Robotics 2

January 25, 2023

Exercise 1

The 2R planar robot in Fig. 1 moves in a vertical plane. The second link has its center of mass on the axis of the second joint. Viscous friction is present at both joints.

a. Derive the dynamic model of this robot in Lagrangian form. Find then a linear parametrization of the model as
$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{\tau},$$
where the vector of dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^p$ has the least dimension p (the gravity acceleration g_0 and the link lengths are assumed to be known).

b. Consider the control law $\boldsymbol{\tau} = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d)$, with diagonal gain matrices $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D \geq 0$ and with constant gravity compensation at $\boldsymbol{q}_d$. Which are the *minimum* values of the four control gains K_{P1}, K_{P2}, K_{D1} and K_{D2} that guarantee global asymptotic stabilization of *any* generic desired equilibrium state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$?

c. With the robot dynamic parameters being unknown (except for the position of the center of mass of the second link), design an adaptive control law that is able to obtain global asymptotic tracking of a desired smooth trajectory $\boldsymbol{q}_d(t)$.

d. Suppose now that: *i)* the robot moves on a horizontal plane, *ii)* friction at the joints is negligible, and *iii)* the motor torques are bounded as $|\tau_i(t)| \leq \tau_{max,i}$, $i = 1, 2$. Consider the rest-to-rest task of moving in minimum time the first joint by $\Delta > 0$, while keeping the second joint *constantly* at its initial value $q_2(0)$. Determine the optimal solution and the minimum time T in analytic form. Sketch the time-optimal profiles of $\dot{q}_1(t)$, $\ddot{q}_1(t)$, $\tau_1(t)$ and $\tau_2(t)$, for $t \in [0, T]$.

Exercise 2

A 2P Cartesian robot on a horizontal plane is equipped with a F/T sensor at the end-effector. The robot should keep contact with a linear surface, which makes an angle $\alpha \in (0, \pi)$ with the x-axis, while moving at a constant tangential speed $v_d > 0$ and applying a constant normal force $f_d > 0$ (see Fig. 2). The environment is compliant with stiffness K_n and frictionless, so that it can provide only normal reaction forces. Design an hybrid force-velocity control law that realizes exponential stabilization of the velocity and force errors in a decoupled way along the two task directions. *Hint: Because of the surface compliance, one can consider in the analysis also a small deformation $\delta_n(t)$ at the contact in the normal task direction, and relate $f_n(t)$ and $v_n(t)$ to it.*

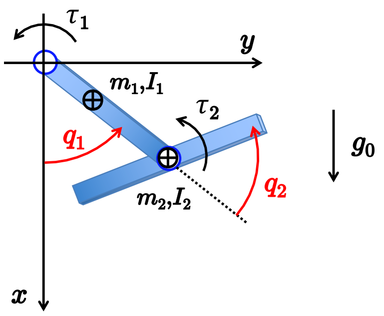

Figure 1: A 2R planar robot with a balanced second link.

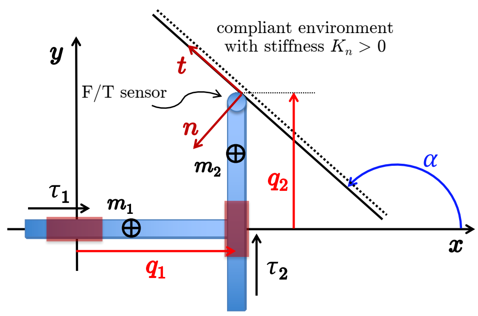

Figure 2: A hybrid force-velocity task to be executed by a 2P Cartesian robot.

[180 minutes; open books]

Solution

January 25, 2023

Exercise 1

a. Dynamic model

Kinetic energy

$$T = T_1 + T_2 = \frac{1}{2}\left(I_1 + m_1 d_{c1}^2\right)\dot{q}_1^2 + \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} I_2 \left(\dot{q}_1 + \dot{q}_2\right)^2$$

where

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \frac{d}{dt}\begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} = \begin{pmatrix} -l_1 s_1 \dot{q}_1 \\ l_1 c_1 \dot{q}_1 \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{v}_{c2}\|^2 = l_1^2 \dot{q}_1^2.$$

Inertia matrix

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}} \quad \Rightarrow \quad \boldsymbol{M} = \begin{pmatrix} I_{tot} & I_2 \\ I_2 & I_2 \end{pmatrix} > 0 \qquad \text{(constant)},$$

with $I_{tot} = I_1 + m_1 d_{c1}^2 + m_2 l_1^2 + I_2 = I_0 + I_2 > I_2$. Coriolis and centrifugal terms are zero.

