Robotics II

June 15, 2010

For the planar RP robot under gravity shown in Fig. 1, consider a class of one-dimensional tasks defined only in terms of the y-component of the end-effector Cartesian position

$$y = p_y(q_1, q_2).$$

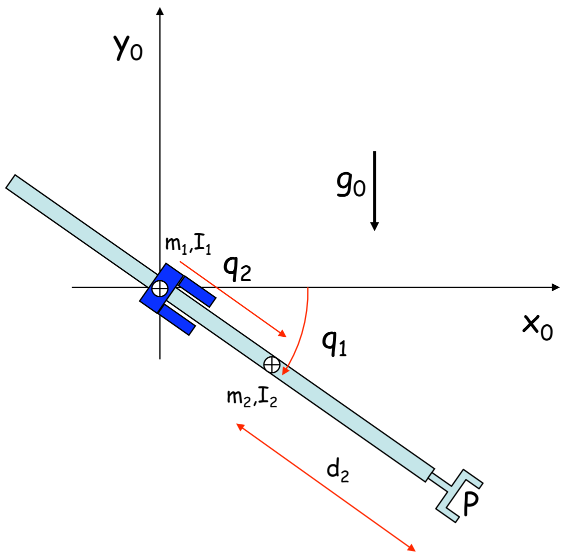

Figure 1: RP robot in the vertical plane, with definition of coordinates ($d_2 > 0$ is a constant)

Noting that the robot is redundant for this class of tasks, determine the explicit expression of the actuation input $\boldsymbol{\tau} = (\tau_1, \tau_2)$ that, at a generic robot state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, realizes a desired $\ddot{y}_d = A$ and has the *minimum norm* property.

[90 minutes; open books]

Solution

June 15, 2010

The dynamic model of the RP robot

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{1}$$

should be obtained first.

With reference to Fig. 1, the robot kinetic energy T is given by

$$p_{c2} = \begin{pmatrix} q_2 c_1 \\ q_2 s_1 \\ 0 \end{pmatrix} \Rightarrow \|\dot{p}_{c2}\| = \dot{q}_2^2 + q_2^2 \dot{q}_1^2$$

$$\begin{aligned}
T_1 &= \frac{1}{2} I_1 \dot{q}_1^2 \\
T_2 &= \frac{1}{2} m_2 \|v_{c2}\|^2 + \frac{1}{2} I_2 \dot{q}_1^2 = \frac{1}{2}(I_2 + m_2 q_2^2)\, \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 \\
T &= T_1 + T_2 = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q}) \dot{\boldsymbol{q}} \quad \Rightarrow \quad \boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} I_1 + I_2 + m_2 q_2^2 & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} b_{11}(q_2) & 0 \\ 0 & b_{22} \end{pmatrix}.
\end{aligned}$$

Using the Christoffel's symbols for the components of the velocity vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$

$$\left(\frac{\partial b_1}{\partial q_1} \; \frac{\partial b_1}{\partial q_2} \right)$$

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}} \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right) + \left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \left(\frac{\partial \boldsymbol{B}(\boldsymbol{q})}{\partial q_i} \right) \right) \qquad i = 1, 2,$$

the Coriolis and centrifugal terms are determined as follows:

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \begin{pmatrix} 0 & m_2 q_2 \\ m_2 q_2 & 0 \end{pmatrix} \quad \Rightarrow \quad c_1(q_2, \dot{q}_1, \dot{q}_2) = 2\, m_2 q_2 \dot{q}_1 \dot{q}_2 \\
\boldsymbol{C}_2(\boldsymbol{q}) &= \begin{pmatrix} -2\, m_2 q_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad c_2(q_1, \dot{q}_1) = -m_2 q_2 \dot{q}_1^2.
\end{aligned}$$

The robot potential energy U is given by

$$\begin{aligned}
U_1 &= U_{10} \qquad U_2 = m_2 g_0\, q_2 \sin q_1 + U_{20} \\
U &= U_1 + U_2 = m_2 g_0\, q_2 \sin q_1 + U_{10} + U_{20} \\
&\Rightarrow \quad \boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} m_2 g_0\, q_2 \cos q_1 \\ m_2 g_0 \sin q_1 \end{pmatrix} = \begin{pmatrix} g_1(q_1, q_2) \\ g_2(q_1) \end{pmatrix},
\end{aligned}$$

with $g_0 = 9.81 > 0$.

The direct kinematics associated to the end-effector position of the RP robot is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} (d_2 + q_2) \cos q_1 \\ (d_2 + q_2) \sin q_1 \end{pmatrix},$$

where $d_2 > 0$ is the constant length shown in Fig. 1. Being the task defined only in terms of the p_y component, it is

$$\dot{p}_y = \begin{pmatrix} (d_2 + q_2) \cos q_1 & \sin q_1 \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

and then

$$\ddot{p}_y = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \begin{pmatrix} \cos q_1 \dot{q}_2 - (d_2 + q_2) \sin q_1 \dot{q}_1 & \cos q_1 \dot{q}_1 \end{pmatrix} \dot{\boldsymbol{q}}. \tag{2}$$

Note that the task Jacobian $\boldsymbol{J}$ is singular if and only if $\sin q_1 = 0$ *and* $q_2 = -d_2$.

Replacing in (2) the accelerations $\ddot{\boldsymbol{q}}$ from (1) yields

$$\ddot{p}_y = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{B}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right) + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Setting then $\ddot{p}_y = A$ and reorganizing terms, we obtain

$$\boldsymbol{M}(\boldsymbol{q})\boldsymbol{\tau} = A - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{B}^{-1}(\boldsymbol{q})\left(\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})\right) =: d(\boldsymbol{q}, \dot{\boldsymbol{q}}),$$

having defined also

$$\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{B}^{-1}(\boldsymbol{q}) = \begin{pmatrix} \dfrac{(d_2 + q_2)\cos q_1}{b_{11}(q_2)} & \dfrac{\sin q_1}{b_{22}} \end{pmatrix}.$$

At a generic robot state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, the question at hand is then formulated as a linear-quadratic optimization problem in the standard form

$$\min \frac{1}{2}\|\boldsymbol{\tau}\|^2 = \frac{1}{2}\left(\tau_1^2 + \tau_2^2\right) \quad \text{s.t.} \quad \boldsymbol{M}\boldsymbol{\tau} = d.$$

The optimal solution is simply

$$\boldsymbol{\tau}^* = \boldsymbol{M}^{\#} d, \tag{3}$$

where all quantities have been already defined. In explicit terms, in case of full (row) rank $\boldsymbol{M}$ we have[1]

$$\boldsymbol{M}^{\#} = \boldsymbol{B}^{-1}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{B}^{-2}\boldsymbol{J}^T\right)^{-1}.$$

In particular, out of the singularities of the 1×2 matrix $\boldsymbol{M}$, which coincide with those of the task Jacobian $\boldsymbol{J}$, the pseudoinverse of $\boldsymbol{M}$ has the explicit expression

$$\boldsymbol{M}^{\#}(\boldsymbol{q}) = \frac{1}{\left(\dfrac{(d_2 + q_2)\cos q_1}{b_{11}(q_2)}\right)^2 + \left(\dfrac{\sin q_1}{b_{22}}\right)^2} \begin{pmatrix} \dfrac{(d_2 + q_2)\cos q_1}{b_{11}(q_2)} \\[2ex] \dfrac{\sin q_1}{b_{22}} \end{pmatrix}.$$

The optimal solution (3) implies that both joints/actuators are typically involved in this one-dimensional task. Although in general the task could have been realized also by actuating only a single joint (the revolute or the prismatic one), the combination results in the minimum actuation effort.

It should be remarked that the norm of $\boldsymbol{\tau}$ has a dimensionality problem. In fact, the first actuation input is a torque (on the revolute joint) and the second is a force (on the prismatic joint), so that physical units are mixed in computing the norm. A way to handle this problem is to introduce a proper scaling in the objective function, i.e., considering a positive definite diagonal matrix $\boldsymbol{W} = \text{diag}\{1, w\} > 0$ and minimizing

$$\frac{1}{2}\boldsymbol{\tau}^T\boldsymbol{W}\boldsymbol{\tau} = \frac{1}{2}\left(\tau_1^2 + w\,\tau_2^2\right),$$

[1] Note also that in general $\boldsymbol{M}^{\#} = (\boldsymbol{J}\boldsymbol{B}^{-1})^{\#} \neq \boldsymbol{B}\boldsymbol{J}^{\#}$. The equality holds if $\boldsymbol{B} = b \cdot \boldsymbol{I}$, for a scalar b.

where the scalar $w > 0$ takes into account how costly a unit of torque is in comparison to a unit of force. The associated solution is then obtained by replacing the pseudoinverse of $\boldsymbol{M}$ in (3) by its weighted pseudoinverse

$$\boldsymbol{M}_{\boldsymbol{W}}^{\#} = \boldsymbol{W}^{-1}\boldsymbol{M}^T \left(\boldsymbol{M}\boldsymbol{W}^{-1}\boldsymbol{M}^T\right)^{-1}.$$

Finally, it is worth mentioning that the above local solution with minimum norm of the actuation inputs is prone to an internal build up of joint velocities, especially for long task trajectories. A countermeasure to this phenomenon is to choose a solution of the form

$$\boldsymbol{\tau} = \boldsymbol{M}^{\#}d + \left(\boldsymbol{I} - \boldsymbol{M}^{\#}\boldsymbol{M}\right)\boldsymbol{\tau}_0, \tag{4}$$

with $\boldsymbol{\tau}_0 = -\boldsymbol{K}_D\dot{\boldsymbol{q}}$ and where $\boldsymbol{K}_D$ is a diagonal, positive definite matrix. The additional torque $\boldsymbol{\tau}_0$ damps the joint velocity $\dot{\boldsymbol{q}}$, without affecting the execution of the task. It is also easy to see that (4) is the solution to the following modified linear-quadratic optimization problem

$$\min \frac{1}{2}\left(\boldsymbol{\tau} - \boldsymbol{\tau}_0\right)^T\left(\boldsymbol{\tau} - \boldsymbol{\tau}_0\right) \quad \text{s.t.} \quad \boldsymbol{M}\boldsymbol{\tau} = d.$$

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Robotics II

July 7, 2010

Consider the one-dimensional mass/spring/damper scheme of a robot in contact with a compliant workpiece and with a force sensing cell in between, as shown in Fig. 1. The robot and the workpiece are represented by the mass $m_r > 0$, with position x_r, and by the mass $m_w > 0$, with position x_w, respectively. The rest positions of the springs modeling the stiffness $k_s > 0$ of the force sensor and the stiffness $k_w > 0$ of the workpiece are $x_r = x_w = 0$. Viscous damping is included between the robot mass m_r and the ground, between the workpiece mass m_w and the ground, and between the two masses, with positive coefficients b_r, b_w, and b_s, respectively. An input force F is applied on the robot mass.

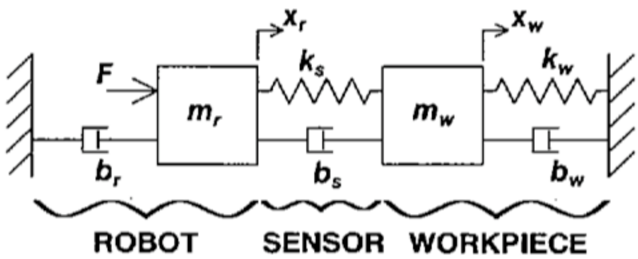

Figure 1: Robot contact model, with two masses, springs and dampers

1. Determine the differential equations of motion for this system.
2. For a contact force control problem, let the input F be specified by a proportional control law of the form $F = k_f(F_d - F_c)$, with gain k_f and desired contact force F_d, and where $F_c = k_s(x_r - x_w)$ is the force measured across the sensor. Determine the unique closed-loop equilibrium position $x_{r,e}$ and $x_{w,e}$ of the two masses m_r and m_w and show that a contact force error with respect to the desired value F_d is present at the equilibrium.
3. Which control actions should be considered in order to reduce or eliminate the presence of a steady-state contact force error?
4. *Optional.* Prove that the closed-loop equilibrium state $x_r = x_{r,e}$, $x_w = x_{w,e}$, $\dot{x}_r = \dot{x}_w = 0$ is exponentially stable for any *positive* value of the gain k_f *(Hint: Use a root locus analysis and/or the Routh criterion in the Laplace domain).*

[120 minutes (150 minutes including the optional item); open books]

Solution

July 7, 2010

The differential equations of motion for the system are:

$$\begin{aligned} m_r\ddot{x}_r + b_r\dot{x}_r + b_s(\dot{x}_r - \dot{x}_w) + k_s(x_r - x_w) &= F \\ m_w\ddot{x}_w + b_w\dot{x}_w + b_s(\dot{x}_w - \dot{x}_r) + k_s(x_w - x_r) + k_w x_w &= 0. \end{aligned} \tag{1}$$

Setting $F = k_f(F_d - F_c) = k_f(F_d - k_s(x_r - x_w))$ and evaluating the system at an equilibrium (i.e., setting $\dot{x}_r = \dot{x}_w = \ddot{x}_r = \ddot{x}_w = 0$), gives

$$\begin{aligned} k_s(x_r - x_w) &= k_f(F_d - k_s(x_r - x_w)) \\ k_s(x_w - x_r) + k_w x_w &= 0. \end{aligned}$$

It is easy to see that the unique solution is

$$x_{r.e} = \frac{k_s + k_w}{k_s k_w}\,\frac{k_f}{1 + k_f}\,F_d \qquad x_{w.e} = \frac{1}{k_w}\,\frac{k_f}{1 + k_f}\,F_d,$$

that provides also the contact force at the equilibrium

$$F_{c,e} = k_s(x_{r,e} - x_{w,e}) = \frac{k_f}{1 + k_f}\,F_d.$$

Therefore, there will be an error on the desired force given by

$$e_F = F_d - F_{c,e} = \frac{1}{1 + k_f}\,F_d.$$

This error can be reduced by amplifying the gain k_f, but cannot be eliminated when using a simple proportional controller. This situation typically asks for the introduction of an integral action of the type

$$F = k_f(F_d - F_c) + k_i\int_0^t (F_d - F_c)d\tau = k_f(F_d - k_s(x_r - x_w)) + k_i\int_0^t (F_d - k_s(x_r - x_w))d\tau.$$

At steady-state ($t \to \infty$), the argument of the integral term should vanish. Therefore,

$$\lim_{t\to\infty}\int_0^t (F_d - k_s(x_r - x_w))d\tau = \mathit{constant} \quad \Rightarrow \quad \lim_{t\to\infty} k_s(x_r - x_w) = F_d.$$

The modified equilibrium (denoted with a prime) will be

$$x'_{r.e} = \frac{k_s + k_w}{k_s k_w}\,F_d \qquad x'_{w.e} = \frac{1}{k_w}\,F_d,$$

and the associated contact force will be desired one,

$$F'_{c,e} = k_s(x'_{r,e} - x'_{w,e}) = F_d,$$

as expected. The same result is obtained if we combine the feedback control action with a constant feedforward term F_d, i.e.

$$F = F_d + k_f(F_d - F_c).$$

The equilibrium configuration should satisfy then

$$\begin{aligned} k_s(x_r - x_w) &= F_d + k_f(F_d - k_s(x_r - x_w)) \\ k_s(x_w - x_r) + k_w x_w &= 0, \end{aligned}$$

leading again to $F'_{c,e} = F_d$ and the same previous positions $x'_{r.e}$ and $x'_{w,e}$.

Indeed, no matter if we keep the proportional control law or add also the integral action or the feedforward term, we need to show that the associated equilibrium is asymptotically stable for the closed-loop system (otherwise it would never be reached from a generic initial state). Since the system is linear, asymptotic stability is equivalent to exponential stability. Moreover, whenever it holds, this result is global.

Considering again the case of a proportional force controller, the easiest way to prove asymptotic stability of the closed-loop state $x_r = x_{r,e}$, $x_w = x_{w,e}$, $\dot{x}_r = \dot{x}_w = 0$ is to use Laplace transform and the root locus method, thanks to the linearity of the system. From eq. (1) we have

$$\begin{aligned} \left(m_r s^2 + (b_r + b_s)s + k_s\right) X_r(s) &= F(s) + (b_s s + k_s)X_w(s) \\ \left(m_w s^2 + (b_w + b_s)s + (k_w + k_s)\right) X_w(s) &= (b_s s + k_s)X_r(s), \end{aligned}$$

where $X_r(s)$, $X_w(s)$, and $F(s)$ are the Laplace transforms of $x_r(t)$, $x_w(t)$, and $F(t)$. Defining for compactness

$$N_s(s) = b_s s + k_s \quad D_r(s) = \left(m_r s^2 + (b_r + b_s)s + k_s\right) \quad D_w(s) = \left(m_w s^2 + (b_w + b_s)s + (k_w + k_s)\right),$$

we can solve for

$$\frac{X_r(s)}{F(s)} = \frac{D_w(s)}{D_r(s)D_w(s) - N_s^2(s)} \qquad \frac{X_w(s)}{F(s)} = \frac{N_s(s)}{D_r(s)D_w(s) - N_s^2(s)} \qquad \frac{X_w(s)}{X_r(s)} = \frac{N_s(s)}{D_w(s)}.$$

Being the output defined as the contact force $F_c(s) = k_s(X_r(s) - X_w(s))$, the transfer function of the (open-loop) system is

$$P(s) = \frac{F_c(s)}{F(s)} = k_s \, \frac{D_w(s) - N_s(s)}{D_r(s)D_w(s) - N_s^2(s)}.$$

Closing the feedback loop with $F(s) = k_f(F_d(s) - F_c(s))$, where $F_d(s)$ is the Laplace transform of the step reference input $F_d(t) = F_d \cdot \delta_{-1}(t)$, we obtain the transfer function

$$W(s) = \frac{F_c(s)}{F_d(s)} = \frac{k_f P(s)}{1 + k_f P(s)} = \frac{k_f k_s (D_w(s) - N_s(s))}{(D_r(s)D_w(s) - N_s^2(s)) + k_f\,(k_s(D_w(s) - N_s(s)))}.$$

Asymptotic stability of the closed-loop system depends on the location on the complex plane s of the poles of $W(s)$, i.e., of the roots of the polynomial equation

$$\left(D_r(s)D_w(s) - N_s^2(s)\right) + k_f\,(k_s(D_w(s) - N_s(s))) = A(s) + k_f B(s) = 0. \tag{2}$$

By varying k_f in (2), we can explore the root locus. For $k_f = 0$, the (four) poles coincide with the poles of the open-loop system $P(s)$, i.e., with the roots of

$$\begin{aligned} A(s) &= D_r(s)D_w(s) - N_s^2(s) \\ &= m_r m_w s^4 + ((b_w + b_s)m_r + (b_r + b_s)m_w)\, s^3 + ((k_s + k_w)m_r + k_s m_w + b_r b_s + b_s b_w + b_r b_w)\, s^2 \\ &\quad + ((b_r + b_w)k_s + (b_r + b_s)k_w)\, s + k_s k_w = 0. \end{aligned}$$

Applying the Routh criterion to the fourth-order polynomial $A(s)$ and using the positivity of all physical coefficients in (1), it is straightforward to check that the four poles of $P(s)$ have all negative real parts (the process itself is asymptotically stable). Moreover, by the standard rules of the root locus, when k_f is increased toward $+\infty$, two of the closed-loop poles converge to the open-loop zeros, i.e., to the roots of

$$B(s) = k_s \left(D_w(s) - N_s(s)\right) = k_s \left(m_w s^2 + b_w s + k_w\right) = 0,$$

which are in the left-hand side of the complex plane. The two other poles will approach the asymptotes of the positive root locus, which are vertical (since the pole-zero excess of $P(s)$ is $n - m = 2$) and are located in the left-hand side of the complex plane s. As a result, the closed-loop system is asymptotically stable for any value of $k_f \geq 0$.

Figure 2 shows the complete positive root locus obtained with Matlab, for some arbitrary (positive) values of the model parameters, confirming the above analysis.

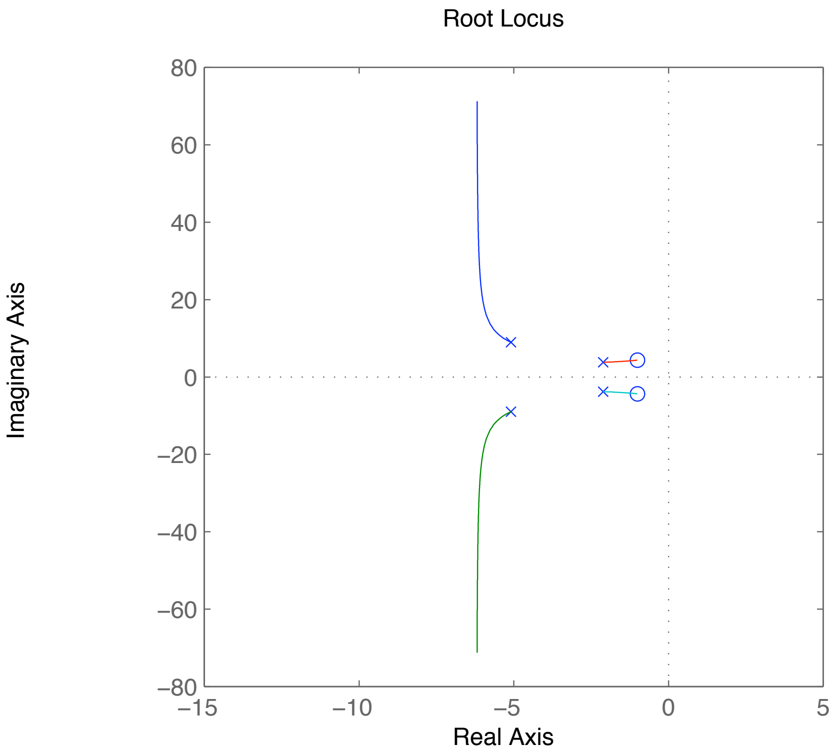

Figure 2: A typical root locus of eq. (2), when varying $k_f \in [0, +\infty)$

Indeed, the same asymptotic stability result could have been proven by applying the Routh criterion to check the locations of roots of the whole polynomial on the left-hand side of (2) (but this would have been more tedious).

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Robotics II

September 15, 2010

Consider the planar robot with three degrees of freedom (RPR) shown in Fig. 1.

$$p_a = \begin{pmatrix} q_2 c_1 \\ q_2 s_1 \\ 0 \end{pmatrix} \quad v_a = \begin{pmatrix} \dot q_2 c_1 - s_1 q_2 \dot q_1 \\ \dot q_2 s_1 + c_1 q_2 \dot q_1 \\ 0 \end{pmatrix}$$

$$\|v_a\|^2 = \dot q_2^2 + q_2^2 \dot q_1^2$$

$$T_1 = \frac{1}{2}(I_1 + m_1 d_{c1}^2)\dot q_1^2$$

$$T_2 = \frac{1}{2} I_2 \dot q_1^2 + \frac{1}{2} m_2 (\dot q_2^2 + q_2^2 \dot q_1^2)$$

$$T_3 = \frac{1}{2} I_3 \dot q_3 + \frac{1}{2} m_3 \|v_{c3}\|^2$$

$$p_{c3} = \begin{pmatrix} (q_2 + d_{c2}) c_1 + d_{c3} c_{12} \\ (q_2 + d_{c2}) s_1 + d_{c3} s_{12} \end{pmatrix}$$

$$\dot p_{c3} = \begin{pmatrix} \dot q_2 c_1 - (q_2 + d_{c2}) s_1 \dot q_1 - d_{c3} s_{12} (\dot q_1 + \dot q_3) \\ \dot q_2 s_1 + (q_2 + d_{c2}) c_1 \dot q_1 + d_{c3} c_{12} (\dot q_1 + \dot q_3) \end{pmatrix}$$

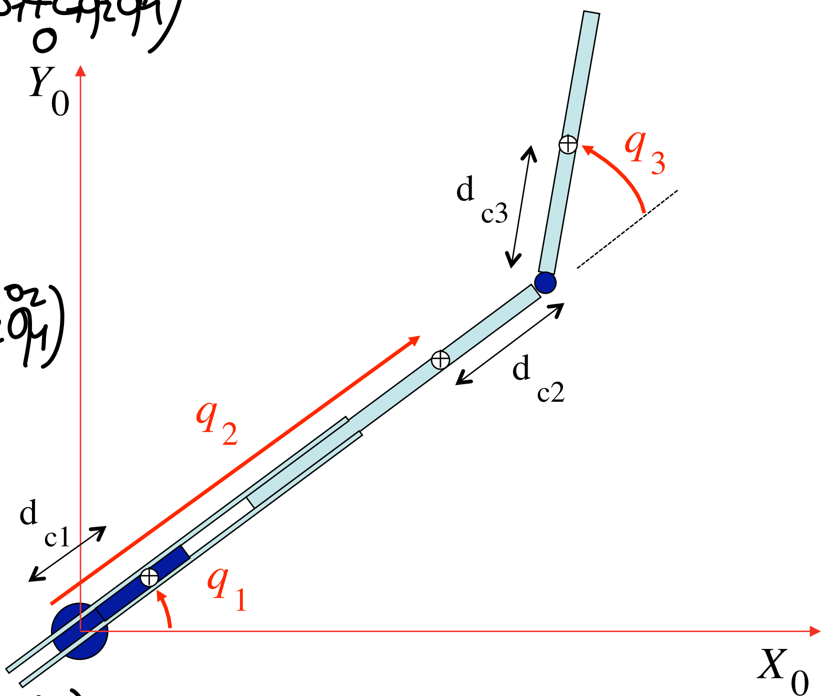

Figure 1: A planar RPR robot

1. Determine the symbolic expression of the robot inertia matrix $\boldsymbol{B}(\boldsymbol{q})$. Explicit all assumptions that are made.

2. Find a set of dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^p$, with a possibly minimal p, that provides a linear parameterization of the inertial term in the dynamic model, i.e., $\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{Y}(\boldsymbol{q}, \ddot{\boldsymbol{q}})\boldsymbol{a}$.

[90 minutes; open books]

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Solution

September 15, 2010

We compute the robot kinetic energy taking into account that the motion is planar: linear velocities are vectors in the plane (x, y), angular velocities are scalars (in the z-direction). With standard notations, for the first link it is:

$$T_1 = \frac{1}{2}\boxed{\left(I_1 + m_1 d_{c1}^2\right)}\dot{q}_1^2.$$

PARALLEL AXIS THEOREM

For the second link, from

$$\boldsymbol{p}_{c2} = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c2} = \begin{pmatrix} \dot{q}_2 \cos q_1 - q_2 \sin q_1 \dot{q}_1 \\ \dot{q}_2 \sin q_1 + q_2 \cos q_1 \dot{q}_1 \end{pmatrix},$$

SINCE ω IS SCALAR (ONLY Z COMPONENT) ⟹ $\omega^T I \omega = I\omega^2$

we have:

$$T_2 = \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}m_2\boldsymbol{v}_{c2}^T\boldsymbol{v}_{c2} = \frac{1}{2}\left(I_2\dot{q}_1^2 + m_2(q_2^2\dot{q}_1^2 + \dot{q}_2^2)\right).$$

I THINK IT'S ω_1

For the third link, from

$$\boldsymbol{p}_{c3} = \begin{pmatrix} (q_2 + d_{c2})\cos q_1 + d_{c3}\cos(q_1 + q_3) \\ (q_2 + d_{c2})\sin q_1 + d_{c3}\sin(q_1 + q_3) \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} \dot{q}_2 \cos q_1 - (q_2 + d_{c2})\sin q_1 \dot{q}_1 - d_{c3}\sin(q_1 + q_3)(\dot{q}_1 + \dot{q}_3) \\ \dot{q}_2 \sin q_1 + (q_2 + d_{c2})\cos q_1 \dot{q}_1 + d_{c3}\cos(q_1 + q_3)(\dot{q}_1 + \dot{q}_3) \end{pmatrix},$$

we have:

$$\begin{aligned} T_3 &= \frac{1}{2}I_3\omega_3^2 + \frac{1}{2}m_3\boldsymbol{v}_{c3}^T\boldsymbol{v}_{c3} = \frac{1}{2}I_3(\dot{q}_1 + \dot{q}_3)^2 \\ &+ \frac{1}{2}m_3\left((q_2 + d_{c2})^2\dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2(\dot{q}_1 + \dot{q}_3)^2 + 2d_{c3}((q_2 + d_{c2})\cos q_3\dot{q}_1 - \sin q_3\dot{q}_2)(\dot{q}_1 + \dot{q}_3)\right). \end{aligned}$$

From

$$T = \sum_{i=1}^{3} T_i = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \text{with} \quad \boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix},$$

we obtain for the single elements of the symmetric inertia matrix:

$$\begin{aligned} b_{11} &= I_1 + m_1 d_{c1}^2 + I_2 + m_2 q_2^2 + I_3 + m_3 d_{c3}^2 + m_3(q_2 + d_{c2})^2 + 2m_3 d_{c3}(q_2 + d_{c2})\cos q_3 \\ &= a_1 + a_2 q_2 + a_3 q_2^2 + 2a_4\cos q_3 + 2a_5 q_2 \cos q_3 \end{aligned}$$

$$b_{12} = -m_3 d_{c3}\sin q_3 = -a_5 \sin q_3$$

$$b_{13} = I_3 + m_3 d_{c3}^2 + m_3 d_{c3}(q_2 + d_{c2})\cos q_3 = a_6 + a_5 q_2 \cos q_3 + a_4 \cos q_3$$

$$b_{22} = m_2 + m_3 = a_3$$

$$b_{23} = -m_3 d_{c3}\sin q_3 = -a_5 \sin q_3$$

$$b_{33} = I_3 + m_3 d_{c3}^2 = a_6.$$

Therefore, the inertia matrix can be compactly rewritten as

$$
\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 + a_2 q_2 + a_3 q_2^2 + 2a_4 \cos q_3 + 2a_5 q_2 \cos q_3 & -a_5 \sin q_3 & a_6 + a_5 q_2 \cos q_3 + a_4 \cos q_3 \\ -a_5 \sin q_3 & a_3 & -a_5 \sin q_3 \\ a_6 + a_5 q_2 \cos q_3 + a_4 \cos q_3 & -a_5 \sin q_3 & a_6 \end{pmatrix}
$$

and the (minimal) parametrization of $\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{Y}(\boldsymbol{q}, \ddot{\boldsymbol{q}})\boldsymbol{a}$ is thus of dimension $p = 6$, with

$$
\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} I_1 + m_1 d_{c1}^2 + I_2 + I_3 + m_3(d_{c2}^2 + d_{c3}^2) \\ 2m_3 d_{c2} \\ m_2 + m_3 \\ m_2 d_{c2} d_{c3} \\ m_3 d_{c3} \\ I_3 + m_3 d_{c3}^2 \end{pmatrix}
$$

and

$$
\boldsymbol{Y}(\boldsymbol{q}, \ddot{\boldsymbol{q}}) = \begin{pmatrix} \ddot{q}_1 & q_2 \ddot{q}_1 & q_2^2 \ddot{q}_1 & \cos q_3 (2\ddot{q}_1 + \ddot{q}_3) & q_2 \cos q_3 (2\ddot{q}_1 + \ddot{q}_3) - \sin q_3 \ddot{q}_2 & \ddot{q}_3 \\ 0 & 0 & \ddot{q}_2 & 0 & -\sin q_3 (\ddot{q}_1 + \ddot{q}_3) & 0 \\ 0 & 0 & 0 & \cos q_3 \ddot{q}_1 & q_2 \cos q_3 \ddot{q}_1 - \sin q_3 \ddot{q}_2 & \ddot{q}_1 + \ddot{q}_3 \end{pmatrix}.
$$

* * * * *

Robotics II

June 17, 2011

Consider a 2R planar robot with equal links of unitary length, uniformly distributed link masses, and moving in the horizontal plane. During *free motion*, the robot dynamics is

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau},$$

where

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2 \cos q_2 & a_3 + a_2 \cos q_2 \\ a_3 + a_2 \cos q_2 & a_3 \end{pmatrix} \qquad \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -a_2 \sin q_2 \, \dot{q}_2(\dot{q}_2 + 2\dot{q}_1) \\ a_2 \sin q_2 \, \dot{q}_1^2 \end{pmatrix},$$

and with the dynamic coefficients a_1, a_2, and a_3 being known (their numerical value is not essential in the following).

In the configuration $\boldsymbol{q} = (0, \pi/4)$ [rad] and with $\dot{\boldsymbol{q}} = (0, -\pi/2)$ [rad/s], the *robot is hit* by an instantaneous planar Cartesian force $\boldsymbol{F} = (-10, 0)$ [N] applied at the midpoint of the second link.

1. Determine the explicit expression of the feedback control law for the joint torque vector $\boldsymbol{\tau}$ that minimizes in norm the resulting instantaneous joint acceleration $\ddot{\boldsymbol{q}}$. Provide the associated expression of $\|\ddot{\boldsymbol{q}}\|$.
2. What are the robot sensing requirements for achieving this result? How could we design a dynamic control law for the torque $\boldsymbol{\tau}$ that approximates this same robot behavior when having only joint position and velocity measurements available?

[120 minutes; open books]

Solution

June 17, 2011

When the 2R planar robot is hit at a given state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ by a force $\boldsymbol{F}$, its dynamics becomes

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau} + \boldsymbol{J}_k^T(\boldsymbol{q})\boldsymbol{F}, \tag{1}$$

where the Jacobian $\boldsymbol{J}_k(\boldsymbol{q})$ is associated to the collision point, in the present case the midpoint of the second link. Taking into account the kinematic data and the collision configuration $\boldsymbol{q} = (0, \pi/4)$, we have

$$\begin{aligned}\boldsymbol{J}_K = \boldsymbol{J}_k(\boldsymbol{q})|_{\boldsymbol{q}=(0,\pi/4)} &= \left.\begin{pmatrix} -\sin q_1 - 0.5\sin(q_1+q_2) & -0.5\sin(q_1+q_2) \\ \cos q_1 + 0.5\cos(q_1+q_2) & 0.5\cos(q_1+q_2) \end{pmatrix}\right|_{\boldsymbol{q}=(0,\pi/4)} \\ &= \begin{pmatrix} -\dfrac{\sqrt{2}}{4} & -\dfrac{\sqrt{2}}{4} \\ 1+\dfrac{\sqrt{2}}{4} & \dfrac{\sqrt{2}}{4} \end{pmatrix}.\end{aligned}$$

The instantaneous joint acceleration $\ddot{\boldsymbol{q}}$ in response to the collision force $\boldsymbol{F}$ and to an applied torque $\boldsymbol{\tau}$ at the joints will be

$$\ddot{\boldsymbol{q}} = \boldsymbol{B}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau} + \boldsymbol{J}_k^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right).$$

Therefore, the control torque

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{F}) = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{J}_k^T(\boldsymbol{q})\boldsymbol{F} \tag{2}$$

will produce $\ddot{\boldsymbol{q}} = \boldsymbol{0}$, which is the minimum possible acceleration ($\|\ddot{\boldsymbol{q}}\| = 0$, and the robot will continue its motion unperturbed by the collision). In order to be realizable, this control law should have access to a (direct or indirect) instantaneous measure of $\boldsymbol{F}$, beyond measuring the internal state of the robot. Moreover, the collision point (assumed to be at the midpoint of the second link in the formulation) should also be known exactly in order to use the correct $\boldsymbol{J}_k^T(\boldsymbol{q})$. These two requirements could be matched in principle if there is a camera observing the scene and/or a surface touch sensor capable of measuring the collision force (if different from the assumed one).

If this can be accomplished, the actual value of the control torque at the collision instant is computed using

$$\begin{aligned}\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})|_{\boldsymbol{q}=(0,\pi/4),\dot{\boldsymbol{q}}=(0,-\pi/2)} &= \left.\begin{pmatrix} -a_2 \sin q_2\, \dot{q}_2(\dot{q}_2 + 2\dot{q}_1) \\ a_2 \sin q_2\, \dot{q}_1^2 \end{pmatrix}\right|_{\boldsymbol{q}=(0,\pi/4),\dot{\boldsymbol{q}}=(0,-\pi/2)} \\ &= \begin{pmatrix} -a_2 \dfrac{\sqrt{2}}{2}\left(\dfrac{\pi}{2}\right)^2 \\ 0 \end{pmatrix}\end{aligned}$$

as

$$\boldsymbol{\tau} = \boldsymbol{c} - \boldsymbol{J}_K^T\boldsymbol{F} = \begin{pmatrix} -a_2 \dfrac{\sqrt{2}}{2}\left(\dfrac{\pi}{2}\right)^2 \\ 0 \end{pmatrix} - \begin{pmatrix} -\dfrac{\sqrt{2}}{4} & * \\ -\dfrac{\sqrt{2}}{4} & * \end{pmatrix}\begin{pmatrix} -10 \\ 0 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 5 - a_2\left(\dfrac{\pi}{2}\right)^2 \\ 5 \end{pmatrix}, \tag{3}$$

where the dynamic coefficient a_2 is known.

Indeed, the above situation is quite restrictive. In order to get rid of extra sensing and measurements, we could use the concept of model-based residuals to estimate at once the global quantity

$$\tau_k = \boldsymbol{J}_k(\boldsymbol{q})^T \boldsymbol{F}$$

needed in the control action (2). In fact, we can generate on line the following (here, two-dimensional) residual signal $\boldsymbol{r}$

$$\boldsymbol{r} = \boldsymbol{K}_I \left(\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} - \int \left(\boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{r} + \boldsymbol{\tau} \right) dt \right) \tag{4}$$

for $\boldsymbol{K}_I > 0$, typically diagonal, and where the matrix $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is a factorization of the Coriolis and centrifugal terms that satisfies

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}, \text{ s.t. } \dot{\boldsymbol{B}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \text{ is skew-symmetric } \Rightarrow \dot{\boldsymbol{B}}(\boldsymbol{q}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{5}$$

Such a matrix always exists and can be computed, e.g., using the Christoffels symbols. From (4), using the dynamics (1) and the property (5), it follows that the evolution of $\boldsymbol{r}$ satisfies

$$\begin{aligned}
\dot{\boldsymbol{r}} &= \boldsymbol{K}_I \left(\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{r} - \boldsymbol{\tau} \right) \\
&= \boldsymbol{K}_I \left(\boldsymbol{\tau} + \boldsymbol{\tau}_k - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{r} - \boldsymbol{\tau} \right) \\
&= \boldsymbol{K}_I \left(\boldsymbol{\tau}_k - \boldsymbol{r} \right).
\end{aligned}$$

Therefore, $\boldsymbol{r}$ is a low-pass stable first-order filter of the unknown signal $\boldsymbol{\tau}_k$ with bandwidth going to infinity for increasingly larger $\boldsymbol{K}_I$. Its evaluation in (4) requires only the proprioceptive measurements $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ (beside, and as before, a reasonable accuracy for the dynamic model of the robot) and the available of the command $\boldsymbol{\tau}$ being applied to the robot.

As a result, the control law (2) can be replaced by

$$\tau = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{r}, \tag{6}$$

and the resulting robot behavior approximated at a low expense.

* * * * *

Robotics II

September 12, 2011

1. A 2R robot moving in the vertical plane and having link lengths $l_1 = l_2 = 1$ [m] is *self-balanced* with respect to gravity in the absence of a payload (see Fig. 1). Provide the mechanical conditions under which the gravity term in the dynamics of this robot is identically zero for any configuration $\boldsymbol{q} = (q_1, q_2)$.

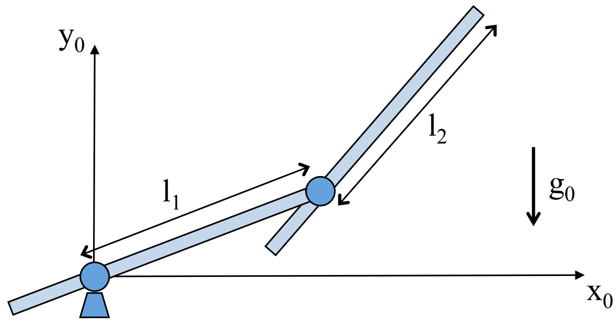

Figure 1: A 2R robot

2. For a regulation task to a desired $\boldsymbol{q}_d$, the following control law (PD + constant gravity compensation)

$$\boldsymbol{u} = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d), \qquad \boldsymbol{K}_P > \boldsymbol{0},\ \boldsymbol{K}_D > \boldsymbol{0}, \tag{1}$$

is applied to this robot, now in the presence of a point-wise payload of known mass M located at the end of the second link.

a) Derive the correct expression of the gravity term $\boldsymbol{g}(\boldsymbol{q}_d)$ in (1).

b) Given the fixed choice of PD gain matrices

$$\boldsymbol{K}_P = \text{diag}\{100, 100\}, \qquad \boldsymbol{K}_D = \text{diag}\{25, 25\}, \tag{2}$$

provide an *upper bound* for the value of M such that the control law (1), with the gains chosen as in (2), certainly guarantees global asymptotic stability of the equilibrium state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$. It is suggested to perform an approximate (conservative) analysis.

[120 minutes; open books]

Solution

September 12, 2011

In the absence of a payload, the potential energy U due to gravity is

$$U = U_1 + U_2, \quad U_1 = m_1 g_0\, d_1 \sin q_1, \quad U_2 = m_2 g_0 \left(l_1 \sin q_1 + d_2 \sin(q_1 + q_2) \right),$$

where m_1 and m_2 are the masses of the two links, $g_0 = 9.81$ [m/s^2] is the gravity acceleration, and d_1 and d_2 are the (oriented) distances of the the center of mass of link 1 and, respectively, of link 2 from the associated joint axes (d_1 and d_2 can be positive, zero, or negative). The gravity term is then

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} (m_1 d_1 + m_2 l_1) g_0 \cos q_1 + m_2 d_2 g_0 \cos(q_1 + q_2) \\ m_2 d_2 g_0 \cos(q_1 + q_2) \end{pmatrix},$$

from which the requested mechanical conditions are:

$$\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0} \quad \Longleftrightarrow \quad \begin{cases} d_2 = 0, \\ d_1 = -\dfrac{m_2}{m_1} l_1 < 0. \end{cases}$$

The center of mass of the second link is then on the axis of joint 2, while the center of mass of the first link is located 'in opposition' to the second link 2 with respect to the joint 2 axis so that the center of gravity of the two masses of the first and second link lies always on the axis of joint 1. In these conditions, it follows that $U \equiv 0$ (or constant).

In the presence of a payload, the additional potential energy U_M due to the point-wise mass M at the end-effector is

$$U_M = M g_0 \left(l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \right),$$

and so the gravity term in the robot dynamics becomes Consider d2 and d1 as above, so g(q) doesn't consider U1 and U2 but only UM

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_M}{\partial \boldsymbol{q}} \right)^T = M g_0 \begin{pmatrix} l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\ l_2 \cos(q_1 + q_2) \end{pmatrix}.$$

Evaluating this at $\boldsymbol{q} = \boldsymbol{q}_d$ yields the term $\boldsymbol{g}(q_d)$ in the control law (1).

For part 2.b, one relies on the property of boundedness for the gradient of the gravity term for all configurations of the robot, i.e.,

$$\left\| \frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} \right\| \leq \alpha, \qquad \forall \boldsymbol{q},$$

where α is a suitable (large enough) positive constant, and the norm of the matrix $\boldsymbol{A} = \dfrac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}$ is defined[1] as the square root of the largest eigenvalue of the (symmetric and positive semi-definite) matrix $\boldsymbol{A}^T \boldsymbol{A}$:

$$\|\boldsymbol{A}\| = \sqrt{\lambda_{\max} \left(\boldsymbol{A}^T \boldsymbol{A} \right)}.$$

[1] This matrix norm is the one induced by (or, naturally associated with) the standard Euclidean norm of vectors: $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^T \boldsymbol{x}}$.

Indeed, both the configuration-dependent norm of matrix $\boldsymbol{A}$ and its constant upper bound α will be functions of (in particular, increase with) the mass M. In order to guarantee that the control law (1) globally asymptotically stabilizes any desired equilibrium state of the robot, with the PD gains chosen as in (2), it will be sufficient to have

$$\alpha < \boldsymbol{K}_{P,\min} = 100. \tag{3}$$

We compute then

$$\boldsymbol{A} = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} = M g_0 \begin{pmatrix} -l_1 \sin q_1 - l_2 \sin(q_1+q_2) & -l_2 \sin(q_1+q_2) \\ -l_2 \sin(q_1+q_2) & -l_2 \sin(q_1+q_2) \end{pmatrix},$$

and

$$\begin{aligned} \boldsymbol{A}^T \boldsymbol{A} \;=\; & (M g_0)^2 \cdot \\ & \begin{pmatrix} (l_1 \sin q_1 + l_2 \sin(q_1+q_2))^2 + (l_2 \sin(q_1+q_2))^2 & (l_1 \sin q_1 + l_2 \sin(q_1+q_2))\, l_2 \sin(q_1+q_2) \\ (l_1 \sin q_1 + l_2 \sin(q_1+q_2))\, l_2 \sin(q_1+q_2) & 2\,(l_2 \sin(q_1+q_2))^2 \end{pmatrix}. \end{aligned}$$

The two (real and non-negative) eigenvalues of $\boldsymbol{A}^T\boldsymbol{A}$ are the roots of

$$\det\left(\lambda \boldsymbol{I} - \boldsymbol{A}^T\boldsymbol{A}\right) = \lambda^2 - b(\boldsymbol{q}, M)\lambda + c(\boldsymbol{q}, M) = 0,$$

with

$$\begin{aligned} b(\boldsymbol{q}, M) &= (M g_0)^4 \left((l_1 \sin q_1 + l_2 \sin(q_1+q_2))^2 + 3\,(l_2 \sin(q_1+q_2))^2 \right) \geq 0, \\ c(\boldsymbol{q}, M) &= (M g_0)^4 \left[2\,(l_2 \sin(q_1+q_2))^4 + (l_2 \sin(q_1+q_2))^2\,(l_1 \sin q_1 + l_2 \sin(q_1+q_2))^2 \right] \geq 0, \end{aligned}$$

and where the inequalities on the right hold for all $\boldsymbol{q}$ and all positive values of M. For a given $M > 0$ and $\boldsymbol{q}$, the largest of the two eigenvalues is written compactly as

$$\lambda_{\max}\left(\boldsymbol{A}^T(\boldsymbol{q}, M)\boldsymbol{A}(\boldsymbol{q}, M)\right) = \frac{b(\boldsymbol{q}, M) + \sqrt{b^2(\boldsymbol{q}, M) - 4\,c(\boldsymbol{q}, M)}}{2}.$$

From the non-negativity of both b and c, an upper bound to this expression is obtained by simply neglecting c, so that

$$\lambda_{\max}\left(\boldsymbol{A}^T(\boldsymbol{q}, M)\boldsymbol{A}(\boldsymbol{q}, M)\right) \leq b(\boldsymbol{q}, M).$$

The maximum value for $b(\boldsymbol{q}, M)$ over all possible $\boldsymbol{q}$ is obtained when simultaneously

$$\sin q_1 = 1 \quad \text{AND} \quad \sin(q_1+q_2) = 1 \qquad (\text{e.g., for } q_1 = \pi/2, q_2 = 0).$$

Plugging the numerical values of the link lengths, $l_1 = l_2 = 1$, and setting $g_0 = 9.81$, we can set for the constant α to be used as an upper bound to $\|\partial \boldsymbol{g}/\partial \boldsymbol{q}\|$

$$\alpha^2 = b(\boldsymbol{q}, M)\big|_{q_1=\pi/2, q_2=0} = 7 \cdot (9.81)^4 \cdot M^4,$$

and thus

$$\alpha = \sqrt{7} \cdot (9.81)^2 \cdot M^2.$$

Therefore, the inequality (3) is satisfied for

$$M < \frac{10}{9.81}\,\frac{1}{\sqrt[4]{7}} \simeq 0.6267\,[\text{kg}].$$

* * * * *

Robotics II

June 11, 2012

Exercise 1

In an image-based visual servoing scheme, a pin-hole camera with constant focal length $\lambda > 0$ is mounted on the end-effector of a robot manipulator. A fixed Cartesian point $P = (X \ Y \ Z)^T$, whose coordinates are expressed in the camera frame and having $Z > 0$, is associated to a point feature $\boldsymbol{f} = (u \ v)^T$ in the image plane. The 2×6 interaction matrix $\boldsymbol{J}_p$ in

$$\dot{\boldsymbol{f}} = \boldsymbol{J}_p(u, v, Z) \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix}$$

provides the velocity of the point feature $\boldsymbol{f}$ when the robot imposes to the camera a linear velocity $\boldsymbol{v} \in \mathbb{R}^3$ and an angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$, both expressed in the camera frame. Determine a basis for all possible motions of the camera that *do not move* the point feature in the image plane, i.e., yielding $\dot{\boldsymbol{f}} = \boldsymbol{0}$. How many independent *pure translation* motions (i.e., with $\boldsymbol{\omega} = \boldsymbol{0}$) of this kind exist for the camera? And how many independent *pure rotation* motions (i.e., with $\boldsymbol{v} = \boldsymbol{0}$)?

Exercise 2

A planar 3R robot with equal uniform thin rod links of unitary length moves in the vertical plane, actuated by the joint torques $\boldsymbol{\tau} \in \mathbb{R}^3$ and subject to a constant contact force $\boldsymbol{F}_c$ applied to the midpoint of the second link and pointing always upward (see Fig. 1).

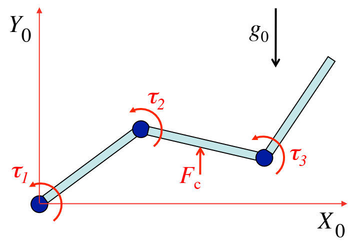

Figure 1: A planar 3R robot with a contact force

- Define a control law for $\boldsymbol{\tau}$ that regulates the robot to a desired generic equilibrium state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$ and give the explicit expression of the kinematic and dynamic terms needed for its implementation. The contact force and its location are assumed to be known, and thus there is no need of a force sensor. Provide a Lyapunov candidate that allows to prove the global asymptotic stability of the closed-loop system using Lyapunov/LaSalle arguments.
- Suppose that the robot is in its desired steady state $(\boldsymbol{q}_d, \boldsymbol{0})$ under the action of the previously defined control law. The contact force is then suddenly doubled in intensity, but the control law is left unchanged. Does the third joint accelerate instantaneously? Assuming that the system remains stable, find a characterization of the new final configuration reached at the end of the transient in terms of the parameters of the controller.
- If the contact force could be measured, would it be possible to preserve the global asymptotic stability of the original state $(\boldsymbol{q}_d, \boldsymbol{0})$ by suitably modifying the control law? If so, how?

[180 minutes; open books]

Solution

June 11, 2012

Exercise 1

The interaction matrix $\boldsymbol{J}_p$ for a point feature takes the form (see lecture slides)

$$\boldsymbol{J}_p = \begin{pmatrix} -\dfrac{\lambda}{Z} & 0 & \dfrac{u}{Z} & \dfrac{uv}{\lambda} & -\dfrac{u^2}{\lambda} - \lambda & v \\ 0 & \dfrac{\lambda}{Z} & \dfrac{v}{Z} & \dfrac{v^2}{\lambda} + \lambda & -\dfrac{uv}{\lambda} & -u \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{\boldsymbol{v}} & \boldsymbol{J}_{\boldsymbol{\omega}} \end{pmatrix}. \qquad (1)$$

Since $\lambda > 0$ and Z is limited (by the range of the camera), the interaction matrix as well as its two (2×3) blocks $\boldsymbol{J}_{\boldsymbol{v}}$ and $\boldsymbol{J}_{\boldsymbol{\omega}}$ will all have rank 2, independently from the values of u and v.

The problem requires to define a basis for the null space of $\boldsymbol{J}_p$ in the six-dimensional space of linear and angular camera velocity $(\boldsymbol{v}, \boldsymbol{\omega})$. From the analysis of the structure of $\boldsymbol{J}_p$ in (1), we have that

$$\dim \mathcal{N}\{\boldsymbol{J}_p\} = 4, \qquad \dim \mathcal{N}\{\boldsymbol{J}_{\boldsymbol{v}}\} = 1, \qquad \dim \mathcal{N}\{\boldsymbol{J}_{\boldsymbol{\omega}}\} = 1.$$

This means that there will be four independent velocity vector of the camera that will not move the feature (u, v) in the image plane, and that one such vector can be given the form of a pure translation $(\boldsymbol{v}, \boldsymbol{0})$ and another one of a pure rotation $(\boldsymbol{0}, \boldsymbol{\omega})$. The other two vectors will be associated to roto-translations. With this in mind, we can solve the problem by inspection or by resorting to the following symbolic Matlab code.

```
% null space of point feature interaction matrix

clear all
clc
syms lam u v Z real

J =
[ -lam/Z, 0, u/Z, (u*v)/lam, - u^2/lam - lam, v]
[ 0, -lam/Z, v/Z, v^2/lam + lam, -(u*v)/lam, -u];

N=null(J)

% check special case for u=v=0 (point at the center of the image plane)

Norigin=subs(N,[u,v],[0,0])

% check alternative solution (a pure rotation) that Matlab does not provide as such

nrot=simplify(u*N(:,2)+v*N(:,3)+lam*N(:,4))
verifynrot=simplify(J*nrot)

% OUTPUT

N =
[ u/lam, (Z*u*v)/lam^2, -(Z*(lam^2 + u^2))/lam^2, (Z*v)/lam]
[ v/lam, (Z*(lam^2 + v^2))/lam^2, -(Z*u*v)/lam^2, -(Z*u)/lam]
[ 1, 0, 0, 0]
[ 0, 1, 0, 0]
[ 0, 0, 1, 0]
[ 0, 0, 0, 1]
```

```
Norigin =
[ 0, 0, -Z, 0]
[ 0, Z, 0, 0]
[ 1, 0, 0, 0]
[ 0, 1, 0, 0]
[ 0, 0, 1, 0]
[ 0, 0, 0, 1]
nrot =
0
0
0
u
v
lam
verifyrot =
0
0
```

The columns of matrix `N` provide the solution. It is apparent that the first vector (conveniently scaled by λ) is a pure translation

$$\lambda \, \mathtt{N}(:,1) = \begin{pmatrix} u \\ v \\ \lambda \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{v} \\ \mathbf{0} \end{pmatrix}.$$

This represents a linear motion of the camera along the ray connecting the point (u, v) on the image plane to the Cartesian point $(X, Y.Z)$. On the other hand, the remaining three basis vectors for the null space of $\boldsymbol{J}_p$ found by Matlab have all the form of roto-translations. However, it is easy to see that their linear combination

$$u \, \mathtt{N}(:,2) + v \, \mathtt{N}(:,3) + \lambda \, \mathtt{N}(:,4) = \begin{pmatrix} \mathbf{0} \\ u \\ v \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\omega} \end{pmatrix}$$

is a pure rotation along the previously defined ray. Indeed, if not properly guided, Matlab will find the null space of a matrix by suitable orthogonalizations and normalizations (see the resulting (4×4) identity matrix in the lower part of `N`), without providing automatically a solution with a desired physical meaning.

Finally, the two remaining null space vectors can be given the form of roto-translations, either both with linear and angular components in the image plane

$$\lambda^2 \, \mathtt{N}(:,2) = \begin{pmatrix} uvZ \\ (v^2+\lambda^2)Z \\ 0 \\ \lambda^2 \\ 0 \\ 0 \end{pmatrix}, \qquad \lambda^2 \, \mathtt{N}(:,3) = \begin{pmatrix} -(u^2+\lambda^2)Z \\ -uvZ \\ 0 \\ 0 \\ \lambda^2 \\ 0 \end{pmatrix},$$

or one still of this type and the other with linear component in the image plane and angular component along the camera main axis

$$\lambda^2 \ (\mathtt{N}(:,2)+\mathtt{N}(:,3)) = \begin{pmatrix} -\left(u^2+\lambda^2-uv\right)Z \\ \left(v^2+\lambda^2-uv\right)Z \\ 0 \\ \lambda^2 \\ \lambda^2 \\ 0 \end{pmatrix}, \qquad \lambda\, \mathtt{N}(:,4) = \begin{pmatrix} vZ \\ -uZ \\ 0 \\ 0 \\ 0 \\ \lambda \end{pmatrix}.$$

Exercise 2

The robot dynamics, under the action of the contact force $\boldsymbol{F}_c$ and the motor control torque $\boldsymbol{\tau}$, is described by

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c,$$

AT EQUILIBRIUM. $q = q_d,\ \dot{q} = 0$

$c = 0,\ \tau = g - J_c^T F_c + K_P 0 + K_D 0$

$= g - J_c^T F_c$

where the Jacobian $\boldsymbol{J}_c(\boldsymbol{q})$ is associated to the linear velocity (in the plane) of the mid point of the second robot link, where the contact force is applied in the upward direction. For a regulation task in the presence of the constant and *known* force $\boldsymbol{F}_c$, we can use the control law

$$\boldsymbol{\tau} = \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\dot{\boldsymbol{q}}. \tag{2}$$

PD + GRAVITY COMP. + F_c COMP.

In fact, by canceling gravity and the effect of the contact force at the joint level, the task becomes in this way one of free regulation in the absence of gravity. This is solvable by a PD joint error action with $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$, yielding global asymptotic stability of the desired equilibrium state $(\boldsymbol{q}_d, \boldsymbol{0})$. The proof uses the Lyapunov candidate

$$V = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\left(\boldsymbol{q}_d - \boldsymbol{q}\right)^T\boldsymbol{K}_P\left(\boldsymbol{q}_d - \boldsymbol{q}\right)$$

and LaSalle analysis.

To implement the control law (2), we need to measure only $(\boldsymbol{q},\dot{\boldsymbol{q}})$ (no force sensing is needed) and compute $\boldsymbol{g}(\boldsymbol{q})$ and $\boldsymbol{J}(\boldsymbol{q})$. From the gravitational potential energy

$$U = U_1 + U_2 + U_3 = g_0\left(m_1\frac{\ell_1}{2}s_1 + m_2\left(\ell_1 s_1 + \frac{\ell_2}{2}s_{12}\right) + m_3\left(\ell_1 s_1 + \ell_2 s_{12} + \frac{\ell_3}{2}s_{123}\right)\right),$$

setting $\ell_1 = \ell_2 = \ell_3 = 1$, we obtain

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \frac{g_0}{2}\begin{pmatrix} m_1 c_1 + m_2\left(2c_1 + c_{12}\right) + m_3\left(2c_1 + 2c_{12} + c_{123}\right) \\ m_2 c_{12} + m_3\left(2c_{12} + c_{123}\right) \\ m_3 c_{123} \end{pmatrix}.$$

From the positional kinematics of the mid point of the second link

$$\boldsymbol{p}_c = \boldsymbol{f}_c(\boldsymbol{q}) = \begin{pmatrix} \ell_1 c_1 + 0.5\,\ell_2 c_{12} \\ \ell_1 s_1 + 0.5\,\ell_2 s_{12} \end{pmatrix},$$

setting $\ell_1 = \ell_2 = 1$, we obtain

$$\boldsymbol{J}_c(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \frac{1}{2}\begin{pmatrix} -\left(2s_1 + s_{12}\right) & -s_{12} & 0 \\ 2c_1 + c_{12} & c_{12} & 0 \end{pmatrix}.$$

Therefore

$$\boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c = \boldsymbol{J}_c^T(\boldsymbol{q})\begin{pmatrix} 0 \\ F_{c,y} \end{pmatrix} = \frac{F_{c,y}}{2}\begin{pmatrix} 2c_1 + c_{12} \\ c_{12} \\ 0 \end{pmatrix}.$$

! Note that no additional control torque is present at the third joint since the contact force is applied at the second link. However, this contact force, if not compensated, will in general accelerate also the third joint because of the inertial coupling.

With the robot at the desired equilibrium state $(\boldsymbol{q}_d, \boldsymbol{0})$ under the action of the control law (2), suppose that the contact force is doubled from $\boldsymbol{F}_c$ to $2\boldsymbol{F}_c$. The dynamic equations become in this state

τ IS STILL $g - J_c F_c$ WHILE F_c DOUBLED

$$\boldsymbol{B}(\boldsymbol{q}_d)\ddot{\boldsymbol{q}} = -\boldsymbol{J}_c^T(\boldsymbol{q}_d)\boldsymbol{F}_c + 2\boldsymbol{J}_c^T(\boldsymbol{q}_d)\boldsymbol{F}_c = \boldsymbol{J}_c^T(\boldsymbol{q}_d)\boldsymbol{F}_c$$

and thus the instantaneous acceleration is

$$\ddot{\boldsymbol{q}} = \boldsymbol{B}^{-1}(\boldsymbol{q}_d)\boldsymbol{J}_c^T(\boldsymbol{q}_d)\boldsymbol{F}_c.$$

Being in general the inertia matrix a full matrix (and so its inverse), all joints will have a non-zero acceleration $\ddot{q}_i$, including $\ddot{q}_3$.

Suppose now that a new steady state is reached, with $\boldsymbol{q} = \bar{\boldsymbol{q}}$ and $\dot{\boldsymbol{q}} = \ddot{\boldsymbol{q}} = \boldsymbol{0}$. In this equilibrium state, it is necessarily

$M \cdot \emptyset + \emptyset + g = \tau + 2J_c^T F_c = J_c^T F_c + K_P(q_D - \bar{q})$

$\underbrace{\quad}_{\emptyset}$ $\; g - J_c^T F_c + K_P(q_D - \bar{q}) + K_D \cdot \emptyset$

$$\boldsymbol{K}_P(\boldsymbol{q}_d - \bar{\boldsymbol{q}}) + \boldsymbol{J}_c^T(\bar{\boldsymbol{q}})\boldsymbol{F}_c = \boldsymbol{0}. \tag{3}$$

We can determine the equilibrium configuration $\bar{\boldsymbol{q}}$ only by solving numerically the nonlinear equation (3). However, assuming that $\boldsymbol{K}_P$ is chosen diagonal as usual, since the last component of $\boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c$ is always zero, the third equation in (3) becomes

$$K_{P,3}\,(q_{d,3} - \bar{q}_3) = 0 \qquad \Rightarrow \qquad \bar{q}_3 = q_{d,3}$$

while $\bar{q}_i \neq q_{d,i}$ for $i = 1, 2$. This means that the third joint will resume its desired value after the perturbation due to the doubling of the contact force acting on the second link.

→ AFTER PERTURBATION = NEXT EQ. $\bar{q}$

Finally, if the contact force is measured (in some way) and used in feedback control, then the same law (2) —now with any value for the measured quantity $\boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c$— will guarantee that $(\boldsymbol{q}_d, \boldsymbol{0})$ will always remain the unique globally asymptotically stable equilibrium for the robot system.

*****

Robotics II

July 5, 2012

Huge portal robots are used in the aeronautical industry for moving and reorienting large surface plates of aircraft bodies. Figure 1 shows a subassembly of one such robots, having three actuated joints and three passive joints (shown in different colors).

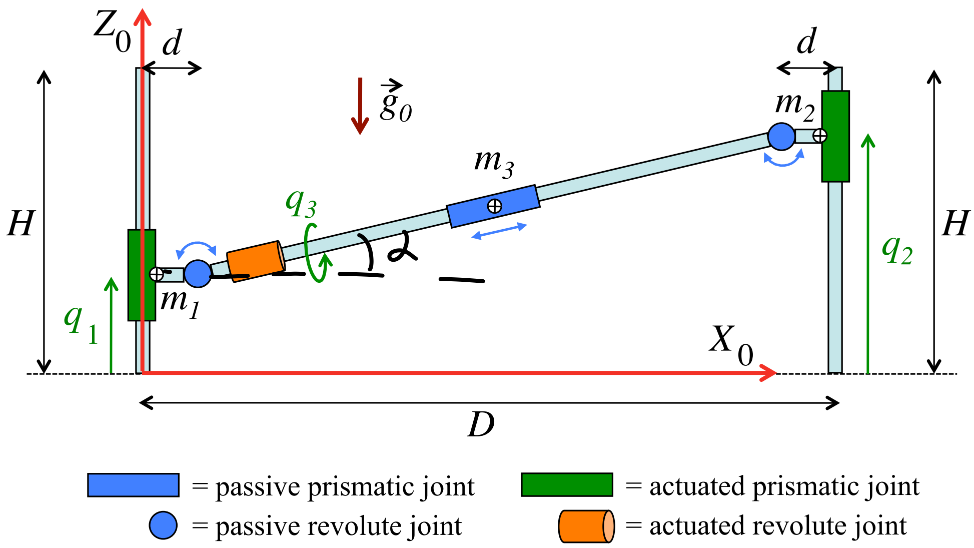

Figure 1: A portal robot used in the aeronautic industry

Two vertical bars of height H are placed at a distance D. Along the vertical bars, two actuated prismatic joints (with variables $q_1 \in [0, H]$ and $q_2 \in [0, H]$) are used to modify the position and the orientation of a connecting bar in the vertical plane $(\boldsymbol{x}_0, \boldsymbol{z}_0)$. The two passive revolute joints, placed at a distance $d << D$ from the vertical bars, transform the linear motion of q_1 and q_2 in a linear and/or angular motion of the connecting bar. The passive prismatic joint along the connecting bar accommodates itself so that the bar changes length consistently with the positions q_1 and q_2. Furthermore, the connecting bar can be rotated by an actuated revolute joint (with angle q_3) along its main geometric axis. The available actuating force/torque $\boldsymbol{u} = (F_1, F_2, \tau_3)$ performs work on the generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$. From an operational point of view, curved plates of the aircraft body are placed and fixed on the connecting bar. Then, the motion of the portal robot can change the orientation of the normals to the plate surface to be worked (e.g., for riveting).

Figure 1 shows also the location of the center of masses of the three main moving bodies (of masses m_1, m_2, and m_3) of the portal robot. The following assumptions are made.

- The mechanism associated to the passive prismatic joint is such that the center of mass of the connecting bar is always kept on the vertical line passing through the center of the portal (i.e., at $x_0 = D/2$), despite the extension or retraction of the connecting bar itself.
- The connecting bar has a diagonal inertia matrix

$$^{b}\boldsymbol{I}_b = \text{diag}\{I_x, I_y, I_z\}, \qquad \text{with} \quad I_y = I_z$$

when expressed in a frame $(\boldsymbol{x}_b, \boldsymbol{y}_b, \boldsymbol{z}_b)$ attached to the bar, with origin at its center of mass and principal axes along its geometric axes of symmetry.

- The values of the scalar inertias I_x and $I_y = I_z$ change with the length of the connecting bar. Since the distance D is much larger than the actual excursion of q_1 and q_2, this effect can be neglected if needed. In this case, for dynamic modeling purposes the bar will be considered of constant length $D - 2d$ (as when it is horizontal or close to this situation).

With the above description in mind, and considering the previous assumptions:

1. Derive the dynamic model of this portal robot using the generalized coordinates $\boldsymbol{q}$ in a Lagrangian approach.
2. Define the simplest control law for $\boldsymbol{u}$ that is able to regulate the robot to a desired constant equilibrium state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$, defining conditions on the control parameters that guarantee global asymptotic stability.
3. *Bonus, or in alternative to 2.* Show that $\boldsymbol{q} = (q_1, q_2, q_3)$ is in fact a set of generalized coordinates for this robot, which has the structure of a closed kinematic chain (closed through the floor). In particular, show that the values of the angular position of the two passive revolute joints and the extension of the connecting bar thanks to the presence of the passive prismatic joint can be uniquely obtained from the values of q_1, q_2, and q_3.

[240 minutes; open books]

Solution

July 5, 2012

Following the Lagrangian approach, we compute the kinetic energy and the potential energy due to gravity for the robot bodies based on the generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$.

The kinetic energy for the first two bodies (moving only up and down) is simply

$$T_1 = \frac{1}{2}\, m_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2}\, m_2 \dot{q}_2^2.$$

For the third body (the connecting bar), it is useful to introduce the variable

$$\alpha = \arctan\left(\frac{q_1 - q_2}{D - 2d}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{1}$$

that represents the angle of the bar w.r.t. to the horizontal (positive counterclockwise around the axis $\boldsymbol{y}_0$). For evaluating the linear contribution to the kinetic energy of this bar

$$T_{3,l} = \frac{1}{2}\, m_3\, \boldsymbol{v}_{c3}^T \boldsymbol{v}_{c3},$$

we compute the position of its center of mass and then differentiate w.r.t. time:

$$\boldsymbol{p}_{c3} = \begin{pmatrix} \dfrac{D}{2} \\ 0 \\ \dfrac{q_1 + q_2}{2} \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\dot{q}_1 + \dot{q}_2}{2} \end{pmatrix}.$$

Therefore,

$$T_{3,l} = \frac{1}{2}\, m_3 \left(\frac{\dot{q}_1 + \dot{q}_2}{2}\right)^2.$$

For evaluating the angular contribution to the kinetic energy

$$T_{3,a} = \frac{1}{2}\, \boldsymbol{\omega}_3^T \boldsymbol{I}_b \boldsymbol{\omega}_3 = \frac{1}{2}\, {}^b\boldsymbol{\omega}_3^T\, {}^b\boldsymbol{I}_b\, {}^b\boldsymbol{\omega}_3$$

it will be convenient to work in the frame $(\boldsymbol{x}_b, \boldsymbol{y}_b, \boldsymbol{z}_b)$ attached to the center of mass and having the $\boldsymbol{x}_b$ along the principal direction of the bar (rotated thus w.r.t. $\boldsymbol{x}_0$ by the angle α around $\boldsymbol{y}_0$), in which the inertia matrix is diagonal. Note that this frame has $\boldsymbol{y}_b$ aligned with $\boldsymbol{y}_0$ when $q_3 = 0$. Its absolute orientation is expressed by the rotation matrix

$${}^0\boldsymbol{R}_b = {}^0\boldsymbol{R}_d(\alpha)\, {}^d\boldsymbol{R}_b(q_3) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos q_3 & -\sin q_3 \\ 0 & \sin q_3 & \cos q_3 \end{pmatrix} = \begin{pmatrix} c_\alpha & s_\alpha s_3 & s_\alpha c_3 \\ 0 & c_3 & -s_3 \\ -s_\alpha & c_\alpha s_3 & c_\alpha c_3 \end{pmatrix}.$$

We will first compute the angular velocity ${}^0\boldsymbol{\omega}_3$, expressed in the base frame, by superposition of the contributions of $\dot{q}_1$, $\dot{q}_2$, and $\dot{q}_3$ (just as in the derivation of the geometric Jacobian). Consider the single contribution of $\dot{q}_1$, while $\dot{q}_2 = \dot{q}_3 = 0$. When the coordinate q_1 moves (say, upward, i.e., with $\dot{q}_1 > 0$), the same linear velocity is applied to the left end of the connecting bar. This can be decomposed in a velocity component along the bar, which will slightly extend it, and a velocity component along the normal to the bar, which is responsible for its rotation (clockwise when looking at the structure from the front side, and thus counterclockwise when seen from $\boldsymbol{y}_0$). When the bar is tilted by α, the resulting angular velocity contribution due to $\dot{q}_1$ will be

$$ {}^0\boldsymbol{\omega}_{3|\dot{q}_1} = \begin{pmatrix} 0 \\ \dfrac{\cos\alpha}{\delta} \\ 0 \end{pmatrix} \dot{q}_1, \qquad \text{with } \delta = \sqrt{(q_1 - q_2)^2 + (D - 2d)^2}, \tag{2} $$

being δ the actual length of the connecting bar. Indeed, as mentioned in the text, we could neglect the (small) contribution by $(q_1 - q_2)^2$ and have simply $\delta = D - 2d > 0$. Proceeding in the general case, the following relation holds from trigonometry:

$$ \begin{pmatrix} q_1 - q_2 \\ D - 2d \end{pmatrix} = \delta \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}. \tag{3} $$

Therefore, we can rewrite the contribution of $\dot{q}_1$ to ${}^0\boldsymbol{\omega}_3$ as

$$ {}^0\boldsymbol{\omega}_{3|\dot{q}_1} = \begin{pmatrix} 0 \\ \dfrac{\delta\cos\alpha}{\delta^2} \\ 0 \end{pmatrix} \dot{q}_1 = \begin{pmatrix} 0 \\ \dfrac{D - 2d}{(q_1 - q_2)^2 + (D - 2d)^2} \\ 0 \end{pmatrix} \dot{q}_1. $$

The same argument can be used for the contribution of $\dot{q}_2$, except that an opposite sign will result. Adding also the contribution of $\dot{q}_3$ to ${}^0\boldsymbol{\omega}_3$, we obtain finally

$$ {}^0\boldsymbol{\omega}_3 = \begin{pmatrix} 0 \\ \dfrac{D - 2d}{(q_1 - q_2)^2 + (D - 2d)^2} \\ 0 \end{pmatrix} \dot{q}_1 - \begin{pmatrix} 0 \\ \dfrac{D - 2d}{(q_1 - q_2)^2 + (D - 2d)^2} \\ 0 \end{pmatrix} \dot{q}_2 + {}^0\boldsymbol{R}_b \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \dot{q}_3. $$

Therefore, in the frame attached to the bar, it is

$$ \begin{aligned} {}^b\boldsymbol{\omega}_3 = {}^0\boldsymbol{R}_b^T\, {}^0\boldsymbol{\omega}_3 &= {}^0\boldsymbol{R}_b^T \begin{pmatrix} 0 \\ \dfrac{D - 2d}{(q_1 - q_2)^2 + (D - 2d)^2} \\ 0 \end{pmatrix} (\dot{q}_1 - \dot{q}_2) + \begin{pmatrix} \dot{q}_3 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \dot{q}_3 \\ \dfrac{(D - 2d)(\dot{q}_1 - \dot{q}_2)}{(q_1 - q_2)^2 + (D - 2d)^2} \cos q_3 \\ -\dfrac{(D - 2d)(\dot{q}_1 - \dot{q}_2)}{(q_1 - q_2)^2 + (D - 2d)^2} \sin q_3 \end{pmatrix}. \end{aligned} \tag{4} $$

By neglecting instead the presence of $(q_1 - q_2)^2$, we have the *reduced* expression

$$ {}^b\boldsymbol{\omega}_{3,r} = \begin{pmatrix} \dot{q}_3 \\ \dfrac{\dot{q}_1 - \dot{q}_2}{D - 2d} \cos q_3 \\ -\dfrac{\dot{q}_1 - \dot{q}_2}{D - 2d} \sin q_3 \end{pmatrix}. \tag{5} $$

When using eq. (4), the angular kinetic energy $T_{3,a}$ of the bar will be

$$
\begin{aligned}
T_{3,a} = \frac{1}{2}\,{}^b\boldsymbol{\omega}_3^T\,{}^b\boldsymbol{I}_b\,{}^b\boldsymbol{\omega}_3 &= \frac{1}{2}\left(I_x\dot{q}_3^2 + \left(\frac{(D-2d)(\dot{q}_1-\dot{q}_2)}{(q_1-q_2)^2+(D-2d)^2}\right)^2 \left(I_y\cos^2 q_3 + I_z\sin^2 q_3\right)\right)\\
&= \frac{1}{2}\left(I_x\dot{q}_3^2 + I_y\left(\frac{D-2d}{(q_1-q_2)^2+(D-2d)^2}\right)^2 (\dot{q}_1-\dot{q}_2)^2\right),
\end{aligned}
$$

where the last expression has been simplified using the assumption $I_y = I_z$ (indeed, a very relevant inertial property!). If eq. (5) is used instead, the (reduced) angular kinetic energy $T_{3,a|r}$ of the bar will be

$$
T_{3,a|r} = \frac{1}{2}\,{}^b\boldsymbol{\omega}_{3,r}^T\,{}^b\boldsymbol{I}_b\,{}^b\boldsymbol{\omega}_{3,r} = \frac{1}{2}\left(I_x\dot{q}_3^2 + \frac{I_y}{(D-2d)^2}(\dot{q}_1-\dot{q}_2)^2\right).
$$

At this stage, the total kinetic energy of the robot takes the usual form

$$
T = T_1 + T_2 + (T_{3,l} + T_{3,a}) = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}
$$

with the robot inertia matrix given by

$$
\boldsymbol{B}(\boldsymbol{q}) = \boldsymbol{B}' + \boldsymbol{B}''(q_1,q_2) = \begin{pmatrix} m_1 + \frac{m_3}{4} & \frac{m_3}{2} & 0\\ \frac{m_3}{2} & m_2 + \frac{m_3}{4} & 0\\ 0 & 0 & I_x \end{pmatrix} + \begin{pmatrix} I_y\,b(q_1,q_2) & -I_y\,b(q_1,q_2) & 0\\ -I_y\,b(q_1,q_2) & I_y\,b(q_1,q_2) & 0\\ 0 & 0 & 0 \end{pmatrix}, \tag{6}
$$

with the notation

$$
b(q_1,q_2) = \left(\frac{D-2d}{(q_1-q_2)^2+(D-2d)^2}\right)^2.
$$

We note that only the (symmetric) matrix term $\boldsymbol{B}''$ of the positive definite robot inertia matrix is configuration dependent (and depends only on the difference $q_1 - q_2$).

The associated Coriolis and centrifugal term

$$
\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(\boldsymbol{q},\dot{\boldsymbol{q}})\\ c_2(\boldsymbol{q},\dot{\boldsymbol{q}})\\ c_3(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{pmatrix}
$$

is computed as usual through the Christoffel symbols

$$
c_i(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T\boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T\left[\left(\frac{\partial\boldsymbol{b}_i(\boldsymbol{q})}{\partial\boldsymbol{q}}\right) + \left(\frac{\partial\boldsymbol{b}_i(\boldsymbol{q})}{\partial\boldsymbol{q}}\right)^T - \left(\frac{\partial\boldsymbol{B}(\boldsymbol{q})}{\partial q_i}\right)\right]\dot{\boldsymbol{q}}, \qquad i = 1,2,3,
$$

being $\boldsymbol{b}_i(\boldsymbol{q})$ the ith column of matrix $\boldsymbol{B}(\boldsymbol{q})$. Thus

$$
\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\begin{pmatrix} I_y\,c(q_1,q_2) & -I_y\,c(q_1,q_2) & 0\\ -I_y\,c(q_1,q_2) & I_y\,c(q_1,q_2) & 0\\ 0 & 0 & 0 \end{pmatrix},
$$

with

$$c(q_1, q_2) = \frac{\partial b(q_1, q_2)}{\partial q_1} = -\frac{4(D-2d)^2(q_1-q_2)}{((q_1-q_2)^2+(D-2d)^2)^3}$$

and where we used the specular dependence of b on q_1 and q_2. Similarly, it is easy to see that $\boldsymbol{C}_2(\boldsymbol{q}) = -\boldsymbol{C}_1(\boldsymbol{q})$. Moreover, $\boldsymbol{C}_3 = \boldsymbol{O}$. As a result, we obtain

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\begin{pmatrix} I_y\, c(q_1, q_2)\,(\dot{q}_1 - \dot{q}_2)^2 \\ -I_y\, c(q_1, q_2)\,(\dot{q}_1 - \dot{q}_2)^2 \\ 0 \end{pmatrix}. \tag{7}$$

If we had used instead the reduced expression for the kinetic energy of the bar, then

$$T = T_1 + T_2 + (T_{3,l} + T_{3,a|r}) = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{B}\,\dot{\boldsymbol{q}}, \tag{8}$$

and the robot inertia matrix would be constant

$$\boldsymbol{B} = \begin{pmatrix} m_1 + \dfrac{m_3}{4} + \dfrac{I_y}{(D-2d)^2} & \dfrac{m_3}{2} - \dfrac{I_y}{(D-2d)^2} & 0 \\ \dfrac{m_3}{2} - \dfrac{I_y}{(D-2d)^2} & m_2 + \dfrac{m_3}{4} + \dfrac{I_y}{(D-2d)^2} & 0 \\ 0 & 0 & I_x \end{pmatrix}. \tag{9}$$

Accordingly, $\boldsymbol{c} = \boldsymbol{0}$ would follow (no Coriolis and centrifugal terms).

We turn now to the computation of the potential energy due to gravity, $U(\boldsymbol{q}) = U_1 + U_2 + U_3$. For this, since ${}^0\boldsymbol{g} = \begin{pmatrix} 0 & 0 & -g_0 \end{pmatrix}^T$ with $g_0 = 9.81$ [m/s^2], we just need to evaluate the height of the centers of masses. Thus

$$U_1 = m_1 g_0 q_1, \qquad U_2 = m_2 g_0 q_2,$$

while, in view of the assumption on the position of the center of mass of the connecting bar,

$$U_3 = m_3 g_0\, \frac{q_1 + q_2}{2}.$$

Therefore,

$$\boldsymbol{g} = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} \left(m_1 + \dfrac{m_3}{2}\right) g_0 \\ \left(m_2 + \dfrac{m_3}{2}\right) g_0 \\ 0 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ 0 \end{pmatrix}. \tag{10}$$

The gravity term is constant, and so $\partial \boldsymbol{g}/\partial \boldsymbol{q} = \boldsymbol{0}$.

Summarizing, in the general case the robot dynamic model takes the form

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g} = \boldsymbol{u}, \tag{11}$$

with inertia matrix, Coriolis and centrifugal term, and gravity term given respectively by eqs. (6), (7), and (10). In the reduced case, the dynamic model simplifies to

$$\boldsymbol{B}\,\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{u}, \tag{12}$$

with constant inertia matrix given by (9) and gravity term as in (10). The dynamic equations (12) are fully linear.

In order to solve the regulation problem for a desired equilibrium state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$ as requested, we can use a decentralized PD control law with *constant* compensation of gravity (which is anyway constant for all configurations!). Therefore, we have

$$\boldsymbol{u} = \boldsymbol{K}_p(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_d\dot{\boldsymbol{q}} + \boldsymbol{g} = \begin{pmatrix} k_{p1}(q_{d,1} - q_1) - k_{d1}\dot{q}_1 + g_1 \\ k_{p2}(q_{d,2} - q_2) - k_{d2}\dot{q}_2 + g_2 \\ k_{p3}(q_{d,3} - q_3) - k_{d3}\dot{q}_3 \end{pmatrix}. \tag{13}$$

Since the Hessian of the gravitational potential energy is identically zero, there will be no strictly positive lower bound for the (diagonal) elements of $\boldsymbol{K}_p$ in this control law: in order to guarantee global asymptotic stability, the sufficient conditions for the control gains are only $\boldsymbol{K}_p > 0$ and $\boldsymbol{K}_d > 0$. Note that this applies to both dynamic models (11) and (12). The only difference is that for (12), global *exponential* stability will be further obtained since the system dynamics is linear. In this case, the following simple modification of (13)

$$\boldsymbol{u} = \boldsymbol{B}\left(\boldsymbol{K}_p(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_d\dot{\boldsymbol{q}}\right) + \boldsymbol{g},$$

i.e., with non-diagonal but still constant PD gains $\boldsymbol{B}\boldsymbol{K}_p$ and $\boldsymbol{B}\boldsymbol{K}_d$, will also provide a fully decoupled dynamics of the joint errors in the closed-loop system.

For the bonus (or alternative to the above) question, consider the sketch in Fig. 2 where, beside the variables q_1, q_2 and q_3 related to the actuated joints, we have assigned also variables β and γ to the two passive revolute joints[1], and δ to the passive prismatic joint. The purpose of our analysis is to show that the set of variables (β, γ, δ) can always be expressed as a function of (q_1, q_2) —this is similar to what we have already shown in eq. (1) for the angle α. Therefore, the robot configuration can be fully described by the minimal set (q_1, q_2, q_3), which is indeed the chosen set of configuration variables used for the Lagrangian dynamics. Because the closed kinematic chain lies always in the plane $(\boldsymbol{x}_0, \boldsymbol{z}_0)$, the variable q_3 plays no role in the analysis, and will then no longer be considered.

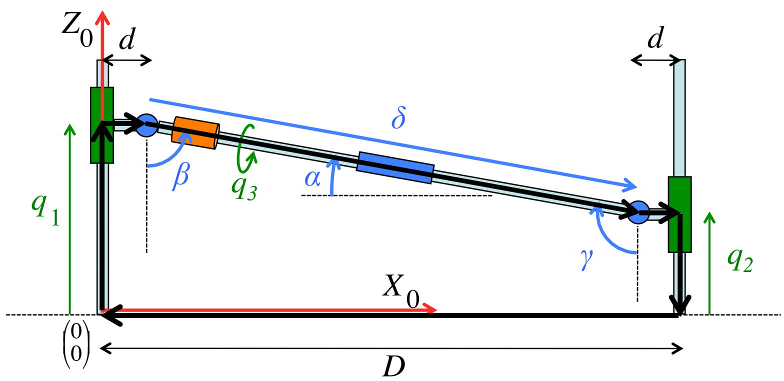

Figure 2: The portal robot as a closed kinematic chain in the plane $(\boldsymbol{x}_0, \boldsymbol{z}_0)$. The passive joints are equipped with the additional variables β, γ, and δ

[1] Note that all angles are conveniently shown in Fig. 2 with an arrow indicating their positive rotation.

The presence of a 'loop closure' (through the floor) in this robotic structure imposes the following kinematic constraints (refer to the vectors shown in black in Fig. 2):

$$\begin{pmatrix} 0 \\ q_1 \end{pmatrix} + \begin{pmatrix} d \\ 0 \end{pmatrix} + \delta \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix} + \begin{pmatrix} d \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -q_2 \end{pmatrix} + \begin{pmatrix} -D \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{14}$$

Moreover, the following relations hold among angles:

$$\beta + \gamma = \pi, \qquad \alpha + \beta = \frac{\pi}{2}. \tag{15}$$

From these, both β and γ can be expressed as a function of α and then, via eq. (1), as a function of $q_1 - q_2$. Equation (14) can be rearranged as

$$\begin{pmatrix} D - 2d \\ q_1 - q_2 \end{pmatrix} = \delta \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix} = \delta \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \tag{16}$$

where the last equality follows from the second relation in (15). This is exactly the trigonometric relation (3). Moreover, squaring and adding the two equations in (16) yields

$$\delta = \sqrt{(q_1 - q_2)^2 + (D - 2d)^2} > 0,$$

which is the variable already defined in (2). Finally, dividing the second equation in (16) by the first one, we obtain

$$\tan\alpha = \frac{q_1 - q_2}{D - 2d},$$

recovering thus the original expression for α introduced in (1).

* * * * *

Robotics II

January 9, 2013

Exercise 1

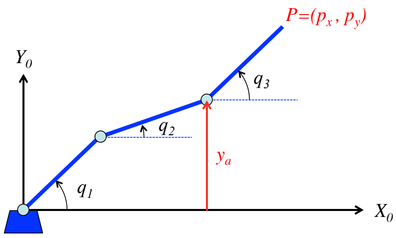

Figure 1: A $3R$ planar robot with unitary link lengths and two sets of task variables

Consider the $3R$ planar robot of Fig. 1, having links of unitary length and with the generalized coordinates defined therein. This robot is redundant for the task of positioning its end-effector at $\boldsymbol{p} = (p_x, p_y)$, as well as for the task of imposing a value to the second link end-point height y_a.

a) For each *separate* task, define the associated task Jacobian and its singularities.

b) Characterize the so-called *algorithmic* singularities (configurations where each task can be executed separately, but not both tasks simultaneously).

c) For the simultaneous execution of both tasks, provide the expression of an inverse differential kinematic solution at the velocity level, based on a *task-priority* strategy that assigns higher priority to the end-effector position task.

Exercise 2

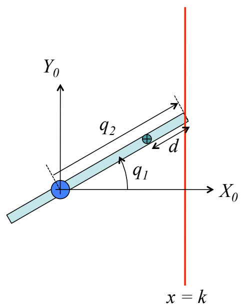

Figure 2: A RP robot moving on a horizontal plane with its end-effector constrained on a line

The end-effector of the RP robot in Fig. 2 is constrained to move on the Cartesian line $x = k$, with $k > 0$. For this operative condition, derive the expression of the *constrained* robot dynamics (in this case, two second-order differential equations, with a dynamically consistent projection matrix acting on forces/torques so as to automatically satisfy the motion constraint in any admissible robot state).

[210 minutes; open books]

Solutions

January 9, 2013

Exercise 1

Being the generalized coordinates q_i ($i = 1, 2, 3$) the absolute angles of the links w.r.t. the $\boldsymbol{x}_0$ axis, the end-effector position is expressed as

$$\boldsymbol{p} = \begin{pmatrix} \cos q_1 + \cos q_2 + \cos q_3 \\ \sin q_1 + \sin q_2 + \sin q_3 \end{pmatrix} = \boldsymbol{f}_1(\boldsymbol{q})$$

The associated task Jacobian is

$$\boldsymbol{J}_1(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_1}{\partial \boldsymbol{q}} = \begin{pmatrix} -\sin q_1 & -\sin q_2 & -\sin q_3 \\ \cos q_1 & \cos q_2 & \cos q_3 \end{pmatrix}$$

and is singular if and only if

$$\sin(q_2 - q_1) = \sin(q_3 - q_2) = 0, \qquad (\Rightarrow\ \sin(q_3 - q_1) = 0) \tag{1}$$

or, in terms of Denavit-Hartenberg relative link angles $\theta_i = q_i - q_{i-1}$ (for $i = 2.3$), when $\sin\theta_2 = \sin\theta_3 = 0$. This occurs only when all three links are folded or stretched along a common radial line originating at the robot base.

The height y_a of the end-point of the second link and its associated task Jacobian are given by

$$y_a = \sin q_1 + \sin q_2 = f_2(\boldsymbol{q}) \quad \Rightarrow \quad \boldsymbol{J}_2(\boldsymbol{q}) = \frac{\partial f_2}{\partial \boldsymbol{q}} = \begin{pmatrix} \cos q_1 & \cos q_2 & 0 \end{pmatrix}.$$

This Jacobian is singular if and only if

$$\cos q_1 = \cos q_2 = 0, \tag{2}$$

namely when the first two links are either folded or stretched *and* the end-point of the second link is on the $\boldsymbol{y}_0$ axis.

When considering the two tasks together, the *Extended* Jacobian is square

$$\boldsymbol{J}_E(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_1(\boldsymbol{q}) \\ \boldsymbol{J}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -\sin q_1 & -\sin q_2 & -\sin q_3 \\ \cos q_1 & \cos q_2 & \cos q_3 \\ \cos q_1 & \cos q_2 & 0 \end{pmatrix}.$$

Algorithmic singularities will occur when both $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are full (row) rank, but

$$\det \boldsymbol{J}_E = -\cos q_3 \cdot \sin(q_2 - q_1) = 0. \tag{3}$$

Comparing eqs. (1–2) with (3), this happens when

- the third link is vertical ($\cos q_3 = 0$), while the first two are not; or,
- the first two links are aligned ($\sin(q_2 - q_1) = 0$) but not vertical, and the third link is not aligned with the first two.

Indeed, the above are only particular conditions for singularity of the Extended Jacobian. In fact, $\boldsymbol{J}_E$ is not invertible as soon as the third link is vertical and/or the first two links are aligned, no matter what is the situation of the other links.

Let $\boldsymbol{v}_d \in \mathbb{R}^2$ be a desired velocity for the robot end-effector and $\dot{y}_{a,d}$ a desired height variation rate for the end-point of the second link. An inverse solution of the form

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_E^{-1}(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{v}_d \\ \dot{y}_{a,d} \end{pmatrix}$$

will blow out as soon as a singularity occurs for $\boldsymbol{J}_E$. A task-priority solution, with the first task (of dimension $m_1 = 2$) of higher priority than the second one (of dimension $m_2 = 1$), is given by

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_1^{\#}(\boldsymbol{q})\, \boldsymbol{v}_d + \left(\boldsymbol{J}_2(\boldsymbol{q}) \left(\boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q}) \boldsymbol{J}_1(\boldsymbol{q}) \right) \right)^{\#} \left(\dot{y}_{a,d} - \boldsymbol{J}_2(\boldsymbol{q}) \boldsymbol{J}_1^{\#}(\boldsymbol{q})\, \boldsymbol{v}_d \right). \tag{4}$$

This will guarantee perfect execution of the first task even when $\boldsymbol{J}_E$ is singular (i.e., eq. (3) holds), provided that eq. (1) is *not* satisfied (in particular, in algorithmic singularities, where eq. (2) is *not* satisfied too).

Using the properties of projection matrices (symmetry and idempotency), and being the matrix $\boldsymbol{J}_2(\boldsymbol{I} - \boldsymbol{J}_1^{\#}\boldsymbol{J}_1)$ a row vector in our case, the solution (4) can also be rewritten as

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_1^{\#}(\boldsymbol{q})\, \boldsymbol{v}_d + \alpha \left(\boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q}) \boldsymbol{J}_1(\boldsymbol{q}) \right) \boldsymbol{J}_2^T(\boldsymbol{q}),$$

with the scalar

$$\alpha = \alpha(\boldsymbol{q}, \boldsymbol{v}_d, \dot{y}_{a,d}) = \frac{\dot{y}_{a,d} - \boldsymbol{J}_2(\boldsymbol{q}) \boldsymbol{J}_1^{\#}(\boldsymbol{q})\, \boldsymbol{v}_d}{\boldsymbol{J}_2(\boldsymbol{q}) \left(\boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q}) \boldsymbol{J}_1(\boldsymbol{q}) \right) \boldsymbol{J}_2^T(\boldsymbol{q})}.$$

Exercise 2

Following the Lagrangian approach, with multipliers $\boldsymbol{\lambda}$ used to weigh the holonomic constraints $\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}$, the dynamic equations (in the absence of gravity) take the form

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u} + \boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{\lambda} \qquad s.t. \quad \boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0},$$

with $\boldsymbol{A}(\boldsymbol{q}) = \partial \boldsymbol{h}(\boldsymbol{q})/\partial \boldsymbol{q}$. By further elaboration, one can eliminate the multipliers (the forces that arise when attempting to violate the constraints) and obtain the so-called *constrained* robot dynamics in the form

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \left(\boldsymbol{I} - \boldsymbol{A}^T(\boldsymbol{q}) \left(\boldsymbol{A}_B^{\#}(\boldsymbol{q}) \right)^T \right) (\boldsymbol{u} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})) - \boldsymbol{B}(\boldsymbol{q}) \boldsymbol{A}_B^{\#}(\boldsymbol{q}) \dot{\boldsymbol{A}}(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

where

$$\boldsymbol{A}_B^{\#}(\boldsymbol{q}) = \boldsymbol{B}^{-1}(\boldsymbol{q}) \boldsymbol{A}^T(\boldsymbol{q}) \left(\boldsymbol{A}(\boldsymbol{q}) \boldsymbol{B}^{-1}(\boldsymbol{q}) \boldsymbol{A}^T(\boldsymbol{q}) \right)^{-1}$$

is the (dynamically consistent) pseudoinverse of $\boldsymbol{A}$, weighted by the robot inertia matrix.

We need thus to provide the robot inertia matrix $\boldsymbol{B}$, the Coriolis and centrifugal vector $\boldsymbol{c}$, the matrix $\boldsymbol{A}$ and its time derivative $\dot{\boldsymbol{A}}$. The kinetic energy[1] is

$$T = T_1 + T_2 = \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2} \left(I_2 \dot{q}_1^2 + m_2 \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} \right).$$

[1] For simplicity, it is assumed that the first link has its center of mass on the axis of the first joint. Otherwise, if the center of mass is at a distance d_{c1}, simply replace I_1 by $I_1 + m_1 d_{c1}^2$ in the following.

Since

$$\boldsymbol{p}_{c2} = \begin{pmatrix} (q_2 - d)\cos q_1 \\ (q_2 - d)\sin q_1 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -(q_2 - d)\sin q_1 \dot{q}_1 + \dot{q}_2 \cos q_1 \\ (q_2 - d)\cos q_1 \dot{q}_1 + \dot{q}_2 \sin q_1 \end{pmatrix},$$

it follows

$$T = \frac{1}{2}\left(I_1 + I_2 + m_2(q_2 - d)^2\right)\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \begin{pmatrix} I_1 + I_2 + m_2(q_2 - d)^2 & 0 \\ 0 & m_2 \end{pmatrix} \dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

From the inertia matrix, using the Christoffel symbols, we obtain

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2m_2(q_2 - d)\dot{q}_1\dot{q}_2 \\ -m_2(q_2 - d)\dot{q}_1^2 \end{pmatrix}.$$

The (scalar) Cartesian constraint on the end-effector is

$$h(\boldsymbol{q}) = q_2 \cos q_1 - k = 0.$$

Thus,

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \end{pmatrix}$$

and

$$\dot{\boldsymbol{A}}(\boldsymbol{q}) = \begin{pmatrix} -\dot{q}_2 \sin q_1 - q_2 \cos q_1 \dot{q}_1 & -\sin q_1 \dot{q}_1 \end{pmatrix}.$$

Since q_2 is never allowed to go to zero (by the constraint $x = k > 0$ on the end-effector), matrix $\boldsymbol{A}$ has always full rank and all expressions in the constrained dynamics hold without singularities. For instance, the dynamically consistent weighted pseudoinverse takes the final expression

$$\boldsymbol{A}_B^{\#}(\boldsymbol{q}) = \frac{m_2(I_1 + I_2 + m_2(q_2 - d)^2)}{I_1 + I_2 + m_2 q_2^2 + m_2 d(d - 2q_2)\cos^2 q_1} \begin{pmatrix} -\dfrac{q_2 \sin q_1}{I_1 + I_2 + m_2(q_2 - d)^2} \\ \dfrac{\cos q_1}{m_2} \end{pmatrix}.$$

* * * * *

Robotics II

February 6, 2013

The end-effector of the RP robot in Fig. 1 is constrained to move on the Cartesian line $x = k$, with $k > 0$.

