Value Iteration

Reinforcement Learning

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Recap

Sequential Decision Making

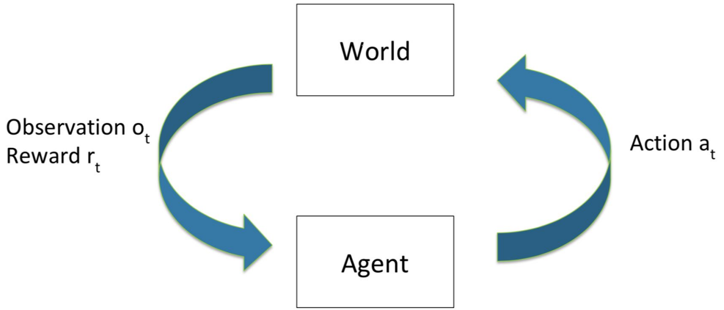
The agent interacts with the environment:

- at discrete timesteps;
- by receiving observations o_t and reward r_t from the environment;
- by taking actions a_t;

The state is a function of the history:

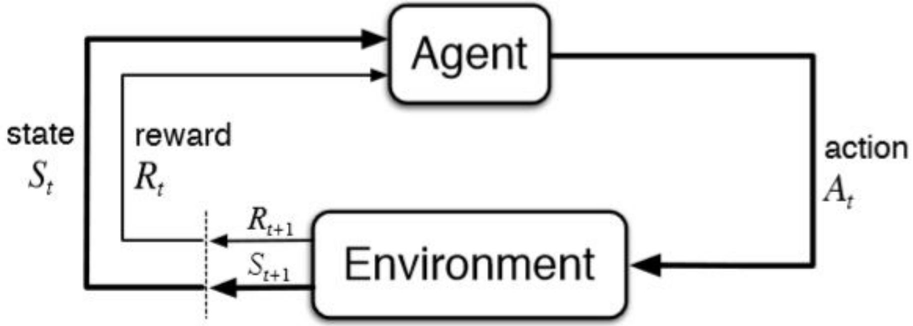
$$s_t = f(h_t)$$

and it is typically hidden or unknown



Markov Decision Process (MDP)

- Set of states S
- Set of actions A



Alternative notation

$s_{t+1} \sim p(.|s_t,a_t)$ or

$s' \sim p(.|s,a)$

Sequential Decision Making under Markov Assumption

- Markovian transition dynamics
- Full Observability
- The transition dynamics *T* is (generally) stochastic $p(s_{t+1}|s_t,a_t)$

Policy

A policy π:

- is a mapping from (all) states to actions;
- determines how agents select actions;
- can be deterministic ($a = \pi(s)$) or stochastic ($\pi(a|s)$ or $p(a|s)$ or $a \sim \pi(.|s)$)

Value Function/Q-Function

- We estimate the goodness of states and actions based on their value
- It’s also a measure to compare policies

$V^{\pi}(s_t) = \mathbb{E}_{\pi}[r_t+\gamma r_{t+1}+\gamma^2 r_{t+2}+\gamma^3 r_{t+3}+...|s_t] = \mathbb{E}[\sum^{\infty}_{h=0}\gamma^h r_h|s_0=s_t,a_h=\pi(s_h),s_{h+1}\sim p(.|s_h,a_h)]$

$Q^{\pi}(s_t,a_t) = \mathbb{E}[\sum^{\infty}_{h=0}\gamma^h r_h|(s_0,a_0)=(s_t,a_t),a_{h+1}=\pi(s_h),s_{h+1}\sim p(.|s_h,a_h)]$

For infinite horizon MDPs there always exists a deterministic policy π^* such that

$$V^{\pi^*}(s)\geq V^{\pi}(s)\ \forall\ s,\pi$$

meaning that π^* (optimal policy) dominates all other policies π in each state

Discount Factor

Setting $\gamma = 1$ for infinite tasks is a bad idea!

