbol{\tau}^* = \boldsymbol{M}^{\#} d, \tag{3}$$

where all quantities have been already defined. In explicit terms, in case of full (row) rank $\boldsymbol{M}$ we have[1]

$$\boldsymbol{M}^{\#} = \boldsymbol{B}^{-1}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{B}^{-2}\boldsymbol{J}^T\right)^{-1}.$$

In particular, out of the singularities of the 1×2 matrix $\boldsymbol{M}$, which coincide with those of the task Jacobian $\boldsymbol{J}$, the pseudoinverse of $\boldsymbol{M}$ has the explicit expression

$$\boldsymbol{M}^{\#}(\boldsymbol{q}) = \frac{1}{\left(\dfrac{(d_2+q_2)\cos q_1}{b_{11}(q_2)}\right)^2 + \left(\dfrac{\sin q_1}{b_{22}}\right)^2} \begin{pmatrix} \dfrac{(d_2+q_2)\cos q_1}{b_{11}(q_2)} \\[2ex] \dfrac{\sin q_1}{b_{22}} \end{pmatrix}.$$

The optimal solution (3) implies that both joints/actuators are typically involved in this one-dimensional task. Although in general the task could have been realized also by actuating only a single joint (the revolute or the prismatic one), the combination results in the minimum actuation effort.

It should be remarked that the norm of $\boldsymbol{\tau}$ has a dimensionality problem. In fact, the first actuation input is a torque (on the revolute joint) and the second is a force (on the prismatic joint), so that physical units are mixed in computing the norm. A way to handle this problem is to introduce a proper scaling in the objective function, i.e., considering a positive definite diagonal matrix $\boldsymbol{W} = \text{diag}\{1, w\} > 0$ and minimizing

$$\frac{1}{2}\boldsymbol{\tau}^T\boldsymbol{W}\boldsymbol{\tau} = \frac{1}{2}\left(\tau_1^2 + w\,\tau_2^2\right),$$

[1]Note also that in general $\boldsymbol{M}^{\#} = (\boldsymbol{J}\boldsymbol{B}^{-1})^{\#} \neq \boldsymbol{B}\boldsymbol{J}^{\#}$. The equality holds if $\boldsymbol{B} = b \cdot \boldsymbol{I}$, for a scalar b.

where the scalar $w > 0$ takes into account how costly a unit of torque is in comparison to a unit of force. The associated solution is then obtained by replacing the pseudoinverse of $\boldsymbol{M}$ in (3) by its weighted pseudoinverse

$$\boldsymbol{M}_{\boldsymbol{W}}^{\#} = \boldsymbol{W}^{-1}\boldsymbol{M}^T \left(\boldsymbol{M}\boldsymbol{W}^{-1}\boldsymbol{M}^T\right)^{-1}.$$

Finally, it is worth mentioning that the above local solution with minimum norm of the actuation inputs is prone to an internal build up of joint velocities, especially for long task trajectories. A countermeasure to this phenomenon is to choose a solution of the form

$$\boldsymbol{\tau} = \boldsymbol{M}^{\#}d + \left(\boldsymbol{I} - \boldsymbol{M}^{\#}\boldsymbol{M}\right)\boldsymbol{\tau}_0, \tag{4}$$

with $\boldsymbol{\tau}_0 = -\boldsymbol{K}_D\dot{\boldsymbol{q}}$ and where $\boldsymbol{K}_D$ is a diagonal, positive definite matrix. The additional torque $\boldsymbol{\tau}_0$ damps the joint velocity $\dot{\boldsymbol{q}}$, without affecting the execution of the task. It is also easy to see that (4) is the solution to the following modified linear-quadratic optimization problem

$$\min \frac{1}{2}\left(\boldsymbol{\tau} - \boldsymbol{\tau}_0\right)^T\left(\boldsymbol{\tau} - \boldsymbol{\tau}_0\right) \quad \text{s.t.} \quad \boldsymbol{M}\boldsymbol{\tau} = d.$$

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