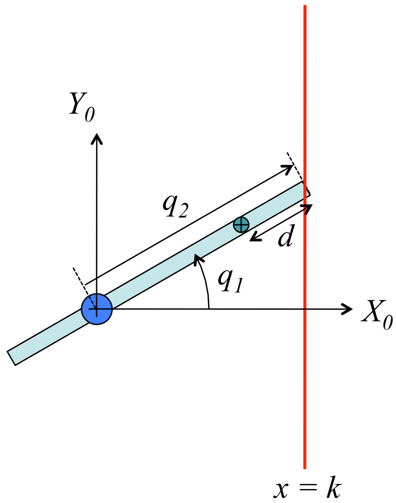

Figure 1: A RP robot moving on a horizontal plane with its end-effector constrained on a line

- Derive the expression of the *reduced* robot dynamics (in this case, a single first-order differential equation), written in terms of pseudoacceleration and automatically satisfying the constraint. Try to provide global validity to this model.
- Design a control law that regulates to constant desired values v_d and λ_d, respectively the tangent velocity and the normal force to the end-effector constraint.

[120 minutes; open books]

Solution

February 6, 2013

Following the Lagrangian approach for a robot with N, with multipliers $\boldsymbol{\lambda}$ used to weigh the M-dimensional holonomic constraints $\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}$, the dynamic equations (in the absence of gravity) take the form

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u} + \boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{\lambda} \qquad s.t. \quad \boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}, \tag{1}$$

with $\boldsymbol{A}(\boldsymbol{q}) = \partial \boldsymbol{h}(\boldsymbol{q})/\partial \boldsymbol{q}$.

The *reduced* dynamic model is obtained by restricting the motion to a R-dimensional configuration space, with $R = N - M$, that is automatically compatible with the constraints $\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}$. The constraints are thus discarded from the formulation. At the same time, it is also possible to eliminate the appearance of the multipliers (i.e., of the forces that arise when attempting to violate the constraints) in the resulting dynamic equations.

We provide first the terms that appear in (1), namely the robot inertia matrix $\boldsymbol{B}$, the Coriolis and centrifugal vector $\boldsymbol{c}$, and the matrix $\boldsymbol{A}$. The kinetic energy[1] is

$$T = T_1 + T_2 = \frac{1}{2} I_1 \dot{q}_1^2 + \frac{1}{2}\left(I_2 \dot{q}_1^2 + m_2 \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2}\right).$$

Since

$$\boldsymbol{p}_{c2} = \begin{pmatrix} (q_2 - d)\cos q_1 \\ (q_2 - d)\sin q_1 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -(q_2 - d)\sin q_1 \dot{q}_1 + \dot{q}_2 \cos q_1 \\ (q_2 - d)\cos q_1 \dot{q}_1 + \dot{q}_2 \sin q_1 \end{pmatrix},$$

it follows

$$T = \frac{1}{2}\left(I_1 + I_2 + m_2(q_2 - d)^2\right)\dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \begin{pmatrix} I_1 + I_2 + m_2(q_2 - d)^2 & 0 \\ 0 & m_2 \end{pmatrix} \dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

From the inertia matrix, using the Christoffel symbols, we obtain

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2m_2(q_2 - d)\dot{q}_1 \dot{q}_2 \\ -m_2(q_2 - d)\dot{q}_1^2 \end{pmatrix}.$$

The (scalar) Cartesian constraint on the end-effector is

$$h(\boldsymbol{q}) = q_2 \cos q_1 - k = 0.$$

Thus,

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \end{pmatrix}.$$

The reduction of the dynamics proceeds then as follows. In the present case, it is $N = 2$ and $M = 1$, so that $R = 1$. Define the matrix $\boldsymbol{D}(\boldsymbol{q})$ (a row in our case) such that

$$\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ d_1(\boldsymbol{q}) & d_2(\boldsymbol{q}) \end{pmatrix} \quad \text{is } \textit{nonsingular}.$$

[1] For simplicity, it is assumed that the first link has its center of mass on the axis of the first joint. Otherwise, if the center of mass is at a distance d_{c1}, simply replace I_1 by $I_1 + m_1 d_{c1}^2$ in the following.

A good choice is

$$\boldsymbol{D}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 & \sin q_1 \end{pmatrix} \quad \Rightarrow \quad \det \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = -q_2.$$

Since the end-effector is constrained to live on $x = k > 0$, it will always be $q_2 \neq 0$. Thus, the requested non-singularity of the matrix holds globally as long as the constraint is enforced. This implies that the following derivations will lead also to a globally defined reduced model.

Define then the pseudovelocity v (a scalar) and its derivative $\dot{v}$ (the pseudoacceleration) as

$$\begin{aligned} v &= \boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}} = q_2 \cos q_1 \dot{q}_1 + \sin q_1 \dot{q}_2, \\ \dot{v} &= \boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{D}}(\boldsymbol{q})\dot{\boldsymbol{q}} = q_2 \cos q_1 \ddot{q}_1 + \sin q_1 \ddot{q}_2 - q_2 \sin q_1 \dot{q}_1^2 + 2 \cos q_1 \dot{q}_1 \dot{q}_2. \end{aligned}$$

To invert these relations, define

$$\begin{pmatrix} \boldsymbol{E}(\boldsymbol{q}) & \boldsymbol{F}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix}^{-1} = \begin{pmatrix} -\dfrac{\sin q_1}{q_2} & \dfrac{\cos q_1}{q_2} \\ \cos q_1 & \sin q_1 \end{pmatrix}.$$

We obtain then

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{F}(\boldsymbol{q})\, v = \begin{pmatrix} \dfrac{\cos q_1}{q_2} \\ \sin q_1 \end{pmatrix} v, \\ \ddot{\boldsymbol{q}} &= \boldsymbol{F}(\boldsymbol{q})\, \dot{v} + \dot{\boldsymbol{F}}(\boldsymbol{q}) v = \begin{pmatrix} \dfrac{\cos q_1}{q_2} \\ \sin q_1 \end{pmatrix} \dot{v} + \begin{pmatrix} -\left(\dfrac{\sin q_1 \dot{q}_1}{q_2} + \dfrac{\cos q_1 \dot{q}_2}{q_2^2} \right) \\ \cos q_1 \dot{q}_1 \end{pmatrix} v. \end{aligned} \tag{2}$$

Based on their definitions, the matrix relations $\boldsymbol{AF} = \boldsymbol{O}$ and $\boldsymbol{AE} = \boldsymbol{I}$ hold for all $\boldsymbol{q}$. Premultiplying (1) by $\boldsymbol{F}^T(\boldsymbol{q})$ and substituting the acceleration $\ddot{\boldsymbol{q}}$ from (2) yields

$$\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\, \dot{v} = \boldsymbol{F}^T(\boldsymbol{q}) \left(\boldsymbol{u} - \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{F}}(\boldsymbol{q}) v - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \right), \tag{3}$$

which is the reduced dynamics that we were looking for —in fact, a scalar first-order differential equation in v. All terms in (3) have been defined, but we can write more explicitly the leading scalar (a pseudoinertia)

$$\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = \left(I_1 + I_2 + m_2 (q_2 - d)^2 \right) \frac{\cos^2 q_1}{q_2^2} + m_2 \sin^2 q_1.$$

Similarly, by premultiplying (1) by $\boldsymbol{E}^T(\boldsymbol{q})$ (a row) we can isolate the scalar multiplier λ. Substituting the acceleration $\ddot{\boldsymbol{q}}$ from (2) yields

$$\lambda = \boldsymbol{E}^T(\boldsymbol{q}) \left(\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\dot{v} + \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{F}}(\boldsymbol{q}) v + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{u} \right). \tag{4}$$

The control task is defined at the end-effector level, and requires the stabilization to zero of the error $e_v = v_d - v$ for the tangential velocity and of the error $e_\lambda = \lambda_d - \lambda$ for the normal force.

The tangential and normal directions are indeed referred to the geometry of the constraint. This hybrid control task is easily achieved in a linear and decoupled way by using a preliminary *feedback linearization* law for the joint-space control input $\boldsymbol{u} \in \mathbb{R}^2$. Defining in eqs. (3) and (4)

$$\boldsymbol{u} = \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{F}}(\boldsymbol{q})v + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{B}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})u_1 - \boldsymbol{A}^T(\boldsymbol{q})u_2$$

leads simultaneously to

$$\dot{v} = u_1, \qquad \lambda = u_2.$$

The control design is completed by choosing

$$u_1 = k_1\, e_v, \qquad u_2 = \lambda_d + k_2 \int_0^t e_\lambda(\tau) d\tau,$$

with $k_1 > 0$, $k_2 > 0$. Beside the feedforward term (λ_d in the force loop, while $\dot{v}_d = 0$ in the velocity loop for constant desired velocity), we have respectively a proportional action on the velocity error and an integral action on the force error. The two error dynamics will be

$$\dot{e}_v + k_1 e_v = 0, \qquad e_\lambda + k_2 \int e_\lambda = 0,$$

both exponentially converging to zero.

* * * * *

Robotics II

July 15, 2013

Consider a planar 3R robot on a horizontal plane, having link lengths ℓ_1, ℓ_2, and ℓ_3. Figure 1 shows three different situations. In each of them, the robot is subject to:

- a *single unknown* external force $\boldsymbol{F}_i = \begin{pmatrix} F_{xi} & F_{yi} \end{pmatrix}^T$, of arbitrary direction and intensity in the plane, applied to a point of the first, second, or third link, at a (*possibly, unknown*) distance ℓ_{ci} from the link i base (with $i = 1, 2, 3$, respectively);
- an associated *known* (measured, or computed by a controller) joint torque $\boldsymbol{\tau} = \begin{pmatrix} \tau_1 & \tau_2 & \tau_3 \end{pmatrix}^T$ that keeps the robot in a static equilibrium $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$, as *measured* by encoders.

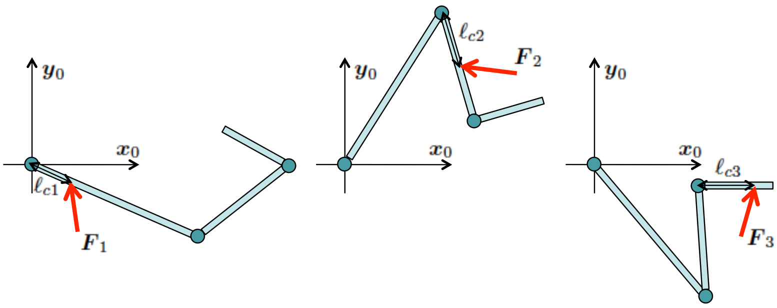

Figure 1: A planar 3R robot subject to an unknown external force $\boldsymbol{F}_i$ applied to the first ($i = 1$, left), second ($i = 2$, center), or third link ($i = 3$, right)

Analyze **for each** of the above three generic situations *if* and *how:*

1. one can identify which link is subject to the external force;
2. knowing the distance ℓ_{ci} of the application point along the link (which can be obtained by means of an external camera, a Kinect, or using distributed tactile sensing), one can *completely* estimate $\boldsymbol{F}_i$, e.g., its direction and intensity;
3. one can estimate $\boldsymbol{F}_i$ *and* the distance $\ell_{c,i}$ of the application point along the link, without any external/extra sensing;
4. the presence of gravity (with the robot being in a vertical plane) makes a difference for the above problems.

Verify the analysis for a 3R robot in a horizontal plane, with the numerical data

$$\ell_1 = 0.5 \text{ [m]}, \quad \ell_2 = 0.3 \text{ [m]}, \quad \ell_3 = 0.2 \text{ [m]}, \qquad \boldsymbol{q} = \begin{pmatrix} 45^\circ & -90^\circ & 60^\circ \end{pmatrix}^T \text{ (in D-H convention)},$$

and for the following two cases of known equilibrium joint torques:

$$a)\ \boldsymbol{\tau}_a = \begin{pmatrix} -0.25 & -0.75 & 0 \end{pmatrix}^T \text{ [Nm]}, \qquad b)\ \boldsymbol{\tau}_b = \begin{pmatrix} 0.7585 & -0.2995 & 0.2000 \end{pmatrix}^T \text{ [Nm]}.$$

Estimate the applied external force in case *a)* and *b)*, respectively. Following the outcome of your analysis, try to work without any a priori knowledge of the application point of the external force. If a case turns out to be under-determined, choose the application point at the *link midpoint.*

[120 minutes; open books]

Solution

July 15, 2013

In the absence of gravity, the joint torque that balances at an equilibrium $\boldsymbol{q}$ a force applied to a point along the robot structure having position $\boldsymbol{p}_c = \boldsymbol{f}_c(\boldsymbol{q})$ is given by

$$\boldsymbol{\tau} = -\left(\frac{\partial \boldsymbol{f}_c}{\partial \boldsymbol{q}}\right)^T \boldsymbol{F} = -\boldsymbol{J}_c^T(\boldsymbol{q})\,\boldsymbol{F}.$$

Planar forces $\boldsymbol{F} \in \mathbb{R}^2$ are expressed here in the base frame $(\boldsymbol{x}_0, \boldsymbol{y}_0)$. From the position of the contact point $\boldsymbol{p}_c \in \mathbb{R}^2$, we can derive the *contact Jacobian* associated to a force $\boldsymbol{F}_i$ acting on link i at a distance ℓ_{ci} from its base, for $i = 1, 2, 3$ (three cases).

For a force $\boldsymbol{F}_1$ acting on the first link, with $\ell_{c1} \in (0, \ell_1]$:

$$\boldsymbol{J}_{c1} = \begin{pmatrix} -\ell_{c1}\sin q_1 & 0 & 0 \\ \ell_{c1}\cos q_1 & 0 & 0 \end{pmatrix}.$$

For a force $\boldsymbol{F}_2$ acting on the second link, with $\ell_{c2} \in (0, \ell_2]$:

$$\boldsymbol{J}_{c2} = \begin{pmatrix} -\ell_1\sin q_1 - \ell_{c2}\sin(q_1+q_2) & -\ell_{c2}\sin(q_1+q_2) & 0 \\ \ell_1\cos q_1 + \ell_{c2}\cos(q_1+q_2) & \ell_{c2}\cos(q_1+q_2) & 0 \end{pmatrix}.$$

For a force $\boldsymbol{F}_3$ acting on the third link, with $\ell_{c3} \in (0, \ell_3]$:

$$\boldsymbol{J}_{c3} = \begin{pmatrix} -\ell_1 s_1 - \ell_2 s_{12} - \ell_{c3} s_{123} & -\ell_2 s_{12} - \ell_{c3} s_{123} & -\ell_{c3} s_{123} \\ \ell_1 c_1 + \ell_2 c_{12} + \ell_{c3} c_{123} & \ell_2 c_{12} + \ell_{c3} c_{123} & \ell_{c3} c_{123} \end{pmatrix},$$

where we used the usual compact notation, e.g., $s_{123} = \sin(q_1 + q_2 + q_3)$.

In order to have a better insight on the contact forces that are felt at the robot joints, it is convenient to express these forces in the local frame attached to each link, i.e., according to their tangential and normal components w.r.t. the geometric link. Since we are dealing with a purely planar problem, the planar rotation matrices of interest are

$${}^0\boldsymbol{R}_1 = \begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix}, \quad {}^0\boldsymbol{R}_2 = \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix}, \quad {}^0\boldsymbol{R}_3 = \begin{pmatrix} c_{123} & -s_{123} \\ s_{123} & c_{123} \end{pmatrix}.$$

we have

$$\boldsymbol{F}_i = {}^0\boldsymbol{R}_i\,{}^i\boldsymbol{F}_i \quad \Rightarrow \quad \boldsymbol{\tau}_i = -\boldsymbol{J}_{ci}^T(\boldsymbol{q})\,\boldsymbol{F}_i = -\boldsymbol{J}_{ci}^T(\boldsymbol{q}){}^0\boldsymbol{R}_i\,{}^i\boldsymbol{F}_i = -\left({}^0\boldsymbol{R}_i^T\boldsymbol{J}_{ci}(\boldsymbol{q})\right)^T {}^i\boldsymbol{F}_i = -{}^i\boldsymbol{J}_{ci}^T(\boldsymbol{q})\,{}^i\boldsymbol{F}_i.$$

Once expressed in the local link frame, the contact Jacobians ${}^i\boldsymbol{J}_{ci}(\boldsymbol{q})$ for the considered cases are:

$${}^1\boldsymbol{J}_{c1} = {}^0\boldsymbol{R}_1^T\boldsymbol{J}_{c1}(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ \ell_{c1} & 0 & 0 \end{pmatrix}, \qquad \textit{always} \text{ of rank } 1;$$

$${}^2\boldsymbol{J}_{c2} = {}^0\boldsymbol{R}_2^T\boldsymbol{J}_{c2}(\boldsymbol{q}) = \begin{pmatrix} \ell_1 s_2 & 0 & 0 \\ \ell_1 c_2 + \ell_{c2} & \ell_{c2} & 0 \end{pmatrix}, \qquad \text{of rank 2 } \textit{if and only if } |s_2| \neq 0;$$

$${}^3\boldsymbol{J}_{c3} = {}^0\boldsymbol{R}_3^T\boldsymbol{J}_{c3}(\boldsymbol{q}) = \begin{pmatrix} \ell_1 s_{23} + \ell_2 s_3 & \ell_2 s_3 & 0 \\ \ell_1 c_{23} + \ell_2 c_3 + \ell_{c3} & \ell_2 c_3 + \ell_{c3} & \ell_{c3} \end{pmatrix}, \text{ of rank 2 } \textit{if and only if } s_2^2 + s_3^2 \neq 0.$$

Therefore, we have the following explicit equations at steady state.

For a force ${}^1\boldsymbol{F}_1 = \begin{pmatrix} {}^1F_{1x} & {}^1F_{1y} \end{pmatrix}^T$ acting on the first link:

$$\begin{aligned} \tau_{1,1} &= -\ell_{c1}\, {}^1F_{1y}, \\ \tau_{1,2} &= 0, \\ \tau_{1,3} &= 0. \end{aligned} \tag{1}$$

For a force ${}^2\boldsymbol{F}_2 = \begin{pmatrix} {}^2F_{2x} & {}^2F_{2y} \end{pmatrix}^T$ acting on the second link:

$$\begin{aligned} \tau_{2,1} &= -\ell_1 s_2 {}^2F_{2x} - (\ell_1 c_2 + \ell_{c2})\, {}^2F_{2y} = -\ell_1 \left(s_2\, {}^2F_{2x} + c_2\, {}^2F_{2y} \right) + \tau_{2,2}, \\ \tau_{2,2} &= -\ell_{c2}\, {}^2F_{2y}, \\ \tau_{2,3} &= 0. \end{aligned} \tag{2}$$

For a force ${}^3\boldsymbol{F}_3 = \begin{pmatrix} {}^3F_{3x} & {}^3F_{3y} \end{pmatrix}^T$ acting on the third link:

$$\begin{aligned} \tau_{3,1} &= -\left(\ell_1 s_{23} + \ell_2 s_3\right) {}^3F_{3x} - \left(\ell_1 c_{23} + \ell_2 c_3 + \ell_{c3}\right) {}^3F_{3y} \\ &= -\ell_1 \left(s_{23}\, {}^3F_{3x} + c_{23}\, {}^3F_{3y} \right) + \tau_{3,2}, \\ \tau_{3,2} &= -\left(\ell_2 s_3\, {}^3F_{3x} + \left(\ell_3 c_3 + \ell_{c3}\right) {}^3F_{3y} \right) = -\ell_2 \left(s_3\, {}^3F_{3x} + c_3\, {}^3F_{3y} \right) + \tau_{3,3}, \\ \tau_{3,3} &= -\ell_{c3}\, {}^3F_{3y}. \end{aligned} \tag{3}$$

Inner recursions from the outer to the inner joints have been used to simplify the expressions. Based on the above, the following series of observations can be made.

1. Identification of which link is subject to the contact force is made using the components of the joint torque vector $\boldsymbol{\tau}_i$, based on the following cascaded (generic) rule:

$$\begin{aligned} \tau_{i,3} = \tau_{i,2} = 0 \quad &\Rightarrow \quad i = 1, \text{ link 1 is involved} \\ \tau_{i,3} = 0, \tau_{i,2} \neq 0 \quad &\Rightarrow \quad i = 2, \text{ link 2 is involved} \\ \text{else} \quad &\Rightarrow \quad i = 3, \text{ link 3 is involved.} \end{aligned}$$

   Note that, if a force is applied at the base of link i (corresponding to the exact location of joint i), it is then attributed to the tip of the previous link $i-1$ (with $\ell_{c,i-1} = \ell_{i-1}$), since $\ell_{ci} \neq 0$ by definition.

2. Knowledge of the joint torque vector $\boldsymbol{\tau}_i$ at an equilibrium configuration $\boldsymbol{q}$ is obtained in one of the following alternative, but equivalent ways (however, note the signs!):

   - from the static measurement by a joint torque sensor, $\boldsymbol{\tau}_m = -\boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c = \boldsymbol{\tau}_i$;
   - as the steady-state value of the residual vector $\boldsymbol{r}$ generated in response to a collision with a constant contact force $\boldsymbol{F}_c$, $\boldsymbol{r} = \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c = -\boldsymbol{\tau}_i$;
   - as the steady-state output of a feedback controller, e.g., with a regulation law designed for a desired position $\boldsymbol{q}_d$ and including a proportional term, $\boldsymbol{\tau}_c = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) = -\boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c = \boldsymbol{\tau}_i$.

3. Since the rank of matrix $\boldsymbol{J}_{c1}$ (and of ${}^1\boldsymbol{J}_{c1}$) is always one, we can *never* estimate completely a contact force acting on link 1. As intuition suggests, from eqs. (1) we can only estimate the force component ${}^1F_{1y}$ that is normal to the link, once we know its application point, as

$${}^1F_{1y} = -\frac{\tau_{1,1}}{\ell_{c1}},$$

whereas ${}^1F_{1x}$ can be arbitrary and is balanced by the internal reaction force of the link structure.

4. When matrix $\boldsymbol{J}_{c2}$ (${}^2\boldsymbol{J}_{c2}$) has full rank, by using eqs. (2) we can estimate completely the contact force $\boldsymbol{F}_2$ acting on link 2, *provided* that ℓ_{c2} is known, as

$$ {}^2F_{2y} = -\frac{\tau_{2,2}}{\ell_{c2}}, \qquad {}^2F_{2x} = -\frac{1}{\ell_1 s_2}\left(\tau_{2,1} - (\ell_1 c_2 + \ell_{c2})\frac{\tau_{2,2}}{\ell_{c2}}\right). $$

To recover the expression of the contact force in the base frame, we use $\boldsymbol{F}_2 = {}^0\boldsymbol{R}_2\, {}^2\boldsymbol{F}_2$.

5. When the second link is aligned with the first one ($s_2 = 0$), $\boldsymbol{J}_{c2}$ loses rank and we recover a similar situation to that in item 3, namely ${}^2F_{2x}$ cannot be estimated. However, in this case the two data $\tau_{2,1}$ and $\tau_{2,2}$ can be used to estimate *both* ${}^2F_{2y}$ and ℓ_{c2} as

$$ {}^2F_{2y} = \pm\,\frac{\tau_{2,1} - \tau_{2,2}}{\ell_1}, \qquad \ell_{c2} = -\frac{\tau_{2,2}}{{}^2F_{2y}}, $$

where the sign '$-$' corresponds to $q_2 = 0$ and the sign '$+$' to $q_2 = \pi$.

6. When matrix ${}^3\boldsymbol{J}_{c3}$ (or $\boldsymbol{J}_{c3}$) has full rank and ℓ_{c3} is known, by using eqs. (3) we can estimate completely the contact force $\boldsymbol{F}_3$ acting on link 3 as

$$ \begin{aligned} {}^3F_{3y} &= -\frac{\tau_{3,3}}{\ell_{c3}}, \\ {}^3F_{3x} &= -\frac{1}{\ell_2 s_3}\left(\tau_{3,2} - (\ell_2 c_3 + \ell_{c3})\frac{\tau_{3,3}}{\ell_{c3}}\right), \qquad \text{if } s_3 \neq 0 \\ \text{or} \qquad {}^3F_{3x} &= -\frac{1}{\ell_2 s_{23}}\left(\tau_{3,1} - \tau_{3,2} - \ell_1 c_{23}\frac{\tau_{3,3}}{\ell_{c3}}\right), \qquad \text{if } s_3 = 0, \text{ but } s_{23} = \pm\, s_2 \neq 0. \end{aligned} $$

To recover the expression of the contact force in the base frame, we use $\boldsymbol{F}_3 = {}^0\boldsymbol{R}_3\, {}^3\boldsymbol{F}_3$. When the robot arm is fully stretched or folded, we can proceed as in item 5 and identify from $\boldsymbol{\tau}_3$ both ${}^3F_{3y}$ and ℓ_{c3}, but not ${}^3F_{3x}$.

7. One interesting feature of having three independent information from the joint torque vector in case of contact on the third link, is that the estimation of the force $\boldsymbol{F}_3$ can be performed even without knowing ℓ_{c3}. In fact, the value ℓ_{c3} has disappeared in the recursive expressions of the first two equations in (3). Therefore, we can evaluate

$$ {}^3\boldsymbol{F}_3 = \begin{pmatrix} {}^3F_{3x} \\ {}^3F_{3y} \end{pmatrix} = -\begin{pmatrix} \ell_1 s_{23} & \ell_1 c_{23} \\ \ell_2 s_3 & \ell_2 c_3 \end{pmatrix}^{-1} \begin{pmatrix} \tau_{3,1} - \tau_{3,2} \\ \tau_{3,2} - \tau_{3,3} \end{pmatrix}, $$

provided that $s_2 \neq 0$ holds (for the invertibility of the coefficient matrix), and then complete the analysis by using the third equation in (3)

$$ \ell_{c3} = -\frac{\tau_{3,3}}{{}^3F_{3y}}. $$

8. The results in the above items 3, 4, and 6 can all be obtained by using the *pseudoinverse* of the associated contact Jacobian, independently from its rank,

$$ \hat{\boldsymbol{F}}_i = -\left(\boldsymbol{J}_{ci}^T(\boldsymbol{q})\right)^{\#}\boldsymbol{\tau}_i \qquad \text{or} \qquad {}^i\hat{\boldsymbol{F}}_i = -\left({}^i\boldsymbol{J}_{ci}^T(\boldsymbol{q})\right)^{\#}\boldsymbol{\tau}_i, \qquad \text{for } i = 1, 2, 3. $$

The 'hat' has been added to express the fact that the estimation may not be complete (e.g., in the contact on the first link, or in the other two cases when the contact Jacobian loses rank). On the other hand, the estimation result in item 7 is obtained by direct inspection of the equations.

9. The presence of a gravity term $\boldsymbol{g}(\boldsymbol{q})$ in the robot dynamics does not change the picture substantially. The main difference is that the joint torque $\boldsymbol{\tau}$ should be replaced in all above formulas by $\boldsymbol{\tau} - \boldsymbol{g}(\boldsymbol{q})$, since at the equilibrium

$$\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}.$$

For the two assigned numerical problems, we have the following results. We note first that the given robot configuration $\bar{\boldsymbol{q}} = \begin{pmatrix} \pi/4 & -\pi/2 & \pi/3 \end{pmatrix}^T$ [rad] does not lead to singularity problems.

a) Contact occurs on the second link (see item 1) and eqs. (2) apply. Since knowledge of the application point is necessary in this case, we set $\ell_{c2} = \ell_2/2 = 0.15$ [m]. The contact Jacobian is then

$$\boldsymbol{J}_{c2}(\bar{\boldsymbol{q}}) = \begin{pmatrix} -0.2475 & 0.1061 & 0 \\ 0.4596 & 0.1061 & 0 \end{pmatrix}.$$

Thus,

$$\boldsymbol{F}_2 = -\left(\boldsymbol{J}_{c2}^T(\bar{\boldsymbol{q}})\right)^{\#}\boldsymbol{\tau}_a = \begin{pmatrix} 4.242 \\ 2.8284 \end{pmatrix}, \quad {}^2\boldsymbol{F}_2 = {}^0\boldsymbol{R}_2^T(\bar{\boldsymbol{q}})\boldsymbol{F}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \quad \text{both expressed in [N].}$$

b) Contact occurs on the third link and eqs. (3) apply. Solving for contact force and distance of its application point from the link base yields

$$\boldsymbol{F}_3 = \begin{pmatrix} 2.6736 \\ -0.3189 \end{pmatrix}, \quad {}^3\boldsymbol{F}_3 = \begin{pmatrix} 2.5 \\ -1 \end{pmatrix}, \quad \text{both expressed in [N];} \quad \ell_{c3} = 0.2\,\text{[m] (i.e., at the tip).}$$

Using the computed ℓ_{c3}, the resulting contact Jacobian is then

$$\boldsymbol{J}_{c3}(\bar{\boldsymbol{q}}) = \begin{pmatrix} 0.1932 & 0.1604 & -0.0518 \\ 0.7589 & 0.4053 & 0.1932 \end{pmatrix}.$$

Finally, one can verify that $\boldsymbol{F}_3 = -\left(\boldsymbol{J}_{c3}^T(\bar{\boldsymbol{q}})\right)^{\#}\boldsymbol{\tau}_b$.

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Robotics 2 - Midterm Test

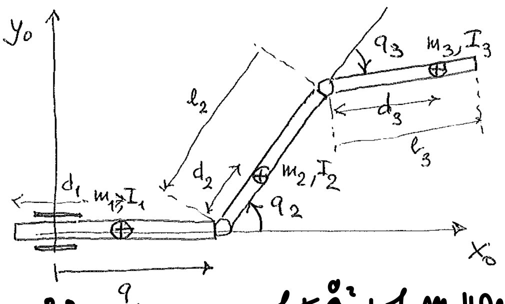
April 13, 2016

Exercise 1

For the PRR planar robot in Fig. 1, determine the symbolic expression of the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ and of the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$. Use the generalized coordinates and the scalar parameters shown in the figure.

$$P_{c2} = \begin{pmatrix} q_1 + d_2 C_2 \\ d_2 S_2 \end{pmatrix}$$

$$\dot{P}_{c2} = \begin{pmatrix} \dot{q}_1 - S_2 d_2 \dot{q}_2 \\ d_2 C_2 \dot{q}_2 \end{pmatrix}$$



$$T_2 = \frac{1}{2} I_2 \dot{q}_2^2 + \frac{1}{2} m_2 \|v_{c2}\|^2$$

$$\|v_{c2}\|^2 = \dot{q}_1^2 + d_2^2 \dot{q}_2^2 - 2 S_2 d_2 \dot{q}_1 \dot{q}_2$$

Figure 1: A planar PRR robot

Exercise 2

The 4R planar robot in Fig. 2 moves under gravity. For each link, the center of mass lies on its longitudinal axis of symmetry, at a generic distance from the driving joint. Determine: *i)* the expression of the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ in the robot dynamic model; *ii)* all equilibrium configurations of the robot (i.e., all $\boldsymbol{q}_e$ such that $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$; *iii)* a linear parametrization of the gravity vector in the form $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}_G(\boldsymbol{q}) \boldsymbol{a}_G$; the particular location of the center of masses of the links such that the gravity vector vanishes (i.e., $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{0}$, for all $\boldsymbol{q}$).

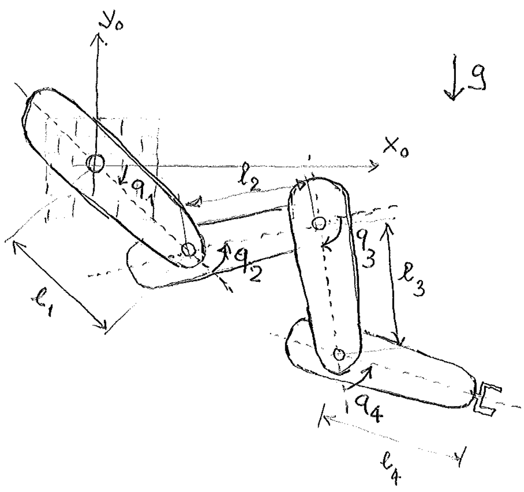


Figure 2: A 4R planar robot under gravity

Exercise 3

The 4R planar robot with all links of equal length ℓ in Fig. 3 needs to realize a motion task defined by a desired linear velocity $\boldsymbol{v}_d$ for its end-effector position $\boldsymbol{p}_e$ and by a desired angular velocity $\dot{\phi}_d$ for the orientation ϕ of its end-effector frame. Characterize first all the singular configurations of the robot for this specific task.

Assume then $\ell = 0.5$ [m], $\boldsymbol{q} = (0\ 0\ \pi/2\ 0)$, $\boldsymbol{v}_d = (1\ 0)$ [m/s], and $\dot{\phi}_d = 0.5$ [rad/s]. Moreover, the joints have limited motion range, i.e., $q_i \in [-2, 2]$ [rad], for $i = 1, \ldots, 4$. Determine the joint velocity $\dot{\boldsymbol{q}}$ that realizes the desired task while decreasing instantaneously the objective function that measures the distance from the midpoint of the joint ranges, i.e., in the form

$$H_{range}(\boldsymbol{q}) = \frac{1}{2N} \sum_{i=1}^{N} \left(\frac{q_i - \bar{q}_i}{q_{M,i} - q_{m,i}} \right)^2 .$$

PROJECTED GRADIENT

$\dot{q} = J^{\#} \dot{r} + (I - J^{\#} J) \dot{q}_0$

with $\dot{q}_0 = -\nabla_q H(q)$

while executing

$\dot{r} = \begin{pmatrix} v_d \\ \dot{\phi}_d \end{pmatrix}$ we increase

$-H(q)$

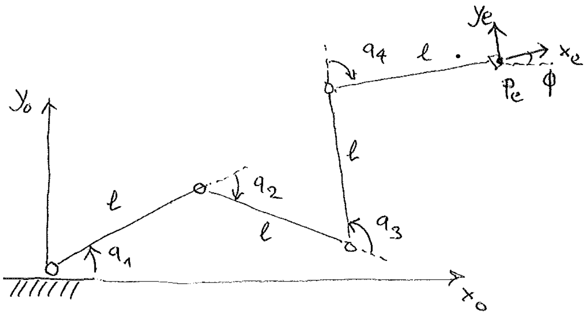


Figure 3: The kinematic skeleton of a planar 4R robot

[150 minutes; open books]

Solution

April 13, 2016

Exercise 1

Since the motion is planar, we will use two-dimensional position and velocity vectors (in the $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ plane) and just the z-component of angular velocities. Also, the usual shorthand notation is adopted for trigonometric quantities, e.g., $s_2 = \sin q_2$, $c_{23} = \cos(q_2 + q_3)$.

Kinetic energy

For link 1, we have (the position of the center of mass on link 1, i.e., d_1, is irrelevant)

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2.$$

For link 2, we compute first the position of the center of mass and its velocity,

$$\boldsymbol{p}_{c2} = \begin{pmatrix} q_1 + d_2 c_2 \\ d_2 s_2 \end{pmatrix} \quad \rightarrow \quad \boldsymbol{v}_{c2} = \begin{pmatrix} \dot{q}_1 - d_2 s_2 \dot{q}_2 \\ d_2 c_2 \dot{q}_2 \end{pmatrix},$$

and then

$$\|\boldsymbol{v}_{c2}\|^2 = \dot{q}_1^2 + d_2^2 \dot{q}_2^2 - 2 d_2 s_2 \dot{q}_1 \dot{q}_2.$$

Since $\omega_{2z} = \dot{q}_2$, we obtain

$$T_2 = \frac{1}{2} m_2 \left(\dot{q}_1^2 + d_2^2 \dot{q}_2^2 - 2 d_2 s_2 \dot{q}_1 \dot{q}_2 \right) + \frac{1}{2} I_2 \dot{q}_2^2.$$

Similarly, for link 3

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_1 + \ell_2 c_2 + d_3 c_{23} \\ \ell_2 s_2 + d_3 s_{23} \end{pmatrix} \quad \rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} \dot{q}_1 - \ell_2 s_2 \dot{q}_2 - d_3 s_{23} (\dot{q}_2 + \dot{q}_3) \\ \ell_2 c_2 \dot{q}_2 + d_3 c_{23} (\dot{q}_2 + \dot{q}_3) \end{pmatrix},$$

and then

$$\|\boldsymbol{v}_{c3}\|^2 = \dot{q}_1^2 + \ell_2^2 \dot{q}_2^2 + d_3 \left(\dot{q}_2 + \dot{q}_3 \right)^2 - 2 \ell_2 s_2 \dot{q}_1 \dot{q}_2 - 2 d_3 s_{23} \dot{q}_1 (\dot{q}_2 + \dot{q}_3) + 2 \ell_2 d_3 \left(s_2 s_{23} + c_2 c_{23} \right) \dot{q}_2 (\dot{q}_2 + \dot{q}_3).$$

Being $\omega_{3z} = \dot{q}_2 + \dot{q}_3$, we obtain (after trigonometric simplification)

$$T_3 = \frac{1}{2} m_3 \left(\dot{q}_1^2 + \ell_2^2 \dot{q}_2^2 + d_3 \left(\dot{q}_2 + \dot{q}_3 \right)^2 - 2 \ell_2 s_2 \dot{q}_1 \dot{q}_2 - 2 d_3 s_{23} \dot{q}_1 (\dot{q}_2 + \dot{q}_3) + 2 \ell_2 d_3 c_3 \dot{q}_2 (\dot{q}_2 + \dot{q}_3) \right) + \frac{1}{2} I_3 \left(\dot{q}_2 + \dot{q}_3 \right)^2.$$

Robot inertia matrix

From

$$T = \sum_{i=1}^{3} T_i = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q}) \dot{\boldsymbol{q}},$$

we obtain the (symmetric) elements $b_{ij} = b_{ji}$ of the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ as

$$\begin{aligned}
b_{11} &= m_1 + m_2 + m_3 =: a_1 \\
b_{22} &= I_2 + m_2 d_2^2 + I_3 + m_3 d_3^2 + m_3 \ell_2^2 + 2 m_3 \ell_2 d_3 c_3 =: a_2 + 2 a_3 c_3 \\
b_{33} &= I_3 + m_3 d_3^2 =: a_4 \\
b_{12} &= -\left(m_2 d_2 + m_3 \ell_2 \right) s_2 - m_3 d_3 s_{23} =: -a_5 s_2 - a_6 s_{23} \\
b_{13} &= -m_3 d_3 s_{23} = -a_6 s_{23} \\
b_{23} &= I_3 + m_3 d_3^2 + m_3 \ell_2 d_3 c_3 = a_4 + a_3 c_3.
\end{aligned}$$

where we have introduced the dynamic coefficients a_i ($i = 1, \ldots, 6$) for the constant factors, in order to have more compact expressions. Thus, the positive definite, symmetric robot inertia matrix can be rewritten as

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 & -\left(a_5 s_2 + a_6 s_{23}\right) & -a_6 s_{23} \\ -\left(a_5 s_2 + a_6 s_{23}\right) & a_2 + 2a_3 c_3 & a_4 + a_3 c_3 \\ -a_6 s_{23} & a_4 + a_3 c_3 & a_4 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1(\boldsymbol{q}) & \boldsymbol{b}_2(\boldsymbol{q}) & \boldsymbol{b}_3(\boldsymbol{q}) \end{pmatrix}. \tag{1}$$

Coriolis and centrifugal vector

From (1) and

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{pmatrix}, \quad c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \quad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left\{ \frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{B}(\boldsymbol{q})}{\partial q_i} \right\} \; (i = 1, 2, 3),$$

we compute

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \frac{1}{2} \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\left(a_5 c_2 + a_6 c_{23}\right) & -a_6 c_{23} \\ 0 & -a_6 c_{23} & -a_6 c_{23} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\left(a_5 c_2 + a_6 c_{23}\right) & -a_6 c_{23} \\ 0 & -a_6 c_{23} & -a_6 c_{23} \end{pmatrix}^T - \boldsymbol{0} \right\} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\left(a_5 c_2 + a_6 c_{23}\right) & -a_6 c_{23} \\ 0 & -a_6 c_{23} & -a_6 c_{23} \end{pmatrix} \\
\boldsymbol{C}_2(\boldsymbol{q}) &= \frac{1}{2} \left\{ \begin{pmatrix} 0 & -\left(a_5 c_2 + a_6 c_{23}\right) & -a_6 c_{23} \\ 0 & 0 & -2a_3 s_3 \\ 0 & 0 & -a_3 s_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -\left(a_5 c_2 + a_6 c_{23}\right) & 0 & 0 \\ -a_6 c_{23} & -2a_3 s_3 & -a_3 s_3 \end{pmatrix} \right. \\
&\qquad \left. - \begin{pmatrix} 0 & -\left(a_5 c_2 + a_6 c_{23}\right) & -a_6 c_{23} \\ -\left(a_5 c_2 + a_6 c_{23}\right) & 0 & 0 \\ -a_6 c_{23} & 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a_3 s_3 \\ 0 & -a_3 s_3 & -a_3 s_3 \end{pmatrix} \\
\boldsymbol{C}_3(\boldsymbol{q}) &= \frac{1}{2} \left\{ \begin{pmatrix} 0 & -a_6 c_{23} & -a_6 c_{23} \\ 0 & 0 & -a_3 s_3 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -a_6 c_{23} & 0 & 0 \\ -a_6 c_{23} & -a_3 s_3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -a_6 c_{23} & -a_6 c_{23} \\ -a_6 c_{23} & -2a_3 s_3 & -a_3 s_3 \\ -a_6 c_{23} & -a_3 s_3 & 0 \end{pmatrix} \right\} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_3 s_3 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

and thus

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -a_5 c_2 \, \dot{q}_2^2 - a_6 c_{23} \left(\dot{q}_2 + \dot{q}_3\right)^2 \\ -a_3 s_3 \left(2\dot{q}_2 + \dot{q}_3\right) \dot{q}_3 \\ a_3 s_3 \, \dot{q}_2^2 \end{pmatrix} = \begin{pmatrix} -\left(m_2 d_2 + m_3 \ell_2\right) c_2 \, \dot{q}_2^2 - m_3 d_3 \, c_{23} \left(\dot{q}_2 + \dot{q}_3\right)^2 \\ -m_3 \ell_2 d_3 \, s_3 \left(2\dot{q}_2 + \dot{q}_3\right) \dot{q}_3 \\ m_3 \ell_2 d_3 \, s_3 \, \dot{q}_2^2 \end{pmatrix}. \tag{2}$$

Exercise 2

Again, the robot motion occurs in a (vertical) plane and we will use for simplicity two-dimensional position vectors in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$. The total potential energy is

$$U = \sum_{i=1}^{4} U_i, \qquad U_i = -m_i \boldsymbol{g}^T \boldsymbol{r}_{0,c_i}, \quad i = 1, \ldots, 4.$$

Since

$$\boldsymbol{g}^T = \begin{pmatrix} 0 & -g_0 & 0 \end{pmatrix}, \qquad g_0 = 9.81\,[\mathrm{m/s^2}],$$

we need to compute only the y-component of the position vector $\boldsymbol{r}_{0,c_i}$ of the center of mass of the link i, for $i = 1, \ldots, 4$. We have

$$\begin{aligned} r_{0,c_{1,y}} &= d_1 s_1 \\ r_{0,c_{2,y}} &= \ell_1 s_1 + d_2 s_{12} \\ r_{0,c_{3,y}} &= \ell_1 s_1 + \ell_2 s_{12} + d_3 s_{123} \\ r_{0,c_{4,y}} &= \ell_1 s_1 + \ell_2 s_{12} + \ell_3 s_{123} + d_4 s_{1234}, \end{aligned}$$

where d_i is the (signed) distance of the center of mass of link i from the axis of joint i $(i = 1, \ldots, 4)$. Thus

$$\begin{aligned} U &= g_0 m_1 d_1 s_1 + g_0 m_2 (\ell_1 s_1 + d_2 s_{12}) + g_0 m_3 (\ell_1 s_1 + \ell_2 s_{12} + d_3 s_{123}) + g_0 m_4 (\ell_1 s_1 + \ell_2 s_{12} + \ell_3 s_{123} + d_4 s_{1234}) \\ &= g_0 \Big\{ [m_1 d_1 + (m_2 + m_3 + m_4)\ell_1]\, s_1 + [m_2 d_2 + (m_3 + m_4)\ell_2]\, s_{12} + [m_3 d_3 + m_4 \ell_3]\, s_{123} + m_4 d_4 s_{1234} \Big\} \\ &=: a_{G1} s_1 + a_{G2} s_{12} + a_{G3} s_{123} + a_{G4} s_{1234}, \end{aligned}$$

where we have introduced the dynamic coefficients a_{Gi} $(i = 1, \ldots, 4)$ for the constant factors related to gravity.

The gravity vector of this robot is then

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} a_{G1} c_1 + a_{G2} c_{12} + a_{G3} c_{123} + a_{G4} c_{1234} \\ a_{G2} c_{12} + a_{G3} c_{123} + a_{G4} c_{1234} \\ a_{G3} c_{123} + a_{G4} c_{1234} \\ a_{G4} c_{1234} \end{pmatrix}, \tag{3}$$

and its linear parametrization is

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} c_1 & c_{12} & c_{123} & c_{1234} \\ 0 & c_{12} & c_{123} & c_{1234} \\ 0 & 0 & c_{123} & c_{1234} \\ 0 & 0 & 0 & c_{1234} \end{pmatrix} \begin{pmatrix} a_{G1} \\ a_{G2} \\ a_{G3} \\ a_{G4} \end{pmatrix} = \boldsymbol{Y}_G(\boldsymbol{q}) \boldsymbol{a}_G. \tag{4}$$

All equilibrium configurations $\boldsymbol{q}_e$ are found by analyzing recursively the vector equation $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$ from the last component backwards:

$$\begin{aligned} g_4(\boldsymbol{q}_e) = 0 \quad &\rightarrow \quad c_{1234} = 0 \\ g_3(\boldsymbol{q}_e) = 0 \quad &\rightarrow \quad \text{being already } c_{1234} = 0 \quad \rightarrow \quad c_{123} = 0 \\ g_2(\boldsymbol{q}_e) = 0 \quad &\rightarrow \quad \text{being already } c_{1234} = 0, c_{123} = 0 \quad \rightarrow \quad c_{12} = 0 \\ g_1(\boldsymbol{q}_e) = 0 \quad &\rightarrow \quad \text{being already } c_{1234} = 0, c_{123} = 0, c_{12} = 0 \quad \rightarrow \quad c_1 = 0. \end{aligned}$$

Thus, the unforced equilibria of the robot (assuming a generic mass distribution) are characterized by

$$q_{e1} = \pm\frac{\pi}{2} \;\cap\; q_{e2} = \{0, \pi\} \;\cap\; q_{e3} = \{0, \pi\} \;\cap\; q_{e4} = \{0, \pi\},$$

namely with the robot being stretched or folded along the vertical direction only.

Finally, perfect balancing in all configurations (i.e., $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{0}$) is obtained for when the mass distribution zeroes the vector of dynamic coefficients, namely $\boldsymbol{a}_G = \boldsymbol{0}$. Starting again from the last component and proceeding backwards, we obtain

$$\begin{aligned} a_{G4} = 0 \quad &\rightarrow \quad d_4 = 0 \\ a_{G3} = 0 \quad &\rightarrow \quad m_3 d_3 + m_4 \ell_3 = 0 \quad \rightarrow \quad d_3 = -\frac{m_4}{m_3}\,\ell_3 \\ a_{G2} = 0 \quad &\rightarrow \quad m_2 d_2 + (m_3 + m_4)\ell_2 = 0 \quad \rightarrow \quad d_2 = -\frac{m_3 + m_4}{m_2}\,\ell_2 \\ a_{G1} = 0 \quad &\rightarrow \quad m_1 d_1 + (m_2 + m_3 + m_4)\ell_1 = 0 \quad \rightarrow \quad d_1 = -\frac{m_2 + m_3 + m_4}{m_1}\,\ell_1. \end{aligned}$$

Exercise 3

The task vector for this 4R planar robot is defined as

$$\boldsymbol{r} = \begin{pmatrix} \boldsymbol{p}_e \\ \phi \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ \phi \end{pmatrix} = \begin{pmatrix} \ell\,(c_1 + c_{12} + c_{123} + c_{1234}) \\ \ell\,(s_1 + s_{12} + s_{123} + s_{1234}) \\ q_1 + q_2 + q_3 + q_4 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}).$$

Differentiating $\boldsymbol{r}$ w.r.t. to time yields

$$\dot{\boldsymbol{r}} = \begin{pmatrix} \boldsymbol{v} \\ \dot{\phi} \end{pmatrix} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with the task Jacobian given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell\,(s_1 + s_{12} + s_{123} + s_{1234}) & -\ell\,(s_{12} + s_{123} + s_{1234}) & -\ell\,(s_{123} + s_{1234}) & -\ell\,s_{1234} \\ \ell\,(c_1 + c_{12} + c_{123} + c_{1234}) & \ell\,(c_{12} + c_{123} + c_{1234}) & \ell\,(c_{123} + c_{1234}) & \ell\,c_{1234} \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{5}$$

For the purpose of singularity analysis, the matrix $\boldsymbol{J}(\boldsymbol{q})$ can be rewritten as

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell\,s_1 & -\ell\,s_{12} & -\ell\,s_{123} & -\ell\,s_{1234} \\ \ell\,c_1 & \ell\,c_{12} & \ell\,c_{123} & \ell\,c_{1234} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \boldsymbol{J}_a(\boldsymbol{q})\,\boldsymbol{T},$$

where the square matrix $\boldsymbol{T}$ is clearly nonsingular. Thus, $\boldsymbol{J}$ and $\boldsymbol{J}_a$ have always the same rank. In particular, the Jacobian $\boldsymbol{J}$ will be full (row) rank if and only if the 2×3 upper left block of matrix $\boldsymbol{J}_a$ will have rank equal to 2. This matrix block corresponds to the well-known Jacobian of a planar 3R robot (with equal links of length ℓ) performing a positional task with its end-effector. The singularities of the 4R arm for the given task occur then if and only if

$$q_2 = \{0, \pi\} \,\cap\, q_3 = \{0, \pi\},$$

namely when its *first three* links are stretched or folded along a single direction.

Plugging the link length $\ell = 0.5$ [m] and the given configuration $\boldsymbol{q} = (0 \;\, 0 \;\, \pi/2 \;\, 0)$ in (5) provides

$$\boldsymbol{J} = \begin{pmatrix} -1 & -1 & -1 & -0.5 \\ 1 & 0.5 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

whose pseudoinverse is computed (by hand or using Matlab) as

$$\boldsymbol{J}^{\#} = \boldsymbol{J}^T \left(\boldsymbol{J}\boldsymbol{J}^T\right)^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0.5 & 1 \\ -1 & 0 & 1 \\ -0.5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3.25 & -1.5 & -3.5 \\ -1.5 & 1.25 & 1.5 \\ -3.5 & 1.5 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 1 & 1/6 \\ -2/3 & 0 & -1/3 \\ -5/3 & -1 & -5/6 \\ 2 & 0 & 2 \end{pmatrix}.$$

The desired velocity task is specified by

$$\dot{\boldsymbol{r}}_d = \begin{pmatrix} \boldsymbol{v}_d \\ \phi_d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0.5 \end{pmatrix}.$$

In view of the separability of the objective function $H_{range}(\boldsymbol{q}) = \sum_{i=1}^{N} H_{range,i}(q_i)$ that measures the distance from the midpoint of the joint ranges, its gradient takes the form

$$\nabla_{\boldsymbol{q}} H_{range}(\boldsymbol{q}) = \left(\frac{\partial H_{range}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T, \qquad \text{with} \quad \frac{\partial H_{range}(\boldsymbol{q})}{\partial q_i} = \frac{\partial H_{range,i}(q_i)}{\partial q_i} = \frac{1}{N}\frac{q_i - \bar{q}_i}{(q_{M,i} - q_{m,i})^2}.$$

With the data $N = 4$, $q_{M,i} = -q_{m,i} = 2$, and thus $\bar{q}_i = 0$, for $i = 1, \ldots, 4$, the gradient at the given configuration $\boldsymbol{q} = (0 \;\; 0 \;\; \pi/2 \;\; 0)$ is

$$\nabla_{\boldsymbol{q}} H_{range} = \frac{1}{64}\begin{pmatrix} 0 \\ 0 \\ \pi/2 \\ 0 \end{pmatrix}$$

The joint velocity solution that realizes the desired task while *decreasing* instantaneously the objective function H_{range} is evaluated then as

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\dot{\boldsymbol{r}}_d - \left(\boldsymbol{I} - \boldsymbol{J}^{\#}\boldsymbol{J}\right)\nabla_{\boldsymbol{q}} H_{range} = -\nabla_{\boldsymbol{q}} H_{range} + \boldsymbol{J}^{\#}\left(\dot{\boldsymbol{r}}_d + \boldsymbol{J}\,\nabla_{\boldsymbol{q}} H_{range}\right) = \begin{pmatrix} 0.4126 \\ -0.8252 \\ -2.0874 \\ 3 \end{pmatrix} \text{[rad/s]}.$$

* * * * *

Robotics II

June 6, 2016

Exercise 1

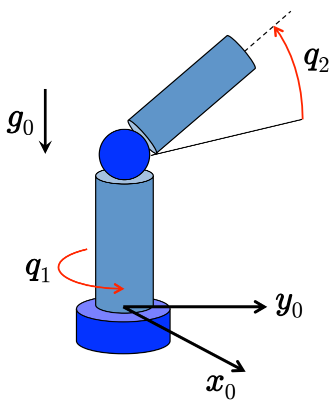


Figure 1: A 2R polar robot

Derive the dynamic model of a 2R polar robot moving in the presence of gravity, using the generalized coordinates $\boldsymbol{q} = (q_1, q_2)$ defined in Fig. 1. Assume that the links have cylindric form (as in the picture) and uniformly distributed mass.

Provide for this robot the explicit expression of the terms of an adaptive control law that guarantees asymptotic tracking of a desired smooth joint trajectory $\boldsymbol{q}_d(t)$, without any a priori knowledge about the robot dynamic parameters. Which is the minimum dimension of such an adaptive controller?

Exercise 2

For the robot in Fig. 1, write down all different symbolic expressions of control laws that you are aware of, which guarantee regulation to a desired (generic) constant configuration $\boldsymbol{q}_d$. Specify for each law the design conditions for success and the type of convergence/stability achieved.

[180 minutes; open books]

Solution

June 6, 2016

Exercise 1

The definition of the joint variables q_1 and q_2 follows the Denavit-Hartenberg convention. We show in Fig. 2 the two DH frames attached to the two moving links (and indexed with 1 and 2), which will be used for defining conveniently their inertial parameters. The rotation matrices between frames 0 and 1 and between frames 1 and 2 are found easily by inspection (without the need of explicitly defining a DH table of parameters) as

$$ {}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 \\ \sin q_1 & 0 & -\cos q_1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 \\ \sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

Since the cylindric links have uniform mass, their center of mass will lie along the $\boldsymbol{y}_1$-axis for link 1 and along the $\boldsymbol{x}_2$-axis for link 2. We denote with $d_2 > 0$ the distance of the center of mass of link 2 from the axis of joint 1 (slightly more than half of the link length). The inertia matrix of each link is diagonal when referred to the kinematic reference frame attached to the link, as well as when referred to the frame with origin in the center of mass and having the same orientation. We denote the link inertia matrices in the latter case as

$$ {}^i\boldsymbol{I}_i = \begin{pmatrix} I_{ix} & & \\ & I_{iy} & \\ & & I_{iz} \end{pmatrix}, \qquad i = 1, 2. $$

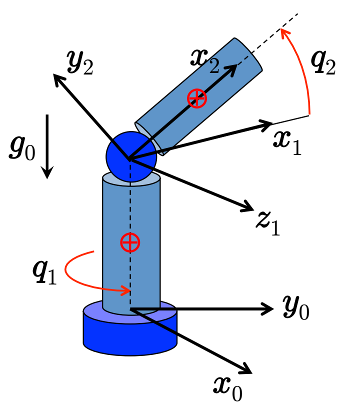


Figure 2: The reference frames used for defining link inertial parameters of the 2R polar robot

We start by computing the various terms in the robot dynamic model, following a Lagrangian approach.

The kinetic energy of the robot is $T = T_1 + T_2$. For the first link,

$$ T_1 = \frac{1}{2} I_{1y}\, \dot{q}_1^2. $$

For the second link, the position of its center of mass is

$$ \boldsymbol{p}_{c2} = \begin{pmatrix} d_2 \cos q_2 \cos q_1 \\ d_2 \cos q_2 \sin q_1 \\ \ell_1 + d_2 \sin q_2 \end{pmatrix}, $$

where ℓ_1 is the length of link 1 (an irrelevant kinematic parameter for what follows). Thus, its velocity is

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -d_2 \cos q_2 \sin q_1 \, \dot{q}_1 - d_2 \cos q_1 \sin q_2 \, \dot{q}_2 \\ d_2 \cos q_2 \cos q_1 \, \dot{q}_1 - d_2 \sin q_1 \sin q_2 \, \dot{q}_2 \\ d_2 \cos q_2 \, \dot{q}_2 \end{pmatrix},$$

and its squared norm simplifies to

$$\|\boldsymbol{v}_{c2}\|^2 = \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} = d_2^2 \left(\dot{q}_2^2 + \cos^2 q_2 \, \dot{q}_1^2 \right).$$

The angular velocity of link 2, when expressed in frame 0, is computed as[1]

$${}^0\boldsymbol{\omega}_2 = {}^0\boldsymbol{z}_0 \, \dot{q}_1 + {}^0\boldsymbol{z}_1 \, \dot{q}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{q}_1 + \begin{pmatrix} \sin q_1 \\ -\cos q_1 \\ 0 \end{pmatrix} \dot{q}_2.$$

In order to use the constant diagonal inertia matrix of link 2, we need to express the angular velocity in frame 2. Since

$${}^1\boldsymbol{z}_0 = {}^0\boldsymbol{R}_1^T(q_1) \, {}^0\boldsymbol{z}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad {}^2\boldsymbol{z}_0 = {}^1\boldsymbol{R}_2^T(q_1) \, {}^1\boldsymbol{z}_0 = \begin{pmatrix} \sin q_2 \\ \cos q_2 \\ 0 \end{pmatrix}$$

and

$${}^1\boldsymbol{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad {}^2\boldsymbol{z}_1 = {}^1\boldsymbol{R}_2^T(q_1) \, {}^1\boldsymbol{z}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we have

$${}^2\boldsymbol{\omega}_2 = {}^2\boldsymbol{z}_0 \, \dot{q}_1 + {}^2\boldsymbol{z}_1 \, \dot{q}_2 = \begin{pmatrix} \sin q_2 \, \dot{q}_1 \\ \cos q_2 \, \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}.$$

As a result,

$$T_2 = \frac{1}{2} m_2 \, \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} + \frac{1}{2} \, {}^2\boldsymbol{\omega}_2^T \, {}^2\boldsymbol{I}_2 \, {}^2\boldsymbol{\omega}_2 = \frac{1}{2} \left(I_{2x} \sin^2 q_2 + \left(I_{2y} + m_2 d_2^2 \right) \cos^2 q_2 \right) \dot{q}_1^2 + \left(I_{2z} + m_2 d_2^2 \right) \dot{q}_2^2.$$

We note that in general $I_{2x} \neq I_{2y}$, whereas it is $I_{2y} = I_{2z}$, due to the cylindric form and uniform mass distribution of the links. Therefore,

$$T = T_1 + T_2 = \frac{1}{2} \dot{\boldsymbol{q}}^T \begin{pmatrix} I_{1y} + I_{2x} \sin^2 q_2 + \left(I_{2y} + m_2 d_2^2 \right) \cos^2 q_2 & 0 \\ 0 & I_{2z} + m_2 d_2^2 \end{pmatrix} \dot{\boldsymbol{q}} = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q}) \dot{\boldsymbol{q}}.$$

Using the following definition of dynamic coefficients

$$a_1' = I_{1y}, \qquad a_2' = I_{2x}, \qquad a_3' = I_{2y} + m_2 d_2^2 = I_{2z} + m_2 d_2^2,$$

we can write the inertia matrix as

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1' + a_2' \sin^2 q_2 + a_3' \cos^2 q_2 & 0 \\ 0 & a_3' \end{pmatrix}.$$

[1] We use here the expression for revolute joints of the columns of the angular part of the geometric Jacobian.

Although there will be no reduction in the number of dynamic coefficients, it is slightly more convenient to use the trigonometric identity $\cos^2 q_2 = 1 - \sin^2 q_2$ and obtain

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 + a_2 \sin^2 q_2 & 0 \\ 0 & a_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_1(q_2) & \boldsymbol{b}_2 \end{pmatrix},$$

with

$$\begin{aligned} a_1 &= (a'_1 + a'_3 =)\, I_{1y} + I_{2y} + m_2 d_2^2 \\ a_2 &= (a'_2 - a'_3 =)\, I_{2x} - I_{2y} - m_2 d_2^2 \\ a_3 &= (a'_3 =)\, I_{2y} + m_2 d_2^2 = I_{2z} + m_2 d_2^2. \end{aligned}$$

For the Coriolis and centrifugal terms, we have

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{pmatrix} = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}, \qquad c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \quad i = 1, 2, \qquad \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \end{pmatrix},$$

where

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left\{ \frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{B}(\boldsymbol{q})}{\partial q_i} \right\}, \qquad i = 1, 2.$$

Note that with the above definition based on Christoffel symbols, the factorization matrix $\boldsymbol{S}$ satisfies automatically the property of skew-symmetry for $\dot{\boldsymbol{B}} - 2\boldsymbol{S}$.

Computing

$$\begin{aligned} \boldsymbol{C}_1(\boldsymbol{q}) &= \frac{1}{2}\left\{ \begin{pmatrix} 0 & 2a_2 \sin q_2 \cos q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 2a_2 \sin q_2 \cos q_2 & 0 \end{pmatrix} - \boldsymbol{0} \right\} \\ &= \begin{pmatrix} 0 & a_2 \sin q_2 \cos q_2 \\ a_2 \sin q_2 \cos q_2 & 0 \end{pmatrix}, \\ \boldsymbol{C}_2(\boldsymbol{q}) &= \cdots = \begin{pmatrix} -a_2 \sin q_2 \cos q_2 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

we have

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2a_2 \sin q_2 \cos q_2\, \dot{q}_1 \dot{q}_2 \\ -a_2 \sin q_2 \cos q_2\, \dot{q}_1^2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = a_2 \sin q_2 \cos q_2 \begin{pmatrix} \dot{q}_2 & \dot{q}_1 \\ -\dot{q}_1 & 0 \end{pmatrix}.$$

The potential energy of the robot is given by

$$U = U_1 + U_2, \qquad U_i = -m_i \boldsymbol{g}_0^T \boldsymbol{r}_{0,c_i}, \quad i = 1, 2.$$

Since

$$\boldsymbol{g}_0^T = \begin{pmatrix} 0 & 0 & -g_0 \end{pmatrix}, \qquad g_0 = 9.81\, [\mathrm{m/s^2}]$$

and the potential energy U_1 is constant, we only need the z-component of the position vector $\boldsymbol{r}_{0,c_2}$ of the center of mass of link 2. We have

$$U_1 = \text{cost}, \qquad U_2 = m_2 g_0 d_2 \sin q_2 = a_4 \sin q_2,$$

where we have introduced a fourth, and last, dynamic coefficient $a_4 = m_2 g_0 d_2$. Therefore,

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ a_4 \cos q_2 \end{pmatrix}.$$

The dynamic model of the robot can thus be written in its linear parametrized form,

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\boldsymbol{a} = \boldsymbol{u},$$

with

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) = \begin{pmatrix} \ddot{q}_1 & \sin^2 q_2\,\ddot{q}_1 + 2\sin q_2\cos q_2\,\dot{q}_1\dot{q}_2 & 0 & 0 \\ 0 & -\sin q_2\cos q_2\,\dot{q}_1^2 & \ddot{q}_2 & \cos q_2 \end{pmatrix}, \qquad \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$$

Defining $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}\boldsymbol{e} = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}(\boldsymbol{q}_d - \boldsymbol{q})$, with a diagonal matrix $\boldsymbol{\Lambda} > 0$, two diagonal gain matrices $\boldsymbol{K}_D > 0$ and $\boldsymbol{K}_P = \boldsymbol{K}_D\boldsymbol{\Lambda}^{-1} > 0$, and a diagonal estimation gain matrix $\boldsymbol{\Gamma} > 0$, the adaptive controller will have dimension 4 (equal to the minimum number of dynamic coefficients to be estimated in this robot) and the expression

$$\begin{aligned}\boldsymbol{u} &= \hat{\boldsymbol{B}}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \hat{\boldsymbol{S}}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \hat{\boldsymbol{g}}(\boldsymbol{q}) + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}} = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\hat{\boldsymbol{a}} + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}} \\ \dot{\hat{\boldsymbol{a}}} &= \boldsymbol{\Gamma}\,\boldsymbol{Y}^T(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}})\,, \qquad \hat{\boldsymbol{a}}(0) = \text{arbitrary},\end{aligned}$$

where

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r) = \begin{pmatrix} \ddot{q}_{r1} & \sin^2 q_2\,\ddot{q}_{r1} + \sin q_2\cos q_2\,(\dot{q}_1\dot{q}_{r2} + \dot{q}_{r1}\dot{q}_2) & 0 & 0 \\ 0 & -\sin q_2\cos q_2\,\dot{q}_1\dot{q}_{r1} & \ddot{q}_{r2} & \cos q_2 \end{pmatrix}, \quad \hat{\boldsymbol{a}} = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \end{pmatrix}.$$

* * * * *

Robotics 2 - Final Test

June 1, 2016

A 2R robot moving in the vertical plane uses the joint coordinates defined in Fig. 1. The inertia matrix of this robot can be written as

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2\cos q_2 & a_3 + a_2 \cos q_2 \\ a_3 + a_2\cos q_2 & a_3 \end{pmatrix}, \tag{1}$$

where the three dynamic coefficients a_i, $i = 1, 2, 3$, satisfy $a_1 > a_3 > a_2 > 0$. The robot is initially at rest and should move under the constraint of keeping the *absolute* orientation of the second link at a constant angle β w.r.t. the $\boldsymbol{x}_0$ axis, with the value $0 < \beta < \pi/2$ being specified by the initial configuration.

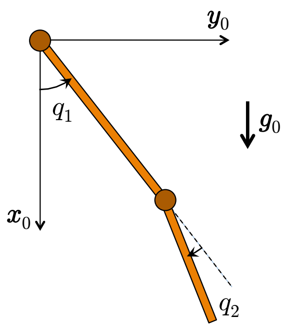


Figure 1: A planar 2R robot

- Derive a one-dimensional *reduced* dynamic model of the robot under the given geometric constraint on its motion. Point out the features of the obtained model. Are there any special (e.g., singular) situations from the dynamic point of view?
- With the robot at rest at $t = 0$ and with $\boldsymbol{q}(0)$ satisfying the geometric constraint, determine *all* feasible torques $\boldsymbol{u}(0)$ that can keep the robot in static equilibrium. Which are the associated values of the scalar constraint force $\lambda(0)$?
- Describe how a simulation with arbitrary joint torque inputs $\boldsymbol{u}(t) \in \mathbb{R}^2$ should be performed when using this model.
- Provide the symbolic expression of the joint torques u_1 and u_2 that should be applied when the robot is at a given generic state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ compatible with the constraint, so that the pseudoacceleration is equal to a desired value $\dot{v}_d \in \mathbb{R}$ and the constraint force is $\lambda = 0$.

[150 minutes; open books]

Solution

June 1, 2016

We first derive the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q}.\dot{\boldsymbol{q}})$ from the expression of $\boldsymbol{B}(\boldsymbol{q})$ in (1), and then compute the potential energy $U(\boldsymbol{q})$ due to gravity and the associated vector $\boldsymbol{g}(\boldsymbol{q})$ in the dynamic model.

From (1) and

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ c_2(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{pmatrix}, \quad c_i(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \quad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left\{\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{B}(\boldsymbol{q})}{\partial q_i}\right\} \ (i = 1, 2),$$

we have

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \frac{1}{2}\left\{\begin{pmatrix} 0 & -2a_2 \sin q_2 \\ 0 & -a_2 \sin q_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -2a_2 \sin q_2 & -a_2 \sin q_2 \end{pmatrix} - \boldsymbol{0}\right\} \\
&= \begin{pmatrix} 0 & -a_2 \sin q_2 \\ -a_2 \sin q_2 & -a_2 \sin q_2 \end{pmatrix} \\
\boldsymbol{C}_2(\boldsymbol{q}) &= \frac{1}{2}\left\{\begin{pmatrix} 0 & -a_2 \sin q_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -a_2 \sin q_2 & 0 \end{pmatrix} - \begin{pmatrix} -2a_2 \sin q_2 & -a_2 \sin q_2 \\ -a_2 \sin q_2 & 0 \end{pmatrix}\right\} \\
&= \begin{pmatrix} a_2 \sin q_2 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

and thus

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} -a_2 \sin q_2 \, (2\dot{q}_1 + \dot{q}_2)\, \dot{q}_2 \\ a_2 \sin q_2 \, \dot{q}_1^2 \end{pmatrix}. \tag{2}$$

The potential energy is given by

$$U = \sum_{i=1}^{2} U_i, \qquad U_i = -m_i \boldsymbol{g}_0^T \boldsymbol{r}_{0,c_i}, \quad i = 1, 2.$$

Since

$$\boldsymbol{g}_0^T = \begin{pmatrix} g_0 & 0 & 0 \end{pmatrix}, \qquad g_0 = 9.81\,[\mathrm{m/s^2}],$$

we need in the computations only the x-component of the position vector $\boldsymbol{r}_{0,c_i}$ of the center of mass of the link i, for $i = 1, 2$. We have

$$\begin{aligned}
U_1 &= -m_1 g_0 \, d_1 \cos q_1 \\
U_2 &= -m_2 g_0 \left(\ell_1 \cos q_1 + d_2 \cos(q_1 + q_2)\right),
\end{aligned}$$

where m_i is the mass of link i, d_i is the distance of the CoM of link i from the axis of joint i, and ℓ_1 is the length of link 1. Therefore,

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} a_4 \sin q_1 + a_5 \sin(q_1 + q_2) \\ a_5 \sin(q_1 + q_2) \end{pmatrix},$$

where we have introduced the dynamic coefficients $a_4 = (m_1 d_1 + m_2 \ell_1) g_0$ and $a_5 = m_2 d_2 g_0$.

The scalar geometric constraint ($m = 1$) imposed on the motion of the robot (with a configuration space of dimension $n = 2$) is to keep the second link oriented at a constant angle β w.r.t. the axis $\boldsymbol{x}_0$. Thus

$$h(\boldsymbol{q}) = q_1 + q_2 - \beta = 0 \qquad \Rightarrow \qquad \boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = 0, \quad \text{with } \boldsymbol{A} = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

We note that matrix $\boldsymbol{A}$ is constant since the constraint is linear in $\boldsymbol{q}$. The constrained dynamic model is then

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u} + \boldsymbol{A}^T\lambda, \qquad \text{s.t. } h(\boldsymbol{q}) = 0.$$

In order to obtain a reduced dynamic model, we can define a matrix $\boldsymbol{D}$ in the following way:

$$\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{(a nonsingular matrix).} \tag{3}$$

From this

$$\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{F} \end{pmatrix},$$

and so we define

$$v = \boldsymbol{D}\dot{\boldsymbol{q}} = \dot{q}_2, \quad \dot{v} = \ddot{q}_2 \qquad \Longleftrightarrow \qquad \dot{\boldsymbol{q}} = \boldsymbol{F}v = \begin{pmatrix} -1 \\ 1 \end{pmatrix} v, \quad \ddot{\boldsymbol{q}} = \boldsymbol{F}\dot{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \dot{v}.$$

Since all these matrices are constant, many simplifications will occur. In particular, the reduced dynamic model (of dimension $n - m = 1$) is expressed by a single differential equation of the form

$$\left(\boldsymbol{F}^T\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F}\right)\dot{v} = \boldsymbol{F}^T\left(\boldsymbol{u} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right), \tag{4}$$

while the expression of the scalar constraint force will be given by

$$\lambda = \boldsymbol{E}^T\left(\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F}\dot{v} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{u}\right). \tag{5}$$

These model expressions will hold everywhere, thanks to the global invertibility of the matrix in (3); no singularity will occur with the chosen representation of the reduced dynamics.

Performing computations, we have

$$\begin{aligned}
&\boldsymbol{F}^T\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F} = a_1 - a_3 > 0 \qquad \text{(constant!)},\\
&\boldsymbol{F}^T\boldsymbol{u} = u_2 - u_1,\\
&-\boldsymbol{F}^T\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})\right) = -a_2\sin q_2\,(\dot{q}_1 + \dot{q}_2)^2 + a_4\sin q_1,\\
&\boldsymbol{E}^T\boldsymbol{B}(\boldsymbol{q})\boldsymbol{F} = a_3 - a_1 - a_2\cos q_2,\\
&\boldsymbol{E}^T\boldsymbol{u} = u_1,\\
&\boldsymbol{E}^T\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})\right) = c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) + g_1(\boldsymbol{q}) = -a_2\sin q_2\,(2\dot{q}_1 + \dot{q}_2)\dot{q}_2 + a_4\sin q_1 + a_5\sin(q_1 + q_2).
\end{aligned}$$

The reduced model (4) has to be initialized at a state $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0))$ that is compatible with the constraint, or

$$q_1(0) + q_2(0) = \beta, \qquad \dot{q}_1(0) + \dot{q}_2(0) = 0. \tag{6}$$

The velocity constraint is indeed satisfied if the robot is initially at rest (although this is not the only case). The constraints (6) will propagate then for all $t > 0$, if we proceed by integrating (4). Thus, $\dot{q}_1 + \dot{q}_2 = 0$ will hold at any time, and the dynamics (4) simplifies finally to

$$(a_1 - a_3)\,\dot{v} = u_2 - u_1 + a_4\sin q_1, \tag{7}$$

while the expression (5) of the constraint force (multiplier) becomes

$$\lambda = (a_3 - a_1 - a_2\cos q_2)\,\dot{v} - a_2\sin q_2\,\dot{q}_1\dot{q}_2 + a_4\sin q_1 + a_5\sin\beta - u_1. \tag{8}$$

The pseudoinertia in (7) is now constant, and the quadratic velocity terms have disappeared as expected. The only residual nonlinearity is due to the gravity.

If the robot starts at rest ($\dot{q}_1(0) = \dot{q}_2(0) = 0$) and satisfies the constraints (6), in order to keep the static equilibrium ($\dot{v} = 0$), from (7) we need to have

$$u_1(0) - u_2(0) = a_4 \sin q_1(0),$$

so that there is (apparently) an infinity of joint torque combinations that will preserve the equilibrium. Associated to each of these, from (8) there is a single constraint force given by

$$\lambda(0) = a_4 \sin q_1(0) + a_5 \sin\beta - u_1(0) = a_5 \sin\beta - u_2(0).$$

Note however that if the constraint $h(\boldsymbol{q}) = 0$ is a *virtual* one, namely it is not imposed by a mechanism but it is enforced only through a control action, then there can be no real constraint forces generated, i.e., $\lambda(0)$ should vanish. In particular, this happens only with the natural choice

$$\boldsymbol{u}(0) = \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} a_4 \sin q_1(0) + a_5 \sin\beta \\ a_5 \sin\beta \end{pmatrix} = \boldsymbol{g}(\boldsymbol{q}(0)).$$

More in general, a simulation with the reduced dynamic model (7) and an arbitrary input torque $\boldsymbol{u}(t)$ proceeds, with an integration step $T > 0$, as follows[1]:

1. The initial state $(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0) = (\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0))$ should satisfy (6). Set $k = 0$.
2. At every sampling instant $t_k = kT$, compute $\dot{v}_k = \dot{v}(t_k)$ as

$$\dot{v}_k = \frac{u_{k,2} - u_{k,1} + a_4 \sin q_{k,1}}{a_1 - a_3},$$

   evaluate $\ddot{\boldsymbol{q}}_k = \boldsymbol{F}\dot{v}_k$, and use your preferred integration routine to obtain $(\boldsymbol{q}_{k+1}, \dot{\boldsymbol{q}}_{k+1})$.

2'. In alternative to step 2, since $\dot{v} = \ddot{q}_2$, compute

$$\ddot{q}_{k,2} = \frac{u_{k,2} - u_{k,1} + a_4 \sin(\beta - q_{k,2})}{a_1 - a_3},$$

   use your preferred integration routine to obtain $(q_{k+1,2}, \dot{q}_{k+1,2})$, and evaluate the remaining components of the state as

$$q_{k+1,1} = \beta - q_{k+1,2}, \qquad \dot{q}_{k+1,1} = -\dot{q}_{k+1,2}.$$

3. Set $k = k + 1$, and cycle over step 2 (or 2').

Finally, to address the last control problem, we evaluate eqs. (7–8) for $\dot{v} = \dot{v}_d$ and $\lambda = 0$ at a generic state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ obtaining the (unique) needed joint torque

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (a_3 - a_1 - a_2 \cos q_2)\,\dot{v}_d - a_2 \sin q_2\, \dot{q}_1 \dot{q}_2 + a_4 \sin q_1 + a_5 \sin\beta \\ u_1 + (a_1 - a_3)\,\dot{v}_d - a_4 \sin q_1 \end{pmatrix}. \tag{9}$$

* * * * *

[1] We use the notation $x_{k,i}$ to denote the i-th component of a vector $\boldsymbol{x}(t)$ evaluated at time $t_k = kT$.

Robotics II

July 11, 2016

Exercise 1

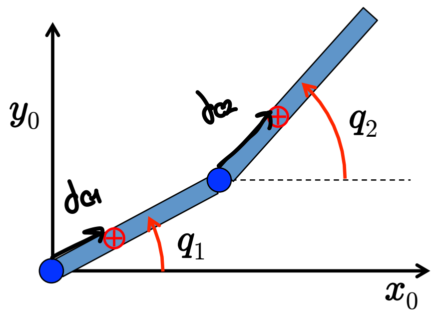


Figure 1: A 2R planar robot moving on a horizontal plane.

Derive the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ of a planar 2R robot using the *absolute* coordinates $\boldsymbol{q} = (q_1, q_2)$ defined in Fig. 1. Will Coriolis and/or centrifugal terms be present in the quadratic velocity term $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of the dynamic model? Why? In the absence of dissipative effects, what is the relation between the input torques $\boldsymbol{u}$ (USING $\dot{q}$ ABSOLUTE) acting in this case on the right-hand side of the Euler-Lagrange equations and the torques $\boldsymbol{u}_\theta = (u_{\theta 1}, u_{\theta 2})$ produced by two motors, one for each joint, directly connected to the respective joint axes?

USING θ_1, θ_2 FROM DH

Exercise 2

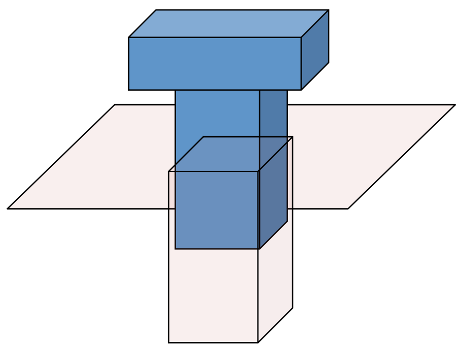


Figure 2: A peg with a square section entering a hole with little or no clearance.

For the task of inserting a peg with a square section into a square hole, as depicted in Fig. 2, define *i)* a suitable *task frame*, *ii)* the *natural constraints* imposed by the geometry of the rigid and frictionless environment on the generalized (i.e., linear and angular) motion/force quantities expressed in this task frame, and *iii)* the *virtual constraints* that can be taken as reference values by a hybrid force-velocity control law for the smooth execution of the task.

Exercise 3

For the peg-in-hole task considered in Exercise 2, consider the problem of regulating the contact force in one direction against a side of the hole. The hole has now a compliant behavior, as modeled by a spring of stiffness $k_e > 0$. With reference to Fig. 3, which displays a horizontal section of the peg in contact with one side of the hole, let $m > 0$ be the mass of the peg and $d > 0$ the position of the undeformed contact along the ${}^t\boldsymbol{x}$ direction.

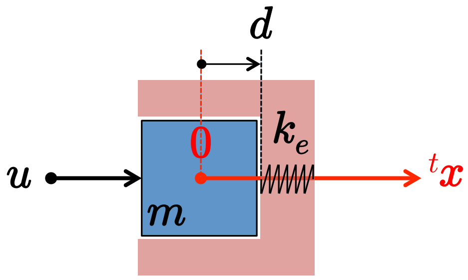


Figure 3: Interaction with a compliant environment at one side of the peg-in-hole of Fig 2.

Design two controllers for the scalar command u such that the contact force F_c is regulated to a desired value $F_d > 0$, with an asymptotically stable transient behavior. In particular:

- suppose first that no force sensing is available and design a *compliance control* law;
- introduce an ideal force sensor (e.g., on the peg surface) and design a *force control* law;
- discuss robustness of the two designs w.r.t. uncertainties in the knowledge of k_e, d and m.

[210 minutes; open books]

Solution

July 11, 2016

Exercise 1

To obtain the robot dynamic model terms, we follow a Lagrangian approach. Since the robot motion occurs at constant potential energy (on a horizontal plane), we need only to compute the kinetic energy $T = T_1 + T_2$. For link i, $i = 1, 2$, let m_i be its mass, d_i the distance of its center of mass from the axis of joint i, and I_i the baricentral inertia of the link around an axis normal to the plane of motion. For the first link, it is

$$T_1 = \frac{1}{2}\left(I_1 + m_1 d_1^2\right)\dot{q}_1^2.$$

For the second link, the position of the center of mass when using the absolute coordinates is

$$\boldsymbol{p}_{c2} = \begin{pmatrix} \ell_1 \cos q_1 + d_2 \cos q_2 \\ \ell_1 \sin q_1 + d_2 \sin q_2 \end{pmatrix},$$

where ℓ_1 is the length of link 1. Thus, its velocity is

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -\ell_1 \sin q_1\, \dot{q}_1 - d_2 \sin q_2\, \dot{q}_2 \\ \ell_1 \cos q_1\, \dot{q}_1 + d_2 \cos q_2\, \dot{q}_2 \end{pmatrix},$$

and its squared norm becomes

$$\|\boldsymbol{v}_{c2}\|^2 = \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} = \ell_1^2 \dot{q}_1^2 + d_2^2 \dot{q}_2^2 + 2\ell_1 d_2 \cos(q_2 - q_1)\, \dot{q}_1 \dot{q}_2.$$

The (scalar) angular velocity of link 2 is simply $\dot{q}_2$. As a result,

$$T_2 = \frac{1}{2}\left(m_2 \ell_1^2 \dot{q}_1^2 + \left(I_2 + m_2 d_2^2\right)\dot{q}_2^2 + 2 m_2 \ell_1 d_2 \cos(q_2 - q_1)\, \dot{q}_1 \dot{q}_2\right).$$

Therefore,

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \begin{pmatrix} I_1 + m_1 d_1^2 + m_2 \ell_1^2 & m_2 \ell_1 d_2 \cos(q_2 - q_1) \\ m_2 \ell_1 d_2 \cos(q_2 - q_1) & I_2 + m_2 d_2^2 \end{pmatrix} \dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Defining for compactness the following dynamic coefficients

$$a_1 = I_1 + m_1 d_1^2 + m_2 \ell_1^2, \qquad a_2 = I_2 + m_2 d_2^2, \qquad a_3 = m_2 \ell_1 d_2,$$

we can rewrite the inertia matrix as

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 & a_3 \cos(q_2 - q_1) \\ a_3 \cos(q_2 - q_1) & a_2 \end{pmatrix}.$$

For the (quadratic) velocity terms in the dynamic model, we have

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{pmatrix}, \qquad c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \quad i = 1, 2,$$

where

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left\{\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{b}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{B}(\boldsymbol{q})}{\partial q_i}\right\}, \qquad i = 1, 2.$$

Computing

$$\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\left\{\begin{pmatrix} 0 & 0 \\ a_3\sin(q_2-q_1) & -a_3\sin(q_2-q_1)\end{pmatrix} + \begin{pmatrix} 0 & a_3\sin(q_2-q_1) \\ 0 & -a_3\sin(q_2-q_1)\end{pmatrix}\right.$$
$$\left. - \begin{pmatrix} 0 & a_3\sin(q_2-q_1) \\ a_3\sin(q_2-q_1) & 0\end{pmatrix}\right\} = \begin{pmatrix} 0 & 0 \\ 0 & -a_3\sin(q_2-q_1)\end{pmatrix},$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \cdots = \begin{pmatrix} a_3\sin(q_2-q_1) & 0 \\ 0 & 0\end{pmatrix},$$

C_{122} $(i=J)$

we finally obtain

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} -a_3\sin(q_2-q_1)\,\dot{q}_2^2 \\ a_3\sin(q_2-q_1)\,\dot{q}_1^2\end{pmatrix}.$$

C_{211} $(i=J)$

There are no Coriolis, but only centrifugal terms in the dynamic equations. This is due to the choice of absolute coordinates, so that a motion of q_1 only, with q_2 kept constant, will not change the orientation of link 2.

Moreover, the transformation from *relative* joint coordinates $\boldsymbol{\theta}$ (those of a classical DH convention) to *absolute* joint coordinates $\boldsymbol{q}$ is

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_1+\theta_2 \end{pmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\dot{\boldsymbol{\theta}} = \boldsymbol{T}\dot{\boldsymbol{\theta}}.$$

Therefore, due to the principle of virtual works, the mapping between the generalized forces $\boldsymbol{u}$ performing work on $\dot{\boldsymbol{q}}$ and the generalized forces $\boldsymbol{u}_\theta$ performing work on $\dot{\boldsymbol{\theta}}$ is given by

$$\boldsymbol{u}_\theta = \begin{pmatrix} u_{\theta 1} \\ u_{\theta 2} \end{pmatrix} = \boldsymbol{T}^T\boldsymbol{u} = \begin{pmatrix} u_1+u_2 \\ u_2 \end{pmatrix} \qquad \Leftrightarrow \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \boldsymbol{T}^{-T}\boldsymbol{u}_\theta = \begin{pmatrix} u_{\theta 1}-u_{\theta 2} \\ u_{\theta 2} \end{pmatrix}.$$

The robot dynamic model written in the $\boldsymbol{q}$ coordinates and driven by the motor torques $\boldsymbol{u}_\theta$ is

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{u} = \boldsymbol{T}^{-T}\boldsymbol{u}_\theta.$$

Exercise 2

With reference to the task frame shown in Fig. 4, the natural (geometric) contraints and the virtual constraints in the peg-hole frictionless interaction are respectively

$$\text{natural:} \quad \begin{cases} {}^t v_x = 0 \\ {}^t v_y = 0 \\ {}^t \omega_x = 0 \\ {}^t \omega_y = 0 \\ {}^t \omega_z = 0 \\ {}^t F_z = 0 \end{cases} \qquad \text{virtual:} \quad \begin{cases} {}^t F_x = F_{xd} \\ {}^t F_y = F_{yd} \\ {}^t M_x = M_{xd} \\ {}^t M_y = M_{yd} \\ {}^t M_z = M_{zd} \\ {}^t v_z = v_{zd} > 0. \end{cases}$$

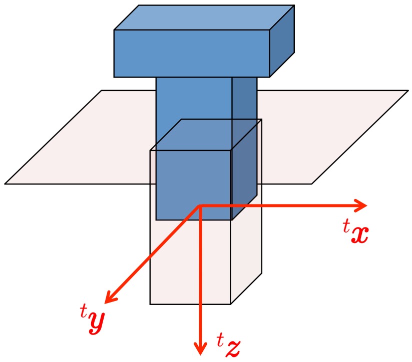


Figure 4: The task frame used in the definition of natural and virtual constraints for hybrid force/motion analysis and control.

Except for the last virtual constraint, which specifies the desired positive speed of insertion of the peg in the hole, all the desired interaction forces and moments can be set to zero, if a smooth behavior is to be realized with minimum mechanical stress on the peg. On the other hand, if a firm contact needs to be maintained with one side of the hole (e.g., in the presence of some uncertain clearance), then the choice of a positive value for F_{xd} and/or F_{yd} would be useful (this is the situation considered, e.g., in Exercise 3).

Exercise 3

Let x be the position along $^t\boldsymbol{x}$ of the peg side that undergoes contact with the hole side. We assume that the peg is already in contact ($x \geq d$). The dynamic model of the peg-environment interaction is then

$$m\ddot{x} + k_e(x - d) = u,$$

whereas the contact force applied by the peg to the hole side is

$$F_c = k_e(x - d), \qquad \text{for } x \geq d$$

Instead, when $x < d$ (no contact) then $F_c = 0$.

If we want to have $F_c = F_d$ at the closed-loop equilibrium, we need to drive the mass m to a position x_d computed as follows:

$$F_c = k_e \left.(x - d)\right|_{x=x_d} = F_d \qquad \Rightarrow \qquad x_d = d + \frac{1}{k_e} F_d. \tag{1}$$

However, a simple *compliance control* law of the PD type

$$u = k_p(x_d - x) - k_d\dot{x}, \qquad \text{with } k_p > 0 \text{ and } k_d > 0, \tag{2}$$

will have a closed-loop equilibrium $x = x_E$ that does not provide the desired contact force F_d at steady state. In fact, the unique equilibrium $x_E (> d)$ will be the solution of

$$k_e(x - d)|_{x=x_E} = k_p(x_d - x)|_{x=x_E} \qquad \Rightarrow \qquad x_E = \frac{k_e d + k_p x_d}{k_e + k_p} = d + \frac{k_p}{k_e(k_e + k_p)} F_d \neq x_d,$$

where (1) has been used. Accordingly, the steady-state contact force is

$$F_E = k_e(x_E - d) = \frac{k_p}{k_e + k_p} F_d,$$

which is never equal to the desired one, even for arbitrary large but finite values of k_p.

A better control design is achieved by adding to the law (2) a term that nominally cancels the interaction force, i.e.,

$$u = k_p(x_d - x) - k_d\dot{x} + k_e(x - d), \qquad \text{with } k_p > 0 \text{ and } k_d > 0. \tag{3}$$

When in contact, the associated closed-loop equation is

$$m\ddot{x} + k_d\dot{x} + k_p x = k_p x_d,$$

which is an asymptotically stable system driven by the constant signal $k_p x_d$, converging at steady state to the desired position $x = x_d$, thus with $F_c = F_d$.

Unfortunately, an uncertain value of the environment stiffness (which appears twice in the control law, namely in the definition of x_d in (1) and in the term compensating the interaction force in (3)) would lead to a residual force error at steady state. In practice, we can implement only

$$u = k_p(\hat{x}_d - x) - k_d\dot{x} + \hat{k}_e(x - d), \qquad \text{with } k_p > 0 \text{ and } k_d > 0. \tag{4}$$

where $\hat{k}_e$ is an estimate of k_e and $\hat{x}_d = d + (F_d/\hat{k}_e)$. For the time being, assume that d is accurately known, so that the last term in (4) is present only when $x \geq d$ (otherwise is zero). When in contact, the closed-loop equation becomes

$$m\ddot{x} + k_d\dot{x} + \left(k_p + k_e - \hat{k}_e\right) x = k_p\hat{x}_d + \left(k_e - \hat{k}_e\right) d = \left(k_p + k_e - \hat{k}_e\right) d + \frac{k_p}{\hat{k}_e} F_d. \tag{5}$$

Provided that the proportional gain k_p in the control law (4) is sufficiently large, i.e.,

$$k_p \geq \alpha > \left|k_e - \hat{k}_e\right| \geq 0, \tag{6}$$

it is easy to see that system (5) will remain asymptotically stable. Its position and contact force converge respectively to

$$x_E = d + \frac{k_p}{\hat{k}_e\left(k_p + k_e - \hat{k}_e\right)} F_d \qquad \text{and} \qquad F_E = \left(\frac{k_e}{\hat{k}_e}\right)\left(\frac{k_p}{k_p + k_e - \hat{k}_e}\right) F_d \neq F_d.$$

The last formula shows that by increasing $k_p \to \infty$ (i.e., to very large values), one can certainly neglect the effect of the estimation error $k_e - \hat{k}_e$, but still *not* the error due to the scaling factor $k_e/\hat{k}_e \neq 1$.

The analysis of the case when the position d of the environment is not accurately known is more complex. Again, the estimate $\hat{d}$ enters twice in place of d within the control law (4), explicitly in the last term and implicitly through the definition of x_d. Not having an accurate estimate of d may lead to a wrong activation/deactivation of the term $\hat{k}_e(x - \hat{d})$, possibly in an inconsistent way with respect to the true contact/no contact situation. A chattering behavior may thus result in the proximity of the contact point. While a more detailed analysis of such a situation is possible, this is out of our scope here.

On the other hand, the above control laws are fully independent from the knowledge of the mass value m, which only affects the behavior of the system during transient phases.

Let us turn the attention to the case when a force sensor is available, whose measure F_m ideally provides the contact force, or

$$F_m = F_c = \begin{cases} k_e(x-d), & x \geq d, \\ 0, & x < d. \end{cases}$$

Then, we can implement a *force control* law of the form

$$u = F_d + k_f(F_d - F_m) - k_d\dot{x}, \qquad \text{with } k_f > 0 \text{ and } k_d > 0. \tag{7}$$

with a constant feedforward, a term proportional to the force error, and a velocity damping term (as before). When in contact, the resulting closed-loop equation is

$$m\ddot{x} + k_d\dot{x} + (1+k_f)\,k_e(x-d) = (1+k_f)\,F_d. \tag{8}$$

Equation (8) represents again an asymptotically stable system, whose position and contact force converge to their desired values, i.e.,

$$x_E = x_d = d + \frac{1}{k_e}\,F_d \qquad \text{and} \qquad F_E = F_d.$$

Note that the same control law (7) can be applied also during a phase of no contact (when $F_m = 0$). In that case, the closed-loop equation takes the form

$$m\ddot{x} + k_d\dot{x} = (1+k_f)\,F_d,$$

and the (approaching) velocity of the peg will converge to the constant value

$$\dot{x}_E = \frac{1+k_f}{k_d}F_d > 0,$$

and proceed in this way until contact is established.

An alternative to the addition of the constant force feedforward F_d in (7) is the use of a PI control law on the force error $e_f = F_d - F_m$ (with $F_m = F_c$), still complemented by a velocity damping term, i.e.,

$$u(t) = k_f(F_d - F_m(t)) + k_i\int_0^t (F_d - F_m(\tau))d\tau - k_d\dot{x}(t), \quad \text{with } k_d > 0, k_i > 0, \text{ and (at least) } k_f > 0. \tag{9}$$

It can be shown that the control scheme obtained by using (9) is equivalent to closing in a unitary feedback loop the transfer function

$$G(s) = \frac{F_m(s)}{e_f(s)} = \frac{k_e\,(k_f s + k_i)}{s\,(ms^2 + k_d s + k_e)}, \tag{10}$$

yielding the closed-loop system $W(s) = F_m(s)/F_d(s) = G(s)/(1+G(s))$. The denominator of $W(s)$ is

$$\text{den}\;W(s) = ms^3 + k_d s^2 + (1+k_f)k_e s + k_i k_e$$

and, by the Routh criterion, its roots are all in the open left-hand side of the complex plane if and only if

$$k_d > 0, \qquad k_i > 0, \qquad k_f > m\,\frac{k_i}{k_d} - 1.$$

Under these conditions, the closed-loop system is asymptotically stable and the force error e_f will converge to zero.

As a matter of fact, both force control laws (7) and (9) achieve always their target, without the need to know any of the parameters k_e, m, or d. The cost of including a force sensor is paid back by the achieved robustness of the control laws that can be designed with the force measurements.

* * * * *

Robotics II

September 12, 2016

Exercise 1

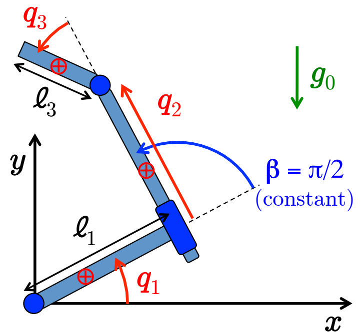


Figure 1: A RPR planar robot moving in a vertical plane.

Derive the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ and the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ in the dynamic model of the planar RPR robot in Fig. 1, using the Lagrangian coordinates $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}^T$ defined therein. Determine all equilibrium configurations $\boldsymbol{q}_0$ under no external or dissipative forces/torques nor actuation inputs.

Exercise 2

$$L = T - U =$$
$$= \frac{1}{2} m_1 \ddot{\theta}^2 + \frac{1}{2} m_2 \ddot{q}^2 -$$
$$- m_1 g_0 q + U_E$$

$$T_1 = \frac{1}{2} m_1 \ddot{\theta}^2$$
$$T_2 = \frac{1}{2} m_2 \ddot{q}^2$$
$$U = U_G + U_E$$
$$U_1 = U_{G1} + U_{E1} = \emptyset$$
$$U_2 = U_{G2} + U_{E2} =$$
$$= -m_1 g_0 q + U_E$$

$$u_1 = \frac{d}{dt}\frac{\partial L}{\partial q_1} - \frac{\partial L}{\partial q_1} \qquad u_2 = \frac{d}{dt}\frac{\partial L}{\partial q_2} - \frac{\partial L}{\partial q_2} = \emptyset \longrightarrow$$

NOW!