Potential energy and gravity terms

$$U = U_1 + U_2 = -m_1 g_0 d_{c1} c_1 - m_2 g_0 l_1 c_1,$$

and so

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} \left(m_1 d_{c1} + m_2 l_1\right) g_0 s_1 \\ 0 \end{pmatrix}.$$

Robot equations

$$\begin{aligned} \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{F}_v \dot{\boldsymbol{q}} &= \boldsymbol{\tau} \\ &\Downarrow \\ I_{tot}\,\ddot{q}_1 + I_2\,\ddot{q}_2 + \left(m_1 d_{c1} + m_2 l_1\right) g_0 s_1 + F_{v1}\,\dot{q}_1 &= \tau_1 \\ I_2\,\ddot{q}_1 + I_2\,\ddot{q}_2 + F_{v2}\,\dot{q}_2 &= \tau_2. \end{aligned} \tag{1}$$

Linear parametrization

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\,\boldsymbol{a} = \begin{pmatrix} \ddot{q}_1 & \ddot{q}_2 & g_0 s_1 & \dot{q}_1 & 0 \\ 0 & \ddot{q}_1 + \ddot{q}_2 & 0 & 0 & \dot{q}_2 \end{pmatrix} \begin{pmatrix} I_{tot} \\ I_2 \\ m_1 d_{c1} + m_2 l_1 \\ F_{v1} \\ F_{v2} \end{pmatrix},$$

with $p = 5$ dynamic coefficients a_i, $i = 1, \ldots, 5$. This is obviously a factorization with the least possible number of dynamic coefficients, although not the only one with 5 coefficients; we may, e.g., replace I_{tot} by I_0 in $\boldsymbol{a}$, obtaining a new regressor matrix $\boldsymbol{Y}$ having $Y_{12} = \ddot{q}_1 + \ddot{q}_2$ as the only changed element.

b. Regulation control law

Under the given assumptions, for the PD plus constant gravity compensation law

$$\boldsymbol{\tau} = \boldsymbol{K}_P\,(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d) \tag{2}$$

the following minimal values for the four control gains are sufficient for the global asymptotic stability of the closed-loop system:

$$K_{P1} > \alpha = a_3 g_0 = (m_1 d_{c1} + m_2 l_1)\, g_0 > 0, \qquad K_{P2} > 0, \qquad K_{D1} = K_{D2} = 0, \tag{3}$$

where $\alpha \geq \|\partial\boldsymbol{g}/\partial\boldsymbol{q}\|$, for all $\boldsymbol{q}$. In fact, the closed-loop system (1),(2) can be rewritten as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + (\boldsymbol{F}_v + \boldsymbol{K}_D)\,\dot{\boldsymbol{q}} + (\boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{g}(\boldsymbol{q}_d)) = \boldsymbol{K}_P\,(\boldsymbol{q}_d - \boldsymbol{q})\,.$$

Thus, being $\boldsymbol{F}_v > 0$, the presence of viscous friction allows to set to zero the derivative gains $\boldsymbol{K}_D$ in the control law, without prejudice for the asymptotic stability. Moreover, at any equilibrium ($\dot{\boldsymbol{q}} = \ddot{\boldsymbol{q}} = \boldsymbol{0}$), we have

$$(\boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{g}(\boldsymbol{q}_d)) = \boldsymbol{K}_P\,(\boldsymbol{q}_d - \boldsymbol{q}) \quad \Leftrightarrow \quad \begin{aligned} a_3 g_0\,(\sin q_1 - \sin q_{d1}) &= K_{P1}\,(q_{d1} - q_1) \\ 0 &= K_{P2}\,(q_{d2} - q_2)\,. \end{aligned}$$

It is clear that these two equilibrium conditions are decoupled each to other. In the first equation, it is sufficient to have $K_{P1} > a_3$ in order to have a *unique* equilibrium solution at $q_1 = q_{d1}$. Instead, in the second equation $K_{P2} > 0$ is already sufficient to guarantee that $q_2 = q_{d2}$ is the unique equilibrium.