Note that $\sum^{\infty}_{h=0}\gamma^{h}$ is a geometric series and for γ in [0,1] this is equivalent to $1/(1-\gamma)$

So, the value of γ approximately determines how many steps ahead we are considering

E.g., $\gamma=0.99$ -> 99 timesteps ahead

Bellman Expectation Equation

The value of a certain state is expanded in terms of the current reward and the value of the next states according to the policy

r here is function of s and π(s)

$V^\pi(s_t) = \mathbb{E}_\pi[r_t+\gamma r_{t+1}+\gamma^2 r_{t+2}+\gamma^3 r_{t+3}+\dots|s_t] = r_t+\gamma\mathbb{E}_{s'\sim p(.|s,\pi(s))}[V^\pi(s')]$

$Q^\pi(s_t,a) = r_t+\gamma\mathbb{E}_{s'\sim p(.|s,a)}[V^\pi(s')]$

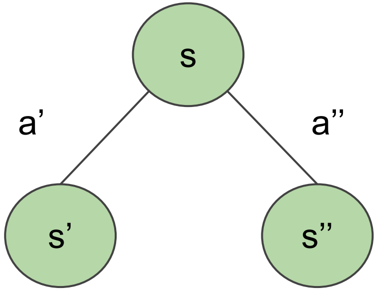
r here is function of s and a

As a result $V(s) = Q(s,\pi(s))$

Bellman Optimality Example

$$V^*(s)=\max_a[r(s,a)+\gamma\mathbb{E}_{s'\sim p(.|s,a)}V^*(s')]$$

- Try a', get r(s,a'), compute $Q^*(s,a')=r(s,a')+\gamma V^*(s')$
- Try a'', get r(s,a''), compute $Q^*(s,a'')=r(s,a'')+\gamma V^*(s'')$



Assume we know V^* at s' and s''

$$V^*(s)=\max_{a',a''}\{Q^*(s,a'),Q^*(s,a'')\}$$

Bellman Optimality (Theorem 1)

$$V^*(s)=\max_a[r(s,a)+\gamma\mathbb{E}_{s'\sim p(.|s,a)}V^*(s')]$$

given $\hat{\pi}=\text{argmax}_a Q^*(s,a)$, we can show $V^{\hat{\pi}}=V^*$

This implies $\pi^*=\text{argmax}_a Q^*(s,a)$ is an optimal policy

Bellman Optimality (Theorem 2)

For any V, if $V(s)=\max_a[r(s,a)+\gamma\mathbb{E}_{s'\sim p(.|s,a)}V(s')]$ for all s, then $V(s)=V^*(s)$

This means we can focus on one step at each time (leaving the remaining "problem" to V(s'), and any V that satisfies this formula is in fact V^*

End - Recap

Policy Evaluation

Question: given

- an MDP (S, A, T, R, γ)
- a policy π

how can we compute the goodness of π, i.e. V^{π}?

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how can we compute the goodness of π, i.e. V^{π}?

WHY IS THIS INTERESTING?

There are A^S possible policies, and we want to find the optimal one! To find it, we need to be able to evaluate it

Exact Policy Evaluation

Given (S, A, T, R, γ) and π, what is V^{π}?

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We know that **for ALL states**, Bellman expectation equation holds

$$V^\pi(s) = r+\gamma\mathbb{E}_{s'\sim p(.|s,\pi(s))}[V^\pi(s')]$$

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How many linear constraints (equations) do we have?

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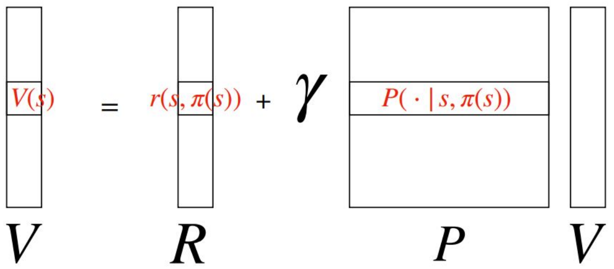
S!