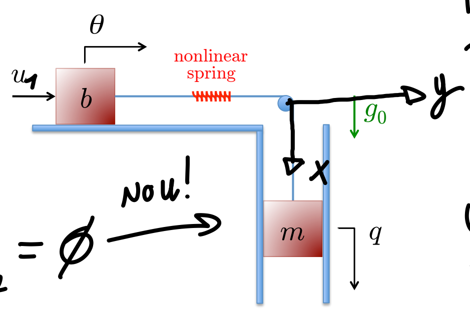


Figure 2: A two-mass system connected by a nonlinear spring, under the action of gravity.

In the mechanical system shown in Fig. 2, the first body (of mass $b > 0$ and position θ) is actuated by a force u and is connected to the second body (of mass $m > 0$ and position q) through a nonlinear spring having potential energy

$$U_e = \frac{1}{2}k(q-\theta)^2 + \frac{1}{4}k_n(q-\theta)^4, \qquad k, k_n > 0.$$

FROM T FROM U

- Derive the dynamic model of this system by following a Lagrangian approach and using as generalized coordinates (θ, q).

$$M\ddot{q} + C + g = u \longrightarrow \text{FROM } L = T - U$$

- Determine the (unique!) equilibrium position $\bar{\theta}$ for the first mass and the required constant input force $\bar{u}$ to be applied in order to keep the second mass at a desired position (height) $\bar{q}$.
- Verify your result by computing the values of $\bar{\theta}$ and $\bar{u}$ for the following numerical data:

$$\bar{q} = 0.1\,[\text{m}], \quad b = m = 3\,[\text{kg}], \quad k = 1000\,[\text{N/m}], \quad k_n = 10000\,[\text{N/m}^3], \quad g_0 = 9.81\,[\text{m/s}^2].$$

[150 minutes; open books]

$$\text{EQ}: \left[(q,\theta) = (\bar{q}, \bar{\theta}),\ (\dot{q}, \dot{\theta}) = (0,0)\right] \longmapsto \ddot{q}, \ddot{\theta} = 0$$

$$g(\bar{q}, \bar{\theta}) = \bar{u}$$

Solution

September 12, 2016

Exercise 1

For link i, $i = 1, 2, 3$, let m_i be its mass and I_i its inertia around an axis normal to the plane of motion and passing through the center of mass. Moreover, for $i = 1$ and $i = 3$, d_i is the distance of the center of mass of link i from the axis of joint i, while for the second link, d_2 will denote the (constant) distance of its center of mass from the axis of joint 3 —see Fig. 3.

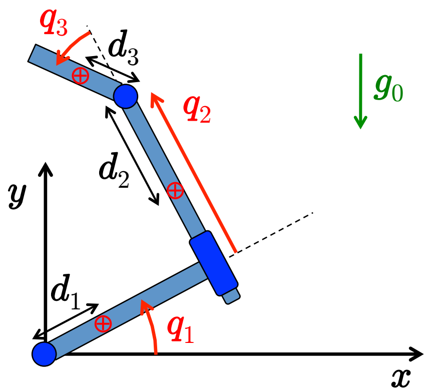


Figure 3: Parameters d_i defining the location of the center mass of the links of the RPR robot.

To derive the robot dynamic model terms, we follow a Lagrangian approach. For obtaining the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$, we compute the kinetic energy $T = T_1 + T_2 + T_3$ of the three robot links. For the first link, it is

$$T_1 = \frac{1}{2} \left(I_1 + m_1 d_1^2\right) \dot{q}_1^2.$$

For the second link, the position of the center of mass is[1]

$$\boldsymbol{p}_{c2} = \begin{pmatrix} \ell_1 \cos q_1 - (q_2 - d_2) \sin q_1 \\ \ell_1 \sin q_1 + (q_2 - d_2) \cos q_1 \end{pmatrix}, \tag{1}$$

where ℓ_1 is the length of link 1. Thus, the velocity of this center of mass is

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -\left(\ell_1 \sin q_1 + (q_2 - d_2) \cos q_1\right) \dot{q}_1 - \sin q_1 \, \dot{q}_2 \\ \left(\ell_1 \cos q_1 - (q_2 - d_2) \sin q_1\right) \dot{q}_1 + \cos q_1 \, \dot{q}_2 \end{pmatrix}$$

and its squared norm is

$$\|\boldsymbol{v}_{c2}\|^2 = \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} = \left(\ell_1^2 + (q_2 - d_2)^2\right) \dot{q}_1^2 + \dot{q}_2^2 + 2\ell_1 \, \dot{q}_1 \dot{q}_2.$$

The (scalar) angular velocity of link 2 is simply $\dot{q}_1$. As a result,

$$T_2 = \frac{1}{2} \left[\left(I_2 + m_2 \left(\ell_1^2 + (q_2 - d_2)^2\right)\right) \dot{q}_1^2 + m_2 \, \dot{q}_2^2 + 2 m_2 \ell_1 \, \dot{q}_1 \dot{q}_2 \right].$$

[1] We take into account the following identities (for $\beta = \pi/2$): $\cos(q_1 + \pi/2) = -\sin q_1$, $\sin(q_1 + \pi/2) = \cos q_1$.

Similarly, the position of the center of mass of the third link is

$$\boldsymbol{p}_{c3} = \begin{pmatrix} \ell_1 \cos q_1 - q_2 \sin q_1 - d_3 \sin(q_1+q_3) \\ \ell_1 \sin q_1 + q_2 \cos q_1 + d_3 \cos(q_1+q_3) \end{pmatrix}. \tag{2}$$

Its velocity is

$$\boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} -(\ell_1 \sin q_1 + q_2 \cos q_1)\,\dot{q}_1 - \sin q_1\,\dot{q}_2 - d_3 \cos(q_1+q_3)(\dot{q}_1+\dot{q}_3) \\ (\ell_1 \cos q_1 - q_2 \sin q_1)\,\dot{q}_1 + \cos q_1\,\dot{q}_2 - d_3 \sin(q_1+q_3)(\dot{q}_1+\dot{q}_3) \end{pmatrix}$$

and the squared norm becomes

$$\|\boldsymbol{v}_{c3}\|^2 = \left(\ell_1^2+q_2^2\right)\dot{q}_1^2+\dot{q}_2^2+d_3^2(\dot{q}_1+\dot{q}_3)^2+2\ell_1\dot{q}_1\dot{q}_2-2d_3\sin q_3(\dot{q}_1+\dot{q}_3)(\ell_1\dot{q}_1+\dot{q}_2)+2d_3q_2\cos q_3(\dot{q}_1+\dot{q}_3)\dot{q}_1.$$

The (scalar) angular velocity of link 3 is simply $(\dot{q}_1+\dot{q}_3)$. As a result,

$$\begin{aligned} T_3 &= \tfrac{1}{2}\left[\left(I_3+m_3d_3^2\right)(\dot{q}_1+\dot{q}_3)^2 + m_3\left(\ell_1^2+q_2^2\right)\dot{q}_1^2 + m_3\,\dot{q}_2^2\right. \\ &\quad \left. + 2m_3\ell_1\,\dot{q}_1\dot{q}_2 + 2m_3d_3(\dot{q}_1+\dot{q}_3)\left(q_2\cos q_3\dot{q}_1 - \sin q_3(\ell_1\dot{q}_1+\dot{q}_2)\right)\right]. \end{aligned}$$

Therefore,

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \begin{pmatrix} b_{11}(q_2,q_3) & b_{12}(q_3) & b_{13}(q_2,q_3) \\ & b_{22} & b_{23}(q_3) \\ \mathit{symm} & & b_{33} \end{pmatrix} \dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

from which the elements b_{ij} of the 3×3, symmetric, positive definite inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ can be extracted:

$$\begin{aligned} b_{11}(q_2,q_3) &= I_1+m_1d_1^2+I_2+m_2d_2^2+I_3+m_3d_3^2+(m_2+m_3)\ell_1^2 \\ &\quad -2m_2d_2q_2+(m_2+m_3)q_2^2+2m_3d_3\left(q_2\cos q_3-\ell_1\sin q_3\right) \\ &= \pi_1+\pi_2\,q_2+\pi_3\,q_2^2+2\pi_4\left(q_2\cos q_3-\ell_1\sin q_3\right) \\ b_{12}(q_3) &= (m_2+m_3)\ell_1-m_3d_3\sin q_3 = \pi_3\,\ell_1-\pi_4\sin q_3 \\ b_{13}(q_2,q_3) &= I_3+m_3d_3^2+m_3d_3\left(q_2\cos q_3-\ell_1\sin q_3\right) = \pi_5+\pi_4\left(q_2\cos q_3-\ell_1\sin q_3\right) \\ b_{22} &= m_2+m_3 = \pi_3 \\ b_{23}(q_3) &= -m_3d_3\sin q_3 = -\pi_4\sin q_3 \\ b_{33} &= I_3+m_3d_3^2 = \pi_5. \end{aligned} \tag{3}$$

In the expressions (3) of the elements of $\boldsymbol{B}(\boldsymbol{q})$, we have assumed that the kinematic parameter ℓ_1 is known and found thus a linear parameterization in terms of five dynamic coefficients π_i, $i=1,\ldots,5$.

For obtaining the vector $\boldsymbol{g}(\boldsymbol{q})$, we compute the potential energy $U=U_1+U_2+U_3$ due to gravity for the three robot links. Using the expressions of the y-component of the vectors $\boldsymbol{p}_{ci}$, $i=1,2,3$ (see also eqs. (1) and (2)), we have

$$\begin{aligned} U_1(q_1) &= m_1g_0\,p_{c1,y} = m_1g_0\,d_1\sin q_1 \\ U_2(q_1,q_2) &= m_2g_0\,p_{c2,y} = m_2g_0\left(\ell_1\sin q_1+(q_2-d_2)\cos q_1\right) \\ U_3(q_1,q_2,q_3) &= m_3g_0\,p_{c3,y} = m_3g_0\left(\ell_1\sin q_1+q_2\cos q_1+d_3\cos(q_1+q_3)\right). \end{aligned}$$

Therefore,

$$
\begin{aligned}
\boldsymbol{g}(\boldsymbol{q}) &= \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T \\
&= \begin{pmatrix} (m_1 d_1 + (m_2+m_3)\ell_1)\, g_0 \cos q_1 - (m_2(q_2-d_2) + m_3 q_2)\, g_0 \sin q_1 - m_3 d_3 g_0 \sin(q_1+q_3) \\ (m_2+m_3) g_0 \cos q_1 \\ -m_3 d_3 g_0 \sin(q_1+q_3) \end{pmatrix}.
\end{aligned}
\tag{4}
$$

Solving for $\boldsymbol{g}(\boldsymbol{q}_0) = \mathbf{0}$ provides all equilibrium configurations $\boldsymbol{q}_0$. From the second component in (4), it follows that $q_{0,1} = \pm\pi/2$, which confirms also the intuition that the prismatic joint axis should be horizontal at the equilibrium. Plugging this into the third equation leads again to $q_{0,3} = \pm\pi/2$. Imposing these two conditions in the first equation, $g_1(\boldsymbol{q}_0) = 0$ provided the additional condition $q_2 = m_2 d_2/(m_2+m_3)$. Thus, all unforced equilibrium configurations for this RPR robot are of the form

$$
\boldsymbol{q}_0 = \begin{pmatrix} \pm\dfrac{\pi}{2} & \dfrac{m_2\, d_2}{m_2+m_3} & \pm\dfrac{\pi}{2} \end{pmatrix}^T.
$$

Exercise 2

The kinetic and potential energies of the mechanical system in Fig. 2 are given by

$$
T = \frac{1}{2} b\, \dot{\theta}^2 + \frac{1}{2} m\, \dot{q}^2, \qquad U = U_e + U_g = \frac{1}{2} k (q-\theta)^2 + \frac{1}{4} k_n (q-\theta)^4 - m g_0 q.
$$

Thus, applying the Euler-Lagrange equations to $L = T - U$ yields the two second-order differential equations

$$
b\ddot{\theta} + k(\theta - q) + k_n(\theta-q)^3 = u \tag{5}
$$

$$
m\ddot{q} - m g_0 + k(q-\theta) + k_n (q-\theta)^3 = 0, \tag{6}
$$

which are nonlinear due to the cubic terms in the deformation $\delta = q - \theta$ of the spring.

For the equilibrium conditions, we set $\ddot{\theta} = \ddot{q} = 0$ in eqs. (5)–(6). From the second one, we obtain the following algebraic equation:

$$
k_n (q-\theta)^3 + k(q-\theta) - m g_0 = 0.
$$

This is a cubic equation of the form

$$
\delta^3 + p\delta + r = 0, \qquad \text{with} \quad p = \frac{k}{k_n}, \quad r = -\frac{m g_0}{k_n}, \tag{7}
$$

which is known to have the single real solution[2]

$$
\delta = \sqrt[3]{-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{p^3}{27}}}. \tag{8}
$$

[2]The (depressed) cubic equation in (7) was studied already in the XVI century. The formula (8) is attributed to the mathematician Gerolamo Cardano, but in fact is due to Scipione del Ferro and Niccolò Fontana (also known as Tartaglia).

Therefore, for a given desired position $q = \bar{q}$ of the mass m, we compute from (8) the spring deformation $\bar{\delta}$ at steady state and set

$$\bar{\theta} = \bar{q} - \bar{\delta}. \tag{9}$$

Moreover, from eq. (5) at steady state we obtain the required input force

$$\bar{u} = k(\bar{\theta} - \bar{q}) + k_n(\bar{\theta} - \bar{q})^3 = k\bar{\delta} + k_n\bar{\delta}^3 = -mg_0. \tag{10}$$

The input force balances the weight of the mass m, as reflected through the elastic force τ_e of the deformed spring. Note that that the mass b plays no role in this analysis.

Using now the given numerical data, and in particular $\bar{q} = 0.1$, $k = 1000$, $k_n = 10000$ and $m = 3$, we compute

$$\delta = 0.0292, \qquad \bar{\theta} = 0.0708, \qquad \bar{u} = -29.4300. \tag{11}$$

As a result, the static deformation of the nonlinear spring at the equilibrium is equal to slightly less than 3 cm. The deformation-force characteristics $\tau_e = k\delta + k_n\delta^3$ of the chosen spring is shown in Fig. 4, where we have indicated also the actual deformation $\bar{\delta}$ at equilibrium. In correspondence to this value, we can easily check that the elastic force is $\tau_e = -\bar{u}$.

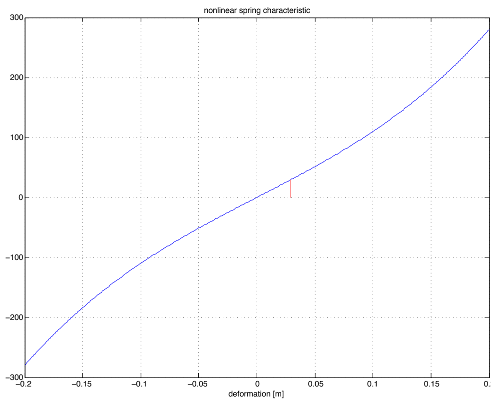


Figure 4: The nonlinear deformation-force characteristics of the spring for $k = 10^3$ [N/m] and $k_n = 10^4$ [N/m^3]. The equilibrium condition for the given problem data is reported in red.

* * * * *

Robotics II

October 28, 2016

Exercise 1

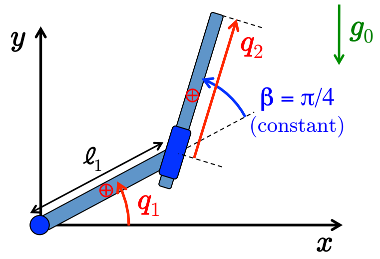


Figure 1: A planar RP robot with skewed prismatic joint.

Derive the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ and the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ in the dynamic model of the planar RP robot in Fig. 1, using the Lagrangian coordinates $\boldsymbol{q} = (q_1, q_2)$ defined therein and assuming uniform mass distribution for the two links. Determine all equilibrium configurations $\boldsymbol{q}_0$ under no external or dissipative forces/torques nor actuation inputs.

Exercise 2

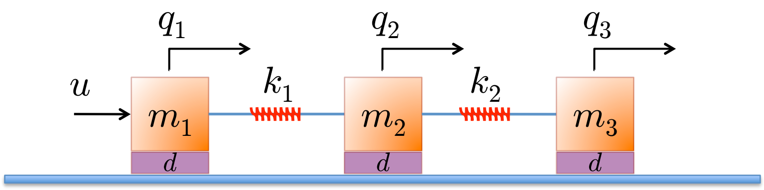


Figure 2: A mechanical system consisting of three masses connected by linear springs.

In the three-body mechanical system shown in Fig. 2, the first body (with mass $m_1 > 0$ and position q_1) is actuated by a force u and is connected to a second body (with mass $m_2 > 0$ and position q_2) through a spring of constant stiffness $k_1 > 0$. The second body is in turn connected to a third one (having mass $m_3 > 0$ and position q_3) through another spring of constant stiffness $k_2 > 0$. Each mass is subject to a dissipative force when moving on the ground, in the form of a viscous friction with coefficient $d > 0$ (equal for all three masses).

- Derive the dynamic model of this system by following a Lagrangian approach and using the set of generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$.
- Determine all equilibrium states, if any, of the system when $u = 0$, as well as all steady-state conditions when a constant force $u = \bar{u} > 0$ is being applied.
- Prove that a control law of the form

$$u = k_p(q_d - q_1), \qquad k_p > 0 \tag{1}$$

  will globally asymptotically stabilize the closed-loop system to a unique equilibrium state (which one?). *Hint: Use either a energy-based Lyapunov approach or exploit the linearity of the system dynamics.*

[150 minutes; open books]

Solution

October 28, 2016

Exercise 1

For $i = 1, 2$, let m_i be the mass of link i and I_i its inertia around an axis normal to the plane of motion, passing through the center of mass (which is at the midpoint of the link, because of the assumption on uniform mass distribution). Moreover, let $d > 0$ be the (constant) distance of the center of mass of link 2 (which is of unspecified length) from its tip, namely from the point characterized by the coordinate q_2. With this in mind, we follow a Lagrangian approach and compute the kinetic and the potential energy of the robot system in order to obtain, respectively, $\boldsymbol{B}(\boldsymbol{q})$ and $\boldsymbol{g}(\boldsymbol{q})$.

The kinetic energy of the first link is

$$T_1 = \frac{1}{2}\left(I_1 + m_1\left(\frac{\ell_1}{2}\right)^2\right)\dot{q}_1^2.$$

For the second link, the position of the center of mass is

$$\boldsymbol{p}_{c2} = \begin{pmatrix} \ell_1 \cos q_1 + (q_2 - d)\cos(q_1 + \pi/4) \\ \ell_1 \sin q_1 + (q_2 - d)\sin(q_1 + \pi/4) \end{pmatrix}.$$

Thus, its velocity is

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -\left(\ell_1 \sin q_1 + (q_2 - d)\sin(q_1 + \pi/4)\right)\dot{q}_1 + \cos(q_1 + \pi/4)\,\dot{q}_2 \\ \left(\ell_1 \cos q_1 + (q_2 - d)\cos(q_1 + \pi/4)\right)\dot{q}_1 + \sin(q_1 + \pi/4)\,\dot{q}_2 \end{pmatrix}.$$

The (scalar) angular velocity of link 2 is simply $\dot{q}_1$. As a result, the kinetic energy of the second link is

$$T_2 = \frac{1}{2}\left(\left(I_2 + m_2\left(\ell_1^2 + (q_2 - d)^2 + 2\ell_1(q_2 - d)\cos(\pi/4)\right)\right)\dot{q}_1^2 + m_2\,\dot{q}_2^2 + 2m_2\ell_1\sin(\pi/4)\,\dot{q}_1\dot{q}_2\right).$$

Therefore,

$$T = T_1 + T_2 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \begin{pmatrix} b_{11}(q_2) & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \dot{\boldsymbol{q}} = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

from which the elements b_{ij} of the 2×2, symmetric, positive definite inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ are obtained as

$$\begin{aligned} b_{11}(q_2) &= I_1 + m_1\left(\frac{\ell_1}{2}\right)^2 + I_2 + m_2\left(\ell_1^2 + (q_2 - d)^2 + \ell_1(q_2 - d)\sqrt{2}\right) \\ b_{12} &= b_{21} = m_2\ell_1\frac{\sqrt{2}}{2} \\ b_{22} &= m_2, \end{aligned}$$

where $\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$ has been used.

For the potential energy $U = U_1 + U_2$ of the two robot links due to gravity, we use the expressions of the y-component of their center of mass:

$$U_1(q_1) = m_1 g_0 \left(\frac{\ell_1}{2}\right) \sin q_1,$$
$$U_2(q_1, q_2) = m_2 g_0 \left(\ell_1 \sin q_1 + (q_2 - d) \sin(q_1 + \pi/4)\right).$$

Therefore, the gravity vector is

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = g_0 \begin{pmatrix} \left(\frac{m_1}{2} + m_2\right) \ell_1 \cos q_1 + m_2(q_2 - d)\cos(q_1 + \pi/4) \\ m_2 \sin(q_1 + \pi/4) \end{pmatrix}.$$

Solving for $\boldsymbol{g}(\boldsymbol{q}_0) = \boldsymbol{0}$ provides all equilibrium configurations $\boldsymbol{q}_0$. For this RP robot, there are two unforced equilibria:

$$\boldsymbol{q}_0 = \begin{pmatrix} -\frac{\pi}{4} & d - \frac{\ell_1\sqrt{2}}{2}\left(1 + \frac{m_1}{2m_2}\right) \end{pmatrix}^T$$

and

$$\boldsymbol{q}_0 = \begin{pmatrix} \frac{3\pi}{4} & d - \frac{\ell_1\sqrt{2}}{2}\left(1 + \frac{m_1}{2m_2}\right) \end{pmatrix}^T.$$

Exercise 2

By following a Lagrangian approach, we compute first the kinetic and the potential (elastic) energy of the mechanical system in Fig. 2:

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$$T = \frac{1}{2}\left(m_1\dot{q}_1^2 + m_2\dot{q}_2^2 + m_3\dot{q}_3^2\right), \qquad U = \frac{1}{2}\left(k_1(q_1 - q_2)^2 + k_2(q_2 - q_3)^2\right).$$

Applying the Euler-Lagrange equations to $L = T - U$ and keeping into account the dissipative forces due to viscous friction, yields the three second-order differential equations

$$\begin{aligned} m_1\ddot{q}_1 + k_1(q_1 - q_2) &= u - d\dot{q}_1 \\ m_2\ddot{q}_2 + k_1(q_2 - q_1) + k_2(q_2 - q_3) &= -d\dot{q}_2 \\ m_3\ddot{q}_3 + k_2(q_3 - q_2) &= -d\dot{q}_3. \end{aligned} \tag{2}$$

Indeed, the same result would have been obtained by Newton's law (balance of forces) applied to the three masses.

For the unforced ($u = 0$) equilibrium configurations, we set $\dot{\boldsymbol{q}} = \ddot{\boldsymbol{q}} = \boldsymbol{0}$ in (2) and obtain an infinity of equilibria $\boldsymbol{q}_0$, all with equal $q_1 = q_2 = q_3$ at a common arbitrary value. The two springs will be undeformed in this rest condition.

For the forced case with $u = \bar{u} > 0$, it is rather tedious but straightforward exercise (using, e.g., Laplace transforms) to verify that the system response to a constant force input will asymptotically reach a steady-state condition, where all masses will be moving at the same constant speed $\dot{q}_1 = \dot{q}_2 = \dot{q}_3 = \bar{u}/3d$. This specific value can be found by setting $\ddot{\boldsymbol{q}} = \boldsymbol{0}$ in (2) and summing up the three equations (the elastic forces cancel each other). In this steady-state condition, the deformation of the two springs can be computed again from the model as $\ddot{q} = 0,\ \dot{q}_i = \frac{\bar{u}}{3d}\ \forall i,\ \bar{u} = \text{cost.}$

$$\delta_{12} = q_1 - q_2 = \frac{1}{k_1}\frac{2\bar{u}}{3}, \qquad \delta_{23} = q_2 - q_3 = \frac{1}{k_2}\frac{\bar{u}}{3},$$

being in general $\delta_{12} \neq \delta_{23}$. Note also that the steady-state deformations are independent from the actual values of the masses (these will influence instead the transient behavior).

We can prove that the control law (1) asymptotically stabilizes the *unique* closed-loop equilibrium state $q_1 = q_2 = q_3 = q_d$, $\dot{\boldsymbol{q}} = \mathbf{0}$, by using a simple Lyapunov/LaSalle argument. Define the Lyapunov candidate as the total system energy $E = T + U$, plus the control energy in the form of a virtual spring of stiffness $k_p > 0$, with rest condition in $q_1 = q_d$:

$$V = \frac{1}{2}\left(m_1\dot{q}_1^2 + m_2\dot{q}_2^2 + m_3\dot{q}_3^2\right) + \frac{1}{2}\left(k_1(q_1 - q_2)^2 + k_2(q_2 - q_3)^2 + k_p(q_d - q_1)^2\right).$$

Indeed $V \geq 0$, whereas $V = 0$ if and only if the closed-loop system is at the desired equilibrium state. Taking the time derivative of V, using (2), and simplifying terms leads to

$$\dot{V} = -d\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) \leq 0.$$

The non-positivity of $\dot{V}$ is thus guaranteed by the presence of viscous friction in the mass-ground contact (otherwise, we could inject dissipation by just adding a damping term $-k_d\dot{q}_1$, $k_d > 0$, in the control law). To complete the proof, we invoke LaSalle's theorem. From $\dot{V} = 0 \iff \dot{\boldsymbol{q}} = \mathbf{0}$, we determine the largest invariant set of states contained in $\dot{V} = 0$. When $\dot{\boldsymbol{q}} \equiv \mathbf{0}$, from the third equation in (2), $\ddot{q}_3 = 0$ necessarily implies $q_3 = q_2$. Substituting backward in the second equation under the same operative conditions yields necessarily $q_2 = q_1$. Finally, from the first equation

$$0 = m_1\ddot{q}_1 = k_p(q_d - q_1) - k_1(q_1 - q_2) - d\dot{q}_1 = k_p(q_d - q_1) \qquad \Rightarrow \qquad q_1 = q_d,$$

and thus the configuration $\boldsymbol{q}_e$ having $q_1 = q_2 = q_3 = q_d$ with velocity $\dot{\boldsymbol{q}} = \mathbf{0}$ is the only invariant state for the closed-loop system. The result is thus proven. Since the system is linear, asymptotic stability will be equivalent to exponential stability. For the same reason, one could have analyzed the characteristic polynomial of the closed-loop linear system, using the Routh criterion to establish stability. However, this would require longer computations in order to prove the same result.

* * * * *

Robotics 2

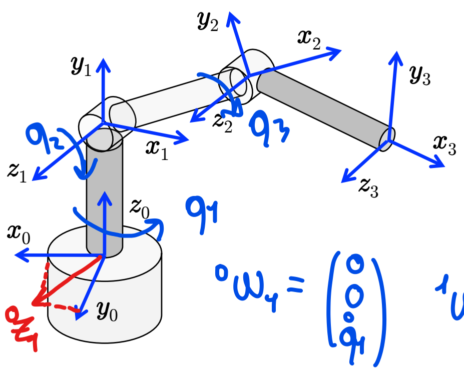
Midterm test in classroom – March 29, 2017

Exercise 1

For the 3R spatial robot in Fig. 1, determine the symbolic expression of the elements of the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ using the recursive algorithm with moving frames for the computation of the kinetic energy of the links. The coordinates $\boldsymbol{q} \in \mathbb{R}^3$ to be used are those of the Denavit-Hartenberg (DH) convention.

$^0z_1 = \begin{pmatrix} \cos(q_1) \\ \sin(q_1) \\ 0 \end{pmatrix} = {}^0z_2$ $\qquad$ $^2\omega_3 = {}^0R_2^T\, {}^0\omega_3$

$\begin{pmatrix} c_1(\dot q_2 + \dot q_3) \\ s_1(\dot q_2 + \dot q_3) \\ \dot q_1 \end{pmatrix} = {}^0\omega_3$



i	α_i	d_i	a_i	θ_i
1	$\pi/2$	L_1	0	q_1
2	0	0	L_2	q_2
3	0	0	L_3	q_3

$\| \quad {}^0\omega_1 + \begin{pmatrix} c_1(\dot q_2 + \dot q_3) \\ s_1(\dot q_2 + \dot q_3) \end{pmatrix} \quad \|$

$^0\omega_1 = \begin{pmatrix} 0 \\ 0 \\ \dot q_1 \end{pmatrix}$ $\qquad$ $^1\omega_2 = \begin{pmatrix} 0 \\ \dot q_1 \\ \dot q_2 \end{pmatrix}$ $\qquad$ $^0\omega_3 = \underbrace{{}^0z_0 \dot q_1}_{^0\omega_1} + {}^0z_1 \dot q_2 + {}^0z_2 \dot q_3$

Figure 1: An elbow-type 3R robot, with associated DH frames and table of parameters.

Besides the masses m_i, $i = 1, 2, 3$, of the links, the other constant dynamic parameters are specified as

$$^1\boldsymbol{r}_{1,c1} = \begin{pmatrix} A \\ -F \\ 0 \end{pmatrix}, \qquad ^2\boldsymbol{r}_{2,c2} = \begin{pmatrix} -C \\ 0 \\ 0 \end{pmatrix}, \qquad ^3\boldsymbol{r}_{3,c3} = \begin{pmatrix} -D \\ 0 \\ E \end{pmatrix}, \tag{1}$$

where A, C, D, E and F take positive values, and

$$^i\boldsymbol{I}_{ci} = \begin{pmatrix} I_{xx,i} & 0 & 0 \\ 0 & I_{yy,i} & 0 \\ 0 & 0 & I_{zz,i} \end{pmatrix}, \qquad i = 1, 2, 3. \tag{2}$$

Once $\boldsymbol{B}(\boldsymbol{q})$ has been obtained, define a set of dynamic coefficients that linearly parametrize the inertia matrix and is of the smallest possible cardinality.

Exercise 2

When the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ is a $m \times n$ matrix with full row rank m, its weighted pseudoinverse $\boldsymbol{J}_W^{\#}(\boldsymbol{q})$ takes the explicit form

$$\boldsymbol{J}_W^{\#}(\boldsymbol{q}) = \boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{q}) \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{q}) \right)^{-1}, \tag{3}$$

where $\boldsymbol{W}$ is a $n \times n$, symmetric, and positive definite matrix. The matrix $\boldsymbol{J}_W^{\#}$ in (3) satisfies three of the four identities that uniquely define a pseudoinverse. Prove that the weighted pseudoinverse takes the following more general form, which holds true even when the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ loses rank:

$$\boldsymbol{J}_W^{\#}(\boldsymbol{q}) = \boldsymbol{W}^{-1/2} \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{W}^{-1/2} \right)^{\#}. \tag{4}$$

Exercise 3

Consider a 4R planar robot with all links of equal length $\ell = 0.5$ [m]. The robot is stretched along the $\boldsymbol{x}_0$ axis, in the DH configuration $\boldsymbol{q} = \boldsymbol{0}$. The end-effector of the robot should execute an instantaneous linear velocity $\boldsymbol{v} = \begin{pmatrix} 0 & 10 \end{pmatrix}^T$ [m/s]. The joint velocities are bounded as $|\dot{q}_i| \leq V_i$, $i = 1, \ldots, 4$, with $V_1 = 4$, $V_2 = 2$, and $V_3 = V_4 = 1$ [rad/s]. Find a feasible joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ that executes the given Cartesian task, either as such or in a scaled way, while satisfying the hard bounds on joint velocity. Scale the task velocity $\boldsymbol{v}$ only if strictly needed. A solution with a lower norm is preferred.
Hint: The following useful expressions hold for the pseudoinverse of a block matrix $\boldsymbol{A}$ (with a submatrix $\boldsymbol{O}$ of zeros) and of a vector $\boldsymbol{u} \neq \boldsymbol{0}$:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{O} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{A}^{\#} = \begin{pmatrix} \boldsymbol{B}^{\#} \\ \boldsymbol{O}^T \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{O} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{A}^{\#} = \begin{pmatrix} \boldsymbol{B}^{\#} & \boldsymbol{O}^T \end{pmatrix}, \tag{5}$$

$$\boldsymbol{u} \in \mathbb{R}^n \quad \Rightarrow \quad \boldsymbol{u}^{\#} = \left(\boldsymbol{u}^T \boldsymbol{u}\right)^{-1} \boldsymbol{u}^T = \frac{\boldsymbol{u}^T}{\|\boldsymbol{u}\|^2}, \qquad \left(\boldsymbol{u}^T\right)^{\#} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|^2}. \tag{6}$$

Exercise 4

Consider a 3R planar robot with links of equal length $\ell = 1$ [m]. The primary task for this robot is to execute an instantaneous Cartesian velocity $\boldsymbol{v} \in \mathbb{R}^2$ with its end-effector. Denote the associated task Jacobian as $\boldsymbol{J}(\boldsymbol{q})$.

- When the robot is in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & \pi/3 & -2\pi/3 \end{pmatrix}^T$, use the Reduced Gradient (RG) method to determine the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ that realizes a desired Cartesian velocity $\boldsymbol{v} = \begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}^T$ [m/s] while maximizing the objective function

$$H(\boldsymbol{q}) = \sin^2 q_2 + \sin^2 q_3. \tag{7}$$

- As an auxiliary task, the robot should move so as to always keep the position $\boldsymbol{p}_2 = (x_2, y_2)$ of the endpoint of its second link on the circle defined by

$$x_2^2 + (y_2 - 1.5)^2 = 0.75. \tag{8}$$

  Determine the Jacobian $\boldsymbol{J}_a(\boldsymbol{q})$ associated to this auxiliary task. When the robot is in the configuration $\boldsymbol{q}_0$ defined above, is the robot in an algorithmic singularity? Can the two requested primary and auxiliary tasks be executed at the same time?

Exercise 5

Consider a 2R planar robot with the nominal kinematics expressed by the DH parameters in Tab. 1.

i	α_i	d_i	a_i	θ_i
1	0	0	L_1	q_1
2	0	0	L_2	q_2

Table 1: Nominal kinematic parameters of a 2R planar robot.

In order to improve the robot accuracy, we would like to perform a large number of calibration experiments in which the robot end-effector position $\boldsymbol{p} \in \mathbb{R}^2$ (in the plane of motion) is measured by an external laser system. To recover the residual errors of the nominal direct kinematics with respect to the external measurements, it is assumed that we require only the fine tuning of the parameters $\boldsymbol{a} \in \mathbb{R}^2$ and of the encoder measurements $\boldsymbol{\theta} \in \mathbb{R}^2$, whose adjustments will be performed simultaneously. What will be the expression of the 2×4 regressor matrix $\boldsymbol{\Phi}$ for a single calibration experiment?

[270 minutes (4.5 hours); open books, but no computer or smartphone]

Solution

March 29, 2017

Exercise 1

From the table of DH parameters given in Fig. 1, we compute the needed rotation matrices

$$
{}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad {}^2\boldsymbol{R}_3(q_3) = \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

with the usual shorthand notation for trigonometric quantities, e.g., $s_2 = \sin q_2$, $c_{23} = \cos(q_2 + q_3)$. Similarly, the following kinematic quantities will be used:

$$
{}^1\boldsymbol{r}_{0,1} = \begin{pmatrix} 0 \\ L_1 \\ 0 \end{pmatrix}, \qquad {}^2\boldsymbol{r}_{1,2} = \begin{pmatrix} L_2 \\ 0 \\ 0 \end{pmatrix}, \qquad {}^3\boldsymbol{r}_{2,3} = \begin{pmatrix} L_3 \\ 0 \\ 0 \end{pmatrix}.
$$

The moving frames algorithm is initialized with ${}^0\boldsymbol{\omega}_0 = \boldsymbol{0}$ and ${}^0\boldsymbol{v}_0 = \boldsymbol{0}$. Also, ${}^i\boldsymbol{z}_i = \boldsymbol{z}_0 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ for $i = 0, 1, 2$ (joint axis 1, 2, and 3, respectively).

Link $i = 1$

$$
{}^1\boldsymbol{\omega}_1 = {}^0\boldsymbol{R}_1^T(q_1)\,{}^0\boldsymbol{\omega}_1 = {}^0\boldsymbol{R}_1^T(q_1)\left({}^0\boldsymbol{\omega}_0 + \dot{q}_1\,{}^0\boldsymbol{z}_0\right) = {}^0\boldsymbol{R}_1^T(q_1)\begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
{}^1\boldsymbol{v}_1 = {}^0\boldsymbol{R}_1^T(q_1)\,{}^0\boldsymbol{v}_1 &= {}^0\boldsymbol{R}_1^T(q_1)\left({}^0\boldsymbol{v}_0 + {}^0\boldsymbol{\omega}_1 \times {}^0\boldsymbol{r}_{0,1}\right) \\
&= {}^0\boldsymbol{R}_1^T(q_1)\,{}^0\boldsymbol{\omega}_1 \times {}^0\boldsymbol{R}_1^T(q_1)\,{}^0\boldsymbol{r}_{0,1} = {}^1\boldsymbol{\omega}_1 \times {}^1\boldsymbol{r}_{0,1} = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ L_1 \\ 0 \end{pmatrix} = \boldsymbol{0}
\end{aligned}
$$

$$
{}^1\boldsymbol{v}_{c1} = {}^1\boldsymbol{v}_1 + {}^1\boldsymbol{\omega}_1 \times {}^1\boldsymbol{r}_{1,c1} = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} A \\ -F \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -A\dot{q}_1 \end{pmatrix}
$$

$$
\Rightarrow \quad T_1 = \frac{1}{2}\,m_1 \left\|{}^1\boldsymbol{v}_{c1}\right\|^2 + \frac{1}{2}\,{}^1\boldsymbol{\omega}_1^T\,{}^1\boldsymbol{I}_{c1}\,{}^1\boldsymbol{\omega}_1 = \frac{1}{2}\left(I_{yy,1} + m_1 A^2\right)\dot{q}_1^2.
$$

Link $i = 2$

$$
{}^2\boldsymbol{\omega}_2 = {}^1\boldsymbol{R}_2^T(q_2)\,{}^1\boldsymbol{\omega}_2 = {}^1\boldsymbol{R}_2^T(q_2)\left({}^1\boldsymbol{\omega}_1 + \dot{q}_2\,{}^1\boldsymbol{z}_1\right) = {}^1\boldsymbol{R}_2^T(q_2)\begin{pmatrix} 0 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} s_2\,\dot{q}_1 \\ c_2\,\dot{q}_1 \\ \dot{q}_2 \end{pmatrix}
$$

$$
\begin{aligned}
{}^2\boldsymbol{v}_2 = {}^1\boldsymbol{R}_2^T(q_2)\,{}^1\boldsymbol{v}_2 &= {}^1\boldsymbol{R}_2^T(q_2)\left({}^1\boldsymbol{v}_1 + {}^1\boldsymbol{\omega}_2 \times {}^1\boldsymbol{r}_{1,2}\right) \\
&= {}^1\boldsymbol{R}_2^T(q_2)\,{}^1\boldsymbol{\omega}_2 \times {}^1\boldsymbol{R}_2^T(q_2)\,{}^1\boldsymbol{r}_{1,2} = {}^2\boldsymbol{\omega}_2 \times {}^2\boldsymbol{r}_{1,2} = \begin{pmatrix} s_2\,\dot{q}_1 \\ c_2\,\dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} L_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ L_2\,\dot{q}_2 \\ -L_2 c_2\,\dot{q}_1 \end{pmatrix}
\end{aligned}
$$

$$
{}^2\boldsymbol{v}_{c2} = {}^2\boldsymbol{v}_2 + {}^2\boldsymbol{\omega}_2 \times {}^2\boldsymbol{r}_{2,c2} = \begin{pmatrix} 0 \\ L_2\,\dot{q}_2 \\ -L_2 c_2\,\dot{q}_1 \end{pmatrix} + \begin{pmatrix} s_2\,\dot{q}_1 \\ c_2\,\dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} -C \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ (L_2 - C)\,\dot{q}_2 \\ -(L_2 - C)\,c_2\,\dot{q}_1 \end{pmatrix}
$$

$$
\begin{aligned}
\Rightarrow \quad T_2 &= \tfrac{1}{2}\,m_2 \left\|{}^2\boldsymbol{v}_{c2}\right\|^2 + \tfrac{1}{2}\,{}^2\boldsymbol{\omega}_2^T\,{}^2\boldsymbol{I}_{c2}\,{}^2\boldsymbol{\omega}_2 \\
&= \tfrac{1}{2}\left(I_{xx,2}s_2^2 + I_{yy,2}c_2^2 + m_2(L_2 - C)^2 c_2^2\right)\dot{q}_1^2 + \tfrac{1}{2}\left(I_{zz,2} + m_2(L_2 - C)^2\right)\dot{q}_2^2.
\end{aligned}
$$

Link $i = 3$

$$ {}^3\boldsymbol{\omega}_3 = {}^2\boldsymbol{R}_3^T(q_3)\,{}^2\boldsymbol{\omega}_3 = {}^2\boldsymbol{R}_3^T(q_3)\left({}^2\boldsymbol{\omega}_2 + \dot q_3\,{}^2\boldsymbol{z}_2\right) = {}^2\boldsymbol{R}_3^T(q_3)\begin{pmatrix} s_2\,\dot q_1 \\ c_2\,\dot q_1 \\ \dot q_2 + \dot q_3 \end{pmatrix} = \begin{pmatrix} s_{23}\,\dot q_1 \\ c_{23}\,\dot q_1 \\ \dot q_2 + \dot q_3 \end{pmatrix} $$

$$ \begin{aligned} {}^3\boldsymbol{v}_3 \;&=\; {}^2\boldsymbol{R}_3^T(q_3)\,{}^2\boldsymbol{v}_3 = {}^2\boldsymbol{R}_3^T(q_3)\left({}^2\boldsymbol{v}_2 + {}^2\boldsymbol{\omega}_3 \times {}^2\boldsymbol{r}_{2,3}\right) = {}^2\boldsymbol{R}_3^T(q_3)\,{}^2\boldsymbol{v}_2 + {}^2\boldsymbol{R}_3^T(q_3)\,{}^2\boldsymbol{\omega}_3 \times {}^2\boldsymbol{R}_3^T(q_3)\,{}^2\boldsymbol{r}_{2,3} \\ &=\; {}^2\boldsymbol{R}_3^T(q_3)\,{}^2\boldsymbol{v}_2 + {}^3\boldsymbol{\omega}_3 \times {}^3\boldsymbol{r}_{2,3} = \begin{pmatrix} L_2 s_3\,\dot q_2 \\ L_2 c_3\,\dot q_2 \\ -L_2 c_2\,\dot q_1 \end{pmatrix} + \begin{pmatrix} s_{23}\,\dot q_1 \\ c_{23}\,\dot q_1 \\ \dot q_2 + \dot q_3 \end{pmatrix} \times \begin{pmatrix} L_3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_2 s_3\,\dot q_2 \\ L_2 c_3\,\dot q_2 + L_3\,(\dot q_2 + \dot q_3) \\ -\left(L_2 c_2 + L_3 c_{23}\right)\,\dot q_1 \end{pmatrix} \end{aligned} $$

$$ \begin{aligned} {}^3\boldsymbol{v}_{c3} = {}^3\boldsymbol{v}_3 + {}^3\boldsymbol{\omega}_3 \times {}^3\boldsymbol{r}_{3,c3} \;&=\; \begin{pmatrix} L_2 s_3\,\dot q_2 \\ L_2 c_3\,\dot q_2 + L_3\,(\dot q_2 + \dot q_3) \\ -\left(L_2 c_2 + L_3 c_{23}\right)\,\dot q_1 \end{pmatrix} + \begin{pmatrix} s_{23}\,\dot q_1 \\ c_{23}\,\dot q_1 \\ \dot q_2 + \dot q_3 \end{pmatrix} \times \begin{pmatrix} -D \\ 0 \\ E \end{pmatrix} \\ &=\; \begin{pmatrix} L_2 s_3\,\dot q_2 + E c_{23}\,\dot q_1 \\ L_2 c_3\,\dot q_2 + (L_3 - D)\,(\dot q_2 + \dot q_3) - E s_{23}\,\dot q_1 \\ -\left(L_2 c_2 + (L_3 - D)\,c_{23}\right)\,\dot q_1 \end{pmatrix} \end{aligned} $$

$$ \begin{aligned} \Rightarrow\quad T_3 \;&=\; \tfrac{1}{2}\,m_3\left\|{}^3\boldsymbol{v}_{c3}\right\|^2 + \tfrac{1}{2}\,{}^3\boldsymbol{\omega}_3^T\,{}^3\boldsymbol{I}_{c3}\,{}^3\boldsymbol{\omega}_3 \\ &=\; \tfrac{1}{2}\,m_3\Big(E^2\dot q_1^2 + \left(L_2 c_2 + (L_3 - D)\,c_{23}\right)^2\dot q_1^2 + L_2^2\dot q_2^2 + (L_3 - D)^2\,(\dot q_2 + \dot q_3)^2 \\ &\qquad + 2\,(L_3 - D)\,(L_2 c_3\,\dot q_2 - E s_{23}\,\dot q_1)\,(\dot q_2 + \dot q_3) - 2 E L_2 s_2\,\dot q_1 \dot q_2\Big) \\ &\quad + \tfrac{1}{2}\left(I_{xx,3} s_{23}^2 + I_{yy,3} c_{23}^2\right)\dot q_1^2 + \tfrac{1}{2}\,I_{zz,3}\,(\dot q_2 + \dot q_3)^2\,. \end{aligned} $$

Robot inertia matrix

From

$$ T = \sum_{i=1}^{3} T_i = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} b_{ij}(\boldsymbol{q})\,\dot q_i \dot q_j, $$

we obtain the elements b_{ij} of the symmetric inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ as

$$ \begin{aligned} b_{11} &= I_{yy,1} + m_1 A^2 + I_{xx,2} s_2^2 + \left(I_{yy,2} + m_2 (L_2 - C)^2\right) c_2^2 \\ &\qquad + m_3 E^2 + I_{xx,3}\,s_{23}^2 + I_{yy,3}\,c_{23}^2 + m_3\left(L_2 c_2 + (L_3 - D)\,c_{23}\right)^2 \\ b_{12} &= -m_3 E\left(L_2 s_2 + (L_3 - D)\,s_{23}\right) \\ b_{13} &= -m_3 E\,(L_3 - D)\,s_{23} \\ b_{21} &= b_{12} \\ b_{22} &= I_{zz,2} + m_2 (L_2 - C)^2 + I_{zz,3} + m_3\,(L_3 - D)^2 + m_3 L_2^2 + 2\,m_3 L_2\,(L_3 - D)\,c_3 \\ b_{23} &= I_{zz,3} + m_3\,(L_3 - D)^2 + m_3 L_2\,(L_3 - D)\,c_3 \\ b_{31} &= b_{13} \\ b_{32} &= b_{23} \\ b_{33} &= I_{zz,3} + m_3\,(L_3 - D)^2\,. \end{aligned} $$

Minimal parametrization

Reorganizing the squares of trigonometric functions, the element b_{11} can be also rewritten as

$$ \begin{aligned} b_{11} = I_{yy,1} &+ m_1 A^2 + I_{yy,2} + m_2 (L_2 - C)^2 + I_{yy,3} + m_3\,(L_3 - D)^2 + m_3\left(L_2^2 + E^2\right) \\ &+ \left(I_{xx,2} - I_{yy,2} - m_2 (L_2 - C)^2 - m_3 L_2^2\right) s_2^2 \\ &+ \left(I_{xx,3} - I_{yy,3} - m_3\,(L_3 - D)^2\right) s_{23}^2 + 2\,m_3 L_2\,(L_3 - D)\,c_2 c_{23}. \end{aligned} \tag{9} $$

Using the expression (9) for b_{11}, and introducing constant dynamic coefficients a_i ($i = 1, \ldots, 8$), the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ takes the more compact, linearly parametrized form

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 + a_2 s_2^2 + a_3 s_{23}^2 + 2a_4 c_2 c_{23} & -a_7 s_2 - a_8 s_{23} & -a_8 s_{23} \\ -a_7 s_2 - a_8 s_{23} & a_5 + 2a_4 c_3 & a_6 + a_4 c_3 \\ -a_8 s_{23} & a_6 + a_4 c_3 & a_6 \end{pmatrix}. \tag{10}$$

Note that the most notable simplification occurs when $E = 0$. In this case, it follows that $a_7 = a_8 = 0$ and the inertia matrix becomes block diagonal.

Exercise 2

One needs to verify that the expression in (4) satisfies the three identities (dropping dependency on $\boldsymbol{q}$):

$$(i)\;\; \boldsymbol{J}\boldsymbol{J}_W^{\#}\boldsymbol{J} = \boldsymbol{J}, \qquad (ii)\;\; \boldsymbol{J}_W^{\#}\boldsymbol{J}\boldsymbol{J}_W^{\#} = \boldsymbol{J}_W^{\#}, \qquad (iii)\;\; \left(\boldsymbol{J}\boldsymbol{J}_W^{\#}\right)^T = \boldsymbol{J}\boldsymbol{J}_W^{\#}.$$

For (i), using the similar property (i) of the pseudoinverse of $\boldsymbol{J}\boldsymbol{W}^{-1/2}$,

$$\begin{aligned} \boldsymbol{J}\boldsymbol{J}_W^{\#}\boldsymbol{J} &= \boldsymbol{J}\left(\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\right)\boldsymbol{J} = \boldsymbol{J}\left(\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\right)\boldsymbol{J}\cdot\left(\boldsymbol{W}^{-1/2}\boldsymbol{W}^{1/2}\right) \\ &= \left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)\boldsymbol{W}^{1/2} = \left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)\boldsymbol{W}^{1/2} = \boldsymbol{J}. \end{aligned}$$

For (ii), using the property (ii) in the definition of the pseudoinverse of $\boldsymbol{J}\boldsymbol{W}^{-1/2}$,

$$\begin{aligned} \boldsymbol{J}_W^{\#}\boldsymbol{J}\boldsymbol{J}_W^{\#} &= \left(\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\right)\boldsymbol{J}\left(\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\right) \\ &= \boldsymbol{W}^{-1/2}\left(\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\boldsymbol{J}\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\right) = \boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#} = \boldsymbol{J}_W^{\#}. \end{aligned}$$

Finally for (iii), the symmetry of the matrix

$$\boldsymbol{J}\boldsymbol{J}_W^{\#} = \boldsymbol{J}\left(\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}\right) = \left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^{\#}$$

follows from the same property of symmetry holding for the pseudoinverse of $\boldsymbol{J}\boldsymbol{W}^{-1/2}$.

Exercise 3

The task Jacobian of the planar 4R robot is given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell\,(s_1 + s_{12} + s_{123} + s_{1234}) & -\ell\,(s_{12} + s_{123} + s_{1234}) & -\ell\,(s_{123} + s_{1234}) & -\ell\, s_{1234} \\ \ell\,(c_1 + c_{12} + c_{123} + c_{1234}) & \ell\,(c_{12} + c_{123} + c_{1234}) & \ell\,(c_{123} + c_{1234}) & \ell\, c_{1234} \end{pmatrix} \tag{11}$$

with

$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{v} \in \mathbb{R}^2, \qquad \dot{\boldsymbol{q}} \in \mathbb{R}^4.$$

When $\boldsymbol{q} = \boldsymbol{0}$ and for $\ell = 0.5$ [m], the Jacobian becomes

$$\boldsymbol{J} := \boldsymbol{J}(\boldsymbol{0}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 1.5 & 1 & 0.5 \end{pmatrix}$$

and is clearly not full rank. However, the desired task velocity lies in the range of $\boldsymbol{J}$,

$$\boldsymbol{v} = \begin{pmatrix} 0 \\ 10 \end{pmatrix} \in \mathcal{R}\{\boldsymbol{J}\} = \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so that it will be realizable at least in direction, possibly in a scaled form in case the joint velocity bounds cannot be satisfied.

Looking for a minimum norm joint velocity, to start with, we derive first the pseudoinverse solution. Using the hints in the text, it is easy to compute the pseudoinverse of $\boldsymbol{J}$ without resorting to a SVD. We have

$$\dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^{\#}\boldsymbol{v} = \begin{pmatrix} 0 & \frac{2}{7.5} \\ 0 & \frac{1.5}{7.5} \\ 0 & \frac{1}{7.5} \\ 0 & \frac{0.5}{7.5} \end{pmatrix} \begin{pmatrix} 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 2.6667 \\ 2.0000 \\ 1.3333 \\ 0.6667 \end{pmatrix} \text{ [rad/s]}.$$

The third joint velocity violates the maximum bound, $\dot{q}_{PS,3} = 1.3333 > 1 = V_3$, so this is not a feasible solution. Thus, we search for an equivalent but feasible solution by using the SNS (Saturation in the Null Space) method, which is particularly simple to apply here.

In step 1 of the SNS method, we saturate the (single) overdriven joint by setting $\dot{q}_3 = V_3 = 1$ [rad/s]. Then, the original task is modified by removing the saturated contribution of the third joint velocity (discarding the associated column of $\boldsymbol{J}$). We rewrite this as

$$\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{J}_3 V_3 = \begin{pmatrix} 0 \\ 10 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1.5 & 0.5 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_4 \end{pmatrix} = \boldsymbol{J}_{-3}\, \dot{\boldsymbol{q}}_{-3},$$

where $\boldsymbol{J}_{-i}$ is the Jacobian obtained by deleting the ith column and, similarly, $\dot{\boldsymbol{q}}_{-i}$ is the vector of joint velocity without the ith component. We recompute next the contribution of the remaining active joints, by pseudoinverting the $\boldsymbol{J}_{-3}$ matrix for the modified task. We obtain

$$\dot{\boldsymbol{q}}_{PS_{-3}} = \boldsymbol{J}_{-3}^{\#}\boldsymbol{v}_1 = \begin{pmatrix} 0 & \frac{2}{6.5} \\ 0 & \frac{1.5}{6.5} \\ 0 & \frac{0.5}{6.5} \end{pmatrix} \begin{pmatrix} 0 \\ 9 \end{pmatrix} = \begin{pmatrix} 2.7692 \\ 2.0769 \\ 0.6923 \end{pmatrix} \text{ [rad/s]}, \qquad \text{(with the additional } \dot{q}_3 = 1 = V_3\text{)}.$$

The second joint velocity violates the maximum bound, $\dot{q}_{PS_{-3},2} = 2.0769 > 2 = V_2$, so this is not yet a feasible solution and we proceed with the SNS method.

In step 2, we saturate also the second overdriven joint by setting $\dot{q}_2 = V_2 = 2$ [rad/s]. The original task is modified by removing both saturated contributions of the second and third joint velocities (discarding the two associated columns of $\boldsymbol{J}$). We rewrite this as

$$\boldsymbol{v}_2 = \boldsymbol{v} - \boldsymbol{J}_2 V_2 - \boldsymbol{J}_3 V_3 = \begin{pmatrix} 0 \\ 10 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0.5 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_4 \end{pmatrix} = \boldsymbol{J}_{-23}\, \dot{\boldsymbol{q}}_{-23},$$

with obvious notation. We recompute next the contribution of the remaining active joints, by pseudoinverting the (now square, but still singular) $\boldsymbol{J}_{-23}$ matrix for the modified task. We obtain

$$\dot{\boldsymbol{q}}_{PS_{-23}} = \boldsymbol{J}_{-23}^{\#}\boldsymbol{v}_2 = \begin{pmatrix} 0 & \frac{2}{4.25} \\ 0 & \frac{0.5}{4.25} \end{pmatrix} \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2.8235 \\ 0.7059 \end{pmatrix} \text{ [rad/s]}, \quad \text{(with } \dot{q}_2 = 2 = V_2,\ \dot{q}_3 = 1 = V_3\text{)}.$$

All bounds are now satisfied and the obtained joint velocity is feasible. Recomposing the complete joint velocity vector, we have the solution

$$\dot{\boldsymbol{q}}^* = \begin{pmatrix} 2.8235 \\ 2.0000 \\ 1.0000 \\ 0.7059 \end{pmatrix} \text{ [rad/s]}, \qquad \text{with } \boldsymbol{J}\dot{\boldsymbol{q}}^* = \boldsymbol{v} \text{ and } \|\dot{\boldsymbol{q}}^*\| = 3.6702.$$

Therefore, there was no need to scale the original task velocity $\boldsymbol{v}$ in order to find a feasible joint velocity solution. Note that the solution $\dot{\boldsymbol{q}}^*$ can be rewritten in the general form

$$\dot{\boldsymbol{q}}^* = (\boldsymbol{J}\boldsymbol{W})^{\#}\, \boldsymbol{v} + \Big(\boldsymbol{I} - (\boldsymbol{J}\boldsymbol{W})^{\#}\, \boldsymbol{J}\Big)\, \dot{\boldsymbol{q}}_N,$$

with

$$\boldsymbol{W} = \operatorname{diag}\{1,0,0,1\} \text{ (the selected active joints)}, \quad \dot{\boldsymbol{q}}_N = \begin{pmatrix} 0 \\ V_2 \\ V_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \text{ (the saturated joint velocities)},$$

which explains also the name given to the method. Indeed, this solution is not unique. However, the SNS method (actually a variant of it, Opt-SNS, which is not needed in this simple case) guarantees also that a feasible minimum norm solution is obtained. For example, another feasible solution is obtained by saturating the first joint velocity to its maximum value ($\dot{q}_1 = V_1 = 4$ [rad/s]) and adjusting the other three joint velocities accordingly. We have

$$\dot{\boldsymbol{q}}_{PS_{-1}} = \boldsymbol{J}_{-1}^{\#}\left(\boldsymbol{v} - \boldsymbol{J}_1 V_1\right) = \begin{pmatrix} 0 & \frac{1.5}{3.5} \\ 0 & \frac{1}{3.5} \\ 0 & \frac{0.5}{3.5} \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.8571 \\ 0.5714 \\ 0.2857 \end{pmatrix} \text{ [rad/s]}, \qquad (\text{with } \dot{q}_1 = 4 = V_1).$$

The complete solution $\dot{\boldsymbol{q}}^{\diamond}$ is feasible, but has a larger norm than the SNS solution $\dot{\boldsymbol{q}}^{*}$:

$$\dot{\boldsymbol{q}}^{\diamond} = \begin{pmatrix} 4.0000 \\ 0.8571 \\ 0.5714 \\ 0.2857 \end{pmatrix} \text{ [rad/s]}, \qquad \text{with } \boldsymbol{J}\dot{\boldsymbol{q}}^{\diamond} = \boldsymbol{v} \text{ and } \|\dot{\boldsymbol{q}}^{\diamond}\| = 4.1404.$$

Exercise 4

Reduced Gradient

The Jacobian of the primary task is similar to that of the previous exercise, see (11)

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\ell\left(s_1 + s_{12} + s_{123}\right) & -\ell\left(s_{12} + s_{123}\right) & -\ell s_{123} \\ \ell\left(c_1 + c_{12} + c_{123}\right) & \ell\left(c_{12} + c_{123}\right) & \ell c_{123} \end{pmatrix}. \tag{12}$$

When evaluated at $\boldsymbol{q} = \boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & \pi/3 & -2\pi/3 \end{pmatrix}^T$ and for $\ell = 1$ [m], the Jacobian (12) becomes

$$\boldsymbol{J} := \boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} -2 & -1 & -0.5 \\ 0 & 0 & \sqrt{3}/2 \end{pmatrix}$$

and is clearly full row rank. However, for the purpose of designing a RG solution, we need to extract from $\boldsymbol{J}$ a non-singular 2×2 submatrix $\boldsymbol{J}_a$, and not every selection will work. In fact, the three possible alternatives (i.e., deleting respectively column 1, 2, or 3) yield

$$\det \boldsymbol{J}_{-1} = -.8660, \qquad \det \boldsymbol{J}_{-2} = -1.7321, \qquad \det \boldsymbol{J}_{-3} = 0 \text{ (singular!)}.$$

We will choose as $\boldsymbol{J}_a$ the minor with the largest determinant (presumably, the best conditioned solution). Thus, $\boldsymbol{q}_a = (q_1, q_3)$, $\boldsymbol{q}_b = q_2$. Accordingly, after a reordering of variables obtained by the unitary matrix $\boldsymbol{T}$ (with $\boldsymbol{T}^{-1} = \boldsymbol{T}^T$)

$$\boldsymbol{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{q} \rightarrow \boldsymbol{T}\dot{\boldsymbol{q}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_3 \\ \hline \dot{q}_2 \end{pmatrix} = \begin{pmatrix} \dot{\boldsymbol{q}}_a \\ \dot{\boldsymbol{q}}_b \end{pmatrix}, \quad \boldsymbol{J} \rightarrow \boldsymbol{J}\boldsymbol{T} = \begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_3 \,|\, \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_a & \boldsymbol{J}_b \end{pmatrix},$$

we have

$$\boldsymbol{J}_a = \begin{pmatrix} -2 & -0.5 \\ 0 & \sqrt{3}/2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{J}_a^{-1} = \begin{pmatrix} -0.5 & -0.2887 \\ 0 & 1.1547 \end{pmatrix}, \qquad \boldsymbol{J}_b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The gradient of the objective function H in (7) evaluated at $\boldsymbol{q} = \boldsymbol{q}_0$ is

$$\nabla_{\boldsymbol{q}} H(\boldsymbol{q}) = \begin{pmatrix} 0 \\ 2\sin q_2 \cos q_2 \\ 2\sin q_3 \cos q_3 \end{pmatrix} \qquad \Rightarrow \qquad \nabla_{\boldsymbol{q}} H := \nabla_{\boldsymbol{q}} H(\boldsymbol{q}_0) = \begin{pmatrix} 0 \\ 0.8660 \\ 0.8660 \end{pmatrix}.$$

The reduced gradient is thus

$$\nabla_{\boldsymbol{q}_b} H' := \nabla_{\boldsymbol{q}_b} H'(\boldsymbol{q}_0) = \begin{pmatrix} -\left(\boldsymbol{J}_a^{-1}\boldsymbol{J}_b\right)^T & 1 \end{pmatrix} \boldsymbol{T} \nabla_{\boldsymbol{q}} H = \begin{pmatrix} -0.5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0.8660 \\ 0.8660 \end{pmatrix} = 0.8660.$$

The solution to the first item is thus

$$\dot{\boldsymbol{q}}_{RG} = \boldsymbol{T}^T \begin{pmatrix} \dot{\boldsymbol{q}}_a \\ \dot{\boldsymbol{q}}_b \end{pmatrix} = \boldsymbol{T}^T \begin{pmatrix} \boldsymbol{J}_a^{-1}\left(\boldsymbol{v} - \boldsymbol{J}_b \nabla_{\boldsymbol{q}_b} H'\right) \\ \nabla_{\boldsymbol{q}_b} H' \end{pmatrix} = \begin{pmatrix} -0.4330 \\ 0.8660 \\ -2.0000 \end{pmatrix} \text{ [rad/s]}.$$

Task augmentation

We consider next the auxiliary task of keeping the endpoint of the second robot link on the circle (8). The endpoint position is

$$\boldsymbol{p}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \ell(c_1 + c_{12}) \\ \ell(s_1 + s_{12}) \end{pmatrix}$$

and its associated Jacobian is

$$\boldsymbol{J}_2(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}_2(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\ell\,(s_1 + s_{12}) & -\ell s_{12} & 0 \\ \ell\,(c_1 + c_{12}) & \ell c_{12} & 0 \end{pmatrix}. \tag{13}$$

We first note that when the robot is in the configuration $\boldsymbol{q}_0 = \begin{pmatrix} \pi/2 & \pi/3 & -2\pi/3 \end{pmatrix}^T$, the position $\boldsymbol{p}_2$ satisfies already the constraint (8), see Fig. 2. Thus, the auxiliary task should constrain the joint velocities so that $\boldsymbol{p}_2$ (when moving or not) will remain on the assigned circle.

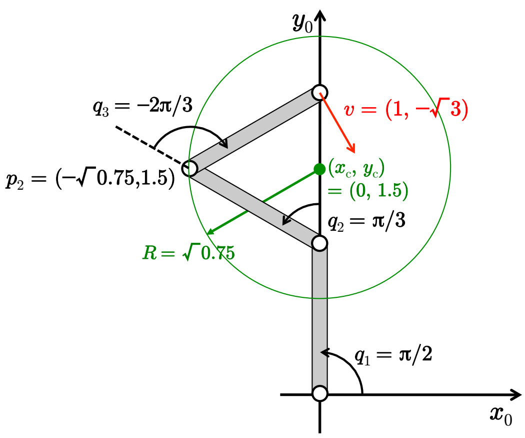


Figure 2: The primary task (in red) and the auxiliary task (in green) for the 3R planar robot.

Differentiating (8) with respect to time yields

$$2x_2\,\dot{x}_2 + 2(y_2 - 1.5)\,\dot{y}_2 = 0.$$

Rearranging this equation so as to isolate the velocity $\dot{\boldsymbol{p}}_2$ and using (13) leads to

$\dot{p}_D = J_2 \dot{q}_D$

$$\begin{pmatrix} 2x_2 & 2(y_2 - 1.5) \end{pmatrix} \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 2x_2 & 2(y_2 - 1.5) \end{pmatrix} \boldsymbol{J}_2(\boldsymbol{q})\dot{\boldsymbol{q}} = 0.$$

The auxiliary task Jacobian is then the 1×3 row vector

$$\begin{aligned} \boldsymbol{J}_a(\boldsymbol{q}) &= \begin{pmatrix} 2x_2 & 2(y_2 - 1.5) \end{pmatrix} \boldsymbol{J}_2(\boldsymbol{q}) = \begin{pmatrix} 2\ell(c_1 + c_{12}) & 2(\ell(s_1 + s_{12}) - 1.5) \end{pmatrix} \begin{pmatrix} -\ell\,(s_1 + s_{12}) & -\ell s_{12} & 0 \\ \ell\,(c_1 + c_{12}) & \ell c_{12} & 0 \end{pmatrix} \\ &= \begin{pmatrix} -3\ell\,(c_1 + c_{12}) & -2\ell^2 s_2 - 3\ell c_{12} & 0 \end{pmatrix}. \end{aligned} \tag{14}$$

Augmenting the primary task Jacobian (12) with the auxiliary task Jacobian (14) leads to a square, 3×3 *extended* Jacobian $\boldsymbol{J}_e(\boldsymbol{q})$ and to an extended task vector $\boldsymbol{v}_e \in \mathbb{R}^3$:

$$\boldsymbol{J}_e(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}(\boldsymbol{q}) \\ \boldsymbol{J}_a(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -\ell\,(s_1 + s_{12} + s_{123}) & -\ell\,(s_{12} + s_{123}) & -\ell s_{123} \\ \ell\,(c_1 + c_{12} + c_{123}) & \ell\,(c_{12} + c_{123}) & \ell c_{123} \\ -3\ell\,(c_1 + c_{12}) & -2\ell^2 s_2 - 3\ell c_{12} & 0 \end{pmatrix}, \quad \boldsymbol{v}_e = \begin{pmatrix} \boldsymbol{v} \\ 0 \end{pmatrix}, \quad \boldsymbol{J}_e(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{v}_e.$$

At the configuration $\boldsymbol{q}_0$, using also $\ell = 1$, we have

$$\boldsymbol{J}_e := \boldsymbol{J}_e(\boldsymbol{q}_0) = \begin{pmatrix} -2 & -1 & -0.5 \\ 0 & 0 & 0.8660 \\ 2.5981 & 0.8660 & 0 \end{pmatrix}. \tag{15}$$

It is easy to see that $\boldsymbol{q}_0$ is not a singularity for the extended task, and thus in particular not an algorithmic singularity. In fact,

$$\operatorname{rank}(\boldsymbol{J}) = 2, \qquad \operatorname{rank}(\boldsymbol{J}_a) = 1, \qquad \text{and } \operatorname{rank}(\boldsymbol{J}_e) = 3 = 2 + 1.$$

Therefore, in this configuration the robot can realize any generic extended task velocity $\boldsymbol{v}_e \in \mathbb{R}^3$ (in particular, with an arbitrary top part $\boldsymbol{v} \in \mathbb{R}^2$). Therefore, the joint velocity

$$\dot{\boldsymbol{q}}^{\dagger} = \boldsymbol{J}_e^{-1}\boldsymbol{v}_e = \boldsymbol{J}_e^{-1} \begin{pmatrix} 1 \\ -\sqrt{3} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \text{ [rad/s]}$$

will instantaneously realize both tasks at the same time. For the particular numerical value assigned as $\boldsymbol{v}$, the simple rotation of the third link around its joint axis will realize the primary task. In this case, the endpoint of the second link will remain at rest, thus satisfying in a trivial way the auxiliary task. In general, with a different desired value of $\boldsymbol{v}$, all robot joints will move so as to realize the extended task, instantaneously keeping the endpoint of the second link on the circle (i.e., its velocity will be tangential to the circle in the current position). For instance,

$$\dot{\boldsymbol{q}}^{\dagger\prime} = \boldsymbol{J}_e^{-1}\boldsymbol{v}_e' = \boldsymbol{J}_e^{-1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.5774 \\ -4.7321 \\ 1.1547 \end{pmatrix} \text{ [rad/s]}.$$

Exercise 5

For the direct kinematics, using the homogeneous transformations defined through Tab. 1, we obtain

$$\begin{aligned} {}^0\boldsymbol{p}_{0,2} &= {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{p}_{1,2} = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\,{}^2\boldsymbol{p}_{2,2}\right) \\ &= {}^0\boldsymbol{A}_1(q_1) \begin{pmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_2c_2 \\ a_2s_2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

The planar position $\boldsymbol{p} \in \mathbb{R}^2$ is given by the (x, y) components extracted from the position vector ${}^0\boldsymbol{p}_{0,2}$. Thus, from

$$\boldsymbol{p} = \begin{pmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{a}, \boldsymbol{\theta})$$

we obtain

$$\boldsymbol{J_a} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{a}} = \begin{pmatrix} c_1 & c_{12} \\ s_1 & s_{12} \end{pmatrix}, \qquad \boldsymbol{J_\theta} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}} = \begin{pmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{pmatrix}.$$

The regressor equation is then

$$\Delta \boldsymbol{p} = \boldsymbol{J_a} \Delta \boldsymbol{a} + \boldsymbol{J_\theta} \Delta \boldsymbol{\theta} = \boldsymbol{\Phi} \Delta \boldsymbol{\phi}, \qquad \Delta \boldsymbol{p} = \boldsymbol{p}_m - \boldsymbol{p} = \boldsymbol{p}_m - \boldsymbol{f}(\boldsymbol{a}_n, \boldsymbol{\theta}_n),$$

with

$$\Delta \boldsymbol{\phi} = \begin{pmatrix} \Delta \boldsymbol{a} \\ \Delta \boldsymbol{\theta} \end{pmatrix} \in \mathbb{R}^4, \qquad \boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{J_a}(\boldsymbol{\theta}) & \boldsymbol{J_\theta}(\boldsymbol{a}, \boldsymbol{\theta}) \end{pmatrix}_{\boldsymbol{a}=\boldsymbol{a}_n, \boldsymbol{\theta}=\boldsymbol{\theta}_n}, \qquad \boldsymbol{a}_n = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \quad \boldsymbol{\theta}_n = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

* * * * *

Robotics 2

Final test in classroom – May 29, 2017

Exercise 1

Let the dynamics of a robot manipulator with n joints be described by the usual Lagrangian model (neglecting dissipative terms):

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}. \tag{1}$$

- List all feedback control laws for $\boldsymbol{\tau}$ that allow regulation to a desired (generic) constant configuration $\boldsymbol{q}_d$. For each control law, specify the design conditions that guarantee success, the type of convergence/stability achieved, and the pros and cons of the method.
- When applied to eq. (1), is it possible that the control law

$$\boldsymbol{\tau} = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\dot{\boldsymbol{q}} \tag{2}$$

  achieves asymptotic stabilization of a desired state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$? If so, under which operative conditions? Would these conditions be only sufficient or also necessary?
- Can it happen that the following control law applied to eq. (1)

$$\boldsymbol{\tau} = \boldsymbol{g}(\boldsymbol{q}_d) + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\dot{\boldsymbol{q}} \tag{3}$$

  does *not* achieve asymptotic stabilization of the state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$? Why (or why not)? What should be done in (3) to validate the reverse statement "does *certainly* achieve ..."?

Exercise 2

- In the context of image-based visual servoing (IBVS), determine the interaction matrix $\boldsymbol{J}_{p,polar}$ associated to a point feature $\boldsymbol{s} = (\rho \;\; \theta)^T \in \mathbb{R}^2$ which is parametrized with polar coordinates in the image plane, i.e.,

$$\dot{\boldsymbol{s}} = \begin{pmatrix} \dot{\rho} \\ \dot{\theta} \end{pmatrix} = \boldsymbol{J}_{p,polar}(\rho, \theta, \lambda, Z) \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix},$$

  where $\lambda > 0$ is the focal length of the camera, $Z > 0$ is the depth coordinate of the Cartesian point $P = (X, Y, Z)$ in the camera frame, and the perspective equations of a pinhole camera model are used.
- Discuss the characteristics of the obtained interaction matrix in terms of decoupled effects on the chosen feature parameters, when only single components of the linear velocity $\boldsymbol{V} \in \mathbb{R}^3$ or of the angular velocity $\boldsymbol{\Omega} \in \mathbb{R}^3$ are active. Compare this analysis with the one on the interaction matrix $\boldsymbol{J}_p$ that uses the Cartesian coordinates (u, v) in the image plane.
- In an IBVS control law that regulates the polar coordinates of a point feature $\boldsymbol{s}$ to a desired constant value $\boldsymbol{s}_d$, what careful actions may be needed when the point P is on or gets close to the optical axis of the camera?

Exercise 3

Consider the planar PRP robot in Fig. 1, moving under gravity. Its dynamic model can be written in the form

$$\boldsymbol{Y_M}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a_M} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}, \tag{4}$$

where the inertial terms and the related Coriolis and centrifugal components are already expressed in a linear factorized form in terms of the dynamic coefficients $\boldsymbol{a_M} \in \mathbb{R}^{p_M}$. In the following, assume that the symbolic expressions of $\boldsymbol{Y_M}$ and $\boldsymbol{a_M}$ are known.

- Derive explicitly the gravity term $\boldsymbol{g}(\boldsymbol{q})$ and an associated minimal factorization in the form $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y_g}(\boldsymbol{q})\, \boldsymbol{a_g}$, with $\boldsymbol{a_g} \in \mathbb{R}^{p_g}$.
- Provide the expression of an adaptive control law that guarantees global asymptotic tracking of a desired joint trajectory $\boldsymbol{q}_d(t) \in C^2$, without any a priori knowledge of the numerical values of the robot dynamic parameters. How many states will have this dynamical controller?
- Assume now that the actual values of the dynamic coefficients $\boldsymbol{a_M}$ are known in advance. Solve the previously stated trajectory tracking problem by designing an adaptive controller that has a reduced number of states. Sketch a formal proof of your result.

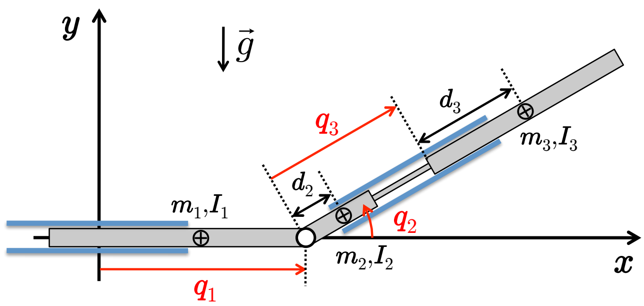


Figure 1: A planar PRP robot, with an associated set of generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$.

Figure 2: Robotized surface polishing of a metallic work piece.

Exercise 4

Consider a robotized polishing task of a large surface area of a metallic work piece, as shown in Fig. 2. The work piece has a complex geometry, with planar and curved surfaces. The polishing tool held by the robot ends with a sphere which is in point-wise contact with the geometric surface of the work piece. The tool needs not to be orthogonal to the local tangent plane to the surface. However, in order to complete an optimal polishing (reducing the roughness of the surface below a specified level —with tolerances in the order of few μm), a specified normal force should be applied to the surface while moving the tool in contact with an assigned constant speed. Assume a perfectly rigid interaction and that friction at the contact can be neglected in this modeling stage.

Define a task frame in which the polishing operation can be correctly defined and executed. Specify accordingly the natural and artificial constraints. Sketch the task frame in two cases, on a flat surface of the work piece and on a curved one.

[240 minutes (4 hours); open books, but no computer or smartphone]

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Solution

May 29, 2017

Exercise 1

The first part is a free text exercise. Completeness, accuracy, and clarity in writing are evaluated.

For the two other specific questions:

- The control law (2) achieves asymptotic stabilization of a desired state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$ also in the presence of gravity if: *i)* $\boldsymbol{g}(\boldsymbol{q}_d) = \boldsymbol{0}$ (the desired configuration is an unforced equilibrium for the open-loop system); *ii)* $\boldsymbol{K}_P$ is symmetric and positive definite, and its minimum eigenvalue $\boldsymbol{K}_{P,m} > \alpha$, where $\alpha > 0$ is a global upper bound on the norm of the Hessian of the potential energy $U_g(\boldsymbol{q})$ due to gravity; *iii)* $\boldsymbol{K}_D$ is symmetric and positive definite. In general, these are only sufficient conditions.

- Even assuming that $\boldsymbol{K}_P$ and $\boldsymbol{K}_D$ are symmetric and positive definite, the control law (3) may still fail because $\boldsymbol{K}_{P,m} \leq \alpha$. Indeed, when this sufficient condition is violated, we cannot predict if we will obtain asymptotic stabilization of the desired equilibrium state or not.

Exercise 2

Using the pinhole model, a 3D Cartesian point $\boldsymbol{P} = (X, Y, Z)$, with coordinates expressed in the camera frame, becomes in the image plane a point feature (u, v)

$$u = \lambda \frac{X}{Z}, \qquad v = \lambda \frac{Y}{Z},$$

when 2D Cartesian coordinates are used, being $\lambda > 0$ the focus length of the camera. These two feature parameters move in the image plane in response to a linear and angular velocity of the camera, respectively $\boldsymbol{V}$ and $\boldsymbol{\Omega}$, as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{Z} & 0 & -\frac{u}{Z} \\ 0 & \frac{\lambda}{Z} & -\frac{v}{Z} \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix},$$

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = -\boldsymbol{V} - \boldsymbol{\Omega} \times \boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 & 0 & -Z & Y \\ 0 & -1 & 0 & Z & 0 & -X \\ 0 & 0 & -1 & -Y & X & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -\frac{\lambda}{Z} & 0 & \frac{u}{Z} & \frac{uv}{\lambda} & -\left(\lambda + \frac{u^2}{\lambda}\right) & v \\ 0 & -\frac{\lambda}{Z} & \frac{v}{Z} & \lambda + \frac{v^2}{\lambda} & -\frac{uv}{\lambda} & -u \end{pmatrix} \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix} = \boldsymbol{J}_p(u, v, \lambda, Z) \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix}.$$

When using instead the polar coordinates as feature parameters in the image plane, we have the transformations

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \rho \cos\theta \\ \rho \sin\theta \end{pmatrix} \qquad \Longleftrightarrow \qquad \begin{pmatrix} \rho \\ \theta \end{pmatrix} = \begin{pmatrix} \sqrt{u^2 + v^2} \\ \text{ATAN2}\{v, u\} \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \dot{\rho} \cos\theta - \rho\dot{\theta} \sin\theta \\ \dot{\rho} \sin\theta + \rho\dot{\theta} \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\rho \sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix} \begin{pmatrix} \dot{\rho} \\ \dot{\theta} \end{pmatrix} = \boldsymbol{J}_t(\rho, \theta) \begin{pmatrix} \dot{\rho} \\ \dot{\theta} \end{pmatrix},$$

with $\det \boldsymbol{J}_t = \rho$. Therefore,

$$\begin{pmatrix} \dot{\rho} \\ \dot{\theta} \end{pmatrix} = \boldsymbol{J}_t^{-1}(\rho, \theta) \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\dfrac{\sin\theta}{\rho} & \dfrac{\cos\theta}{\rho} \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}, \qquad \text{provided that } \rho = \sqrt{u^2 + v^2} \neq 0.$$

Evaluating then

$$\begin{aligned} \boldsymbol{J}_{p,t}(\rho, \theta, \lambda, Z) &= \boldsymbol{J}_p(u, v, \lambda, Z)\Big|_{\substack{u=\rho\cos\theta \\ v=\rho\sin\theta}} \\ &= \begin{pmatrix} -\dfrac{\lambda}{Z} & \boxed{0} & \dfrac{\rho\cos\theta}{Z} & \dfrac{\rho^2\sin\theta\cos\theta}{\lambda} & -\left(\lambda + \dfrac{\rho^2\cos^2\theta}{\lambda}\right) & \rho\sin\theta \\ \boxed{0} & -\dfrac{\lambda}{Z} & \dfrac{\rho\sin\theta}{Z} & \lambda + \dfrac{\rho^2\sin^2\theta}{\lambda} & -\dfrac{\rho^2\sin\theta\cos\theta}{\lambda} & -\rho\cos\theta \end{pmatrix}, \end{aligned}$$

we obtain

$$\begin{pmatrix} \dot{\rho} \\ \dot{\theta} \end{pmatrix} = \boldsymbol{J}_t^{-1}(\rho, \theta)\boldsymbol{J}_{p,t}(\rho, \theta, \lambda, Z) \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix} = \boldsymbol{J}_{p,polar}(\rho, \theta, \lambda, Z) \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix},$$

with

$$\boldsymbol{J}_{p,polar}(\rho, \theta, \lambda, Z) = \begin{pmatrix} -\dfrac{\lambda\cos\theta}{Z} & -\dfrac{\lambda\sin\theta}{Z} & \dfrac{\rho}{Z} & \dfrac{(\lambda^2+\rho^2)\sin\theta}{\lambda} & -\dfrac{(\lambda^2+\rho^2)\cos\theta}{\lambda} & \boxed{0} \\ \dfrac{\lambda\sin\theta}{\rho Z} & -\dfrac{\lambda\cos\theta}{\rho Z} & \boxed{0} & \dfrac{\lambda\cos\theta}{\rho} & \dfrac{\lambda\sin\theta}{\rho} & -1 \end{pmatrix}.$$

Comparing now the structural zeros of $\boldsymbol{J}_p(u, v, \lambda, Z)$ and $\boldsymbol{J}_{p,polar}(\rho, \theta, \lambda, Z)$, we see that in the first case the effects of V_x and V_y respectively on $\dot{u}$ and $\dot{v}$ are decoupled, whereas in the second case the decoupled effects of V_z and Ω_z are respectively on $\dot{\rho}$ and $\dot{\theta}$.

Finally, when the 3D Cartesian point $\boldsymbol{P}$ is sufficiently close to the optical axis ($\rho \to 0$), matrix $\boldsymbol{J}_t$ becomes ill-conditioned, meaning that small motions of the Cartesian point may then lead to huge variations of the chosen polar parameters —in particular of θ— in the image plane. To recover a regular behavior close to and across $\rho = 0$, the inverse of $\boldsymbol{J}_t$ may be replaced, for instance, by a damped least squares matrix with damping factor $\sigma^2 > 0$,

$$\begin{aligned} \boldsymbol{J}_t^{DLS}(\rho, \theta) &= \boldsymbol{J}_t^T(\rho, \theta)\left(\sigma^2 \boldsymbol{I}_{2\times 2} + \boldsymbol{J}_t(\rho, \theta)\boldsymbol{J}_t^T(\rho, \theta)\right)^{-1} \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{pmatrix} \begin{pmatrix} \sigma^2 + \cos^2\theta + \rho^2\sin^2\theta & (1-\rho^2)\sin\theta\cos\theta \\ (1-\rho^2)\sin\theta\cos\theta & \sigma^2 + \sin^2\theta + \rho^2\cos^2\theta \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \dfrac{\cos\theta}{1+\sigma^2} & \dfrac{\sin\theta}{1+\sigma^2} \\ -\dfrac{\rho\sin\theta}{\rho^2+\sigma^2} & \dfrac{\rho\cos\theta}{\rho^2+\sigma^2} \end{pmatrix}, \end{aligned}$$

with $\det(\boldsymbol{J}_t^{DLS}) = \rho/[\rho^2 + \sigma^2)(1 + \sigma^2)] > 0$ for $\sigma^2 > 0$ (and equal to $1/\rho$ for $\sigma^2 = 0$).

Exercise 3

We compute first the gravitational potential energy $U_g(\boldsymbol{q}) = U_1 + U_2 + U_3$, with

$$U_1 = 0, \qquad U_2 = m_2 g_0 d_2 \sin q_2, \qquad U_3 = m_3 g_0 (q_3 + d_3)\sin q_2.$$

Therefore, the gravity vector and its (minimal) linear parametrization are given by

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ g_0\,(m_2 d_2 + m_3(q_3 + d_3))\cos q_2 \\ g_0 m_3 \sin q_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ g_0\, q_3 \cos q_2 & g_0 \cos q_2 \\ g_0 \sin q_2 & 0 \end{pmatrix} \begin{pmatrix} m_3 \\ m_2 d_2 + m_3 d_3 \end{pmatrix} = \boldsymbol{Y_g}(\boldsymbol{q})\,\boldsymbol{a_g},$$

with $p_g = \dim(\boldsymbol{a_g}) = 2$. The dynamic model (4) can thus be rewritten as

$$\boldsymbol{Y_M}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a_M} + \boldsymbol{Y_g}(\boldsymbol{q})\,\boldsymbol{a_g} = \begin{pmatrix} \boldsymbol{Y_M}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) & \boldsymbol{Y_g}(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \boldsymbol{a_M} \\ \boldsymbol{a_g} \end{pmatrix} = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{\tau}.$$

! Note that the linear parametrization of the dynamic model by $\boldsymbol{a}$ may not be minimal, since there could be some overlapping between the coefficients in $\boldsymbol{a}_M$ and those defined in $\boldsymbol{a}_g$.

Assuming that the values of all dynamic coefficients $\boldsymbol{a}$ are unknown, the adaptive trajectory tracking control takes the usual form

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\hat{\boldsymbol{a}} + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}), \qquad \boldsymbol{K}_P, \boldsymbol{K}_D > 0 \\ \dot{\hat{\boldsymbol{a}}} &= \boldsymbol{\Gamma}\boldsymbol{Y}^T(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\left(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}\right), \qquad \boldsymbol{\Gamma} > 0, \end{aligned} \tag{5}$$

where $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}(\boldsymbol{q}_d - \boldsymbol{q})$, $\boldsymbol{\Lambda} > 0$, and

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r) = \begin{pmatrix} \boldsymbol{Y_M}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r) & \boldsymbol{Y_g}(\boldsymbol{q}) \end{pmatrix}\hat{\boldsymbol{a}} = \hat{\boldsymbol{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \hat{\boldsymbol{C}}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \hat{\boldsymbol{g}}(\boldsymbol{q}).$$

The number of states of the adaptive controller (5), i.e., the number of its defining differential equations, is equal to the number of updating dynamic coefficients: $\dim(\hat{\boldsymbol{a}}) = p_M + p_g = p_M + 2$.

Assuming that the values of the dynamic coefficients $\boldsymbol{a}_M$ are already known, one can define an adaptive trajectory control with the same global tracking properties of (5), but with a reduced number of states equal to $p_g = 2$. In fact, it is natural to define this controller as

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$$\begin{aligned} \boldsymbol{\tau} &= \boxed{\boldsymbol{Y}_M(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\boldsymbol{a_M}} + \boldsymbol{Y_g}(\boldsymbol{q})\boxed{\hat{\boldsymbol{a}}_{\boldsymbol{g}}} + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}), \qquad \boldsymbol{K}_P, \boldsymbol{K}_D > 0 \\ \dot{\hat{\boldsymbol{a}}}_{\boldsymbol{g}} &= \boldsymbol{\Gamma_g}\boldsymbol{Y}_{\boldsymbol{g}}^T(\boldsymbol{q})\left(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}\right), \qquad \boldsymbol{\Gamma_g} > 0, \end{aligned} \tag{6}$$

where $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}(\boldsymbol{q}_d - \boldsymbol{q})$, $\boldsymbol{\Lambda} > 0$, and $\boldsymbol{Y_g}(\boldsymbol{q})\hat{\boldsymbol{a}}_{\boldsymbol{g}} = \hat{\boldsymbol{g}}(\boldsymbol{q})$.

The proof of global asymptotic stability of the tracking error follows the same lines of the proof holding for (5). The 'trick' in (6) that leads to this sufficient result stands in preserving the passivity structure also in the known term: namely, adopting $\dot{\boldsymbol{q}}_r$ and $\ddot{\boldsymbol{q}}_r$ in the evaluation of $\boldsymbol{Y_M}$ (and *not* $\dot{\boldsymbol{q}}$ only (or $\dot{\boldsymbol{q}}_d$) and $\ddot{\boldsymbol{q}}_d$), as well as using the skew-symmetric property of a suitable factorization $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ of the Coriolis and centrifugal terms. More in detail, take the Lyapunov candidate

$$V_g = \frac{1}{2}\boldsymbol{s}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{s} + \boldsymbol{e}^T\boldsymbol{K}_P\boldsymbol{e} + \frac{1}{2}\tilde{\boldsymbol{a}}_{\boldsymbol{g}}^T\boldsymbol{\Gamma_g}\tilde{\boldsymbol{a}}_{\boldsymbol{g}} \geq 0,$$

with $\boldsymbol{s} = \dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}$, $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$, $\boldsymbol{R} > 0$, and $\tilde{\boldsymbol{a}}_{\boldsymbol{g}} = \boldsymbol{a_g} - \hat{\boldsymbol{a}}_{\boldsymbol{g}}$. Its time derivative is

$$\dot{V}_g = \frac{1}{2}\boldsymbol{s}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{s} + \boldsymbol{s}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}} + 2\boldsymbol{e}^T\boldsymbol{K}_P\dot{\boldsymbol{e}} - \tilde{\boldsymbol{a}}_{\boldsymbol{g}}^T\boldsymbol{\Gamma_g}\dot{\hat{\boldsymbol{a}}}_{\boldsymbol{g}}.$$

The closed-loop system under (6) satisfies the differential equation

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \hat{\boldsymbol{g}}(\boldsymbol{q}) + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}}$$

or

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{s} = \tilde{\boldsymbol{g}}(\boldsymbol{q}) - \boldsymbol{K}_P\boldsymbol{e} - \boldsymbol{K}_D\dot{\boldsymbol{e}}, \tag{7}$$

with $\tilde{\boldsymbol{g}}(\boldsymbol{q}) = \boldsymbol{g}(\boldsymbol{q}) - \hat{\boldsymbol{g}}(\boldsymbol{q}) = \boldsymbol{Y}_{\boldsymbol{g}}(\boldsymbol{q})\boldsymbol{a}_{\boldsymbol{g}} - \boldsymbol{Y}_{\boldsymbol{g}}(\boldsymbol{q})\hat{\boldsymbol{a}}_{\boldsymbol{g}} = \boldsymbol{Y}_{\boldsymbol{g}}(\boldsymbol{q})\tilde{\boldsymbol{a}}_{\boldsymbol{g}}$. When evaluated along the trajectories of (7), $\dot{V}_g$ becomes

$$\begin{aligned}
\dot{V}_g &= \frac{1}{2}\boldsymbol{s}^T\left(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)\boldsymbol{s} + \boldsymbol{s}^T\left(\tilde{\boldsymbol{g}}(\boldsymbol{q}) - \boldsymbol{K}_P\boldsymbol{e} - \boldsymbol{K}_D\dot{\boldsymbol{e}}\right) + 2\,\boldsymbol{e}^T\boldsymbol{K}_P\dot{\boldsymbol{e}} - \tilde{\boldsymbol{a}}_{\boldsymbol{g}}^T\boldsymbol{\Gamma}_{\boldsymbol{g}}\dot{\hat{\boldsymbol{a}}}_{\boldsymbol{g}}\\
&= \boldsymbol{s}^T\left(\boldsymbol{Y}_{\boldsymbol{g}}(\boldsymbol{q})\tilde{\boldsymbol{a}}_{\boldsymbol{g}} - \boldsymbol{K}_P\boldsymbol{e} - \boldsymbol{K}_D\dot{\boldsymbol{e}}\right) + 2\,\boldsymbol{e}^T\boldsymbol{K}_P\dot{\boldsymbol{e}} - \tilde{\boldsymbol{a}}_{\boldsymbol{g}}^T\boldsymbol{Y}_{\boldsymbol{g}}^T(\boldsymbol{q})\boldsymbol{s}\\
&= -\left(\dot{\boldsymbol{e}} + \boldsymbol{\Lambda}\boldsymbol{e}\right)^T\left(\boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}}\right) + 2\,\boldsymbol{e}^T\boldsymbol{K}_P\dot{\boldsymbol{e}}\\
&= -\left(\dot{\boldsymbol{e}} + \boldsymbol{K}_D^{-1}\boldsymbol{K}_P\boldsymbol{e}\right)^T\left(\boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}}\right) + 2\,\boldsymbol{e}^T\boldsymbol{K}_P\dot{\boldsymbol{e}} = -\boldsymbol{e}^T\boldsymbol{K}_P\boldsymbol{K}_D^{-1}\boldsymbol{K}_P\boldsymbol{e} - \dot{\boldsymbol{e}}^T\boldsymbol{K}_D\dot{\boldsymbol{e}} \le 0,
\end{aligned}$$

where the skew-symmetry of $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$, the updating law for $\hat{\boldsymbol{a}}_{\boldsymbol{g}}$ in (6), and $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^T = \boldsymbol{K}_D^{-1}\boldsymbol{K}_P > 0$ (with diagonal PD gains) have been used. The rest of the proof follows as in the general case, using Barbalat lemma and LaSalle theorem.

Exercise 4

With reference to Fig. 3, the task frame is placed at the contact point between the spherical tool and the surface. The $\boldsymbol{z}_t$-axis is *always* aligned with the downward normal to the surface, no matter if the surface is locally flat or curved. One of the two other axes, say $\boldsymbol{x}_t$, can be chosen to be aligned with the desired polishing direction on the metallic surface. Remember that, in the modeling phase, the tool-surface contact is assumed to be rigid and *frictionless*. Therefore, the spherical tool can also slide without rolling in contact. Moreover, being the contact point-wise, there cannot be any moment applied by the tool at the contact.

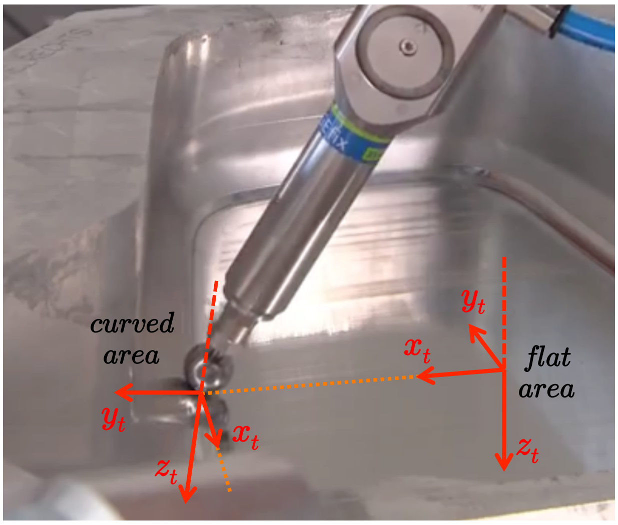


Figure 3: Task frame definition for the surface polishing of a metallic work piece.

According to the chosen task frame, the natural (geometric) constraints of this interaction problem are

$$v_z = 0, \quad F_x = 0, \quad F_y = 0, \quad M_x = 0, \quad M_y = 0, \quad M_z = 0,$$

while the artificial constraints are

$$F_z = F_{z,d} > 0, \quad v_x = v_{x,d} > 0, \quad v_y = v_{y,d} = 0, \quad \omega_x = \omega_{x,d}, \quad \omega_y = \omega_{y,d}, \quad \omega_z = \omega_{z,d}.$$

Therefore, a hybrid force-motion controller designed based on feedback linearization and decoupling will include five scalar position control loops and only one force control loop. The value $v_{y,d} = 0$ is related to the choice of the $\boldsymbol{x}_t$-axis being aligned with the tangent to the polishing path, which is executed at the speed $v_{x,d}$, that should be followed on the surface. Moreover, when there is no need to change the orientation of the tool (e.g., for avoiding collision with some lateral side of the work piece), we set $\boldsymbol{\omega}_d = \begin{pmatrix} \omega_{x,d} & \omega_{y,d} & \omega_{z,d} \end{pmatrix}^T = \mathbf{0}$. Finally, the positive value $F_{z,d}$ specifies the normal force needed for obtaining a satisfactory polishing result (by pressure).

* * * * *

Robotics II

June 6, 2017

Exercise 1

Consider a planar 3R robot with unitary link lengths as in Fig. 1, where the generalized coordinates $\boldsymbol{q}$ are defined as the *absolute* angles of the links w.r.t. the $\boldsymbol{x}$-axis. The position of the robot end-effector $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q})$, as obtained through the direct kinematics, should follow the desired trajectory

$\dot{p}_d = \begin{pmatrix} 2\cos(3t)\cdot 3 \\ -\sin(3t+\frac{\pi}{2})\cdot 3 \end{pmatrix}$

$$\boldsymbol{p}_d(t) = \begin{pmatrix} 1 + 2\sin 3t \\ 2 + \cos\left(3t + \frac{\pi}{2}\right) \end{pmatrix}, \qquad \text{for } t \geq 0. \tag{1}$$

$t=0 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$t=0$ $\begin{pmatrix} 6 \\ -3 \end{pmatrix}$

The robot is kinematically redundant for this task.