It is easy to see that the original Lyapunov proof showing global asymptotic stability of the desired state $(\boldsymbol{q}_d, \boldsymbol{0})$ with the PD+ control (2) works as well in the present case under the gain assumptions (3). As a result, the standard sufficient condition

$$K_{Pm} = \min\{K_{P1}, K_{P2}\} \geq \alpha \quad \Rightarrow \quad K_{P1} \geq \alpha, \quad K_{P2} \geq \alpha > 0$$

is relaxed: it is sufficient to have just a positive proportional gain $K_{P2} > 0$ at joint 2, without any strictly positive lower bound. Moreover, the conditions (3) on the positional gains become also *necessary* for global asymptotic stabilization if we consider that the same control law (2) should work for *any* chosen $\boldsymbol{q}_d$. In particular, the necessity of $K_{P1} > \alpha = a_3 g_0$ follows from the local analysis of the behavior of the closed-loop system linearized around $q_{d1} = \pi$.

c. Adaptive control law for trajectory tracking

Based on the previous results, an adaptive control law for tracking a desired smooth trajectory $\boldsymbol{q}_d(t)$, with global asympotic stability of the tracking error $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$, takes the following expression:

$$\begin{aligned} \boldsymbol{\tau} &= \hat{\boldsymbol{M}}\ddot{\boldsymbol{q}}_r + \hat{\boldsymbol{g}}(\boldsymbol{q}) + \hat{\boldsymbol{F}}_v\,\dot{\boldsymbol{q}}_r + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}} \\ \dot{\hat{\boldsymbol{a}}} &= \begin{pmatrix} \dot{\hat{I}}_{tot} \\ \dot{\hat{I}}_2 \\ \dot{\widehat{m_1 d_{c1} + m_2 l_1}} \\ \dot{\hat{F}}_{v1} \\ \dot{\hat{F}}_{v2} \end{pmatrix} = \boldsymbol{\Gamma}\boldsymbol{Y}^T(\boldsymbol{q}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\boldsymbol{s} = \boldsymbol{\Gamma} \begin{pmatrix} \ddot{q}_{r1} & 0 \\ \ddot{q}_{r2} & \ddot{q}_{r1} + \ddot{q}_{r2} \\ g_0 s_1 & 0 \\ \dot{q}_{r1} & 0 \\ 0 & \dot{q}_{r2} \end{pmatrix} (\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}), \end{aligned}$$

with (diagonal) $\boldsymbol{K}_P > 0$, $\boldsymbol{K}_D > 0$ and $\boldsymbol{\Gamma} > 0$, a modified reference velocity $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}\boldsymbol{e}$, and the choice $\boldsymbol{\Lambda} = \boldsymbol{K}_D^{-1}\boldsymbol{K}_P$.

d. Rest-to-rest motion in minimum time

On the horizontal plane and without dissipative terms, the dynamic model (1) reduces to

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{\tau} \quad \Rightarrow \quad \begin{aligned} I_{tot}\,\ddot{q}_1 + I_2\,\ddot{q}_2 &= \tau_1 \\ I_2\,\ddot{q}_1 + I_2\,\ddot{q}_2 &= \tau_2. \end{aligned} \tag{4}$$

The specified motion task requires the second joint to remain at rest in the same initial configuration[1], thus imposing $\dot{q}_2 = \ddot{q}_2 = 0$. The two equations (4) are then rewritten in a direct/inverse dynamic form as

$$\ddot{q}_1 = \frac{1}{I_{tot}}\,\tau_1, \qquad \tau_2 = I_2\,\ddot{q}_1 = \frac{I_2}{I_{tot}}\,\tau_1 = \frac{I_2}{I_2 + I_0}\,\tau_1 < \tau_1. \tag{5}$$