Exact Policy Evaluation

We know that **for ALL states,** Bellman equation holds

$$V^{\pi}(s) = r+\gamma\mathbb{E}_{s'\sim p(.|s,\pi(s))}[V^{\pi}(s')]$$

We can combine all the constraints together:

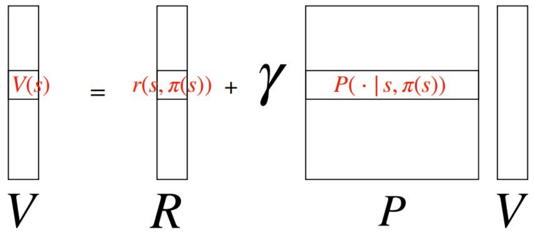


Credits: Wen Sun

Exact Policy Evaluation

Since we have this set of constraints

$$V = R + \gamma PV$$



we can solve for V as

$$V = (I - \gamma P)^{-1}R$$

Exact Policy Evaluation

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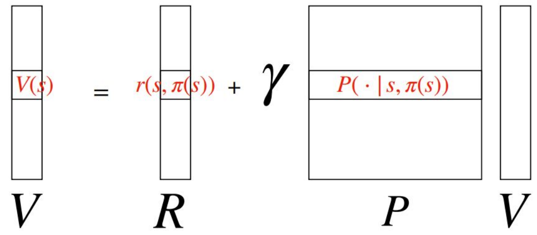

we can solve for V as

$$V = (I - \gamma P)^{-1} R$$

:(Nice but computationally expensive: inverting the matrix is $O(S^3)$

Fixed-Point Iteration

What is a fixed-point? A point where holds

$$x = f(x)$$

How can we find such points?

- Initialize x_0
- Repeat $x_{i+1} = f(x_i)$
- Stop at convergence where x is found and does not change anymore

Contractions

Convergence to a fixed-point is possible thanks to the existence of **contraction mappings**

f: M->M (M is a metric space) is a contraction mapping if:

$$|f(x) - f(x')| \leq k|x-x'| \text{ for } k \text{ in } [0, 1)$$

A contraction mapping has at most one fixed point

Contraction Operator

In the simplest case the contraction mapping can be an operator as simple as a matrix, e.g. O:

$$|OV - OV'| \leq \gamma|V - V'|$$

(we can replace k with γ as they have the same range)

Iterative Policy Evaluation

We assume rewards are in [0, 1]

- Initialize V_0 in $[0, 1/(1-\gamma)]$ (typically 0)
- Until convergence:

$$V_{i+1} = R + \gamma P V_i$$

(note: this is using matricial form because it's doing it for all states)

Iterative Policy Evaluation

- Initialize V_0 in [0, 1/(1-γ)] (typically 0)
- Until convergence:

$$T(V_i) = V_{i+1} = R + \gamma P V_i$$

(note: this is using matricial form because it's doing it for all states)

We are using the Bellman expectation backup - T(V)

Iterative Policy Evaluation Theorem

At the end we have, for all s in S

$$||T(U(s))-T(V(s))|| \leq \gamma||U(s)-V(s)||$$

Infinity norm: $||V|| = \max_s |V(s)|$

(the largest difference between state values)

$$V^{t+1}(s) = r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(s,\pi(s))} V^t(s')$$

Iterative Policy Evaluation Theorem

$$V^{\pi}(s) = r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(s,\pi(s))} V^{\pi}(s')$$

At the end we have, for all s in S

$$||T(V^t(s)) - T(V^{\pi}(s))|| = ||V^{t+1}(s) - V^{\pi}(s)|| \leq \gamma^{t+1} ||V^0 - V^{\pi}||$$

Hence the Bellman expectation backup is a contraction mapping and V^{π} is the fixed point

$$V^{t+1}(s) = r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(s, \pi(s))} V^t(s')$$