- Define a differential inversion scheme at the level of joint jerk commands $\dddot{\boldsymbol{q}}$ such that the squared norm $\|\dddot{\boldsymbol{q}}\|^2$ is locally minimized and the trajectory can be executed exactly right from the initial time $t = 0$.
- Provide numerical values for the initial joint position $\boldsymbol{q}(0)$, joint velocity $\dot{\boldsymbol{q}}(0)$, and joint acceleration $\ddot{\boldsymbol{q}}(0)$ such that there is a perfect initial matching with the desired trajectory. Provide also the numerical value of the initial locally optimal command $\dddot{\boldsymbol{q}}(0)$.
- Suppose that there is no perfect matching between the initial kinematic conditions of the robot and the trajectory at time $t = 0$. How can we modify the command law for $\dddot{\boldsymbol{q}}$ such that the error $\boldsymbol{e}(t) = \boldsymbol{p}_d(t) - \boldsymbol{p}(t)$ and all its time derivatives will exponentially converge to zero?

$\ddot{p}_d = \begin{pmatrix} -6\sin(3t)\cdot 3 \\ -\cos(3t+\frac{\pi}{2})\cdot 9 \end{pmatrix}$

$t=0$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\dddot{p} = \begin{pmatrix} -18\cdot 3\cos(3t) \\ 9\cdot 3\sin(3t+\frac{\pi}{2}) \end{pmatrix}$

$t=0$ $\begin{pmatrix} -54 \\ 27 \end{pmatrix}$

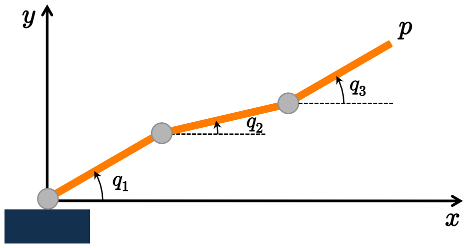


Figure 1: A planar 3R robot with absolute angles as generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$.

Exercise 2

For the same robot in Fig. 1, and using the same coordinates defined therein, assume that the three links have equal, uniformly distributed mass $m_i = m = 10$ kg, for $i = 1, 2, 3$. Each torque τ_i delivered by the motors and performing work on the absolute coordinate q_i is bounded as $|\tau_i| \leq T_{max} = 300$ Nm, for $i = 1, 2, 3$. Consider the Cartesian regulation control law

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\, \boldsymbol{K}_P \Big(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q})\Big) - \boldsymbol{K}_D \dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}), \qquad \text{with } \boldsymbol{p}_d = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{2}$$

$\dot{q} = 0$

where the gain matrices $\boldsymbol{K}_P$ and $\boldsymbol{K}_D$ are diagonal and positive definite. Let the robot starts <u>at rest</u> at $t = 0$ in the configuration $\boldsymbol{q}(0) = \begin{pmatrix} \pi/2 & 0 & 0 \end{pmatrix}^T$.

- If the gain matrices are of the form $\boldsymbol{K}_P = k_P \cdot \boldsymbol{I}_{2\times 2}$ and $\boldsymbol{K}_D = k_D \cdot \boldsymbol{I}_{2\times 2}$, provide the largest values for the scalars k_P and k_D such that $\boldsymbol{\tau}(0)$ in (2) does not violate its bounds.
- Let now the positional gain matrix be $\boldsymbol{K}_P = \text{diag}\{k_{Px}, k_{Py}\}$, while $\boldsymbol{K}_D$ is as before. Provide the largest values for the scalars k_{Px}, k_{Py}, and k_d such that $\boldsymbol{\tau}(0)$ in (2) does not violate its bounds.
- How would things change if the bounds were set as $|\tau_{\theta,i}| \leq T_{max} = 300$ Nm, where $\tau_{\theta,i}$ is the torque delivered by the motors and performing work on the *relative* (Denavit-Hartenberg) coordinate θ_i, for $i = 1, 2, 3$?

[Turn sheet for the next exercise]

Exercise 3

Consider the planar PRP robot in Fig. 2.

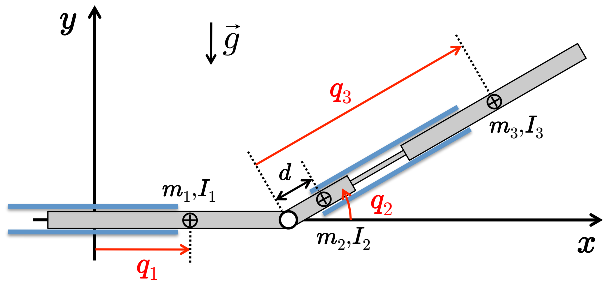


Figure 2: A planar PRP robot moving in a vertical plane, with definition of the generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ to be used.

- Determine the expressions of the inertial, Coriolis and centrifugal, and gravity terms in the dynamic model expressed in the usual Lagrangian form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}.$$

- Find a factorization $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}.\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ such that $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is a skew-symmetric matrix.
- Find all equilibrium configurations $\boldsymbol{q}_e$ (i.e., such that $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$), if any.
- Provide symbolic expressions for the scalar coefficients $\alpha > 0$ and $\beta > 0$ such that the following global linear bound holds for the Hessian of the gravitational potential energy $U_g(\boldsymbol{q})$:

$$\left\| \frac{\partial^2 U_g(\boldsymbol{q})}{\partial \boldsymbol{q}^2} \right\| = \left\| \frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} \right\| \leq \alpha + \beta\,|q_3|, \qquad \forall \boldsymbol{q} \in \mathbb{R}^3.$$

[240 minutes; open books but no computer or smartphone]

Solution

June 6, 2017

Exercise 1

The direct kinematics of the planar 3R robot with unitary link lengths using absolute coordinates (i.e., the link angles w.r.t. the $\boldsymbol{x}$-axis) is

$$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q}) = \begin{pmatrix} c_1 + c_2 + c_3 \\ s_1 + s_2 + s_3 \end{pmatrix}.$$

The associated first-order differential kinematics, with the Jacobian matrix, is

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \begin{pmatrix} -s_1 & -s_2 & -s_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\dot{\boldsymbol{q}}.$$

The second-order differential kinematics, with the first time-derivative $\dot{\boldsymbol{J}}$ of the Jacobian, is

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \begin{pmatrix} -c_1\dot{q}_1 & -c_2\dot{q}_2 & -c_3\dot{q}_3 \\ -s_1\dot{q}_1 & -s_2\dot{q}_2 & -s_3\dot{q}_3 \end{pmatrix}\dot{\boldsymbol{q}}.$$

The third-order differential kinematics, including the second time-derivative $\ddot{\boldsymbol{J}}$ of the Jacobian, is

$$\dddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\,\dddot{\boldsymbol{q}} + 2\dot{\boldsymbol{J}}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \ddot{\boldsymbol{J}}(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\,\dddot{\boldsymbol{q}} + 2\dot{\boldsymbol{J}}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \begin{pmatrix} s_1\dot{q}_1^2 - c_1\ddot{q}_1 & s_2\dot{q}_2^2 - c_2\ddot{q}_2 & s_3\dot{q}_3^2 - c_3\ddot{q}_3 \\ -c_1\dot{q}_1^2 - s_1\ddot{q}_1 & -c_2\dot{q}_2^2 - s_2\ddot{q}_2 & -c_3\dot{q}_3^2 - s_3\ddot{q}_3 \end{pmatrix}\dot{\boldsymbol{q}}.$$

When the initial conditions of the robot are perfectly matched with the desired end-effector trajectory,

$$\boldsymbol{p}(\boldsymbol{q}(0)) = \boldsymbol{p}_d(0), \qquad \boldsymbol{J}(\boldsymbol{q}(0))\,\dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{p}}_d(0), \qquad \boldsymbol{J}(\boldsymbol{q}(0))\,\ddot{\boldsymbol{q}}(0) + \dot{\boldsymbol{J}}(\boldsymbol{q}(0))\,\dot{\boldsymbol{q}}(0) = \ddot{\boldsymbol{p}}_d(0), \tag{3}$$

the nominal solution for executing $\boldsymbol{p}_d(t)$ with minimum norm of the joint jerk is (dropping dependencies)

$$\dddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\left(\dddot{\boldsymbol{p}}_d - 2\dot{\boldsymbol{J}}\,\ddot{\boldsymbol{q}} - \ddot{\boldsymbol{J}}\,\dot{\boldsymbol{q}}\right). \tag{4}$$

From (1), we have

$$\dot{\boldsymbol{p}}_d = \begin{pmatrix} 6\cos 3t \\ -3\sin\left(3t + \frac{\pi}{2}\right) \end{pmatrix}, \qquad \ddot{\boldsymbol{p}}_d = \begin{pmatrix} -18\sin 3t \\ -9\cos\left(3t + \frac{\pi}{2}\right) \end{pmatrix}, \qquad \dddot{\boldsymbol{p}}_d = \begin{pmatrix} -54\cos 3t \\ 27\sin\left(3t + \frac{\pi}{2}\right) \end{pmatrix}.$$

Thus

$$\boldsymbol{p}_d(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \dot{\boldsymbol{p}}_d(0) = \begin{pmatrix} 6 \\ -3 \end{pmatrix}, \qquad \ddot{\boldsymbol{p}}_d(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \dddot{\boldsymbol{p}}_d(0) = \begin{pmatrix} -54 \\ 27 \end{pmatrix}.$$

It is easy to find an initial configuration $\boldsymbol{q}_0 = \boldsymbol{q}(0)$ that is matched with the initial position of the trajectory:

$$\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & \pi/2 \end{pmatrix}^T \text{ [rad]} \qquad \Rightarrow \qquad \boldsymbol{p}(\boldsymbol{q}_0) = \boldsymbol{p}_d(0).$$

In this configuration, the Jacobian is full rank and its pseudoinverse is easily computed as

$$\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{J}_0^{\#} = \boldsymbol{J}_0^T\left(\boldsymbol{J}_0\boldsymbol{J}_0^T\right)^{-1} = \begin{pmatrix} 0 & 1 \\ -0.5 & 0 \\ -0.5 & 0 \end{pmatrix}$$

The associated initial joint velocity $\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{q}}(0)$ and acceleration $\ddot{\boldsymbol{q}}_0 = \ddot{\boldsymbol{q}}(0)$ can be computed as minimum norm solutions at their differential level. We have

$$\dot{\boldsymbol{q}}_0 = \boldsymbol{J}_0^{\#}\dot{\boldsymbol{p}}_d(0) = \begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix} \text{ [rad/s]}.$$

From this, evaluating

$$\dot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 3 \end{pmatrix}\begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix} = \begin{pmatrix} -9 \\ -18 \end{pmatrix},$$

we obtain also

$$\ddot{\boldsymbol{q}}_0 = \boldsymbol{J}_0^{\#}\left(\ddot{\boldsymbol{p}}_d(0) - \dot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0\right) = -\boldsymbol{J}_0^{\#}\dot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 18 \\ -4.5 \\ -4.5 \end{pmatrix} \text{ [rad/s}^2\text{]}.$$

Evaluating now

$$\dot{\boldsymbol{J}}_0\,\ddot{\boldsymbol{q}}_0 = \begin{pmatrix} 54 \\ -27 \end{pmatrix}, \qquad \ddot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0 = \begin{pmatrix} -18 & 9 & 9 \\ -9 & 4.5 & 4.5 \end{pmatrix}\begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \boldsymbol{0},$$

from eq. (4) we finally obtain the jerk command at time $t = 0$:

$$\dddot{\boldsymbol{q}}(0) = \boldsymbol{J}_0^{\#}\left(\dddot{\boldsymbol{p}}_d(0) - 2\dot{\boldsymbol{J}}_0\,\ddot{\boldsymbol{q}}_0 - \ddot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0\right) = \begin{pmatrix} 0 & 1 \\ -0.5 & 0 \\ -0.5 & 0 \end{pmatrix}\left(\begin{pmatrix} -54 \\ 27 \end{pmatrix} - 2\begin{pmatrix} 54 \\ -27 \end{pmatrix}\right) = \begin{pmatrix} 81 \\ 81 \\ 81 \end{pmatrix} \text{ [rad/s}^3\text{]}.$$

Instead, when the initial conditions of the robot are not matched with the desired end-effector trajectory (i.e., if one or more of the identities in (3) is violated), in order to obtain exponential tracking of $\boldsymbol{p}_d(t)$, the solution with minimum norm of the joint jerk can be modified as (dropping dependencies)

$$\dddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\left(\dddot{\boldsymbol{p}}_d + k_2\left(\ddot{\boldsymbol{p}}_d - \boldsymbol{J}\ddot{\boldsymbol{q}} - \dot{\boldsymbol{J}}\dot{\boldsymbol{q}}\right) + k_1\left(\dot{\boldsymbol{p}}_d - \boldsymbol{J}\dot{\boldsymbol{q}}\right) + k_0\left(\boldsymbol{p}_d - \boldsymbol{p}\right) - 2\dot{\boldsymbol{J}}\,\ddot{\boldsymbol{q}} - \ddot{\boldsymbol{J}}\,\dot{\boldsymbol{q}}\right), \tag{5}$$

where the scalars k_0, k_1, and k_2 are such that

$$k(s) = s^3 + k_2 s^2 + k_1 s + k_0$$

is a Hurwitz polynomial, namely it has all roots in the left-hand side of the complex plane. From Routh criterion, this happens if and only if

$$k_0 > 0, \qquad k_1 > \frac{k_0}{k_2} > 0, \qquad k_2 > 0. \tag{6}$$

To show the transient properties of the control law (5), let the Cartesian position error be defined as $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p} \in \mathbb{R}^2$. From

$$\dddot{\boldsymbol{e}} = \dddot{\boldsymbol{p}}_d - \dddot{\boldsymbol{p}} = \dddot{\boldsymbol{p}}_d - \left(\boldsymbol{J}\,\dddot{\boldsymbol{q}} + 2\dot{\boldsymbol{J}}\,\ddot{\boldsymbol{q}} + \ddot{\boldsymbol{J}}\,\dot{\boldsymbol{q}}\right)$$

using (5) and being $\boldsymbol{J}\boldsymbol{J}^{\#} = \boldsymbol{I}_{2\times 2}$, it is easy to see that the following *linear* differential equation holds:

$$\dddot{\boldsymbol{e}} + k_2\,\ddot{\boldsymbol{e}} + k_1\,\dot{\boldsymbol{e}} + k_0\,\boldsymbol{e} = \boldsymbol{0}.$$

Under the conditions (6), the evolution of $\boldsymbol{e}(t)$ and of its time derivatives is that of the modes of an asymptotically stable linear system, namely exponentially or pseudo-exponentially converging to zero.

Exercise 2

We compute first the gravitational potential energy $U_g(\boldsymbol{q}) = U_1 + U_2 + U_3$. We have

$$U_1 = m_1 g_0 d_1 \sin q_1, \qquad U_2 = m_2 g_0 \left(\ell_1 \sin q_1 + d_2 \sin q_2\right),$$
$$U_3 = m_3 g_0 \left(\ell_1 \sin q_1 + \ell_2 \sin q_2 + d_3 \sin q_3\right).$$

Since $d_i = \ell_i/2 = 0.5$, for $i = 1, 2, 3$, it is

$$U_g(\boldsymbol{q}) = g_0 \left(\frac{m_1}{2} + m_2 + m_3\right) \sin q_1 + g_0 \left(\frac{m_2}{2} + m_3\right) \sin q_2 + g_0 \frac{m_3}{2} \sin q_3$$

and

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0 \left((m_1/2) + m_2 + m_3\right) \cos q_1 \\ g_0 \left((m_2/2) + m_3\right) \cos q_2 \\ g_0 (m_3/2) \cos q_3 \end{pmatrix}.$$

Using the expressions of $\boldsymbol{p}(\boldsymbol{q})$ and $\boldsymbol{J}(\boldsymbol{q})$ from Exercise 1 and the mass data, we evaluate the control law (2) with $\boldsymbol{K}_P = k_P \cdot \boldsymbol{I}_{2\times 2}$, at the initial time $t = 0$, when $\boldsymbol{q}(0) = \begin{pmatrix} \pi/2 & 0 & 0 \end{pmatrix}^T$ and $\dot{\boldsymbol{q}}(0) = \boldsymbol{0}$:

$$\begin{aligned} \boldsymbol{\tau}(0) &= k_p \, \boldsymbol{J}^T(\boldsymbol{q}(0)) \left(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q}(0))\right) + \boldsymbol{g}(\boldsymbol{q}(0)) \\ &= k_p \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 15 g_0 \\ 5 g_0 \end{pmatrix} = \begin{pmatrix} k_p \\ k_p + 15 g_0 \\ k_p + 5 g_0 \end{pmatrix} = \begin{pmatrix} 300 \\ 300 \\ 300 \end{pmatrix}, \quad k_p > 0, \; g_0 = 9.81 > 0. \end{aligned} \tag{7}$$

Therefore, the largest value $k_p > 0$ that satisfies the bounds on the joint torques, $|\tau_i| \le T_{max} = 300$ Nm, for $i = 1, 2, 3$, is the one that saturates the second torque component, i.e.,

$$\tau_2(0) = k_p + 15 g_0 = 300 \text{ [Nm]} \qquad \Rightarrow \qquad k_p = 300 - 15 g_0 \simeq 152.85.$$

If $\boldsymbol{K}_P = \text{diag}\{k_{Px}, k_{Py}\}$ and all the rest is as before, the control law (2) is evaluated again as

$$\boldsymbol{\tau}(0) = \boldsymbol{J}^T(\boldsymbol{q}(0)) \, \text{diag}\{k_{Px}, k_{Py}\} \left(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q}(0))\right) + \boldsymbol{g}(\boldsymbol{q}(0)) = \begin{pmatrix} k_{Px} \\ k_{Py} + 15 g_0 \\ k_{Py} + 5 g_0 \end{pmatrix} = \begin{pmatrix} 300 \\ 300 \\ 300 \end{pmatrix}, \quad k_{Px} > 0, \; k_{Py} > 0. \tag{8}$$

Therefore, we can take as the largest gain values those that saturate the first two components of the torque $\boldsymbol{\tau}$, i.e.,

$$k_{Px} = \tau_1(0) = 300 \text{ [Nm]}, \qquad k_{Py} = 300 - 15 g_0 \simeq 152.85 \text{ [Nm]}.$$

In both cases, the value of $\boldsymbol{K}_D = k_D \cdot \boldsymbol{I}_{2\times 2}$ does not play any role (as long as $\dot{\boldsymbol{q}} = \boldsymbol{0}$).

Finally, consider the case of torque bounds in the form $|\tau_{\theta,i}| \le T_{max} = 300$ Nm, for $i = 1, 2, 3$, where $\boldsymbol{\tau_\theta}$ are the torques producing work on the relative coordinates $\boldsymbol{\theta}$ (of the Denavit-Hartenberg convention). Since

$$\boldsymbol{q} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \boldsymbol{\theta} = \boldsymbol{T}\boldsymbol{\theta} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}} = \boldsymbol{T}\dot{\boldsymbol{\theta}},$$

$$\tau^T(T\dot{\theta}) = \tau_\theta^T \dot{\theta} \Rightarrow T^T \tau = \tau_\theta$$

from the principle of virtual work ($\boldsymbol{\tau}^T \dot{\boldsymbol{q}} = \boldsymbol{\tau}_{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}}$) we have

$$\boldsymbol{\tau_\theta} = \boldsymbol{T}^T \boldsymbol{\tau} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{\tau} \qquad \Rightarrow \qquad \boldsymbol{\tau} = \boldsymbol{T}^{-T} \boldsymbol{\tau_\theta} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{\tau_\theta}. \tag{9}$$

Therefore, taking for example the gain structure in (7), it follows that

$$\begin{pmatrix} -300 \\ -300 \\ -300 \end{pmatrix} \le \boldsymbol{\tau_\theta}(0) = \boldsymbol{T}^T \boldsymbol{\tau}(0) = \boldsymbol{T}^T \begin{pmatrix} k_p \\ k_p + 15 g_0 \\ k_p + 5 g_0 \end{pmatrix} = \begin{pmatrix} 3k_p + 20 g_0 \\ 2k_p + 20 g_0 \\ k_p + 5 g_0 \end{pmatrix} \le \begin{pmatrix} 300 \\ 300 \\ 300 \end{pmatrix}.$$

The largest value $k_p > 0$ that satisfies all the above bounds is obtained then from the first component:

$$k_p = \frac{300 - 20 g_0}{3} \simeq 34.6 \text{ [Nm]}.$$

Note also that, from the linear transformations (9), a feasible cube of side $2T_{max} = 600$ Nm centered in the origin of the $\boldsymbol{\tau_\theta}$-space becomes a skewed parallelepiped in the $\boldsymbol{\tau}$-space (and vice versa).

Exercise 3

Following a Lagrangian approach, we compute first the kinetic energy $T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T_1 + T_2 + T_3$. We have

$$T_1 = \tfrac{1}{2} m_1 \dot{q}_1^2, \qquad T_2 = \tfrac{1}{2} m_2 \left(\dot{q}_1^2 + d^2 \dot{q}_2^2 - 2d \sin q_2 \, \dot{q}_1 \dot{q}_2 \right) + \tfrac{1}{2} I_2 \, \dot{q}_2^2,$$
$$T_3 = \tfrac{1}{2} m_3 \left(\dot{q}_1^2 + q_3^2 \, \dot{q}_2^2 + \dot{q}_3^2 - 2 q_3 \sin q_2 \, \dot{q}_1 \dot{q}_2 + 2 \cos q_2 \, \dot{q}_1 \dot{q}_3 \right) + \tfrac{1}{2} I_3 \, \dot{q}_2^2.$$

Thus

$$T = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \frac{1}{2} \dot{\boldsymbol{q}}^T \begin{pmatrix} m_1 + m_2 + m_3 & -(m_2 d + m_3 q_3) \sin q_2 & m_3 \cos q_2 \\ & I_2 + m_2 d^2 + I_3 + m_3 q_3^2 & 0 \\ & symm & m_3 \end{pmatrix} \dot{\boldsymbol{q}}.$$

The components of the Coriolis and centrifugal vector are computed using the Christoffel's symbols

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial \boldsymbol{q}_i} \right),$$

being $\boldsymbol{m}_i$ the ith column of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. We have

$$\begin{aligned} \boldsymbol{C}_1(\boldsymbol{q}) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(m_2 d + m_3 q_3) \cos q_2 & -m_3 \sin q_2 \\ 0 & -m_3 \sin q_2 & 0 \end{pmatrix} \\ \Rightarrow \qquad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -(m_2 d + m_3 q_3) \cos q_2 \, \dot{q}_2^2 - 2\, m_3 \sin q_2 \, \dot{q}_2 \dot{q}_3. \end{aligned}$$

Similarly

$$\boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m_3 q_3 \\ 0 & m_3 q_3 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 2\, m_3 q_3 \, \dot{q}_2 \dot{q}_3 \,,$$

and

$$\boldsymbol{C}_3(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -m_3 q_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 q_3 \, \dot{q}_2^2 \,.$$

A factorization of the Coriolis and centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}}$ that satisfies the skew-symmetric property is given by

$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_3(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} 0 & -(m_2 d + m_3 q_3) \cos q_2 \, \dot{q}_2 - m_3 \sin q_2 \, \dot{q}_3 & -m_3 \sin q_2 \, \dot{q}_2 \\ 0 & m_3 q_3 \, \dot{q}_3 & m_3 q_3 \, \dot{q}_2 \\ 0 & -m_3 q_3 \, \dot{q}_2 & 0 \end{pmatrix}.$$

Being

$$\dot{\boldsymbol{M}}(\boldsymbol{q}) = \begin{pmatrix} 0 & -(m_2 d + m_3 q_3)\cos q_2 \dot{q}_2 - m_3 \sin q_2\, \dot{q}_3 & -m_3 \sin q_2\, \dot{q}_2 \\ -(m_2 d + m_3 q_3)\cos q_2\, \dot{q}_2 - m_3 \sin q_2\, \dot{q}_3 & 2\, m_3 q_3\, \dot{q}_3 & 0 \\ -m_3 \sin q_2\, \dot{q}_2 & 0 & 0 \end{pmatrix},$$

it is easy to check that the matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is skew-symmetric.

For the potential energy due to gravity, $U_g(\boldsymbol{q}) = U_1 + U_2 + U_3$, we have (up to a constant)

$$U_1 = 0, \qquad U_2 = m_2 g_0\, d \sin q_2, \qquad U_3 = m_3 g_0\, q_3 \sin q_2.$$

Thus

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ (m_2 d + m_3 q_3) g_0 \cos q_2 \\ m_3 g_0 \sin q_2 \end{pmatrix}.$$

The unforced equilibrium configurations are

$$\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0} \quad \Rightarrow \quad q_{e,1} = any, \quad q_{e,2} = \{0, \pi\}, \quad q_{e,3} = -\frac{m_2}{m_3}\, d.$$

Taking a further partial derivative of $\boldsymbol{g}$ w.r.t. $\boldsymbol{q}$, we obtain the matrix

$$\frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \frac{\partial^2 U_g(\boldsymbol{q})}{\partial \boldsymbol{q}^2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(m_2 d + m_3 q_3) g_0 \sin q_2 & m_3 g_0 \cos q_2 \\ 0 & m_3 g_0 \cos q_2 & 0 \end{pmatrix} = \boldsymbol{A}(\boldsymbol{q}).$$

Matrix $\boldsymbol{A}$ is symmetric, thus it has real eigenvalues. To have all non-negative eigenvalues (so that we can order them and find their maximum, as requested by the definition of norm of a matrix that we use), we compute the semi-positive definite matrix

$$\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & g_0^2\left((m_2 d + m_3 q_3)^2 \sin^2 q_2 + m_3^2 \cos^2 q_2\right) & -g_0^2 m_3\,(m_2 d + m_3 q_3) \sin q_2 \cos q_2 \\ 0 & -g_0^2 m_3\,(m_2 d + m_3 q_3) \sin q_2 \cos q_2 & g_0^2 m_3^2 \cos^2 q_2 \end{pmatrix},$$

which has clearly one zero eigenvalue. Denote by $\boldsymbol{B}$ the lower 2×2 block on the diagonal of this matrix. The characteristic polynomial of $\boldsymbol{A}^T\boldsymbol{A}$ is then

$$\det\left(\lambda \boldsymbol{I} - \boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right) = \lambda \cdot \det\left(\lambda \boldsymbol{I} - \boldsymbol{B}(\boldsymbol{q})\right) = \lambda\left(\lambda^2 - \mathrm{trace}\{\boldsymbol{B}(\boldsymbol{q})\}\lambda + \det\{\boldsymbol{B}(\boldsymbol{q})\}\right)$$

with $\mathrm{trace}\{\boldsymbol{B}(\boldsymbol{q})\} > 0$ and $\det\{\boldsymbol{B}(\boldsymbol{q})\} > 0$. Therefore, the maximum eigenvalue of $\boldsymbol{A}^T\boldsymbol{A}$ is

$$\lambda_{max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right) = \frac{1}{2}\,\mathrm{trace}\{\boldsymbol{B}(\boldsymbol{q})\} + \frac{1}{2}\sqrt{(\mathrm{trace}\{\boldsymbol{B}(\boldsymbol{q})\})^2 - 4\det\{\boldsymbol{B}(\boldsymbol{q})\}}$$

Since we are looking for a bound on the norm of $\boldsymbol{A}(\boldsymbol{q})$, we can write the chain of inequalities

$$\begin{aligned} \lambda_{max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right) &\leq \mathrm{trace}\{\boldsymbol{B}(\boldsymbol{q})\} = g_0^2\left((m_2 d + m_3 q_3)^2 \sin^2 q_2 + 2\, m_3^2 \cos^2 q_2\right) \\ &< g_0^2\left((m_2 d + m_3 q_3)^2 + 2\, m_3^2\right) < g_0^2 (m_2 d + m_3 |q_3| + \sqrt{2}\, m_3)^2. \end{aligned}$$

Therefore, we finally obtain the requested bound

$$\left\|\frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right\| = \|\boldsymbol{A}(\boldsymbol{q})\| = \sqrt{\lambda_{max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right)} < g_0\,(m_2 d + m_3 |q_3| + m_3\sqrt{2}) = \alpha + \beta\,|q_3|, \quad \forall \boldsymbol{q} \in \mathbb{R}^3,$$

with

$$\alpha = g_0(m_2 d + m_3\sqrt{2}), \qquad \beta = g_0 m_3.$$

* * * * *

Robotics II

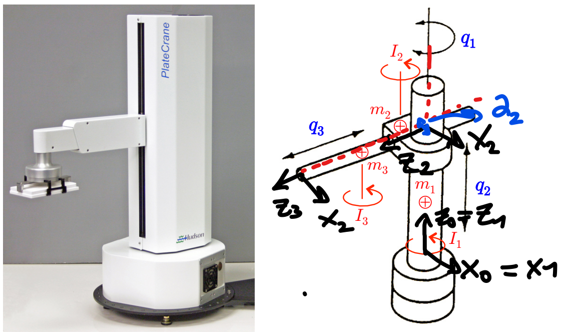
July 11, 2017

Exercise 1

For the RPP cylindrical robot in Fig. 1, using the generalized coordinates defined therein, provide the symbolic expression of each term of the dynamic model that appears in the control law

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}_d + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_d + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}), \qquad \boldsymbol{K}_P > 0, \quad \boldsymbol{K}_D > 0, \tag{1}$$

so that global asymptotic tracking of a desired joint trajectory $\boldsymbol{q}_d(t) \in C^2$ is guaranteed. Joint axis 3 has a DH offset $a_2 \neq 0$ from joint axis 2. Moreover, the center of mass of links 1 and 3 is placed on the joint axis having the same index, while the center of mass of link 2 is at a distance $r_2 > 0$ from joint axis 2.



α	a	d	q
0	0	0	q_1
$\frac{\pi}{2}$	0	q_2	0
0	$-a_2$	q_3	0

$${}^1\omega_1 = \begin{pmatrix} 0 \\ 0 \\ \dot q_1 \end{pmatrix}$$

$${}^0p_{c2} = \begin{pmatrix} r_2 S_1 \\ r_2 C_1 \\ q_2 \end{pmatrix} \qquad {}^0v_{c2} = \begin{pmatrix} r_2 C_1 \dot q_1 \\ -r_2 S_1 \dot q_1 \\ \dot q_2 \end{pmatrix}$$

$$\|v_{c2}\|^2 = r_2^2 \dot q_1^2 + \dot q_2^2$$

Figure 1: A RPP cylindrical robot with its generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$.

Exercise 2

Consider a Cartesian robot moving in a vertical place and having its end effector constrained to an ideal (rigid, frictionless) linear surface tilted by an acute angle $\alpha > 0$ from the $\boldsymbol{x}$-axis, as in Fig. 2. For this robot in constrained motion, provide the explicit symbolic expressions of

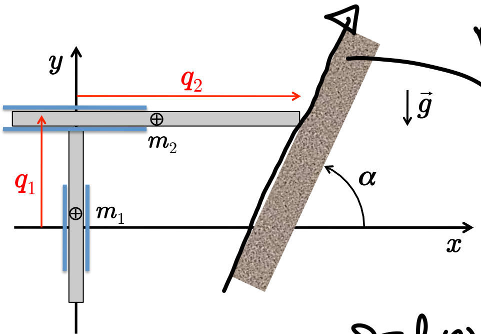
- the *reduced dynamics*, i.e., the differential equation relating the pseudo-acceleration $\dot{v} \in \mathbb{R}$ to the pseudo-velocity $v \in \mathbb{R}$, the configuration $\boldsymbol{q} \in \mathbb{R}^2$, and the input forces at the joints $\boldsymbol{u} \in \mathbb{R}^2$;
- the *multiplier* $\lambda \in \mathbb{R}$, i.e., the scalar reaction force that would act on the robot end effector when attempting to violate the geometric constraint.

Is it possible, by a suitable choice of $\boldsymbol{u}$, to realize a uniform motion with constant velocity $v = V$ on the surface, while inducing no reaction force, i.e., with $\lambda = 0$? If so, when and how?

$$T_1 = \frac{1}{2} m_1 \dot q_1^2$$

$$T_2 = \frac{1}{2} m_2 (\dot q_2^2 + \dot q_1^2)$$

$$T = \frac{1}{2}\left(\dot q_1^2 (m_1 + m_2) + m_2 \dot q_2^2\right)$$



$$r = \begin{pmatrix} q_2 \\ q_1 \end{pmatrix}$$

$$r_y = m r_x + q \qquad m \to \tan(\alpha), \quad q \to \beta$$

$$\beta \cos(\alpha)$$

$$q_1 = \tan(\alpha) q_2 + \beta$$

$$q_1 \cos(\alpha) = q_2 \sin(\alpha) + h_0$$

So $h(q) = -q_1 C_\alpha + q_2 S_\alpha + h_0 = 0$

$$M = \begin{pmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{pmatrix} \qquad C = (0 \ \ 0)$$

Figure 2: A PP robot with the end effector in constrained motion on a linear surface.

$$g = \begin{pmatrix} g_0 (m_1 + m_2) \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A \\ D \end{pmatrix} = \begin{pmatrix} -C_\alpha & S_\alpha \\ S_\alpha & C_\alpha \end{pmatrix}$$

Exercise 3

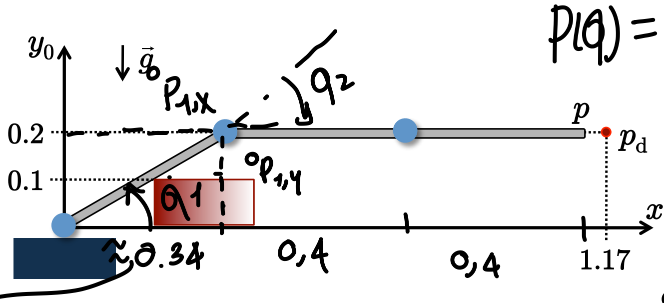
Consider a planar 3R robot, with equal link lengths $\ell = 0.4$ m and equal, uniformly distributed link masses $m = 2.5$ kg, that moves in a vertical plane. The generalized coordinates $\boldsymbol{q} \in \mathbb{R}^3$ are defined by a standard Denavit-Hartenberg convention. Under the action of the Cartesian control law (with gravity cancelation)

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\,\boldsymbol{K}_P\Big(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q})\Big) - \boldsymbol{K}_D\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}), \tag{2}$$

where $\boldsymbol{p}_d = \begin{pmatrix} 1.17 & 0.2 \end{pmatrix}^T$, $\boldsymbol{K}_P = 400 \cdot \boldsymbol{I}_{2\times 2}$, $\boldsymbol{K}_D = 40 \cdot \boldsymbol{I}_{2\times 2}$, and $\boldsymbol{p}(\boldsymbol{q})$ is the direct kinematics of the end-effector position, the robot has reached the equilibrium condition shown in Fig. 3, in which the first link is in contact with a rigid obstacle. In this steady state, determine the numerical values of

- the control torque $\boldsymbol{\tau} \in \mathbb{R}^3$ at the joints;
- the joint torque $\boldsymbol{\tau}_c \in \mathbb{R}^3$ at the joints due to the contact force $\boldsymbol{F}_c \in \mathbb{R}^2$ acting on the first link;
- the momentum-based residual $\boldsymbol{r} \in \mathbb{R}^3$ for collision detection/isolation, when $\boldsymbol{K}_I = 10 \cdot \boldsymbol{I}_{3\times 3}$;
- if possible, the components of the contact force $\boldsymbol{F}_c$ acting on the first link (expressed in frame RF_0 or in frame RF_1).

$$p(q) = \begin{pmatrix} \ell(c_1 + c_{12} + c_{123}) \\ \ell(s_1 + s_{12} + s_{123}) \end{pmatrix}$$



$$^0P_1 = \begin{pmatrix} 0.34 \\ 0.2 \end{pmatrix} = \begin{pmatrix} \ell c_1 \\ \ell s_1 \end{pmatrix} \mapsto$$

$$q_{1e} = \mathrm{ATAN2}(0.34, 0.2) = \frac{\pi}{6}$$

PITHAGORAS THEOREM

Figure 3: A planar 3R in contact with a rigid obstacle.

$$^0P_2 = \begin{pmatrix} 0.74 \\ 0.2 \end{pmatrix} = \begin{pmatrix} \ell(c_1 + c_{12}) \\ \ell(s_1 + s_{12}) \end{pmatrix}$$

[210 minutes; open books but no computer or smartphone]

$$\frac{0.74}{\ell} - 0.86 = \cos\left(q_2 + \frac{\pi}{6}\right)$$

$$\frac{0.2}{\ell} - 0.5 = \sin\left(q_2 + \frac{\pi}{6}\right)$$

$$q_{2e} = \mathrm{ATAN2}\left(\frac{0.2}{\ell} - 0.5, \frac{0.74}{\ell} - 0.86\right) - \frac{\pi}{6} = -\frac{\pi}{6}$$

$$q_{3e} = 0$$

We compute $J(q_e)$, then we can compute:

$$\tau = J(q_e)^T \boxed{K_p} \underbrace{(P_d - P(q_e))}_{\text{we have}} - \boxed{K_D}\boxed{\dot q} + \underbrace{g(q_e)}_{\text{easy to compute}}$$

$\dot q = 0$

PROBLEM GAVE US THEM

GRAVITY COMPENSATION

τ_c COMPENSATION

$$\text{SO} \quad \tau = \boxed{J^T F_e} + g$$

Solution

July 11, 2017

Exercise 1

Following a Lagrangian approach, we compute first the kinetic energy $T = T_1 + T_2 + T_3$. We have

$$\begin{array}{c} T_1 = \frac{1}{2} I_1 \, \dot{q}_1^2 \qquad T_2 = \frac{1}{2} I_2 \, \dot{q}_1^2 + \frac{1}{2} m_2 \left(r_2^2 \, \dot{q}_1^2 + \dot{q}_2^2 \right) \\ T_3 = \frac{1}{2} I_3 \, \dot{q}_1^2 + \frac{1}{2} m_3 \left(\left(a_2^2 + q_3^2 \right) \dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2 a_2 \dot{q}_1 \dot{q}_3 \right) \end{array} \quad \Rightarrow \quad T = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}},$$

with

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_1 + I_2 + I_3 + m_2 r_2^2 + m_3 a_2^2 + m_3 q_3^2 & 0 & m_3 a_2 \\ 0 & m_2 + m_3 & 0 \\ m_3 a_2 & 0 & m_3 \end{pmatrix}. \tag{3}$$

In order to guarantee global asymptotic trajectory tracking for the control law (1), the factorization $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}}$ of the Coriolis and centrifugal terms should be such that $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is a skew-symmetric matrix. This is automatically guaranteed if the components of the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are computed using the Christoffel's symbols, i.e.,

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial \boldsymbol{q}_i} \right), \qquad i = 1, 2, 3, \tag{4}$$

being $\boldsymbol{m}_i$ the ith column of the inertia matrix $\boldsymbol{M}$. Define $I_0 = I_1 + I_2 + I_3 + m_2 r_2^2 + m_3 a_2^2$, so that $m_{11}(\boldsymbol{q}) = I_0 + m_3 q_3^2$. I_0 is one of the three dynamic coefficients in the complete robot model, the other two being m_3 (a_2 is a kinematic parameter supposed to be known) and $(m_2 + m_3)$. Using (4), we obtain

$$\begin{aligned} \boldsymbol{C}_1(\boldsymbol{q}) &= \begin{pmatrix} 0 & 0 & m_3 q_3 \\ 0 & 0 & 0 \\ m_3 q_3 & 0 & 0 \end{pmatrix} \qquad &\Rightarrow \qquad & c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 2 m_3 q_3 \, \dot{q}_1 \dot{q}_3, \\ \boldsymbol{C}_2(\boldsymbol{q}) &= \boldsymbol{0} &\Rightarrow \qquad & c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \\ \boldsymbol{C}_3(\boldsymbol{q}) &= \begin{pmatrix} -m_3 q_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} &\Rightarrow \qquad & c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 q_3 \, \dot{q}_1^2. \end{aligned}$$

A factorization that satisfies the skew-symmetric property is then given by

$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_3(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} m_3 q_3 \, \dot{q}_3 & 0 & m_3 q_3 \, \dot{q}_1 \\ 0 & 0 & 0 \\ -m_3 q_3 \, \dot{q}_1 & 0 & 0 \end{pmatrix}. \tag{5}$$

For the potential energy due to gravity, $U_g = U_1 + U_2 + U_3$, we have (up to a constant)

$$U_1 = 0, \qquad U_2 = m_2 g_0 \, q_2, \qquad U_3 = m_3 g_0 \, q_2,$$

and thus

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} 0 \\ (m_2 + m_3) g_0 \\ 0 \end{pmatrix}. \tag{6}$$

Summarizing, the terms from the robot dynamics used in the control law (1) are given by (3), (5), and (6).

Exercise 2

The dynamic model of the PP robot in free motion is

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{u} \qquad \Longleftrightarrow \qquad \begin{aligned} (m_1+m_2)\ddot{q}_1 + (m_1+m_2)g_0 &= u_1 \\ m_2\ddot{q}_2 &= u_2. \end{aligned}$$

The linear constraint on the end-effector position is expressed by its implicit form[1] in terms of the angle α as (note the exchanged order of joint variables!)

$$\boldsymbol{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} q_2 \\ q_1 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}), \qquad k(\boldsymbol{r}) = r_x \sin\alpha - r_y \cos\alpha + k_0 = 0$$

$$\Rightarrow \quad h(\boldsymbol{q}) = k\left(\boldsymbol{f}(\boldsymbol{q})\right) = -q_1 \cos\alpha + q_2 \sin\alpha + k_0 = 0.$$

Accordingly, the constraint Jacobian $\boldsymbol{A}$ is a constant, 1×2 matrix (always of full rank)

$$\boldsymbol{A} = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\cos\alpha & \sin\alpha \end{pmatrix}$$

The kinematic constraint is

$$\boldsymbol{A}\,\dot{\boldsymbol{q}} = 0 \qquad \Rightarrow \qquad -\dot{q}_1 \cos\alpha + \dot{q}_2 \sin\alpha = 0.$$

The dynamic model of the PP robot constrained to the surface becomes

$$\begin{aligned} &\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{u} + \boldsymbol{A}^T\lambda \\ &\text{s.t.} \quad h(\boldsymbol{q}) = 0 \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} (m_1+m_2)\ddot{q}_1 + (m_1+m_2)g_0 &= u_1 - \lambda\cos\alpha \\ m_2\ddot{q}_2 &= u_2 + \lambda\sin\alpha \\ \text{s.t.} \ -q_1\cos\alpha + q_2\sin\alpha + k_0 &= 0. \end{aligned}$$

To proceed with the reduced dynamics approach, we can border $\boldsymbol{A}$ with a row matrix $\boldsymbol{D}$, so as to build a (globally) nonsingular matrix:

$$\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix} = \begin{pmatrix} -\cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix}^{-1} = \begin{pmatrix} -\cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{F} \end{pmatrix}.$$

The pseudo-velocity v that automatically satisfies the differential constraint is given by

$$v = \boldsymbol{D}\dot{\boldsymbol{q}} = \dot{q}_1 \sin\alpha + \dot{q}_2 \cos\alpha,$$

whereas the admissible joint velocities and accelerations are given by

$$\dot{\boldsymbol{q}} = \boldsymbol{F}v = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} v, \qquad \ddot{\boldsymbol{q}} = \boldsymbol{F}\dot{v} = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} \dot{v}.$$

Since

$$\begin{aligned} \boldsymbol{F}^T\boldsymbol{M}\boldsymbol{F} &= \begin{pmatrix} \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} = m_1 \sin^2\alpha + m_2 \\ \boldsymbol{F}^T\boldsymbol{g} &= \begin{pmatrix} \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} (m_1+m_2)g_0 \\ 0 \end{pmatrix} = (m_1+m_2)g_0 \sin\alpha \end{aligned}$$

the reduced dynamic model is given by

$$\left(\boldsymbol{F}^T\boldsymbol{M}\boldsymbol{F}\right)\dot{v} + \boldsymbol{F}^T\boldsymbol{g} = \boldsymbol{F}^T\boldsymbol{u} \qquad \Longleftrightarrow \qquad \left(m_1\sin^2\alpha + m_2\right)\dot{v} + (m_1+m_2)g_0\sin\alpha = u_1\sin\alpha + u_2\cos\alpha, \quad (7)$$

[1] The geometric expression of a line in a plane (r_x, r_y) having an angular coefficient $m = \tan\alpha$, for $\alpha \in (0, \pi/2)$, is $r_y = (\tan\alpha)\, r_x + \beta$. Multiplying by $\cos\alpha \neq 0$, one easily obtains the expression $k(\boldsymbol{r}) = 0$, where $k_0 = \beta\cos\alpha$.

while, being $\boldsymbol{E}^T = \begin{pmatrix} -\cos\alpha & \sin\alpha \end{pmatrix}$, the force multiplier is given by

$$\lambda = \boldsymbol{E}^T \left(\boldsymbol{M}\boldsymbol{F}\,\dot{v} + \boldsymbol{g} - \boldsymbol{u}\right) \quad \Longleftrightarrow \quad \lambda = -m_1 \cos\alpha \sin\alpha\,\dot{v} - (m_1+m_2)g_0\cos\alpha + u_1\cos\alpha - u_2\sin\alpha. \quad (8)$$

Note that for $\alpha = \pi/2$, equations (7–8) collapse to

$$(m_1+m_2)\,\dot{v} + (m_1+m_2)g_0 = u_1, \qquad \lambda = -u_2,$$

showing in particular that each force applied by the second actuator will result in an equal reaction force in the opposite direction.

Finally, it is possible to maintain a uniform motion with constant velocity $v = V$ on the linear surface without having a reaction force $\lambda = 0$ at all times, under following (necessary and sufficient) conditions:

- the robot is in an initial configuration $\boldsymbol{q}_0$ satisfying $h(\boldsymbol{q}_0) = 0$, i.e., the end effector is already on the linear surface;
- the robot has an initial velocity $\dot{\boldsymbol{q}}_0$ satisfying $\boldsymbol{A}\dot{\boldsymbol{q}}_0 = 0$ and such that $\|\dot{\boldsymbol{r}}_0\| = V$ or

  $$\dot{\boldsymbol{q}}_0 = V\begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{r}}_0 = V\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

  i.e, the end-effector velocity is tangential to the surface and has the right module;
- the input $\boldsymbol{u}$ is computed by solving eqs. (7–8) for $\dot{v} = 0$ and $\lambda = 0$, or

  $$(m_1+m_2)g_0\begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} = \begin{pmatrix} \sin\alpha & \cos\alpha \\ \cos\alpha & -\sin\alpha \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (m_1+m_2)g_0 \\ 0 \end{pmatrix};$$

  namely, it is sufficient that the first actuator sustains the total gravity load of the robot.

Exercise 3

The direct kinematics for the end-effector position of the planar 3R robot with equal link lengths ℓ and standard DH angles is

$$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q}) = \ell\begin{pmatrix} \cos q_1 + \cos(q_1+q_2) + \cos(q_1+q_2+q_3) \\ \sin q_1 + \sin(q_1+q_2) + \sin(q_1+q_2+q_3) \end{pmatrix} = \ell\begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{pmatrix},$$

where a shorthand notation has been introduced. The associated analytic Jacobian matrix is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \ell\begin{pmatrix} -(s_1+s_{12}+s_{123}) & -(s_{12}+s_{123}) & -s_{123} \\ c_1+c_{12}+c_{123} & c_{12}+c_{123} & c_{123} \end{pmatrix}.$$

The equilibrium configuration of the robot in Fig. 2 is easily recognized to be $\boldsymbol{q} = \begin{pmatrix} \pi/6 & -\pi/6 & 0 \end{pmatrix}^T$. Using this configuration and the data of the problem, we obtain numerical values for the following terms in the control law (2):

$$\boldsymbol{p} = \begin{pmatrix} 1.1464 \\ 0.2 \end{pmatrix}, \qquad \boldsymbol{F}_e = \boldsymbol{K}_P\left(\boldsymbol{p}_d - \boldsymbol{p}\right) = \begin{pmatrix} 9.4359 \\ 0 \end{pmatrix}, \qquad \boldsymbol{J}^T = \begin{pmatrix} -0.2 & 1.1464 \\ 0 & 0.8 \\ 0 & 0.4 \end{pmatrix}.$$

To compute the gravity vector, we start from the potential energy $U_g(\boldsymbol{q}) = U_1 + U_2 + U_3$. Having each link the same uniformly distributed mass m, we have

$$U_1 = m\,g_0\ell\,\frac{1}{2}\,s_1, \qquad U_2 = m\,g_0\ell\left(s_1 + \frac{1}{2}\,s_{12}\right), \qquad U_3 = m\,g_0\ell\left(s_1 + s_{12} + \frac{1}{2}\,s_{123}\right),$$

and thus

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = m\, g_0 \ell \begin{pmatrix} 2.5\, c_1 + 1.5\, c_{12} + 0.5\, c_{123} \\ 1.5\, c_{12} + 0.5\, c_{123} \\ 0.5\, c_{123} \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{g} = \begin{pmatrix} 40.8593 \\ 19.6200 \\ 4.9050 \end{pmatrix}.$$

Therefore, being $\dot{\boldsymbol{q}} = \boldsymbol{0}$, the control torque at steady state is

$$\boldsymbol{\tau} = \boldsymbol{J}^T \boldsymbol{F}_e + \boldsymbol{g} = \begin{pmatrix} 38.9721 \\ 19.6200 \\ 4.9050 \end{pmatrix} \text{ [Nm]}.$$

The contact with the obstacle occurs at the midpoint of link 1, with

$$\boldsymbol{p}_c = \frac{\ell}{2}\begin{pmatrix} c_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 0.1732 \\ 0.1 \end{pmatrix}, \qquad \boldsymbol{J}_c^T = \frac{\ell}{2}\begin{pmatrix} -s_1 & c_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -0.1 & 0.1732 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The unknown contact force $\boldsymbol{F}_c$ generates a joint torque $\boldsymbol{\tau}_c = \boldsymbol{J}_c^T \boldsymbol{F}_c$, which will have a non-zero component τ_{c1} only at joint 1, the only joint preceding the contact point. Moreover, because gravity is being compensated by control (and thus removed from this analysis) and the robot is in a static equilibrium, from the balance of joint torques it follows that

$$\boldsymbol{\tau}_c + \boldsymbol{\tau}_e = \boldsymbol{0} \qquad \Longleftrightarrow \qquad \boldsymbol{J}_c^T \boldsymbol{F}_c + \boldsymbol{J}^T \boldsymbol{F}_e = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{\tau}_c = -\boldsymbol{J}^T \boldsymbol{F}_e = \begin{pmatrix} 1.8872 \\ 0 \\ 0 \end{pmatrix} \text{ [Nm]}.$$

This is also the value reached by the momentum-based residual at steady state, i.e.,

$$\dot{\boldsymbol{r}} = \boldsymbol{K}_I\,(\boldsymbol{\tau}_c - \boldsymbol{r}) \qquad \Rightarrow \qquad \boldsymbol{r}_\infty = \lim_{t\to\infty} \boldsymbol{r}(t) = \boldsymbol{\tau}_c = \begin{pmatrix} 1.8872 \\ 0 \\ 0 \end{pmatrix} \text{ [Nm]}.$$

Note that the internal structure of $\boldsymbol{r}$ supports its *isolation* property: in fact, the components of $\boldsymbol{r}$ associated to joints that are beyond the contacting link are always zero. This is true not only at steady state, but also in dynamic conditions, when the robot is moving and may hit an obstacle.

At this stage, knowing exactly the contact point on link 1, we can recover at least the component ${}^nF_c$ of the contact force $\boldsymbol{F}_c$ that is normal to the first link. Instead, the internal force component ${}^tF_c$ that is aligned with link 1 produces no torque at joint 1, and thus cannot be reconstructed from τ_{c1} only. Expressing the contact force in the frame RF_1 rotated with link 1, we have

$${}^1\boldsymbol{F}_c = \begin{pmatrix} {}^tF_c \\ {}^nF_c \end{pmatrix}, \qquad \tau_{c1} = {}^nF_c\,\frac{\ell}{2} \qquad \Rightarrow \qquad {}^nF_c = \frac{2\tau_{c1}}{\ell} = 9.4359 \text{ [N]}.$$

This partial result can be obtained also from pseudo-inversion of the static relation $\boldsymbol{\tau}_c = \boldsymbol{J}_c^T \boldsymbol{F}_c$. Note first that the contact Jacobian in frame 1 is

$${}^1\boldsymbol{J}_c(\boldsymbol{q}) = \mathbf{Rot}^T(q_1)\,\boldsymbol{J}_c(\boldsymbol{q}) = \begin{pmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{pmatrix} \frac{\ell}{2} \begin{pmatrix} -s_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \ell/2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0 \end{pmatrix}.$$

From

$$\boldsymbol{\tau}_c = \boldsymbol{J}_c^T \boldsymbol{F}_c = \left({}^1\boldsymbol{J}_c\right)^T \cdot {}^1\boldsymbol{F}_c$$

we obtain

$${}^1\boldsymbol{F}_c = \left({}^1\boldsymbol{J}_c^T\right)^{\#} \boldsymbol{\tau}_c = \begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1.8872 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 9.4359 \end{pmatrix} = \begin{pmatrix} {}^tF_c \\ {}^nF_c \end{pmatrix}.$$

Since pseudo-inversion yields always a minimum norm solution, the obtained tangential component of the contact force is ${}^tF_c = 0$, i.e., is automatically set to zero. Finally, the contact force expressed in the base reference frame RF_0 is computed as

$$\boldsymbol{F}_c = \mathbf{Rot}(q_1)\,{}^1\boldsymbol{F}_c = \begin{pmatrix} -4.7180 \\ 8.1718 \end{pmatrix} \text{ [N]}.$$

Indeed, $\|\boldsymbol{F}_c\| = \left\|{}^1\boldsymbol{F}_c\right\| = 9.4359$.

* * * * *

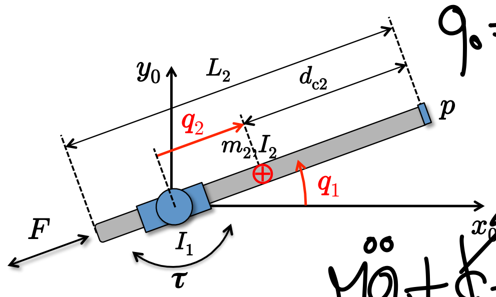
Robotics II

September 21, 2017

Exercise 1

Consider the RP planar robot in Fig. 1, with the coordinates $\boldsymbol{q} = (q_1, q_2)$, the kinematic parameter L_2, and the dynamic parameters d_{c2}, m_2, I_1 and I_2 defined therein.

$$\ddot{q}_0 = M^{-1}\left(\begin{pmatrix}\tau\\ F\end{pmatrix} - g\right)$$



$$M\ddot{q}_0 + \cancel{c} + g = \begin{pmatrix}\tau\\ F\end{pmatrix}$$

Figure 1: A RP planar robot with the relevant variables and parameters.

- Provide the symbolic expression of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$, of the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, and of the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ when the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ is inclined with respect to the horizontal plane by an angle $\alpha \in [0, \pi/2]$ around the $\boldsymbol{x}_0$ axis.
- Determine the symbolic expression of $\ddot{\boldsymbol{q}}_0 \in \mathbb{R}^2$, the joint acceleration when the robot starts from rest and the two actuators apply a torque τ and a force F as command inputs.
- Next, assume that

  i) $\alpha = 0$ and the robot is at rest;

  ii) the second link is a uniform thin rod with mass m_2 and inertia $I_2 = (m_2 L_2^2)/12$;

  iii) the torque and the force provided by the motors are bounded: $|\tau| \le T_{max}$, $|F| \le F_{max}$;

  iv) the prismatic joint has a limited symmetric range, with $q_2 \in [-L_2, L_2]$.

  In these conditions:

  a. Provide the expression of the squared norm $\|\ddot{\boldsymbol{p}}_0\|^2$, where $\ddot{\boldsymbol{p}}_0 \in \mathbb{R}^2$ is the end-effector acceleration when the robot starts from rest. Verify that this quantity is a function of $\boldsymbol{q}$ and sketch graphically this dependence.

  b. Analyze at least qualitatively how the configurations $\boldsymbol{q}^*_{min}$ and $\boldsymbol{q}^*_{max}$ that provide, respectively, the minimum and maximum of $\|\ddot{\boldsymbol{p}}_0\|^2$ change, when the inertia I_1 of the first link is either much larger or much smaller than I_2 (by 1-2 orders of magnitude).

Exercise 2

A lightweight 6R robot with a spherical wrist operates in a working environment where a human is occasionally present. During normal operation, the robot task is to track accurately a desired smooth trajectory $\boldsymbol{p}_d(t)$ for the end-effector position $\boldsymbol{p} = \boldsymbol{f}_p(\boldsymbol{q}) \in \mathbb{R}^3$ and an associated desired trajectory $\boldsymbol{\phi}_d(t)$ for a minimal representation of the end-effector orientation $\boldsymbol{\phi} = \boldsymbol{f}_\phi(\boldsymbol{q}) \in \mathbb{R}^3$. Assume that:

- The complete dynamic model of the robot in free motion is perfectly known, and is described (with the usual notations) by the following equations
$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} - \boldsymbol{\tau}_f(\dot{\boldsymbol{q}}), \tag{1}$$
where the friction term $\boldsymbol{\tau}_f$ denotes a dissipative action at the joints.
- The direct kinematic functions $\boldsymbol{f}_p$ and $\boldsymbol{f}_\phi$ are known, as well as the 6×6 analytic Jacobian associated to the end-effector task
$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \frac{\partial \boldsymbol{f}_p(\boldsymbol{q})}{\partial \boldsymbol{q}} \\ \frac{\partial \boldsymbol{f}_\phi(\boldsymbol{q})}{\partial \boldsymbol{q}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_p(\boldsymbol{q}) \\ \boldsymbol{J}_\phi(\boldsymbol{q}) \end{pmatrix}, \tag{2}$$
where the two matrices $\boldsymbol{J}_p$ and $\boldsymbol{J}_\phi$ have dimension 3×6, and matrix $\boldsymbol{J}$ is nonsingular in the region of interest.
- The robot is equipped only with encoders at the joints, and the environment is *not* monitored by any external sensor.

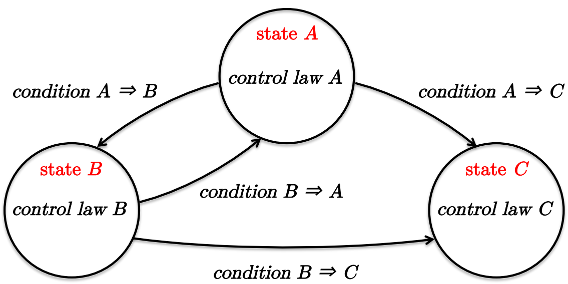


Figure 2: A state diagram of robot control operation in collision-aware tasks.

With reference to the state diagram in Fig. 2, the following collision-aware behavior for safe Human-Robot Interaction (HRI) should be realized through a suitable set of robot control laws and conditions for the transitions:

- During normal operation (state A in the diagram), if a mild contact occurs and is detected, the robot keeps the three-dimensional position task but relaxes the orientation task, trying to accommodate in this way a reflex reaction to the contact (state B).
- Instead, when a severe collision occurs during normal operation, the robot abandons the task completely by bouncing away from the collision area (state C) ad then stops.
- While in state B, the robot may either switch back to normal operation when the contact is no longer present, or abandon also the orientation task and switch to state C in case the interaction forces will increase further.

Specify the control laws and the transition conditions to be used in the state diagram of Fig. 2.

[150 minutes; open books]

Solution

September 21, 2017

Exercise 1

Following a Lagrangian approach, we compute first the kinetic energy $T = T_1 + T_2$. We have

$$T_1 = \frac{1}{2} I_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2} m_2 \left\| \frac{d}{dt} \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix} \right\|^2 + \frac{1}{2} I_2 \dot{q}_1^2 = \frac{1}{2} \left(I_2 + m_2 q_2^2\right) \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2,$$

and thus the diagonal inertia matrix

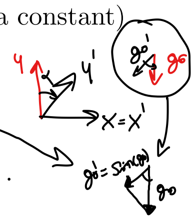
$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_1 + I_2 + m_2 q_2^2 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{3}$$

Using the Christoffel symbols, the Coriolis and centrifugal terms are easily computed from (3) as

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2 m_2 q_2 \, \dot{q}_1 \dot{q}_2 \\ -m_2 q_2 \, \dot{q}_1^2 \end{pmatrix}.$$

For the potential energy due to gravity, $U_g = U_1 + U_2$, we have (up to a constant)

$$U_1 = 0, \qquad U_2 = m_2 \boxed{\left(g_0 \sin \alpha\right)} q_2 \sin q_1,$$



and thus

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T = m_2 \, g_0 \sin \alpha \begin{pmatrix} q_2 \cos q_1 \\ \sin q_1 \end{pmatrix}. \tag{4}$$

When the robot is at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$), the joint acceleration takes the expression

$$\ddot{\boldsymbol{q}}_0 = \ddot{\boldsymbol{q}}\,|_{\dot{\boldsymbol{q}}=\boldsymbol{0}} = \boldsymbol{M}^{-1}(\boldsymbol{q}) \left(\begin{pmatrix} \tau \\ F \end{pmatrix} - \boldsymbol{g}(\boldsymbol{q}) \right) = \begin{pmatrix} \dfrac{\tau - m_2 g_0 \sin \alpha \, q_2 \cos q_1}{I_1 + I_2 + m_2 q_2^2} \\ \dfrac{F - m_2 g_0 \sin \alpha \sin q_1}{m_2} \end{pmatrix} = \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix}. \tag{5}$$

The end-effector position and its velocity are

$$\boldsymbol{p} = (q_2 + d_{c2}) \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \qquad \dot{\boldsymbol{p}} = \dot{q}_2 \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} + (q_2 + d_{c2}) \, \dot{q}_1 \begin{pmatrix} -\sin q_1 \\ \cos q_1 \end{pmatrix}.$$

Thus, the end-effector acceleration at *zero* joint velocity is

$$\begin{aligned} \ddot{\boldsymbol{p}}_0 = \ddot{\boldsymbol{p}}\,|_{\dot{\boldsymbol{q}}=\boldsymbol{0}} &= \ddot{q}_2 \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} + (q_2 + d_{c2}) \, \ddot{q}_1 \begin{pmatrix} -\sin q_1 \\ \cos q_1 \end{pmatrix} \\ &= \begin{pmatrix} \cos q_1 & -\sin q_1 \\ \sin q_1 & \cos q_1 \end{pmatrix} \begin{pmatrix} \ddot{q}_2 \\ (q_2 + d_{c2}) \, \ddot{q}_1 \end{pmatrix} = \boldsymbol{R}(q_1) \begin{pmatrix} \ddot{q}_2 \\ (q_2 + d_{c2}) \, \ddot{q}_1 \end{pmatrix}, \end{aligned} \tag{6}$$

CAN ALSO BE FOUND AS $\ddot{p}_0 = J\ddot{q}_0 + \emptyset \rightarrow \dot{q}_0 = 0$

where $\boldsymbol{R}(\cdot)$ is a 2×2 (planar) rotation matrix. From (6), we have

$$\|\ddot{\boldsymbol{p}}_0\|^2 = \ddot{\boldsymbol{p}}_0^T \, \ddot{\boldsymbol{p}}_0 = \left\| \begin{matrix} \ddot{q}_2 \\ (q_2 + d_{c2}) \, \ddot{q}_1 \end{matrix} \right\|^2 = (q_2 + d_{c2})^2 \, \ddot{q}_1^2 + \ddot{q}_2^2. \tag{7}$$

$$2m_2 q_2 (q_2+d_{c2})\tau^2 = 0$$

$$q_2(q_2+d_{c2}) = 0$$

$$q_2 = 0 \quad q_2 = -d_{c2}$$

$$\cancel{k(q_2+d_{c2})}\,\dot{q}_2 = 2m_2 \cdot k\, q_2 \dot{q}_2 (q_2+d_{c2})^2 \tau^2$$

$$\cancel{\dot{q}_2} = 2m_2 q_2 \cancel{\dot{q}_2} (q_2+d_{c2})\tau^2$$

$$\frac{\left[2(q_2+d_{c2})\cdot \dot{q}_2 (I_1+I_2+m_2q_2^2)\right]}{((I_1+I_2+m_2q_2^2))^4} - \left[2(I_1+I_2+m_2q_2^2)\cdot m_2 \cdot 2q_2\dot{q}_2\,(q_2+d_{c2})^2\right]\tau^2$$

Using (5) for $\alpha = 0$ in (7), we obtain

$$\|\ddot{\boldsymbol{p}}_0\|^2 = \frac{1}{m_2^2} F^2 + \frac{(q_2+d_{c2})^2}{(I_1+I_2+m_2q_2^2)^2}\tau^2, \qquad (8)$$

which shows an actual dependence only on the prismatic joint variable q_2. The two addends in (8) are separately driven by the two motors: the first one is a *radial* contribution due to F, which is scaled just by m_2^2 and is independent from the robot configuration; the second one is the *tangential* contribution (normal to the second link) due to τ, which depends in a nonlinear fashion on q_2, as well as on m_2, d_{c2}, I_1, and I_2 (and their relative values).

It is easy to see that the minimum of $\|\ddot{\boldsymbol{p}}_0\|^2$ is obtained at $q_{2,min}^* = -d_{c2}$ (with arbitrary q_1^*), namely when the end-effector position is at the origin (on the axis of joint 1). Note also that this value is independent from the dynamic parameters m_2, I_1, and I_2.

From the expression (8), it follows that the maximum value H of the squared norm of the acceleration is given by

$$H = \frac{F_{max}^2}{m_2^2} + \max_{q_2\in[-L_2,L_2]} \frac{(q_2+d_{c2})^2}{(I_1+I_2+m_2q_2^2)^2}\tau_{max}^2,$$

where the maximum bounds on the inputs have been used. Under the given assumption on the mass distribution of link 2, in order to find the absolute maximum of the tangential contribution in $\|\ddot{\boldsymbol{p}}_0\|^2$ one should study the behavior of the positive function

$$h(q_2) = \left(\frac{q_2+d_{c2}}{I_1+I_2+m_2q_2^2}\right)^2 = \left(\frac{q_2+\frac{L_2}{2}}{I_1+m_2\left(\frac{L_2^2}{12}+q_2^2\right)}\right)^2$$

for q_2 in the closed interval $[-L_2, L_2]$. The stationary points of h satisfy the necessary condition

$$\frac{dh(q_2)}{dq_2} = 0 \quad\Longleftrightarrow\quad 2\left(\frac{q_2+\frac{L_2}{2}}{I_1+m_2\left(\frac{L_2^2}{12}+q_2^2\right)}\right)\frac{I_1+m_2\left(\frac{L_2^2}{12}+q_2^2\right)-2m_2q_2\left(q_2+\frac{L_2}{2}\right)}{\left(I_1+m_2\left(\frac{L_2^2}{12}+q_2^2\right)\right)^2} = 0$$

$$\Longleftrightarrow\quad \frac{\left(q_2+\frac{L_2}{2}\right)\left(m_2\,q_2^2+m_2L_2\,q_2-\left(I_1+m_2\frac{L_2^2}{12}\right)\right)}{\left(I_1+m_2\left(\frac{L_2^2}{12}+q_2^2\right)\right)^3} = 0.$$

The zeros of the derivative occur where one of the two polynomial factors (one linear, the other quadratic) at the numerator vanishes. This occurs at[1]

$$q_2 = q_{2,min}^* = -\frac{L_2}{2} \qquad \Rightarrow \quad \text{a minimum of } h(q_2)$$

and at

$$q_2 = q_{2,max}^* = -\frac{L_2}{2} + \sqrt{\left(\frac{L_2}{2}\right)^2 + \frac{I_1}{m_2} + \frac{L_2^2}{12}} \qquad \Rightarrow \quad \text{a maximum —only if} \in [-L_2, L_2].$$

For very large values of the ratio $I_1/I_2 \propto I_1/m_2$, this second expression will be larger than L_2, and thus outside the closed interval of definition for q_2. Therefore, the maximum will occur at the

[1] The second root of the quadratic factor is always strictly lower than $-L_2$, thus outside the interval $[-L_2, L_2]$.

upper limit, i.e., $q^*_{2,max} = L_2$. On the other hand, for very small values of I_1/m_2, neglecting this term and using a Taylor expansion yields $q^*_{2,max} \approx L_2/12$.

As a numerical example, Fig. 3 shows the plots of $h(q_2)$ for various ratios of I_1/I_2, when the second link is a uniform thin rod of mass $m_2 = 1$ [kg] and length $L_2 = 0.5$ [m]. For instance, when $I_1/I_2 = 50$ (red profile on the left), the maximum is at $q^*_{2,max} = L_2 = 0.5$. On the other hand, when $I_1/I_2 = 0.01$ (red profile on the right), the maximum is at $q^*_{2,max} \approx L_2/12 = 0.04$ [m].

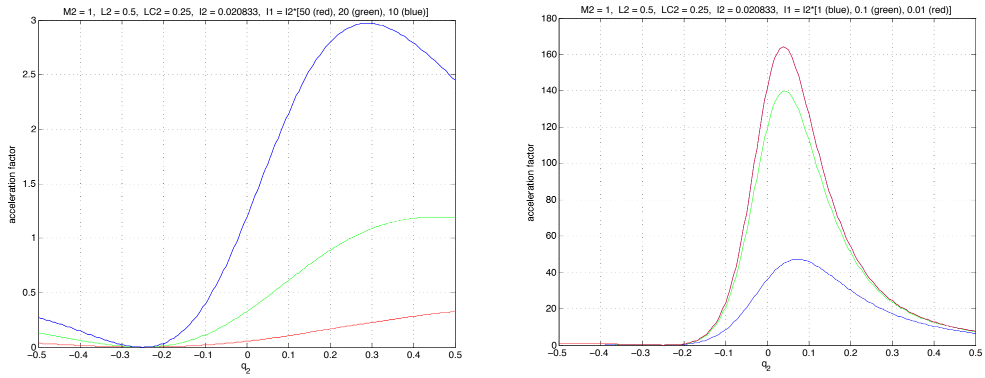


Figure 3: Behavior of the $h(q_2)$ function, for various ratios I_1/I_2: high ratios $50, 20, 10$ (left) and low ratios $= 1, 0.1, 0.01$ (right).

The physical explanation of these behaviors is as follows. When the inertia of the first link is very large, this constant inertia will dominate the effort needed by the first motor to accelerate the robot structure and so the maximum tangential component of the end-effector acceleration will be obtained when the second link is fully stretched. On the other hand, when the first link inertia can be assumed as negligible in the picture, the maximum tangential acceleration of the end-effector will result from the trade-off between two contrasting effects: the amplification of the joint acceleration due to a longer radial extension of the second link and its reduction due to the associated larger inertia seen by the first motor torque. Thus, qualitatively speaking, the peak will be somewhere in between $q_2 = 0$ and $q_2 = L_2$.

Note finally that when the location of the center of mass of the second link (with non-uniformly distributed mass) approaches the tip of the link ($d_{c2} = 0$), the above qualitative behavior remains the same, but the plots in Fig. 3 will become symmetric w.r.t. $q_2 = 0$, with the single minimum at $q^*_{2,min} = 0$ and two maxima in $\pm|q^*_{2,max}| \in [-L_2, L_2]$.

Exercise 2

The problem can be solved by using the residual vector $\boldsymbol{r}$ as a collision monitoring signal, together with a number of ordered positive thresholds on its norm $\|\boldsymbol{r}\|$ to be used in the switching conditions, and suitable control laws for each state.

Based on the known model (1), the residual $\boldsymbol{r} \in \mathbb{R}^6$ can be defined as

$$\boldsymbol{r}(t) = \boldsymbol{K}\left(\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} - \int_0^t \left(\boldsymbol{\tau} + \boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{\tau}_f(\dot{\boldsymbol{q}}) + \boldsymbol{r}\right) ds\right), \quad \text{with } \boldsymbol{K} > 0 \text{ (diagonal)}, \tag{9}$$

where $\boldsymbol{\tau}$ is the actual control torque applied in any of the robot states A, B, or C. Using (1), equation (9) implies the dynamic behavior

$$\dot{\boldsymbol{r}} = \boldsymbol{K}\left(\boldsymbol{\tau}_c - \boldsymbol{r}\right), \tag{10}$$

where $\boldsymbol{\tau}_c \in \mathbb{R}^6$ is the joint torque resulting from a collision force/moment occurring anywhere along the robot structure. Indeed, if at some time t the torque τ_c returns to zero, then each component of $\boldsymbol{r}$ will decay exponentially to zero as well. Moreover, for a sufficiently large $\boldsymbol{K}$, from (10) we can use the approximation $\boldsymbol{\tau}_C \simeq \boldsymbol{r}$ and use the residual vector $\boldsymbol{r}$ as a proxy of the unknown joint torque $\boldsymbol{\tau}_c$ due to collision.

With reference to Fig. 2, in the following suitable control laws will be defined for each state.

- **Control in state** A. Define the desired task trajectory as $\boldsymbol{x}_d(t) = \left(\boldsymbol{p}_d^T(t)\ \boldsymbol{\phi}_d^T(t)\right)^T \in \mathbb{R}^6$. In order to accurately follow this smooth trajectory, we use the Cartesian feedback linearization controller

$$\begin{aligned}\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\Big(&\ddot{\boldsymbol{x}}_d + \boldsymbol{K}_D\left(\dot{\boldsymbol{x}}_d - \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \boldsymbol{K}_P\left(\boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{q})\right) - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\Big)\\ &+ \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{\tau}_f(\dot{\boldsymbol{q}}),\end{aligned} \tag{11}$$

  with 6×6 (typically diagonal) gain matrices $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$. Within this law, the presence of a PD action on the task error allows to recover exponentially transient errors. This is necessary, e.g., when the complete task is partially abandoned and then resumed (in case we are coming back to state A from state B).

- **Control in state** B. In this case, the orientation part of the desired task will be relaxed, while the position task $\boldsymbol{p}_d(t) \in \mathbb{R}^3$ for the robot end-effector should be kept. Therefore, the robot becomes kinematically redundant since the task has dimension $m = 3$ while the robot has $n = 6$ control commands available; the degree of redundancy is thus $n - m = 3$. We continue to achieve Cartesian position tracking, e.g., by using a dynamically consistent redundancy resolution scheme. This control scheme uses the 3×6 Jacobian $\boldsymbol{J}_p$ in a partially feedback linearizing law that is weighted by the inverse of the task inertia matrix $\boldsymbol{\Lambda}(\boldsymbol{q})$ and adds a suitable torque $\boldsymbol{\tau}_0 \in \mathbb{R}^6$ projected in the dynamic null space of the task. We have thus

$$\begin{aligned}\boldsymbol{\tau} = \boldsymbol{J}_p^T(\boldsymbol{q})\boldsymbol{\Lambda}(\boldsymbol{q})\Big(&\ddot{\boldsymbol{p}}_d + \boldsymbol{K}_{D,p}\left(\dot{\boldsymbol{p}}_d - \boldsymbol{J}_p(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \boldsymbol{K}_{P,p}\left(\boldsymbol{p}_d - \boldsymbol{f}_p(\boldsymbol{q})\right) - \dot{\boldsymbol{J}}_p(\boldsymbol{q})\dot{\boldsymbol{q}}\\ &+ \boldsymbol{J}_p(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{\tau}_f(\dot{\boldsymbol{q}})\right)\Big),\\ &+ \Big(\boldsymbol{I} - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{\Lambda}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\Big)\boldsymbol{\tau}_0,\end{aligned} \tag{12}$$

  with 3×3 (typically diagonal) gain matrices $\boldsymbol{K}_{P,p} > 0$ and $\boldsymbol{K}_{D,p} > 0$, and the 3×3 inertia matrix reduced to the task

$$\boldsymbol{\Lambda}(\boldsymbol{q}) = \left(\boldsymbol{J}_p(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}_p^T(\boldsymbol{q})\right)^{-1}.$$

  In (12), the torque $\boldsymbol{\tau}_0 = \boldsymbol{K}_r\boldsymbol{r}$ is used, with $\boldsymbol{K}_r > 0$, so as to obtain a reaction to the collision torque $\boldsymbol{\tau}_c \simeq \boldsymbol{r}$ which is consistent with the remaining Cartesian position task.

- **Control in state** C. In this case, the complete original task is abandoned. The robot reacts to the collision in a stronger or weaker way depending on the intensity (and direction in the joint space) of $\boldsymbol{r}$, which is a proxy of the severity of the collision. Moreover, to avoid bias in the reaction due to the gravity, this term should be cancelled. As a result

$$\boldsymbol{\tau} = \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_r \boldsymbol{r} \tag{13}$$

with $\boldsymbol{K}_r > 0$. Once the contact is lost, $\boldsymbol{r}$ will go to zero. As a result, thanks of the presence of friction, the robot will come to a stop in a zero-gravity condition. Joint velocity damping can be added so as to anticipate the instant when the robot is finally at rest, but this will limit quick reaction to collisions.

Transitions between the states in Fig. 2 will be driven by the actual value of $\|\boldsymbol{r}\| \geq 0$. To this end, define a sequence of positive thresholds for this variable:

$$0 < r_{low} < r_{mild} < r_{severe}.$$

The value r_{low} is the minimum threshold that should be crossed by $\|\boldsymbol{r}\|$ in order to detect reliably contact/collision events (i.e., obtaining few false positives, or eliminating them). The detection instant $t_{detect} \geq 0$ is the first instant at which $\|\boldsymbol{r}(t_{detect})\| \geq r_{low}$. For the choice of this lowest threshold, one takes into account the presence of noise in position sensing and in the generation of an estimate of the velocity $\dot{\boldsymbol{q}}$ by numerical differentiation of the position measures $\boldsymbol{q}$, as well as the remaining model uncertainties. For the two other thresholds, the rationale is that mild collisions will generate small values of the norm of the residual and, conversely, severe collisions will be associated to large values of $\boldsymbol{r}$. The value r_{mild} is chosen only slightly above r_{low}, so that the control system may detect a contact but not yet consider it as a collision, letting thus the robot continue the original motion task. With this in mind, the following switching conditions correctly realize the desired behavior:

- **condition** $A \Rightarrow B$: $r_{mild} \leq \|\boldsymbol{r}\| < r_{severe}$;
- **condition** $A \Rightarrow C$: $\|\boldsymbol{r}\| \geq r_{severe}$;
- **condition** $B \Rightarrow C$: $\|\boldsymbol{r}\| \geq r_{severe}$;
- **condition** $B \Rightarrow A$: $\|\boldsymbol{r}\| < r_{low}$.

Note that the last condition may be replaced also by $\|\boldsymbol{r}\| < r_{mild}$. However, using the more conservative value r_{low} introduces some hysteresis, so that the robot will avoid switching several times between the states A and B when the norm of the residual is oscillating around r_{mild}.

* * * * *

Robotics II

January 11, 2018

Exercise 1

The RP planar robot in Fig. 1, with coordinates $\boldsymbol{q} = (q_1, q_2)$ and parameters m_2, d_{c2}, I_1 and I_2 defined therein, should execute a task defined by a time-varying trajectory $y_d(t) \in \mathbb{R}$ for the height of its end-effector.

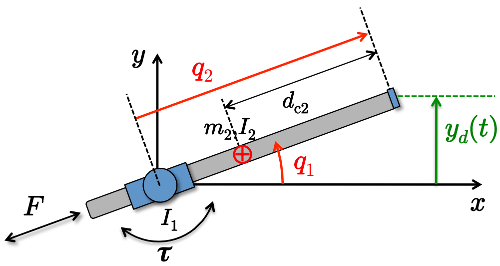


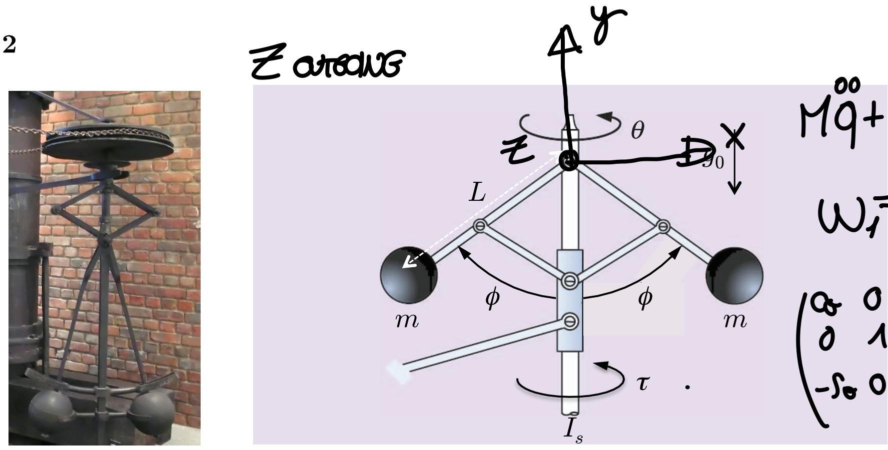
Figure 1: A RP planar robot with the relevant parameters and variables.

Assuming as input command the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^2$, determine the explicit expressions of the kinematic control laws that execute the task in nominal conditions, recover exponentially from any task error, and

- minimize $\frac{1}{2}\|\dot{\boldsymbol{q}}\|^2$: which is the theoretical pitfall of this solution?
- minimize the weighted norm $\frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W} \dot{\boldsymbol{q}}$, with constant $\boldsymbol{W} = \text{diag}\{w_1, w_2\} > 0$; what happens for very large ratios w_1/w_2 (in the limit $\to \infty$); and for $w_2/w_1 \to \infty$?
- minimize the kinetic energy $T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$, being $\boldsymbol{M}(\boldsymbol{q}) > 0$ the robot inertia matrix.

Exercise 2

Z AROUNG



$M\ddot{q} + C + g - F_v = \tau$

$\omega_1 = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} = \omega_2$

$\begin{pmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{pmatrix} \begin{pmatrix} L\sin(\phi) \\ L\cos(\phi) \\ 0 \end{pmatrix} = \begin{pmatrix} LC_\theta S_\phi \\ LC_\phi \\ -LS_\phi S_\theta \end{pmatrix}$

$\|$ THIS THING

X

Figure 2: The Boulton-Watt governor and a scheme with definition of parameters and variables.

Figure 2 shows a picture and a simplified scheme of the famous Boulton-Watt centrifugal governor, a system invented to regulate the rotational speed of a steam engine by a mechanical leverage (feedback) opening a valve that provides steam under pressure to the engine. We consider here only the so-called *open-loop* dynamic behavior of the system, under the action of an external torque $\tau \in \mathbb{R}$ applied to the main rotating shaft.

$T_{1M} = \frac{1}{2} m_1 \|\dot{X}\|^2 \quad = \quad T_{2M} = \frac{1}{2} m_2 \|\dot{X}\|^2$

$T_S = \frac{1}{2} I_S \dot{\theta}^2 \longrightarrow$ FOR THE SHAFT

OTHER WAY OF SEEING IT

$T_M = T_{\text{ROTATION AROUND } y\ (\theta)} + T_{\text{ROTATION AROUND } z\ (\phi)}$

$\frac{1}{2} m (L\sin\phi\,\dot{\theta})^2$ $v = r \cdot \omega$

$\frac{1}{2} m (L \cdot \dot{\phi})^2$ $v = r \cdot \omega$

Assume that:

- the main shaft has an inertia I_s around its rotation axis
- the two balls have identical mass m that is concentrated at the end of a link of length L
- the links and all other linkages have negligible masses
- a viscous friction torque with coefficient $f_v > 0$ is acting on the main shaft
- all other frictional effects are negligible.

Derive the complete dynamic model of this system using a Lagrangian formalism. Assuming knowledge of the geometric parameter L, provide a linear parametrization of the dynamics in terms of its dynamic coefficients. Find the value of the constant torque τ_Ω to be applied for sustaining a steady-state rotation at a given angular speed $\Omega > 0$. Finally, design a nonlinear feedback for τ so as to achieve partial feedback linearization of the system, i.e., exact linearization by feedback of only part of the closed-loop dynamics, in this case of one of the two coordinates.

Exercise 3

Consider the design of impedance control laws and force control laws for the 1-dof example, shown in Fig. 3, namely a single mass m that moves on a frictionless horizontal plane under the action of a commanded force $f \in \mathbb{R}$ and of a contact force $f_c \in \mathbb{R}$.

$X_1 = X_d$ $\quad \dot{X}_1 = X_2$

$X_2 = \dot{X}_d$ $\quad \dot{X}_2 = \ddot{X}_d = \frac{F+F_c}{m}$

$F = m\ddot{X}_d - F_c$

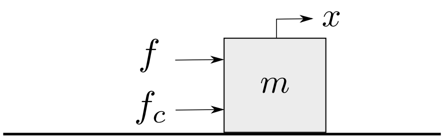


Figure 3: A mass m subject to a commanded force f and a contact force f_c.

In particular:

- The impedance controllers should work with a generic time-varying, smooth position reference $x_d(t)$, either with or without the use of a load cell that can measure the contact force f_c. Illustrate the properties of the obtained closed-loop systems.
- What happens when $x_d(t)$ degenerates to a constant? What happens during free motion, when $f_c = 0$?
- For $m = 5$ [kg], design the control parameters of the impedance law so that the dynamics of the position error $e = x_d - x$ in the closed-loop system is characterized by a pair of asymptotically stable complex poles with natural frequency $\omega_n = 10$ [rad/s] and critical damping ratio $\zeta = 0.7071$.
- On the other hand, the force controllers should be able to regulate the (measured) contact force f_c to a constant value f_d, using any combination of desired force feedforward and force error feedback. Illustrate the properties of the obtained closed-loop systems.
- What happens during free motion, when $f_c = 0$ and a constant contact force f_d is desired?

[150 minutes; open books]

Solution

January 11, 2018

Exercise 1

The problem deals with kinematic redundancy since the RP robot has $n = 2$ joints and the required task is scalar $m = 1$. The task output function and its Jacobian are

$$y(\boldsymbol{q}) = q_2 \sin q_1, \quad \boldsymbol{J}(\boldsymbol{q}) = \frac{\partial y(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} q_2 \cos q_1 & \sin q_1 \end{pmatrix}. \tag{1}$$

The 1×2 task Jacobian loses rank (vanishes) iff $q_1 = \{0, \pi\}$ and $q_2 = 0$ simultaneously.

The minimization of the squared norm of $\dot{\boldsymbol{q}}$ is achieved by the use of the pseudoinverse of the task Jacobian. Out of singularities, $\boldsymbol{J}^{\#} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}$ and the kinematic control law takes the expression

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\,(\dot{y}_d + k(y_d - y(\boldsymbol{q}))) = \frac{1}{s_1^2 + q_2^2 c_1^2}\begin{pmatrix} q_2 c_1 \\ s_1 \end{pmatrix}(\dot{y}_d + k(y_d - q_2 \sin q_1)), \tag{2}$$

where $k > 0$ is a control gain that guarantees exponential recovery from transient errors, i.e., $\dot{e}(t) = -ke(t)$, with $e = y_d - q_2 \sin q_1 \neq 0$, during task execution. The pitfall of (2) is that the norm $\|\dot{\boldsymbol{q}}\|$ involves mixed angular (the revolute joint velocity $\dot{q}_1$) and linear (the prismatic joint velocity $\dot{q}_2$) quantities, so its straight minimization is ill-defined conceptually. In fact, the denominator in (2) contains the sum of an non-dimensional term (s_1^2) and of a term with (squared) length units. Stated differently, changing the representing units (e.g., from 1 m to 100 cm) will change the 'optimal' solution.

The minimization of the weighted norm $\frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{W}\dot{\boldsymbol{q}}$, leading to weighted pseudoinversion of the task Jacobian, may solve this theoretical issue. In particular, the units of the (positive) elements in the diagonal of $\boldsymbol{W}$ can be used to make terms non-dimensional (e.g., by choosing w_1 in (squared) length units). Out of singularities, $\boldsymbol{J}^{\#}_{\boldsymbol{W}} = \boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T)^{-1}$ and the kinematic control law takes the expression

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}_{\boldsymbol{W}}(\boldsymbol{q})\,(\dot{y}_d + k(y_d - y(\boldsymbol{q}))) = \frac{1}{\dfrac{q_2^2 c_1^2}{w_1} + \dfrac{s_1^2}{w_2}}\begin{pmatrix} \dfrac{q_2 c_1}{w_1} \\ \dfrac{s_1}{w_2} \end{pmatrix}(\dot{y}_d + k(y_d - q_2 \sin q_1)), \tag{3}$$

with $k > 0$ as before. Indeed, different values of the weights w_1 and w_2 will lead to different joint velocity solutions. It is easy to verify that is the relative ratio between w_1 and w_2 that really matters. For very large ratios w_1/w_2, the cost of moving the (revolute) joint 1 will be dominant and therefore the solution (3) will tend to minimize its motion while performing the task. In the limit, when $w_1 \to \infty$, it follows from (3) that $\dot{q}_1 \to 0$, while $\dot{q}_2 \propto 1/s_1$: therefore, executing the task will become more and more problematic as the second link gets closer to the horizontal. Similarly, for $w_2/w_1 \to \infty$ the second (prismatic) joint will be very expensive to move, while $\dot{q}_1 \propto 1/q_2 c_1$: the control effort will increase dramatically when the second link is close to being vertical ($c_1 \simeq 0$) and/or fully retracted ($q_2 \simeq 0$).

For the third objective, we need first to derive the inertia matrix of the RP robot. From the expression of the kinetic energy $T = T_1 + T_2$, with

$$T_1 = \frac{1}{2}I_1\dot{q}_1^2, \quad T_2 = \frac{1}{2}m_2\left\|\frac{d}{dt}\begin{pmatrix}(q_2 - d_{c2})\cos q_1 \\ (q_2 - d_{c2})\sin q_1\end{pmatrix}\right\|^2 + \frac{1}{2}I_2\dot{q}_1^2 = \frac{1}{2}\left(I_2 + m_2(q_2 - d_{c2})^2\right)\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2,$$

we obtain a diagonal inertia matrix as

$$
\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} \overbrace{I_1 + I_2 + m_2(q_2 - d_{c2})^2}^{[\mathrm{kgm}^2]} & 0 \\ 0 & m_2 \end{pmatrix} = \begin{pmatrix} m_{11}(q_2) & 0 \\ 0 & m_{22} \end{pmatrix}. \tag{4}
$$

The minimization of the kinetic energy T is then a special case of a weighted pseudoinversion of the task Jacobian, with one weight being configuration dependent. Thus, out of singularities, the inertia-weighted kinematic control law takes the expression

$$
\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}_{\boldsymbol{M}}(\boldsymbol{q})\left(\dot{y}_d + k(y_d - y(\boldsymbol{q}))\right) = \frac{1}{\dfrac{q_2^2 c_1^2}{\underbrace{m_{11}(q_2)}_{\mathrm{kgm}^2}} + \dfrac{s_1^2}{m_{22}}} \begin{pmatrix} \dfrac{q_2 c_1}{m_{11}(q_2)} \\ \dfrac{s_1}{m_{22}} \end{pmatrix} \left(\dot{y}_d + k(y_d - q_2 \sin q_1)\right). \tag{5}
$$

Note that the two addends in the first denominator have both consistent units of [kg^{-1}].

Exercise 2

Let $\boldsymbol{q} = (\theta, \phi)$. Following a Lagrangian approach, under the given assumptions, we compute the kinetic energy $T = T_s + 2T_m$ for the main shaft and the two equal balls. We have

$$
T_s = \frac{1}{2}\, I_s\, \dot{\theta}^2, \qquad T_m = \frac{1}{2}\, mL^2 \left(\dot{\phi}^2 + \dot{\theta}^2 \sin^2\phi\right),
$$

and thus the diagonal inertia matrix

$$
\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_s + 2mL^2 \sin^2\phi & 0 \\ 0 & 2mL^2 \end{pmatrix}. \tag{6}
$$

Using the Christoffel symbols, the Coriolis and centrifugal terms are easily computed from (6) as

$$
\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 4mL^2 \sin\phi\cos\phi\; \dot{\theta}\,\dot{\phi} \\ -2mL^2 \sin\phi\cos\phi\; \dot{\theta}^2 \end{pmatrix} = mL^2 \sin(2\phi) \begin{pmatrix} 2\,\dot{\theta}\,\dot{\phi} \\ -\,\dot{\theta}^2 \end{pmatrix} \tag{7}
$$

For the potential energy due to gravity, $U = U_s + 2U_m$, we have (up to a constant)

$$
U_s = 0, \qquad U_m = -mg_0 L\cos\phi,
$$

and thus

$$
\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ 2mg_0 L \sin\phi \end{pmatrix}. \tag{8}
$$

Including also viscous friction on the main shaft, the dynamic equations are

$$
\begin{aligned}
\left(I_s + 2mL^2 \sin^2\phi\right)\ddot{\theta} + 4mL^2 \sin\phi\cos\phi\; \dot{\theta}\,\dot{\phi} + f_v\dot{\theta} \;&=\; \tau \\
2mL^2\, \ddot{\phi} - 2mL^2 \sin\phi\cos\phi\; \dot{\theta}^2 + 2mg_0 L\sin\phi \;&=\; 0.
\end{aligned} \tag{9}
$$

Assuming knowledge of the geometric parameter L, equation (9) can be expressed in the linearly parametrized form

$$
\begin{pmatrix} \ddot{\theta} & 2L^2 \sin^2\phi\; \ddot{\theta} + 2L^2 \sin(2\phi)\, \dot{\theta}\,\dot{\phi} & \dot{\theta} \\ 0 & 2L^2\, \ddot{\phi} - L^2 \sin(2\phi)\, \dot{\theta}^2 + 2g_0 L\sin\phi & 0 \end{pmatrix} \begin{pmatrix} I_s \\ m \\ f_v \end{pmatrix} = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{\pi} = \begin{pmatrix} \tau \\ 0 \end{pmatrix}, \tag{10}
$$

with the vector $\boldsymbol{\pi} \in \mathbb{R}^3$ of dynamic coefficients.

In a steady-state equilibrium with constant angular velocity $\dot{\theta} = \Omega > 0$, we have $\ddot{\theta} = 0$ and $\ddot{\phi} = \dot{\phi} = 0$. This yields from (9)

$$\tau_\Omega = f_v \Omega, \qquad L \sin\phi \cos\phi \; \Omega^2 + g_0 \sin\phi = 0 \qquad \Rightarrow \qquad \cos\phi_e = \frac{g_0}{L\,\Omega^2}. \tag{11}$$

The input torque τ_Ω has to compensate just for the energy loss due to friction, in order to keep a uniform motion via constant angular velocity. Moreover, the equilibrium angle ϕ_e results from the balance of the gravity force and the centrifugal force. Its value increases (in the range $(0, \pi/2)$) together with Ω.

Finally, by applying the nonlinear feedback law

$$\tau = \left(I_s + 2mL^2 \sin^2\phi\right) a + 4mL^2 \sin\phi \cos\phi \; \dot{\theta}\,\dot{\phi} + f_v \dot{\theta} \tag{12}$$

where $a \in \mathbb{R}$ is the new control input (an acceleration), system (9) is transformed into

$$\begin{aligned} \ddot{\theta} &= a \\ \ddot{\phi} - \sin\phi \cos\phi \, \dot{\theta}^2 + \frac{g_0}{L} \sin\phi &= 0. \end{aligned} \tag{13}$$

The dynamics of θ is now exactly linear (a double integrator), while partial control of the motion of ϕ can be achieved only through the centrifugal term in the second equation, being $\dot{\theta}^2 = \left(\int a \, dt\right)^2$.

Exercise 3

The dynamic equation of the system in Fig. 3 is

$$m\ddot{x} = f + f_c. \tag{14}$$

Impedance control. The so-called inverse dynamics control law becomes in this simple case

$$f = ma - f_c, \tag{15}$$

and transforms system (14) into the double integrator

$$\ddot{x} = a. \tag{16}$$

The auxiliary input a has to be designed so that the controlled mass m, under the action of the contact force f_c, matches the behavior of an impedance model characterized by a desired (apparent) mass $m_d > 0$, desired damping $k_d > 0$, and desired stiffness $k_p > 0$, all acting with respect to a smooth motion reference $x_d(t)$, or

$$m_d \left(\ddot{x} - \ddot{x}_d\right) + k_d \left(\dot{x} - \dot{x}_d\right) + k_p \left(x - x_d\right) = f_c. \tag{17}$$

Equating $\ddot{x}$ in (16) and in the reference behavior (17), solving for a and substituting in (15) yields the control force

$$f = \frac{m}{m_d} \left(\ddot{x}_d + k_d \left(\dot{x}_d - \dot{x}\right) + k_p \left(x_d - x\right)\right) + \left(\frac{m}{m_d} - 1\right) f_c. \tag{18}$$

The feedback law (18) requires in general a measure of the contact force f_c.

In the reference model (17), the position error $e = x_d - x$ does not converge to zero if there is a contact force f_c. Otherwise, e will asymptotically go to zero —indeed exponentially, in view of the linearity of the system dynamics. In particular, for $k_d^2 < 4k_p m_d$, the obtained second-order linear system (17) is characterized by a pair of asymptotically stable complex poles with natural frequency and damping ratio given by

$$\omega_n = \sqrt{\frac{k_p}{m_d}}, \qquad \zeta = \frac{k_d}{2\sqrt{k_p m_d}}. \tag{19}$$

Reducing the desired mass m_d, for given values of stiffness and damping, will increase both the natural frequency ω_n and the damping ratio ζ, and thus improve transients. On the other hand, for a given mass m_d, an increase of the stiffness k_p should be accompanied by an increase of the damping k_d in order to prevent more oscillatory transients. If the desired mass equals the natural (original) mass, i.e., $m_d = m$, a measure of the contact force f_c is no longer needed in the impedance controller (18).