The rest-to-rest motion in minimum time for the first joint will be a bang-bang profile in acceleration (and torque). Similarly, because of the coupling between the two commands in (5), also the second torque that keeps joint 2 at rest will have a bang-bang profile. However, only one between the two commanded torques is allowed to reach its bound, depending on the relative values of $\tau_{max,1}$ and $\tau_{max,2}$ and on the robot inertias. In fact, introducing a scalar parameter α to possibly scale the maximum torque at joint 1, we have

$$\tau_1 = \alpha\,\tau_{max,1}, \;\; \alpha \in (0,1] \quad \Rightarrow \quad \tau_2 = \frac{I_2}{I_2 + I_0}\,\alpha\,\tau_{max,1} \le \tau_{max,2} \quad \Rightarrow \quad \alpha \le \frac{I_2 + I_0}{I_2}\,\frac{\tau_{max,2}}{\tau_{max,1}}. \tag{6}$$

Therefore, the maximum torque that can be applied at joint 1 (complying with both bounds) is

$$\bar{\tau}_1 = \min\left\{1, \frac{I_2 + I_0}{I_2}\,\frac{\tau_{max,2}}{\tau_{max,1}}\right\}\tau_{max,1} = \min\left\{\tau_{max,1}, \frac{I_2 + I_0}{I_2}\,\tau_{max,2}\right\}. \tag{7}$$

Accordingly, the torque needed at joint 2 to keep it at rest will be

$$\bar{\tau}_2 = \frac{I_2}{I_2 + I_0}\,\bar{\tau}_1 = \min\left\{\frac{I_2}{I_2 + I_0}\,\tau_{max,1}, \tau_{max,2}\right\}. \tag{8}$$

In order to perform in minimum time the desired rest-to-rest displacement $\Delta > 0$ of the first joint without moving the second, we apply the torques (symmetric in time, so $T_s = T/2$)

$$\tau_1(t) = \begin{cases} \bar{\tau}_1, & t \in [0, T/2] \\ -\bar{\tau}_1, & t \in [T/2, T], \end{cases} \qquad \tau_2(t) = \begin{cases} \bar{\tau}_2, & t \in [0, T/2] \\ -\bar{\tau}_2, & t \in [T/2, T], \end{cases}$$

where T is the minimum motion time, yet to be determined. The acceleration and the velocity of joint 1 will have, respectively, a bang-bang and a triangular time profile:

$$\ddot{q}_1(t) = \begin{cases} \dfrac{\bar{\tau}_1}{I_{tot}}, & t \in [0, T/2] \\ -\dfrac{\bar{\tau}_1}{I_{tot}}, & t \in [T/2, T], \end{cases} \qquad \dot{q}_1(t) = \begin{cases} \dfrac{\bar{\tau}_1}{I_{tot}}\,t, & t \in [0, T/2] \\ \dfrac{\bar{\tau}_1}{I_{tot}}\,T - \dfrac{\bar{\tau}_1}{I_{tot}}\,t, & t \in [T/2, T]. \end{cases}$$

Finally, the minimum time T is obtained by equating the area below the (positive) velocity profile to the displacement $\Delta > 0$. We obtain

$$\frac{\bar{\tau}_1}{I_{tot}}\,\frac{T}{2}\cdot\frac{T}{2} = \Delta \qquad \Rightarrow \qquad T = \sqrt{\frac{4\Delta I_{tot}}{\bar{\tau}_1}}.$$

[1] The actual value of $q_2(0)$ is irrelevant for what follows.