Iterative Policy Evaluation Theorem

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At the end we have, for all s in S

$$||T(V^t(s)) - T(V^{\pi}(s))|| = ||V^{t+1}(s) - V^{\pi}(s)|| \leq \gamma^{t+1} ||V^0 - V^{\pi}||$$

$$\forall s, \left| V^{t+1}(s) - V^{\pi}(s) \right|$$

We are not using inf-norm, but we are saying it holds for all states, largest difference included

$$= \left| r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(\cdot|s, \pi(s))} V^t(s') - \left(r(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim P(\cdot|s, \pi(s))} V^{\pi}(s') \right) \right|$$

$$= \gamma \left| \mathbb{E}_{s' \sim P(\cdot|s, \pi(s))} V^t(s') - \mathbb{E}_{s' \sim P(\cdot|s, \pi(s))} V^{\pi}(s') \right|$$

Apply Jensen's inequality (and every norm and expectation is convex)

$$\leq \gamma \mathbb{E}_{s' \sim P(\cdot|s, \pi(s))} \left| V^t(s') - V^{\pi}(s') \right|$$

Average is always smaller than max(E|f(x)| <= max_x|f(x)|)

$$\leq \gamma \left\| V^t - V^{\pi} \right\|_{\infty}$$

Iterative Policy Evaluation Theorem

At the end we have, for all s in S

$$||V^{t+1}(s)-V^{\pi}(s)|| \leq \gamma^{t+1}||V^{0}-V^{\pi}||$$

$$\| V^{t+1} - V^{\pi} \|_{\infty} \leq \gamma \| V^{t} - V^{\pi} \|_{\infty} \leq \gamma^{t+1} \| V^{0} - V^{\pi} \|_{\infty}$$

Iterative Policy Evaluation: Iterations

For iterative PE to find an ϵ accurate value function, we need a number of iterations n, with computational cost $O(S^2 \ln(1/\epsilon))$:

$$\gamma^n \|V^0 - V^\pi\| \leq \epsilon$$

$$\ln\left(\frac{\| V^0 - V^\star \|_\infty}{\epsilon}\right) \Big/ \ln(1/\gamma)$$

How to Find the Optimal Policy?

Now, what we're really interested in is finding the optimal policy π^*

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Naive approach: we know how to do policy evaluation, then

- For each possible policy, for all states
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How to Find the Optimal Policy?

Now, what we're really interested in is finding the optimal policy π^*

Let's use Bellman optimality equation!

$$V^*(s)=\max_a[r(s,a)+\gamma\mathbb{E}_{s'\sim p(.|s,a)}V^*(s')]$$

Bellman Optimality Backup is a Contraction

- Infinity norm: $||V|| = \max_s |V(s)|$
- Set $\gamma < 1$
- Define the (non-linear) BV operator as a Bellman optimality equation applied to V:

$$BV = \max_a (r(s,a) + \gamma \mathbb{E}_{s' \sim p(.|s,a)}[V(s')])$$

Alternative notation TV

Bellman Backup is a Contraction

$$
\begin{aligned}
\|BV_k - BV_j\| &= \left\| \max_a \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right) - \max_{a'} \left(R(s,a') + \gamma \sum_{s' \in S} P(s'|s,a') V_j(s') \right) \right\| \\
&\leq \max_a \left\| \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') - R(s,a) - \gamma \sum_{s' \in S} P(s'|s,a) V_j(s') \right) \right\| \\
&= \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a)(V_k(s') - V_j(s')) \right\| \\
&\leq \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a) \|V_k - V_j\|) \right\| \\
&= \max_a \left\| \gamma \|V_k - V_j\| \sum_{s' \in S} P(s'|s,a)) \right\| \\
&= \gamma \|V_k - V_j\|
\end{aligned}
$$

Bellman Backup is a Contraction

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&\leq \max_a \left\| \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') - R(s,a) - \gamma \sum_{s' \in S} P(s'|s,a) V_j(s') \right) \right\| \\
&= \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a)(V_k(s') - V_j(s')) \right\| \\
&\leq \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a) \|V_k - V_j\|) \right\| \\
&= \max_a \left\| \gamma \|V_k - V_j\| \sum_{s' \in S} P(s'|s,a)) \right\| \\
&= \gamma \|V_k - V_j\|
\end{aligned}
$$

If you apply B to two different value functions, distance between value functions shrinks after applying Bellman optimality equation to each