Wishing to achieve $\omega_n = 10$ and $\zeta = 0.7071 = 1/\sqrt{2}$, equations (19) provide

$$k_p = 100\, m_d, \qquad k_d = 10\sqrt{2}\, m_d, \qquad \text{for any } m_d > 0. \tag{20}$$

Being $m = 5$ [kg], if we take in particular $m_d = m = 5$, we obtain as gains

$$k_p = 500, \qquad k_d = 50\sqrt{2} = 70.71, \tag{21}$$

and a measure of f_c will not be needed.

In regulation tasks (with $x_d(t) = x_d =$ constant), by choosing again $m_d = m$, the control law 18) collapses to just a PD action on the position error e,

$$f = k_p\,(x_d - x) - k_d\dot{x}. \tag{22}$$

This scheme is also called *compliance control*, since the main design parameter left is the desired stiffness k_p. Also in this case, the system will converge to $x = x_d$ if (and only if) there is no contact force. With $f_c \neq 0$ but constant, the position $x_e \neq x_d$ that satisfies

$$k_p(x_d - x_e) + f_c = 0 \quad \Rightarrow \quad x_e = x_d + \frac{f_c}{k_p} \tag{23}$$

will be an asyptotically (exponentially) stable closed-loop equilibrium, as can be possibly checked with the Lyapunov candidate $V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_p(x - x_e)^2 \geq 0$ (using in this case LaSalle theorem for the analysis).

Force control. If we desire to regulate explicitly the contact force to a desired constant value f_d, it is necessary to build a force error $e_f = f_d - f_c$ into the control law. After using (15), define the auxiliary input a as

$$a = \frac{1}{m_d}\left(k_f\,(f_d - f_c) - k_d\dot{x}\right), \tag{24}$$

with force error gain $k_f > 0$ and velocity damping coefficient $k_d > 0$. The associated control force is then

$$f = \frac{m}{m_d}\left(k_f\,(f_d - f_c) - k_d\dot{x}\right) - f_c. \tag{25}$$

A contact force measure is needed in this case, even if we choose $m_d = m$. The closed-loop system becomes

$$m_d\ddot{x} + k_d\dot{x} = k_f(f_d - f_c). \tag{26}$$

During free motion, i.e., as long as $f_c = 0$, the mass will eventually move at the constant speed $\dot{x}_e = k_f f_d / k_d$. Therefore, the gain k_d can be tuned so as to keep this speed low (say, during an approaching phase before contacting a hard environment).

An analysis of the general behavior of system (26) for $f_c \neq 0$ is impossible without assigning a model that describes the source of the contact force f_c. Even if we can measure it, as assumed when designing (25), we do not know the evolution of this disturbance nor can impose a desired behavior to it. Should the force error e_f converge to zero at steady state, it follows from eq. (26) that also the mass velocity $\dot{x}$ would go to zero. However, the position x_e reached at the equilibrium would depend on the actual history of the external contact force (see an example in Appendix).

Assume then that contact forces are generated by a compliant environment with stiffness $k_c > 0$, placed beyond the (undeformed) position $x = x_c > 0$. Then, the model for the reaction force of the environment is

$$f_e = \begin{cases} -k_c(x - x_c), & \text{for } x \geq x_c, \\ 0, & \text{else.} \end{cases} \tag{27}$$

During contact, the force applied to the mass is $f_c = -f_e$. Thus, from (26) and (27) it follows

$$m_d \ddot{x} + k_d \dot{x} = k_f \left(f_d - k_c(x - x_c) \right) \qquad \Rightarrow \qquad m_d \ddot{x} + k_d \dot{x} + k_f k_c x = k_f \left(f_d + k_c x_c \right). \tag{28}$$

The steady-state position reached by the second-order asymptotically stable system (28) in response to the (positive) step input $k_f (f_d + k_c x_c)$ and the associated steady-state contact force will be

$$x_e = x_c + \frac{f_d}{k_c} \qquad \Rightarrow \qquad f_c = \Big(-f_e = k_c(x_e - x_c) \Big) = f_d. \tag{29}$$

A slight variant of the force control law (25) is obtained by replacing the cancelation of the actual contact force in (15) by a compensation/feedforward of the desired contact force, i.e., $f = ma - f_d$. Using again (24), we obtain

$$f = \frac{m}{m_d} \left(k_f \left(f_d - f_c \right) - k_d \dot{x} \right) - f_d, \tag{30}$$

and, as a result, the closed-loop system

$$m_d \ddot{x} + k_d \dot{x} = \left(k_f - \frac{m_d}{m} \right) (f_d - f_c). \tag{31}$$

Using the contact force model (27) leads finally to

$$m_d \ddot{x} + k_d \dot{x} + \left(k_f - \frac{m_d}{m} \right) k_c x = \left(k_f - \frac{m_d}{m} \right) \left(f_d + k_c x_c \right). \tag{32}$$

It is immediate to see that the analysis of (32) can be completed as for (28), provided that the slightly more restrictive design condition $k_f > m_d/m > 0$ is satisfied. Under this hypothesis, the steady-state conditions for the asymptotically stable system (32) are the same given in (29).

* * * * *

Appendix (extra material to Exercise 3)

Consider a scheme for the contact force generation modeled by

$$\dot{f}_c = \alpha(f_d - f_c), \qquad \text{with } \alpha > 0, \tag{33}$$

and assume, e.g., $f_c(0) = f_{c0} > f_d$ (the initial contact force is larger than the one desired). Then

$$f_c(t) = f_d - (f_d - f_{c0}) \exp^{-\alpha t} \quad \text{and} \quad e_f(t) = f_d - f_c(t) = (f_d - f_{c0}) \exp^{-\alpha t} = e_{f0} \exp^{-\alpha t}. \tag{34}$$

Assuming $x(0) = \dot{x}(0) = 0$ and discarding the special case $\alpha = k_d/m_d$, the solution of (26) can be found by Laplace techniques and is given by the following position trajectory

$$x(t) = \frac{k_f e_{f0}}{k_d\, \alpha} + \frac{k_f e_{f0}}{k_d - \alpha\, m_d} \left(\frac{m_d}{k_d} \exp^{-\frac{k_d}{m_d} t} - \frac{1}{\alpha} \exp^{-\alpha t} \right), \tag{35}$$

and associated velocity

$$\dot{x}(t) = \frac{k_f e_{f0}}{k_d - \alpha\, m_d} \left(\exp^{-\alpha t} - \exp^{-\frac{k_d}{m_d} t} \right). \tag{36}$$

It follows from (35) that, at steady state,

$$x_e = \lim_{t \to \infty} x(t) = \frac{k_f e_{f0}}{k_d\, \alpha}, \tag{37}$$

which shows an explicit dependence on the parameter α of the contact force model (33). Figure 4 shows two possible evolutions of the applied force error term $k_f(f_d - f_c)$ (in blue) and of the resulting mass position x (in green), for $\alpha = 2$ and $\alpha = 3$, with the other parameters being $f_d = 3$ [N], $f_{c0} = 2$ [N] (and thus, $e_f = f_d - f_{c0} = 1$ [N]), $k_f = 1.4$, $m_d = 1$ [kg], and $k_d = 1$ [kg/s].

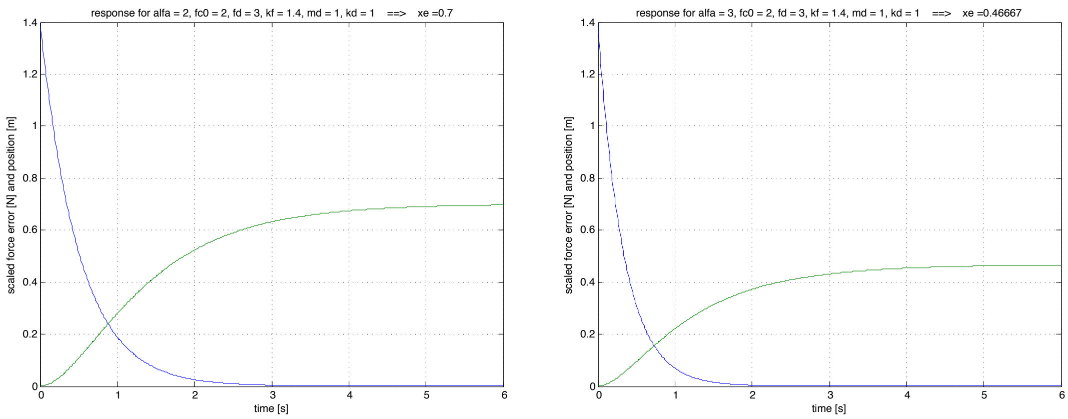


Figure 4: Simulation results of (26) of a controlled mass m_d subject to the contact force f_c in (34), for $\alpha = 2$ [left] and $\alpha = 3$ [right]. The plots are the position x (shown in green) and the force error term $k_f(f_d - f_c) = k_f e_f$ (in blue). The reached position x_e is the one computed in (37).

* * * * *

Robotics II

February 5, 2018

Exercise 1

Consider a robot manipulator with $\boldsymbol{q} \in \mathbb{R}^n$ joint variables that is redundant with respect to a task described by $\boldsymbol{r} \in \mathbb{R}^m$ variables, with $m < n$. The $m \times n$ task Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ relates task and joint velocities, i.e., $\dot{\boldsymbol{r}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$. For task regulation problems, kinematic control typically defines a law of the form

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{K}\boldsymbol{e} + \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\dot{\boldsymbol{q}}_0, \qquad \boldsymbol{K} > 0\,(\text{diagonal}), \quad \boldsymbol{e} = \boldsymbol{r}_d - \boldsymbol{r}, \tag{1}$$

where $\boldsymbol{r}_d$ is the desired value for the task variables. The first term in (1) leads to $\dot{\boldsymbol{e}} = -\boldsymbol{K}\boldsymbol{e}$, so that exponential convergence of the error $\boldsymbol{e}$ to zero is guaranteed (out of task singularities). The second term allows shaping the robot configuration during motion without affecting task execution, using a joint velocity $\dot{\boldsymbol{q}}_0 \in \mathbb{R}^n$ projected in the null space of the task Jacobian. When $m = n$, there is no null space to explore (the second term vanishes); when $m > n$, the pseudoinverse command $\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{K}\boldsymbol{e}$ guarantees only that a minimum error (in norm) is achieved, but still with $\boldsymbol{e} \neq \boldsymbol{0}$.

With the above in mind, consider a visual servoing problem that uses as task variables M point features, namely the coordinates $\boldsymbol{f}_i = (u_i \; v_i)^T \in \mathbb{R}^2$, $i = 1, \ldots, M$, of M points on the 2D image plane of an eye-in-hand camera. Let the task vector be $\boldsymbol{r} = \boldsymbol{f} \in \mathbb{R}^m$, with $m = 2M$, while vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ (both in $\mathbb{R}^M$) collect the coordinates of the image points. The task Jacobian $\boldsymbol{J}$ is the product $\boldsymbol{J}_p(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{Z})\boldsymbol{J}_m(\boldsymbol{q})$, where $\boldsymbol{J}_p$ is the $m \times 6$ interaction matrix of the M point features (depending on the sensed image and on the depths $Z_i > 0$ of the 3D points), and $\boldsymbol{J}_m$ is the $6 \times n$ geometric Jacobian of the manipulator carrying the camera. The rank of $\boldsymbol{J}$ will be equal to $\rho \leq \min\{m, 6, n\}$, no matter how large n is, and the dimension of the null space $\mathcal{N}\{\boldsymbol{J}\}$ will be $n - \rho$.

To handle critical issues related to the lack of full row rank for the task Jacobian, it was recently proposed to modify the way the regulation task is accomplished. Instead of considering the original task as a m-dimensional vector, so as to achieve the goal $\boldsymbol{r} = \boldsymbol{r}_d$, or $\boldsymbol{e} = \boldsymbol{0}$, by driving each and every component to its desired value, we can define the task as the norm of the error $\boldsymbol{e} \in \mathbb{R}^m$,

$$\eta = \|\boldsymbol{e}\|, \tag{2}$$

and achieve the goal by specifying $\eta = 0$ as the desired task value. This is indeed equivalent to obtaining $\boldsymbol{e} = \boldsymbol{0}$. Thanks to the modified definition, the task is one-dimensional and the null space of the associated task Jacobian will always be of dimension equal to (at least) $n - 1$.

Formulate and address the problem of task-based kinematic control for $\eta \in \mathbb{R}$. In particular:

- determine the general form of the $1 \times n$ task Jacobian matrix $\boldsymbol{J}_\eta$ associated to (2);
- analyze the singularities and/or the problems of definition of $\boldsymbol{J}_\eta$;
- write the explicit expression of the pseudoinverse $\boldsymbol{J}_\eta^{\#}$ in the control law (1), assuming that $\boldsymbol{e} \neq 0$ and $\boldsymbol{J}_\eta$ is full (row) rank;
- specify the main terms needed in this type of task-based kinematic control for the case of a visual servoing task with $M = 2$ point features.

[Turn sheet for next exercises]

Exercise 2

For a robot of the cylindrical type with a sequence PRP of joints and mounted on a vertical wall, provide the dynamic model in the usual form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{F}\dot{\boldsymbol{q}} = \boldsymbol{\tau}, \tag{3}$$

using the generalized coordinates $\boldsymbol{q} \in \mathbb{R}^3$ and the dynamic coefficients defined in Fig. 1. In (3), $\boldsymbol{F} > 0$ is a diagonal matrix of viscous coefficients. Make reasonable assumptions on the zero values of the variables q_i, $i = 1, 2, 3$, and neglect the small offset between joint axes 2 and 3 (i.e., assume that these two axes intersect). Moreover, assume that the center of mass of each link is placed on the joint axis having the same index.

Without any a priori knowledge of dynamic parameters, define all the terms needed in the design of an adaptive control law for this robot so as to achieve global asymptotic tracking of a desired joint trajectory $\boldsymbol{q}_d(t)$, with $q_{di}(t) = q_{0i} + A_i(1 - \cos(2\pi t/T))^\nu$ $(i = 1, 2, 3)$ of amplitude A_i and period T, for an unlimited time $t \geq 0$. Which is the minimum value of the integer $\nu \in \mathbb{N}$ that allows asymptotic exact tracking?

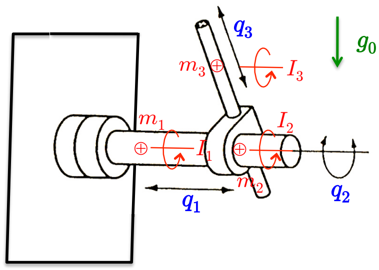


Figure 1: A cylindrical PRP-type robot mounted on a vertical wall, with its generalized coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ and relevant dynamic parameters.

Exercise 3

Consider an actuated pendulum that moves in the vertical plane under a joint torque u. The single link is a uniform thin rod of mass $m = 10$ kg and length $L = 2$ m. The downward equilibrium is at $\theta = 0$. Suppose that the link is initially at rest in $\theta_0 = 0$, and that we want to regulate its angular position to $\theta_d = \pi/3$ rad using the control scheme

$$u = k_p(\theta_d - \theta) - k_d\dot{\theta} + u_{i-1}, \qquad i = 1, 2, \ldots \tag{4}$$

with $k_p = 500$ and $k_d = 45$. The constant feedforward u_{i-1} in (4) is updated at every new reached equilibrium configuration θ_i, $i = 1, 2, \ldots$, as

$$u_i = u_{i-1} + k_p(\theta_d - \theta_i), \quad i = 1, 2, \ldots, \qquad \text{with } u_0 = 0. \tag{5}$$

Will the error $e_i = \theta_d - \theta_i$ converge to zero for $i \to \infty$ with this iterative control scheme? Why, or why not? If (or when) it does, can you predict which is the minimum number $i_{\min} > 0$ of iterations guaranteeing that the error will satisfy $|e_i| < \varepsilon = 0.01$ rad for all $i \geq i_{\min}$?

[150 minutes; open books]

Solution

February 5, 2018

Exercise 1

We first rewrite more explicitly eq. (2) as

$$\eta = \|\boldsymbol{e}\| = \sqrt{\boldsymbol{e}^T\boldsymbol{e}} = \|\boldsymbol{r}_d - \boldsymbol{r}\|. \tag{6}$$

Since $\boldsymbol{r}_d$ is constant, taking the time derivative of (6) yields

$$\dot{\eta} = \frac{1}{2}\,\frac{2\,\boldsymbol{e}^T\dot{\boldsymbol{e}}}{\sqrt{\boldsymbol{e}^T\boldsymbol{e}}} = -\frac{\boldsymbol{e}^T\dot{\boldsymbol{r}}}{\eta} = -\frac{\boldsymbol{e}^T\boldsymbol{J}(\boldsymbol{q})}{\|\boldsymbol{e}\|}\,\dot{\boldsymbol{q}} = \boldsymbol{J}_\eta(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{7}$$

where $\boldsymbol{J}$ is the Jacobian matrix associated to the original task variables $\boldsymbol{r}$. Thus, the new $1 \times n$ task Jacobian is

$$\underset{1\times n}{\boldsymbol{J}_\eta(\boldsymbol{q})} = -\frac{\overset{1\times m}{\boldsymbol{e}^T}\,\overset{m\times n}{\boldsymbol{J}(\boldsymbol{q})}}{\|\boldsymbol{e}\|}. \tag{8}$$

This matrix has rank one whenever $\boldsymbol{e} \notin \mathcal{N}\{\boldsymbol{J}^T(\boldsymbol{q})\}$. On the other hand, the condition $\boldsymbol{e} = \boldsymbol{0}$ (i.e., exactly where the task is accomplished!) is critical because both the numerator and the denominator go to zero, so that a further analysis is needed (which is out of the scope of this exercise). Far from these situations, and similarly to (1), we define the control law as

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_\eta^\#(\boldsymbol{q})\,k\,\eta + \left(\boldsymbol{I} - \boldsymbol{J}_\eta^\#(\boldsymbol{q})\boldsymbol{J}_\eta(\boldsymbol{q})\right)\dot{\boldsymbol{q}}_0, \tag{9}$$

with the pseudoinverse $\boldsymbol{J}_\eta^\#$ of the row vector $\boldsymbol{J}_\eta$ being the column vector computed as

$$\boldsymbol{J}_\eta^\#(\boldsymbol{q}) = \boldsymbol{J}_\eta^T(\boldsymbol{q})\left(\boldsymbol{J}_\eta(\boldsymbol{q})\boldsymbol{J}_\eta^T(\boldsymbol{q})\right)^{-1} = -\frac{\|\boldsymbol{e}\|}{\boldsymbol{e}^T\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{e}}\,\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{e}. \tag{10}$$

It is easy to see that (9) will work properly: in fact, plugging this $\dot{\boldsymbol{q}}$ in (7) leads to the exponentially stable (scalar) error system

$$\dot{\eta} = -k\,\eta. \tag{11}$$

Moreover, the projection matrix $\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{J}_\eta^\#\boldsymbol{J}_\eta$ in (9) has rank one, and the null space in which we can now accommodate a desired extra motion $\dot{\boldsymbol{q}}_0$ is now $(n-1)$-dimensional.

For a visual servoing task with $M = 2$ point features, $\boldsymbol{r} = \boldsymbol{f} \in \mathbb{R}^4$, we need first to define the 4×6 interaction matrix $\boldsymbol{J}_p$, as well as the (generic) $6 \times n$ geometric Jacobian $\boldsymbol{J}_m$, so that

$$\dot{\boldsymbol{f}} = \boldsymbol{J}_p(\boldsymbol{u},\boldsymbol{v},\boldsymbol{Z})\begin{pmatrix} V \\ \Omega \end{pmatrix}, \qquad \begin{pmatrix} V \\ \Omega \end{pmatrix} = \boldsymbol{J}_m(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad \Rightarrow \qquad \boldsymbol{J} = \boldsymbol{J}_p\boldsymbol{J}_m, \tag{12}$$

being $V \in \mathbb{R}^3$ and $\Omega \in \mathbb{R}^3$, respectively the linear and angular velocity of the eye-in-hand camera. The interaction matrix takes the expression

$$\boldsymbol{J}_p(\boldsymbol{u},\boldsymbol{v},\boldsymbol{Z}) = \begin{pmatrix} \boldsymbol{J}_{p1}(u_1,v_1,Z_1) \\ \boldsymbol{J}_{p2}(u_2,v_2,Z_2) \end{pmatrix} = \begin{pmatrix} -\dfrac{\lambda}{Z_1} & 0 & \dfrac{u_1}{Z_1} & \dfrac{u_1v_1}{\lambda} & -\left(\lambda + \dfrac{u_1^2}{\lambda}\right) & v_1 \\ 0 & -\dfrac{\lambda}{Z_1} & \dfrac{v_1}{Z_1} & \lambda + \dfrac{v_1^2}{\lambda} & -\dfrac{u_1v_1}{\lambda} & -u_1 \\ -\dfrac{\lambda}{Z_2} & 0 & \dfrac{u_2}{Z_2} & \dfrac{u_2v_2}{\lambda} & -\left(\lambda + \dfrac{u_2^2}{\lambda}\right) & v_2 \\ 0 & -\dfrac{\lambda}{Z_2} & \dfrac{v_2}{Z_2} & \lambda + \dfrac{v_2^2}{\lambda} & -\dfrac{u_2v_2}{\lambda} & -u_2 \end{pmatrix}, \tag{13}$$

with the camera focal length $\lambda > 0$. Moreover, from

$$\boldsymbol{f} = \begin{pmatrix} u_1 & v_1 & u_2 & v_2 \end{pmatrix}^T, \qquad \boldsymbol{f}_d = \begin{pmatrix} u_{1d} & v_{1d} & u_{2d} & v_{2d} \end{pmatrix}^T, \qquad \boldsymbol{e} = \boldsymbol{f}_d - \boldsymbol{f}, \tag{14}$$

the scalar task takes the expression

$$\eta = \|\boldsymbol{e}\| = \sqrt{(u_{1d} - u_1)^2 + (v_{1d} - v_1)^2 + (u_{2d} - u_2)^2 + (v_{2d} - v_2)^2}. \tag{15}$$

The associated task Jacobian is compactly written as

$$\boldsymbol{J}_\eta = -\frac{1}{\eta}\, \boldsymbol{e}^T \begin{pmatrix} \boldsymbol{J}_{p1}(u_1, v_1, Z_1) \\ \boldsymbol{J}_{p2}(u_2, v_2, Z_2) \end{pmatrix} \boldsymbol{J}_m(\boldsymbol{q}). \tag{16}$$

Finally, the pseudoinverse (10) takes the form

$$\boldsymbol{J}_\eta^{\#} = -\eta \, \frac{\boldsymbol{J}_m^T(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{J}_{p1}^T(u_1, v_1, Z_1) & \boldsymbol{J}_{p2}^T(u_2, v_2, Z_2) \end{pmatrix} \boldsymbol{e}}{\boldsymbol{e}^T \begin{pmatrix} \boldsymbol{J}_{p1}(u_1, v_1, Z_1) \\ \boldsymbol{J}_{p2}(u_2, v_2, Z_2) \end{pmatrix} \boldsymbol{J}_m(\boldsymbol{q})\, \boldsymbol{J}_m^T(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{J}_{p1}^T(u_1, v_1, Z_1) & \boldsymbol{J}_{p2}^T(u_2, v_2, Z_2) \end{pmatrix} \boldsymbol{e}}. \tag{17}$$

Exercise 2

Following a Lagrangian approach, we compute first the kinetic energy $T = T_1 + T_2 + T_3$. Since the position of the center of mass of the third link will be an unknown dynamic parameter, we need to define q_3 in a purely kinematic way as the radial position of the distal end of the third link with respect to the joint axis 2. The radial position of the center of mass of link 3 will then be given by $q_3 - d_3$, being $d_3 > 0$ the distance of the center of mass from the link end. With this and under the other given assumptions, considering the sequence PRP of joint types we have

$$\left.\begin{array}{c} T_1 = \frac{1}{2} m_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2} m_2 \dot{q}_1^2 + \frac{1}{2} I_2\, \dot{q}_2^2, \\ T_3 = \frac{1}{2} m_3 \left(\dot{q}_1^2 + (q_3 - d_3)^2\, \dot{q}_2^2 + \dot{q}_3^2 \right) + \frac{1}{2} I_3\, \dot{q}_2^2 \end{array}\right. \quad \Rightarrow \quad T = \tfrac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}},$$

with

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1 + m_2 + m_3 & 0 & 0 \\ 0 & I_2 + I_3 + m_3 (q_3 - d_3)^2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{18}$$

For the Coriolis and centrifugal terms, the requested adaptive control law will use the factorization $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}}$ such that $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is a skew-symmetric matrix. This is automatically guaranteed if the components of the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are computed using the Christoffel's symbols, i.e.,

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \quad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial \boldsymbol{q}_i} \right), \quad i = 1, 2, 3, \tag{19}$$

being $\boldsymbol{m}_i$ the ith column of the inertia matrix $\boldsymbol{M}$. Using (18) and (19), we obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \boldsymbol{0} \qquad \Rightarrow \qquad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0,$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & m_3(q_3 - d_3) \\ 0 & m_3(q_3 - d_3) & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 2 m_3 (q_3 - d_3)\, \dot{q}_2 \dot{q}_3,$$

$$\boldsymbol{C}_3(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -m_3(q_3 - d_3) & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 (q_3 - d_3)\, \dot{q}_2^2.$$

A factorization that satisfies the skew-symmetric property is then given by

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_3(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_3(q_3-d_3)\,\dot{q}_3 & m_3(q_3-d_3)\,\dot{q}_2 \\ 0 & -m_3(q_3-d_3)\,\dot{q}_2 & 0 \end{pmatrix}. \tag{20}$$

For the potential energy due to gravity, $U_g = U_1 + U_2 + U_3$, we have (up to a constant)

$$U_1 = 0, \qquad U_2 = 0, \qquad U_3 = -m_3 g_0\,(q_3 - d_3)\cos q_2,$$

where we assumed that the third link is vertical and points downward for $q_2 = 0$. Thus

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ m_3 g_0\,(q_3-d_3)\sin q_2 \\ -m_3 g_0 \cos q_2 \end{pmatrix}. \tag{21}$$

The dynamic model of the robot, including the viscous friction term with $\boldsymbol{F} = \operatorname{diag}\{f_1, f_2, f_3\}$, can thus be written in the linear parametrized form,

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{F}\dot{\boldsymbol{q}} = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{\tau}, \tag{22}$$

with the 3×7 regressor

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) = \begin{pmatrix} \ddot{q}_1 & 0 & 0 & 0 & \dot{q}_1 & 0 & 0 \\ 0 & \ddot{q}_2 & q_3^2\,\ddot{q}_2 + 2q_3\,\dot{q}_2\dot{q}_3 + g_0\sin q_2 & -2q_3\,\ddot{q}_2 - 2\,\dot{q}_2\dot{q}_3 - g_0\sin q_2 & 0 & \dot{q}_2 & 0 \\ 0 & 0 & \ddot{q}_3 - q_3\,\dot{q}_2^2 - g_0\cos q_2 & \dot{q}_2^2 & 0 & 0 & \dot{q}_3 \end{pmatrix} \tag{23}$$

and the vector of dynamic coefficients

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} m_1 + m_2 + m_3 \\ I_2 + I_3 + m_3 d_3^2 \\ m_3 \\ m_3 d_3 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} \in \mathbb{R}^7. \tag{24}$$

Defining $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}\boldsymbol{e} = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}(\boldsymbol{q}_d - \boldsymbol{q})$, with a diagonal matrix $\boldsymbol{\Lambda} > 0$, two diagonal gain matrices $\boldsymbol{K}_D > 0$ and $\boldsymbol{K}_P = \boldsymbol{K}_D\boldsymbol{\Lambda} > 0$, and a diagonal estimation gain matrix $\boldsymbol{\Gamma} > 0$, the adaptive controller will have dimension 7 and the expression

$$\begin{aligned} \boldsymbol{u} &= \hat{\boldsymbol{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \hat{\boldsymbol{C}}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \hat{\boldsymbol{g}}(\boldsymbol{q}) + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}} = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\hat{\boldsymbol{a}} + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}} \\ \dot{\hat{\boldsymbol{a}}} &= \boldsymbol{\Gamma}\,\boldsymbol{Y}^T(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}), \qquad \hat{\boldsymbol{a}}(0) = \text{arbitrary}, \end{aligned} \tag{25}$$

where

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r) =$$
$$\begin{pmatrix} \ddot{q}_{r1} & 0 & 0 & 0 & \dot{q}_{r1} & 0 & 0 \\ 0 & \ddot{q}_{r2} & q_3^2\ddot{q}_{r2} + q_3(\dot{q}_2\dot{q}_{r3} + \dot{q}_{r2}\dot{q}_3) + g_0\sin q_2 & -2q_3\ddot{q}_{r2} - (\dot{q}_2\dot{q}_{r3} + \dot{q}_{r2}\dot{q}_3) - g_0\sin q_2 & 0 & \dot{q}_{r2} & 0 \\ 0 & 0 & \ddot{q}_{r3} - q_3\,\dot{q}_2\dot{q}_{r2} - g_0\cos q_2 & \dot{q}_2\dot{q}_{r2} & 0 & 0 & \dot{q}_{r3} \end{pmatrix} \tag{26}$$

and

$$\hat{\boldsymbol{a}} = \begin{pmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \hat{a}_4 & \hat{a}_5 & \hat{a}_6 & \hat{a}_7 \end{pmatrix}^T. \tag{27}$$

Finally, the desired trajectory is sufficiently smooth already with $\nu = 1$ and guarantees thus permanent exact tracking when the error $\boldsymbol{e}(t) = \boldsymbol{q}_d(t) - \boldsymbol{q}(t)$ asymptotically vanishes. In particular, we shall need the quantities

$$q_{di}(t) = q_{0i} + A_i \left(1 - \cos\frac{2\pi t}{T}\right), \quad \dot{q}_{di}(t) = \frac{2\pi A_i}{T}\sin\frac{2\pi t}{T}, \quad \ddot{q}_{di}(t) = \frac{4\pi^2 A_i}{T^2}\cos\frac{2\pi t}{T} \tag{28}$$

for $i = 1, 2, 3$.

Exercise 3

The dynamic equation of the pendulum is

$$\left(I_0 + md^2\right)\ddot{\theta} + mdg_0 \sin\theta = u, \tag{29}$$

with $d = L/2 = 1$ m and $I_0 = mL^2/12$, which is the inertia of a uniform thin rod of mass m and length L around an orthogonal axis passing through its center of mass. However, note that the value of I_0 (as well as that of the total inertia $I_0 + md^2$) will be irrelevant in the solution of our problem. Same for the gain k_d.

The gradient of the gravity term can be easily bounded as

$$\left\|\frac{\partial g(\theta)}{\partial\theta}\right\| = |mdg_0\cos\theta| \le mdg_0 = 98.1 = \alpha. \tag{30}$$

Being

$$k_p = 500 > 196.2 = 2\alpha, \tag{31}$$

the iterative scheme (4–5) will certainly converge to $e = \theta_d - \theta = 0$. Moreover, we can take out from k_p a factor $1/\beta$ in the following way

$$k_p = \frac{k_p'}{\beta} = 500 \qquad \Rightarrow \qquad k_p' = 100 > 98.1 = \alpha, \qquad 0 < \beta = \frac{1}{5} \le \frac{1}{2}, \tag{32}$$

so that we recognize the sufficient conditions for contraction of the iterative learning control (4–5). From the proof of the related theorem and the value of β in (32), we have

$$\|e_i\| < \frac{\beta}{1-\beta}\|e_{i-1}\| \qquad \Rightarrow \qquad |\theta_d - \theta_i| = |e_i| < \frac{\frac{1}{5}}{1-\frac{1}{5}}\,|e_{i-1}| = 0.25\,|\theta_d - \theta_{i-1}|\,. \tag{33}$$

As a result, we know in advance that the error will be reduced *at least* by a factor 4 from one iteration to the other. Thus, starting with the known initial error

$$e_0 = \theta_d - \theta_0 = \frac{\pi}{3} = 1.0472, \tag{34}$$

we can iteratively estimate upper bounds $\hat{e}_i$ for the absolute errors $|e_i|$:

$$\begin{aligned}
|e_1| &= |\theta_d - \theta_1| < 0.25\,|e_0| = \frac{\pi}{12} = 0.2618 = \hat{e}_1, \\
|e_2| &= |\theta_d - \theta_2| < 0.25\,|e_1| = \frac{\pi}{48} = 0.0654 = \hat{e}_2, \\
|e_3| &= |\theta_d - \theta_3| < 0.25\,|e_2| = \frac{\pi}{208} = 0.0164 = \hat{e}_3, \\
|e_4| &= |\theta_d - \theta_4| < 0.25\,|e_3| = \frac{\pi}{832} = 0.0041 = \hat{e}_4 < 0.01 = \varepsilon.
\end{aligned} \tag{35}$$

We can conclude that the absolute error with respect to θ_d will be reduced and kept below the required tolerance ε starting with the iteration $i_{\min} = 4$. Note that the control scheme tolerates a large uncertainty for the bound α on the gravity term. In the present case, we could handle a link mass which is up to 250% larger than the nominal value and would still converge, though progressively slower, without modifying the chosen proportional control gain (nor anything else).

We can do an exact calculation of the solution sequence of angles θ_i and feedforward torques u_i, even without performing a dynamic simulation but just knowing in advance the value $M = mdg_0 = 98.1$. In fact, every new equilibrium configuration $\theta = \theta_i$ will have to satisfy the nonlinear equation

$$mg_0 d \sin\theta = k_p(\theta_d - \theta) + u_{i-1}, \qquad i = 1, 2, \dots \tag{36}$$

We can solve numerically (36) using, e.g., the matlab function `fsolve`

```
theta(i) = fsolve(@(theta) M*sin(theta) - kp*(thetad - theta) - u(i - 1), theta(i - 1))
```

providing each time θ_{i-1} as initial guess. We update then the feedforward torque u_i using the recursion (5). Table 1 shows the actual convergence of the iterative control process, and also a comparison of the actual errors vs. their estimated bounds in (35). Indeed, the actual error converges to zero faster than its estimated bound.

i	θ_i	e_i	$\hat{e}_i$	u_i
0 (init)	0	1.0472	1.0472	0
1	0.8942	0.1530	0.2618	76.4904
2	1.0318	0.0154	0.0654	84.1916
3	1.0458	0.0014	0.0164	84.888
4 (stop)	1.0471	0.0001	0.0041	84.9510
true	1.0472	0	–	84.9571

Table 1: Iterative learning process for regulation at $\theta_d = \pi/3 = 1.0472$ rad with gravity torque estimation. Iterations are stopped when $\hat{e}_i \leq \varepsilon = 0.01$. Angles/errors in [rad], torques in [Nm].

* * * * *

Robotics II

March 27, 2018

Exercise 1

An automated crane can be seen as a mechanical system with two degrees of freedom that moves along a horizontal rail subject to the actuation force F, and that transports a swinging link connected with a passive and frictionless revolute joint, as sketched in Fig. 1. With reference to the kinematic variables and dynamic parameters defined therein:

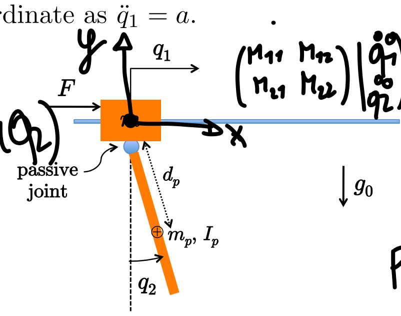
- derive the dynamic model of this system using a Lagrangian formalism;
- provide a linear parameterization of the obtained model in terms of a minimal number of dynamic coefficients;
- provide a linear approximation of the nonlinear model for small variations around the unforced equilibrium state $\boldsymbol{x}_0 = \begin{pmatrix} q_1 & q_2 & \dot{q}_1 & \dot{q}_2 \end{pmatrix}^T = \boldsymbol{0}$;
- find the nonlinear state feedback law for the force $F = F(\boldsymbol{x}, a)$ that linearizes exactly the dynamics of the first coordinate as $\ddot{q}_1 = a$.

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

$U_1 = 0$

$U_2 = -m_p g_0 d_p \cos(q_2)$

$$g = \begin{pmatrix} 0 \\ +m_p g_0 d_p \sin(q_2) \end{pmatrix}$$



$P_{cm} = \begin{pmatrix} q_1 \\ 0 \end{pmatrix}$ $\quad \omega_m = 0$

$P_{cp} = \begin{pmatrix} d_p \cdot \sin(q_2) \\ -d_p \cdot \cos(q_2) \end{pmatrix}$ $\quad \omega_p = \dot{q}_2$

Figure 1: An automated crane with the relevant kinematic and dynamic definitions.

Exercise 2

The end-effector of a PPR robot moving on a horizontal plane and equipped with a 2D force sensor should follow a stiff and frictionless linear surface tilted by $\alpha > 0$ w.r.t. the absolute $\boldsymbol{y}$ axis, starting at time $t = t_0$ in the position $\boldsymbol{p}_s = \begin{pmatrix} x_s & y_s \end{pmatrix}^T$, with a tangential speed $V_t = V_t(t_0) + A_t(t - t_0)$ (with $V_t(t_0) > 0$ and a constant $A_t > 0$), and applying a constant normal force $F_n > 0$ (see Fig. 2). Assuming full knowledge of geometric, kinematic, and dynamic parameters, provide the symbolic expressions of the initial robot state and explicitly of all terms in the force/torque commands at the joints that will guarantee perfect execution of the desired task in nominal conditions. Is the solution unique? If not, provide the simplest one.

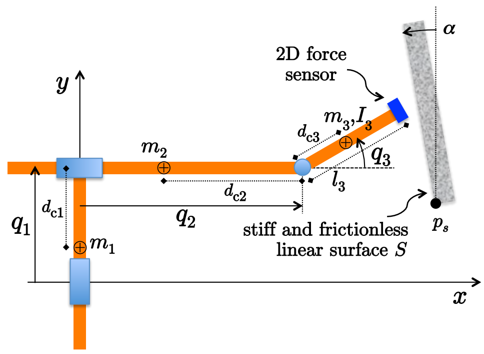


Figure 2: A PPR robot should move in contact with a stiff and frictionless linear surface.

[150 minutes; open books]

Solution

March 27, 2018

Exercise 1

The crane is an underactuated mechanical system with $n = 2$ degrees of freedom, but with only a single control command ($p = 1$). To derive its dynamic model, we can follow a Lagrangian approach. For this, the position and velocity of the center of mass of the swinging link are[1]

$$\boldsymbol{p}_c = \begin{pmatrix} q_1 + d_p \sin q_2 \\ -\, d_p \cos q_2 \end{pmatrix}, \qquad \boldsymbol{v}_c = \dot{\boldsymbol{p}}_c = \begin{pmatrix} \dot{q}_1 + d_p \cos q_2 \, \dot{q}_2 \\ d_p \sin q_2 \, \dot{q}_2 \end{pmatrix}.$$

The kinetic energy of the system is

$$T = T_m + T_p = \frac{1}{2} m \, \dot{q}_1^2 + \frac{1}{2} m_p \|\boldsymbol{v}_c\|^2 + \frac{1}{2} I_p \dot{q}_2^2 = \frac{1}{2}\left((m + m_p)\dot{q}_1^2 + (I_p + m_p d_p^2)\dot{q}_2^2 + 2 m_p d_p \cos q_2 \dot{q}_1 \dot{q}_2 \right),$$

and thus

$$T = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}} \qquad \Rightarrow \qquad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m + m_p & m_p d_p \cos q_2 \\ m_p d_p \cos q_2 & I_p + m_p d_p^2 \end{pmatrix}.$$

Using the Christoffel's symbols, we found only a centrifugal term

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -m_p d_p \sin q_2 \, \dot{q}_2^2 \\ 0 \end{pmatrix}.$$

The potential energy and the associated gravity vector are

$$U = U_0 - m_p g_0 d_p \cos q_2 \qquad \Rightarrow \qquad \boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} 0 \\ m_p g_0 d_p \sin q_2 \end{pmatrix},$$

with $g_0 = 9.81$ [m/s^2]. Assuming the possible presence of a viscous friction term (with a viscous coefficient $f_v \geq 0$) on the movement along the rail, the dynamic equations take the scalar form

$$\begin{aligned} (m + m_p)\ddot{q}_1 + m_p d_p \cos q_2 \, \ddot{q}_2 - m_p d_p \sin q_2 \, \dot{q}_2^2 + f_v \dot{q}_1 &= F \\ m_p d_p \cos q_2 \, \ddot{q}_1 + (I_p + m_p d_p^2)\ddot{q}_2 + m_p g_0 d_p \sin q_2 &= 0 \end{aligned} \tag{1}$$

Equations (1) can be rewritten in the linearly parametrized form

$$\begin{pmatrix} \ddot{q}_1 & 0 & \cos q_2 \, \ddot{q}_2 - \sin q_2 \, \dot{q}_2^2 & \dot{q}_1 \\ 0 & \ddot{q}_2 & \cos q_2 \, \ddot{q}_1 + g_0 \sin q_2 & 0 \end{pmatrix} \begin{pmatrix} m + m_p \\ I_p + m_p d_p^2 \\ m_p d_p \\ f_v \end{pmatrix} = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\boldsymbol{\pi} = \begin{pmatrix} F \\ 0 \end{pmatrix}.$$

The linear approximation of the dynamic equations of the crane around the (stable) equilibrium state $\boldsymbol{x}_0 = \begin{pmatrix} \boldsymbol{q}_0^T & \dot{\boldsymbol{q}}_0^T \end{pmatrix}^T = \begin{pmatrix} q_1 & q_2 & \dot{q}_1 & \dot{q}_2 \end{pmatrix}^T = \boldsymbol{0}$, which satisfies (1) with $F = F_0 = 0$ (unforced), is obtained by setting in (1)

$$\boldsymbol{q} = \boldsymbol{q}_0 + \boldsymbol{\Delta q} = \boldsymbol{\Delta q}, \quad \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_0 + \boldsymbol{\Delta} \dot{\boldsymbol{q}} = \boldsymbol{\Delta} \dot{\boldsymbol{q}}, \quad \ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}_0 + \boldsymbol{\Delta} \ddot{\boldsymbol{q}} = \boldsymbol{\Delta} \ddot{\boldsymbol{q}}, \quad F = F_0 + \Delta F = \Delta F,$$

[1] We have taken the x-axis along the rail, and the y-axis in the vertical upward direction.

and neglecting second- and higher-order increments (e.g., setting $\sin\Delta q_2 \simeq \Delta q_2$ and $\cos\Delta q_2 \simeq 1$):

$$\begin{pmatrix} m+m_p & m_p d_p \\ m_p d_p & I_p + m_p d_p^2 \end{pmatrix}\begin{pmatrix} \Delta\ddot{q}_1 \\ \Delta\ddot{q}_2 \end{pmatrix} + \begin{pmatrix} f_v & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Delta\dot{q}_1 \\ \Delta\dot{q}_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & m_p g_0 d_p \end{pmatrix}\begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix} = \begin{pmatrix} \Delta F \\ 0 \end{pmatrix}.$$

Finally, partial feedback linearization of the crane dynamics as concerns the motion of q_1 is obtained as follows. Solve (globally!) for the revolute joint acceleration $\ddot{q}_2$ from the second equation in (1),

$$\ddot{q}_2 = -\frac{1}{I_p + m_p d_p^2}\left(m_p d_p \cos q_2\, \ddot{q}_1 + m_p d_p g_0 \sin q_2\right)$$

and substitute it in the first one, yielding

$$\left((m+m_p) - \frac{m_p^2 d_p^2 \cos^2 q_2}{I_p + m_p d_p^2}\right)\ddot{q}_1 - \frac{m_p^2 d_p^2\, g_0 \sin q_2 \cos q_2}{I_p + m_p d_p^2} - m_p d_p \sin q_2\, \dot{q}_2^2 + f_v \dot{q}_1 = F.$$

From this, it is immediate to see that the nonlinear state feedback law

$$F = \frac{(m+m_p)I_p + m\, m_p d_p^2 + m_p^2 d_p^2 \sin^2 q_2}{I_p + m_p d_p^2}\, a - \frac{m_p^2 d_p^2\, g_0 \sin q_2 \cos q_2}{I_p + m_p d_p^2} - m_p d_p \sin q_2\, \dot{q}_2^2 + f_v \dot{q}_1$$

yields (again, globally) $\ddot{q}_1 = a$. Accordingly, the second equation in (1) becomes

$$(I_p + m_p d_p^2)\ddot{q}_2 + m_p g_0 d_p \sin q_2 = -m_p d_p \cos q_2\, a$$

Exercise 2

Noting that q_1 affects the y-coordinate and q_2 the x-coordinate, the direct/differential kinematics of the end-effector position, velocity, and acceleration are given respectively by

$$\boldsymbol{p} = \begin{pmatrix} q_2 + l_3 \cos q_3 \\ q_1 + l_3 \sin q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}),$$

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} 0 & 1 & -l_3 \sin q_3 \\ 1 & 0 & l_3 \cos q_3 \end{pmatrix}\dot{\boldsymbol{q}} \tag{2}$$

and

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \begin{pmatrix} 0 & 0 & -l_3 \cos q_3\, \dot{q}_3 \\ 0 & 0 & -l_3 \sin q_3\, \dot{q}_3 \end{pmatrix}\dot{\boldsymbol{q}}, \tag{3}$$

where the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ has been introduced.

The robot has $n = 3$ joints and the (hybrid) planar task has dimension $m = 2$ (one in force, the other in motion/velocity). In the presence of $n - m = 1$ degree of redundancy, the task can be executed in an infinite number of ways, beginning right from the different initial choices of an inverse kinematic configuration $\boldsymbol{q}(t_0)$ at time t_0, among those associated to the initial Cartesian point $\boldsymbol{p}(t_0) = \boldsymbol{p}_s$, and of the initial joint velocity $\dot{\boldsymbol{q}}(t_0)$, among those associated to the initial end-effector velocity

$$\dot{\boldsymbol{p}}(t_0) = V_t(t_0)\begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}.$$

It is then possible to parametrize the joint-space motion in terms of one variable. In this case, the easy choice is to pick the third joint variable q_3 as the parametrizing one. We set an arbitrary (but sufficiently smooth) time profile for it

$$q_3(t) = \beta(t), \qquad \dot{q}_3(t) = \dot{\beta}(t), \qquad \forall t \geq t_0,$$

and thus

$$q_3(t_0) = \beta(t_0), \qquad \dot{q}_3(t_0) = \dot{\beta}(t_0).$$

As a result, the two prismatic joints will be initialized at

$$\begin{pmatrix} q_1(t_0) \\ q_2(t_0) \end{pmatrix} = \begin{pmatrix} y_s - l_3 \sin\beta(t_0) \\ x_s - l_3 \cos\beta(t_0) \end{pmatrix},$$

with initial velocity

$$\begin{pmatrix} \dot{q}_1(t_0) \\ \dot{q}_2(t_0) \end{pmatrix} = \begin{pmatrix} V_t(t_0)\cos\alpha - l_3\,\dot{\beta}(t_0)\cos\beta(t_0) \\ -V_t(t_0)\sin\alpha - l_3\,\dot{\beta}(t_0)\sin\beta(t_0) \end{pmatrix}.$$

Moreover, we can also invert the second-order differential kinematics (3) in a parametrized way as

$$\ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \begin{pmatrix} \ddot{p}_y + l_3\left(-\cos\beta\,\ddot{\beta} + \sin\beta\,\dot{\beta}^2\right) \\ \ddot{p}_x + l_3\left(\sin\beta\,\ddot{\beta} + \cos\beta\,\dot{\beta}^2\right) \\ \ddot{\beta} \end{pmatrix}. \tag{4}$$

With this in mind, the dynamic model of the planar PPR robot (in the absence of gravity and without dissipative effects), when in contact with a stiff environment environment takes the form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{\tau} + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F}. \tag{5}$$

where $\boldsymbol{F} \in \mathbb{R}^2$ is the contact force applied by the environment on the robot end-effector (equal and opposite to the one applied by the robot on the environment) and $\boldsymbol{J}$ has been defined in (2). We provide next the explicit symbolic expressions of the dynamic terms appearing in (5). Note first that the position and velocity of the center of mass of the third link are

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_2 + d_{c3}\cos q_3 \\ q_1 + d_{c3}\sin q_3 \end{pmatrix}, \qquad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} \dot{q}_2 - d_{c3}\sin q_3\,\dot{q}_3 \\ \dot{q}_1 + d_{c3}\cos q_3\,\dot{q}_3 \end{pmatrix}.$$

Following a Lagrangian approach, we compute the total kinetic energy $T = T_1 + T_2 + T_3$ as:

$$\begin{array}{c} T_1 = \frac{1}{2}m_1\dot{q}_1^2, \qquad T_2 = \frac{1}{2}m_2\left(\dot{q}_1^2 + \dot{q}_2^2\right), \\ T_3 = \frac{1}{2}m_3\left(\dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2\,\dot{q}_3^2 + 2d_{c3}\dot{q}_3\left(\cos q_3\dot{q}_1 - \sin q_3\dot{q}_2\right)\right) + \frac{1}{2}I_3\,\dot{q}_3^2 \end{array} \quad \Rightarrow \quad T = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1 + m_2 + m_3 & 0 & m_3 d_{c3}\cos q_3 \\ 0 & m_2 + m_3 & -m_3 d_{c3}\sin q_3 \\ m_3 d_{c3}\cos q_3 & -m_3 d_{c3}\sin q_3 & I_3 + m_3 d_{c3}^2 \end{pmatrix}.$$

The Coriolis and centrifugal terms $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})$ are derived using the Christoffel's symbols, i.e., for each component

$$c_i(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T\boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \quad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial \boldsymbol{q}_i}\right), \quad i = 1,2,3,$$

being $\boldsymbol{m}_i$ the ith column of the inertia matrix $\boldsymbol{M}$. We obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m_3 d_{c3} \sin q_3 \end{pmatrix} \qquad \Rightarrow \qquad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 d_{c3} \sin q_3\, \dot{q}_3^2,$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m_3 d_{c3} \cos q_3 \end{pmatrix} \qquad \Rightarrow \qquad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 d_{c3} \cos q_3\, \dot{q}_3^2,$$

$$\boldsymbol{C}_3(\boldsymbol{q}) = \boldsymbol{0} \qquad \Rightarrow \qquad c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0.$$

We note that there are only centrifugal terms and no Coriolis torques.
Applying now to (5) the feedback linearizing control law

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} \tag{6}$$

will transform the system into a set of decoupled input-output integrators

$$\ddot{\boldsymbol{q}} = \boldsymbol{a}.$$

For the specified hybrid task, the desired end-effector acceleration and contact force are respectively

$$\ddot{\boldsymbol{p}}_d = A_t \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}, \qquad \boldsymbol{F}_d = F_n \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}.$$

From the desired end-effector acceleration, using (4), we obtain also the desired joint acceleration in parametrized form

$$\ddot{\boldsymbol{q}}_d = \begin{pmatrix} A_t \cos\alpha + l_3 \left(-\cos\beta\, \ddot{\beta} + \sin\beta\, \dot{\beta}^2 \right) \\ -A_t \sin\alpha + l_3 \left(\sin\beta\, \ddot{\beta} + \cos\beta\, \dot{\beta}^2 \right) \\ \ddot{\beta} \end{pmatrix}$$

Substituting $\boldsymbol{a} = \ddot{\boldsymbol{q}}_d$ and $\boldsymbol{F} = -\boldsymbol{F}_d$ in the feedback linearizing law (6), yields the desired nominal control commands

$$\boldsymbol{\tau}_d = \boldsymbol{M}(\boldsymbol{q}) \begin{pmatrix} A_t \cos\alpha + l_3 \left(-\cos\beta\, \ddot{\beta} + \sin\beta\, \dot{\beta}^2 \right) \\ -A_t \sin\alpha + l_3 \left(\sin\beta\, \ddot{\beta} + \cos\beta\, \dot{\beta}^2 \right) \\ \ddot{\beta} \end{pmatrix} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{J}^T(\boldsymbol{q}) \begin{pmatrix} F_n \cos\alpha \\ F_n \sin\alpha \end{pmatrix}, \tag{7}$$

where the dependence of the inertia matrix $\boldsymbol{M}$ and of the Jacobian $\boldsymbol{J}$ is actually only on $q_3 = \beta$ and that of the centrifugal terms $\boldsymbol{c}$ is only on $q_3 = \beta$ and $\dot{q}_3 = \dot{\beta}$. This can be made more explicit by rewriting (7) in extended form as

$$\boldsymbol{\tau}_d = \begin{pmatrix} (m_1 + m_2 + m_3) \left(A_t \cos\alpha + l_3 \left(-\cos\beta\, \ddot{\beta} + \sin\beta\, \dot{\beta}^2 \right) \right) + m_3 d_{c3} \left(\cos\beta\, \ddot{\beta} - \sin\beta\, \dot{\beta}^2 \right) + F_n \sin\alpha \\ (m_1 + m_3) \left(-A_t \sin\alpha + l_3 \left(\sin\beta\, \ddot{\beta} + \cos\beta\, \dot{\beta}^2 \right) \right) - m_3 d_{c3} \left(\sin\beta\, \ddot{\beta} + \cos\beta\, \dot{\beta}^2 \right) + F_n \cos\alpha \\ (I_3 + m_3 d_{c3}^2)\, \ddot{\beta} + m_3 d_{c3} \left(A_t \cos(\alpha - \beta) - l_3 \ddot{\beta} \right) + F_n\, l_3 \sin(\alpha - \beta) \end{pmatrix}.$$

Note that the first two components of $\boldsymbol{\tau}_d$ are forces (the units of all terms are [N] = [kg·m/s^2]), while the last component is a torque (units in [Nm]). Moreover, thanks to the fact that the initial robot state is matched with the task at the initial time $t = t_0$, there will be no need of a feedback action on task errors in the nominal control commands (7) in order to execute the entire task in ideal conditions.

The above parametrized control law is one of the many realizing the desired task, depending on the choice of the time evolution $\beta(t)$ for the variable q_3 of the revolute joint. Indeed, simplifications arise for specific choices. The simplest one is choosing to keep q_3 at a constant value β, with $\dot{\beta} = \ddot{\beta} = 0$. We obtain

$$\boldsymbol{\tau}_d = \begin{pmatrix} \tau_{d1} \\ \tau_{d2} \\ \tau_{d3} \end{pmatrix} = \begin{pmatrix} A_t(m_1 + m_2 + m_3)\cos\alpha + F_n \sin\alpha \\ -A_t(m_1 + m_3)\sin\alpha + F_n \cos\alpha \\ A_t\, m_3 d_{c3} \cos(\alpha - \beta) + F_n\, l_3 \sin(\alpha - \beta) \end{pmatrix}.$$

Having chosen to keep the third joint at rest for the entire motion, the robot behaves kinematically as a 2P robot. In particular, when placing the third robot link normal to the frictionless surface, we have $\beta = \alpha$ and the third control component reduces to $\tau_{d3} = A_t\, m_3 d_{c3}$.

* * * * *

Robotics 2

Midterm test in classroom – April 26, 2018

Exercise 1

Consider the 4-dof planar robot in Fig. 1. The robot has the first two joints prismatic and the last two joints revolute, and moves in a vertical plane.

- Using the generalized coordinates $\boldsymbol{q} \in \mathbb{R}^4$ and the dynamic parameters defined in Fig. 1, determine the symbolic expression of the dynamic model of this robot using a Lagrangian formulation and considering also the presence of viscous friction at the joints.
- Assume that all kinematic parameters as well as the acceleration of gravity ($g_0 = 9.81$) are known. Provide a linear parameterization of the obtained dynamic model in the form

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{u}, \tag{1}$$

  where $\boldsymbol{u} \in \mathbb{R}^4$ is the generalized force provided by the motors at the joints, and the vector of dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^p$ has the minimum possible dimension p.

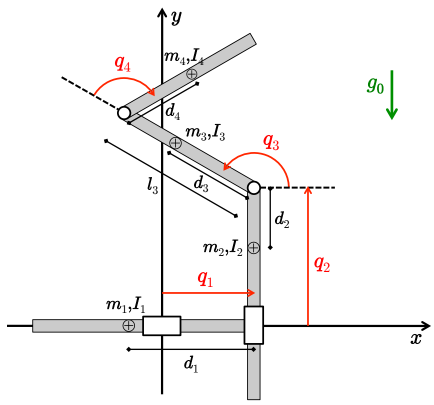


Figure 1: A 4-dof planar 2P2R robot, with associated coordinates $\boldsymbol{q}$ and dynamic data.

Exercise 2

A 2-dof robot has the inertia matrix

INERTIAL COUPLING

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2 \cos q_2 & a_3 + a_2 \cos q_2 \\ a_3 + a_2 \cos q_2 & a_3 \end{pmatrix}. \tag{2}$$

- Find two matrices $\boldsymbol{S}_1(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\boldsymbol{S}_2(\boldsymbol{q}, \dot{\boldsymbol{q}})$ factorizing the Coriolis and centrifugal terms $\boldsymbol{c}(\boldsymbol{q}.\dot{\boldsymbol{q}})$ (i.e., $\boldsymbol{S}_i(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, for $i = 1, 2$), such that $\dot{\boldsymbol{B}} - 2\boldsymbol{S}_1$ is skew-symmetric, while $\dot{\boldsymbol{B}} - 2\boldsymbol{S}_2$ is not.
- Assume that the robot is on a horizontal plane. The second joint should be moved along a cubic trajectory by an angle Δq_2 from rest to rest in a given time $T > 0$, while the first link is kept still. Provide the symbolic expression of the torque $\boldsymbol{\tau}(0) \in \mathbb{R}^2$ that needs to be applied at initial time $t = 0$ in order to start correctly the execution of this motion.
- Give the value of $\boldsymbol{\tau}(0)$ for the following numerical data

$$a_1 = 17, \quad a_2 = 5, \quad a_3 = 3, \quad q_2(0) = -\frac{\pi}{2}, \quad \Delta q_2 = \frac{\pi}{2}, \quad T = 2 \text{ s},$$

  and explain the physical meaning of the signs of its two components.

Exercise 3

Having defined a n-dimensional vector $\boldsymbol{q}$ of generalized coordinates, the dynamic model of a rigid robot in a Lagrangian formulation is obtained writing the Euler-Lagrange equations for $L = T - U$, where T is the kinetic energy and U is the potential energy of the robot. In the frictionless case, the model takes the usual second-order differential expression

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u}. \tag{3}$$

On the other hand, one can also rely on the use of the total energy of a robot, $H = T + U$, the so-called Hamiltonian of the system. Using the generalized momentum $\boldsymbol{p} = \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} \in \mathbb{R}^n$, prove that the (first-order) dynamic equations of the robot in a state-space format can be written in the so-called Hamiltonian form

$$\begin{pmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I}_n \\ -\boldsymbol{I}_n & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \nabla_{\boldsymbol{q}} H \\ \nabla_{\boldsymbol{p}} H \end{pmatrix} + \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{I}_n \end{pmatrix} \boldsymbol{u}, \tag{4}$$

where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix and $\nabla_{\boldsymbol{x}} H = (\partial H/\partial \boldsymbol{x})^T$ is a n-dimensional (column) vector.

Exercise 4

Consider the PPR planar robot in Fig. 2 and the coordinates $\boldsymbol{q} \in \mathbb{R}^3$ given therein. The length of the third link is $l > 0$. Assume that the Cartesian task to be performed by this robot is defined in terms of the position $\boldsymbol{p} \in \mathbb{R}^2$ of its end effector. Denote the associated task Jacobian as $\boldsymbol{J}(\boldsymbol{q})$.

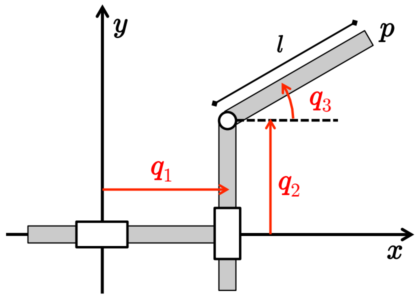


Figure 2: A 3-dof planar PPR robot, with associated coordinates $\boldsymbol{q}$.

- Find the joint velocity $\dot{\boldsymbol{q}}$ that minimizes $\frac{1}{2}\|\dot{\boldsymbol{q}}\|^2$ while realizing a desired task velocity $\dot{\boldsymbol{p}}$. When the linear quantities in the problem are first expressed in [m] and [m/s] units and then in [cm] and [cm/s] units, the solution does not remain the same. Why? Illustrate this by computing the numerical solution with the following data: $l = 0.5$ [m] $= 50$ [cm], $q_3 = \pi/6$ [rad], $\dot{\boldsymbol{p}} = (-1 \;\; 1)^T$ [m/s] $= (-100 \;\; 100)^T$ [cm/s].
- With a weighted pseudoinverse solution, may the above issue be resolved using a suitable diagonal weighting matrix $\boldsymbol{W} > 0$? Illustrate your conclusion by reconsidering the same numerical example of the previous item. Provide also an interpretation of the role of weighting.
- Using again another weighted pseudoinverse, show than it is possible to realize a generic Cartesian task velocity $\dot{\boldsymbol{p}}$ by moving almost only the first two joints of the PPR robot. Illustrate this on the same previous numerical example.

[270 minutes (4.5 hours); open books, but no computer or smartphone]

Solution

April 26, 2018

Exercise 1

Kinetic energy

First two links:

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2} m_2 \left(\dot{q}_1^2 + \dot{q}_2^2\right).$$

Third link:

$$\boldsymbol{p}_{c3} = \begin{pmatrix} p_{c3,x} \\ p_{c3,y} \end{pmatrix} = \begin{pmatrix} q_1 + d_3 \cos q_3 \\ q_2 + d_3 \sin q_3 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} \dot{q}_1 - d_3 \sin q_3 \, \dot{q}_3 \\ \dot{q}_2 + d_3 \cos q_3 \, \dot{q}_3 \end{pmatrix}, \qquad \omega_3 = \dot{q}_3$$

$$\Rightarrow \quad T_3 = \frac{1}{2} m_3 \left(\dot{q}_1^2 + \dot{q}_2^2 + d_3^2 \dot{q}_3^2 + 2 d_3 \dot{q}_3 \left(\dot{q}_2 \cos q_3 - \dot{q}_1 \sin q_3\right)\right) + \frac{1}{2} I_3 \, \dot{q}_3^2.$$

Fourth link:

$$\boldsymbol{p}_{c4} = \begin{pmatrix} p_{c4,x} \\ p_{c4,y} \end{pmatrix} = \begin{pmatrix} q_1 + l_3 \cos q_3 + d_4 \cos(q_3 + q_4) \\ q_2 + l_3 \sin q_3 + d_4 \sin(q_3 + q_4) \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{v}_{c4} = \dot{\boldsymbol{p}}_{c4} = \begin{pmatrix} \dot{q}_1 - l_3 \sin q_3 \, \dot{q}_3 - d_4 \sin(q_3 + q_4)(\dot{q}_3 + \dot{q}_4) \\ \dot{q}_2 + l_3 \cos q_3 \, \dot{q}_3 + d_4 \cos(q_3 + q_4)(\dot{q}_3 + \dot{q}_4) \end{pmatrix}, \qquad \omega_4 = \dot{q}_3 + \dot{q}_4$$

$$\begin{aligned} \Rightarrow \quad T_4 = \frac{1}{2} m_4 \Big(& \dot{q}_1^2 + \dot{q}_2^2 + l_3^2 \dot{q}_3^2 + d_4^2 \left(\dot{q}_3 + \dot{q}_4\right)^2 + 2 l_3 \dot{q}_3 \left(\dot{q}_2 \cos q_3 - \dot{q}_1 \sin q_3\right) + 2 l_3 d_4 \cos q_4 \, \dot{q}_3 \left(\dot{q}_3 + \dot{q}_4\right) \\ & + 2 d_4 \left(\dot{q}_3 + \dot{q}_4\right) \left(\dot{q}_2 \cos(q_3 + q_4) - \dot{q}_1 \sin(q_3 + q_4)\right)\Big) + \tfrac{1}{2} I_4 \left(\dot{q}_3 + \dot{q}_4\right)^2. \end{aligned}$$

Robot inertia matrix

From

$$T = \sum_{i=1}^{4} T_i = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} b_{ij}(\boldsymbol{q}) \, \dot{q}_i \dot{q}_j,$$

we obtain the elements $b_{ij} = b_{ji}$ of the symmetric inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ as

$$\begin{aligned}
b_{11} &= m_1 + m_2 + m_3 + m_4 \\
b_{12} &= 0 \\
b_{13} &= -\left(m_3 d_3 + m_4 l_3\right) \sin q_3 - m_4 d_4 \sin(q_3 + q_4) \\
b_{14} &= -m_4 d_4 \sin(q_3 + q_4) \\
b_{22} &= m_2 + m_3 + m_4 \\
b_{23} &= \left(m_3 d_3 + m_4 l_3\right) \cos q_3 + m_4 d_4 \cos(q_3 + q_4) \\
b_{24} &= m_4 d_4 \cos(q_3 + q_4) \\
b_{33} &= I_3 + m_3 d_3^2 + I_4 + m_4 d_4^2 + m_4 l_3^2 + 2 m_4 d_4 l_3 \cos q_4 \\
b_{34} &= I_4 + m_4 d_4^2 + m_4 d_4 l_3 \cos q_4 \\
b_{44} &= I_4 + m_4 d_4^2.
\end{aligned}$$

Note that the inertia matrix is only a function of q_3 and q_4.

Minimal parametrization of the inertia matrix

We can collect the dynamic parameters appearing in the robot inertia matrix into 6 dynamic coefficients, which are defined as follows:

$$\begin{aligned} a_1 &= m_1 + m_2 + m_3 + m_4 \\ a_2 &= m_2 + m_3 + m_4 \\ a_3 &= I_3 + m_3 d_3^2 + I_4 + m_4 d_4^2 + m_4 l_3^2 \\ a_4 &= I_4 + m_4 d_4^2 \\ a_5 &= m_4 d_4 \\ a_6 &= m_3 d_3 + m_4 l_3. \end{aligned} \tag{5}$$

As a result, the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$ takes the more compact, linearly parametrized form

$$\boldsymbol{B}(\boldsymbol{q}) = \begin{pmatrix} a_1 & 0 & -a_6 s_3 - a_5 s_{34} & -a_5 s_{34} \\ 0 & a_2 & a_6 c_3 + a_5 c_{34} & a_5 c_{34} \\ -a_6 s_3 - a_5 s_{34} & a_6 c_3 + a_5 c_{34} & a_3 + 2 a_5 l_3 c_4 & a_4 + a_5 l_3 c_4 \\ -a_5 s_{34} & a_5 c_{34} & a_4 + a_5 l_3 c_4 & a_4 \end{pmatrix}, \tag{6}$$

where the shorthand notation for trigonometric functions was used (e.g., $s_{34} = \sin(q_3 + q_4)$). Indeed, l_3 could also be incorporated into the coefficient a_6, but this would lead, as we shall see, to an additional dynamic coefficient in the definition of the gravity terms.

Coriolis and centrifugal terms

Defining by $\boldsymbol{b}_i$ the ith column of the inertia matrix $\boldsymbol{B}(\boldsymbol{q})$, we compute the components of the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ using the Christoffel symbols:

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{b}_i}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{b}_i}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{B}}{\partial q_i}\right), \qquad i = 1, \ldots, 4.$$

This is tedious, but straightforward. For the four components we obtain:

$$\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_6 c_3 - a_5 c_{34} & -a_5 c_{34} \\ 0 & 0 & -a_5 c_{34} & -a_5 c_{34} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_6 c_3 - a_5 c_{34} & -a_5 c_{34} \\ 0 & 0 & -a_5 c_{34} & -a_5 c_{34} \end{pmatrix} - \boldsymbol{O}\right)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_6 c_3 - a_5 c_{34} & -a_5 c_{34} \\ 0 & 0 & -a_5 c_{34} & -a_5 c_{34} \end{pmatrix} \quad \Rightarrow \quad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -a_5 c_{34} \left(\dot{q}_3 + \dot{q}_4\right)^2 - a_6 c_3\, \dot{q}_3^2$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_6 s_3 - a_5 s_{34} & -a_5 s_{34} \\ 0 & 0 & -a_5 s_{34} & -a_5 s_{34} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_6 s_3 - a_5 s_{34} & -a_5 s_{34} \\ 0 & 0 & -a_5 s_{34} & -a_5 s_{34} \end{pmatrix} - \boldsymbol{O}\right)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_6 s_3 - a_5 s_{34} & -a_5 s_{34} \\ 0 & 0 & -a_5 s_{34} & -a_5 s_{34} \end{pmatrix} \quad \Rightarrow \quad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -a_5 s_{34} \left(\dot{q}_3 + \dot{q}_4\right)^2 - a_6 s_3\, \dot{q}_3^2$$

$$
\boldsymbol{C}_3(\boldsymbol{q}) = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & -a_6c_3 - a_5c_{34} & -a_5c_{34} \\ 0 & 0 & -a_6s_3 - a_5s_{34} & -a_5s_{34} \\ 0 & 0 & 0 & -2a_5l_3s_4 \\ 0 & 0 & 0 & -a_5l_3s_4 \end{pmatrix}\right.
$$

$$
+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_6c_3 - a_5c_{34} & -a_6s_3 - a_5s_{34} & 0 & 0 \\ -a_5c_{34} & -a_5s_{34} & -2a_5l_3s_4 & -a_5l_3s_4 \end{pmatrix}
$$

$$
\left. - \begin{pmatrix} 0 & 0 & -a_6c_3 - a_5c_{34} & -a_5c_{34} \\ 0 & 0 & -a_6s_3 - a_5s_{34} & -a_5s_{34} \\ -a_6c_3 - a_5c_{34} & -a_6s_3 - a_5s_{34} & 0 & 0 \\ -a_5c_{34} & -a_5s_{34} & 0 & 0 \end{pmatrix}\right)
$$

$$
= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_5l_3s_4 \\ 0 & 0 & -a_5l_3s_4 & -a_5l_3s_4 \end{pmatrix} \quad \Rightarrow \quad c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -a_5l_3s_4\,\dot{q}_4\,(2\dot{q}_3 + \dot{q}_4)
$$

$$
\boldsymbol{C}_4(\boldsymbol{q}) = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & -a_5c_{34} & -a_5c_{34} \\ 0 & 0 & -a_5s_{34} & -a_5s_{34} \\ 0 & 0 & 0 & -a_5l_3s_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_5c_{34} & -a_5s_{34} & 0 & 0 \\ -a_5c_{34} & -a_5s_{34} & -a_5l_3s_4 & 0 \end{pmatrix}\right.
$$

$$
\left. - \begin{pmatrix} 0 & 0 & -a_5c_{34} & -a_5c_{34} \\ 0 & 0 & -a_5s_{34} & -a_5s_{34} \\ -a_5c_{34} & -a_5s_{34} & -2a_5l_3s_4 & -a_5l_3s_4 \\ -a_5c_{34} & -a_5s_{34} & -a_5l_3s_4 & 0 \end{pmatrix}\right)
$$

$$
= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_5l_3s_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad c_4(\boldsymbol{q}, \dot{\boldsymbol{q}}) = a_5l_3s_4\,\dot{q}_3^2.
$$

Summarizing, we have a very short final expression:

$$
\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -a_5c_{34}\,(\dot{q}_3 + \dot{q}_4)^2 - a_6c_3\,\dot{q}_3^2 \\ -a_5s_{34}\,(\dot{q}_3 + \dot{q}_4)^2 - a_6s_3\,\dot{q}_3^2 \\ -a_5l_3s_4\,\dot{q}_4\,(2\dot{q}_3 + \dot{q}_4) \\ a_5l_3s_4\,\dot{q}_3^2 \end{pmatrix}. \tag{7}
$$

Potential energy and gravity terms

$$
U_1 = 0, \qquad U_2 = m_2g_0(q_2 - d_2), \qquad U_3 = m_3g_0\,p_{c3,y} = m_3g_0\,(q_2 + d_3\sin q_3)\,,
$$

$$
U_4 = m_4g_0\,p_{c4,y} = m_4g_0\,(q_2 + l_3\sin q_3 + d_4\sin(q_3 + q_4)) \qquad \text{[all defined up to a constant]}.
$$

From $U = \sum_1^4 U_i$, we have

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ (m_2 + m_3 + m_4)\, g_0 \\ (m_3 d_3 + m_4 l_3)\, g_0 c_3 + m_4 d_4 g_0 c_{34} \\ m_4 d_4 g_0 c_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ a_2 g_0 \\ a_6 g_0 c_3 + a_5 g_0 c_{34} \\ a_5 g_0 c_{34} \end{pmatrix}, \tag{8}$$

where the three previously defined dynamic coefficents a_2, a_5, and a_6 have been used.

Complete dynamic model with viscous friction

Using the expressions eqs. (6), (7), and (8), and considering the presence of viscous friction acting at the individual joints, we have

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) + \begin{pmatrix} f_{v1}\dot{q}_1 \\ f_{v2}\dot{q}_2 \\ f_{v3}\dot{q}_3 \\ f_{v4}\dot{q}_4 \end{pmatrix} = \boldsymbol{u}. \tag{9}$$

By defining the four more dynamic coefficients

$$a_7 = f_{v1}, \qquad a_8 = f_{v2}, \qquad a_9 = f_{v3}, \qquad a_{10} = f_{v4},$$

we conclude that the robot dynamic model depends linearly on a total of $p = 10$ dynamic coefficients, which can be organized in the vector $\boldsymbol{a} \in \mathbb{R}^{10}$. This is also the minimal number of coefficients in the present case.

Linear parametrization

The robot dynamic model (9) can be expressed in the linearly parametrized form

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}.\ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{u}$$

where the 4×10 matrix $\boldsymbol{Y}$ is given by

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 & 0 & 0 & 0 & & & \dot{q}_1 & 0 & 0 & 0 \\ 0 & \ddot{q}_2 + g_0 & 0 & 0 & \boldsymbol{Y}_5 & \boldsymbol{Y}_6 & 0 & \dot{q}_2 & 0 & 0 \\ 0 & 0 & \ddot{q}_3 & \ddot{q}_4 & & & 0 & 0 & \dot{q}_3 & 0 \\ 0 & 0 & 0 & \ddot{q}_3 + \ddot{q}_4 & & & 0 & 0 & 0 & \dot{q}_4 \end{pmatrix}. \tag{10}$$

The expressions of the fifth and sixth columns of this matrix are

$$\boldsymbol{Y}_5 = \begin{pmatrix} -s_{34}\,(\ddot{q}_3 + \ddot{q}_4) - c_{34}\,(\dot{q}_3 + \dot{q}_4)^2 \\ c_{34}\,(\ddot{q}_3 + \ddot{q}_4) - s_{34}\,(\dot{q}_3 + \dot{q}_4)^2 \\ l_3 c_4\,(2\ddot{q}_3 + \ddot{q}_4) + c_{34}\,(\ddot{q}_2 + g_0) - s_{34}\ddot{q}_1 - l_3 s_4 \dot{q}_4\,(2\dot{q}_3 + \dot{q}_4) \\ l_3 c_4 \ddot{q}_3 + c_{34}\,(\ddot{q}_2 + g_0) - s_{34}\ddot{q}_1 + l_3 s_4 \dot{q}_3^2 \end{pmatrix}$$

and

$$\boldsymbol{Y}_6 = \begin{pmatrix} -s_3\ddot{q}_3 - c_3\dot{q}_3^2 \\ c_3\ddot{q}_3 - s_3\dot{q}_3^2 \\ c_3\,(\ddot{q}_2 + g_0) - s_3\ddot{q}_1 \\ 0 \end{pmatrix}.$$

Exercise 2

From the inertia matrix (2) of the robot, the matrices of Christoffel symbols are computed as

$$C_1(q) = \begin{pmatrix} 0 & -a_2 \sin q_2 \\ -a_2 \sin q_2 & -a_2 \sin q_2 \end{pmatrix}, \qquad C_2(q) = \begin{pmatrix} a_2 \sin q_2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{11}$$

leading to the Coriolis and centrifugal terms

$$c(q, \dot{q}) = \begin{pmatrix} \dot{q}^T C_1(q)\, \dot{q} \\ \dot{q}^T C_2(q)\, \dot{q} \end{pmatrix} = \begin{pmatrix} -a_2 \sin q_2 \left(2\dot{q}_1 \dot{q}_2 + \dot{q}_2^2\right) \\ a_2 \sin q_2\, \dot{q}_1^2 \end{pmatrix}. \tag{12}$$

Moreover, the time derivative of the inertia matrix is

$$\dot{B} = \begin{pmatrix} -2a_2 \sin q_2\, \dot{q}_2 & -a_2 \sin q_2\, \dot{q}_2 \\ -a_2 \sin q_2\, \dot{q}_2 & 0 \end{pmatrix}. \tag{13}$$

A factorization $S(q, \dot{q})\dot{q}$ of (12) that satisfies the skew-symmetry of $\dot{B} - 2S$ is found using directly the matrices (11) of Christoffel symbols. It is easy to check that the matrix

$$S_1(q, \dot{q}) = \begin{pmatrix} \dot{q}^T C_1(q) \\ \dot{q}^T C_2(q) \end{pmatrix} = \begin{pmatrix} -a_2 \sin q_2\, \dot{q}_2 & -a_2 \sin q_2\, (\dot{q}_1 + \dot{q}_2) \\ a_2 \sin q_2\, \dot{q}_1 & 0 \end{pmatrix} \tag{14}$$

provides

$$\dot{B} - 2S_1 = \begin{pmatrix} 0 & a_2 \sin q_2 (2\dot{q}_1 + \dot{q}_2) \\ -a_2 \sin q_2\, (2\dot{q}_1 + \dot{q}_2) & 0 \end{pmatrix},$$

satisfying the desired skew-symmetric property. On the other hand, a feasible factorization that uses the matrix

$$S_2(q, \dot{q}) = \begin{pmatrix} -2a_2 \sin q_2\, \dot{q}_2 & -a_2 \sin q_2\, \dot{q}_2 \\ a_2 \sin q_2\, \dot{q}_1 & 0 \end{pmatrix} \tag{15}$$

provides

$$\dot{B} - 2S_2 = \begin{pmatrix} 2a_2 \sin q_2\, \dot{q}_2 & a_2 \sin q_2\, \dot{q}_2 \\ -a_2 \sin q_2\, (2\dot{q}_1 + \dot{q}_2) & 0 \end{pmatrix},$$

which is clearly not a skew-symmetric matrix.

For the second part of the exercise, we need to solve a simple inverse dynamics problem. Since the robot moves on a horizontal plane, we have $g(q) \equiv 0$. Moreover, the desired motion starts at $t = 0$ with zero joint velocity, so that $c(q(0), \dot{q}(0)) = 0$ as well. Finally, the first link should not move, so that $\ddot{q}_1 \equiv 0$. The dynamic equations at the initial instant become then

$$B(q(0)) \begin{pmatrix} 0 \\ \ddot{q}_2(0) \end{pmatrix} = \tau(0).$$ $M\ddot{q} + \cancel{c} + \cancel{g} = \tau \rightarrow$ in $t=0$

Since the second joint should perform a rest-to-rest motion of Δq_2 in time T with a cubic trajectory, we have

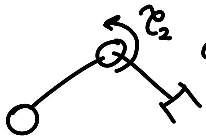
$$q_2(t) = q_2(0) + \Delta q_2 \left(3 \left(\frac{t}{T}\right)^2 - 2 \left(\frac{t}{T}\right)^3 \right) \quad \Rightarrow \quad \ddot{q}_2(0) = \frac{6\Delta q_2}{T^2}.$$

Therefore, the required initial torque at the two joints is

$\ddot{q}_2(t) = \Delta q_2 \frac{6}{T^2} \left(1 - 2\left(\frac{t}{T}\right)\right)$

$$\tau(0) = \begin{pmatrix} a_3 + a_2 \cos q_2(0) \\ a_3 \end{pmatrix} \frac{6\Delta q_2}{T^2}. \tag{16}$$

Replacing the given numerical data in (16), we obtain



IF SECOND LINK GOES UP DUE TO τ_2 THE FIRST LINK GOES DOWN SO WE NEED τ_1 TO COMPENSATE

$$\boldsymbol{\tau}(0) = \begin{pmatrix} 7.0686 \\ 7.0686 \end{pmatrix} \text{ [Nm]}.$$

Since $a_3 > 0$ (always), the torque at the second joint will have the same sign of the desired displacement/acceleration of that joint. In the present case, also the first torque is positive (and equal to the second one, being $\cos q_2(0) = 0$). This positive (counterclockwise) torque is needed to contrast the clockwise motion that the first link would otherwise have, due to the inertial coupling with the instantaneous acceleration of the second link.

Exercise 3

In a Lagrangian formulation, we have

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U(\boldsymbol{q}) = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} - U(\boldsymbol{q})$$

and thus

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)^T - \left(\frac{\partial L}{\partial \boldsymbol{q}}\right)^T = \boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\operatorname{col}\left\{\dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{B}}{\partial q_i}\,\dot{\boldsymbol{q}}\right\} + \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u}, \tag{17}$$

where

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\operatorname{col}\left\{\dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{B}}{\partial q_i}\,\dot{\boldsymbol{q}}\right\} = \frac{1}{2}\operatorname{col}\left\{\dot{\boldsymbol{q}}^T \left[\frac{\partial \boldsymbol{b}_i}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{b}_i}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{B}}{\partial q_i}\right]\dot{\boldsymbol{q}}\right\}, \quad \boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T,$$

being $\boldsymbol{x} = \operatorname{col}\{x_i\}$ a n-dimensional column vector with x_i as the ith component, for $i = 1, \ldots, n$.

Since $\dot{\boldsymbol{q}} = \boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{p}$, it is easy to see that the Hamiltonian can be rewritten as a function of the generalized coordinates $\boldsymbol{q}$ and the generalized momentum $\boldsymbol{p}$ only:

$$H(\boldsymbol{q}, \boldsymbol{p}) = T(\boldsymbol{q}, \boldsymbol{p}) + U(\boldsymbol{q}) = \frac{1}{2}\,\boldsymbol{p}^T \boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{p} + U(\boldsymbol{q}).$$

We compute then the partial derivatives

$$\begin{aligned}
\nabla_{\boldsymbol{p}} H &= \left(\frac{\partial H}{\partial \boldsymbol{p}}\right)^T = \nabla_{\boldsymbol{p}} T = \boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{p}, \\
\nabla_{\boldsymbol{q}} H &= \left(\frac{\partial H}{\partial \boldsymbol{q}}\right)^T = \nabla_{\boldsymbol{q}} T + \nabla_{\boldsymbol{q}} U = \frac{1}{2}\operatorname{col}\left\{\boldsymbol{p}^T \frac{\partial \boldsymbol{B}^{-1}}{\partial q_i}\,\boldsymbol{p}\right\} + \boldsymbol{g}(\boldsymbol{q}).
\end{aligned} \tag{18}$$

Differentiating w.r.t. time $\boldsymbol{p} = \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}$ and using (17), it is

$$\dot{\boldsymbol{p}} = \boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{u} + \frac{1}{2}\operatorname{col}\left\{\dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{B}}{\partial q_i}\,\dot{\boldsymbol{q}}\right\} - \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u} + \frac{1}{2}\operatorname{col}\left\{\boldsymbol{p}^T \boldsymbol{B}^{-1} \frac{\partial \boldsymbol{B}}{\partial q_i}\,\boldsymbol{B}^{-1}\boldsymbol{p}\right\} - \boldsymbol{g}(\boldsymbol{q}). \tag{19}$$

Moreover, from the identity

$$\boldsymbol{B}(\boldsymbol{q})\boldsymbol{B}^{-1}(\boldsymbol{q}) = \boldsymbol{I} \quad \Rightarrow \quad \frac{\partial}{\partial q_i}\left(\boldsymbol{B}(\boldsymbol{q})\boldsymbol{B}^{-1}(\boldsymbol{q})\right) = \left(\frac{\partial}{\partial q_i}\boldsymbol{B}(\boldsymbol{q})\right)\boldsymbol{B}^{-1}(\boldsymbol{q}) + \boldsymbol{B}(\boldsymbol{q})\left(\frac{\partial}{\partial q_i}\boldsymbol{B}^{-1}(\boldsymbol{q})\right) = \boldsymbol{O},$$

we obtain the general property

$$\frac{\partial \boldsymbol{B}^{-1}}{\partial q_i} = -\boldsymbol{B}^{-1}\frac{\partial \boldsymbol{B}}{\partial q_i}\boldsymbol{B}^{-1}. \tag{20}$$

Replacing (20) in (19) leads to

$$\dot{\boldsymbol{p}} = \boldsymbol{u} - \frac{1}{2}\,\mathrm{col}\left\{\boldsymbol{p}^T\frac{\partial \boldsymbol{B}^{-1}}{\partial q_i}\boldsymbol{p}\right\} - \boldsymbol{g}(\boldsymbol{q}).$$

As a result, using also eq. (18) one can write

$$\begin{aligned}\dot{\boldsymbol{q}} &= \boldsymbol{B}^{-1}(\boldsymbol{q})\boldsymbol{p} \;=\; \nabla_{\boldsymbol{p}} H\\ \dot{\boldsymbol{p}} &= -\left(\frac{1}{2}\,\mathrm{col}\left\{\boldsymbol{p}^T\frac{\partial \boldsymbol{B}^{-1}}{\partial q_i}\boldsymbol{p}\right\} + \boldsymbol{g}(\boldsymbol{q})\right) + \boldsymbol{u} \;=\; -\nabla_{\boldsymbol{q}} H + \boldsymbol{u},\end{aligned}$$

which can be immediately rearranged in the form of the stated equations (4).

Exercise 4

The task Jacobian of the planar PPR robot is given by

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} 1 & 0 & -l\sin q_3\\ 0 & 1 & l\cos q_3\end{pmatrix}, \tag{21}$$

with

$$\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \dot{\boldsymbol{p}}\in\mathbb{R}^2, \qquad \dot{\boldsymbol{q}}\in\mathbb{R}^3.$$

Clearly, the Jacobian (21) is always of maximum rank $\rho = 2$. Therefore, the minimum norm joint velocity that realizes a desired $\dot{\boldsymbol{p}}$ is provided by

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\dot{\boldsymbol{p}}, \tag{22}$$

where the pseudoinverse matrix takes the explicit expression

$$\boldsymbol{J}^{\#}(\boldsymbol{q}) = \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1} = \frac{1}{1+l^2}\begin{pmatrix} 1+l^2\cos^2 q_3 & l^2\sin q_3\cos q_3\\ l^2\sin q_3\cos q_3 & 1+l^2\sin^2 q_3\\ -l\sin q_3 & l\cos q_3\end{pmatrix}, \tag{23}$$

being $\det(\boldsymbol{J}\boldsymbol{J}^T) = 1+l^2$. When looking at the elements of the matrix in (23), it is easy to realize that there is a problem of unit inconsistency. In particular, the expression of the determinant of $\boldsymbol{J}\boldsymbol{J}^T$ consists of the sum of a non-dimensional quantity and of one having dimension of length to the square. As a matter of fact, the robot has joints of different nature. Therefore, the norm $\|\dot{\boldsymbol{q}}\|$ has mixed expressions and its minimization leads in general to different solutions, depending on the units chosen for the linear and the angular quantities.

For illustration, if we evaluate the solution (22) with the given numerical data using [m] as linear unit,

$$l = 0.5 \text{ [m]}, \qquad q_3 = \pi/6 \text{ [rad]}, \qquad \dot{\boldsymbol{p}} = \begin{pmatrix}-1\\ 1\end{pmatrix} \text{ [m/s]},$$

we obtain

$$\dot{\boldsymbol{q}} = \begin{pmatrix} 0.95 & 0.0866\\ 0.0866 & 0.85\\ -0.2 & 0.3464\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} = \begin{pmatrix}-0.8634\\ 0.7634\\ 0.5464\end{pmatrix}\begin{matrix}\text{[m/s]}\\ \text{[m/s]}\\ \text{[rad/s]}\end{matrix}. \tag{24}$$

Instead, when using [cm] as the linear unit,

$$l = 50 \text{ [cm]}, \qquad q_3 = \pi/6 \text{ [rad]}, \qquad \dot{\boldsymbol{p}} = \begin{pmatrix} -100 \\ 100 \end{pmatrix} \text{ [cm/s]},$$

we obtain

$$\dot{\boldsymbol{q}} = \begin{pmatrix} 0.7501 & 0.4328 \\ 0.4328 & 0.2503 \\ -0.01 & 0.0173 \end{pmatrix} \begin{pmatrix} -100 \\ 100 \end{pmatrix} = \begin{pmatrix} -31.72 \\ -18.25 \\ 2.731 \end{pmatrix} \begin{matrix} \text{[cm/s]} \\ \text{[cm/s]} \\ \text{[rad/s]} \end{matrix} = \begin{pmatrix} -0.3172 \\ -0.1825 \\ 2.731 \end{pmatrix} \begin{matrix} \text{[m/s]} \\ \text{[m/s]} \\ \text{[rad/s]} \end{matrix}, \tag{25}$$

which is completely different (in all components) with respect to (24).

Consider now the use of a weighted pseudoinverse to define a solution as

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}_{\boldsymbol{W}}(\boldsymbol{q})\dot{\boldsymbol{p}}, \tag{26}$$

where, being the task Jacobian always of full rank, we can use the explicit expression

$$\boldsymbol{J}^{\#}_{\boldsymbol{W}}(\boldsymbol{q}) = \boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{q}) \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}. \tag{27}$$

Since the issue here is the different nature of the first two joints (prismatic) with respect to the last one (revolute), it makes sense to focus our attention to a diagonal weighting matrix of the form

$$\boldsymbol{W} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & w \end{pmatrix}, \qquad \text{with } w > 0. \tag{28}$$

When computing the inverse matrix in (27), one gets

$$\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1} = \begin{pmatrix} 1 + \dfrac{l^2}{w}\sin^2 q_3 & \dfrac{l^2}{w}\sin q_3 \cos q_3 \\ \dfrac{l^2}{w}\sin q_3 \cos q_3 & 1 + \dfrac{l^2}{w}\cos^2 q_3 \end{pmatrix} \quad \text{and} \quad \det\left(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T\right) = 1 + \frac{l^2}{w}.$$

Therefore, the choice $w = l^2$ in (28) would eliminate the presence of terms with dimensional units in this key expression (in particular, the determinant would become equal to 2). In fact, with this choice the final expression of such a properly weighted pseudoinverse is

$$\boldsymbol{J}^{\#}_{\boldsymbol{W}}(\boldsymbol{q}) = \frac{1}{2} \begin{pmatrix} 1 + \cos^2 q_3 & \sin q_3 \cos q_3 \\ \sin q_3 \cos q_3 & 1 + \sin^2 q_3 \\ -\dfrac{\sin q_3}{l} & \dfrac{\cos q_3}{l} \end{pmatrix}. \tag{29}$$

Evaluating the solution (26) with $w = l^2$ in the previous numerical example, and using [m] as linear unit,

$$l = 0.5 \text{ [m]}, \qquad q_3 = \pi/6 \text{ [rad]}, \qquad \dot{\boldsymbol{p}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ [m/s]},$$

we obtain

$$\dot{\boldsymbol{q}} = \begin{pmatrix} 0.875 & 0.2165 \\ 0.2165 & 0.625 \\ -0.5 & 0.866 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.6585 \\ 0.4085 \\ 1.366 \end{pmatrix} \begin{matrix} \text{[m/s]} \\ \text{[m/s]} \\ \text{[rad/s]} \end{matrix}. \tag{30}$$

When using [cm] as the linear unit,

$$l = 50 \text{ [cm]}, \qquad q_3 = \pi/6 \text{ [rad]}, \qquad \dot{\boldsymbol{p}} = \begin{pmatrix} -100 \\ 100 \end{pmatrix} \text{ [cm/s]},$$

we obtain in this case

$$\dot{\boldsymbol{q}} = \begin{pmatrix} 0.875 & 0.2165 \\ 0.2165 & 0.625 \\ -0.005 & 0.0087 \end{pmatrix} \begin{pmatrix} -100 \\ 100 \end{pmatrix} = \begin{pmatrix} -65.8484 \\ -40.8494 \\ 1.366 \end{pmatrix} \begin{matrix} \text{[cm/s]} \\ \text{[cm/s]} \\ \text{[rad/s]} \end{matrix} = \begin{pmatrix} -0.6585 \\ 0.4085 \\ 1.366 \end{pmatrix} \begin{matrix} \text{[m/s]} \\ \text{[m/s]} \\ \text{[rad/s]} \end{matrix}, \tag{31}$$

which is exactly the solution (30) already obtained with the other linear units.

The role of the suitable weighting $w = l^2$ on the velocity of the third joint can be given the following interpretation. The general solution (26) minimizes the objective

$$H = \frac{1}{2}\|\dot{\boldsymbol{q}}\|^2_{\boldsymbol{W}} = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W} \dot{\boldsymbol{q}} = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + w\dot{q}_3^2\right) = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + (l\dot{q}_3)^2\right),$$

subject to the task constraint $\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}$. Therefore, this (unique!) weight will equalize the cost for a Cartesian displacement of the robot end-effector of a given length, no matter if this is achieved by one of the two prismatic joints or by the third revolute joint. Stated differently, a linear path achieved by translating along the first or second prismatic joint at 1 [m/s] for 1 second will cost in the objective function H the same as an arc of a circle of radius l achieved by rotating the third joint at $1/l$ [rad/s] for 1 second.

On the other hand, selecting a very large weight $w \gg 1$ in (28) will penalize the motion of the third (revolute) joint with respect to the prismatic ones. The PPR robot will then try to achieve the desired Cartesian velocity moving mainly the first two joints. For instance, using $w = 1000$ in the previous numerical example (and using [m] for the linear units) provides

$$\dot{\boldsymbol{q}} = \begin{pmatrix} 0.9999 & 0.0001 \\ 0.0001 & 0.9998 \\ -0.0002 & 0.0004 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.9998 \\ 0.9997 \\ 0.0007 \end{pmatrix} \begin{matrix} \text{[m/s]} \\ \text{[m/s]} \\ \text{[rad/s]} \end{matrix}, \tag{32}$$

namely a velocity that involves in practice only the first two prismatic joints.

* * * * *

Robotics II

June 11, 2018

Exercise 1

The dynamic model of the planar RP robot in Fig. 1 moving in a vertical plane can be written in the usual form as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}. \tag{1}$$

- Define two different matrices $\boldsymbol{S}_1(\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\boldsymbol{S}_2(\boldsymbol{q},\dot{\boldsymbol{q}})$ that factorize the Coriolis and centrifugal terms (i.e., yielding $\boldsymbol{S}_i(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})$, for $i = 1, 2$), such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}_1$ is skew-symmetric, while $\dot{\boldsymbol{M}} - 2\boldsymbol{S}_2$ is not.
- Give the explicit symbolic expressions of the terms appearing in the definition of the momentum-based residual vector $\boldsymbol{r} \in \mathbb{R}^2$ that allows detection and isolation of collisions.
- Are there situations in which collision forces $\boldsymbol{F}_K \in \mathbb{R}^2$ in the plane of motion lead to poor or no detection, or to incorrect isolation of the involved link? Discuss the issue.

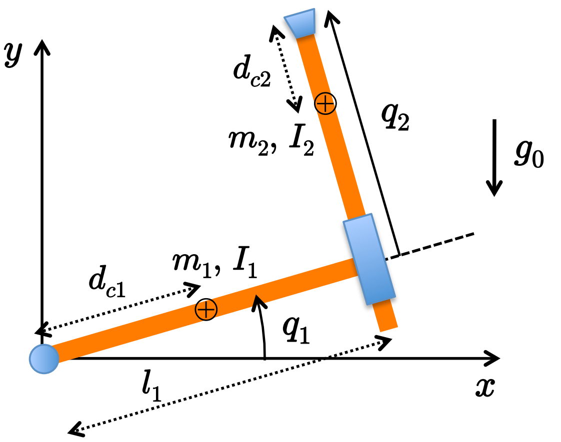


Figure 1: A planar RP robot, with the definition of joint variables and parameters.

Exercise 2

NOT TO BE ESTIMATED!

Consider an actuated pendulum with a link of (known) length l that moves without friction in the vertical plane. The pendulum is driven by a DC motor at the base and carries a heavy and *unknown* payload. Assume that the motor inertia and the mass and inertia of the link are negligible with respect to the payload, which should be seen as a concentrated mass m at the tip of the pendulum. The drive gain k_i of the current-to-torque relation $\tau = k_i\, i_m$ of the motor is *unknown*, and only the motor current i_m can be commanded.

Design an adaptive control law for i_m that achieves global asymptotic tracking of a smooth desired trajectory $\theta_d(t)$ for the joint angle θ of the pendulum. Provide a (sketch of) proof of your result.