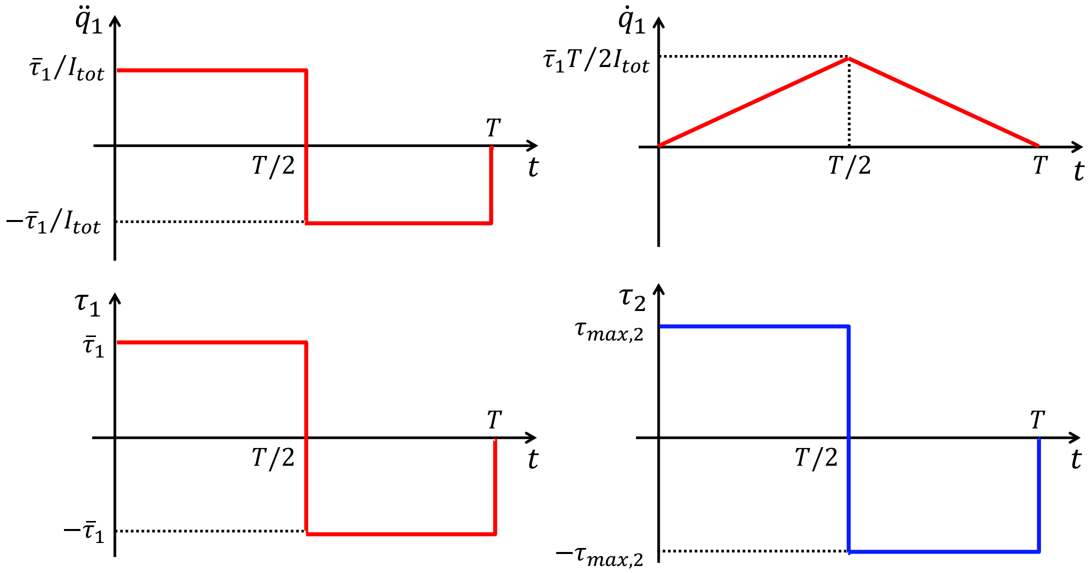

Figure 3: Minimum-time profiles for a displacement $\Delta > 0$ of q_1, when $\tau_2(t) = \pm\tau_{max,2}$: $\ddot{q}_1(t)$ and $\dot{q}_1(t)$ [top]; $\tau_1(t)$ and $\tau_2(t)$ [bottom].

Figure 3 shows the minimum-time profiles of $\ddot{q}_1(t)$ and $\dot{q}_1(t)$, and of the torques $\tau_1(t)$ and $\tau_2(t)$, assuming that the torque at joint 2 is the one that saturates its bound in (7) and (8), i.e., $\alpha < 1$ in (6). Thus, $\bar{\tau}_1 = (I_2 + I_0)\,\tau_{max,2}/I_2 < \tau_{max,1}$.

Exercise 2

The dynamic model of the Cartesian robot in contact with the environment is

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{\tau} + \boldsymbol{\tau}_f \quad \Rightarrow \quad \begin{pmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} \tau_1 + \tau_{f1} \\ \tau_2 + \tau_{f2} \end{pmatrix}, \tag{9}$$

where $\boldsymbol{\tau}_f$ are the joint torques resulting from the forces exerted by the environment on the robot (and performing work on $\boldsymbol{q}$).

The orientation of the task frame shown in Fig. 2 is given by a 2×2 constant rotation matrix $\boldsymbol{R}$ in the plane. Accordingly, the following relationships between task velocities $\boldsymbol{v}$ and forces $\boldsymbol{f}$ and joint velocities $\dot{\boldsymbol{q}}$ and torques $\boldsymbol{\tau}_f$ hold, all vectors being in $\mathbb{R}^2$:

$$\boldsymbol{R} = \begin{pmatrix} c_\alpha & -s_\alpha \\ s_\alpha & c_\alpha \end{pmatrix}, \qquad \boldsymbol{v} = \boldsymbol{R}^T\dot{\boldsymbol{q}}, \qquad \boldsymbol{v} = \begin{pmatrix} v_t \\ v_n \end{pmatrix}, \qquad \boldsymbol{\tau}_f = \boldsymbol{R}\,\boldsymbol{f}, \qquad \boldsymbol{f} = \begin{pmatrix} 0 \\ f_n \end{pmatrix}.$$

Since the environment is frictionless, we have set $f_t = 0$: applied contact forces can be balanced by reaction forces f_n only along the normal to the environment. On the other hand, being the environment compliant, a non-zero (though small) normal velocity v_n may also be present at the contact. Let δ_n be the deformation at the contact point along the environment normal. Then

$$v_n = \dot{\delta}_n, \qquad f_n = K_n \delta_n. \tag{10}$$

as it was a spring: stiffness * deformation

With the above notations, we rewrite the dynamic model (9) in the task space as

$$\boldsymbol{R}^T\boldsymbol{M}\boldsymbol{R}\,\dot{\boldsymbol{v}} = \boldsymbol{R}^T\boldsymbol{\tau} + \boldsymbol{R}^T\boldsymbol{\tau}_f \qquad \Rightarrow \qquad \bar{\boldsymbol{M}}\dot{\boldsymbol{v}} = \bar{\boldsymbol{\tau}} + \boldsymbol{f}, \tag{11}$$

with

$$\bar{\boldsymbol{M}} = \begin{pmatrix} m_1 c_\alpha^2 + m_2 & -m_1 c_\alpha s_\alpha \\ -m_1 c_\alpha s_\alpha & m_1 s_\alpha^2 + m_2 \end{pmatrix}, \qquad \bar{\boldsymbol{\tau}} = \begin{pmatrix} c_\alpha \tau_1 + s_\alpha \tau_2 \\ -s_\alpha \tau_1 + c_\alpha \tau_2 \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} 0 \\ f_n \end{pmatrix}.$$