Bellman Backup is a Contraction

$$
\begin{aligned}
\|BV_k - BV_j\| &= \left\| \max_a \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') \right) - \max_{a'} \left(R(s,a') + \gamma \sum_{s' \in S} P(s'|s,a') V_j(s') \right) \right\| \\
&\leq \max_a \left\| \left(R(s,a) + \gamma \sum_{s' \in S} P(s'|s,a) V_k(s') - R(s,a) - \gamma \sum_{s' \in S} P(s'|s,a) V_j(s') \right) \right\| \\
&= \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a)(V_k(s') - V_j(s')) \right\| \\
&\leq \max_a \left\| \gamma \sum_{s' \in S} P(s'|s,a) \|V_k - V_j\|) \right\| \\
&= \max_a \left\| \gamma \|V_k - V_j\| \sum_{s' \in S} P(s'|s,a)) \right\| \\
&= \gamma \|V_k - V_j\|
\end{aligned}
$$

This operator is a γ-contraction, i.e. it makes value functions closer by at least γ,

Bellman Operator for Q

$$TQ(s,a) = r(s,a)+\gamma\mathbb{E}_{s'\sim p(.|s,a)}\max_{a'}[Q(s',a')])$$

Since Q: S x A -> $\mathbb{R}$, then also TQ: S x A -> $\mathbb{R}$

Value Iteration

All of this also holds for V^*

We can obtain $Q^* = TQ^*$, since Q^* is a fixed-point solution to $Q = TQ$

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We can obtain $Q^* = TQ^*$, since Q^* is a fixed-point solution to $Q = TQ$

- Initialize $||Q_0||$ in $[0, 1/(1-\gamma)]$ (typically 0)
- Until convergence, for all states and actions:

$$Q_{i+1} = TQ_i$$

We know the Bellman operator is a contraction!

Value Iteration

All of this also holds for V^*

We can obtain $Q^* = TQ^*$, since Q^* is a fixed-point solution to $Q = TQ$

- Initialize $||Q_0||$ in $[0, 1/(1-\gamma)]$ (typically 0)
- Until convergence, for all states and actions:

$$Q_{i+1} = TQ_i$$

$$||Q_{i+1} - Q^*|| = ||TQ_i - TQ^*|| \leq \gamma||Q_i - Q^*|| \leq \gamma^{i+1}||Q_0 - Q^*||$$

Policy from Value Iteration

We know that $\pi^*(s) = \text{argmax}_a Q^*(s,a)$, and since $Q_i(s,a) \cong Q^*(s,a)$ we could choose

$$\pi_i(s) = \text{argmax}_a Q_i(s,a)$$

Policy from Value Iteration

$$\pi_i(s) = \text{argmax}_a Q_i(s,a)$$

What is the quality of such policy? For all states

$$V^{\pi_i}(s) \geq V^*(s) - 2\gamma^i/(1-\gamma)\|Q_0 - Q^*\|$$

Policy from Value Iteration

$\pi_i(s) = \text{argmax}_a Q_i(s,a)$

What is the quality of such policy? For all states

$V^{\pi_i}(s) \geq V^\star(s) - 2\gamma^i/(1-\gamma)||Q_0 - Q^\star||$

$$
\begin{aligned}
V^{\pi^t}(s) - V^\star(s) &= Q^{\pi^t}(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&= Q^{\pi^t}(s, \pi^t(s)) - Q^\star(s, \pi^t(s)) + Q^\star(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&= \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))} \left(V^{\pi^t}(s') - V^\star(s') \right) + Q^\star(s, \pi^t(s)) - Q^\star(s, \pi^\star(s)) \\
&\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))} \left(V^{\pi^t}(s') - V^\star(s') \right) + Q^\star(s, \pi^t(s)) - Q^t(s, \pi^t(s)) + Q^t(s, \pi^\star(s)) - Q^\star(s, \pi^\star(s)) \\
&\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))} \left(V^{\pi^t}(s') - V^\star(s') \right) - 2\gamma^t \|Q^0 - Q^\star\|_\infty
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Policy from Value Iteration