Exercise 3

With reference to Fig. 2, a mass $m > 0$ moves under the action of a control force F, in the presence of viscous friction with coefficient $d > 0$, and interacts with a stiff environment. We would like to regulate the contact force F_c to a constant desired value $F_d > 0$. The contact force is measured by a load cell of stiffness $k_s > 0$, i.e., $F_c = k_s x$ where $x = 0$ corresponds to the initial contact position with the environment. Consider a class of control laws of the form

$$F = \alpha k_f (F_d - F_c) + \beta F_d, \tag{2}$$

where $k_f > 0$, and with:

1. $(\alpha, \beta) = (0, 1)$ [pure feedforward];
2. $(\alpha, \beta) = (1, 0)$ [pure proportional feedback];
3. $(\alpha, \beta) = (1, 1)$ [combined feedback/feedforward].

- For each of the three above control cases, check the system equilibrium and verify its stability properties, giving a proof of your statements (e.g., via Lyapunov/LaSalle, or using Laplace analysis in view of the linearity of the system) and briefly discussing the benefits and limitations of each law.
- How robust are these results with respect to uncertainty in the knowledge of the physical parameters m. d, and k_s?
- Explain what happens under the action of the above control laws when there is no environment present ($F_c \equiv 0$). Would the system reach some form of steady state?

$\ddot{x} = 0$

$m\ddot{x} - k_s x - d\dot{x} = F$

case 1)

$m\ddot{x} + k_s x + d\dot{x} = F_d$

$x_1 = x$

$\dot{x}_1 = x_2$

$\ddot{x} = \dot{x}_2 = \frac{F - k_s x - d\dot{x}}{m} = \frac{F_d - k_s x_1 - d x_2}{m}$

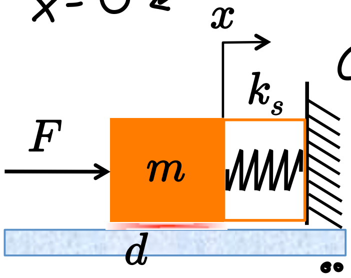


Figure 2: A mass in contact through a load cell with a stiff environment.

$A = \begin{pmatrix} 0 & 1 \\ -\frac{k_s}{m} & -\frac{d}{m} \end{pmatrix}$

EQ. when ACC $\ddot{x}$ AND VEL $\dot{x} = 0$

[240 minutes; open books, but no computer or smartphone]

$b = \begin{pmatrix} 0 \\ +F_d/m \end{pmatrix}$

$\dot{x}_1 = 0 = x_2$

$\dot{x}_2 = \frac{F_d - k_s x_1 - d\,0}{m} = \frac{F_d - k_s x_1}{m} = 0$

$x_1 = \frac{F_d}{k_s}$ eq. point

Solution

June 11, 2018

Exercise 1

For later use, we derive the terms in the dynamic model (1) following a Lagrangian approach. The kinetic energy of the first link is

$$T_1 = \frac{1}{2}(I_1 + m_1 d_{c1}^2)\dot{q}_1^2.$$

The position and velocity of the center of mass of the second link are, respectively,

$$\boldsymbol{p}_{c2} = \begin{pmatrix} l_1 \cos q_1 - (q_2 - d_{c2}) \sin q_1 \\ l_1 \sin q_1 + (q_2 - d_{c2}) \cos q_1 \end{pmatrix}, \qquad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -\sin q_1 (l_1 \dot{q}_1 + \dot{q}_2) - (q_2 - d_{c2}) \cos q_1 \, \dot{q}_1 \\ \cos q_1 (l_1 \dot{q}_1 + \dot{q}_2) - (q_2 - d_{c2}) \sin q_1 \, \dot{q}_1 \end{pmatrix}.$$

Therefore, the kinetic energy of the second link is

$$T_2 = \frac{1}{2} I_2 \, \dot{q}_1^2 + \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 = \frac{1}{2}(I_2 + m_2 (q_2 - d_{c2})^2)\dot{q}_1^2 + \frac{1}{2} m_2 (l_1 \dot{q}_1 + \dot{q}_2)^2.$$

The kinetic energy of the system is

$$T = T_1 + T_2 = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}},$$

with inertia matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_1 + m_1 d_{c1}^2 + I_2 + m_2 l_1^2 + m_2 (q_2 - d_{c2})^2 & m_2 l_1 \\ m_2 l_1 & m_2 \end{pmatrix}.$$

The generallzed momentum of the robot is then

$$\boldsymbol{p} = \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \begin{pmatrix} \left(I_1 + m_1 d_{c1}^2 + I_2 + m_2 (q_2 - d_{c2})^2\right) \dot{q}_1 + m_2 l_1 (l_1 \dot{q}_1 + \dot{q}_2) \\ m_2 (l_1 \dot{q}_1 + \dot{q}_2) \end{pmatrix}. \tag{3}$$

The Coriolis and centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are derived using the Christoffel symbols, i.e., for each component

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial \boldsymbol{q}_i} \right), \qquad i = 1, 2,$$

being $\boldsymbol{m}_i$ the ith column of the inertia matrix $\boldsymbol{M}$. We obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & m_2 (q_2 - d_{c2}) \\ m_2 (q_2 - d_{c2}) & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 2 m_2 (q_2 - d_{c2}) \dot{q}_1 \dot{q}_2$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} -m_2 (q_2 - d_{c2}) & 0 \\ 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_2 (q_2 - d_{c2}) \dot{q}_1^2,$$

and thus

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2 m_2 (q_2 - d_{c2}) \dot{q}_1 \dot{q}_2 \\ -m_2 (q_2 - d_{c2}) \dot{q}_1^2 \end{pmatrix}.$$

The potential energy of the robot is

$$U = U_1 + U_2 = m_1 g_0\, p_{c1,y} + m_2 g_0\, p_{c2,y} = m_1 g_0\, d_{c1} \sin q_1 + m_2 g_0 \left(l_1 \sin q_1 + (q_2 - d_{c2}) \cos q_1\right),$$

with $g_0 = 9.81$ [m/s^2]. The associated gravity vector is

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0 \left((m_1 d_{c1} + m_2 l_1) \cos q_1 - m_2 (q_2 - d_{c2}) \sin q_1\right) \\ m_2 g_0 \cos q_1 \end{pmatrix}, \tag{4}$$

As for the factorizations of the Coriolis and centrifugal terms, using again the definition of Christoffel symbols, we compute

$$\boldsymbol{S}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} m_2(q_2 - d_{c2})\dot{q}_2 & m_2(q_2 - d_{c2})\dot{q}_1 \\ -m_2(q_2 - d_{c2})\dot{q}_1 & 0 \end{pmatrix}, \tag{5}$$

which guarantees the skew-symmetry of the matrix

$$\begin{aligned} \dot{\boldsymbol{M}} - 2\boldsymbol{S}_1 &= \begin{pmatrix} 2m_2(q_2 - d_{c2})\dot{q}_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 2m_2(q_2 - d_{c2})\dot{q}_2 & 2m_2(q_2 - d_{c2})\dot{q}_1 \\ -2m_2(q_2 - d_{c2})\dot{q}_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2m_2(q_2 - d_{c2})\dot{q}_1 \\ 2m_2(q_2 - d_{c2})\dot{q}_1 & 0 \end{pmatrix}. \end{aligned}$$

On the other hand, the alternative choice

$$\boldsymbol{S}_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 0 & 2m_2(q_2 - d_{c2})\dot{q}_1 \\ -m_2(q_2 - d_{c2})\dot{q}_1 & 0 \end{pmatrix},$$

which produces another feasible factorization $\boldsymbol{S}_2(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, leads to the matrix

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}_2 = \begin{pmatrix} 2m_2(q_2 - d_{c2})\dot{q}_2 & -4m_2(q_2 - d_{c2})\dot{q}_1 \\ 2m_2(q_2 - d_{c2})\dot{q}_1 & 0 \end{pmatrix}$$

that is clearly *not* skew symmetric. This implies, e.g., that matrix $\boldsymbol{S}_2$ *cannot* be used in the definition of the residual vector $\boldsymbol{r}$ for collision detection and isolation.

The definition of the residual is

$$\boldsymbol{r}(t) = \boldsymbol{K}_I \left(\boldsymbol{p}(t) - \int_0^t \left(\boldsymbol{\tau} + \boldsymbol{S}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{r}\right) ds - \boldsymbol{p}(0)\right), \qquad \boldsymbol{K}_I > 0, \tag{6}$$

where $\boldsymbol{p} = \boldsymbol{M}\dot{\boldsymbol{q}}$ is given by (3), $\boldsymbol{p}(0) = \boldsymbol{0}$ iff $\dot{\boldsymbol{q}}(0) = \boldsymbol{0}$ (the robot starts at rest), $\boldsymbol{g}$ is given by (4), and matrix $\boldsymbol{S}$ should factorize the Coriolis and centrifugal terms so that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is skew symmetric. Choosing in particular $\boldsymbol{S} = \boldsymbol{S}_1$ in (5), we have in this case

$$\boldsymbol{S}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{pmatrix} m_2(q_2 - d_{c2})\dot{q}_2 & -m_2(q_2 - d_{c2})\dot{q}_1 \\ m_2(q_2 - d_{c2})\dot{q}_1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ m_2(q_2 - d_{c2})\dot{q}_1^2 \end{pmatrix}.$$

The residual $\boldsymbol{r}$ in (6) will be affected by a non-zero contact force $\boldsymbol{F}_K \in \mathbb{R}^2$ lying in the plane $(\boldsymbol{x}, \boldsymbol{y})$ and acting on one of the robot links through the joint torque $\boldsymbol{\tau}_K = \boldsymbol{J}_K^T(\boldsymbol{q})\boldsymbol{F}_K$, except for some singular cases. Essentially, these are directions along which the contact point cannot be given by means of $\dot{\boldsymbol{q}}$ a linear velocity in the plane of motion. In the following, we distinguish between collisions on the first or on the second link (see Fig. 3).

- **Collision on link 1**. The position of the contact point along the first link and the associated contact Jacobian are

$$\boldsymbol{p}_{K1} = \rho_1 \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \quad \text{with } \rho_1 \in [0, l_1] \quad \Rightarrow \quad \boldsymbol{J}_{K1} = \begin{pmatrix} -\sin q_1 & 0 \\ \cos q_1 & 0 \end{pmatrix},$$

  and thus collision is not detected when

$$\boldsymbol{F}_{K1} = \alpha \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \quad \text{for arbitrary } \alpha = \|\boldsymbol{F}_{K1}\| \quad \Rightarrow \quad \boldsymbol{J}_{K1}^T \boldsymbol{F}_{K1} = \boldsymbol{0},$$

  namely, when the contact force is aligned with the first link (Fig. 3a). The closer is the alignment of $\boldsymbol{F}_{K1}$ with the axis of link 1, the poorer will be the detection.

- **Collision on link 2**. The position of the contact point along the second link is[1]

$$\boldsymbol{p}_{K2} = \begin{pmatrix} l_1 \cos q_1 - (q_2 - \rho_2) \sin q_1 \\ l_1 \sin q_1 + (q_2 - \rho_2) \cos q_1 \end{pmatrix} = \mathrm{Rot}_{2\times 2}(q_1) \begin{pmatrix} l_1 \\ q_2 - \rho_2 \end{pmatrix}, \quad \text{with } \rho_2 \in [0, l_{2,max}]$$

  and the associated contact Jacobian is

$$\Rightarrow \quad \boldsymbol{J}_{K2} = \begin{pmatrix} -\left(l_1 \sin q_1 + (q_2 - \rho_2) \cos q_1\right) & -\sin q_1 \\ l_1 \cos q_1 - (q_2 - \rho_2) \sin q_1 & \cos q_1 \end{pmatrix}, \qquad \det \boldsymbol{J}_{K2} = \rho_2 - q_2.$$

  Thus, collision is not detected when $\rho_2 = q_2$ and

$$\boldsymbol{F}_{K2} = \alpha \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \quad \text{for arbitrary } \alpha = \pm \|\boldsymbol{F}_{K2}\| \quad \Rightarrow \quad \boldsymbol{J}_{K2|_{q_2=\rho_2}}^T \boldsymbol{F}_{K2} = \boldsymbol{0},$$

  namely, when the contact occurs at the second joint location and the force is orthogonal to the second link (Fig. 3b). On the other hand, we obtain still detection but wrong isolation when the contact force is orthogonal to the second link, as before, and the contact point is not along the first link axis ($\rho_2 \neq q_2$). In this case (see Fig. 3c), we would have

$$\boldsymbol{r} \simeq \boldsymbol{\tau}_{K2} = \boldsymbol{J}_{K2}^T \boldsymbol{F}_{K2} = \alpha \cdot \boldsymbol{J}_{K2}^T \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} = \begin{pmatrix} \alpha\,(q_2 - \rho_2) \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix},$$

  indicating incorrectly that a collision occurred on link 1, rather than on link 2. Finally, if

$$\boldsymbol{F}'_{K2} = \beta \begin{pmatrix} l_1 \cos q_1 - (q_2 - \rho_2) \sin q_1 \\ l_1 \sin q_1 + (q_2 - \rho_2) \cos q_1 \end{pmatrix}, \quad \text{for arbitrary } \beta \neq 0,$$

  namely, when the line of action of the contact force passes through the axis of joint 1, we obtain (see Fig. 3d)

$$\boldsymbol{r} \simeq \boldsymbol{\tau}'_{K2} = \boldsymbol{J}_{K2}^T \boldsymbol{F}'_{K2} = \begin{pmatrix} 0 \\ \beta(q_2 - \rho_2) \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix},$$

  indicating correctly that the collision is on link 2 (the largest index with a non-zero component in $\boldsymbol{r}$), although the first component of the residual is vanishing ($r_1 = 0$).

[1] We assume that the prismatic joint has a maximum excursion of $q_2 \in [0, l_{2,max}]$.

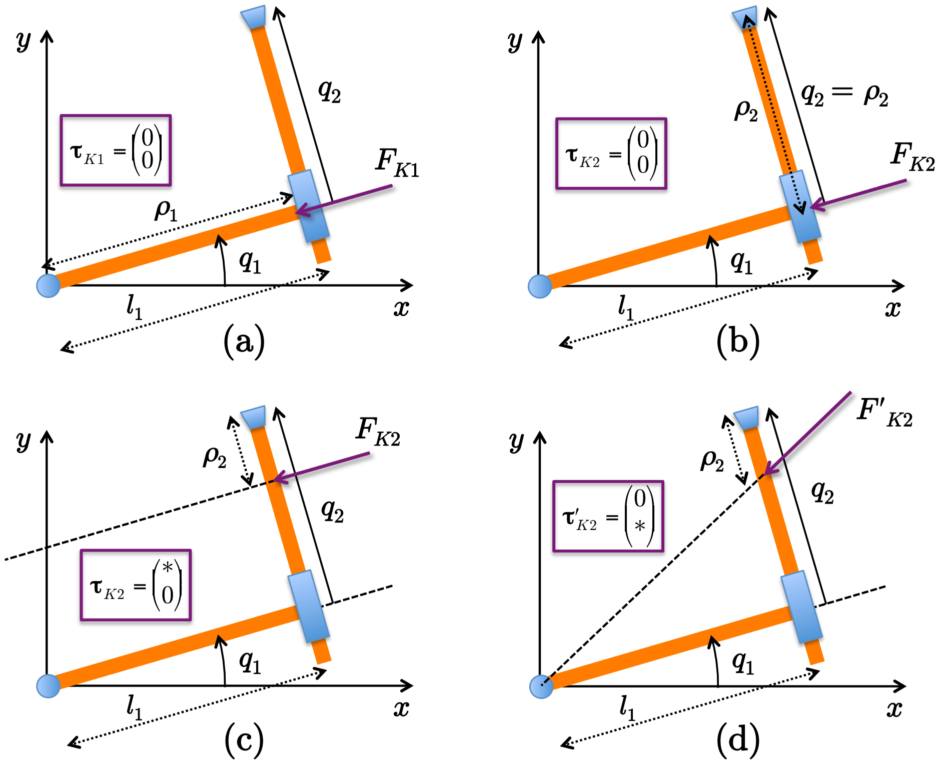


Figure 3: Possible collision situations leading to no detection or wrong isolation: (a) no detection of a contact force on link 1; (b) no detection of a contact force on link 2; (c) wrong isolation of a contact force on link 2; (d) isolation of a contact force on link 2, despite the residual at joint 1 is not being affected.

Exercise 2

Define as coordinate of the pendulum link its angle θ from the downward vertical (positive if counterclockwise). Under the simplifying assumptions made, the Lagrangian dynamics of the actuated pendulum is computed from the kinetic and potential energy

$$T = \frac{1}{2} m l^2 \dot{\theta}^2, \qquad U = -m g_0 \, l \cos\theta,$$

as

$$m l^2 \ddot{\theta} + m g_0 \, l \sin\theta = \tau = k_i i_m, \tag{7}$$

where the drive gain of the DC motor has been included. The unknowns in (7) are the payload (concentrated) mass m and the drive gain k_i. Dividing by $k_i > 0$, this equation can be rewritten in the standard linearly parametrized form (and without unknown parameters affecting the input)

wow!

$$\frac{m}{k_i} l^2 \ddot{\theta} + \frac{m}{k_i} g_0 \, l \sin\theta = \left(l^2 \ddot{\theta} + g_0 \, l \sin\theta \right) \frac{m}{k_i} = Y(\theta, \ddot{\theta})\, a = i_m, \tag{8}$$

being $a = m/k_i > 0$ the only dynamic coefficient that matters ($p = 1$). We note that a plays the role of a scaled mechanical inertia. Being the motor current i_m the input to the system, it is easy to derive from eq. (8) an adaptive control law for trajectory tracking that mimics the classical one derived when the input is directly the motor torque τ. Given a twice-differentiable desired trajectory $\theta_d(t)$ for the joint variable, define such adaptive law as

$$\begin{aligned} i_m &= Y(\theta,\ddot{\theta}_r)\,\hat{a} + k_p e + k_d\dot{e}, \qquad k_p>0, k_d>0, \qquad e=\theta_d-\theta, \qquad \dot{\theta}_r = \dot{\theta}_d + \lambda e = \dot{\theta}_d + \frac{k_p}{k_d}e,\\ \dot{\hat{a}} &= \gamma\, Y(\theta,\ddot{\theta}_r)\,(\dot{\theta}_r-\dot{\theta}), \qquad \gamma>0, \qquad \dot{\theta}_r-\dot{\theta} = \dot{e}+\lambda e, \end{aligned} \tag{9}$$

with

$$Y(\theta,\ddot{\theta}_r) = l^2\ddot{\theta}_r + g_0\, l\sin\theta, \qquad \hat{a} = \widehat{\left(\frac{m}{k_i}\right)} \in \mathbb{R}.$$

The global asymptotic tracking of the smooth trajectory $\theta_d(t)$ can be proven by following the same arguments as in the classical case, i.e., via a Lyapunov candidate and the use of Barbalat lemma and LaSalle theorem. However, one should carefully define the candidate by considering the scaled mechanical inertia of the system. Therefore, noting the absence of dissipative terms, define the candidate function

$$V = \frac{1}{2}\frac{ml^2}{k_i}(\dot{\theta}_r-\dot{\theta})^2 + \frac{1}{2}R\,e^2 + \frac{1}{2\gamma}\tilde{a}^2 \geq 0,$$

with $R = 2k_p > 0$ and $\tilde{a} = a - \hat{a}$. We have that $V=0$ iff $e=\dot{e}=\tilde{a}=0$. For the closed-loop system (8–9), we can write

$$Y(\theta,\ddot{\theta})\,a = \left(l^2\ddot{\theta} + g_0\, l\sin\theta\right)\frac{m}{k_i} = \left(l^2\ddot{\theta}_r + g_0\, l\sin\theta\right)\widehat{\left(\frac{m}{k_i}\right)} + k_p e + k_d\dot{e} = Y(\theta,\ddot{\theta}_r)\,\hat{a} + k_p e + k_d\dot{e}.$$

Subtracting both sides of this equality from $Y(\theta,\ddot{\theta}_r)\,a$, one obtains

$$Y(\theta,\ddot{\theta}_r)\,a - Y(\theta,\ddot{\theta})\,a = \frac{ml^2}{k_i}(\ddot{\theta}_r-\ddot{\theta}) = \left(l^2\ddot{\theta}_r + g_0\, l\sin\theta\right)\tilde{a} - k_p e - k_d\dot{e} = Y(\theta,\ddot{\theta}_r)\,\tilde{a} - k_p e - k_d\dot{e}. \tag{10}$$

Using (10), the time derivative of V computed along the trajectories of the closed-loop system (8–9) is evaluated as

$$\begin{aligned} \dot{V} &= \frac{ml^2}{k_i}(\ddot{\theta}_r-\ddot{\theta})\,(\dot{\theta}_r-\dot{\theta}) + 2k_p\, e\,\dot{e} - \frac{1}{\gamma}\,\tilde{a}\,\dot{\hat{a}}\\ &= \left(Y(\theta,\ddot{\theta}_r)\,\tilde{a} - k_p e - k_d\dot{e}\right)(\dot{\theta}_r-\dot{\theta}) + 2k_p\, e\,\dot{e} - \frac{1}{\gamma}\,\gamma\, Y(\theta,\ddot{\theta}_r)\,\tilde{a}\,(\dot{\theta}_r-\dot{\theta})\\ &= -\,(k_p e + k_d\dot{e})\,(\dot{e} + \frac{k_p}{k_d}e) + 2k_p\, e\,\dot{e}\\ &= -\,k_d\,\dot{e}^2 - \frac{k_p^2}{k_d}\,e^2 \leq 0. \end{aligned}$$

The rest of the proof is completed just like in the classical case.

Exercise 3

The dynamic model of the system in Fig. 2 is

$$m\ddot{x} = F - F_c - d\dot{x},$$

with all non-conservative forces performing work on x on the right-hand side. Since $F_c = k_s x$ (as the single compliant element in the contact), this equation can be rewritten as

$$m\ddot{x} + d\dot{x} + k_s x = F, \tag{11}$$

with all physical coefficients being positive. Applying the class of force control laws (2) yields

$$m\ddot{x} + d\dot{x} + k_s x = \alpha k_f (F_d - F_c) + \beta F_d = \alpha k_f (F_d - k_s x) + \beta F_d. \tag{12}$$

At the equilibrium, $\dot{x} = \ddot{x} = 0$, we have

$$k_s x = \alpha k_f (F_d - k_s x) + \beta F_d,$$

which is solved for a position $x = x_e$ and an associated contact force $F_c = F_e = k_s x_e$ as

$$x_e = \frac{\beta + \alpha k_f}{k_s(1 + \alpha k_f)} F_d, \qquad F_e = \frac{\beta + \alpha k_f}{1 + \alpha k_f} F_d \quad \Rightarrow \quad e_f = F_d - F_e = \frac{1 - \beta}{1 + \alpha k_f} F_d. \tag{13}$$

Therefore, the correct desired contact force F_d is obtained at the equilibrium if and only if $\beta = 1$ (presence of the constant feedforward), no matter if $\alpha = 0$ (no feedback) or $\alpha = 1$ (combined situation). In such case, it is in fact

$$x_e = \frac{1}{k_s} F_d, \qquad F_e = F_d \quad \Rightarrow \quad e_f = 0. \tag{14}$$

On the other hand, for $\beta = 0$ and $\alpha = 1$ (pure proportional feedback), we have at the equilibrium

$$x_e = \frac{k_f}{k_s(1 + k_f)} F_d, \qquad F_e = \frac{k_f}{1 + k_f} F_d \quad \Rightarrow \quad e_f = \frac{1}{1 + k_f} F_d, \tag{15}$$

which shows that only for $k_f \to \infty$ (or, for large proportional gains) we can drive the force error to zero (or, below a given tolerance).

Indeed, we need to show that the above equilibria are asymptotically stable. In view of the linearity of the system, whenever this property holds, the equilibrium will also be a global, exponentially stable one.

Consider first the case of a pure feedforward command $F = F_d$ ($\alpha = 0$, $\beta = 1$). The system dynamics is

$$m\ddot{x} + d\dot{x} + k_s x = F_d$$

In order to study the asymptotic stability of the equilibrium (14), we choose as Lyapunov candidate

$$V_1 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k_s (x - x_e)^2 \geq 0 \tag{16}$$

Its time derivative is evaluated on the controlled system as

$$\dot{V}_1 = m\dot{x}\ddot{x} + k_s(x - x_e)\dot{x} = \dot{x}(-d\dot{x} - k_s x_e + F_d) = -d\dot{x}^2 \leq 0.$$

We have $\dot{V}_1 = 0$ if and only if $\dot{x} = 0$. The system behaves then as $m\ddot{x} = F_d - k_s x$, showing that there will be an acceleration iff the contact force $F_c = k_s x$ is different from the desired one. By LaSalle theorem, we conclude the asymptotic stability of the equilibrium state $x = x_e$, $\dot{x} = 0$. Stated differently, by pushing constantly on the mass with the desired force $F_d > 0$, a steady state is reached with the desired contact force (thanks to the asymptotic stability of the original open-loop system). However, the transient behavior will be specified only by the actual physical mass m, sensor stiffness k_s, and viscous (damping) coefficient d. Moreover, the pure feedforward scheme is highly sensitive to unmodeled disturbance forces acting on the system.

Consider now the case of a pure proportional feedback of the force error, namely $F = k_f (F_d - F_c)$ ($\alpha = 1$, $\beta = 0$). The closed-loop system is then

$$m\ddot{x} + d\dot{x} + k_s x = k_f (F_d - F_c) = k_f (F_d - k_s x) \quad \Rightarrow \quad m\ddot{x} + d\dot{x} + k_s (1 + k_f) x = k_f F_d$$

In order to study the asymptotic stability of the (incorrect) equilibrium (15), we choose as Lyapunov candidate

$$V_2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_s(1+k_f)(x-x_e)^2 \geq 0. \tag{17}$$

Its time derivative, evaluated along the trajectories of the closed-loop system, is

$$\dot{V}_2 = m\dot{x}\ddot{x}+k_s(1+k_f)(x-x_e)\dot{x} = \dot{x}\left(-d\dot{x} - k_s(1+k_f)x + k_f F_d + k_s(1+k_f)(x-x_e)\right) = -d\dot{x}^2 \leq 0,$$

where we replaced the expression of x_e in (15) in order to simplify terms. The conclusion about asymptotic stability follows from a LaSalle analysis similar to the previous case. Indeed, the steady-state force error $e_f \neq 0$ can be decreased by increasing k_f, modifying accordingly the transient behavior. However, when increasing k_f the system response will become faster but soon underdamped. As a matter of fact, a useful additional damping action of the form $-d_c\dot{x}$, with $d_c > 0$, is actually missing in the considered control law.

On the other hand, when combining the feedforward and feedback actions in the control law ($\alpha = \beta = 1$), the steady-state error will vanish without the need of increasing the feedback gain k_f. As a result, this can be tuned so as to obtain the best transient behavior and possibly reduce the effects of extra disturbing forces. The analysis of the asymptotic stability of x_e in (14) can be conducted as before, using the same Lyapunov candidate V_2 in (17).

For those more acquainted with Laplace transformation methods in linear control systems, it is worth mentioning that the above stability analyses could have been conducted more easily (and quickly) by looking at the location of poles (with Routh criterion, or even with simpler methods) of suitable transfer functions in the Laplace domain s, both for the open-loop system

$$\frac{F_c(s)}{F(s)} = \frac{k_s}{ms^2+ds+k_s},$$

and for the closed-loop system, e.g., under combined feedback/feedforward

$$\frac{F_c(s)}{F_d(s)} = \frac{k_s(1+k_f)}{ms^2+ds+k_s(1+k_f)}.$$

We note also that all the obtained stability results are completely independent from the parameters m, d, and k_s (as long as they remain physically meaningful, i.e., positive). These quantities were invoked in the analysis, but are never used for force control design, which inherits therefore some intrinsic robustness. Nonetheless, the values of these parameters will affect the quality of the transient behavior in response to reference values F_d.

At last, when $F_c = 0$ (no interaction with the environment), all controllers will behave in a similar way. We would have in that case

$$m\ddot{x} + d\dot{x} = F = \begin{cases} F_d & \text{[pure feedforward]} \\ k_f F_d & \text{[pure proportional feedback]} \\ (1+k_f)F_d & \text{[combined feedback/feedforward]} \end{cases} \tag{18}$$

and the mass would always reach a constant steady-state velocity (with $\ddot{x} = 0$) equal to

$$\dot{x}_{ss} = \frac{F}{d}$$

with the constant F as specified in (18).

* * * * *

Robotics II

July 11, 2018

Exercise 1

Consider a 2R planar robot, with the two links of length l_1 and l_2, having the actuating motors mounted on the axes of the two revolute joints. Each motor delivers its torque to the driven link through an elastic transmission, modeled as a torsional spring of stiffness $k_i > 0$, for $i = 1, 2$. The robot has no motion reduction elements. In the ith motor-link assembly, for $i = 1, 2$, let m_{m_i} and I_{m_i} be, respectively, the balanced mass and the inertia of the rotor of the motor around its spinning axis, and m_i, d_{c_i}, and I_i the link mass, the distance of the center of mass of the link from the preceding joint axis, and the link inertia around its center of mass. Using as generalized coordinates the angle θ_{m_i} of the rotor of motor i w.r.t. the preceding link axis, and the angle θ_i of link i w.r.t. the preceding link axis, for $i = 1, 2$, define the 4×4 inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of the robot, where $\boldsymbol{q} = (\boldsymbol{\theta}^T \ \boldsymbol{\theta}_m^T)^T$. State explicitly any simplifying assumption that you may wish to use. Moreover, find a linear parametrization of the inertial term $\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} = \boldsymbol{Y}_M(\boldsymbol{q}, \ddot{\boldsymbol{q}})\,\boldsymbol{a}$ of the robot dynamic model in terms of a minimal set of p suitable dynamic coefficients a_i, $i = 1, \ldots, p$.

Exercise 2

$\ddot{q} = J^{\#}(a - \dot{J}\dot{q}) \xrightarrow{\text{AT REST}} \ddot{q} = J^{\#} a$

Consider a 4R planar robot with all links of equal length $\ell = 0.2$ [m]. The robot is in the DH configuration $\boldsymbol{q} = (\ 0 \ \ \pi/2 \ \ 0 \ \ \pi/2\ )^T$ [rad] and at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$). In this state, we should assign to the end-effector a desired linear acceleration $\boldsymbol{a} = (\ 5 \ \ 0\ )^T$ [m/s^2]. The joint accelerations are taken as input commands, and are bounded as $|\ddot{q}_i| \le A_i$, $i = 1, \ldots, 4$, with the limits $A_1 = 9$, $A_2 = 6$, $A_3 = 4$, and $A_4 = 2$ [rad/s^2]. Find, if possible, a feasible joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^4$ that executes instantaneously the desired Cartesian task, while satisfying these hard bounds. A solution with a lower norm is preferred, and could be obtained by a straightforward variation of the SNS method moved to the acceleration level.

Exercise 3

In a visual servoing scheme, n point features with coordinates (u_i, v_i), for $i = 1, \ldots, n$, can be extracted from the image. Define the 2×6 interaction matrix $\bar{\boldsymbol{J}}$ between the 6D vector of linear velocity $\boldsymbol{V} \in \mathbb{R}^3$ and angular velocity $\boldsymbol{\Omega} \in \mathbb{R}^3$ of the camera and the time derivative of the coordinates $(\bar{u}, \bar{v})$ of the average position of the n point features in the image plane. State all variables that matrix $\bar{\boldsymbol{J}}$ depends upon. NOTE: $\bar{J} \neq J(\bar{u}, \bar{v}, \bar{Z})$

Exercise 4

For a robot with n degrees of freedom, partition the generalized coordinates as $\boldsymbol{q} = (\boldsymbol{q}_a, \boldsymbol{q}_b)$, where $\boldsymbol{q}_a$ has n_a components, $\boldsymbol{q}_b$ has n_b components, and $n_a + n_b = n$. Provide the explicit expressions of the n_a-dimensional reduced robot dynamics and of the constraint-preserving forces $\boldsymbol{\lambda} \in \mathbb{R}^{n_b}$, when the geometric constraint $\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{q}_b - \boldsymbol{q}_{b,d} = \boldsymbol{0}$ is imposed at all times, with $\boldsymbol{q}_{b,d}$ being constant.

[240 minutes; open books, but no computer or smartphone]

$\underset{n\times 1}{q} = \begin{pmatrix} q_a \\ q_b \end{pmatrix}_{(n_a+n_b)\times 1}$

$\underset{n_b\times 1}{h(q)} = q_b - q_{b,d}$

$\dfrac{\partial h(q)}{\partial q} = \underset{n_b\times n}{A} = \begin{pmatrix} \underset{n_b\times n_a}{O} & \underset{n_b\times n_b}{I} \end{pmatrix}$

$\begin{pmatrix} A \\ D \end{pmatrix}_{(n_a+n_b)\times n}$ with $\underset{n_a\times n}{D} = \begin{pmatrix} \underset{n\times n_a}{I} & \underset{n\times n_b}{O} \end{pmatrix}$

So $\begin{pmatrix} A \\ D \end{pmatrix}$ is not singular

$\underset{n\times n}{\begin{pmatrix} A \\ D \end{pmatrix}^{-1}} = (E \ \ F)$

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Solution

July 11, 2018

Exercise 1

We need to compute the kinetic energy of the two motors and the two links, all in planar motion. For the motors, we have

$$T_{m_1} = \frac{1}{2} I_{m_1} \dot{\theta}_{m_1}^2, \qquad T_{m_2} = \frac{1}{2} m_{m_2} l_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_{m_2} (\dot{\theta}_1 + \dot{\theta}_{m_2})^2,$$

since the first motor is balanced and its center of mass does not move, while the center of mass of the second motor is placed on the second joint axis at a distance equal to the link length l_1.
For the links, we have

$$T_{l_1} = \frac{1}{2} \left(I_1 + m_1 d_{c_1}^2 \right) \dot{\theta}_1^2, \qquad T_{l_2} = \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} I_2 (\dot{\theta}_1 + \dot{\theta}_2)^2,$$

with

$$\boldsymbol{p}_{c_2} = \begin{pmatrix} l_1 \cos\theta_1 + d_{c_2} \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + d_{c_2} \sin(\theta_1 + \theta_2) \end{pmatrix}, \quad \boldsymbol{v}_{c_2} = \dot{\boldsymbol{p}}_{c_2} = \begin{pmatrix} -(l_1 \dot{\theta}_1 \sin\theta_1 + d_{c_2} (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2)) \\ l_1 \dot{\theta}_1 \cos\theta_1 + d_{c_2} (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) \end{pmatrix},$$

and thus

$$\|\boldsymbol{v}_{c_2}\|^2 = l_1^2 \dot{\theta}_1^2 + d_{c_2}^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 l_1 d_{c_2} \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \cos\theta_2.$$

Therefore, having set $\boldsymbol{q} = \begin{pmatrix} \boldsymbol{\theta}^T & \boldsymbol{\theta}_m^T \end{pmatrix}^T = \begin{pmatrix} \theta_1 & \theta_2 & \theta_{m_1} & \theta_{m_2} \end{pmatrix}^T$. we can write the total kinetic energy as

$$T = T_{m_1} + T_{l_1} + T_{m_2} + T_{l_2} = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}},$$

with the 4×4 inertia matrix of the robot given by

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} \begin{array}{c} I_1 + m_1 d_{c_1}^2 + I_2 + m_2 d_{c_2}^2 + m_2 l_1^2 \\ + I_{m_2} + m_{m_2} l_1^2 + 2 m_2 l_1 d_{c_2} \cos\theta_2 \end{array} & I_2 + m_2 d_{c_2}^2 + m_2 l_1 d_{c_2} \cos\theta_2 & 0 & I_{m_2} \\ I_2 + m_2 d_{c_2}^2 + m_2 l_1 d_{c_2} \cos\theta_2 & I_2 + m_2 d_{c_2}^2 & 0 & 0 \\ 0 & 0 & I_{m1} & 0 \\ I_{m_2} & 0 & 0 & I_{m_2} \end{pmatrix}.$$

Note that, if we assume that in the kinetic energy of the second motor the contribution of the angular velocity due to the previous link carrying the motor can be neglected in comparison with the spinning velocity of the rotor of the motor itself, we would have

$$T_{m_2} = \cdots + \frac{1}{2} I_{m_2} (\dot{\theta}_1 + \dot{\theta}_{m_2})^2 \simeq \cdots + \frac{1}{2} I_{m_2} \dot{\theta}_{m_2}^2, \tag{1}$$

and the off-diagonal terms $M_{14} = M_{41} = I_{m_2}$ of the inertia matrix would disappear. As a result, the matrix would become block diagonal, with two 2×2 blocks (the second being diagonal) that pertain to the link kinetic energy and, respectively, to the motor kinetic energy. This assumption is quite realistic when the motors are connected to the driven links via transmissions with large reduction ratios (which is not, however, the present case), independently from the presence or not of elasticity in the transmissions.

The robot inertia matrix can be rewritten compactly using the following $p = 5$ dynamic coefficients

$$\begin{aligned} a_1 &= I_1 + m_1 d_{c_1}^2 + I_2 + m_2 l_1^2 + I_{m_2} + m_{m_2} l_1^2 \\ a_2 &= m_2 l_1 d_{c_2} \\ a_3 &= I_2 + m_2 d_{c_2}^2 \\ a_4 &= I_{m_1} \\ a_5 &= I_{m_2}, \end{aligned}$$

as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2 \cos\theta_2 & a_3 + a_2\cos\theta_2 & 0 & a_5 \\ a_3 + a_2 \cos\theta_2 & a_3 & 0 & 0 \\ 0 & 0 & a_4 & 0 \\ a_5 & 0 & 0 & a_5 \end{pmatrix}.$$

Thus, the inertial terms in the robot dynamic model can be given a linearly parametrized form as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot\theta_1 & (2\ddot\theta_1 + \ddot\theta_2)\cos\theta_2 & \ddot\theta_2 & 0 & \ddot\theta_{m2} \\ 0 & \ddot\theta_1 \cos\theta_2 & \ddot\theta_1 + \ddot\theta_2 & 0 & 0 \\ 0 & 0 & 0 & \ddot\theta_{m1} & 0 \\ 0 & 0 & 0 & 0 & \ddot\theta_1 + \ddot\theta_{m2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \boldsymbol{Y}_M(\boldsymbol{q}, \ddot{\boldsymbol{q}})\boldsymbol{a}.$$

We remark that the simplifying assumption (1) would eliminate from $\boldsymbol{Y}_M$ the presence of $\ddot\theta_{m2}$ in element $Y_{M,15}$ and of $\ddot\theta_1$ in $Y_{M,45}$, but not reduce the number of dynamic coefficients: as a matter of fact, $p = 5$ is the smallest possible number of such coefficients.

Exercise 2

The task Jacobian of the planar 4R robot is given by

$$\boldsymbol{J}(\boldsymbol{q}) = \ell \begin{pmatrix} -(s_1 + s_{12} + s_{123} + s_{1234}) & -(s_{12} + s_{123} + s_{1234}) & -(s_{123} + s_{1234}) & -s_{1234} \\ c_1 + c_{12} + c_{123} + c_{1234} & c_{12} + c_{123} + c_{1234} & c_{123} + c_{1234} & c_{1234} \end{pmatrix}, \quad (2)$$

and is used, together with its time derivative $\dot{\boldsymbol{J}}(\boldsymbol{q})$, in the first- and second-order differential mappings

$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{a} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{a} \in \mathbb{R}^2, \qquad \ddot{\boldsymbol{q}} \in \mathbb{R}^4.$$

Since a minimum norm solution is being sought at the acceleration level, we solve the second-order differential kinematics in the least squares sense using pseudoinversion as

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q}) \left(\boldsymbol{a} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right). \quad (3)$$

When $\boldsymbol{q} = \bar{\boldsymbol{q}} = (\, 0 \;\; \pi/2 \;\; 0 \;\; \pi/2 \,)^T$ [rad] and for $\ell = 0.2$ [m], the Jacobian becomes

$$\boldsymbol{J} := \boldsymbol{J}(\bar{\boldsymbol{q}}) = \begin{pmatrix} -0.4 & -0.4 & -0.2 & 0 \\ 0 & -0.2 & -0.2 & -0.2 \end{pmatrix}$$

Having $\boldsymbol{J}$ full rank[1], any desired task acceleration $\boldsymbol{a}$ in (3) will be exactly realized in the absence of bounds, or realized at least in direction (possibly in a scaled form) in case the joint accelerations

[1]This property is particularly strong in this case, since *all* 2×2 minors are nonsingular in this configuration.

bounds cannot be satisfied. Moreover, since the robot is at rest ($\dot{\boldsymbol{q}} = 0$) in $\boldsymbol{q} = \bar{\boldsymbol{q}}$, the relation (3) collapses into

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\boldsymbol{a}. \tag{4}$$

In this setting, the SNS (Saturation in the Null Space) method presented originally at the velocity level can be applied without any modification, except for the acceleration limits A_i's replacing the velocity ones.

The pseudoinverse solution in (4) provides the joint acceleration with minimum norm. We have

$$\ddot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^{\#}\boldsymbol{a} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}\boldsymbol{a} = \begin{pmatrix} -1.6667 & 1.6667 \\ -0.8333 & -0.8333 \\ 0 & -1.6667 \\ 0.8333 & -2.5 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -8.3333 \\ -4.1667 \\ 0 \\ 4.1667 \end{pmatrix} \text{ [rad/s}^2\text{]}. \tag{5}$$

The fourth joint acceleration violates the maximum limit, $\ddot{q}_{PS,4} = 4.1667 > 2 = A_4$, so this is not a feasible solution. Thus, we search for a feasible solution by using the SNS method.

In step 1 of the SNS method, we saturate the overdriven joint by setting $\ddot{q}_4 = A_3 = 2$ [rad/s^2]. Then, the original task is modified by removing the saturated contribution of the fourth joint acceleration (and discarding the associated column of $\boldsymbol{J}$). We rewrite this as

$$\boldsymbol{a}_1 = \boldsymbol{a} - \boldsymbol{J}_4 A_4 = \begin{pmatrix} 5 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ -0.2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0.4 \end{pmatrix} = \begin{pmatrix} -0.4 & -0.4 & -0.2 \\ 0 & -0.2 & -0.2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \boldsymbol{J}_{-4}\,\ddot{\boldsymbol{q}}_{-4},$$

where $\boldsymbol{J}_{-i}$ is the Jacobian obtained by deleting the ith column and, similarly, $\ddot{\boldsymbol{q}}_{-i}$ is the vector of joint accelerations without the ith component. We recompute next the contribution of the remaining active joints, by pseudoinverting the $\boldsymbol{J}_{-4}$ matrix for the modified task. We obtain

$$\ddot{\boldsymbol{q}}_{PS_{-4}} = \boldsymbol{J}_{-4}^{\#}\,\boldsymbol{a}_1 = \begin{pmatrix} -2.2222 & 3.3333 \\ -0.5556 & -1.6667 \\ 0.5556 & -3.3333 \end{pmatrix} \begin{pmatrix} 5 \\ 0.4 \end{pmatrix} = \begin{pmatrix} -9.7778 \\ -3.4444 \\ 1.4444 \end{pmatrix} \text{ [rad/s}^2\text{]},$$

to be completed with the additional choice $\ddot{q}_4 = A_4 = 2$. The first joint acceleration violates now its limit (on the negative side), $\ddot{q}_{PS_{-4},1} = -9.7778 < -9 = -A_1$. So. this is not yet a feasible solution and we proceed with the SNS method.

In step 2, we saturate also the first overdriven joint by setting $\ddot{q}_1 = -A_1 = -9$ [rad/s^2]. The original task is modified by removing both saturated acceleration contributions by the first and fourth joints (discarding the two associated columns of $\boldsymbol{J}$). We rewrite this as

$$\boldsymbol{a}_2 = \boldsymbol{a} - \boldsymbol{J}_4 A_4 - \boldsymbol{J}_1(-A_1) = \boldsymbol{a}_1 + \boldsymbol{J}_1 A_1 = \begin{pmatrix} 5 \\ 0.4 \end{pmatrix} + 9\begin{pmatrix} -0.4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.4 \\ 0.4 \end{pmatrix}$$

and

$$\boldsymbol{a}_2 = \begin{pmatrix} 1.4 \\ 0.4 \end{pmatrix} = \begin{pmatrix} -0.4 & -0.2 \\ -0.2 & -0.2 \end{pmatrix} \begin{pmatrix} \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \boldsymbol{J}_{-14}\,\ddot{\boldsymbol{q}}_{-14},$$

with obvious notation. We recompute next the contribution of the remaining active joints, by pseudoinverting the (now square and nonsingular) matrix $\boldsymbol{J}_{-23}$ for the modified task. We obtain

$$\ddot{\boldsymbol{q}}_{PS_{-14}} = \boldsymbol{J}_{-14}^{\#}\,\boldsymbol{a}_2 = \boldsymbol{J}_{-14}^{-1}\,\boldsymbol{a}_2 = \begin{pmatrix} -5 & 5 \\ 5 & -10 \end{pmatrix} \begin{pmatrix} 1.4 \\ 0.4 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \end{pmatrix} \text{ [rad/s}^2\text{]},$$

with $\ddot{q}_1 = -A_1 = -9$ and $\ddot{q}_4 = A_4 = 2$. All bounds are now satisfied and the obtained joint acceleration is feasible. Recomposing the complete vector, we have the solution

$$\ddot{\boldsymbol{q}}_{SNS} = \begin{pmatrix} -9 \\ -5 \\ 3 \\ 2 \end{pmatrix} \ [\mathrm{rad/s^2}], \qquad \text{with } \boldsymbol{J}\ddot{\boldsymbol{q}}_{SNS} = \boldsymbol{a} \ \text{ and } \ \|\ddot{\boldsymbol{q}}_{SNS}\| = 10.9087. \tag{6}$$

This feasible solution is the one having the least possible norm.

The solution (6) is not the only feasible one. As a matter of fact, one could have attempted a heuristic procedure to find a (set of) solution(s) in a reasonable but otherwise arbitrary way, e.g., by fixing one component of the input acceleration to one of its (upper or lower) limits, and working out then the rest of the solution. For this, reconsider the original equation to be solved, written explicitly in terms of a linear system in the joint accelerations (scaling the coefficients so as to become all integers):

$$\boldsymbol{J}(\bar{\boldsymbol{q}})\,\ddot{\boldsymbol{q}} = \boldsymbol{a} \quad \Rightarrow \quad \begin{pmatrix} -0.4 & -0.4 & -0.2 & 0 \\ 0 & -0.2 & -0.2 & -0.2 \end{pmatrix} \ddot{\boldsymbol{q}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{array}{r} 2\ddot{q}_1 + 2\ddot{q}_2 + \ddot{q}_3 = -25 \\ \ddot{q}_2 + \ddot{q}_3 + \ddot{q}_4 = 0. \end{array}$$

By inspection, we find that choosing $\ddot{q}_1 = -A_1 = -9$ will contribute at best to the solution of the first scalar equation, being this variable present only in this equation and having the largest coefficient (and thus the highest sensitivity). In addition, set parametrically $\ddot{q}_4 = \alpha$ in the second scalar equation. We have then

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \begin{pmatrix} -25 + 2A_1 \\ -\ddot{q}_4 \end{pmatrix} = \begin{pmatrix} -7 \\ -\alpha \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \begin{pmatrix} -7 + \alpha \\ 7 - 2\alpha \end{pmatrix},$$

with the components of the parametric solution that need to satisfy the bounds

$$\begin{array}{l} |\ddot{q}_2| \leq A_2 = 6 \\ |\ddot{q}_3| \leq A_3 = 4 \\ |\ddot{q}_4| \leq A_4 = 2 \end{array} \qquad \Longleftrightarrow \qquad \begin{array}{c} -6 \leq -7 + \alpha \leq 6 \\ -4 \leq 7 - 2\alpha \leq 4 \\ -2 \leq \alpha \leq 2 \end{array} \qquad \Rightarrow \quad \alpha \in [1.5, 2].$$

The feasible interval for α comes from the simultaneous intersection of the set of inequalities. Therefore, we have a parametrized family of feasible solutions in the form

$$\ddot{\boldsymbol{q}}(\alpha) = \begin{pmatrix} -9 \\ -7 + \alpha \\ 7 - 2\alpha \\ \alpha \end{pmatrix} \ [\mathrm{rad/s^2}], \qquad \text{with } \|\ddot{\boldsymbol{q}}(\alpha)\| = \sqrt{6\alpha^2 - 42\alpha + 179}, \qquad \alpha \in [1.5, 2]. \tag{7}$$

We see immediately that $\ddot{\boldsymbol{q}}_{SNS} = \ddot{\boldsymbol{q}}(\alpha = 2)$. Moreover, the quadratic polynomial in the norm of $\ddot{\boldsymbol{q}}(\alpha)$ has an unconstrained minimum at $\alpha = 3.5$, which is outside the interval $[1.5, 2]$ of feasibility for α. Therefore, the minimum norm is obtained at the upper limit $\alpha = 2$ of this closed interval, i.e., with $\ddot{\boldsymbol{q}}_{SNS}$. Any other feasible solution will have a larger norm than $\ddot{\boldsymbol{q}}_{SNS}$.

We finally remark that, in order to find a feasible joint acceleration, a different (say, more conventional) solution would have been to use pseudo-inversion with a scaling of the original task acceleration $\boldsymbol{a}$ (in intensity, but without a change in direction). This is done as follows. From (5), we compute the necessary scaling factor $s > 1$ as

$$s = \max\left\{ \frac{|\ddot{q}_{PS,i}|}{A_i}, i = 1, \ldots, 4 \right\} = \max\left\{ \frac{8.3333}{9}, \frac{4.1667}{6}, \frac{0}{4}, \frac{4.1667}{2} \right\} = 2.0833.$$

This value is imposed by the exceeding acceleration of the fourth joint. We compute then

$$\boldsymbol{a}_{scaled} = \frac{\boldsymbol{a}}{s} = \begin{pmatrix} 2.4 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \ddot{\boldsymbol{q}}_{PS,scaled} = \boldsymbol{J}^{\#}\boldsymbol{a}_{scaled} = \begin{pmatrix} -4 \\ -2 \\ 0 \\ 2 \end{pmatrix}.$$

Indeed, the obtained joint acceleration has lower norm than $\ddot{\boldsymbol{q}}_{SNS}$, but realizes in fact only $2.4/5 = 48\%$ of the desired task acceleration of the original problem.

Exercise 3

The interaction matrix of a generic point feature with image coordinates (u_i, v_i) is known to be

$$\boldsymbol{J}_{p_i}(u_i, v_i, Z_i) = \begin{pmatrix} -\dfrac{\lambda}{Z_i} & 0 & \dfrac{u_i}{Z_i} & \dfrac{u_i v_i}{Z_i} & -\left(\lambda + \dfrac{u_i^2}{\lambda}\right) & v_i \\ 0 & -\dfrac{\lambda}{Z_i} & \dfrac{v_i}{Z_i} & \lambda + \dfrac{v_i^2}{\lambda} & -\dfrac{u_i v_i}{Z_i} & -u_i \end{pmatrix},$$

with

$$\begin{pmatrix} \dot{u}_i \\ \dot{v}_i \end{pmatrix} = \boldsymbol{J}_{p_i}(u_i, v_i, Z_i) \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix},$$

where the parameter $\lambda > 0$ is the constant focal length of the camera and Z_i is the depth of the Cartesian point $\boldsymbol{P}_i \in \mathbb{R}^3$ in the pre-image of (u_i, v_i). The average position of n point features in the image plane has coordinates

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{n} \sum_{i=1}^n u_i \\ \dfrac{1}{n} \sum_{i=1}^n v_i \end{pmatrix} = \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} u_i \\ v_i \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} \dot{\bar{u}} \\ \dot{\bar{v}} \end{pmatrix} = \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} \dot{u}_i \\ \dot{v}_i \end{pmatrix} = \left(\frac{1}{n} \sum_{i=1}^{n} \boldsymbol{J}_{p_i}(u_i, v_i, Z_i) \right) \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix} = \bar{\boldsymbol{J}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{Z}) \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\lambda}{n} \sum_{i=1}^n \frac{1}{Z_i} & 0 & \frac{1}{n} \sum_{i=1}^n \frac{u_i}{Z_i} & \frac{1}{n} \sum_{i=1}^n \frac{u_i v_i}{Z_i} & -\left(\lambda + \frac{1}{n} \sum_{i=1}^n \frac{u_i^2}{\lambda}\right) & \bar{v} \\ 0 & -\frac{\lambda}{n} \sum_{i=1}^n \frac{1}{Z_i} & \frac{1}{n} \sum_{i=1}^n \frac{v_i}{Z_i} & \lambda + \frac{1}{n} \sum_{i=1}^n \frac{v_i^2}{\lambda} & -\frac{1}{n} \sum_{i=1}^n \frac{u_i v_i}{Z_i} & -\bar{u} \end{pmatrix} \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix},$$

with a dependence of the interaction matrix $\bar{\boldsymbol{J}}$ on the components of $\boldsymbol{u} = (u_1 \dots u_n)^T \in \mathbb{R}^n$, $\boldsymbol{v} = (v_1 \dots v_n)^T \in \mathbb{R}^n$, and $\boldsymbol{Z} = (Z_1 \dots Z_n)^T \in \mathbb{R}^n$.

Exercise 4

The n_b-dimensional geometric constraint

$$\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{q}_b - \boldsymbol{q}_{b,d} = \boldsymbol{0} \tag{8}$$

has a simple associated Jacobian

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial \boldsymbol{h}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \end{pmatrix}.$$

Therefore, considering the decomposition $\boldsymbol{q} = (\boldsymbol{q}_a, \boldsymbol{q}_b)$, the dynamic model of the constrained robot can be partitioned as follows

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{\tau}+\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{\lambda} \quad \Rightarrow \quad \begin{pmatrix} \boldsymbol{M}_{aa}(\boldsymbol{q}) & \boldsymbol{M}_{ab}(\boldsymbol{q}) \\ \boldsymbol{M}_{ab}^T(\boldsymbol{q}) & \boldsymbol{M}_{bb}(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{q}}_a \\ \ddot{\boldsymbol{q}}_b \end{pmatrix} + \begin{pmatrix} \boldsymbol{n}_a(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ \boldsymbol{n}_b(\boldsymbol{q},\dot{\boldsymbol{q}}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\tau}_a \\ \boldsymbol{\tau}_b \end{pmatrix} + \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{\lambda} \end{pmatrix},$$

where $\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})$ collects all non-inertial terms in the model and $\boldsymbol{\lambda} \in \mathbb{R}^{n_b}$ is the vector of Lagrange multipliers associated to the geometric constraints.
To obtain a reduced dynamic model (with only $n - n_b = n_a$ independent coordinates), we proceed in the general way by bordering $\boldsymbol{A}(\boldsymbol{q})$ with the rows of a matrix $\boldsymbol{D}(\boldsymbol{q})$, so as to obtain a square and nonsingular transformation matrix. The situation is particularly simple since $\boldsymbol{A}$ is constant, and so can be chosen $\boldsymbol{D}$. A globally valid choice is then

$$\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{O} \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{O} \end{pmatrix} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{F} \end{pmatrix}.$$

Thus, the following bidirectional mappings are established between the generalized velocity $\dot{\boldsymbol{q}}$ (and acceleration $\ddot{\boldsymbol{q}}$) and the pseudo-velocity $\boldsymbol{v}$ (and pseudo-acceleration $\dot{\boldsymbol{v}}$):

$$\boldsymbol{v} = \boldsymbol{D}\dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \end{pmatrix} \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_a, \qquad \dot{\boldsymbol{v}} = \ddot{\boldsymbol{q}}_a \qquad \Longleftrightarrow \qquad \dot{\boldsymbol{q}} = \boldsymbol{F}\boldsymbol{v} = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{O} \end{pmatrix} \boldsymbol{v}, \qquad \ddot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{O} \end{pmatrix} \dot{\boldsymbol{v}}.$$

Dropping dependencies, the reduced inertia matrix and the reduced non-inertial dynamic terms are computed as

$$\boldsymbol{F}^T\boldsymbol{M}\boldsymbol{F} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \end{pmatrix} \boldsymbol{M} \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{O} \end{pmatrix} = \boldsymbol{M}_{aa},$$
$$\boldsymbol{F}^T(\boldsymbol{\tau} - \boldsymbol{n}) = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \end{pmatrix} \begin{pmatrix} \boldsymbol{\tau}_a - \boldsymbol{n}_a \\ \boldsymbol{\tau}_b - \boldsymbol{n}_b \end{pmatrix} = \boldsymbol{\tau}_a - \boldsymbol{n}_a.$$

Therefore, taking into account that $\boldsymbol{q}_b = \boldsymbol{q}_{b,d}$ and $\dot{\boldsymbol{q}}_b = \ddot{\boldsymbol{q}}_b = \boldsymbol{0}$ from (8), the reduced dynamic model becomes

$$\boldsymbol{M}_{aa}(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\,\ddot{\boldsymbol{q}}_a + \boldsymbol{n}_a(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}) = \boldsymbol{\tau}_a,$$

while the Lagrange multipliers (i.e., the forces that will preserve the geometric constraints when attempting their violation) takes the expression

$$\boldsymbol{\lambda} = \boldsymbol{E}^T(\boldsymbol{M}\boldsymbol{F}\dot{\boldsymbol{v}} + \boldsymbol{n} - \boldsymbol{\tau}) = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{I} \end{pmatrix} \left(\begin{pmatrix} * \\ \boldsymbol{M}_{ab}^T \end{pmatrix} \ddot{\boldsymbol{q}}_a + \begin{pmatrix} * \\ \boldsymbol{n}_b \end{pmatrix} - \begin{pmatrix} * \\ \boldsymbol{\tau}_b \end{pmatrix} \right),$$

or, by expliciting the dependencies,

$$\boldsymbol{\lambda} = \boldsymbol{M}_{ab}^T(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\,\ddot{\boldsymbol{q}}_a + \boldsymbol{n}_b(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}) - \boldsymbol{\tau}_b.$$

We conclude with two extra comments. The following torque command, expressed as a function of the constrained robot state and of the arbitrary input torque $\boldsymbol{\tau}_a$,

$$\begin{aligned} \boldsymbol{\tau}_b &= \boldsymbol{M}_{ab}^T(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\,\ddot{\boldsymbol{q}}_a + \boldsymbol{n}_b(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}) \\ &= \boldsymbol{M}_{ab}^T(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\boldsymbol{M}_{aa}^{-1}(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\left(\boldsymbol{\tau}_a - \boldsymbol{n}_a(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0})\right) + \boldsymbol{n}_b(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}), \end{aligned}$$

will guarantee $\boldsymbol{\lambda} \equiv \boldsymbol{0}$ at all times, resulting in a feasible motion with minimal internal effort. On the other hand, the feedback linearizing control law that achieves (in a decoupled way) a desired value $\boldsymbol{a}$ for the acceleration $\ddot{\boldsymbol{q}}_a$ of the free variables and a desired constraint force $\boldsymbol{\lambda} = \boldsymbol{\lambda}_d$ is given by

$$\boldsymbol{\tau} = \begin{pmatrix} \boldsymbol{M}_{aa}(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}) \\ \boldsymbol{M}_{ab}^T \boldsymbol{q}_a, \boldsymbol{q}_{b,d}) \end{pmatrix} \boldsymbol{a} + \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{I} \end{pmatrix} \boldsymbol{\lambda}_d + \boldsymbol{n}(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}),$$

or

$$\begin{aligned} \boldsymbol{\tau}_a &= \boldsymbol{M}_{aa}(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\, \boldsymbol{a} + \boldsymbol{n}_a(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}), \\ \boldsymbol{\tau}_b &= \boldsymbol{M}_{ab}^T(\boldsymbol{q}_a, \boldsymbol{q}_{b,d})\, \boldsymbol{a} + \boldsymbol{n}_b(\boldsymbol{q}_a, \boldsymbol{q}_{b,d}, \dot{\boldsymbol{q}}_a, \boldsymbol{0}) + \boldsymbol{\lambda}_d. \end{aligned}$$

* * * * *

Robotics 2

Midterm test in classroom – April 29, 2019

Exercise 1

Determine which of the following four 2×2 matrices can be, under the stated conditions, the inertia matrix associated to a real 2-dof robot with coordinates q_1 and q_2, and which not (and why):

DH COORDINATES

MUST NOT BE q_1 DEPEND.

$$M_A = \begin{pmatrix} a_1 + a_2 q_1^2 & a_2 \\ a_2 & a_2 \end{pmatrix}, \qquad a_1 > a_2 > 0;$$

$\det = a_2(a_2 q_1^2 + a_1 - a_2) > 0$ ALWAYS IF $a_1 > a_2 > 0$

ABS. COORDINATES

$$M_B = \begin{pmatrix} a_1 & a_3 \cos(q_2 - q_1) \\ a_3 \cos(\boxed{q_2} - \boxed{q_1}) & a_2 \end{pmatrix}, \qquad a_1 > 0, \quad a_1 a_2 > a_3^2 > 0;$$

$\rightarrow$ THIS IS CORRECT

$\det = a_1 a_2 - a_3^2 c_{12}^2 \geq a_1 a_2 - a_3^2 > 0$

since $a_1 a_2 > a_3^2$

$$M_C = \begin{pmatrix} a_1 + 2a_2 \cos q_2 & a_1 + a_2 \cos q_2 \\ a_1 + a_2 \cos q_2 & a_1 \end{pmatrix}, \qquad a_1 > 2a_2 > 0;$$

> 0

BUT $\det \leq 0$ ALWAYS

$$M_D = \begin{pmatrix} m_1 + m_2 & -0.5m_2 \\ -0.5m_2 & m_2 \end{pmatrix}, \qquad m_1 > 0, \quad m_2 > 0.$$

$m_1 + m_2 > 0$ $\det > 0$ $\rightarrow$ THIS IS CORRECT

For each case that is feasible, sketch the possible structure of the associated robot.

Exercise 2

Consider the 3-dof robot in Fig. 1, moving on a horizontal plane.

- Using the generalized coordinates $\boldsymbol{q} \in \mathbb{R}^3$ and the dynamic parameters defined in Fig. 1, determine the dynamic model of this robot using a Lagrangian formulation.
- Assume that the kinematic parameters are known. Provide a linear parameterization of the dynamic model in the form

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{u},$$

being $\boldsymbol{u} \in \mathbb{R}^3$ the generalized force provided by the motors at the joints, such that the vector of dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^p$ has the minimum possible dimension p.

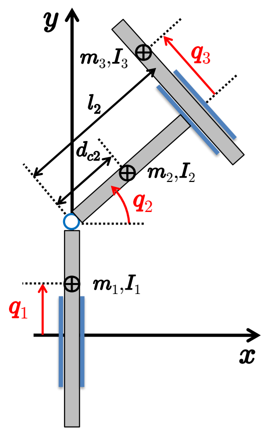


Figure 1: A 3-dof planar PRP robot, with its associated coordinates $\boldsymbol{q}$ and dynamic data.

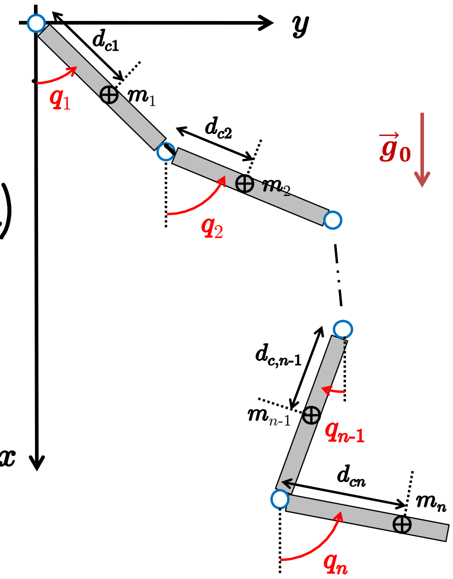
Exercise 3

$\vec{g}_0 = \begin{pmatrix} g_0 \\ 0 \end{pmatrix}$

The nR planar robot in Fig. 2 moves in the vertical plane under gravity. Each link has length $l_i > 0$ and has its center of mass on the kinematic axis at a distance $d_{ci} \gtreqless 0$ from the driving joint. Using the absolute coordinates $\boldsymbol{q}$ shown in Fig. 2, determine: *i)* the generic expressions of the components of the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ in the robot dynamic model; *ii)* all equilibrium configurations of the robot (i.e., all $\boldsymbol{q}_e \in \mathbb{R}^n$ such that $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$); and, *iii)* the generic conditions on the center of mass of each link such that the gravity vector vanishes identically (i.e., $\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}, \forall \boldsymbol{q}$).

$U = \sum_{i=1}^{n} -m_i g_0 P_{ci,x}$

$P_{ci,x} = \sum_{j=1}^{i-1} l_j \cdot \cos(q_j) + d_{ci} \cos(q_i)$



$\frac{\partial U}{\partial q} = \frac{\partial}{\partial q} \sum_{i=1}^{n} -m_i g_0 P_{ci,x}$

$= \sum_{i=1}^{n} -m_i g_0 \frac{\partial}{\partial q} P_{ci,x}$

Figure 2: A nR planar robot, with its associated absolute coordinates $\boldsymbol{q}$

Exercise 4

a) A 2-dof robot has the following inertia matrix:

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 & a_2 \sin q_2 \\ a_2 \sin q_2 & a_3 \end{pmatrix}. \tag{1}$$

Find two matrices $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ that factorize the centrifugal/Coriolis terms in $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ (i.e., with $\boldsymbol{S}_i(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, for $i = 1, 2$), such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}_1$ is skew-symmetric and $\dot{\boldsymbol{M}} - 2\boldsymbol{S}_2$ is not.

b) Consider a robot with $n = 3$ joints, inertia matrix $\boldsymbol{M}(\boldsymbol{q}) > 0$, and centrifugal/Coriolis terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$, where the 3×3 factorizing matrix $\boldsymbol{S}$ is obtained through the Christoffel symbols. Show that one can always find another factorizing matrix $\boldsymbol{S}' \neq \boldsymbol{S}$ that satisfies $\boldsymbol{S}'(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and leads to a skew-symmetric matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}'$.

Exercise 5

For a 2R planar robot with links of length l_1 and l_2 and inertia matrix $J^{\#}_{M} = M^{-1} J^{T} (J M^{-1} J^{T})^{-1}$

$$M(q) = \begin{pmatrix} a_1 + 2a_2 \cos q_2 & a_3 + a_2 \cos q_2 \\ a_3 + a_2 \cos q_2 & a_3 \end{pmatrix}, \tag{2}$$

the task requires the end-effector to be moved so as to change its distance $\rho = \rho(q)$ from the base according to a desired smooth timing law $\rho_d(t)$. Define the instantaneous joint velocity $\dot{q}$ that realizes the task while minimizing the robot kinetic energy, and provide the symbolic expression of all the terms needed for its evaluation. Compute then the numerical value of $\dot{q}$ with the data:

$$q_1 = 0, \quad q_2 = \frac{\pi}{2}, \quad l_1 = l_2 = 1 \text{ [m]}, \qquad a_1 = 10, \quad a_2 = 2.5, \quad a_3 = \frac{5}{3}, \qquad \dot{\rho}_d = 0.5 \text{ [m/s]}. \tag{3}$$

Exercise 6

A 3R planar robot with links of unitary length is at rest in the Denavit-Hartenberg configuration $q = \begin{pmatrix} \pi/6 & \pi/6 & \pi/6 \end{pmatrix}$ [rad] and should instantaneously accelerate its end effector at $\ddot{p} = \begin{pmatrix} 4 & 2 \end{pmatrix}$ [m/s^2]. Find, if possible, a joint acceleration $\ddot{q} \in \mathbb{R}^3$ with the least possible norm that perfectly realizes the task under the bounds

$$|\ddot{q}_1| \leq 2.8 \text{ [rad/s}^2\text{]}, \qquad |\ddot{q}_2| \leq 3.6 \text{ [rad/s}^2\text{]}, \qquad |\ddot{q}_3| \leq 4 \text{ [rad/s}^2\text{]}. \tag{4}$$

[210 minutes (3.5 hours); open books, computer, but no internet and no smartphone]

Solution

April 29, 2019

Exercise 1

A) Matrix $\boldsymbol{M}_A$ is not a robot inertia matrix because it is a function of the first coordinate q_1 at the robot base. This can never be the case: the definition of q_1 is in fact arbitrary, as is the choice of a base frame for the robot. The inertia matrix is instead an intrinsic property of the manipulator structure.

B) $\boldsymbol{M}_B$ is the inertia matrix of a 2-dof robot with a parallelogram structure (see left of Fig. 3, which is taken from the lecture slides), in which we have $a_1 = I_{c1,zz} + m_1 l_{c1}^2 + I_{c3,zz} + m_3 l_{c3}^2 + m_4 l_1^2$, $a_2 = I_{c2,zz} + m_2 l_{c2}^2 + I_{c4,zz} + m_4 l_{c3}^2 + m_3 l_2^2$, and $a_3 = m_3 l_2 l_{c3} - m_4 l_1 l_{c2}$. Absolute coordinates have been used therein. The dependence of $\boldsymbol{M}_B$ only on the difference $q_2 - q_1$ confirms that a robot inertia matrix can only be a function of the internal configuration (thus, it does not depend on the choice of the base reference frame). This can be seen also analytically, by applying the linear change of coordinates

$$\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \boldsymbol{J}\boldsymbol{q}.$$

In the new coordinates $\boldsymbol{p} \in \mathbb{R}^2$, the transformed inertia matrix becomes

$$\widetilde{\boldsymbol{M}}_B = \boldsymbol{J}^{-T} \boldsymbol{M}_B \boldsymbol{J}^{-1} = \begin{pmatrix} a_1 + 2a_3 \cos p_2 & a_2 + a_3 \cos p_2 \\ a_2 + a_3 \cos p_2 & a_3 \end{pmatrix},$$

which is a function of the second coordinate p_2 only.

C) Matrix $\boldsymbol{M}_C$ resembles the inertia matrix of the usual 2R planar robot, but is not. A dynamic coefficient $a_3 > 0$ is missing in $M(1,1)$, and this destroys the positive definiteness for every $\boldsymbol{q}$, as it should be instead for an inertia matrix. In fact, the determinant

$$\det \boldsymbol{M}_C = -a_2^2 \cos^2 q_2$$

is never positive.

D) $\boldsymbol{M}_D$ is the inertia matrix of a 2P robot, with the second prismatic joint having a twist angle of $\alpha = \pm 120°$ w.r.t. the first prismatic joint. The first link has mass m_1 and the second link has mass m_2. A sketch of this robot is shown on the right of Fig. 3.

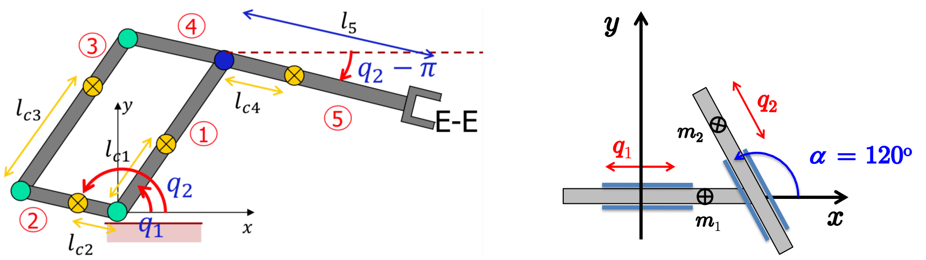


Figure 3: [left] 2-dof parallelogram robot; [right] PP robot with second axis twisted by $\alpha = 120°$.

Exercise 2

Following a Lagrangian approach, we derive the kinetic energy of the robot, $T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T_1 + T_2 + T_3$. Using König theorem[1],

$$T_i = \frac{1}{2}\, m_i \|\boldsymbol{v}_{ci}\|^2 + \frac{1}{2}\, \boldsymbol{\omega}_i^T \boldsymbol{I}_i \boldsymbol{\omega}_i,$$

and performing computations for each link, one has

$$T_1 = \tfrac{1}{2}\, m_1 \dot{q}_1^2,$$

$$\boldsymbol{p}_{c2} = \begin{pmatrix} d_{c2}\cos q_2 \\ q_1 + k_1 + d_{c2}\sin q_2 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -d_{c2}\sin q_2\, \dot{q}_2 \\ \dot{q}_1 + d_{c2}\cos q_2\, \dot{q}_2 \end{pmatrix},$$

$$T_2 = \tfrac{1}{2}\, m_2 \left(\dot{q}_1^2 + d_{c2}^2\, \dot{q}_2^2 + 2\, d_{c2}\cos q_2\, \dot{q}_1\dot{q}_2\right) + \tfrac{1}{2}\, I_2\, \dot{q}_2^2,$$

where k_1 is the distance from the center of mass of link 1 to the axis of joint 2 (an irrelevant constant), and

$$\boldsymbol{p}_{c3} = \begin{pmatrix} l_2\cos q_2 - q_3\sin q_2 \\ q_1 + k_1 + l_2\sin q_2 + q_3\cos q_2 \end{pmatrix} \Rightarrow \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} -\left(l_2\sin q_2 + q_3\cos q_2\right)\dot{q}_2 - \sin q_2\,\dot{q}_3 \\ \dot{q}_1 + \left(l_2\cos q_2 - q_3\sin q_2\right)\dot{q}_2 + \cos q_2\,\dot{q}_3 \end{pmatrix},$$

$$T_3 = \tfrac{1}{2}\, m_3 \left(\dot{q}_1^2 + \left(l_2^2 + q_3^2\right)\dot{q}_2^2 + \dot{q}_3^2 + 2\left(l_2\cos q_2 - q_3\sin q_2\right)\dot{q}_1\dot{q}_2 + 2\cos q_2\,\dot{q}_1\dot{q}_3 + 2 l_2\,\dot{q}_2\dot{q}_3\right) + \tfrac{1}{2}\, I_3\, \dot{q}_2^2.$$

Thus

$$T = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \begin{pmatrix} m_1 + m_2 + m_3 & m_3\left(l_2\cos q_2 - q_3\sin q_2\right) + m_2 d_{c2}\cos q_2 & m_3\cos q_2 \\ & I_2 + m_2 d_{c2}^2 + I_3 + m_3\left(l_2^2 + q_3^2\right) & m_3 l_2 \\ \mathit{symm} & & m_3 \end{pmatrix} \dot{\boldsymbol{q}}.$$

Introduce now a (minimal) parametrization of the robot inertia matrix, collecting the 4 dynamic coefficients that appear in $\boldsymbol{M}(\boldsymbol{q})$ and defined as follows[2]:

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} m_1 + m_2 + m_3 \\ I_2 + m_2 d_{c2}^2 + I_3 \\ m_3 \\ m_2 d_{c2} \end{pmatrix}. \tag{5}$$

As a result, the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ takes the linearly parametrized form

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 & a_3\left(l_2\cos q_2 - q_3\sin q_2\right) + a_4\cos q_2 & a_3\cos q_2 \\ a_3\left(l_2\cos q_2 - q_3\sin q_2\right) + a_4\cos q_2 & a_2 + a_3\left(l_2^2 + q_3^2\right) & a_3 l_2 \\ a_3\cos q_2 & a_3 l_2 & a_3 \end{pmatrix}. \tag{6}$$

The components of the Coriolis and centrifugal vector are computed from (6) using the Christoffel's symbols

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{M}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{M}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i}\right),$$

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[1] Being the motion planar, we restrict all linear vectors (e.g., $\boldsymbol{p}_{c2}$ or $\boldsymbol{v}_{c2}$) to be two-dimensional (living in the plane $(\boldsymbol{x}, \boldsymbol{y})$), while angular velocities are just scalars (the component of $\boldsymbol{\omega}_i$ along the $\boldsymbol{z}$-axis normal to the plane).

[2] As indicated in the text, we consider the kinematic parameter l_2 to be known.

being $\boldsymbol{M}_i$ the ith column of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. We have

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & -a_3\sin q_2 \\ 0 & -a_3\sin q_2 & 0 \end{pmatrix}\right. \\
&\qquad \left. + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & -a_3\sin q_2 \\ 0 & -a_3\sin q_2 & 0 \end{pmatrix} - \mathbf{0}\right) \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & -a_3\sin q_2 \\ 0 & -a_3\sin q_2 & 0 \end{pmatrix} \\
\Rightarrow \qquad c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= -\left(a_3(l_2\sin q_3 + q_3\cos q_2) + a_4\sin q_2\right)\dot{q}_2^2 - 2\,a_3\sin q_2\,\dot{q}_2\dot{q}_3.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\boldsymbol{C}_2(\boldsymbol{q}) &= \frac{1}{2}\left(\begin{pmatrix} 0 & -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & -a_3\sin q_2 \\ 0 & 0 & 2a_3q_3 \\ 0 & 0 & 0 \end{pmatrix}\right. \\
&\qquad + \begin{pmatrix} 0 & 0 & 0 \\ -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & 0 & 0 \\ -a_3\sin q_2 & 2a_3q_3 & 0 \end{pmatrix} \\
&\left. - \begin{pmatrix} 0 & -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & -a_3\sin q_2 \\ -a_3(l_2\sin q_2 + q_3\cos q_2) - a_4\sin q_2 & 0 & 0 \\ -a_3\sin q_2 & 0 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_3q_3 \\ 0 & a_3q_3 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 2\,a_3q_3\,\dot{q}_2\dot{q}_3
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{C}_3(\boldsymbol{q}) &= \frac{1}{2}\left(\begin{pmatrix} 0 & -a_3\sin q_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -a_3\sin q_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -a_3\sin q_2 & 0 \\ -a_3\sin q_2 & 2a_3q_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_3q_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -a_3q_3\,\dot{q}_2^2.
\end{aligned}$$

Summarizing, we have the final expression:

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -\left(a_3(l_2\sin q_2 + q_3\cos q_2) + a_4\sin q_2\right)\dot{q}_2^2 - 2\,a_3\sin q_2\,\dot{q}_2\dot{q}_3 \\ 2\,a_3q_3\,\dot{q}_2\dot{q}_3 \\ -a_3q_3\,\dot{q}_2^2 \end{pmatrix}. \tag{7}$$

Using the inertia matrix in (6) and the quadratic velocity vector in (7), the complete robot dynamic model can be expressed in the linearly parametrized form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}.\ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{u},$$

where $\boldsymbol{a} \in \mathbb{R}^4$ is defined in (5) and the 3×4 matrix $\boldsymbol{Y}$ is given by

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 & 0 & & \cos q_2\, \ddot{q}_2 - \sin q_2\, \dot{q}_2^2 \\ 0 & \ddot{q}_2 & \boldsymbol{Y}_3 & \cos q_2\, \ddot{q}_1 \\ 0 & 0 & & 0 \end{pmatrix}, \tag{8}$$

being the third column

$$\boldsymbol{Y}_3 = \begin{pmatrix} (l_2 \cos q_2 - q_3 \sin q_2)\, \ddot{q}_2 + \cos q_2\, \ddot{q}_3 - (l_2 \sin q_2 + q_3 \cos q_2)\, \dot{q}_2^2 - 2 \sin q_2\, \dot{q}_2 \dot{q}_3 \\ (l_2 \cos q_2 - q_3 \sin q_2)\, \ddot{q}_1 + \left(l_2^2 + q_3^2\right) \ddot{q}_2 + l_2\, \ddot{q}_3 + 2q_3\, \dot{q}_2 \dot{q}_3 \\ \cos q_2\, \ddot{q}_1 + l_2\, \ddot{q}_2 + \ddot{q}_3 - q_3\, \dot{q}_2^2 \end{pmatrix}.$$

Exercise 3

The use of absolute coordinates makes it simpler to derive the solution for the gravity term in the general case of a planar (serial) robot arm with n revolute joints. Since

$$\boldsymbol{g}_0 = \begin{pmatrix} g_0 \\ 0 \\ 0 \end{pmatrix}, \qquad g_0 = 9.81 \ [\mathrm{m/s^2}],$$

the potential energy due to gravity of link j of the robot, for $j = 1, \dots, n$, is

$$U_j(\boldsymbol{q}) = -m_j \boldsymbol{g}_0^T \boldsymbol{r}_{0,cj} = -m_j g_0 \left(\sum_{k=1}^{j-1} l_k \cos q_k + d_{cj} \cos q_j \right). \tag{9}$$

The total potential energy due to gravity is then

$$\begin{aligned} U(\boldsymbol{q}) = \sum_{j=1}^{n} U_j(\boldsymbol{q}) &= -g_0 \sum_{j=1}^{n} m_j \left(\sum_{k=1}^{j-1} l_k \cos q_k + d_{cj} \cos q_j \right) \\ &= -g_0 \sum_{i=1}^{n} \left(\left(\sum_{j=i+1}^{n} m_j \right) l_i + m_i d_{ci} \right) \cos q_i \,. \end{aligned} \tag{10}$$

For instance, setting $n = 5$, we have from eq. (9)

$$\begin{aligned} U_1 &= -m_1 g_0\, d_{c1} \cos q_1 \\ U_2 &= -m_2 g_0\, (l_1 \cos q_1 + d_{c2} \cos q_2) \\ U_3 &= -m_3 g_0\, (l_1 \cos q_1 + l_2 \cos q_2 + d_{c3} \cos q_3) \\ U_4 &= -m_4 g_0\, (l_1 \cos q_1 + l_2 \cos q_2 + l_3 \cos q_3 + d_{c4} \cos q_4) \\ U_5 &= -m_5 g_0\, (l_1 \cos q_1 + l_2 \cos q_2 + l_3 \cos q_3 + l_4 \cos q_4 + d_{c5} \cos q_5) \\ U(\boldsymbol{q}) &= \sum_{j=1}^{5} U_j, \end{aligned}$$

where the structure of the last expression of $U(\boldsymbol{q})$ in eq. (10) can be easily recognized.

According to (10), the components $g_i(\boldsymbol{q})$ of the gravity vector $\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^n$ are

$$g_i(\boldsymbol{q}) = \frac{\partial U(\boldsymbol{q})}{\partial q_i} = g_0 \left(\left(\sum_{j=i+1}^{n} m_j \right) l_i + m_i d_{ci} \right) \sin q_i, \qquad i = 1, \ldots, n. \tag{11}$$

The unforced equilibrium configurations of the robot (independently of the values of kinematic and dynamic parameters) are then characterized by

$$q_i = q_{e,i} = \{0, \pi\}, \qquad \forall i \in \{1, \ldots, n\} \qquad \Rightarrow \qquad \boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}.$$

All these configurations correspond to the robot links being stretched or folded along the vertical axis $\boldsymbol{x}$. The total number of such equilibria is $N_e = 2^n$.

In order to have a robot always balanced with respect to gravity, i.e., $\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}$, the following conditions should hold from (11):

$$\left(\sum_{j=i+1}^{n} m_j \right) l_i + m_i d_{ci} = 0 \qquad \Longleftrightarrow \qquad d_{ci} = -\frac{\left(\sum_{j=i+1}^{n} m_j \right) l_i}{m_i} < 0, \qquad i = 1, \ldots, n-1,$$

and

$$d_{cn} = 0.$$

In words, the mass and location of the center of mass of each link should balance (at the associated joint axis) the total mass of the following links in the chain, as if it were concentrated at the end of the link. Any configuration $\boldsymbol{q} \in \mathbb{R}^n$ would then be an equilibrium.

Exercise 4

a) From the inertia matrix (1) of this robot, which is a PR planar arm already treated in the lecture slides, the matrices of Christoffel symbols and the velocity vector (containing only a centrifugal term in the present case) are computed as

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 \\ 0 & a_2 \cos q_2 \end{pmatrix}, \qquad \boldsymbol{C}_2(\boldsymbol{q}) = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} a_2 \cos q_2\, \dot{q}_2^2 \\ 0 \end{pmatrix} \tag{12}$$

with $a_2 = -m_2 d_{c2}$. The time derivative of the inertia matrix (1) is

$$\dot{\boldsymbol{M}} = \begin{pmatrix} 0 & a_2 \cos q_2\, \dot{q}_2 \\ a_2 \cos q_2\, \dot{q}_2 & 0 \end{pmatrix}.$$

A factorization $\boldsymbol{S}_1 \dot{\boldsymbol{q}}$ of vector $\boldsymbol{c}$ in (12) leading to the skew symmetry of $\dot{\boldsymbol{M}} - 2\boldsymbol{S}_1$ is found by using the matrices of Christoffel symbols (or just by trivial inspection). It is easy to check that the matrix

$$\boldsymbol{S}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} 0 & a_2 \cos q_2\, \dot{q}_2 \\ 0 & 0 \end{pmatrix} \tag{13}$$

provides

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}_1 = \begin{pmatrix} 0 & -a_2 \cos q_2\, \dot{q}_2 \\ a_2 \cos q_2\, \dot{q}_2 & 0 \end{pmatrix},$$

which satisfies the desired skew-symmetric property. On the other hand, a different feasible factorization that uses the matrix

$$\begin{aligned}\boldsymbol{S}_2(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \begin{pmatrix} \dot{q}_2 & a_2\cos q_2\,\dot{q}_2-\dot{q}_1 \\ 0 & 0 \end{pmatrix} \\ &= \boldsymbol{S}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) + \begin{pmatrix} \dot{q}_2 & -\dot{q}_1 \\ 0 & 0 \end{pmatrix} = \boldsymbol{S}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{S}_0(\dot{\boldsymbol{q}}), \quad \text{with } \boldsymbol{S}_0(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{0},\end{aligned} \tag{14}$$

satisfies $\boldsymbol{S}_2\dot{\boldsymbol{q}} = \boldsymbol{c}$ and provides

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}_2 = \begin{pmatrix} -2\dot{q}_2 & -a_2\sin q_2\,\dot{q}_2 + 2\dot{q}_1 \\ a_2\sin q_2\,\dot{q}_2 & 0 \end{pmatrix},$$

which is clearly not a skew-symmetric matrix.

b) We use here the simple fact that, when $\dot{\boldsymbol{q}} \in \mathbb{R}^3$, one has

$$\dot{\boldsymbol{q}} \times \dot{\boldsymbol{q}} = \boldsymbol{S}_0(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{pmatrix} 0 & -\dot{q}_3 & \dot{q}_2 \\ \dot{q}_3 & 0 & -\dot{q}_1 \\ -\dot{q}_2 & \dot{q}_1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \boldsymbol{0},$$

where $\boldsymbol{S}_0$ is by construction a skew-symmetric matrix. If a factorization $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ leading to a skew symmetric matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is available, then matrix $\boldsymbol{S}' = \boldsymbol{S} + \boldsymbol{S}_0 \neq \boldsymbol{S}$ will be another feasible factorization, since

$$\boldsymbol{S}'(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \left(\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{S}_0(\dot{\boldsymbol{q}})\right)\dot{\boldsymbol{q}} = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}).$$

Moreover, the matrix

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}' = \left(\dot{\boldsymbol{M}} - 2\boldsymbol{S}\right) + (-2\boldsymbol{S}_0)$$

is also skew symmetric, being the sum of two skew-symmetric matrices. Indeed, matrix $\boldsymbol{S}'$ is not obtained using the Christoffel symbols only, and it is also clear that an infinite number of such feasible factorizations exists, all leading to the skew-symmetric property.

This construction can be easily generalized to the case of arbitrary $n \geq 3$, by considering for instance a velocity vector $\dot{\boldsymbol{q}}_0 \in \mathbb{R}^n$ with all zero components but three consecutive ones, and an associated $n \times n$ skew-symmetric matrix $\boldsymbol{S}_0(\dot{\boldsymbol{q}}_0)$ with a single non-zero 3×3 block at the proper place on the diagonal, constructed as in the three-dimensional case.

Exercise 5

The direct kinematics of the 2R planar robot is

$$\boldsymbol{p} = \begin{pmatrix} l_1\cos q_1 + l_2\cos(q_1+q_2) \\ l_1\sin q_1 + l_2\sin(q_1+q_2) \end{pmatrix}.$$

Therefore, the distance of the robot end effector from the base is

$$\rho(\boldsymbol{q}) = \|\boldsymbol{p}\| = \sqrt{l_1^2 + l_2^2 + 2\,l_1 l_2\cos q_2}.$$

The task Jacobian is then a 1×2 matrix

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial\rho(\boldsymbol{q})}{\partial\boldsymbol{q}} = \begin{pmatrix} 0 & -\dfrac{l_1 l_2\sin q_2}{\|\boldsymbol{p}\|} \end{pmatrix}, \tag{15}$$

which, for $l_1 \neq l_2$, is full rank except when $q_2 = \{0, \pi\}$. For $l_1 = l_2$, the rank of $\boldsymbol{J}$ drops only at $q_2 = 0$, whereas the element $J_{12}(q_2)$ has a discontinuity at $q_2 = \pm\pi$ (the two limits for $q_2 \to +\pi$ and $q_2 \to -\pi$ are non-zero and different).

A joint velocity $\dot{\boldsymbol{q}}$ that realizes the task while minimizing the robot kinetic energy uses the inertia-weighted pseudoinverse of the task Jacobian,

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_{WPs} = \boldsymbol{J}^{\#}_{\boldsymbol{M}}(\boldsymbol{q})\,\dot{\rho}_d = \boldsymbol{M}^{-1}(\boldsymbol{q})\,\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{M}^{-1}(\boldsymbol{q})\,\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\dot{\rho}_d, \tag{16}$$

where the last equality holds provided that $\boldsymbol{J}$ is full rank. In this case, using in (16) the symbolic expressions (2) and (15), we obtain

$$\dot{\boldsymbol{q}}_{WPs} = -\frac{l_1 l_2 \sin q_2}{\|\boldsymbol{p}\| \cdot \det \boldsymbol{M}(\boldsymbol{q})}\begin{pmatrix} -(a_3 + a_2\cos q_2) \\ a_1 + 2a_2\cos q_2 \end{pmatrix} \cdot \left(\frac{(l_1 l_2 \sin q_2)^2 (a_1 + 2a_2\cos q_2)}{\|\boldsymbol{p}\|^2 \cdot \det \boldsymbol{M}(\boldsymbol{q})}\right)^{-1}\dot{\rho}_d,$$

where $\det \boldsymbol{M}(\boldsymbol{q}) = a_3(a_1 - a_3) - a_2^2\cos^2 q_2 > 0$. Simplifying, this yields

$$\dot{\boldsymbol{q}}_{WPs} = \frac{\|\boldsymbol{p}\|}{l_1 l_2 \sin q_2}\begin{pmatrix} \dfrac{a_3 + a_2\cos q_2}{a_1 + 2a_2\cos q_2} \\ -1 \end{pmatrix}\dot{\rho}_d. \tag{17}$$

Substituting in (17) the numerical data of the problem, we obtain finally

$$\dot{\boldsymbol{q}}_{WPs} = \begin{pmatrix} 0.1179 \\ -0.7071 \end{pmatrix}. \tag{18}$$

An alternative (and simpler) solution would have been to seek for the minimum norm joint velocity, i.e., using the pseudoinverse of $\boldsymbol{J}$,

$$\dot{\boldsymbol{q}}_{Ps} = \boldsymbol{J}^{\#}(\boldsymbol{q})\,\dot{\rho}_d = \begin{pmatrix} 0 \\ -\dfrac{\|\boldsymbol{p}\|}{l_1 l_2 \sin q_2} \end{pmatrix}\dot{\rho}_d,$$

leading to the numerical value

$$\dot{\boldsymbol{q}}_{Ps} = \begin{pmatrix} 0 \\ -0.7071 \end{pmatrix}. \tag{19}$$

Indeed, a different target is reached by the two solutions $\dot{\boldsymbol{q}}_{WPs}$ in (18) and $\dot{\boldsymbol{q}}_{Ps}$ in (19) in terms of the objective function that is being minimized. We have in fact

$$\frac{1}{2}\,\dot{\boldsymbol{q}}^T_{WPs}\,\boldsymbol{M}\,\dot{\boldsymbol{q}}_{WPs} = \min_{\dot{\boldsymbol{q}}:\,\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\rho}_d}\frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}\,\dot{\boldsymbol{q}} = 0.3472 < 0.4167 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T_{Ps}\,\boldsymbol{M}\,\dot{\boldsymbol{q}}_{Ps}$$

and, viceversa,

$$\frac{1}{2}\,\dot{\boldsymbol{q}}^T_{Ps}\,\dot{\boldsymbol{q}}_{Ps} = \min_{\dot{\boldsymbol{q}}:\,\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\rho}_d}\frac{1}{2}\,\dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}} = 0.25 < 0.2569 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T_{WPs}\,\dot{\boldsymbol{q}}_{WPs}.$$

Exercise 6

Using Denavit-Hartenberg coordinates, the Jacobian of a 3R planar robot with unitary links is

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix},$$

with the usual compact notation. At the configuration $\boldsymbol{q} = \begin{pmatrix} \pi/6 & \pi/6 & \pi/6 \end{pmatrix}^T$ [rad], it becomes

$$\boldsymbol{J} = \begin{pmatrix} -2.3660 & -1.8660 & -1 \\ 1.3660 & 0.5 & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_2 & \boldsymbol{J}_3 \end{pmatrix}. \tag{20}$$

Note that rank $\boldsymbol{J} = 2$. Moreover, its three 2×2 minors are all different from zero in this case. Since the robot is at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$), the second-order differential mapping to be inverted is simply

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}\ddot{\boldsymbol{q}}, \qquad \text{for} \quad \ddot{\boldsymbol{p}} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \ [\text{m/s}^2].$$

In order to get a solution that is feasible with respect to the hard bounds (4) and has the least possible norm, we can apply the SNS (Saturation in the Null Space) method on joint accelerations[3].

To start with, we look for a solution with minimum acceleration norm,

$$\ddot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^{\#}\ddot{\boldsymbol{p}} = \begin{pmatrix} 0.1715 & 0.9832 \\ -0.4686 & -0.6861 \\ -0.5314 & -1.0460 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.6524 \\ -3.2466 \\ -4.2175 \end{pmatrix}. \tag{21}$$

The third joint acceleration violates its maximum bound, $\ddot{q}_{PS,3} = -4.2175 < -4 = -A_3$, so this solution is not feasible. Next, we search for a feasible solution by saturating the (single) overdriven joint, i.e., setting $\ddot{q}_3 = -A_3 = -4$ [rad/s^2] (this is step 1 of the SNS algorithm). The original task is modified by removing the contribution of the saturated acceleration of the third joint (and by discarding the associated column of $\boldsymbol{J}$). We rewrite this as

$$\ddot{\boldsymbol{p}}_1 = \ddot{\boldsymbol{p}} - \boldsymbol{J}_3(-A_3) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot (-4) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2.3660 & -1.8660 \\ 1.3660 & 0.5 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \boldsymbol{J}_{-3}\, \ddot{\boldsymbol{q}}_{-3},$$

where $\boldsymbol{J}_{-i}$ is the matrix obtained by deleting the ith column from the Jacobian in (20) and, accordingly, $\ddot{\boldsymbol{q}}_{-i}$ is the vector of joint accelerations without the ith component. For the modified task, we compute the contribution of the two active joints by inverting matrix $\boldsymbol{J}_{-3}$, which is now square and nonsingular. Thus, we obtain the unique solution

$$\ddot{\boldsymbol{q}}_{-3} = (\boldsymbol{J}_{-3})^{-1}\, \ddot{\boldsymbol{p}}_1 = \begin{pmatrix} 0.3660 & 1.3660 \\ -1 & -1.7321 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.7321 \\ -3.4641 \end{pmatrix} \ [\text{rad/s}^2].$$

All bounds are now satisfied and the resulting joint acceleration, with the third component fixed at $\ddot{q}_3 = -4 = -A_3$, is feasible. Thus, there is no need to scale down the original task acceleration $\ddot{\boldsymbol{p}}$. Recomposing the complete joint acceleration vector, we have the solution

$$\ddot{\boldsymbol{q}}^* = \begin{pmatrix} 2.7321 \\ -3.4641 \\ -4 \end{pmatrix} \ [\text{rad/s}^2], \qquad \text{with } \boldsymbol{J}\ddot{\boldsymbol{q}}^* = \ddot{\boldsymbol{p}} \quad \text{and} \quad \|\ddot{\boldsymbol{q}}^*\| = 5.9552. \tag{22}$$

This solution is not unique, but the underlying SNS method which has been followed guarantees that a feasible solution of minimum norm has been obtained.

For instance, another feasible solution can be obtained by setting the acceleration of the second joint to its lower bound ($\ddot{q}_2 = -A_2 = -3.6$ [rad/s^2]), which is the saturation level that is closer

[3]The solution is quite intuitive in this case and could also be obtained without any knowledge of the SNS method.

to the value of this component in the unconstrained minimum norm solution (21), and adjusting accordingly the other two joint accelerations. Since

$$\ddot{\boldsymbol{q}}_{-2} = (\boldsymbol{J}_{-2})^{-1}\left(\ddot{\boldsymbol{p}} - \boldsymbol{J}_2 \cdot (-A_2)\right) = \begin{pmatrix} 0 & 0.7321 \\ -1 & -1.7321 \end{pmatrix} \begin{pmatrix} -2.7177 \\ 3.8 \end{pmatrix} = \begin{pmatrix} 2.7818 \\ -3.8641 \end{pmatrix} \text{ [rad/s}^2\text{]},$$

the complete solution $\ddot{\boldsymbol{q}}^{\diamond}$ is feasible but has slightly larger norm than the SNS solution $\ddot{\boldsymbol{q}}^{*}$ in (22):

$$\ddot{\boldsymbol{q}}^{\diamond} = \begin{pmatrix} 2.7818 \\ -3.6 \\ -3.8641 \end{pmatrix} \text{ [rad/s]}, \qquad \text{with } \boldsymbol{J}\ddot{\boldsymbol{q}}^{\diamond} = \ddot{\boldsymbol{p}} \quad \text{and} \quad \|\dot{\boldsymbol{q}}^{\diamond}\| = 5.9691.$$

Finally, it is easy to see that a solution having the acceleration of the first joint saturated at its upper bound, $\ddot{\boldsymbol{q}} = \begin{pmatrix} 2.8 & -3.6497 & -3.8144 \end{pmatrix}^T$, would instead be unfeasible (the second component is out of bounds).