The dynamics (11) in the task space is still linear but coupled between the t and n axes. We proceed then with the following decoupling control law:

$$\bar{\boldsymbol{\tau}} = \bar{\boldsymbol{M}} \begin{pmatrix} K_v\,(v_d - v_t) \\ K_f\,((f_d - f_n) - K_d v_n) \end{pmatrix} - \begin{pmatrix} 0 \\ f_n \end{pmatrix}, \qquad K_v > 0,\ K_f > 0,\ K_d > 0. \tag{12}$$

The closed-loop system (11),(12) becomes

$$\begin{aligned} \dot{v}_t &= K_v\,(v_d - v_t) \\ \dot{v}_n &= K_f\,((f_d - f_n) - K_d v_n). \end{aligned} \tag{13}$$

Along the tangential direction of the task space, the control action is proportional to the velocity error $e_v = v_d - v_t$. Since v_d is constant, $\dot{e}_v = -\dot{v}_t$ and the first equation in (13) is rewritten as

$$\dot{e}_v = -K_v e_v \quad \Rightarrow \quad e_v(t) = e_v(0) \exp\left(-K_v\, t\right),$$

which shows exponential stabilization to zero of the tangential velocity error.

Along the normal direction of the task space, the control law cancels any (measured) contact force f_n, adds a proportional action on the force error $e_f = f_d - f_n$, and includes a velocity damping $-K_d v_n$. For analysis, using the relationships (10) of the compliant environment, the second equation in (13) can be expressed in terms of the deformation δ_n as

$$M_f \ddot{\delta}_n + K_d \dot{\delta}_n + K_n \delta_n = f_d, \qquad \text{with } M_f = \frac{1}{K_f} > 0. \tag{14}$$

Thus, an impedance-like behavior has been obtained, where the apparent mass M_f and damping K_d can be chosen freely, while the stiffness K_n is the one of the environment. Moreover, the forcing term on the right-hand side is the desired contact force f_d, rather than the actual one f_n as in a standard impedance design. This setting is indeed appropriate. In fact, the second-order dynamics (14) is stable and converges exponentially[2], as $t \to \infty$, to the constant equilibrium deformation $\bar{\delta}_n = f_d / K_n$. In turn, this implies that the normal force at steady state is the desired one:

$$\lim_{t \to \infty} f_n(t) = K_n \bar{\delta}_n = f_d.$$

Finally, the control torque in the joint space is obtained from (12) as

$$\boldsymbol{\tau} = \boldsymbol{R}\,\bar{\boldsymbol{\tau}} = \boldsymbol{M}\boldsymbol{R} \begin{pmatrix} K_v\,(v_d - v_t) \\ K_f\,((f_d - f_n) - K_d v_n) \end{pmatrix} - \boldsymbol{\tau}_f \qquad \left(\text{being } \boldsymbol{\tau}_f = \begin{pmatrix} -s_\alpha \\ c_\alpha \end{pmatrix} f_n \right).$$

* * * * *

[2]Other than by setting $\dot{\delta}_n = \ddot{\delta}_n = 0$ in (14), the steady-state deformation $\bar{\delta}_n$ can also be computed by analyzing in the Laplace domain the system response to a step input f_d. It is

$$W(s) = \frac{\delta_n(s)}{f_d(s)} = \frac{1}{M_f s^2 + F_d s + K_n} \quad \Rightarrow \quad \bar{\delta}_n = \lim_{t \to \infty} \delta_n(t) = \lim_{s \to 0} s\,\delta_n(s) = \lim_{s \to 0} s\,W(s) \cdot \frac{f_d}{s} = W(0)\,f_d = \frac{f_d}{K_n}.$$