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\end{aligned}
$$

Add and subtract

Policy from Value Iteration

$\pi_i(s) = \text{argmax}_a Q_i(s,a)$

What is the quality of such policy? For all states

$V^{\pi_i}(s) \geq V^\star(s) - 2\gamma^i/(1-\gamma)\|Q_0 - Q^\star\|$

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&= \gamma \mathbb{E}_{s'\sim P(s,\pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s,\pi^t(s)) - Q^\star(s,\pi^\star(s)) \\
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\end{aligned}
$$

Expand and get rid of r

Policy from Value Iteration

$$\pi_i(s) = \text{argmax}_a Q_i(s,a)$$

What is the quality of such policy? For all states

$$V^{\pi_i}(s) \geq V^*(s) - 2\gamma^i/(1-\gamma)||Q_0 - Q^*||$$

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&\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))} \left(V^{\pi^t}(s') - V^\star(s') \right) - 2\gamma^t \|Q^0 - Q^\star\|_\infty
\end{aligned}
$$

...and you're left with this

Policy from Value Iteration

$\pi_i(s) = \text{argmax}_a Q_i(s,a)$

What is the quality of such policy? For all states

$V^{\pi_i}(s) \geq V^*(s) - 2\gamma^i/(1-\gamma)||Q_0-Q^*||$

$$
\begin{aligned}
V^{\pi^t}(s) - V^\star(s) &= Q^{\pi^t}(s,\pi^t(s)) - Q^\star(s,\pi^\star(s)) \\
&= Q^{\pi^t}(s,\pi^t(s)) - Q^\star(s,\pi^t(s)) + Q^\star(s,\pi^t(s)) - Q^\star(s,\pi^\star(s)) \\
&= \gamma \mathbb{E}_{s'\sim P(s,\pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) + Q^\star(s,\pi^t(s)) - Q^\star(s,\pi^\star(s)) \\
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&\geq \gamma \mathbb{E}_{s'\sim P(s,\pi^t(s))}\left(V^{\pi^t}(s') - V^\star(s')\right) - 2\gamma^t \|Q^0 - Q^\star\|_\infty
\end{aligned}
$$

negative, because $\pi_i(s) = \text{argmax}_a Q_i(s,a)$

Policy from Value Iteration

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\end{aligned}
$$

by definition of Q^*

Policy from Value Iteration

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\end{aligned}
$$

same as before

SAPIENZA
UNIVERSITÀ DI ROMA

Policy from Value Iteration

$$\pi_i(s) = \text{argmax}_a Q_i(s,a)$$

What is the quality of such policy? For all states

$$V^{\pi^i}(s) \geq V^*(s) - 2\gamma^i/(1-\gamma)||Q_0-Q^*||$$

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\end{aligned}$$

just exploit this

$$||Q_i - Q^*|| \leq \gamma^i||Q_0 - Q^*||$$

Policy from Value Iteration

$$\pi_i(s) = \text{argmax}_a Q_i(s,a)$$

What is the quality of such policy? For all states

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\end{aligned}
$$

and again

$$||Q_i - Q^*|| \leq \gamma^i ||Q_0 - Q^*||$$

Policy from Value Iteration

$$\pi_i(s) = \text{argmax}_a Q_i(s,a)$$

What is the quality of such policy? For all states

$$V^{\pi_i}(s) \geq V^*(s) - 2\gamma^t/(1-\gamma)||Q_0 - Q^*||$$

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&\geq \gamma \mathbb{E}_{s' \sim P(s, \pi^t(s))} \left(V^{\pi^t}(s') - V^\star(s') \right) - 2\gamma^t \|Q^0 - Q^\star\|_\infty
\end{aligned}$$

repeat and get $1/(1-\gamma)$

Policy from Value Iteration

If we want an ϵ error on the quality of the policy, to determine the number of iterations i we can just solve for it in this equation

$$2\gamma^i/(1-\gamma)||Q_0-Q^*|| \leq \epsilon$$