* * * * *

Robotics II

June 17, 2019

Exercise 1

Consider the Kawasaki S030 robot with six revolute joints and a spherical wrist shown in Fig. 1. For every link mass m_i, $i = 1, \ldots, 6$, the location of the center of mass is shown graphically in the different views (see also the distributed extra sheet in larger size; please, disregard any numerical value therein). Note that the position of the center of mass of the fifth link (m_5) as well as that of the sixth link (m_6) coincide with the wrist center —a simplifying assumption. The robot has been calibrated and all kinematic quantities are thus known.

Determine the symbolic expression of the gravity vector $\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^6$ and a possible linear parametrization in terms of the unknown dynamic coefficients $\boldsymbol{a}_g \in \mathbb{R}^p$, with the smallest value of p. Find all equilibrium configurations $\boldsymbol{q}_e$ of the robot (i.e., such that $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$).

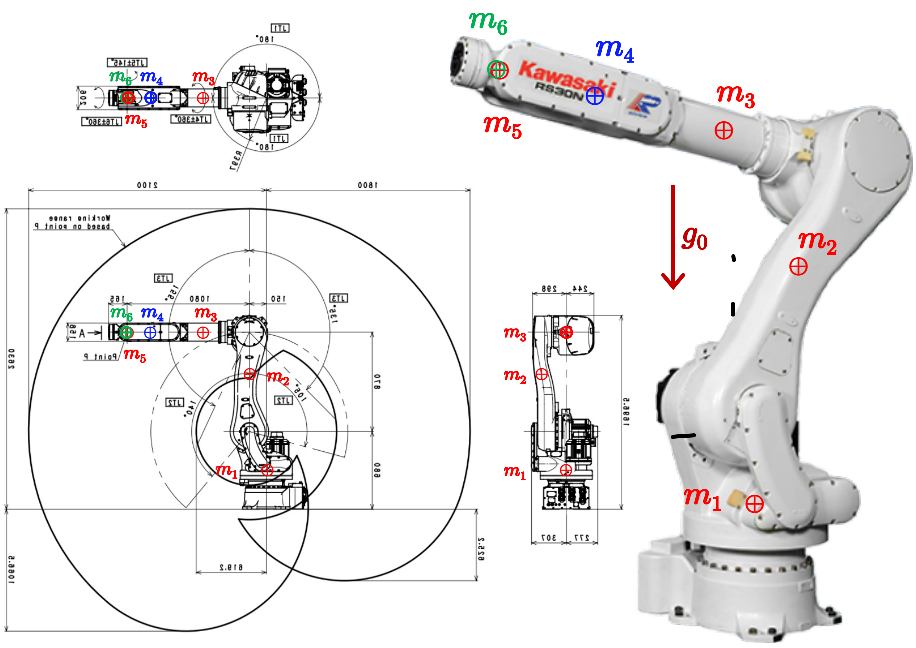


Figure 1: 6R Kawasaki S030 robot: Localization of the centers of mass of the six links.

Exercise 2

Consider the 3×3 matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + (a_2 \cos q_2 + a_3 \cos(q_2 + q_3))^2 & 0 & 0 \\ 0 & a_4 + a_5 + 2a_6 \cos q_3 & a_5 + a_6 \cos q_3 \\ 0 & a_5 + a_6 \cos q_3 & a_5 \end{pmatrix}.$$

Check whether this can be the inertia matrix of a 3-dof serial robot manipulator and, if so, under which conditions this holds true for the dynamic coefficients a_i ($i = 1, \ldots, 6$) appearing in $\boldsymbol{M}(\boldsymbol{q})$.

Exercise 3

A robot with $n > 3$ joints, parametrized by $\boldsymbol{q}$, is redundant w.r.t. a positional task $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{q})$ of dimension $m = 2$ or $m = 3$ (so, always with $m < n$), having task Jacobian $\boldsymbol{J}(\boldsymbol{q}) = \partial \boldsymbol{f}(\boldsymbol{q})/\partial \boldsymbol{q}$. At a configuration $\bar{\boldsymbol{q}}$ where rank $\boldsymbol{J}(\bar{\boldsymbol{q}}) = r < m$, consider the following two cases of joint velocity commands associated to a desired task velocity $\dot{\boldsymbol{x}}_d$:

$$\dot{\boldsymbol{q}}_A = \boldsymbol{J}^{\#}(\bar{\boldsymbol{q}})\,\dot{\boldsymbol{x}}_d, \qquad\qquad \dot{\boldsymbol{q}}_B = \boldsymbol{J}^T(\bar{\boldsymbol{q}})\,\dot{\boldsymbol{x}}_d.$$

Taking advantage of the Singular Value Decomposition of the matrix $\bar{\boldsymbol{J}} = \boldsymbol{J}(\bar{\boldsymbol{q}})$, show that:

i. In both cases, the actual $\dot{\boldsymbol{x}}$ can be different from the desired $\dot{\boldsymbol{x}}_d$, but the vectors $\dot{\boldsymbol{x}}_d$ and $\dot{\boldsymbol{x}}$ make always a relative angle that is smaller than $\pi/2$;

ii. When $\dot{\boldsymbol{x}}_d \in \mathcal{R}\left(\bar{\boldsymbol{J}}\right)$, $\dot{\boldsymbol{q}}_A$ gives no task velocity error, while $\dot{\boldsymbol{q}}_B$ leads in general to an error $\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}}_d - \dot{\boldsymbol{x}} \neq \boldsymbol{0}$.

Exercise 4

For a 2R robot moving on a horizontal plane, determine (in symbolic or numerical form) all terms in the expression of a control law producing a joint torque $\boldsymbol{\tau} \in \mathbb{R}^2$ that is able to regulate the robot end-effector position $\boldsymbol{p} \in \mathbb{R}^2$ to a desired constant value $\boldsymbol{p}_d$, with a transient error $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p}$ which globally satisfies, up to kinematic singularities, the differential equations

$$\ddot{\boldsymbol{e}} + \begin{pmatrix} 20 & 0 \\ 0 & 10 \end{pmatrix} \dot{\boldsymbol{e}} + \begin{pmatrix} 100 & 0 \\ 0 & 50 \end{pmatrix} \boldsymbol{e} = \boldsymbol{0}. \qquad .$$

When starting at rest from an initial configuration $\boldsymbol{q}_0$ that is not associated to $\boldsymbol{p}_d$, will the Cartesian behavior of the robot end-effector be oscillatory during the transient?

Exercise 5

A robot should slide a cube, firmly held by its end-effector gripper, on a flat surface, following an arbitrary path (see Fig. 2). For modeling purposes, assume that the surface is infinitely stiff and frictionless.

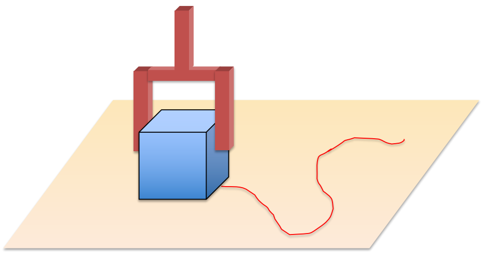


Figure 2: A cube sliding along a path on a flat surface.

Provide the set of natural constraints and a suitable set of artificial constraints for this interaction task with the environment. Consider both cases of a constant or arbitrary time-varying orientation of the cube while moving along the path. How many control loops of the generalized force or motion type are needed to achieve a perfectly linear and decoupled behavior in the task space? How many degrees of freedom are necessary for the robot in order to fulfil all control specifications? Can at least some of the desired control tasks be performed by a Scara robot? And by a 3R planar robot? If so, under which conditions?

[open books, 240 minutes]

Solution

June 17, 2019

Exercise 1

We need to define first a set of generalized coordinates $\boldsymbol{q}$, either by following a DH assignment of frames or by direct inspection of Fig. 1. The latter is more convenient here. In fact, it is rather easy to see that the definition of the link variables q_1 and (q_4, q_5, q_6) is irrelevant for the computation of the gravity term $\boldsymbol{g}(\boldsymbol{q})$ in the robot dynamics because:

- the rotation q_1 of joint 1 will not affect the height of the center of mass of any link;
- the position of the center of mass of link 4 will not depend on q_4, since m_4 lies exactly on the axis of joint 4;
- the position of the two center of masses m_5 and m_6 (which are assumed to be coincident[1] with the center of the spherical wrist) is independent from the joint variables of the wrist.

As a matter of fact, the potential energy of the system due to gravity, and thus the dynamic term $\boldsymbol{g}(\boldsymbol{q})$, will be a function of q_2 and q_3 only. Moreover, we will have $g_1 = g_4 = g_5 = g_6 \equiv 0$ for the components of $\boldsymbol{g}(q_2, q_3)$.

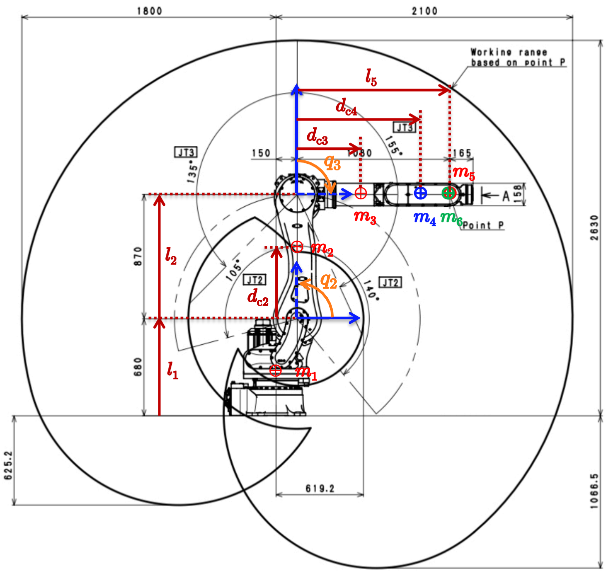


Figure 3: One possible choice of variables q_2 and q_3 for the second and third links of the Kawasaki robot, together with the definition kinematic/dynamic parameters.

With reference to Fig. 3, where the two joint variables q_2 and q_3 are defined (mimicking the classical assignment for the planar 2R case), together with the kinematic/dynamic parameters l_1, d_{c2}, l_2, d_{c3}, d_{c4} and l_5, we obtain

$$U_1 = \text{constant}, \quad U_2 = m_2 g_0 (l_1 + d_{c2} \sin q_2), \quad U_3 = m_3 g_0 (l_1 + l_2 \sin q_2 + d_{c3} \sin(q_2 + q_3)),$$

[1]This simplifying assumption is a very strong one, in particular concerning the location of m_6.

$$U_4 = m_4 g_0 (l_1 + l_2 \sin q_2 + d_{c4} \sin(q_2+q_3))\,, \quad U_5 + U_6 = (m_5+m_6) g_0 (l_1 + l_2 \sin q_2 + l_5 \sin(q_2+q_3))\,.$$

As a result,

$$\begin{aligned} U = \textstyle\sum_{i=1}^{6} U_i &= g_0\,(m_2 d_{c2} + (m_3+m_4+m_5+m_6) l_2) \sin q_2 \\ &\quad + g_0\,(m_3 d_{c3} + m_4 d_{c4} + (m_5+m_6) l_5) \sin(q_2+q_3) + \text{constants} \\ &= a_1 \sin q_2 + a_2 \sin(q_2+q_3) + \text{constants}. \end{aligned}$$

Thus,

$$\begin{aligned} \boldsymbol{g}(q_2,q_3) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T &= \begin{pmatrix} 0 \\ a_1 \cos q_2 + a_2 \cos(q_2+q_3) \\ a_2 \cos(q_2+q_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \cos q_2 & \cos(q_2+q_3) \\ 0 & cos(q_2+q_3) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \boldsymbol{Y_g}(q_2,q_3)\,\boldsymbol{a}, \end{aligned} \tag{1}$$

$q_2 = \pm\frac{\pi}{2}$

$q_2+q_3 = \pm\frac{\pi}{2} \mapsto q_3 = \pm\frac{\pi}{2} \pm\frac{\pi}{2}$

π OR 0

with a linear parametrization expressed in terms of only $p=2$ dynamic coefficients.

All equilibrium configurations $\boldsymbol{q}_e$ are found by setting $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$ in (1). We obtain

$$\boldsymbol{q}_e = \begin{pmatrix} \text{any} & \pm\pi/2 & 0 \text{ or } \pi & \text{any} & \text{any} & \text{any} \end{pmatrix}^T.$$

Exercise 2

The given matrix $\boldsymbol{M}(\boldsymbol{q})$ is symmetric and does not depend on q_1 —both are necessary conditions for being the inertia matrix of a serial manipulator. In order to be a positive definite matrix, it is necessary that all diagonal elements are strictly positive for all $\boldsymbol{q}$. This implies

$$a_1 > 0, \qquad a_4 + a_5 > 2|a_6| > 0, \qquad a_5 > 0. \tag{2}$$

The necessary and sufficient condition for positive definiteness of a symmetric matrix (Sylvester criterion) is that the leading minors are strictly positive (for all $\boldsymbol{q}$). Under (2), this boils down in checking that

$$\det \boldsymbol{M}_{[2:3]} = \det \begin{pmatrix} a_4 + a_5 + 2a_6 \cos q_3 & a_5 + a_6 \cos q_3 \\ a_5 + a_6 \cos q_3 & a_5 \end{pmatrix} > 0, \qquad \forall \boldsymbol{q}.$$

We have

$$\det \boldsymbol{M}_{[2:3]} = a_4 a_5 - a_6^2 \cos^2 q_3 > 0 \quad \Rightarrow \quad a_4 a_5 > a_6^2 \geq 0 \quad \Rightarrow \quad a_4 > 0, \tag{3}$$

the latter being implied by the previous condition $a_5 > 0$. Joining conditions (2) and (3) leads to the necessary and sufficient conditions[2]

$$a_1 > 0, \qquad a_4 > 0, \qquad a_5 > 0, \qquad a_4 + a_5 > 2|a_6|, \qquad a_4 a_5 > a_6^2. \tag{4}$$

[2] Matrix $\boldsymbol{M}(\boldsymbol{q})$ is in fact the inertia matrix of the robot considered in the midterm test of Robotics 2 during the academic year 2016/17, with some additional simplifying assumptions. As such, the conditions (4) are automatically satisfied in that case by the explicit expressions of the dynamic coefficients.

Exercise 3

We use the Singular Value Decomposition of the $m \times n$ matrix $\bar{\boldsymbol{J}} = \boldsymbol{J}(\bar{\boldsymbol{q}})$ (with $m < n$). From

$$\bar{\boldsymbol{J}} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \boldsymbol{U}\begin{pmatrix} \operatorname{diag}\{\sigma_1, \dots, \sigma_r\} & \boldsymbol{O}_{r\times(m-r)} & \multirow{2}{*}{$\boldsymbol{O}_{m\times(n-m)}$} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \dots, 0\} & \end{pmatrix}\boldsymbol{V}^T,$$

where $\boldsymbol{U} = \begin{pmatrix} \boldsymbol{u}_1 \ \boldsymbol{u}_2 \ \dots \ \boldsymbol{u}_r \ \boldsymbol{u}_{r+1} \ \dots \ \boldsymbol{u}_m \end{pmatrix}$ and $\boldsymbol{V}$ are two orthonormal matrices, respectively of dimension m and n, and the singular values of $\bar{\boldsymbol{J}}$ have been ordered as $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$, $\sigma_{r+1} = \dots = \sigma_m = 0$, we have

$$\bar{\boldsymbol{J}}^{\#} = \boldsymbol{V}\boldsymbol{\Sigma}^{\#}\boldsymbol{U}^T = \boldsymbol{V}\begin{pmatrix} \operatorname{diag}\left\{\frac{1}{\sigma_1}, \dots, \frac{1}{\sigma_r}\right\} & \boldsymbol{O}_{r\times(m-r)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \dots, 0\} \\ \multicolumn{2}{c}{\boldsymbol{O}_{(n-m)\times n}} \end{pmatrix}\boldsymbol{U}^T$$

and

$$\bar{\boldsymbol{J}}^T = \boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{U}^T = \boldsymbol{V}\begin{pmatrix} \operatorname{diag}\{\sigma_1, \dots, \sigma_r\} & \boldsymbol{O}_{r\times(m-r)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \dots, 0\} \\ \multicolumn{2}{c}{\boldsymbol{O}_{(n-m)\times n}} \end{pmatrix}\boldsymbol{U}^T.$$

Thus, the result of the two command choices $\dot{\boldsymbol{q}}_A = \bar{\boldsymbol{J}}^{\#}\dot{\boldsymbol{x}}_d$ and $\dot{\boldsymbol{q}}_B = \bar{\boldsymbol{J}}^T\dot{\boldsymbol{x}}_d$ is

$$\begin{aligned} \dot{\boldsymbol{x}}_A = \bar{\boldsymbol{J}}\dot{\boldsymbol{q}}_A &= \bar{\boldsymbol{J}}\bar{\boldsymbol{J}}^{\#}\dot{\boldsymbol{x}}_d = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{\Sigma}^{\#}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\#}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d \\ &= \boldsymbol{U}\begin{pmatrix} \boldsymbol{I}_{r\times r} & \boldsymbol{O}_{r\times(m-r)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \dots, 0\} \end{pmatrix}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d = \begin{pmatrix} \boldsymbol{u}_1 \ \boldsymbol{u}_2 \ \dots \ \boldsymbol{u}_r \ \boldsymbol{0} \ \dots \ \boldsymbol{0} \end{pmatrix}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d \end{aligned}$$

and

$$\begin{aligned} \dot{\boldsymbol{x}}_B = \bar{\boldsymbol{J}}\dot{\boldsymbol{q}}_B &= \bar{\boldsymbol{J}}\bar{\boldsymbol{J}}^T\dot{\boldsymbol{x}}_d = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T\boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{U}^T\dot{\boldsymbol{x}}_d = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\boldsymbol{U}^T\dot{\boldsymbol{x}}_d \\ &= \boldsymbol{U}\begin{pmatrix} \operatorname{diag}\{\sigma_1^2, \dots, \sigma_r^2\} & \boldsymbol{O}_{r\times(m-r)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \dots, 0\} \end{pmatrix}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d = \begin{pmatrix} \sigma_1^2\boldsymbol{u}_1 \ \sigma_2^2\boldsymbol{u}_2 \ \dots \ \sigma_r^2\boldsymbol{u}_r \ \boldsymbol{0} \ \dots \ \boldsymbol{0} \end{pmatrix}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d. \end{aligned}$$

Based on these expressions, we can immediately see that[3]

$$\dot{\boldsymbol{x}}_d^T\dot{\boldsymbol{x}}_A = \dot{\boldsymbol{x}}_d^T\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\#}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d = \dot{\boldsymbol{x}}_d^T\boldsymbol{U}\begin{pmatrix} \boldsymbol{I}_{r\times r} & \boldsymbol{O}_{r\times(m-r)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \dots, 0\} \end{pmatrix}^2\boldsymbol{U}^T\dot{\boldsymbol{x}}_d = \boldsymbol{w}^T\boldsymbol{w} = \|\boldsymbol{w}\|^2 \geq 0,$$

having set

$$\boldsymbol{w} = \begin{pmatrix} \boldsymbol{I}_{r\times r} & \boldsymbol{O}_{r\times(m-r)} \\ \boldsymbol{O}_{(m-r)\times r} & \operatorname{diag}\{0, \dots, 0\} \end{pmatrix}\boldsymbol{U}^T\dot{\boldsymbol{x}}_d.$$

Similarly, one can show that $\dot{\boldsymbol{x}}_B^T\dot{\boldsymbol{x}}_d \geq 0$. From the definition of the scalar products it follows that

$$\dot{\boldsymbol{x}}_d^T\dot{\boldsymbol{x}}_A = \|\dot{\boldsymbol{x}}_d\| \cdot \|\dot{\boldsymbol{x}}_A\|\cos\alpha_A \geq 0, \qquad \dot{\boldsymbol{x}}_d^T\dot{\boldsymbol{x}}_B = \|\dot{\boldsymbol{x}}_d\| \cdot \|\dot{\boldsymbol{x}}_B\|\cos\alpha_B \geq 0.$$

Therefore, each of the obtained Cartesian velocities $\dot{\boldsymbol{x}}_A$ and $\dot{\boldsymbol{x}}_B$ will form an angle $\alpha_i \leq \pi/2$, $i = A, B$, with the desired $\dot{\boldsymbol{x}}_d$.

[3] Matrix $\boldsymbol{A} := \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\#}$ is diagonal and idempotent. Thus we can write $\boldsymbol{A} = \boldsymbol{A}^2 = \boldsymbol{A}^T\boldsymbol{A}$.

Furthermore, when the desired Cartesian velocity $\dot{\boldsymbol{x}}_d$ is in the image of the Jacobian $\bar{\boldsymbol{J}}$, it is always spanned by the first r colums of the orthonormal matrix $\boldsymbol{U}$, namely

$$\dot{\boldsymbol{x}}_d \in \mathcal{R}\left(\bar{\boldsymbol{J}}\right) \qquad \Longrightarrow \qquad \dot{\boldsymbol{x}}_d = \sum_{i=1}^{r} \lambda_i \boldsymbol{u}_i = \left(\, \boldsymbol{u}_1 \; \boldsymbol{u}_2 \; \ldots \; \boldsymbol{u}_r \,\right) \boldsymbol{\lambda},$$

for a generic vector $\boldsymbol{\lambda} = \left(\, \lambda_1 \; \lambda_2 \; \ldots \; \lambda_r \,\right)^T \in \mathbb{R}^m$. In this case, the choice $\dot{\boldsymbol{q}}_A$ yields

$$\begin{aligned}
\dot{\boldsymbol{x}}_A = \bar{\boldsymbol{J}}\dot{\boldsymbol{q}}_A &= \left(\boldsymbol{u}_1 \; \boldsymbol{u}_2 \; \ldots \; \boldsymbol{u}_r \; \boldsymbol{0} \; \ldots \; \boldsymbol{0}\right) \boldsymbol{U}^T \dot{\boldsymbol{x}}_d \\
&= \left(\boldsymbol{u}_1 \; \boldsymbol{u}_2 \; \ldots \; \boldsymbol{u}_r \; \boldsymbol{0} \; \ldots \; \boldsymbol{0}\right) \begin{pmatrix} \boldsymbol{u}_1^T \\ \ldots \\ \boldsymbol{u}_r^T \\ \boldsymbol{u}_{r+1}^T \\ \ldots \\ \boldsymbol{u}_m^T \end{pmatrix} \left(\, \boldsymbol{u}_1 \; \boldsymbol{u}_2 \; \ldots \; \boldsymbol{u}_r \,\right) \boldsymbol{\lambda} \\
&= \left(\boldsymbol{u}_1 \; \boldsymbol{u}_2 \; \ldots \; \boldsymbol{u}_r \; \boldsymbol{0} \; \ldots \; \boldsymbol{0}\right) \begin{pmatrix} \boldsymbol{I}_{r\times r} \\ \boldsymbol{O}_{(m-r)\times r} \end{pmatrix} \boldsymbol{\lambda} = \left(\boldsymbol{u}_1 \; \boldsymbol{u}_2 \; \ldots \; \boldsymbol{u}_r\right) \boldsymbol{\lambda} = \dot{\boldsymbol{x}}_d
\end{aligned}$$

with no Cartesian velocity error. On the other hand, with the choice $\dot{\boldsymbol{q}}_B$ we do not generate the desired $\dot{\boldsymbol{x}}_d$:

$$\begin{aligned}
\dot{\boldsymbol{x}}_B = \bar{\boldsymbol{J}}\dot{\boldsymbol{q}}_B &= \left(\sigma_1^2 \, \boldsymbol{u}_1 \; \sigma_2^2 \, \boldsymbol{u}_2 \; \ldots \; \sigma_r^2 \, \boldsymbol{u}_r \; \boldsymbol{0} \; \ldots \; \boldsymbol{0}\right) \boldsymbol{U}^T \dot{\boldsymbol{x}}_d \\
&= \ldots = \left(\sigma_1^2 \, \boldsymbol{u}_1 \quad \sigma_2^2 \, \boldsymbol{u}_2 \quad \ldots \quad \sigma_r^2 \, \boldsymbol{u}_r\right) \boldsymbol{\lambda} \neq \dot{\boldsymbol{x}}_d.
\end{aligned}$$

Exercise 4

In order to achieve the desired linear and decoupled dynamics for the Cartesian error $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p}$ at the end-effector level, the torque command $\boldsymbol{\tau}$ in

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau}$$

for the controlled 2R planar robot cannot be chosen as the simple Cartesian PD regulation law

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q}) \left(\boldsymbol{K}_P \boldsymbol{e} - \boldsymbol{K}_D \dot{\boldsymbol{p}}\right), \qquad \boldsymbol{K}_P, \boldsymbol{K}_D > 0,$$

where $\dot{\boldsymbol{e}} = -\dot{\boldsymbol{p}}$ being $\dot{\boldsymbol{p}}_d = \boldsymbol{0}$. Rather, we should resort to a feedback linearization control law in the Cartesian space. In the considered case of a square and (assumed) nonsingular robot Jacobian, this law can be designed in two equivalent ways. Either by

$$\left.\begin{array}{l} \boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \quad \Rightarrow \quad \ddot{\boldsymbol{q}} = \boldsymbol{a} \\ \ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{K}_P \boldsymbol{e} - \boldsymbol{K}_D \dot{\boldsymbol{p}} \end{array}\right\} \quad \Rightarrow \quad \boldsymbol{a} = \boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{p}) - \boldsymbol{K}_D \dot{\boldsymbol{p}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right),$$

and thus

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q})) - \boldsymbol{K}_D \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{5}$$

with the matrix control gains chosen as

$$\boldsymbol{K}_P = \begin{pmatrix} 100 & 0 \\ 0 & 50 \end{pmatrix} > 0, \qquad \boldsymbol{K}_D = \begin{pmatrix} 20 & 0 \\ 0 & 10 \end{pmatrix} > 0. \tag{6}$$

Or, by using the Cartesian dynamics of the robot

$$\boldsymbol{M}_p(\boldsymbol{q})\ddot{\boldsymbol{p}} + \boldsymbol{c}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{F},$$

with

$$\boldsymbol{M}_p(\boldsymbol{q}) = \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}, \quad \boldsymbol{c}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{J}^{-T}(\boldsymbol{q}) \left(\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right),$$

and choosing

$$\left.\begin{array}{l} \boldsymbol{F} = \boldsymbol{M}_p(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{c}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}) \ \Rightarrow\ \ddot{\boldsymbol{p}} = \boldsymbol{a} \\ \ddot{\boldsymbol{p}} = \boldsymbol{K}_P\boldsymbol{e} - \boldsymbol{K}_D\dot{\boldsymbol{p}}, \quad \boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} \end{array}\right\} \Rightarrow \boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{M}_p(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{p}) - \boldsymbol{K}_D\dot{\boldsymbol{p}}\right) + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{c}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}).$$

By elaborating further the latter expression of $\boldsymbol{\tau}$,

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{p}) - \boldsymbol{K}_D\dot{\boldsymbol{p}}\right) + \left(\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) \\ &= \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{p}) - \boldsymbol{K}_D\dot{\boldsymbol{p}}\right) + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}, \end{aligned}$$

we recover (5) as expected.

The requested symbolic form of the terms in (5) are easily obtained for a 2R planar robot (see lecture slides). The kinematic terms are

$$\begin{aligned} \boldsymbol{p}(\boldsymbol{q}) &= \begin{pmatrix} l_1 \cos q_1 + l_2 \cos(q_1 + q_2) \\ l_1 \sin q_1 + l_2 \sin(q_1 + q_2) \end{pmatrix}, \\ \boldsymbol{J}(\boldsymbol{q}) &= \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -(l_1 \sin q_1 + l_2 \sin(q_1 + q_2)) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos q_1 + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{pmatrix}, \\ \dot{\boldsymbol{J}}(\boldsymbol{q}) &= -\begin{pmatrix} l_1 \cos q_1 \dot{q}_1 + l_2 \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) & l_2 \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) \\ l_1 \sin q_1 \dot{q}_1 + l_2 \sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) & l_2 \sin(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) \end{pmatrix}. \end{aligned}$$

The dynamic terms are

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q}) &= \begin{pmatrix} a_1 + 2a_2 \cos q_2 & a_3 + a_2 \cos q_2 \\ a_3 + a_2 \cos q_2 & a_3 \end{pmatrix}, \\ \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \begin{pmatrix} -a_2 \sin q_2 \left(2\dot{q}_1 + \dot{q}_2\right) \dot{q}_2 \\ a_2 \sin q_2 \, \dot{q}_1^2 \end{pmatrix}, \end{aligned}$$

with dynamic coefficients $a_1 = I_{c1,zz} + m_1 d_{c1}^2 + I_{c2,zz} + m_2 d_{c2}^2 + m_2 l_1^2 > 0$, $a_2 = m_2 l_1 d_{c2}$ and $a_3 = I_{c2,zz} + m_2 d_{c2}^2 > 0$. The numerical values used in (5) are those of the matrix gains given by (6).

In order to study the characteristics of the transient behavior of the error $\boldsymbol{e}(t) \to \boldsymbol{0}$, one may compute the roots of the following two algebraic equations in the Laplace domain:

$$\begin{aligned} \left(s^2 + 20s + 100\right) e_x(s) &= (s + 10)^2 \, e_x(s) = 0, \\ \left(s^2 + 10s + 50\right) e_y(s) &= (s + 5 - 5i)\,(s + 5 + 5i)\, e_y(s) = 0. \end{aligned}$$

As a result, when starting at rest with an initial $e_x(0) \neq 0$, $e_x(t)$ will converge to zero without overshoot (double negative real root), whereas $e_y(t)$ will converge to zero from an initial $e_y(0) \neq 0$

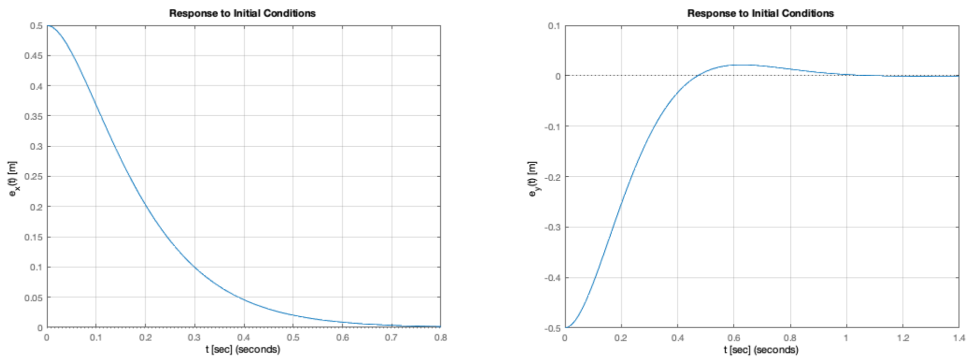


Figure 4: The evolution of $e_x(t)$ [left] and $e_y(t)$ [right] when the robot under control (5) starts at rest in a configuration $\boldsymbol{q}_0$ with initial Cartesian position error $\boldsymbol{e}(0) = (e_x(0), e_y(0)) = (0.5, -0.5) \neq \boldsymbol{0}$.

always changing its sign during the transient (a pair of complex conjugate roots, with negative real part). See the numerical example in Fig. 4.

Exercise 5

With reference to the task frame $(\boldsymbol{x}_t, \boldsymbol{y}_t, \boldsymbol{z}_t)$ shown in Fig. 5, the six natural constraints on the task are:

$$v_z = 0, \quad \omega_x = 0, \quad \omega_y = 0, \quad F_x = 0, \quad F_y = 0, \quad M_z = 0.$$

The six complementary artificial constraints specify the way in which the interaction task should be executed:

$$F_z = F_{z,d} < 0, \quad M_x = M_{x,d} = 0, \quad M_y = M_{y,d} = 0, \quad v_x = v_{x,d}, \quad v_y = v_{y,d}, \quad \omega_z = \omega_{z,d}\,.$$

Here, $|F_{z,d}| \neq 0$ is the intensity of the normal force to the plane that the robot should apply to the cube so as to keep one of its faces in permanent contact. The desired moments around the axes $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$ are both set to zero, so as to minimize the actual strain on the cube. The desired trajectory on the plane will be followed with a scalar speed $\dot{s} = \sqrt{v_{x,d}^2 + v_{y,d}^2} > 0$. Finally, the choice of either a constant or arbitrary time-varying orientation of the cube while moving along the path is made by setting either $\omega_{z,d} = 0$ or, respectively, $\omega_{z,d} = \omega_{z,d}(t)$.

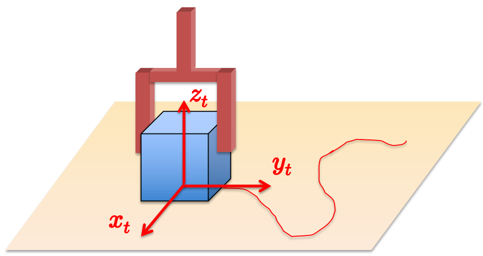


Figure 5: The instantaneous task frame associated to the cube moving on a flat surface.

As a result, in a hybrid force-velocity formulation, there will be three control loops on the generalized force components and three control loops on the planar motion components. Out of singularities, hybrid force-velocity control will achieve a perfectly linear and decoupled behavior of these six controlled outputs associated to the task space. In general, a robot with six degrees of freedom will be necessary in order to fulfil all control specifications.

With a Scara-type robot (four parallel joint axes, three revolute joints providing motion on a plane and a prismatic joint acting orthogonally), only four control specifications can be satisfied. If the joint axes of the robot are (perfectly) normal to the plane of motion of the cube, then the remaining two specifications $M_{x,d} = M_{y,d} = 0$ are automatically satisfied (although any desired value different from zero for these quantities could not be realized).

When using a 3R planar robot (with all revolute joint axes normal to the plane of motion of the cube), the task specification of a non-zero normal force along the axis $\boldsymbol{z}_t$ cannot be accomplished. The dofs of this robot are instead necessary and sufficient to execute a complete planar motion, with arbitrary values of $v_{x,d}$, $v_{y,d}$ and $\omega_{z,d}$.

* * * * *

Robotics II

July 11, 2019

Exercise 1

The 3R planar robot in Fig. 1 is commanded at the joint velocity level. The robot has to perform two tasks simultaneously, if possible. The first task is to keep the second link vertical and upwards at any time. The second task is to follow a desired cyclic Cartesian trajectory $\boldsymbol{p}_d(t) \in \mathbb{R}^2$, $t \in [0, T]$, for the end-effector position. Provide the actual expressions of all terms in a task priority control law, with the given order of tasks. Determine the robot configurations for which both tasks can be perfectly executed together, and define accordingly the region of the plane where this can happen. Which would be the control law in this case? With link lengths $L_1 = L_2 = L_3 = 0.5$ [m], compute the numerical value of $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ using the task priority law at $\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & -\pi/2 \end{pmatrix}^T$ [rad] for $\dot{\boldsymbol{p}}_d = \begin{pmatrix} 0.1 & -0.5 \end{pmatrix}^T$ [m/s]. Finally, when errors are present during the execution of these tasks, how should the control law be modified in order to reduce them?

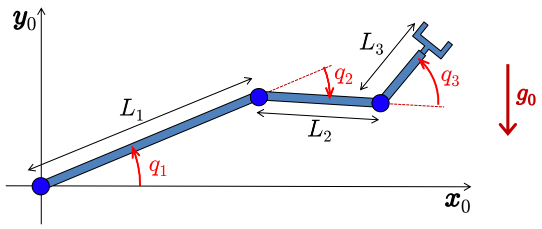


Figure 1: A 3R planar robot with its joint variables and generic link lengths.

Exercise 2

The RP planar robot shown in Fig. 2 lives in a vertical plane and may collide with some (human) obstacle when in motion. Its controller is therefore equipped with a momentum-based collision detection algorithm that generates a residual vector $\boldsymbol{r} \in \mathbb{R}^2$ as monitoring signal. Provide the explicit symbolic expressions of the two scalar components of $\boldsymbol{r}$ (introduce the needed kinematic and/or dynamic quantities). Suppose that, at time $t = t_c$, a collision occurs on the robot tip with an impact force $\boldsymbol{F}_c$ that is purely normal to the second link. What will be the instantaneous value of the time derivative of the residual vector $\dot{\boldsymbol{r}}(t_c)$?

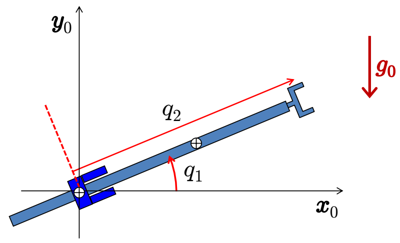


Figure 2: A RP robot moving in a vertical plane.

Exercise 3

An actuated pendulum under gravity should perform a rest-to-rest swing-up maneuver from the downward position $\theta(0) = 0$ to the upward position $\theta(T) = \pi$ in a total time T, using a bang-coast-bang acceleration profile with symmetric acceleration and deceleration phases, each of duration $T_s = T/4$. The link of the pendulum is a thin rod of length $l = 2$ [m], with uniformly distributed mass $m = 10$ [kg] and baricentral inertia $I_c = ml^2/12$ [kg·m^2]. The motor at the link base can deliver a maximum absolute torque $\tau_{max} = 200$ [Nm]. Determine the minimum time T_{min} in the chosen class of trajectories such that the motion is feasible. Sketch the resulting angular position, velocity, acceleration, and torque profiles.

[open books, 210 minutes]

Solution

July 11, 2019

Exercise 1

We will use throughout the DH coordinates indicated in Fig. 2. The first task, i.e., keeping the second link vertical and upwards, is one-dimensional ($m_1 = 1$) and is specified by

$$f_1(\boldsymbol{q}) = q_1 + q_2 = r_{d1} = \frac{\pi}{2} \qquad \Rightarrow \qquad \boldsymbol{J}_1 = \frac{\partial f_1(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T, \quad \dot{r}_{d1} = 0.$$

The second task, i.e., following a desired cyclic Cartesian trajectory $\boldsymbol{p}_d(t)$ with the robot tip, is two-dimensional ($m_2 = 2$) and is specified by

$$\boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} L_1 c_1 + L_2 c_{12} + L_3 c_{123} \\ L_1 s_1 + L_2 s_{12} + L_3 s_{123} \end{pmatrix} = \boldsymbol{r}_{d2} = \boldsymbol{p}_d(t)$$

$$\Rightarrow \quad \boldsymbol{J}_2(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_2(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -(L_1 s_1 + L_2 s_{12} + L_3 s_{123}) & -(L_2 s_{12} + L_3 s_{123}) & -L_3 s_{123} \\ L_1 c_1 + L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_3 c_{123} & L_3 c_{123} \end{pmatrix}, \quad \dot{\boldsymbol{r}}_{d2} = \dot{\boldsymbol{p}}_d.$$

with the usual compact notation for the trigonometric functions (e.g., $c_{12} = \cos(q_1 + q_2)$).

The basic Task Priority (TP) method for two ordered tasks provides

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_1^{\#} \dot{r}_{d1} + \left(\boldsymbol{I} - \boldsymbol{J}_1^{\#} \boldsymbol{J}_1\right) \boldsymbol{v}_1, \qquad \text{with} \quad \boldsymbol{v}_1 = \left(\boldsymbol{J}_2(\boldsymbol{q})(\boldsymbol{I} - \boldsymbol{J}_1^{\#} \boldsymbol{J}_1)\right)^{\#} \left(\dot{\boldsymbol{r}}_{d2} - \boldsymbol{J}_2(\boldsymbol{q}) \boldsymbol{J}_1^{\#} \dot{r}_{d1}\right), \tag{1}$$

where $\boldsymbol{P}_1 = \boldsymbol{I} - \boldsymbol{J}_1^{\#} \boldsymbol{J}_1$ is the (here, constant) projection matrix in the null space of the first task and no extra term has been used in the null space of the second task ($\boldsymbol{v}_2 = \boldsymbol{0}$). Since $\dot{r}_{d1} = 0$ in this case, and being $\boldsymbol{P}(\boldsymbol{J}\boldsymbol{P})^{\#} = (\boldsymbol{J}\boldsymbol{P})^{\#}$ for any projection matrix $\boldsymbol{P}$, equation (1) simplifies to

$$\dot{\boldsymbol{q}} = \left(\boldsymbol{J}_2(\boldsymbol{q}) \boldsymbol{P}_1\right)^{\#} \dot{\boldsymbol{p}}_d. \tag{2}$$

From

$$\boldsymbol{J}_1^{\#} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix}, \qquad \boldsymbol{P}_1 = \begin{pmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$\boldsymbol{J}_2(\boldsymbol{q}) \boldsymbol{P}_1 = \begin{pmatrix} -0.5 L_1 s_1 & 0.5 L_1 s_1 & -L_3 s_{123} \\ 0.5 L_1 c_1 & -0.5 L_1 c_1 & L_3 c_{123} \end{pmatrix}. \tag{3}$$

While the first two columns of the matrix in (3) are always dependent, it is easy to see that its rank is full unless $\sin(q_2 + q_3) = 0$. With the joint velocity command (2), the first task will always be satisfied if the constraint $f_1(\boldsymbol{q}) = \pi/2$ holds at the start, whereas the second task will be satisfied either exactly or in a least squares sense, depending on the current robot configuration and on the direction of the desired velocity $\dot{\boldsymbol{p}}_d$.

In order to verify when both tasks can be achieved simultaneously, we impose $q_1(t) + q_2(t) \equiv \pi/2$ at all times. From the direct kinematics of the robot tip $\boldsymbol{p} = \boldsymbol{f}_2(\boldsymbol{q})$, one obtains then the reduced form

$$\boldsymbol{p}_{\text{red}} = \boldsymbol{f}_2(\boldsymbol{q})|_{q_1+q_2=\pi/2} = \begin{pmatrix} L_1 c_1 - L_3 s_3 \\ L_1 s_1 + L_2 + L_3 c_3 \end{pmatrix} = \boldsymbol{p}_{\text{red}}(q_1, q_3).$$

In order to keep the constraint on the first task satisfied, we need to have $\dot{q}_2 = -\dot{q}_1$ for the second joint command. The two remaining joints $\boldsymbol{q}_{\text{red}} = \begin{pmatrix} q_1 & q_3 \end{pmatrix}^T$ will produce a tip velocity

$$\dot{\boldsymbol{p}}_{\text{red}} = \boldsymbol{J}_{\text{red}}(\boldsymbol{q}_{\text{red}}) \dot{\boldsymbol{q}}_{\text{red}}, \qquad \text{with} \quad \boldsymbol{J}_{\text{red}}(\boldsymbol{q}_{\text{red}}) = \frac{\partial \boldsymbol{p}_{\text{red}}(\boldsymbol{q}_{\text{red}})}{\partial \boldsymbol{q}_{\text{red}}} = \begin{pmatrix} -L_1 s_1 & -L_3 c_3 \\ L_1 c_1 & -L_3 s_3 \end{pmatrix}, \quad \dot{\boldsymbol{q}}_{\text{red}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_3 \end{pmatrix}.$$

As a result, the robot will be able to generate also any desired $\dot{\boldsymbol{p}}_{\text{red}} = \dot{\boldsymbol{p}}_d \in \mathbb{R}^2$, provided that

$$\det \boldsymbol{J}_{\text{red}} = L_1 L_3 \cos(q_3 - q_1) \neq 0 \qquad \Longleftrightarrow \qquad q_3 \neq q_1 \pm \frac{\pi}{2}. \tag{4}$$

The actual region of the plane where the two tasks can be performed simultaneously is illustrated in Fig. 3 for some specific but arbitrary values of the link lengths. The second link is always kept vertical and upwards. The circular annulus has outer radius R_{out}, inner radius R_{in}, and center C_{WS} on the axis $\boldsymbol{y}_0$, computed by simple geometric reasoning as

$$R_{\text{out}} = \frac{(L_1 + L_2 + L_3) - (-L_1 + L_2 - L_3)}{2} = L_1 + L_3, \qquad R_{\text{in}} = R_{\text{out}} - 2L_3 = |L_1 - L_3|,$$

and

$$C_{\text{WS}} = \frac{(L_1 + L_2 + L_3) + (-L_1 + L_2 - L_3)}{2} = L_2.$$

For $L_1 = L_2 = L_3 = L$, this is a full circle ($R_{\text{in}} = 0$) of radius $R_{\text{out}} = 2L$, centered at $C_{\text{WS}} = L$ on axis $\boldsymbol{y}_0$.

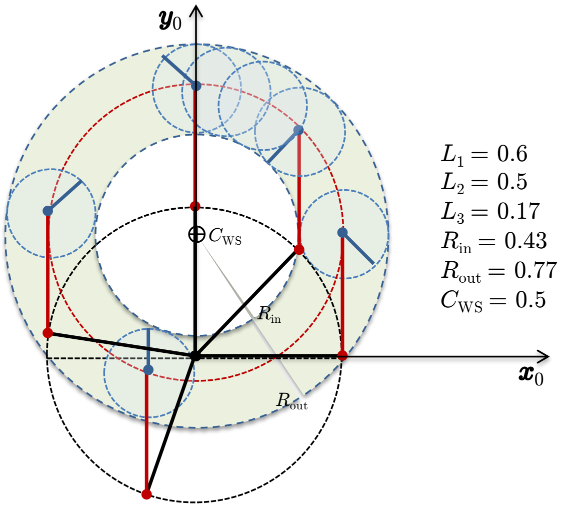


Figure 3: The Cartesian region of compatibility for both tasks (drawn for a specific set of link lengths).

As mentioned, when $q_1 + q_2 = \pi/2$ and (4) hold, then the TP method (2) will generate the exact (and unique) solution for both tasks. In these conditions, the same solution is obtained with $\dot{\boldsymbol{q}}_{\text{red}} = \boldsymbol{J}_{\text{red}}^{-1}(\boldsymbol{q}_{\text{red}})\dot{\boldsymbol{p}}_d$ and $\dot{q}_2 = -\dot{q}_{red,1}$. Equivalently, because of the assumed consistency of the two tasks, the problem can be solved also by the Extended Jacobian method (since $n = m_1 + m_2 = 3$):

$$\dot{\boldsymbol{r}} = \begin{pmatrix} \dot{p}_1 \\ \dot{\boldsymbol{p}}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2(\boldsymbol{q}) \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}_E(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad \Rightarrow \qquad \dot{\boldsymbol{q}} = \boldsymbol{J}_E^{-1}(\boldsymbol{q})\dot{\boldsymbol{r}}_d = \boldsymbol{J}_E^{-1}(\boldsymbol{q}) \begin{pmatrix} 0 \\ \dot{\boldsymbol{p}}_d \end{pmatrix}. \tag{5}$$

We have in fact $\det \boldsymbol{J}_E(\boldsymbol{q}) = -L_1 L_3 \sin(q_2 + q_3)$. So, when the second link is kept vertical and upwards ($q_1 + q_2 = \pi/2$), the two singularities of the Extended Jacobian matrix ($q_2 + q_3 = \{0, \pi\}$) correspond exactly to having $q_3 = q_1 \pm \pi/2$, i.e., the violation of condition (4).

With the link lengths $L_1 = L_2 = L_3 = 0.5$ [m] and for the given desired tip velocity $\dot{\boldsymbol{p}}_d = \begin{pmatrix} 0.1 & -0.5 \end{pmatrix}^T$ [m/s], when the robot is, e.g., in the configuration $\boldsymbol{q}_b = \begin{pmatrix} 0 & \pi/2 & \pi/3 \end{pmatrix}^T$ (condition (4) holds), then

$$\boldsymbol{J}_E(\boldsymbol{q}_b) = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2(\boldsymbol{q}_b) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -0.75 & -0.75 & -0.25 \\ 0.067 & -0.433 & -0.433 \end{pmatrix}, \quad (\boldsymbol{J}_2(\boldsymbol{q}_b)\boldsymbol{P}_1)^{\#} = \begin{pmatrix} -3.4641 & 2 \\ 3.4641 & -2 \\ -4 & 0 \end{pmatrix},$$

and the joint velocity provided by (2) or by (5) is

$$\dot{\boldsymbol{q}}_b = (\boldsymbol{J}_2(\boldsymbol{q}_b)\boldsymbol{P}_1)^{\#}\,\dot{\boldsymbol{p}}_d = \boldsymbol{J}_E^{-1}(\boldsymbol{q}_b)\begin{pmatrix} 0 \\ \dot{\boldsymbol{p}}_d \end{pmatrix} = \begin{pmatrix} -1.3464 \\ 1.3464 \\ -0.4 \end{pmatrix} \text{ [rad/s]} \quad \Rightarrow \quad \begin{cases} \boldsymbol{J}_1\dot{\boldsymbol{q}}_b = 0 = \dot{r}_{d1} \\ \boldsymbol{J}_2(\boldsymbol{q}_b)\dot{\boldsymbol{q}}_b = \begin{pmatrix} 0.1 \\ -0.5 \end{pmatrix} = \dot{\boldsymbol{r}}_{d2}. \end{cases}$$

On the other hand, when the robot is in the requested configuration $\boldsymbol{q}_0 = \begin{pmatrix} 0 & \pi/2 & -\pi/2 \end{pmatrix}^T$ [rad], the two tasks are inconsistent (condition (4) is violated). In this situation, the robot end effector is on the outer boundary of the Cartesian region of compatibility, and the desired tip velocity points outside. The task priority law (2) provides in this case

$$\dot{\boldsymbol{q}}_0 = (\boldsymbol{J}_2(\boldsymbol{q}_0)\boldsymbol{P}_1)^{\#}\,\dot{\boldsymbol{p}}_d = \begin{pmatrix} 0 & 2/3 \\ 0 & -2/3 \\ 0 & 4/3 \end{pmatrix}\begin{pmatrix} 0.1 \\ -0.5 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \\ -2/3 \end{pmatrix} \text{ [rad/s]} \;\Rightarrow\; \begin{cases} \boldsymbol{J}_1\dot{\boldsymbol{q}}_0 = 0 = \dot{p}_{d1} \\ \boldsymbol{J}_2(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 0 \\ -0.5 \end{pmatrix} = \dot{\boldsymbol{p}}_0 \neq \dot{\boldsymbol{p}}_d. \end{cases}$$

Note that the computed solution $\dot{\boldsymbol{q}}_0$ will realize only part of the desired tip velocity $\dot{\boldsymbol{p}}_d$ requested as secondary task, namely the component of $\dot{\boldsymbol{p}}_d \in \mathcal{R}\left\{\boldsymbol{J}_2(\boldsymbol{q}_0)\right\}$ (see Fig. 4).

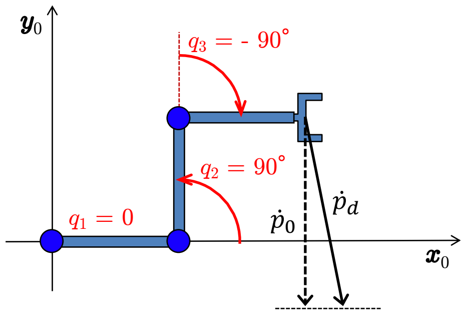


Figure 4: The specified secondary task velocity $\dot{\boldsymbol{p}}_d$ for the actual 3R planar robot in the configuration $\boldsymbol{q}_0$ and the realized one $\dot{\boldsymbol{p}}_0$.

Finally, suppose that errors $e_1 = r_{d1} - f_1(\boldsymbol{q}) = \pi/2 - (q_1 + q_2) \neq 0$ and/or $\boldsymbol{e}_2 = \boldsymbol{r}_{d2} - \boldsymbol{f}_2(\boldsymbol{q}) = \boldsymbol{p}_d - \boldsymbol{f}_2(\boldsymbol{q}) \neq \boldsymbol{0}$ are present during the simultaneous execution of the tasks. The task priority scheme (1) will be modified by introducing an error feedback term in both tasks as

$$\begin{aligned} \dot{\boldsymbol{q}}_c &= \boldsymbol{J}_1^{\#}\left(\dot{r}_{d1} + k_1 e_1\right) + \boldsymbol{P}_1\left(\boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{P}_1\right)^{\#}\left(\dot{\boldsymbol{r}}_{d2} + \boldsymbol{K}_2\boldsymbol{e}_2 - \boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{J}_1^{\#}\left(\dot{r}_{d1} + k_1 e_1\right)\right) \\ &= \boldsymbol{J}_1^{\#} k_1 e_1 + \left(\boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{P}_1\right)^{\#}\left(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_2\boldsymbol{e}_2 - \boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{J}_1^{\#} k_1 e_1\right), \end{aligned} \tag{6}$$

with a scalar gain $k_1 > 0$ and a (typically, diagonal) matrix gain $\boldsymbol{K}_2 > 0$. Since $\boldsymbol{J}_1\dot{\boldsymbol{q}}_c = k_1 e_1$, we always have $\dot{e}_1 = -k_1 e_1$ and the error on the first task will exponentially converge to zero. On the other hand, the control law (6) will generate the largest possible reduction (or, in the worst case, the smallest increase) of the error on the second task, without ever affecting the first task.

Exercise 2

Based on the dynamic model of the RP planar robot

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}, \qquad \text{with } \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}, \tag{7}$$

we need to derive the dynamic elements that appear in the expression of the residual vector

$$\boldsymbol{r}(t) = \boldsymbol{K}_I \left[\boldsymbol{p}(t) - \int_0^t \left(\boldsymbol{\tau} + \boldsymbol{S}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{r} \right) ds - \boldsymbol{p}(0) \right], \tag{8}$$

where $\boldsymbol{p} = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$ is the generalized momentum and matrix $\boldsymbol{K}_I > 0$ is diagonal. Without loss of generality, we can assume that the robot is at rest at the beginning of the experiment, i.e, $\boldsymbol{p}(0) = \boldsymbol{0}$.

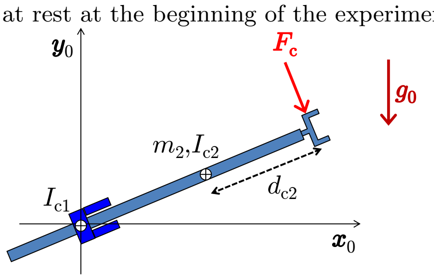


Figure 5: Definition of the relevant dynamic parameters for the RP robot of Fig. 2. Also shown is a collision force $\boldsymbol{F}_c$ acting at the tip along the normal direction to the second link.

With reference to the dynamic parameters defined in Fig. 5, for the kinetic energy

$$T = T_1 + T_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

since $\boldsymbol{p}_{c2} = (q_2 - d_{c2}) \begin{pmatrix} \cos q_1 & \sin q_1 \end{pmatrix}^T$ and $\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2}$, we have

$$T_1 = \frac{1}{2} I_{c1}\, \dot{q}_1^2, \qquad T_2 = \frac{1}{2} I_{c2}\, \dot{q}_1^2 + \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 = \frac{1}{2}\left(I_{c2} + m_2 (q_2 - d_{c2})^2 \right) \dot{q}_1^2 + \frac{1}{2} m_2\, \dot{q}_2^2.$$

The robot inertia matrix is then

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_{c1} + I_{c2} + m_2 (q_2 - d_{c2})^2 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{9}$$

From this, we compute the Coriolis/centrifugal terms using the matrices of Christoffel symbols

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left[\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right) + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \left(\frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i} \right) \right], \qquad i = 1, 2.$$

We obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & m_2 (q_2 - d_{c2}) \\ m_2 (q_2 - d_{c2}) & 0 \end{pmatrix}, \qquad \boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} -m_2 (q_2 - d_{c2}) & 0 \\ 0 & 0 \end{pmatrix},$$

and thus

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \dot{\boldsymbol{q}} \end{pmatrix} = \begin{pmatrix} 2 m_2\, (q_2 - d_{c2})\, \dot{q}_1 \dot{q}_2 \\ -m_2\, (q_2 - d_{c2})\, \dot{q}_1^2 \end{pmatrix}. \tag{10}$$

A factorization $\boldsymbol{c}(\boldsymbol{q}. \dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}}$ in (7) such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is skew-symmetric is given by

$$\boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} m_2\, (q_2 - d_{c2})\, \dot{q}_2 & m_2\, (q_2 - d_{c2})\, \dot{q}_1 \\ -m_2\, (q_2 - d_{c2})\, \dot{q}_1 & 0 \end{pmatrix}. \tag{11}$$

For the potential energy

$$U = U_1 + U_2 = U(\boldsymbol{q}),$$

we have

$$U_1 = \text{constant}, \qquad U_2 = m_2 g_0\, (q_2 - d_{c2}) \sin q_1,$$

and so

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} m_2 g_0 \,(q_2 - d_{c2}) \cos q_1 \\ m_2 g_0 \sin q_1 \end{pmatrix}. \tag{12}$$

From the expressions (9) and (11–12), we finally obtain

$$r_1(t) = k_{I1}\left[\left(I_{c1} + I_{c2} + m_2(q_2 - d_{c2})^2\right)\dot{q}_1 - \int_0^t \left(\tau_1 - m_2 g_0\,(q_2 - d_{c2})\cos q_1 + r_1\right) ds\right],$$

and

$$r_2(t) = k_{I2}\left[m_2 \dot{q}_2 - \int_0^t \left(\tau_2 + m_2\,(q_2 - d_{c2})\,\dot{q}_1^2 - m_2 g_0 \sin q_1 + r_2\right) ds\right].$$

Suppose now that, at time $t = t_c$, a collision force $\boldsymbol{F}_c$ acts at the robot tip in the orthogonal direction to the second link and with an intensity $F \neq 0$ (see again Fig. 5). The Jacobian $\boldsymbol{J}_c(\boldsymbol{q})$ associated to the contact point and the contact force are then

$$\boldsymbol{J}_c(\boldsymbol{q}) = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix}, \qquad \boldsymbol{F}_c = F \begin{pmatrix} -\sin q_1 \\ \cos q_1 \end{pmatrix},$$

while the resulting torque at the joint is computed as

$$\boldsymbol{\tau}_c = \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c = \begin{pmatrix} F\,q_2 \\ 0 \end{pmatrix}.$$

From the nominal behavior of the residual vector $\boldsymbol{r}$, being $\boldsymbol{r}(t) = \boldsymbol{0}$ for all $t \in [0, t_c]$, it follows that

$$\dot{\boldsymbol{r}}(t_c) = \boldsymbol{K}_I\left(\boldsymbol{\tau}_c(t_c) - \boldsymbol{r}(t_c)\right) \qquad \Rightarrow \qquad \begin{cases} \dot{r}_1(t_c) = k_{I1} F(t_c)\, q_2(t_c) \\ \dot{r}_2(t_c) = 0. \end{cases}$$

Although the collision occurs on the second link, the second component of the residual will not be affected immediately; in fact, $\boldsymbol{F}_c$ is not producing work on q_2, due to the specific direction assumed for the impact force.

Exercise 3

The acceleration profile for the rest-to-rest motion trajectory $\theta(t)$ is *assigned* to be of the bang-coast-bang type, having symmetric initial and final acceleration/deceleration phases, each of duration $T_s = T/4$ and with $\ddot{\theta} = \pm A$, and a cruising phase that lasts for half of the motion time, i.e., $T/2$, with constant velocity $\dot{\theta} = V$. From this motion structure, it is easy to compute the following quantities:

$T - 2T_s = T - 2\frac{T}{4} = \frac{T}{2}$

FROM

$\ddot{\theta} = A$ $\qquad \theta(0) + \frac{1}{2}At^2$

$T_s = \frac{V}{A}$

$$V = \dot{\theta}\left(\frac{T}{4}\right) = A\,\frac{T}{4}, \qquad \Delta\theta_s = \theta\left(\frac{T}{4}\right) = \frac{1}{2}A\left(\frac{T}{4}\right)^2 = \frac{AT^2}{32}, \qquad \Delta\theta = \theta(T) = 2\Delta\theta_s + V\boxed{\frac{T}{2}} = \frac{3AT^2}{16}.$$

Thus, for a desired total displacement $\Delta\theta > 0$ and a given motion time T, we have for the acceleration A and cruise velocity V

USING $A = \frac{4V}{T}$

$$A = \frac{16\Delta\theta}{3T^2} > 0 \qquad \Rightarrow \qquad V = \frac{4\Delta\theta}{3T} > 0. \tag{13}$$

The swing-up maneuver from $\theta(0) = 0$ to $\Delta\theta = \theta(T) = \pi$ in time T needs then an acceleration/deceleration $A = \pm 16\pi/(3T^2)$ in the first and third motion phases. Note that, when the acceleration phase ends at time $t = T_s = T/4$, the performed motion will be $\Delta\theta_s = \Delta\theta/6 = \pi/6$. By symmetry, when the deceleration phase begins at time $t = T - T_s = 3T/4$, the performed motion completed so far will be $\Delta\theta - \Delta\theta_s = 5\Delta\theta/6 = 5\pi/6$.

With the above in mind, consider the dynamics of the actuated pendulum

$$I\ddot{\theta} + m g_0 d \sin\theta = \tau, \tag{14}$$

where $\theta = 0$ corresponds to the downward equilibrium and the dynamic parameters are given by

$$d = \frac{l}{2} = 1 \text{ [m]}, \; mg_0 d = 98.1 \text{ [kg·m}^2\text{]}, \; I = I_c + md^2 = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{3} = \frac{40}{3} = 13.33 \text{ [kg·m}^2\text{]}.$$

By inverse dynamics on (14), the torque needed to perform the desired motion during the three phases is:

$$\tau(t) = \begin{cases} IA + mg_0 d \sin\theta(t), & \theta \in [0, \pi/6), & \text{phase I: } t \in [0, T/4), \\ mg_0 d \sin\theta(t), & \theta \in [\pi/6, 5\pi/6), & \text{phase II: } t \in [T/4, 3T/4), \\ -IA + mg_0 d \sin\theta(t), & \theta \in [\pi/6, \pi], & \text{phase III: } t \in [3T/4, T]. \end{cases} \tag{15}$$

The gravity contribution to the inverse dynamics torque is maximum at the midpoint of motion, i.e., at $\theta = \pi/2$, is independent of the total motion time, and is equal to $\tau_g = mg_0 d < \tau_{max}$. Note that if it were $\tau_g > \tau_{max}$, then actuation would be too weak to perform the intended task (even when moving the pendulum very slowly, with an arbitrarily long motion time T).

Further, from (13) and (15) it is easy to see that, when speeding up motion by uniformly reducing T, the inertial torque component in the first phase will increase quadratically and the maximum required torque will be attained at the end of the first phase, where the gravity contribution is the largest (and has the same sign of the acceleration). Thus, for feasibility we require that

$$IA + mg_0 d \sin\Delta\theta_s = \frac{16\pi I}{3T^2} + mg_0 d \sin\frac{\pi}{6} \leq \tau_{max},$$

and the optimal motion time will be defined as the lower bound for all feasible motion times,

$$T \geq \sqrt{\frac{16\pi I}{3\left(\tau_{max} - mg_0 d \sin(\pi/6)\right)}} = T_{min}.$$

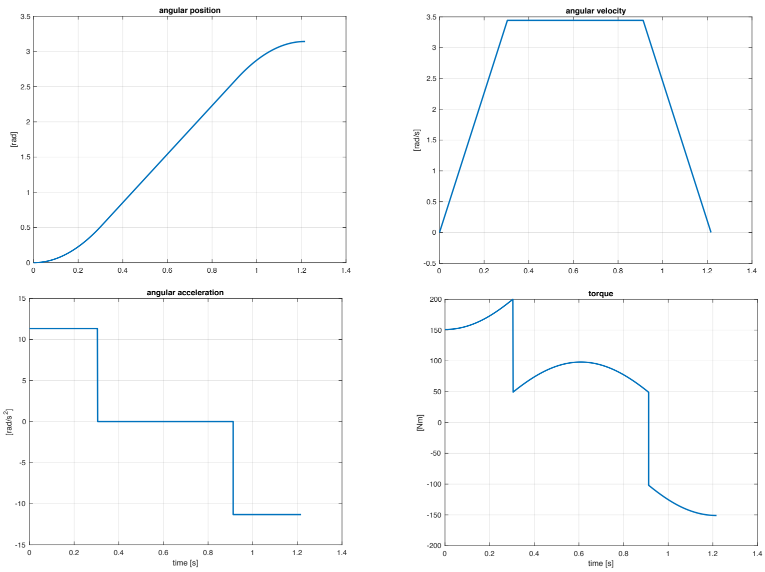


Figure 6: Kinematic (position, velocity, and acceleration) and dynamic (torque) profiles of the minimum time rest-to-rest swing-up maneuver.

Plugging in the numerical data, we find the optimal time $T_{min} = 1.2165$ [s]. The maximum torque during motion is indeed $\tau_{max} = 200$ [Nm], reached at the single instant $t = T_{min}/4 = 0.3041$ [s]. Accordingly, we obtain from (13) $A = 11.3212$ [rad/s^2] and $V = 3.4432$ [rad/s]. Figure 6 shows the resulting time profiles of the angular position, velocity and acceleration, and of the commanded torque $\tau(t)$.

* * * * *

Robotics II

September 11, 2019

Exercise 1

Consider the 3R robot in Fig. 1, moving on a horizontal plane. The robot has identical links (each of length L, uniformly distributed mass m, and inertia $I_L = mL^2/12$ around the barycentral vertical axis) and is commanded at the joint level by torques $\boldsymbol{\tau}(t) \in \mathbb{R}^3$. Neglect in the following any dissipative/friction effects. With the system at $t = 0$ in a generic initial state $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)) = (\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0)$ with $\dot{\boldsymbol{q}}_0 \neq \mathbf{0}$, we want to control the robot so that its kinetic energy $T = T(\boldsymbol{q}, \dot{\boldsymbol{q}})$ in the closed-loop dynamics satisfies the following desired target equation:

$$\frac{dT}{dt} = -\gamma T, \qquad \text{with } \gamma > 0.$$

Determine the expression of the control law $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ that realizes this behavior. For $L = 0.2$ [m], $m = 3$ [kg], $\boldsymbol{q}_0 = (0, \pi/2, \pi/2)$ [rad], $\dot{\boldsymbol{q}}_0 = (0, -\pi, -\pi)$ [rad/s] and $\gamma = 1$, compute the numerical value of such a control torque at $t = 0$, i.e., $\boldsymbol{\tau}(0)$.

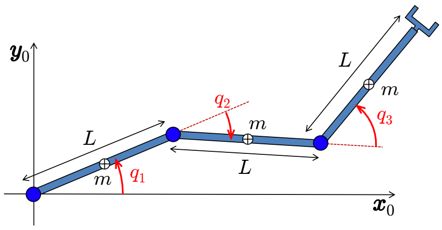


Figure 1: A 3R robot moving on a horizontal plane, and its coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$.

Exercise 2

The RP planar robot shown in Fig. 2 should execute a rest-to-rest motion task in minimum time under torque/force bounds $|\tau_i| \leq \tau_{max,i} > 0$, $i = 1, 2$, with its end-effector moving along a circular path of radius $R > d$ by an angle α from A to B. Determine the analytic expression of the minimum time T^* in terms of the task data and of the robot dynamic parameters. Draw the profile of the two components of the time-optimal command $\boldsymbol{\tau}^*(t)$, for $t \in [0, T^*]$.

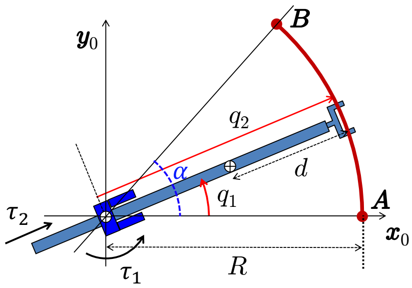


Figure 2: A RP robot moving its end-effector along a circular path on a horizontal plane.

Exercise 3

With reference to Fig. 3, a mass m_1 is moving at constant speed $v_0 > 0$ and collides at some time $t = t_c$ with a mass m_2 which is initially at rest. Assume a purely ideal situation: there is no dissipation due to friction and the collision is perfectly elastic. Therefore, the total kinetic energy T and the total (scalar) momentum P of the two masses will both remain constant over time. Determine the expressions of the velocities $v_1(t_c^+)$ and $v_2(t_c^+)$ of the two masses after the collision. Describe what happens when $m_1 > m_2$, $m_1 = m_2$, or $m_1 < m_2$, and in the limit cases when $m_2 \to 0$ or $m_2 \to \infty$.

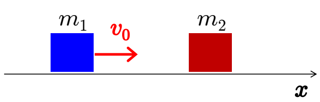


Figure 3: A mass m_1 in motion collides with a second mass m_2 initially at rest.

[open books, 180 minutes]

Solution

September 11, 2019

Exercise 1

The dynamic model of a frictionless robot in the absence of gravity is given by

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} = \boldsymbol{\tau}, \tag{1}$$

where any factorization matrix $\boldsymbol{S}$ can be used for the (quadratic) Coriolis and centrifugal terms. From the expression of the kinetic energy $T = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$, we obtain

$$\dot{T} = \frac{dT}{dt} = \dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^T\left(\boldsymbol{\tau} - \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}\right) + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^T\boldsymbol{\tau}, \tag{2}$$

where we have used (1) and the principle of energy conservation (implying $\dot{\boldsymbol{q}}^T\left(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)\dot{\boldsymbol{q}} \equiv 0$, $\forall(\boldsymbol{q},\dot{\boldsymbol{q}})$). In order to impose the desired behavior to the Kinetic energy, it follows immediately that

$$\dot{T} = \dot{\boldsymbol{q}}^T\boldsymbol{\tau} = -\gamma\,T = -\frac{\gamma}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad \Longrightarrow \qquad \boldsymbol{\tau} = -\frac{\gamma}{2}\,\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{3}$$

The control law should apply a torque that is the (scaled) negative value of the current generalized momentum $\boldsymbol{p} = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$ of the robot.

To realize (3), one needs to derive only the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ for the 3R planar robot at hand. The kinetic energy is given by

$$T = \sum_{I=1}^{3} T_i, \qquad T_i = \frac{1}{2}\,m\,\|\boldsymbol{v}_{ci}\|^2 + \frac{1}{2}\,I_L\,\omega_{z,i}^2, \qquad i = 1,2,3.$$

We compute first

$$T_1 = \frac{1}{2}\,m\left(\frac{L}{2}\,\dot{q}_1\right)^2 + \frac{1}{2}\,I_L\,\dot{q}_1^2 \quad \left(\cdots = \frac{1}{2}\,m\,\frac{L^2}{3}\,\dot{q}_1^2\right).$$

Then, from

$$\boldsymbol{p}_{c2} = \begin{pmatrix} L\cos q_1 + (L/2)\cos(q_1+q_2) \\ L\sin q_1 + (L/2)\sin(q_1+q_2) \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \begin{pmatrix} -L\sin q_1\,\dot{q}_1 - (L/2)\sin(q_1+q_2)(\dot{q}_1+\dot{q}_2) \\ L\cos q_1\,\dot{q}_1 + (L/2)\cos(q_1+q_2)(\dot{q}_1+\dot{q}_2) \end{pmatrix}$$

and

$$\boldsymbol{p}_{c3} = \begin{pmatrix} L(\cos q_1 + \cos(q_1+q_2)) + (L/2)\cos(q_1+q_2+q_3) \\ L(\sin q_1 + \sin(q_1+q_2)) + (L/2)\sin(q_1+q_2+q_3) \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} -L(\sin q_1\,\dot{q}_1 + \sin(q_1+q_2)(\dot{q}_1+\dot{q}_2)) - (L/2)\sin(q_1+q_2+q_3)(\dot{q}_1+\dot{q}_2+\dot{q}_3) \\ L(\cos q_1\,\dot{q}_1 + \cos(q_1+q_2)(\dot{q}_1+\dot{q}_2)) + (L/2)\cos(q_1+q_2+q_3)(\dot{q}_1+\dot{q}_2+\dot{q}_3) \end{pmatrix},$$

we obtain

$$T_2 = \frac{1}{2}\,m\left(L^2\dot{q}_1^2 + \frac{L^2}{4}(\dot{q}_1+\dot{q}_2)^2 + L^2\cos q_2\,\dot{q}_1(\dot{q}_1+\dot{q}_2)\right) + \frac{1}{2}\,I_L\,(\dot{q}_1+\dot{q}_2)^2$$

and

$$\begin{aligned} T_3 \;=\; & \frac{1}{2}\,m\Big(L^2\dot{q}_1^2 + L^2(\dot{q}_1+\dot{q}_2)^2 + 2L^2\cos q_2\,\dot{q}_1(\dot{q}_1+\dot{q}_2) + \frac{L^2}{4}(\dot{q}_1+\dot{q}_2+\dot{q}_3)^2 \\ & + L^2\left(\cos(q_2+q_3)\dot{q}_1 + \cos q_3(\dot{q}_1+\dot{q}_2)\right)(\dot{q}_1+\dot{q}_2+\dot{q}_3)\Big) + \frac{1}{2}\,I_L\,(\dot{q}_1+\dot{q}_2+\dot{q}_3)^2. \end{aligned}$$

Therefore, using the compact notation for trigonometric functions and substituting for $I_L = mL^2/12$, the inertia matrix is

$$\boldsymbol{M}(\boldsymbol{q}) = mL^2 \begin{pmatrix} 4 + 3c_2 + c_3 + c_{23} & \frac{5}{3} + \frac{3}{2}c_2 + c_3 + \frac{1}{2}c_{23} & \frac{1}{3} + \frac{1}{2}\left(c_3 + c_{23}\right) \\ & \frac{5}{3} + c_3 & \frac{1}{3} + \frac{1}{2}c_3 \\ symm & & \frac{1}{3} \end{pmatrix}. \tag{4}$$

Finally, evaluating the control law at $\boldsymbol{q}_0 = (0, \pi/2, \pi/2)$ [rad] and $\dot{\boldsymbol{q}}_0 = (0, -\pi, -\pi)$ [rad/s] and with the data $L = 0.2$ [m], $m = 3$ [kg] (thus $I_L = 0.01$ [kg·m^2]) and $\gamma = 1$, gives

$$\boldsymbol{\tau}(0) = -\frac{1}{2}\,\boldsymbol{M}(\boldsymbol{q}_0)\,\dot{\boldsymbol{q}}_0 = -\frac{1}{2}\cdot\frac{3}{25}\begin{pmatrix} 3 & \frac{7}{6} & -\frac{1}{6} \\ \frac{7}{6} & \frac{5}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 \\ -\pi \\ -\pi \end{pmatrix} = \frac{\pi}{2}\begin{pmatrix} 0.12 \\ 0.24 \\ 0.08 \end{pmatrix} = \begin{pmatrix} 0.1885 \\ 0.3770 \\ 0.1257 \end{pmatrix} \text{ [Nm]}. \tag{5}$$

Exercise 2

We start by deriving the dynamic model of the RP planar robot in Fig. 2. For the kinetic energy

$$T = T_1 + T_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

since $\boldsymbol{p}_{c2} = (q_2 - d)\begin{pmatrix}\cos q_1 & \sin q_1\end{pmatrix}^T$ and $\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2}$, we have

$$T_1 = \frac{1}{2}I_{c1}\,\dot{q}_1^2, \qquad T_2 = \frac{1}{2}I_{c2}\,\dot{q}_1^2 + \frac{1}{2}m_2\,\|\boldsymbol{v}_{c2}\|^2 = \frac{1}{2}\left(I_{c2} + m_2(q_2-d)^2\right)\dot{q}_1^2 + \frac{1}{2}m_2\,\dot{q}_2^2,$$

with an obvious interpretation of the dynamic parameters. The robot inertia matrix is then

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_{c1} + I_{c2} + m_2(q_2-d)^2 & 0 \\ 0 & m_2 \end{pmatrix}. \tag{6}$$

From this, we compute the Coriolis/centrifugal terms using the matrices of Christoffel symbols

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right) + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \left(\frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i}\right)\right], \qquad i = 1, 2.$$

We obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & m_2(q_2-d) \\ m_2(q_2-d) & 0 \end{pmatrix}, \qquad \boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} -m_2(q_2-d) & 0 \\ 0 & 0 \end{pmatrix},$$

and thus

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T\boldsymbol{C}_1(\boldsymbol{q})\dot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}^T\boldsymbol{C}_2(\boldsymbol{q})\dot{\boldsymbol{q}} \end{pmatrix} = \begin{pmatrix} 2m_2\,(q_2-d)\,\dot{q}_1\dot{q}_2 \\ -m_2\,(q_2-d)\,\dot{q}_1^2 \end{pmatrix}. \tag{7}$$

From (6) and (7), we write the (unconstrained) dynamic equations in their scalar form as

$$\left(I_{c1} + I_{c2} + m_2(q_2-d)^2)\right)\ddot{q}_1 + 2m_2\,(q_2-d)\,\dot{q}_1\dot{q}_2 \;=\; \tau_1, \tag{8}$$

$$m_2\ddot{q}_2 - m_2\,(q_2-d)\,\dot{q}_1^2 \;=\; \tau_2. \tag{9}$$

In order to execute the task, the second joint variable should remain constant at all times, namely $q_2 = R$, $\dot{q}_2 = \ddot{q}_2 = 0$. Therefore, from (8) with $q_2 = R$ and $\dot{q}_2 = 0$, the robot dynamics along the path can be described by

$$I_0\,\ddot{q}_1 = \tau_1, \qquad \text{with } I_0 = I_{c1} + I_{c2} + m_2(R-d)^2 > 0, \tag{10}$$

whereas, from (9) used as inverse dynamics with $q_2 = R$ and $\ddot{q}_2 = 0$, the second motor should apply the force

$$\tau_2(t) = -m_2\,(R-d)\,\dot{q}_1^2(t) \tag{11}$$

in order to have the end-effector remaining perfectly on the path. Equations (10–11) are the core of the solution. Based on the linear dynamics (10), to perform the desired rest-to-rest motion task in minimum time, the first motor should apply a bang-bang torque profile $\tau_1(t)$ (with maximum positive and negative torque $\pm\tau_{max,1}$, each applied for half of the motion interval). The total motion time should be sufficient to complete the rotation $\Delta q_1 = \alpha > 0$. Again from (10), this corresponds to using a maximum (absolute) acceleration bound in the definition of the time-optimal motion of joint 1, i.e,

$$|\ddot{q}_1| \le A_{max,1} = \frac{\tau_{max,1}}{I_0}. \tag{12}$$

While doing so, however, the velocity $\dot{q}_1(t)$ of the first joint will increase linearly and, according to (11), the force that the second motor needs to apply in order to keep the robot end-effector on the path will increase quadratically. As a result, the second actuator may exceed its dynamic capabilities. Therefore, the bound $|\tau_2| \le \tau_{max,2}$ will impose also a bound $V_{max,1}$ on the (absolute) velocity that the first joint can reach. We have[1]

$$|\tau_2| = m_2\,(R-d)\,\dot{q}_1^2 \le \tau_{max,2} \qquad \Longrightarrow \qquad |\dot{q}_1| \le V_{max,1} = \sqrt{\frac{\tau_{max,2}}{m_2\,(R-d)}}. \tag{13}$$

Under the combined velocity/torque (viz. velocity/acceleration) bounds for the motion of joint 1, the minimum time solution will have in general a bang-coast-bang profile for the first torque (and its acceleration as well). The motion time T^* is computed then from known formulas.

[1] Note that $R - d > 0$ by assumption, so the argument of the square root is positive.

If $\alpha > V_{max,1}^2/A_{max,1}$, a coast phase will exist. Then

$$T_s = \frac{V_{max,1}}{A_{max,1}} \quad \Longrightarrow \quad (T^* - T_s)V_{max,1} = \alpha \quad \Longrightarrow \quad T^* = \frac{\alpha}{V_{max,1}} + \frac{V_{max,1}}{A_{max,1}}, \tag{14}$$

where one should replace the definitions of bounds in (12) and (13). The (qualitative) plots of the resulting torque/force vector $\boldsymbol{\tau}^*(t)$ are reported in Fig. 4. The second joint force $\tau_2^*(t)$ follows from (11), with a quadratic time profile where the velocity of the first joint is linear in time and a constant value where $\dot{q}_1$ is constant. The other special cases (with pure bang-bang commands) are treated similarly.

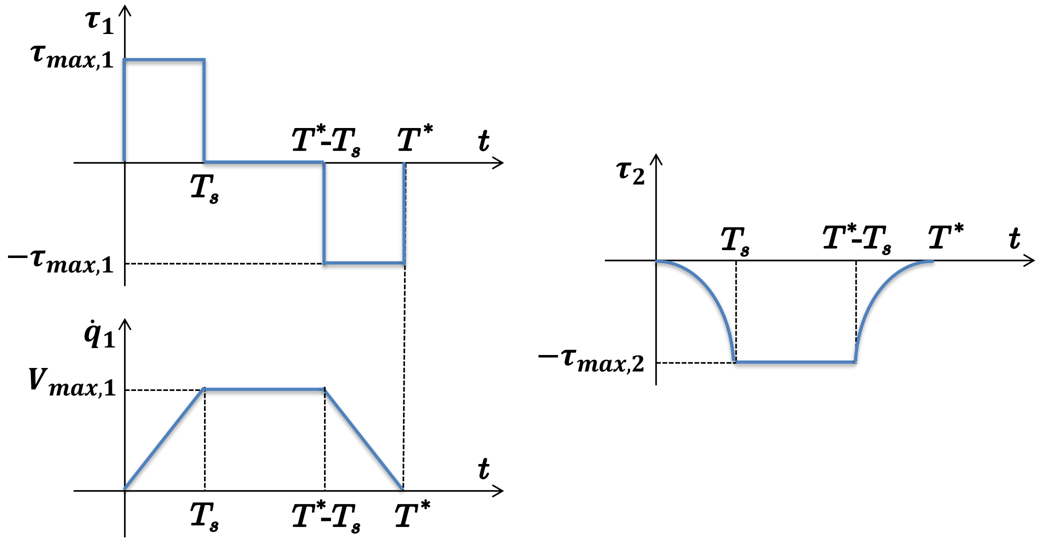


Figure 4: Optimal profiles of the torque τ_1^*, of the related velocity $\dot{q}_1^*$, and of the force τ_2^* for the requested rest-to-rest minimum time motion of the RP robot in Fig. 2.

Exercise 3

This is a simple application of conservation principles of the total kinetic energy T and total momentum P (along the direction x) for the system with the two masses m_1 and m_2. In formulas,

$$T(t) = \frac{1}{2}m_1 v_1^2(t) + \frac{1}{2}m_2 v_2^2(t) = \text{constant}, \qquad P(t) = m_1 v_1(t) + m_2 v_2(t) = \text{constant}, \qquad \forall t.$$

We apply these identities around the collision time $t = t_c$, just before ($t = t_c^-$) and just after ($t = t_c^+$). Let

$$v_1 = v_1(t_c^+), \qquad v_1(t_c^-) = v_0 > 0, \qquad v_2 = v_2(t_c^+), \qquad v_2(t_c^-) = 0,$$

where v_1 and v_2 are the unknowns of our problem. Thus,

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_0^2 \tag{15}$$

and

$$m_1 v_1 + m_2 v_2 = m_1 v_0. \tag{16}$$

Equations (15) and (16) are rewritten respectively as

$$m_1(v_1^2 - v_0^2) + m_2 v_2^2 = m_1(v_1 - v_0)(v_1 + v_0) + m_2 v_2^2 = 0 \tag{17}$$

and

$$m_1(v_1 - v_0) = -m_2 v_2. \tag{18}$$

Substituting (18) in (17) and simplifying yields

$$v_2 = v_1 + v_0. \tag{19}$$

Plugging (19) back into (16) leads to

$$m_1 v_1 + m_2(v_1 + v_0) = m_1 v_0 \qquad \Longrightarrow \qquad v_1 = \frac{m_1 - m_2}{m_1 + m_2}\, v_0. \tag{20}$$

Finally, substituting v_1 in (19) gives

$$v_2 = \frac{2m_1}{m_1 + m_2}\, v_0. \tag{21}$$

From (20–21), we conclude that:

$$\begin{cases} m_2 \to 0 & \Longrightarrow & v_1 = v_0 > 0, & v_2 = 2v_0 > 0, \\ m_2 < m_1 & \Longrightarrow & v_0 > v_1 > 0, & v_2 > v_0 > 0, \\ m_2 = m_1 & \Longrightarrow & v_1 = 0, & v_2 = v_0 > 0, \\ m_2 > m_1 & \Longrightarrow & -v_0 < v_1 < 0, & 0 < v_2 < v_0, \\ m_2 \to \infty & \Longrightarrow & v_1 = -v_0 < 0, & v_2 = 0. \end{cases}$$

* * * * *

Robotics 2

Remote Midterm Test – April 15, 2020

The test has the form of a Questionnaire. Please answer with texts and formulas and write clearly. You may also use the 'Reply Sheet' in the `Exam.net` *environment to type in some answers. Take pictures of each page of your handwritten answers and upload them in the system before submitting. Try to follow the same order of the questions. Number your replies accordingly.*

Question #1

When and why is it convenient to choose a two-stage calibration procedure for the uncertain Denavit-Hartenberg parameters in the kinematic model of a manipulator?

Question #2

The position $\boldsymbol{p} \in \mathbb{R}^3$ of the origin O_n of the last frame of a n-dof serial manipulator is computed in homogeneous coordinates through the direct kinematics as

$$\begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1\, {}^1\boldsymbol{A}_2 \ldots {}^{i-1}\boldsymbol{A}_i \ldots {}^{n-2}\boldsymbol{A}_{n-1}\, {}^{n-1}\boldsymbol{A}_n \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix},$$

where 4×4 Denavit-Hartenberg homogeneous transformation matrices are used. Suppose that the only uncertainty in the kinematic model is on the value of the twist angle α_i of the ith homogeneous matrix around its nominal value $\alpha_i^{nom} = \pi/2$. Write the expression of the 3×1 regressor matrix $\boldsymbol{\Phi}$ in the basic equation $\Delta\boldsymbol{p} = \boldsymbol{\Phi}\,\Delta\alpha_i$ that is used for calibration at a generic configuration $\boldsymbol{q} \in \mathbb{R}^n$.

Question #3

The differential kinematics of a 3-dof robot performing a two-dimensional task $\boldsymbol{x}$ is expressed by $\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{x}}$. Suppose that, in a given configuration $\boldsymbol{q} \in \mathbb{R}^3$, we have the following values for the task Jacobian $\boldsymbol{J}$ and the desired task velocity $\dot{\boldsymbol{x}}$:

$$\boldsymbol{J} = \begin{pmatrix} 3 & 1 & 2 \\ 1.5 & 0.5 & 1 \end{pmatrix}, \qquad \dot{\boldsymbol{x}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Find the joint velocity $\dot{\boldsymbol{q}}^*$ of minimum norm that realizes at best the desired instantaneous task. Does the task velocity error vanish or not? Find another $\dot{\boldsymbol{q}}' \neq \dot{\boldsymbol{q}}^*$ providing the same task velocity error and show that $\|\dot{\boldsymbol{q}}^*\| < \|\dot{\boldsymbol{q}}'\|$.

Question #4

A 3R planar robot with links of unitary length moving in a vertical plane has to perform two tasks: *i)* follow a trajectory with its end-effector position, and *ii)* keep its last link upwards. At $\boldsymbol{q} = \begin{pmatrix} \pi/4 & 0 & \pi/4 \end{pmatrix}^T$ [rad], the desired end-effector linear velocity is $\boldsymbol{v}_p = \begin{pmatrix} 2 & -1 \end{pmatrix}^T$ [m/s]. Does there exist a joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ that executes both tasks simultaneously? If not, find a joint velocity $\dot{\boldsymbol{q}}_{TP}$ with the Task Priority method, giving higher priority to the last link orientation task.

Question #5

A 3R planar robot is moving on a horizontal plane. At a given instant of time t, the robot is in the configuration $\boldsymbol{q}(t) = \begin{pmatrix} 0 & \pi/2 & \pi/4 \end{pmatrix}^T$ [rad], with velocity $\dot{\boldsymbol{q}}(t) = \begin{pmatrix} \pi/2 & -\pi/4 & \pi/8 \end{pmatrix}^T$ [rad/s]. If the applied torque is $\boldsymbol{u}(t) = \begin{pmatrix} 1.5 & 0 & -4 \end{pmatrix}^T$ [Nm], will the instantaneous total energy E of the robot increase, stay the same, or decrease? And what about the Lagrangian function L?

Question #6

Given the inertia matrix of a 2R polar robot

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + a_2 \sin^2 q_2 + a_3 \cos^2 q_2 & 0 \\ 0 & a_4 \end{pmatrix},$$

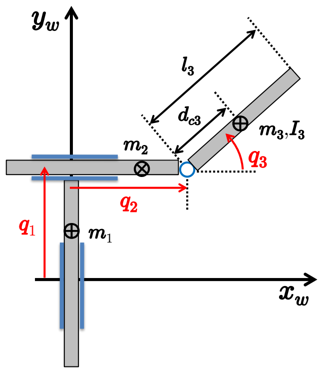
find two factorizations of the associated Coriolis/centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}'(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{S}''(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ such that the matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}'$ is skew symmetric, while the matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}''$ is not.

Question #7

Consider the PPR planar robot in the figure below. Using the coordinates $\boldsymbol{q} \in \mathbb{R}^3$ and the dynamic parameters defined therein, determine the expression of the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. Provide then a linear parametrization only of the inertial terms in the dynamic model, i.e., such that

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{Y}_M(\boldsymbol{q}, \ddot{\boldsymbol{q}})\, \boldsymbol{a}_M,$$

where the $3 \times p$ regressor matrix $\boldsymbol{Y}_M$ and the vector of dynamic coefficients $\boldsymbol{a}_M \in \mathbb{R}^p$ have the least possible dimension p.



Question #8

Provide the inertia matrix $\boldsymbol{M}_p(\boldsymbol{p})$ of the robot considered in Question **#7** when using for the Lagrangian dynamic modeling the new set of coordinates $\boldsymbol{p} = \begin{pmatrix} x & y & \alpha \end{pmatrix}^T \in \mathbb{R}^3$, where (x, y) are the components of the Cartesian position of the robot end-effector in world coordinates and α is the angle of the last link w.r.t. the $\boldsymbol{x}_w$ axis of the world frame.

Question #9

A single link moving under gravity is modeled by the differential equation $I\ddot{\theta} + mg_0 d \sin\theta = u$, with $m = 3$ [kg], $d = 0.5$ [m], $I = 1$ [kgm^2], and $g_0 = 9.81$ [m/s^2]. The motor torque is bounded by $|u| \leq U = 25$ [Nm]. The desired task is a rest-to-rest swing-up maneuver from $\theta(0) = 0$ to $\theta(T) = \pi$ [rad] in $T = 1$ [s], to be done with a bang-bang acceleration profile. Is the torque bound satisfied? If not, find the minimum uniform time scaling to execute the task in a feasible way.

Question #10

Assume that we have available the Newton-Euler routine $NE_\alpha(\text{arg}_1, \text{arg}_2, \text{arg}_3)$, equipped with the kinematic and dynamic data of a n-dof serial manipulator. How can we compute the kinetic energy T in a generic state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of this robot by just one call of this routine and one scalar product?

[180 minutes (3 hours); open books]

Solution

April 15, 2020

Question #1

When and why is it convenient to choose a two-stage calibration procedure for the uncertain Denavit-Hartenberg parameters in the kinematic model of a manipulator?

Reply #1

When it is expected that subsets of Denavit-Hartenberg parameters will have a very different uncertainty range (some with large, some with small uncertainty), the calibration procedure is performed in a first stage only for the set of parameters with large uncertainty, holding the others at their nominal values. In a second stage, calibration is completed for all parameters at the same time. In this stage, one starts with the nominal values for the original parameters with small uncertainty and with the updated values for those that have been partially calibrated in the first stage (and thus have now also a small residual uncertainty). This two-stage procedure improves the accuracy of the pseudoinverse solution of the regressor equation by equalizing the numerical conditioning of the regressor matrix. Normalizing a set of equations in this way is very common in optimization and in engineering practice. ■

Question #2

The position $\boldsymbol{p} \in \mathbb{R}^3$ of the origin O_n of the last frame of a n-dof serial manipulator is computed in homogeneous coordinates through the direct kinematics as

$$\begin{pmatrix} \boldsymbol{p} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1\, {}^1\boldsymbol{A}_2 \ldots {}^{i-1}\boldsymbol{A}_i \ldots {}^{n-2}\boldsymbol{A}_{n-1}\, {}^{n-1}\boldsymbol{A}_n \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix},$$

where 4×4 Denavit-Hartenberg homogeneous transformation matrices are used. Suppose that the only uncertainty in the kinematic model is on the value of the twist angle α_i of the ith homogeneous matrix around its nominal value $\alpha_i^{nom} = \pi/2$. Write the expression of the 3×1 regressor matrix $\boldsymbol{\Phi}$ in the basic equation $\Delta \boldsymbol{p} = \boldsymbol{\Phi}\, \Delta\alpha_i$ that is used for calibration at a generic configuration $\boldsymbol{q} \in \mathbb{R}^n$.

Reply #2

We need to evaluate the sensitivity of $\boldsymbol{p} \in \mathbb{R}^3$ with respect to the single scalar parameter α_i, which appears only in the ith Denavit-Hartenberg (DH) homogeneous transformation matrix ${}^{i-1}\boldsymbol{A}_i$. Therefore, by rewriting the direct kinematics in compact form, we have

$$\frac{\partial \boldsymbol{p}_{hom}}{\partial \alpha_i} = \begin{pmatrix} \dfrac{\partial \boldsymbol{p}}{\partial \alpha_i} \\ 0 \end{pmatrix} = {}^0\boldsymbol{A}_{i-1} \frac{\partial \left({}^{i-1}\boldsymbol{A}_i\right)}{\partial \alpha_i}\, {}^i\boldsymbol{A}_n \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix},$$

where the sensitivity of the i-th DH matrix is

$$\begin{aligned} \frac{\partial \left({}^{i-1}\boldsymbol{A}_i\right)}{\partial \alpha_i} &= \frac{\partial}{\partial \alpha_i} \begin{pmatrix} \cos\theta_i & -\sin\theta_i \cos\alpha_i & \sin\theta_i \sin\alpha_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cos\alpha_i & -\cos\theta_i \sin\alpha_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sin\theta_i \sin\alpha_i & \sin\theta_i \cos\alpha_i & 0 \\ 0 & -\cos\theta_i \sin\alpha_i & -\cos\theta_i \cos\alpha_i & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The first-order Taylor expansion of the direct kinematics around the nominal value of α_i is

$$\boldsymbol{p}_{hom}^{nom} + \left.\frac{\partial \boldsymbol{p}_{hom}}{\partial \alpha_i}\right|_{\alpha_i=\alpha_i^{nom}} (\alpha_i - \alpha_i^{nom})$$

$$= {}^0\boldsymbol{A}_{i-1}^{nom} \left({}^{i-1}\boldsymbol{A}_i^{nom} + \left.\frac{\partial \left({}^{i-1}\boldsymbol{A}_i\right)}{\partial \alpha_i}\right|_{DH_i=DH_i^{nom}} (\alpha_i - \alpha_i^{nom}) \right) {}^i\boldsymbol{A}_n^{nom} \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}.$$

Eliminating the nominal identities on the left and right side, we obtain the regressor matrix $\boldsymbol{\Phi}$ (actually, a vector here) as

$$\begin{pmatrix} \boldsymbol{\Phi} \\ 0 \end{pmatrix} = {}^0\boldsymbol{A}_{i-1}^{nom} \left.\frac{\partial \left({}^{i-1}\boldsymbol{A}_i\right)}{\partial \alpha_i}\right|_{DH_i=DH_i^{nom}} {}^i\boldsymbol{A}_n^{nom} \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix},$$

Being $\alpha_i^{nom} = \pi/2$, the evaluation of the sensitivity matrix $\left(\partial^{i-1}\boldsymbol{A}_i\right)/\partial\alpha_i$ in nominal conditions yields

$$\left.\begin{pmatrix} 0 & \sin\theta_i \sin\alpha_i & \sin\theta_i \cos\alpha_i & 0 \\ 0 & -\cos\theta_i \sin\alpha_i & -\cos\theta_i \cos\alpha_i & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right|_{DH_i=DH_i^{nom}} = \begin{pmatrix} 0 & \sin\theta_i^{nom} & 0 & 0 \\ 0 & -\cos\theta_i^{nom} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, the regressor equation is

$$\boldsymbol{\Phi}\, \Delta\alpha_i = \Delta\boldsymbol{p},$$

with $\Delta\alpha_i = \alpha_i - \alpha_i^{nom} \in \mathbb{R}$ and $\Delta\boldsymbol{p} \in \mathbb{R}^3$ being the end-effector position error measured in a generic experiment. ■

Question #3

The differential kinematics of a 3-dof robot performing a two-dimensional task $\boldsymbol{x}$ is expressed by $\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{x}}$. Suppose that, in a given configuration $\boldsymbol{q} \in \mathbb{R}^3$, we have the following values for the task Jacobian $\boldsymbol{J}$ and the desired task velocity $\dot{\boldsymbol{x}}$:

$$\boldsymbol{J} = \begin{pmatrix} 3 & 1 & 2 \\ 1.5 & 0.5 & 1 \end{pmatrix}, \qquad \dot{\boldsymbol{x}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Find the joint velocity $\dot{\boldsymbol{q}}^*$ of minimum norm that realizes at best the desired instantaneous task. Does the task velocity error vanish or not? Find another $\dot{\boldsymbol{q}}' \neq \dot{\boldsymbol{q}}^*$ providing the same task velocity error and show that $\|\dot{\boldsymbol{q}}^*\| < \|\dot{\boldsymbol{q}}'\|$.

Reply #3

It is easy to see that $\text{rank}(\boldsymbol{J}) = 1$, but also that $\dot{\boldsymbol{x}} \in \text{range}\{\boldsymbol{J}\}$ so that we can find (∞^2!) solutions to this underdetermined system of linear equations. The minimum norm solution $\dot{\boldsymbol{q}}^*$ is the one based on the pseudoinverse of $\boldsymbol{J}$, i.e., $\dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^{\#}\dot{\boldsymbol{x}}$, and will yield in this case zero task velocity error (i.e., $\dot{\boldsymbol{x}} - \boldsymbol{J}\dot{\boldsymbol{q}}^{PS} = \boldsymbol{0}$). Since we can discard one of the two equations in $\boldsymbol{J}\dot{\boldsymbol{q}} = \dot{\boldsymbol{x}}$ (because of their linear dependence and consistency), the pseudoinverse solution is easily computed from

$$\boldsymbol{J}_1\dot{\boldsymbol{q}} = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix} \dot{\boldsymbol{q}} = 2 = \dot{x}_1 \quad \Rightarrow \quad \dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}_1^{\#}\dot{x}_1 = \frac{1}{14}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \cdot 2 = \begin{pmatrix} 3/7 \\ 1/7 \\ 2/7 \end{pmatrix} = \begin{pmatrix} 0.4286 \\ 0.1429 \\ 0.2857 \end{pmatrix}.$$

Another solution is found by simple inspection. For instance, being the third column of $\boldsymbol{J}$ equal to $\dot{\boldsymbol{x}}$, the joint velocity $\dot{\boldsymbol{q}}' = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$ is also a solution. Indeed, $\|\dot{\boldsymbol{q}}_{PS}\| = 0.5345 < 1 = \|\dot{\boldsymbol{q}}'\|$. ■

Question #4

A 3R planar robot with links of unitary length moving in a vertical plane has to perform two tasks: i) *follow a trajectory with its end-effector position, and* ii) *keep its last link upwards. At* $\boldsymbol{q} = \begin{pmatrix} \pi/4 & 0 & \pi/4 \end{pmatrix}^T$ *[rad], the desired end-effector linear velocity is* $\boldsymbol{v}_p = \begin{pmatrix} 2 & -1 \end{pmatrix}^T$ *[m/s]. Does there exist a joint velocity* $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ *that executes both tasks simultaneously? If not, find a joint velocity* $\dot{\boldsymbol{q}}_{TP}$ *with the Task Priority method, giving higher priority to the last link orientation task.*

Reply #4

Since

$$\alpha = f_1(\boldsymbol{q}) = q_1 + q_2 + q_3, \qquad \boldsymbol{p} = \boldsymbol{f}_2(\boldsymbol{q}) = \begin{pmatrix} \cos q_1 + \cos(q_1+q_2) + \cos(q_1+q_2+q_3) \\ \sin q_1 + \sin(q_1+q_2) + \sin(q_1+q_2+q_3) \end{pmatrix},$$

the two Jacobians of the link orientation task and, respectively, of the position task are

$$\boldsymbol{J}_1 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \qquad \boldsymbol{J}_2(\boldsymbol{q}) = \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix},$$

with the usual shorthand notation for trigonometric quantities (e.g., $s_{12} = \sin q_1 + \sin(q_1+q_2)$). At $\boldsymbol{q} = \begin{pmatrix} \pi/4 & 0 & \pi/4 \end{pmatrix}^T$, the orientation of the third link is already upwards ($\alpha = \pi/2$), and this would means that no task velocity is needed for keeping the correct link orientation, or $v_\alpha = 0$. The complete task Jacobian matrix and the associated task velocity vector are thus

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_1 \\ \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -2.4142 & -1.7071 & -1 \\ 1.4142 & 0.7071 & 0 \end{pmatrix}, \qquad \boldsymbol{v} = \begin{pmatrix} v_1 \\ \boldsymbol{v}_2 \end{pmatrix} = \begin{pmatrix} v_\alpha \\ \boldsymbol{v}_p \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

The Jacobian $\boldsymbol{J}$ is singular (the sum of its rows is zero), while $\boldsymbol{v} \notin \text{range}\{\boldsymbol{J}\}$ (in fact, the sum of the scalar components of $\boldsymbol{v}$ is 1). This means that the two tasks are in conflict and cannot be executed simultaneously without an error. If attempting a solution with, for instance, the pseudoinverse of $\boldsymbol{J}$ (rather than with the forbidden inverse), we would get

$$\dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^\# \boldsymbol{v} = \begin{pmatrix} -0.8873 \\ -0.1111 \\ 0.6650 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{e}_{v,PS} = \boldsymbol{v} - \boldsymbol{J}^\# \dot{\boldsymbol{q}}_{PS} = \begin{pmatrix} 0.3333 \\ 0.3333 \\ 0.3333 \end{pmatrix},$$

spamming equally the error on all components of both velocity tasks. Instead, consider the Task Priority (TP) method for the two tasks, each with its assigned priority. For the highest priority task, we have

$$\boldsymbol{J}_1^\# = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{P}_1 = \boldsymbol{I} - \boldsymbol{J}_1^\# \boldsymbol{J}_1 = \begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Considering that $v_1 = 0$, the TP method simplifies to

$$\dot{\boldsymbol{q}}_{TP} = \boldsymbol{J}_1^\# v_1 + (\boldsymbol{J}_2 \boldsymbol{P}_1)^\# \left(\boldsymbol{v}_2 - \boldsymbol{J}_2 \boldsymbol{J}_1^\# v_1 \right) = (\boldsymbol{J}_2 \boldsymbol{P}_1)^\# \, \boldsymbol{v}_2,$$

yielding

$$
\begin{aligned}
\dot{\boldsymbol{q}}_{TP} &= \left(\begin{pmatrix} -2.4142 & -1.7071 & 1 \\ 1.4142 & 0.7071 & 0 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \right)^{\#} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} -0.7071 & 0 & 0.7071 \\ 0.7071 & 0 & -0.7071 \end{pmatrix}^{\#} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -0.3536 & 0.3536 \\ 0 & 0 \\ 0.3536 & -0.3536 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1.0607 \\ 0 \\ 1.0607 \end{pmatrix}.
\end{aligned}
$$

Thus, joints 1 and 3 will move with the same speed but in opposite directions so as to satisfy the first task, pushing the error only to the second task. In fact, we have

$$
\boldsymbol{e}_{v,TP} = \boldsymbol{v} - \boldsymbol{J}^{\#} \dot{\boldsymbol{q}}_{TP} = - \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \end{pmatrix},
$$

with $e_{v_1,TP} = 0$. Note, however, that the TP method leads to a larger norm of the error on the linear velocity task than the PS method: $\|\boldsymbol{e}_{v_2,TP}\| = 0.7071 > 0.4243 = \|\boldsymbol{e}_{v_2,PS}\|$. ■

Question #5

A 3R planar robot is moving on a horizontal plane. At a given instant of time t, the robot is in the configuration $\boldsymbol{q}(t) = \begin{pmatrix} 0 & \pi/2 & \pi/4 \end{pmatrix}^T$ [rad], with velocity $\dot{\boldsymbol{q}}(t) = \begin{pmatrix} \pi/2 & -\pi/4 & \pi/8 \end{pmatrix}^T$ [rad/s]. If the applied torque is $\boldsymbol{u}(t) = \begin{pmatrix} 1.5 & 0 & -4 \end{pmatrix}^T$ [Nm], will the instantaneous total energy E of the robot increase, stay the same, or decrease? And what about the Lagrangian function L?

Reply #5

Since the robot moves with constant potential energy U, we have $\dot{U} = 0$. Then, the instantaneous variation $\dot{E}$ of the total energy $E = T + U$ and the instantaneous variation $\dot{L}$ of the Lagrangian function $L = T - U$ will be the same. At the time instant t, we have

$$
\dot{E}(t) = \dot{L}(t)\, (= \dot{T}(t)) = \dot{\boldsymbol{q}}^T(t)\boldsymbol{u}(t) = \begin{pmatrix} \pi/2 & -\pi/4 & \pi/8 \end{pmatrix}^T \begin{pmatrix} 1.5 \\ 0 \\ -4 \end{pmatrix} = 0.7854 > 0.
$$

Thus, the total energy of the robot and its Lagrangian will instantaneously increase. ■

Question #6

Given the inertia matrix of a 2R polar robot

$$
\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + a_2 \sin^2 q_2 + a_3 \cos^2 q_2 & 0 \\ 0 & a_4 \end{pmatrix},
$$

find two factorizations of the associated Coriolis/centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}'(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{S}''(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ such that the matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}'$ is skew symmetric, while the matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}''$ is not.

Reply #6

We compute the velocity terms using the matrices $\boldsymbol{C}_i$ of Christoffel's symbols. These are also helpful for defining a factorization that satisfies the requested skew-symmetric property. For the components of vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, we have:

$$
c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \quad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{M}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{M}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial \boldsymbol{q}_i} \right), \quad \text{for } i = 1, 2,
$$

being $\boldsymbol{M}_i$ the ith column of the inertia matrix $\boldsymbol{M}$. We obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\begin{pmatrix} 0 & 2(a_2 - a_3)\sin q_2 \cos q_2 \\ 2(a_2 - a_3)\sin q_2 \cos q_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2}(a_2 - a_3)\sin(2q_2) \\ \frac{1}{2}(a_2 - a_3)\sin(2q_2) & 0 \end{pmatrix}$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = -\frac{1}{2}\begin{pmatrix} 2(a_2 - a_3)\sin q_2 \cos q_2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(a_2 - a_3)\sin(2q_2) & 0 \\ 0 & 0 \end{pmatrix},$$

leading to

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{pmatrix} = \begin{pmatrix} (a_2 - a_3)\sin(2q_2)\dot{q}_1\dot{q}_2 \\ -\frac{1}{2}(a_2 - a_3)\sin(2q_2)\dot{q}_1^2 \end{pmatrix}.$$

We need at this point the time derivative of the inertia matrix, i.e.,

$$\dot{\boldsymbol{M}} = \begin{pmatrix} (a_2 - a_3)\sin(2q_2)\dot{q}_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

A factorization $\boldsymbol{S}'$ that satisfies the skew-symmetric property is then given by

$$\boldsymbol{S}'(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(a_2 - a_3)\sin(2q_2)\,\dot{q}_2 & \frac{1}{2}(a_2 - a_3)\sin(2q_2)\,\dot{q}_1 \\ -\frac{1}{2}(a_2 - a_3)\sin(2q_2)\,\dot{q}_1 & 0 \end{pmatrix},$$

being $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}'(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ and

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}' = \begin{pmatrix} 0 & -(a_2 - a_3)\sin(2q_2)\,\dot{q}_1 \\ (a_2 - a_3)\sin(2q_2)\,\dot{q}_1 & 0 \end{pmatrix}.$$

A possible factorization $\boldsymbol{S}''$ that, on the contrary, fails to satisfy the skew-symmetric property is

$$\boldsymbol{S}''(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} (a_2 - a_3)\sin(2q_2)\,\dot{q}_2 & 0 \\ -\frac{1}{2}(a_2 - a_3)\sin(2q_2)\,\dot{q}_1 & 0 \end{pmatrix}.$$

In fact, one can verify that $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}''(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$, but the matrix

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}'' = \begin{pmatrix} -(a_2 - a_3)\sin(2q_2)\dot{q}_1 & 0 \\ (a_2 - a_3)\sin(2q_2)\,\dot{q}_1 & 0 \end{pmatrix},$$

is not skew-symmetric. ■

Question #7

Consider the PPR planar robot in the figure below. Using the coordinates $\boldsymbol{q} \in \mathbb{R}^3$ and the dynamic parameters defined therein, determine the expression of the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. Provide then a linear parametrization only of the inertial terms in the dynamic model, i.e., such that

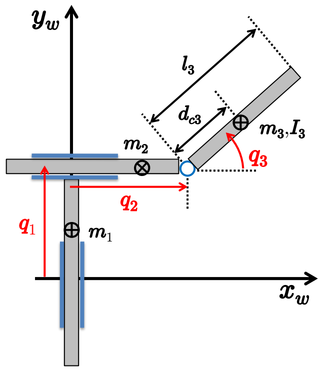
$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{Y}_M(\boldsymbol{q}, \ddot{\boldsymbol{q}})\,\boldsymbol{a}_M,$$

where the $3 \times p$ regressor matrix $\boldsymbol{Y}_M$ and the vector of dynamic coefficients $\boldsymbol{a}_M \in \mathbb{R}^p$ have the least possible dimension p.

Reply #7

The first two simple contributions to the robot kinetic energy are

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2 \qquad \text{and} \qquad T_2 = \frac{1}{2} m_2 \left(\dot{q}_1^2 + \dot{q}_2^2 \right).$$



For T_3, we compute first (in the plane)

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_2 + d_{c3} \cos q_3 \\ q_1 + d_{c3} \sin q_3 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} \dot{q}_2 - d_{c3} \sin q_3 \, \dot{q}_3 \\ \dot{q}_1 + d_{c3} \cos q_3 \, \dot{q}_3 \end{pmatrix}$$

$$\Rightarrow \quad \left\| \boldsymbol{v}_{c3} \right\|^2 = \dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2 \dot{q}_3^2 + 2 d_{c3} \left(\cos q_3 \, \dot{q}_1 - \sin q_3 \, \dot{q}_2 \right) \dot{q}_3$$

and then

$$T_3 = \frac{1}{2} I_3 \dot{q}_3^2 + \frac{1}{2} m_3 \left(\dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2 \dot{q}_3^2 + 2 d_{c3} \left(\cos q_3 \, \dot{q}_1 - \sin q_3 \, \dot{q}_2 \right) \dot{q}_3 \right).$$

The total kinetic energy of the robot is thus

$$T = T_1 + T_2 + T_3 = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \frac{1}{2} \dot{\boldsymbol{q}}^T \begin{pmatrix} m_1 + m_2 + m_3 & 0 & m_3 d_{c3} \cos q_3 \\ 0 & m_2 + m_3 & -m_3 d_{c3} \sin q_3 \\ m_3 d_{c3} \cos q_3 & -m_3 d_{c3} \sin q_3 & I_3 + m_3 d_{c3}^2 \end{pmatrix} \dot{\boldsymbol{q}}.$$

By introducing a vector $\boldsymbol{a}_M \in \mathbb{R}^4$ of dynamic coefficients, the inertia matrix can be rewritten as

$$\boldsymbol{a}_M = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} m_1 + m_2 + m_3 \\ m_2 + m_3 \\ I_3 + m_3 d_{c3}^2 \\ m_3 d_{c3} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 & 0 & a_4 \cos q_3 \\ 0 & a_2 & -a_4 \sin q_3 \\ a_4 \cos q_3 & -a_4 \sin q_3 & a_3 \end{pmatrix}.$$

Clearly, $p = 4$ is the minimum number of dynamic coefficients for this robot. The linear parametrization of the inertial terms is

$$\boldsymbol{M}(\boldsymbol{q}) \ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot{q}_1 & 0 & 0 & \cos q_3 \, \ddot{q}_3 \\ 0 & \ddot{q}_2 & 0 & -\sin q_3 \, \ddot{q}_3 \\ 0 & 0 & \ddot{q}_3 & \cos q_3 \, \ddot{q}_1 - \sin q_3 \, \ddot{q}_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \boldsymbol{Y}_M(\boldsymbol{q}, \ddot{\boldsymbol{q}}) \, \boldsymbol{a}_M. \qquad \blacksquare$$

Question #8

Provide the inertia matrix $\boldsymbol{M}_p(\boldsymbol{p})$ of the robot considered in Question **#7** *when using for the Lagrangian dynamic modeling the new set of coordinates $\boldsymbol{p} = \begin{pmatrix} x & y & \alpha \end{pmatrix}^T \in \mathbb{R}^3$, where (x, y) are the components of the Cartesian position of the robot end-effector in world coordinates and α is the angle of the last link w.r.t. the $\boldsymbol{x}_w$ axis of the world frame.*

Reply #8

The change of coordinates

$$\boldsymbol{p} = \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix} = \begin{pmatrix} q_2 + l_3 \cos q_3 \\ q_1 + l_3 \sin q_3 \\ q_3 \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q})$$

represents the desire to use Cartesian variables for describing the dynamics of a PPR robot. The change of coordinates is here a diffeomorphism (i.e., a differentiable mapping with a unique and differentiable inverse) in $\mathbb{R}_2 \times SO(1)$. Its inverse is

$$\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{p}) = \begin{pmatrix} p_2 - l_3 \sin p_3 \\ p_1 - l_3 \cos p_3 \\ p_3 \end{pmatrix},$$

while the Jacobian matrix of the transformation (and its inverse) takes the form

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & 1 & -l_3 \sin q_3 \\ 1 & 0 & l_3 \cos q_3 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{J}^{-1}(\boldsymbol{q}) = \begin{pmatrix} 0 & 1 & -l_3 \cos q_3 \\ 1 & 0 & l_3 \sin q_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

The inertia matrix of the PPR robot in the new coordinates is obtained as

$$\boldsymbol{M}_p(\boldsymbol{p}) = \left(\boldsymbol{J}^{-T}(\boldsymbol{q}) \boldsymbol{M}(\boldsymbol{q}) \boldsymbol{J}^{-1}(\boldsymbol{q}) \right) \Big|_{\boldsymbol{q}=\boldsymbol{f}^{-1}(\boldsymbol{p})}$$

in which all quantities have been already defined. ■

Additional remark: Further elaboration of the above expression is straightforward but lengthy (and beyond the scope of the present question). Nonetheless, using the Matlab Symbolic Toolbox, it can be shown that the explicit expression of $\boldsymbol{M}_p$ as a function of $\boldsymbol{p}$ only can be rewritten as

$$\boldsymbol{M}_p(\boldsymbol{p}) = \begin{pmatrix} a_{p2} & 0 & -a_{p4} \sin p_3 \\ 0 & a_{p1} & a_{p5} \cos p_3 \\ -a_{p4} \sin p_3 & a_{p5} \cos p_3 & a_{p3} + (a_{p1} - a_{p2})\, l_3^2 \cos^2 p_3 \end{pmatrix},$$

where a new set of dynamic coefficients $\boldsymbol{a}_p \in \mathbb{R}^5$ has been introduced for compactness, defined in terms of the dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^4$ already present in $\boldsymbol{M}(\boldsymbol{q})$. These new dynamic coefficients are

$$\boldsymbol{a}_p = \begin{pmatrix} a_1 \\ a_2 \\ a_3 + a_2 l_3^2 - 2 a_4 l_3 \\ a_4 - a_2 l_3 \\ a_4 - a_1 l_3 \end{pmatrix}.$$

Indeed, note that the 5 coefficients in $\boldsymbol{a}_p$ are not a minimal dynamic set: they can be expressed in fact as linear combinations of the 4 coefficients in $\boldsymbol{a}$, provided that the length l_3 of the third link (a kinematic quantity) is known.

Question #9

A single link moving under gravity is modeled by the differential equation $I\ddot{\theta} + mg_0 d \sin\theta = u$*, with* $m = 3$ *[kg],* $d = 0.5$ *[m],* $I = 1$ *[kgm*^{2}*], and* $g_0 = 9.81$ *[m/s*^{2}*]. The motor torque is bounded by* $|u| \leq U = 25$ *[Nm]. The desired task is a rest-to-rest swing-up maneuver from* $\theta(0) = 0$ *to* $\theta(T) = \pi$ *[rad] in* $T = 1$ *[s], to be done with a bang-bang acceleration profile. Is the torque bound satisfied? If not, find the minimum uniform time scaling to execute the task in a feasible way.*

Reply #9

We start by determining the value A of the piecewise constant (bang-bang) acceleration profile requested for executing the desired trajectory, given $\Delta\theta = \theta(T) - \theta(0) = \pi$ and $T = 1$. Starting at rest, the velocity $\dot{\theta}$ will grow linearly up to the midtime $t = T/2$, reaching a value $V = A \cdot T/2$, and then returning linearly to zero at $t = T$. The area underlying the triangular velocity profile is equal to the angular displacement $\Delta\theta$. Thus,

$$\Delta\theta = \int_0^T \dot{\theta}\, d\tau = \frac{V \cdot T}{2} = \frac{A \cdot T^2}{4} \qquad \Rightarrow \qquad A = \frac{4\Delta\theta}{T^2} = 4\pi \text{ [rad/s}^2\text{]}.$$

The inertial term in the dynamic model will have a constant value $u_i = I\ddot{\theta}(t) = IA = 4\pi$ in the first half of the motion, $t \in [0, T/2]$, until the link reaches the midpoint $\Delta/2 = \Pi/2$ of the motion trajectory; during the second half, $t \in (T/2, T]$, this inertial term will have the same amplitude but a negative sign. On the other hand, the gravitational torque $u_g = mg_0 d \sin\theta(t) = 14.715 \sin\theta(t)$ will grow from zero to its maximum at $t = T/2$, when $\theta(T/2) = \Delta\theta/2 = \pi/2$ and $u_{g,max} = 14.7150$, and return then symmetrically to zero. As a result, the maximum (positive) torque requested by the desired trajectory is attained at $t = T/2 = 1$ [s] and is equal to $u_{max} = u_i + u_{g,max} = 27.2814 > 25 = U$ [Nm], exceeding so the motor torque bound. The original trajectory is unfeasible. We need then to uniformly slow down motion by a factor $k > 1$, in order to reduce the inertial acceleration component of the inverse dynamics torque (the gravitational torque u_g is unaffected by any time scaling). Since the inertial torque scales with k^2 (quadratically), the minimum scaling factor k is computed as

$$k = \max\left\{1, \sqrt{\frac{u_{max} - u_{g,max}}{U - u_{g,max}}}\right\} = \max\left\{1, \sqrt{\frac{4\pi}{10.2850}}\right\} = 1.1054.$$

The new motion time will be $T_s = kT = 1.1054$ [s] and the peak of the total torque will be again assumed at $t = T_s/2$, where $\theta(T_s/2) = \pi/2$. Without the need of a new inverse dynamics analysis, this is computed as

$$u_{max,s} = \frac{u_{max} - u_{g,max}}{k^2} + u_{g,max} = \frac{u_i}{k^2} + u_{g,max} = (U - u_{g,max}) + u_{g,max} = U = 25 \text{ [Nm]}. \qquad \blacksquare$$

Additional material: Plots of the relevant quantities obtained using Matlab are reported at the end for the original trajectory (Figure 1) and for the scaled, feasible one (Figure 2).

Question #10

Assume that we have available the Newton-Euler routine $\mathrm{NE}_\alpha(\mathrm{arg}_1, \mathrm{arg}_2, \mathrm{arg}_3)$*, equipped with the kinematic and dynamic data of a n-dof serial manipulator. How can we compute the kinetic energy* T *in a generic state* $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ *of this robot by just one call of this routine and one scalar product?*

Reply #10

We compute first the Newton-Euler routine output $\boldsymbol{y} = \frac{1}{2}\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = NE_0\left(\boldsymbol{q}, \boldsymbol{0}, \frac{1}{2}\dot{\boldsymbol{q}}\right)$ and then obtain the result with a scalar product: $\dot{\boldsymbol{q}}^T\boldsymbol{y} = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = T(\boldsymbol{q}, \dot{\boldsymbol{q}})$. $\blacksquare$

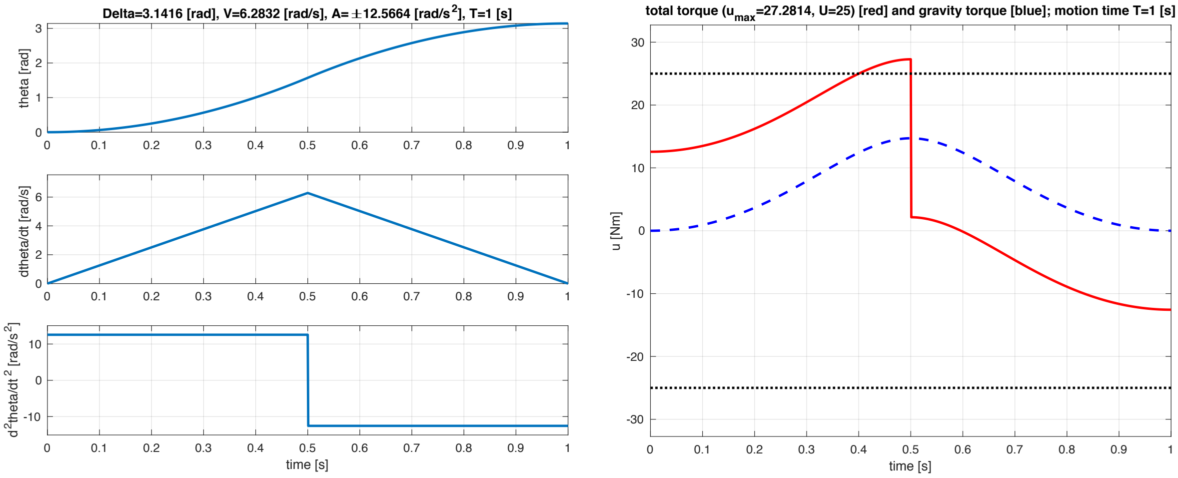


Figure 1: Position, velocity and acceleration profiles [left] and total and gravitational torques [right] for the original unfeasible trajectory with $T = 1$ [s].

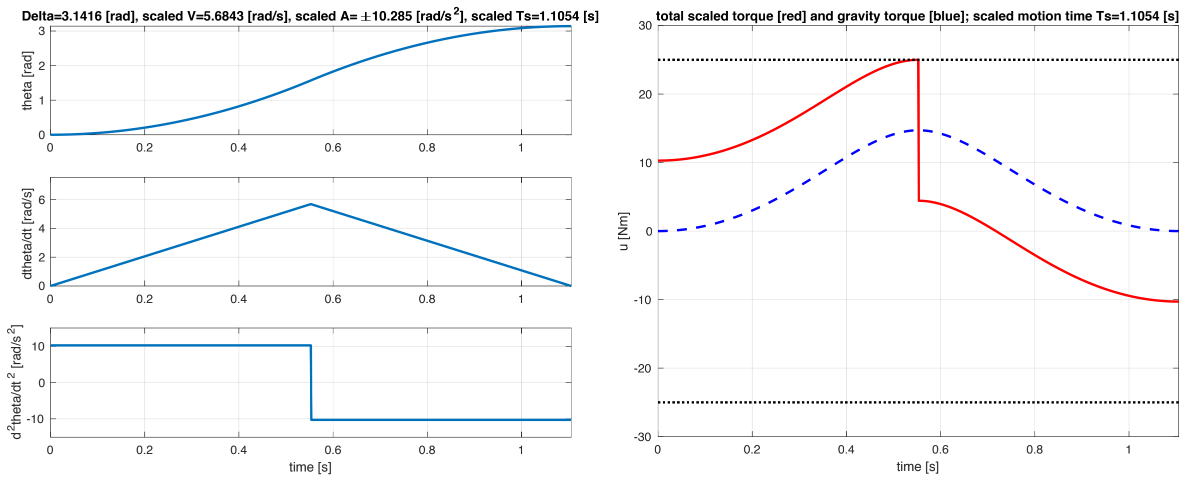


Figure 2: Position, velocity and acceleration profiles [left] and total and gravitational torques [right] for the scaled feasible trajectory with $T_s = 1.1054$ [s].

* * * * *

Robotics 2

Remote Midterm Test – April 14, 2021

Exercise #1

The 2R robot in Fig. 1 moves in a vertical plane. The two links have, respectively, kinematic lengths l_1 and l_2, masses m_1 and m_2, and barycentric inertias I_1 and I_2 (around the axis normal to the motion plane). The position of the center of mass (CoM) of each link with respect to the attached link frame is given by $\boldsymbol{r}_{ci} = \begin{pmatrix} r_{ci,x} & r_{ci,y} & 0 \end{pmatrix}^T$, with $r_{ci,x} \neq -l_i$ and $r_{ci,y} \neq 0$, for $i = 1, 2$.

A) Determine the robot dynamic model, $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u}$, neglecting dissipative effects.

B) Provide a linear parametrization of the model, $\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{u}$, in terms of a regressor matrix $\boldsymbol{Y} \in \mathbb{R}^{2\times p}$ and a vector $\boldsymbol{a} \in \mathbb{R}^p$ of dynamic coefficients.

Exercise #2

Consider the planar 3R robot with links of unitary length in Fig. 2 and use the absolute coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ defined therein. The robot is commanded with the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$. Starting from the configuration $\boldsymbol{q}(0) = (\pi/4, 0, 0)$, the robot should simultaneously move its end-effector point P along a vertical line parallel to $\boldsymbol{y}_0$ with a constant speed $v > 0$, while keeping the second link horizontal. Determine the first encountered configuration $\boldsymbol{q}_s$ at which these two tasks run into an algorithmic singularity. In $\boldsymbol{q} = \boldsymbol{q}_s$ and for $v = 1$ [m/s], compute the following three commands:

a) $\dot{\boldsymbol{q}}_{PS}$ using pseudoinversion of the extended Jacobian of the two tasks;

b) $\dot{\boldsymbol{q}}_{DLS}$ using damped least squares on the extended Jacobian, with damping parameter $\mu^2 = 0.25$;

c) $\dot{\boldsymbol{q}}_{TP}$ using the task priority method, with the end-effector task having the highest priority.

Compare in the three cases the norm of the resulting joint velocity, the norm of the end-effector velocity error $\dot{\boldsymbol{e}}_P \in \mathbb{R}^2$, and the absolute value of the second joint velocity error $\dot{e}_{q_2} \in \mathbb{R}$.

Exercise #3

The dynamic model of a PR robot moving on a horizontal plane is given in the lecture slides[1]. The end-effector point P should trace in minimum time a circular path of radius $R = l_2$

$$\boldsymbol{p}(s) = \begin{pmatrix} k + R\cos(s-\alpha) \\ R\sin(s-\alpha) \end{pmatrix}, \qquad s = [0, 2\alpha] \qquad (0 < \alpha < \frac{\pi}{2}),$$

from rest to rest between P_i and P_f (see Fig. 3), under bounded input force/torque $|u_i| \leq U_{i,max}$,, $i = 1, 2$. Assuming that the second bound $U_{2,max}$ is the only limiting factor, provide the expression of the needed input $\boldsymbol{u}_d(t) \in \mathbb{R}^2$ and of the minimum time T. Sketch the time profiles of the inputs.

[180 minutes (3 hours); open books]

[1] Block `03_LagrangianDynamics_1.pdf`, slide #25.

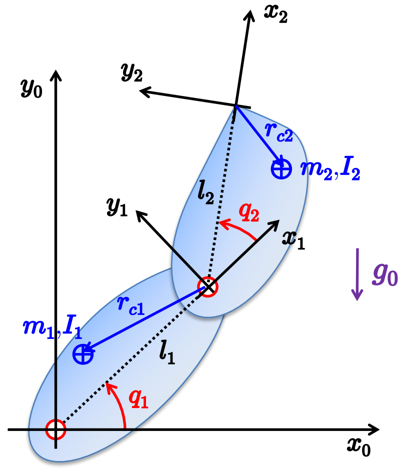


Figure 1: A planar 2R robot having the link CoMs in generic positions.

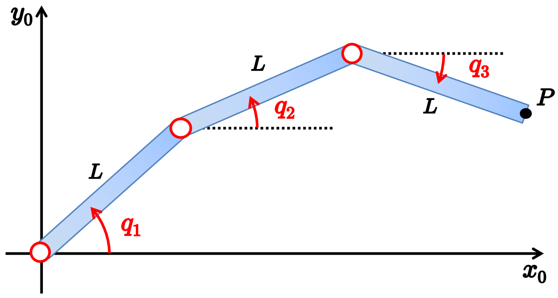


Figure 2: A planar 3R robot, with absolute coordinates $\boldsymbol{q}$ and equal links of length $L = 1$ m.

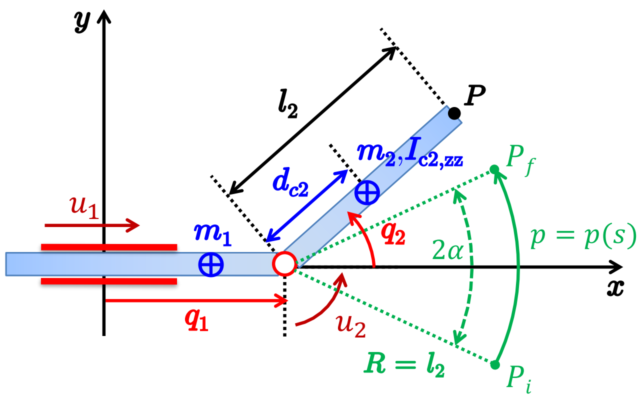


Figure 3: The assigned motion task for a planar PR robot.

Solution

April 14, 2021

Exercise #1

The special feature of this planar 2R arm is the generic location of the CoM of the links, not necessarily placed on the kinematic link axis. A Lagrangian approach is followed for dynamic modeling. We can work either with vectors in 3D or in 2D, considering the planar nature of the problem[2]. In the first case, we may also use the recursive algorithm with moving frames (the result is indeed the same).

Kinetic energy. For the first link, we have

$$T_1 = \frac{1}{2}\, m_1 \|\boldsymbol{v}_{c1}\|^2 + \frac{1}{2}\, \boldsymbol{\omega}_1^T \boldsymbol{I}_1\, \boldsymbol{\omega}_1 = \frac{1}{2}\, m_1 \left((l_1 + r_{c1,x})^2 + r_{c1,y}^2 \right) \dot{q}_1^2 + \frac{1}{2}\, I_1 \dot{q}_1^2,$$

where the coefficient in parentheses multiplying m_1 is the squared distance of the CoM of link 1 from the axis of joint 1. For the second link, we have

$$T_2 = \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} \boldsymbol{\omega}_2^T \boldsymbol{I}_2\, \boldsymbol{\omega}_2 = \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} I_2 \left(\dot{q}_1 + \dot{q}_2\right)^2.$$

The velocity of the CoM of link 2 can be computed in two alternative ways. We can start from the absolute position of the CoM in 2D,

$$^0\boldsymbol{p}_{c2} = \begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} + {}^0\bar{\boldsymbol{R}}_2(q_1, q_2) \begin{pmatrix} l_2 + r_{c2,x} \\ r_{c2,y} \end{pmatrix}, \qquad {}^0\bar{\boldsymbol{R}}_2(q_1, q_2) = \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix},$$

leading to

$$^0\boldsymbol{v}_{c2} = {}^0\dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -l_1 s_1 \\ l_1 c_1 \end{pmatrix} \dot{q}_1 + \begin{pmatrix} -\left(l_2 + r_{c2,x}\right) s_{12} - r_{c2,y}\, c_{12} \\ \left(l_2 + r_{c2,x}\right) c_{12} - r_{c2,y}\, s_{12} \end{pmatrix} (\dot{q}_1 + \dot{q}_2).$$

Or we can work with velocities in 3D and rely on moving frames; in this case, using

$$^1\boldsymbol{\omega}_2 = {}^2\boldsymbol{\omega}_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{pmatrix}, \qquad {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we compute

$$\begin{aligned} ^2\boldsymbol{v}_2 &= {}^1\boldsymbol{R}_2^T(q_2) \left({}^1\boldsymbol{v}_1 + {}^1\boldsymbol{\omega}_2 \times {}^1\boldsymbol{r}_{12} \right) = {}^1\boldsymbol{R}_2^T(q_2) \begin{pmatrix} 0 \\ l_1 \dot{q}_1 \\ 0 \end{pmatrix} + {}^1\boldsymbol{R}_2^T(q_2) {}^1\boldsymbol{\omega}_2 \times {}^1\boldsymbol{R}_2^T(q_2) {}^1\boldsymbol{r}_{12} \\ &= {}^2\boldsymbol{v}_1 + {}^2\boldsymbol{\omega}_2 \times {}^2\boldsymbol{r}_{12} = \begin{pmatrix} l_1 s_2\, \dot{q}_1 \\ l_1 c_2\, \dot{q}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} l_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 s_2\, \dot{q}_1 \\ l_1 c_2\, \dot{q}_1 + l_2\left(\dot{q}_1 + \dot{q}_2\right) \\ 0 \end{pmatrix}, \end{aligned}$$

and then

$$^2\boldsymbol{v}_{c2} = {}^2\boldsymbol{v}_2 + {}^2\boldsymbol{\omega}_2 \times {}^2\boldsymbol{r}_{c2} = {}^2\boldsymbol{v}_2 + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{pmatrix} \times \begin{pmatrix} r_{c2,x} \\ r_{c2,y} \\ 0 \end{pmatrix} = \begin{pmatrix} l_1 s_2\, \dot{q}_1 - r_{c2,y}\left(\dot{q}_1 + \dot{q}_2\right) \\ l_1 c_2\, \dot{q}_1 + \left(l_2 + r_{c2,x}\right)\left(\dot{q}_1 + \dot{q}_2\right) \\ 0 \end{pmatrix}.$$

[2] We use the compact trigonometric notation throughout this exercise, e.g., $c_{12} = \cos(q_1 + q_2)$.

As a result

$$\left\|{}^{2}\boldsymbol{v}_{c2}\right\|^{2} = l_1^2\,\dot{q}_1^2 + \left(\left(l_2 + r_{c2,x}\right)^2 + r_{c2,y}^2\right)\left(\dot{q}_1 + \dot{q}_2\right)^2 + 2l_1\left(\left(l_2 + r_{c2,x}\right)c_2 - r_{c2,y}s_2\right)\dot{q}_1\left(\dot{q}_1 + \dot{q}_2\right).$$

Indeed, it is $\left\|{}^{2}\boldsymbol{v}_{c2}\right\|^{2} = \left\|{}^{0}\boldsymbol{v}_{c2}\right\|^{2}$. But computations (and simplifications) are easier when using the moving frames. The total kinetic energy is thus

$$\begin{aligned} T = T_1 + T_2 = &\ \frac{1}{2}\left(I_1 + m_1\left(\left(l_1 + r_{c1,x}\right)^2 + r_{c1,y}^2\right) + m_2 l_1^2 + I_2 + m_2\left(\left(l_2 + r_{c2,x}\right)^2 + r_{c2,y}^2\right)\right. \\ &\quad \left. + 2m_2 l_1\left(\left(l_2 + r_{c2,x}\right)c_2 - r_{c2,y}s_2\right)\right)\dot{q}_1^2 + \frac{1}{2}\left(I_2 + m_2\left(\left(l_2 + r_{c2,x}\right)^2 + r_{c2,y}^2\right)\right)\dot{q}_2^2 \\ &+ \left(I_2 + m_2\left(\left(l_2 + r_{c2,x}\right)^2 + r_{c2,y}^2\right) + m_2 l_1\left(\left(l_2 + r_{c2,x}\right)c_2 - r_{c2,y}s_2\right)\right)\dot{q}_1\dot{q}_2 \\ = &\ \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}. \end{aligned}$$

Inertia matrix. One can organize the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ in a compact way, by introducing the following dynamic coefficients:

$$\begin{aligned} a_1 &= I_1 + m_1\big((l_1 + r_{c1,x})^2 + r_{c1,y}^2\big) + m_2 l_1^2 + I_2 + m_2\big((l_2 + r_{c2,x})^2 + r_{c2,y}^2\big) \\ a_2 &= m_2 l_1 (l_2 + r_{c2,x}) \\ a_3 &= -m_2 l_1 r_{c2,y} \\ a_4 &= I_2 + m_2\big((l_2 + r_{c2,x})^2 + r_{c2,y}^2\big) \end{aligned} \tag{1}$$

As a result,

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2c_2 + 2a_3s_2 & a_4 + a_2c_2 + a_3s_2 \\ a_4 + a_2c_2 + a_3s_2 & a_4 \end{pmatrix} \tag{2}$$

This compact form is useful for the following derivation of the velocity terms in the dynamic model. Note that the asymmetric location of the CoM of link 2 w.r.t. the link axis $\boldsymbol{x}_2$ (i.e., $r_{c2,y} \neq 0$) has introduced the extra dynamic coefficient a_3 and modified the two coefficients a_1 and a_4. On the other hand, asymmetry in the CoM of link 1 (i.e., $r_{c1,y} \neq 0$) modifies only a_1.

Coriolis and centrifugal terms. Denoting by $\boldsymbol{M}_i$ the ith column of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$, we compute the components of the Coriolis/centrifugal vector $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})$ using the Christoffel symbols:

$$c_i(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T\boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{M}_i}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{M}_i}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}}{\partial q_i}\right), \qquad i = 1, 2.$$

We obtain

$$\begin{aligned} \boldsymbol{C}_1(\boldsymbol{q}) &= \frac{1}{2}\left(\begin{pmatrix} 0 & -2a_2s_2 + 2a_3c_2 \\ 0 & -a_2s_2 + a_3c_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -2a_2s_2 + 2a_3c_2 & -a_2s_2 + a_3c_2 \end{pmatrix} - \boldsymbol{O}\right) \\ &= \begin{pmatrix} 0 & -a_2s_2 + a_3c_2 \\ -a_2s_2 + a_3c_2 & -a_2s_2 + a_3c_2 \end{pmatrix} \;\Rightarrow\; \begin{aligned} c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \left(a_3c_2 - a_2s_2\right)\left(2\dot{q}_1 + \dot{q}_2\right)\dot{q}_2 \\ &= -m_2 l_1\left(r_{c2,y}c_2 + (l_2 + r_{c2,x})s_2\right)\left(2\dot{q}_1 + \dot{q}_2\right)\dot{q}_2 \end{aligned} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{C}_2(\boldsymbol{q}) &= \frac{1}{2}\left(\begin{pmatrix} 0 & -a_2s_2 + a_3c_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -a_2s_2 + a_3c_2 & 0 \end{pmatrix} - \begin{pmatrix} 2a_3c_2 - 2a_2s_2 & a_3c_2 - a_2s_2 \\ a_3c_2 - a_2s_2 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} -(a_3c_2 - a_2s_2) & 0 \\ 0 & 0 \end{pmatrix} \;\Rightarrow\; \begin{aligned} c_2(\boldsymbol{q},\dot{\boldsymbol{q}}) &= -\left(a_3c_2 - a_2s_2\right)\dot{q}_1^2 \\ &= m_2 l_1\left(r_{c2,y}c_2 + (l_2 + r_{c2,x})s_2\right)\dot{q}_1^2 \end{aligned} \end{aligned}$$

Thus, the final expression of the quadratic velocity terms in the model is

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = (a_3 c_2 - a_2 s_2)\begin{pmatrix} \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2 \\ -\dot{q}_1^2 \end{pmatrix}. \tag{3}$$

Potential energy and gravity term. From the expression of the potential energy of a generic link i in the serial chain,

$$U_i = -m_i\,\boldsymbol{g}_0^T \boldsymbol{r}_{0,ci},$$

we obtain for the first link

$$U_1 = -m_1 \begin{pmatrix} 0 & -g_0 & 0 \end{pmatrix} \begin{pmatrix} * \\ (l_1 + r_{c1,x})\, s_1 + r_{c1,y} c_1 \\ * \end{pmatrix} = m_1 g_0 \big((l_1 + r_{c1,x})\, s_1 + r_{c1,y} c_1\big),$$

and for the second link

$$U_2 = -m_2 \begin{pmatrix} 0 & -g_0 & 0 \end{pmatrix} \begin{pmatrix} * \\ l_1 s_1 + (l_2 + r_{c2,x})\, s_{12} + r_{c2,y} c_{12} \\ * \end{pmatrix} = m_2 g_0 \big(l_1 s_1 + (l_2 + r_{c2,x})\, s_{12} + r_{c2,y} c_{12}\big).$$

From $U = U_1 + U_2$, we have

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0\left((m_1 (l_1 + r_{c1,x}) + m_2 l_1)\, c_1 - m_1 r_{c1,y} s_1 + m_2 (l_2 + r_{c2,x})\, c_{12} - m_2 r_{c2,y} s_{12}\right) \\ m_2 g_0 \big((l_2 + r_{c2,x})\, c_{12} - r_{c2,y} s_{12}\big) \end{pmatrix}.$$

The gravity vector in the dynamic model is rewritten more compactly as

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} a_5 c_1 + a_6 s_1 + a_7 c_{12} + a_8 s_{12} \\ a_7 c_{12} + a_8 s_{12} \end{pmatrix} \tag{4}$$

by introducing the additional dynamic coefficients

$$\begin{aligned} a_5 &= g_0 \big(m_1 (l_1 + r_{c1,x}) + m_2 l_1\big) \\ a_6 &= -m_1 g_0 r_{c1,y} \\ a_7 &= m_2 g_0 (l_2 + r_{c2,x}) \\ a_8 &= -m_2 g_0 r_{c2,y}. \end{aligned} \tag{5}$$

In the gravity term, the asymmetric location of the CoM of each link introduces a single extra dynamic coefficient, namely a_6 for the first link and a_8 for the second. Instead, the two other gravity coefficients of a 2R robot with symmetric CoMs are not modified.

Linear parametrization. The complete dynamic model of the considered 2R robot,

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u}, \tag{6}$$

is obtained by using (2), (3), and (4). We have already introduced the inertia-related dynamic coefficients in (1) and the gravity-related ones in (5). The linear factorization of the model (6),

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a}, \tag{7}$$

is immediately obtained in terms of the coefficient vector $\boldsymbol{a} \in \mathbb{R}^8$. The regressor matrix in (7) is

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) = \begin{pmatrix} \ddot{q}_1 & \begin{matrix} c_2 (2\ddot{q}_1 + \ddot{q}_2) \\ -\, s_2 \dot{q}_2 (2\dot{q}_1 + \dot{q}_2) \end{matrix} & \begin{matrix} s_2 (2\ddot{q}_1 + \ddot{q}_2) \\ +\, c_2 \dot{q}_2 (2\dot{q}_1 + \dot{q}_2) \end{matrix} & \ddot{q}_2 & c_1 & s_1 & c_{12} & s_{12} \\ 0 & c_2 \ddot{q}_1 + s_2 \dot{q}_1^2 & s_2 \ddot{q}_1 - c_2 \dot{q}_1^2 & \ddot{q}_1 + \ddot{q}_2 & 0 & 0 & c_{12} & s_{12} \end{pmatrix}. \tag{8}$$

We finally note that, assuming both the link length l_1 and the gravity acceleration g_0 to be known, the number of independent dynamic coefficients reduces from $p = 8$ to $p = 6$. In facts, two pairs of dynamic coefficients collapse:

$$\left.\begin{aligned} a_2 &= m_2\, l_1\, (l_2 + r_{c2,x}) = l_1\, a_2' \\ a_7 &= m_2\, g_0\, (l_2 + r_{c2,x}) = g_0\, a_2' \end{aligned}\right\} \iff a_2' = m_2\, (l_2 + r_{c2,x})\,,$$

$$\left.\begin{aligned} a_3 &= -m_2\, l_1\, r_{c2,y} = l_1\, a_3' \\ a_8 &= -m_2\, g_0\, r_{c2,y} = g_0\, a_3' \end{aligned}\right\} \iff a_3' = -m_2\, r_{c2,y}.$$

The first merging is present also in the 2R robot with symmetric CoMs. The second is related to the asymmetric case only.

Exercise #2

Taking into account the use of absolute coordinates (link angles w.r.t. the $\boldsymbol{x}_0$ axis) for this planar robot with $n = 3$ and unitary link lengths, the kinematics of the first task (of dimension $m_1 = 2$) involving the position $\boldsymbol{p}$ of the end-effector point P is

$$\boldsymbol{r}_1 = \boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \cos q_1 + \cos q_2 + \cos q_3 \\ \sin q_1 + \sin q_2 + \sin q_3 \end{pmatrix} = \boldsymbol{f}_1(\boldsymbol{q}), \tag{9}$$

with associated Jacobian

$$\boldsymbol{J}_1(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_1(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\sin q_1 & -\sin q_2 & -\sin q_3 \\ \cos q_1 & \cos q_2 & \cos q_3 \end{pmatrix}. \tag{10}$$

From (9), the end-effector position in the initial configuration $\boldsymbol{q}(0) = (\pi/4, 0, 0)$ is

$$\boldsymbol{p}(0) = \boldsymbol{f}_1(\boldsymbol{q}(0)) = \begin{pmatrix} 2 + \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 2.7071 \\ 0.7071 \end{pmatrix}.$$

The desired behavior is to move the point P from $\boldsymbol{p}(0)$ along a vertical line parallel to $\boldsymbol{y}_0$ with a constant speed $v > 0$. Thus

$$\boldsymbol{r}_{1d}(t) = \boldsymbol{p}(0) + \begin{pmatrix} 0 \\ vt \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{r}}_{1d} = \dot{\boldsymbol{p}}_d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} v.$$

The second task (of dimension $m_2 = 1$) is to keep the second link always horizontal (as in the initial configuration $\boldsymbol{q}(0)$). It is described by

$$r_2 = q_2 = f_2(\boldsymbol{q}) \quad \Rightarrow \quad \boldsymbol{J}_2 = \frac{\partial f_2(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \qquad r_{2d}(t) = 0 \quad \Rightarrow \quad \dot{r}_{2d} = \dot{q}_{2d} = 0. \tag{11}$$

The two tasks are simultaneously executed using the extended Jacobian $\boldsymbol{J}_E(\boldsymbol{q})$ (a square matrix of size $m = m_1 + m_2 = 3 = n$) and the extended task velocity defined by

$$\boldsymbol{J}_E(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_1(\boldsymbol{q}) \\ \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} -\sin q_1 & -\sin q_2 & -\sin q_3 \\ \cos q_1 & \cos q_2 & \cos q_3 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \dot{\boldsymbol{r}}_{E,d} = \begin{pmatrix} \dot{\boldsymbol{r}}_{1d} \\ \dot{r}_{2d} \end{pmatrix} \in \mathbb{R}^3. \tag{12}$$

Therefore, out of singularities, the joint velocity will be commanded in an unique way by

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_E^{-1}(\boldsymbol{q})\, \dot{\boldsymbol{r}}_{E,d} = \boldsymbol{J}_E^{-1}(\boldsymbol{q}) \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}. \tag{13}$$

In the initial configuration $\boldsymbol{q}(0)$, the extended Jacobian $\boldsymbol{J}_E(\boldsymbol{q}(0))$ has full rank. From (13), we obtain $\dot{\boldsymbol{q}}(0) = (0, 0, 1)$ [rad/s], with only the third link rotating counterclockwise. It is rather intuitive that, when the robot moves its end effector upwards vertically and keeps its second link horizontal ($q_2 = 0$), the first link will rotate clockwise (decreasing its orientation from $\pi/4$) and the third link counterclockwise (increasing its absolute orientation from 0). This motion will continue until a singular configuration $\boldsymbol{q}_s$ is first encountered for the extended Jacobian in (12). To determine $\boldsymbol{q}_s$, we impose at the same time

$$\det \boldsymbol{J}_E(\boldsymbol{q}_s) = \sin(q_{s3} - q_{s1}) = 0$$

and that the end effector is still on the initial vertical path (with the second link horizontal), or

$$p_x(\boldsymbol{q}_s)|_{q_{s2}=0} = \cos q_{s1} + \cos q_{s2}|_{q_{s2}=0} + \cos q_{s3} = \cos q_{s1} + 1 + \cos q_{s3} = 2 + \frac{\sqrt{2}}{2} = p_x(\boldsymbol{q}(0)).$$

These two equations are solved as

$$q_{s3} = q_{s1} \quad \Rightarrow \quad 2\cos q_{s1} = 1 + \frac{\sqrt{2}}{2} \quad \Rightarrow \quad q_{s1} = \arccos\left(\frac{2+\sqrt{2}}{4}\right) = 0.5480 \text{ [rad]}.$$

The singular configuration and the associated end-effector position are thus (see Fig. 4)

$$\boldsymbol{q}_s = \begin{pmatrix} 0.5480 \\ 0 \\ 0.5480 \end{pmatrix} \text{[rad]} \qquad \Rightarrow \qquad \boldsymbol{p}_s = \boldsymbol{f}(\boldsymbol{q}_s) = \begin{pmatrix} 2.7071 \\ 1.0420 \end{pmatrix} \text{[m]}.$$

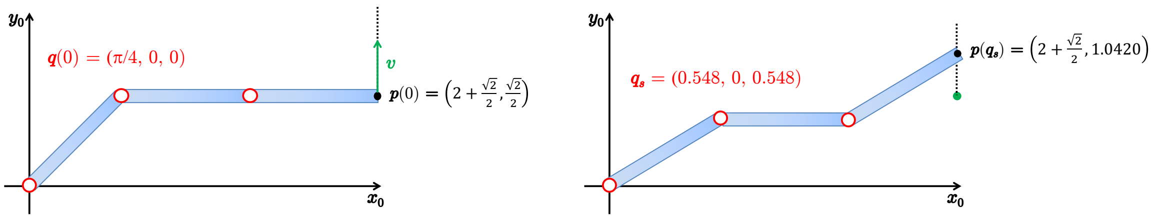


Figure 4: The 3R robot in its initial configuration $\boldsymbol{q}(0)$ and in the singular configuration $\boldsymbol{q}_s$.

The extended Jacobian is evaluated as

$$\boldsymbol{J}_E(\boldsymbol{q}_s) = \begin{pmatrix} \boldsymbol{J}_1(\boldsymbol{q}_s) \\ \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} -0.5210 & 0 & -0.5210 \\ 0.8536 & 1 & 0.8536 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since

$$\operatorname{rank} \boldsymbol{J}_1(\boldsymbol{q}_s) = 2 = m_1, \quad \operatorname{rank} \boldsymbol{J}_2 = 1 = m_2, \qquad \text{but} \qquad \operatorname{rank} \boldsymbol{J}_E(\boldsymbol{q}_s) = 2 < 3 = m = m_1 + m_2,$$

the configuration $\boldsymbol{q}_s$ is a true *algorithmic* singularity. Moreover, the extended task cannot be realized in this configuration, since

$$\dot{\boldsymbol{r}}_d = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} \notin \mathcal{R}\left\{\boldsymbol{J}_E(\boldsymbol{q}_s)\right\}, \qquad \forall v \neq 0.$$

We evaluate then the three requested joint velocity commands, setting in particular $v = 1$ [m/s]. Using the pseudoinverse method, we have

$$\dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}_E^{\#}(\boldsymbol{q}_s)\,\dot{\boldsymbol{r}}_d = \begin{pmatrix} -0.4098 & 0.3357 & -0.3357 \\ 0.3498 & 0.2135 & 0.7865 \\ -0.4098 & 0.3357 & -0.3357 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.3357 \\ 0.2135 \\ 0.3357 \end{pmatrix}. \tag{14}$$

Using instead the damped least squares method, with damping parameter $\mu^2 = 0.25$, we obtain

$$\dot{\boldsymbol{q}}_{DLS} = \boldsymbol{J}_E^T(\boldsymbol{q}_s)\left(\mu^2\boldsymbol{I} + \boldsymbol{J}_E(\boldsymbol{q}_s)\boldsymbol{J}_E^T(\boldsymbol{q}_s)\right)^{-1}\dot{\boldsymbol{r}}_d = \begin{pmatrix} -0.3251 & 0.2959 & -0.2367 \\ 0.2467 & 0.2199 & 0.6241 \\ -0.3251 & 0.2959 & -0.2367 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.2959 \\ 0.2199 \\ 0.2959 \end{pmatrix}. \tag{15}$$

Note that the two velocities $\dot{\boldsymbol{q}}_{PS}$ and $\dot{\boldsymbol{q}}_{DLS}$ are quite similar. In particular, in both commands the first and the third joint move with the same speed (different in the two methods). Finally, to apply the task priority method when the end-effector task is given the highest priority, we need to compute[3]

$$\dot{\boldsymbol{q}}_{TP} = \boldsymbol{J}_1^{\#}(\boldsymbol{q}_s)\,\dot{\boldsymbol{r}}_{d1} + \left(\boldsymbol{J}_2\boldsymbol{P}_1(\boldsymbol{q}_s)\right)^{\#}\left(\dot{r}_{d2} - \boldsymbol{J}_2\boldsymbol{J}_1^{\#}(\boldsymbol{q}_s)\dot{\boldsymbol{r}}_{d1}\right), \tag{16}$$

where, being $\boldsymbol{J}_1(\boldsymbol{q}_s)$ a full rank matrix, the pseudoinverse of the first task Jacobian is evaluated as

$$\boldsymbol{J}_1^{\#}(\boldsymbol{q}_s) = \boldsymbol{J}_1^T(\boldsymbol{q}_s)\left(\boldsymbol{J}_1(\boldsymbol{q}_s)\boldsymbol{J}_1^T(\boldsymbol{q}_s)\right)^{-1} = \begin{pmatrix} -0.9597 & 0 \\ 1.6383 & 1 \\ -0.9597 & 0 \end{pmatrix},$$

and the associated projector in the null space $\mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}_s)\}$ becomes

$$\boldsymbol{P}_1(\boldsymbol{q}_s) = \boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q}_s)\boldsymbol{J}_1(\boldsymbol{q}_s) = \begin{pmatrix} 0.5 & 0 & -0.5 \\ 0 & 0 & 0 \\ -0.5 & 0 & 0.5 \end{pmatrix}.$$

It is easy then to see the vanishing of the term

$$\boldsymbol{J}_2\boldsymbol{P}_1(\boldsymbol{q}_s) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \left(\boldsymbol{J}_2\boldsymbol{P}_1(\boldsymbol{q}_s)\right)^{\#} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \boldsymbol{0}. \tag{17}$$

As a result, the task priority method (16) collapses here into the simple use of the pseudoinverse of the first task Jacobian

$$\dot{\boldsymbol{q}}_{TP} = \boldsymbol{J}_1^{\#}(\boldsymbol{q}_s)\,\dot{\boldsymbol{r}}_{d1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{18}$$

[3]Here, we set $\dot{r}_{d2} = 0$. However, one should not care too much about the terms inside the last parenthesis in (16): these will be premultiplied anyway by zero —see eq. (17).

Only the second link rotates, fully violating the second task but perfectly realizing the first one. The task executions obtained with the three methods (14), (15), and (18) are

$$\left.\begin{array}{r}\dot{\boldsymbol{r}}_{PS} = \boldsymbol{J}_E(\boldsymbol{q}_s)\,\dot{\boldsymbol{q}}_{PS} = \begin{pmatrix} -0.3498 \\ 0.7865 \\ 0.2135 \end{pmatrix} \\[2ex] \dot{\boldsymbol{r}}_{DLS} = \boldsymbol{J}_E(\boldsymbol{q}_s)\,\dot{\boldsymbol{q}}_{DLS} = \begin{pmatrix} -0.3084 \\ 0.7251 \\ 0.2199 \end{pmatrix} \\[2ex] \dot{\boldsymbol{r}}_{TP} = \boldsymbol{J}_E(\boldsymbol{q}_s)\,\dot{\boldsymbol{q}}_{TP} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\end{array}\right\} \quad \Longleftrightarrow \quad \dot{\boldsymbol{r}}_{E,d} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

For comparison, the norms of the joint velocity $\dot{\boldsymbol{q}}_{method}$ obtained with the three methods, together with the norms of the end-effector velocity error $\dot{\boldsymbol{e}}_P = \dot{\boldsymbol{p}}_d - \boldsymbol{J}_1(\boldsymbol{q})\dot{\boldsymbol{q}}_{method}$ (task 1), and the absolute value of the velocity error of the second joint $\dot{e}_{q_2} = \dot{q}_{2,method}$ (task 2) are reported in Table 1.

Table 1: Comparison of results with the three methods.

method	$\|\dot{\boldsymbol{q}}_{method}\|$ [rad/s]	$\|\dot{\boldsymbol{e}}_P\|$ [m/s]	$\lvert\dot{e}_{q_2}\rvert$ [rad/s]
PS	0.5205	0.4098	0.2135
DLS	0.4728	0.4131	0.2199
TP	1	0	1

Exercise #3

The dynamic model of the PR robot in Fig. 3 is

$$(m_1 + m_2)\,\ddot{q}_1 - m_2 d_{c2} \sin q_2\, \ddot{q}_2 - m_2 d_{c2} \cos q_2\, \dot{q}_2^2 \;=\; u_1, \tag{19}$$

$$-m_2 d_{c2} \sin q_2\, \ddot{q}_1 + \left(I_{c2,zz} + m_2 d_{c2}^2\right) \ddot{q}_2 \;=\; u_2. \tag{20}$$

The geometry of the desired Cartesian motion task is very peculiar to this robot: an arc of a circle with center C on the joint axis 2 and radius R equal to the length l_2 of the second link. This requires simply no motion for the first joint, namely

$$q_{1d} = k \quad \text{(this value is irrelevant)}, \qquad \dot{q}_{1d} = \ddot{q}_{1d} = 0.$$

Accordingly, the inverse dynamics obtained from (19–20) yields

$$u_{1d} \;=\; -m_2 d_{c2} \sin q_{2d}\, \ddot{q}_{2d} - m_2 d_{c2} \cos q_{2d}\, \dot{q}_{2d}^2, \tag{21}$$

$$u_{2d} \;=\; I\, \ddot{q}_{2d}, \tag{22}$$

where we set for compactness $I = I_{c2,zz} + m_2 d_{c2}^2 > 0$. Therefore, in order to trace the arc of the circle from rest to rest ($\dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}(T) = \boldsymbol{0}$) in minimum time, equation (22) implies that a bang-bang torque $\pm U_{2,max}$ will be applied at the second (revolute) joint, with switching at the middle time $t = T/2$ of the motion. The force u_{1d} in (21) at the prismatic joint is needed to keep the first link at rest. The assumption that the bound $U_{2,max}$ is the only limiting factor when reducing as much as possible the motion time implies that the input force $|u_{1d}(t)|$ will never exceed $U_{1,max}$.

The motion profile $q_{2d}(t)$ of the second joint is then easily defined. Setting $A_{2,max} = U_{2,max}/I$ as the maximum acceleration of the second joint, and taking into account that $q_{2d}(0) = -\alpha$ and $\dot{q}_{2d}(0) = 0$, by successive integration and boundary condition satisfaction we get

$$
\begin{aligned}
\ddot{q}_{2d}(t) &= \begin{cases} A_{2,max}, & t \in \left[0, \frac{T}{2}\right] \\ -A_{2,max}, & t \in \left[\frac{T}{2}, T\right] \end{cases} \\
\dot{q}_{2d}(t) &= \begin{cases} A_{2,max}\, t, & t \in \left[0, \frac{T}{2}\right] \\ A_{2,max}\,(T-t)\,, & t \in \left[\frac{T}{2}, T\right] \end{cases} \\
q_{2d}(t) &= \begin{cases} -\alpha + \frac{1}{2}\, A_{2,max}\, t^2, & t \in \left[0, \frac{T}{2}\right] \\ -\alpha + A_{2,max}\left(\frac{T}{2}\right)^2 - \frac{1}{2}\, A_{2,max}\,(T-t)^2\,, & t \in \left[\frac{T}{2}, T\right]. \end{cases}
\end{aligned}
\tag{23}
$$

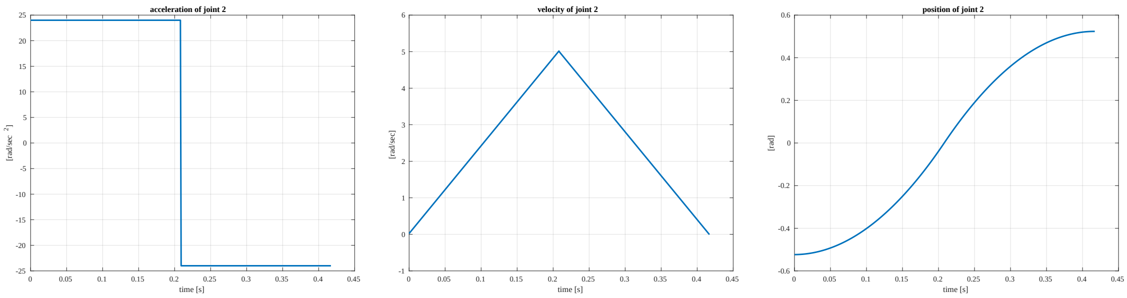


Figure 5: Time-optimal acceleration, velocity, and position profiles for the second joint.

The motion time T is determined by imposing that the area of the (triangular and symmetric) velocity profile $\dot{q}_{2d}(t)$ in $[0, T]$ is equal to the required joint displacement $\Delta q_2 = 2\alpha$. Thus

$$
\dot{q}_{2d}\,(T/2) \cdot \frac{T}{2} = A_{2,max}\,\frac{T}{2} \cdot \frac{T}{2} = 2\alpha \qquad \Rightarrow \qquad T = \sqrt{\frac{8\alpha}{A_{2,max}}}. \tag{24}
$$

Figure 5 shows representative kinematic profiles of the second joint motion[4]. As for the first input, we have from eqs. (21) and (23)

$$
\begin{aligned}
u_{1d}(t) &= -m_2 d_{c2} \sin q_{2d}(t)\, \ddot{q}_{2d}(t) - m_2 d_{c2} \cos q_{2d}(t)\, \dot{q}_{2d}^2(t) \\
&= u_{1d,acceleration}(t) + u_{1d,centripetal}(t) \\
&= -m_2 d_{c2}\, A_{2,max} \left(\sin q_{2d}(t) + \cos q_{2d}(t) A_{2,max}\, t^2\right),
\end{aligned}
\tag{25}
$$

where the last identity holds for the first half of the motion, i.e., for $t \in \left[0, \frac{T}{2}\right]$. The behavior in the second half of the motion, for $t \in \left[\frac{T}{2}, T\right]$, is perfectly specular.

The analysis of the two contributions to $u_{1d}(t)$ in (25) is simple —see also Fig. 6. The acceleration term is always non-negative, with a sinusoidal profile that has its maximum at $t = 0$ and $t = T$, where

$$
u_{1d,acceleration}(0) = u_{1d,acceleration}(T) = m_2 d_{c2}\, A_{2,max} \cdot \sin\alpha, \tag{26}
$$

while $u_{1d,acceleration}(T/2) = 0$. Vice versa, the centripetal term is never positive, it is zero at $t = 0$ and $t = T$, and takes its maximum (negative) value at $t = T/2$, when $q_{2d}(T/2) = 0$, with

$$
u_{1d,centripetal}(T/2) = -m_2 d_{c2}\, A_{2,max}^2 \left(\frac{T}{2}\right)^2 = m_2 d_{c2}\, A_{2,max} \cdot 2\alpha, \tag{27}
$$

[4] To generate these plots, the data reported in (29–30) have been used.

where (24) has been used. It is easy to see that the maximum value in (27) always dominates (26). Therefore,

$$|u_{1d}(t)| \le U_{1,max}, \quad \forall t \in [0,T] \quad \Longleftrightarrow \quad \max_{t\in[0,T]} |u_{1d}(t)| = 2\alpha\, m_2 d_{c2}\, A_{2,max} \le U_{1,max}.$$

For this inequality to be verified with the assumed time-optimal solution for joint 2 (i.e., for the assumption in the text to hold true), the bounds on the two inputs should satisfy

$$A_{2,max} = \frac{U_{2,max}}{I} \quad \Rightarrow \quad \frac{2\alpha\, m_2 d_{c2}}{I}\, U_{2,max} \le U_{1,max}. \tag{28}$$

While it is straightforward to sketch the input profiles of u_{1d} (approximately) and u_{2d} (exactly, being this a bang-bang torque), we conclude instead with a numerical evaluation using MATLAB. Setting for the arc of the circle the value $\alpha = \pi/6 = 30°$ and using the robot data

$$m_1 = m_2 = 2 \text{ [kg]}, \quad l_2 = 0.5,\ d_{c2} = 0.25 \text{ [m]}, \quad I = 0.1667 \text{ [kg m}^2\text{]}, \tag{29}$$

with bounds

$$U_{1,max} = 14 \text{ [N]}, \qquad U_{2,max} = 4 \text{ [Nm]} \qquad \Rightarrow \qquad A_{2,max} = 24 \text{ [rad/s}^2\text{]}, \tag{30}$$

the minimum motion time is found by (24) as $T = 0.4178$ [s]. Note that the input bounds in (30) satisfy inequality (28), being $u_{1d}(T/2) = -12.587$ [N]. The associated profile of the force input on the prismatic joint is shown in Fig. 6, together with those of its two contributions. The two time-optimal inputs profiles and their assigned bounds are reported in Fig. 7.

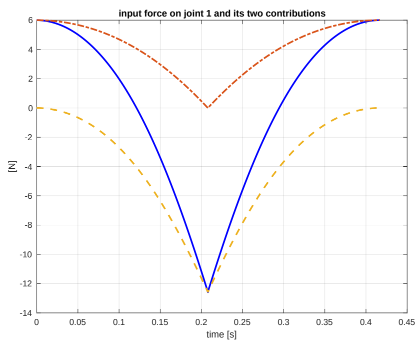


Figure 6: Input force $u_{1d}(t)$, with its acceleration (dotted-dashed) and centripetal (dashed) terms.

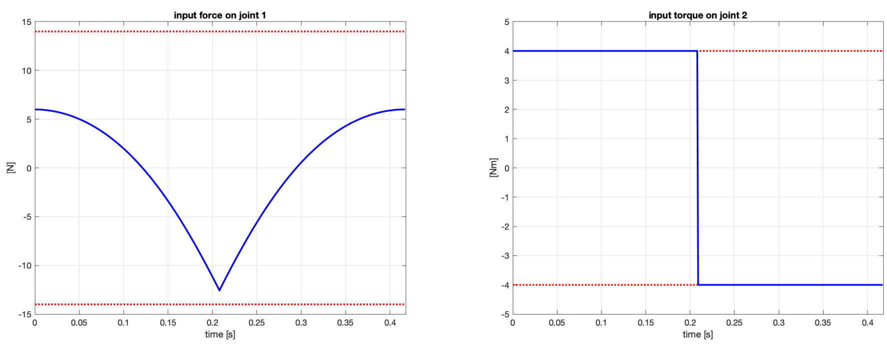


Figure 7: Time-optimal input profiles along the assigned path and their maximum bounds.

* * * * *

Robotics 2

Midterm Test – April 13, 2022

Exercise #1

We need to calibrate the link lengths of a planar 2R robot, whose nominal values are $\hat{l}_1 = \hat{l}_2 = 1$ [m]. All other kinematic parameters are assumed to be good enough. At four different Denavit-Hartenberg configurations $\boldsymbol{q}$, the following data (in [m]) for the position $\boldsymbol{p} \in \mathbb{R}^2$ of the robot end-effector are collected by an accurate external measurement system:

$$
\begin{aligned}
\boldsymbol{q}_a &= (0, 0) && \Rightarrow && \boldsymbol{p}_a = (2, 0)\\
\boldsymbol{q}_b &= (\pi/2, 0) && \Rightarrow && \boldsymbol{p}_b = (0, 2)\\
\boldsymbol{q}_c &= (\pi/4, -\pi/4) && \Rightarrow && \boldsymbol{p}_c = (1.6925, 0.7425)\\
\boldsymbol{q}_d &= (0, \pi/4) && \Rightarrow && \boldsymbol{p}_d = (1.7218, 0.6718).
\end{aligned}
$$

Provide the best estimate of the actual lengths l_1 and l_2 of the two robot links, using the above information. Is this calibration problem linear or nonlinear?

Exercise #2

A robot is driven by joint acceleration commands $\ddot{\boldsymbol{q}} \in \mathbb{R}^n$ which are kept constant for a (sufficiently small) sampling time T_c, i.e., $\ddot{\boldsymbol{q}}(t) = \ddot{\boldsymbol{q}}_k$, for $t \in [t_k, t_{k+1}) = [t_k, t_k + T_c)$. Thus, the next velocity at time $t = t_{k+1}$ can be expressed as $\dot{\boldsymbol{q}}_{k+1} = \dot{\boldsymbol{q}}(t_{k+1}) = \dot{\boldsymbol{q}}_k + T_c\,\ddot{\boldsymbol{q}}_k$. At time $t = t_k$, the robot is in the state $(\boldsymbol{q}_k, \dot{\boldsymbol{q}}_k)$ and has to realize a desired task acceleration $\ddot{\boldsymbol{r}}_{d,k} \in \mathbb{R}^m$, with $m < n$, being the task function $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q})$. What is the expression of the command $\ddot{\boldsymbol{q}}_k$ that executes the task while minimizing the squared norm of the joint velocity at the *next* sampled instant t_{k+1}?

Exercise #3

Consider the spatial 3R robot in Fig. 1. Using the D-H generalized coordinates defined therein, compute the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. Assume that the links have their center of mass on $\boldsymbol{x}_1$, $\boldsymbol{y}_2$, and $\boldsymbol{x}_3$, respectively, and that the barycentric link inertia matrices are diagonal, i.e., ${}^{i}\boldsymbol{I}_{ci} = \mathrm{diag}\{I_{ci,xx}, I_{ci,yy}, I_{ci,zz}\}$, $i = 1, 2, 3$.

a_i	α_i	d_i	θ_i
L_1	0	0	q_1
0	$\frac{\pi}{2}$	L_2	q_2
L_3	0	0	q_3

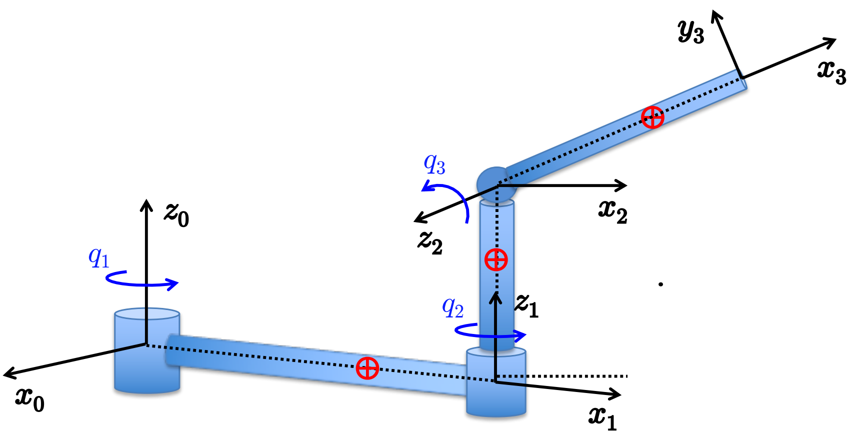


Figure 1: A spatial 3R robot, with D-H frames assigned to each link.

Exercise #4

A planar 3R robot with unitary link lengths is commanded by a joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ with components bounded as $|\dot{q}_i| \le 2$ [rad/s], $i = 1, 2, 3$. The D-H joint variables have limited ranges specified by

$$q_1 \in [-\pi/2, \pi/2], \qquad q_2 \in [0, 2\pi/3], \qquad q_3 \in [-\pi/4, \pi/4].$$

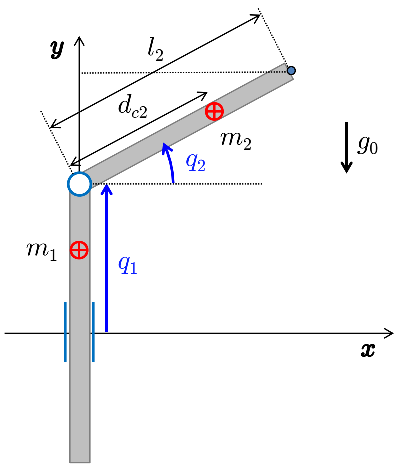
At the configuration $\widehat{\boldsymbol{q}} = (2\pi/5, \pi/2, -\pi/4)$, the robot should move its end-effector horizontally with a speed $v_x = -3$ [m/s], while trying to keep the joints close to their midranges. Compute the value of the instantaneous joint velocity $\dot{\boldsymbol{q}}$ that performs the Cartesian task while improving at best the criterion $H_{range}(\boldsymbol{q})$. Check if this joint velocity is feasible and, if not, perform the least end-effector task scaling to recover feasibility.

Exercise #5

Figure 2 shows a PR robot and its inertia matrix, already expressed in terms of three dynamic coefficients a, b and c. The robot moves in a vertical plane. A task trajectory $y_d(t) \in \mathbb{R}$ is assigned to the coordinate y of the end-effector position. With the robot being at rest in the configuration $\bar{\boldsymbol{q}} = \begin{pmatrix} 1 & \pi/2 \end{pmatrix}^T$, provide the joint force/torque inputs $\boldsymbol{\tau}_A \in \mathbb{R}^2$ and $\boldsymbol{\tau}_B \in \mathbb{R}^2$ executing the desired task that instantaneously minimize, respectively,

$$H_A = \frac{1}{2}\|\boldsymbol{\tau}\|^2 \qquad \text{or} \qquad H_B = \frac{1}{2}\|\boldsymbol{\tau}\|^2_{\boldsymbol{M}^{-2}(\bar{\boldsymbol{q}})}.$$

Which of the two solutions $\boldsymbol{\tau}_A$ and $\boldsymbol{\tau}_B$ has the largest first component in absolute value?



$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a & b\cos q_2 \\ b\cos q_2 & c \end{pmatrix} > 0$$

Figure 2: A planar PR robot and its inertia matrix.

Exercise #6

For the same PR robot in Fig. 2, determine the gravity term $\boldsymbol{g}(\boldsymbol{q})$ in the dynamic model and define a tight upper bound $\alpha > 0$ on the norm of the square matrix $\partial \boldsymbol{g}(\boldsymbol{q})/\partial \boldsymbol{q}$, for any value of $\boldsymbol{q}$.

[180 minutes (3 hours); open books]

Solution

April 13, 2022

Exercise #1

This calibration task is formulated as a *linear* least squares problem. In fact, the relevant measurement equations for the planar 2R robot can be written as

$$\Delta\boldsymbol{p} = \boldsymbol{p} - \hat{\boldsymbol{p}} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \end{pmatrix} - \begin{pmatrix} \hat{l}_1 c_1 + \hat{l}_2 c_{12} \\ \hat{l}_1 s_1 + \hat{l}_2 s_{12} \end{pmatrix} = \begin{pmatrix} \Delta l_1\, c_1 + \Delta l_2\, c_{12} \\ \Delta l_1\, s_1 + \Delta l_2\, s_{12} \end{pmatrix} = \begin{pmatrix} c_1 & c_{12} \\ s_1 & s_{12} \end{pmatrix} \begin{pmatrix} \Delta l_1 \\ \Delta l_2 \end{pmatrix},$$

or

$$\Delta\boldsymbol{p} = \boldsymbol{\Phi}(\boldsymbol{q})\, \Delta\boldsymbol{l}, \qquad \text{with } \boldsymbol{\Phi}(\boldsymbol{q}) = \begin{pmatrix} c_1 & c_{12} \\ s_1 & s_{12} \end{pmatrix},$$

without the need of any local approximation because the link lengths appear linearly in the direct kinematics of the robot. From the nominal model, we compute in the chosen configurations

$$\hat{\boldsymbol{p}}_a = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \qquad \hat{\boldsymbol{p}}_b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \qquad \hat{\boldsymbol{p}}_c = \begin{pmatrix} 1.7071 \\ 0.7071 \end{pmatrix}, \qquad \hat{\boldsymbol{p}}_d = \begin{pmatrix} 1.7071 \\ 0.7071 \end{pmatrix}.$$

Note that the first two nominal positions of the end-effector correspond to the measured ones. Stacking the results of the four experiments, we obtain the overdetermined linear system of equations

$$\Delta\bar{\boldsymbol{p}} = \begin{pmatrix} \Delta\boldsymbol{p}_a \\ \Delta\boldsymbol{p}_b \\ \Delta\boldsymbol{p}_c \\ \Delta\boldsymbol{p}_c \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Phi}(\boldsymbol{q}_a) \\ \boldsymbol{\Phi}(\boldsymbol{q}_b) \\ \boldsymbol{\Phi}(\boldsymbol{q}_c) \\ \boldsymbol{\Phi}(\boldsymbol{q}_d) \end{pmatrix} \Delta\boldsymbol{l} = \bar{\boldsymbol{\Phi}}\, \Delta\boldsymbol{l},$$

or

$$\Delta\bar{\boldsymbol{p}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -0.0146 \\ 0.0354 \\ 0.0146 \\ -0.0354 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0.7071 & 1 \\ 0.7071 & 0 \\ 1 & 0.7071 \\ 0 & 0.7071 \end{pmatrix} \Delta\boldsymbol{l} = \bar{\boldsymbol{\Phi}}\, \Delta\boldsymbol{l}.$$

By pseudoinversion of the 8×2 matrix $\bar{\boldsymbol{\Phi}}$, we obtain the value that minimizes the estimation error in a least squares sense,

$$\Delta\boldsymbol{l} = \bar{\boldsymbol{\Phi}}^{\#} \Delta\bar{\boldsymbol{p}} = \begin{pmatrix} 0.05 \\ -0.05 \end{pmatrix} = \begin{pmatrix} \Delta l_1 \\ \Delta l_2 \end{pmatrix}. \tag{1}$$

Therefore, the resulting estimates of the lengths of the two links are

$$l_1 = \hat{l}_1 + \Delta l_1 = 1.05, \qquad l_2 = \hat{l}_2 + \Delta l_2 = 0.95 \qquad \text{[m]}.$$

We finally note that the second and third regressor equations provide no information (all zeros!), whereas the fourth equation is a repetition of the first one. These phenomena are related to the singularity of the $\boldsymbol{\Phi}(\boldsymbol{q})$ matrix when $\sin q_2 = 0$ (e.g., in the configurations $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$ —not the best choices for calibration!). Therefore, these rows can be safely eliminated from the computation without any change in the final result.

Exercise #2

We are in the presence of redundancy ($m < n$). The objective function to be minimized at time $t = t_k$ is a complete quadratic function of the joint acceleration $\ddot{\boldsymbol{q}}_k$, the input to be chosen. We have

$$H(\ddot{\boldsymbol{q}}_k) = \frac{1}{2}\left\|\dot{\boldsymbol{q}}_{k+1}\right\|^2 = \frac{1}{2}\left\|\dot{\boldsymbol{q}}_k + T_c\,\ddot{\boldsymbol{q}}_k\right\|^2 = \frac{T_c^2}{2}\,\ddot{\boldsymbol{q}}_k^T\ddot{\boldsymbol{q}}_k + T_c\,\dot{\boldsymbol{q}}_k^T\ddot{\boldsymbol{q}}_k + c,$$

with the constant $c = \frac{1}{2}\,\dot{\boldsymbol{q}}_k^T\dot{\boldsymbol{q}}_k$. The unconstrained minimization of $H(\ddot{\boldsymbol{q}}_k)$ would yield the *preferred* acceleration $\ddot{\boldsymbol{q}}_k = -\dot{\boldsymbol{q}}_k/T_c$, which produces in fact a zero value for the non-negative objective function H. However, the required robot task is expressed by imposing the equality constraint

$$\boldsymbol{J}(\boldsymbol{q}_k)\,\ddot{\boldsymbol{q}}_k = \ddot{\boldsymbol{r}}_{d,k} - \dot{\boldsymbol{J}}(\boldsymbol{q}_k)\dot{\boldsymbol{q}}_k,$$

which is linear in the joint acceleration. Thus, the problem is in the standard form of LQ optimization and the solution is found by applying the general formula with $\boldsymbol{x} = \ddot{\boldsymbol{q}}_k$, $\boldsymbol{W} = T_c^2\,\boldsymbol{I}$, $\boldsymbol{x}_0 = -\dot{\boldsymbol{q}}_k/T_c$, and $\boldsymbol{y} = \ddot{\boldsymbol{r}}_{d,k} - \dot{\boldsymbol{J}}(\boldsymbol{q}_k)\dot{\boldsymbol{q}}_k$ (see the slides). Assuming a full rank Jacobian, we obtain

$$\begin{aligned}
\ddot{\boldsymbol{q}}_k &= -\frac{\dot{\boldsymbol{q}}_k}{T_c} + \frac{1}{T_c^2}\,\boldsymbol{J}^T(\boldsymbol{q}_k)\left(\frac{1}{T_c^2}\,\boldsymbol{J}(\boldsymbol{q}_k)\,\boldsymbol{J}^T(\boldsymbol{q}_k)\right)^{-1}\left(\ddot{\boldsymbol{r}}_{d,k} - \dot{\boldsymbol{J}}(\boldsymbol{q}_k)\dot{\boldsymbol{q}}_k - \boldsymbol{J}(\boldsymbol{q}_k)\left(-\frac{\dot{\boldsymbol{q}}_k}{T_c}\right)\right)\\
&= -\frac{\dot{\boldsymbol{q}}_k}{T_c} + \boldsymbol{J}^T(\boldsymbol{q}_k)\left(\boldsymbol{J}(\boldsymbol{q}_k)\,\boldsymbol{J}^T(\boldsymbol{q}_k)\right)^{-1}\left(\ddot{\boldsymbol{r}}_{d,k} - \dot{\boldsymbol{J}}(\boldsymbol{q}_k)\dot{\boldsymbol{q}}_k + \boldsymbol{J}(\boldsymbol{q}_k)\frac{\dot{\boldsymbol{q}}_k}{T_c}\right)\\
&= \boldsymbol{J}^{\#}(\boldsymbol{q}_k)\left(\ddot{\boldsymbol{r}}_{d,k} - \dot{\boldsymbol{J}}(\boldsymbol{q}_k)\dot{\boldsymbol{q}}_k\right) - \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q}_k)\,\boldsymbol{J}(\boldsymbol{q}_k)\right)\frac{\dot{\boldsymbol{q}}_k}{T_c}.
\end{aligned} \tag{2}$$

Exercise #3

We compute the kinetic energy of the three links. Denote by m_i the mass of link i, by l_i its length (i.e., the parameter d_i or a_i of the D-H convention), and by ${}^i\boldsymbol{I}_{ci} = \mathrm{diag}\{I_{ci,xx}, I_{ci,yy}, I_{ci,zz}\}$ its inertia matrix, for $i = 1,2,3$. Moreover, let $d_{ci} > 0$ be the distance of the center of mass (CoM) of link i from the axis of joint i; because of the assumption on the location of the CoM of each link, only one scalar is needed for each link[1].

Link 1

$$T_1 = \frac{1}{2}\left(I_{c1,zz} + m_1 d_{c1}^2\right)\dot{q}_1^2.$$

Link 2

$$T_2 = \frac{1}{2}\,m_2 l_1^2\,\dot{q}_1^2 + \frac{1}{2}\,I_{c2,yy}\left(\dot{q}_1 + \dot{q}_2\right)^2.$$

Link 3

$$\boldsymbol{p}_{c3} = \begin{pmatrix} l_1c_1 + d_{c3}c_3c_{12}\\ l_1s_1 + d_{c3}c_3s_{12}\\ l_2 + d_{c3}s_3\end{pmatrix} \quad\Rightarrow\quad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} -\left(l_1s_1\dot{q}_1 + d_{c3}c_3s_{12}\left(\dot{q}_1+\dot{q}_2\right) + d_{c3}s_3c_{12}\dot{q}_3\right)\\ l_1c_1\dot{q}_1 + d_{c3}c_3c_{12}\left(\dot{q}_1+\dot{q}_2\right) - d_{c3}s_3s_{12}\dot{q}_3\\ d_{c3}c_3\dot{q}_3\end{pmatrix}$$

$${}^1\boldsymbol{\omega}_1 = \begin{pmatrix}0\\0\\\dot{q}_1\end{pmatrix} \Rightarrow {}^2\boldsymbol{\omega}_2 = \begin{pmatrix}0\\\dot{q}_1+\dot{q}_2\\0\end{pmatrix} \Rightarrow {}^3\boldsymbol{\omega}_3 = {}^2\boldsymbol{R}_3^T(q_3)\left({}^2\boldsymbol{\omega}_2 + \begin{pmatrix}0\\0\\\dot{q}_3\end{pmatrix}\right) = \begin{pmatrix}s_3\left(\dot{q}_1+\dot{q}_2\right)\\c_3\left(\dot{q}_1+\dot{q}_2\right)\\\dot{q}_3\end{pmatrix}$$

[1] If using the moving frames algorithm for the computation of ${}^i\boldsymbol{v}_{ci}$ in the kinetic energy, it will be convenient to define the constant vectors of CoM positions in each of the local frame as follows: ${}^1\boldsymbol{r}_{c1} = (-l_1 + d_{c1}, 0, 0)$, ${}^2\boldsymbol{r}_{c2} = (0, -l_2 + d_{c2}, 0)$ —although this is not relevant in ${}^2\boldsymbol{v}_{c2}$, and ${}^3\boldsymbol{r}_{c3} = (-l_3 + d_{c3}, 0, 0)$. These symbolic choices in the recursive algorithm provide the same result as with the direct computations used in the text.

$$
\begin{aligned}
T_3 &= \frac{1}{2}\, m_3\, \boldsymbol{v}_{c3}^T \boldsymbol{v}_{c3} + \frac{1}{2}\, {}^3\boldsymbol{\omega}_3^T\, {}^3\boldsymbol{I}_{c3}\, {}^3\boldsymbol{\omega}_3 \\
&= \frac{1}{2} m_3 \left(l_1^2 \dot{q}_1^2 + d_{c3}^3 c_3^2 \left(\dot{q}_1 + \dot{q}_2\right)^2 + 2 l_1 d_{c3} \left(c_2 c_3\, \dot{q}_1 \left(\dot{q}_1 + \dot{q}_2\right) - s_2 s_3\, \dot{q}_1 \dot{q}_3\right)\right) \\
&\quad + \frac{1}{2} \left(I_{c3,xx} s_3^2 + I_{c3,yy} c_3^2\right) \left(\dot{q}_1 + \dot{q}_2\right)^2 + \frac{1}{2} I_{c3,zz}\, \dot{q}_3^2.
\end{aligned}
$$

Inertia matrix

$$
\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_{11}(q_2,q_3) & m_{12}(q_2,q_3) & m_{13}(q_2,q_3) \\ m_{12}(q_2,q_3) & m_{22}(q_3) & 0 \\ m_{13}(q_2,q_3) & 0 & m_{33} \end{pmatrix} \tag{3}
$$

with

$$
\begin{aligned}
m_{11}(q_2,q_3) &= I_{c1,zz} + m_1 d_{c1}^2 + I_{c2,yy} + (m_2+m_3) l_1^2 + m_3 d_{c3}^2 c_3^2 + 2 m_3 l_1 d_{c3}\, c_2 c_3 + \left(I_{c3,xx} s_3^2 + I_{c3,yy} c_3^2\right) \\
m_{12}(q_2,q_3) &= I_{c2,yy} + m_3 d_{c3}^2 c_3^2 + m_3 l_1 d_{c3}\, c_2 c_3 + \left(I_{c3,xx} s_3^2 + I_{c3,yy} c_3^2\right) \\
m_{13}(q_2,q_3) &= -m_3 l_1 d_{c3}\, s_2 s_3 \\
m_{22}(q_3) &= I_{c2,yy} + m_3 d_{c3}^2 c_3^2 + \left(I_{c3,xx} s_3^2 + I_{c3,yy} c_3^2\right) \\
m_{33} &= I_{c3,zz} + m_3 d_{c3}^2.
\end{aligned}
$$

Note finally that one can remove the presence of s_3^2 by replacing it everywhere with $(1-c_3^2)$. This is also what MATLAB does when applying a `simplify` instruction to the symbolic expressions. The affected elements of $\boldsymbol{M}(\boldsymbol{q})$ become then

$$
\begin{aligned}
m_{11}(q_2,q_3) &= I_{c1,zz} + m_1 d_{c1}^2 + I_{c2,yy} + (m_2+m_3) l_1^2 + I_{c3,xx} + 2 m_3 l_1 d_{c3}\, c_2 c_3 + \left(I_{c3,yy} + m_3 d_{c3}^2 - I_{c3,xx}\right) c_3^2 \\
m_{12}(q_2,q_3) &= I_{c2,yy} + I_{c3,xx} + m_3 l_1 d_{c3}\, c_2 c_3 + \left(I_{c3,yy} + m_3 d_{c3}^2 c_3^2 - I_{c3,xx}\right) c_3^2 \\
m_{22}(q_3) &= I_{c2,yy} + I_{c3,xx} + \left(I_{c3,yy} + m_3 d_{c3}^2 - I_{c3,xx}\right) c_3^2.
\end{aligned}
$$

Exercise #4

The planar 3R robot ($n=3$) is redundant for the Cartesian position task ($m=2$). When the joint limits are not regarded as hard constraints, the solution to the stated problem is

$$
\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}(\boldsymbol{q})\dot{\boldsymbol{r}} - \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right) \nabla_q H_{range}(\boldsymbol{q}),
$$

where the task velocity is

$$
\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} \qquad \Rightarrow \qquad \dot{\boldsymbol{r}} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix},
$$

and the associated Jacobian, evaluated at $\widehat{\boldsymbol{q}} = (2\pi/5, \pi/2, -\pi/4)$, is given by

$$
\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -\left(s_1 + s_{12} + s_{123}\right) & -\left(s_{12} + s_{123}\right) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix} \ \Rightarrow\ \boldsymbol{J} = \begin{pmatrix} -2.1511 & -1.2000 & -0.8910 \\ -1.0960 & -1.4050 & -0.4540 \end{pmatrix}.
$$

For each joint i, we have a range $[q_{m,i}, q_{M,i}]$ and a midrange $\bar{q}_i = \left(q_{M,i} + q_{m,i}\right)/2$. As a result, the objective function to be minimized is

$$
H_{range}(\boldsymbol{q}) = \frac{1}{2n} \sum_{i=1}^{n} \frac{\left(q_i - \bar{q}_i\right)^2}{\left(q_{M,i} - q_{m,i}\right)^2} = \frac{1}{6} \left(\frac{q_1^2}{\pi^2} + \frac{\left(q_2 - (\pi/3)\right)^2}{\left(2\pi/3\right)^2} + \frac{q_3^2}{\left(\pi/2\right)^2} \right).
$$

Its gradient evaluated at $\widehat{\boldsymbol{q}} = (2\pi/5, \pi/2, -\pi/4)$ is

$$\nabla_q H_{range}(\boldsymbol{q}) = \frac{1}{3}\begin{pmatrix} q_1/\pi^2 \\ (q_2 - \pi/3)/\left(2\pi/3\right)^2 \\ q_3/\left(\pi/2\right)^2 \end{pmatrix} \quad \Rightarrow \quad \nabla_q H_{range} = \begin{pmatrix} 0.0424 \\ 0.0398 \\ -0.1061 \end{pmatrix}.$$

As a result, the two terms of the solution are separately evaluated as

$$\dot{\boldsymbol{q}}_r = \boldsymbol{J}^{\#}\dot{\boldsymbol{r}} = \begin{pmatrix} 2.1076 \\ -1.9261 \\ 0.8730 \end{pmatrix}, \qquad \dot{\boldsymbol{q}}_n = -\left(\boldsymbol{I} - \boldsymbol{J}^{\#}\boldsymbol{J}\right)\nabla_q H_{range} = \begin{pmatrix} -0.0437 \\ 0 \\ 0.1056 \end{pmatrix},$$

yielding thus

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_r + \dot{\boldsymbol{q}}_n = \begin{pmatrix} 2.0638 \\ -1.9261 \\ 0.9786 \end{pmatrix}. \tag{4}$$

The first component of the solution exceeds the (positive) velocity bound. This is true as well for the minimum norm solution $\dot{\boldsymbol{q}}_r$; the first component of the null space term $\dot{\boldsymbol{q}}_n$, being negative, mildens the situation but is not sufficient to recover feasibility. Therefore, the largest scaling factor $k < 1$ of the task velocity $\dot{\boldsymbol{r}}$ that allows to obtain a feasible solution w.r.t. the joint velocity bounds (uniformly equal to $\dot{q}_{max} = 2$ [rad/s] for all joints) is computed as follows:

$$\dot{\boldsymbol{r}} \to k\,\dot{\boldsymbol{r}} \;\Rightarrow\; \dot{\boldsymbol{q}} \to k\,\dot{\boldsymbol{q}}_r + \dot{\boldsymbol{q}}_n \;\Rightarrow\; k\,\dot{q}_{r,1} + \dot{q}_{n,1} \overset{\downarrow}{=} \dot{q}_{max} \;\Rightarrow\; k^* = \frac{\dot{q}_{max} - \dot{q}_{n,1}}{\dot{q}_{r,1}} = \frac{2 + 0.0437}{2.1076} = 0.9697.$$

Therefore, the scaled task velocity and the scaled joint velocity that recovers feasibility are

$$\dot{\boldsymbol{r}}_s = k^*\dot{\boldsymbol{r}} = \begin{pmatrix} -2.9091 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_s = k^*\dot{\boldsymbol{q}}_r + \dot{\boldsymbol{q}}_n = \begin{pmatrix} 2 \\ -1.8678 \\ 0.9521 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{J}\dot{\boldsymbol{q}}_s = \begin{pmatrix} -2.9091 \\ 0 \end{pmatrix}. \tag{5}$$

It should be noted that, in this particular case, we could have chosen a larger step $\alpha > 1$ (rather than $\alpha = 1$) along the negative gradient direction of H_{range} within the term $\dot{\boldsymbol{q}}_n$, thus recovering feasibility of the solution without the need of task scaling. On the other hand, a direct application of the SNS method to recover feasibility would not be correct, since the solution $\dot{\boldsymbol{q}}$ in (4) contains also a null-space term that does not scale with the task velocity $\dot{\boldsymbol{r}}$.

Exercise #5

The planar PR robot ($n = 2$) is redundant with respect to a task of dimension $m = 1$. For the specified (scalar) task, we have

$$r = y = q_1 + l_2 s_2 \qquad \Rightarrow \qquad \dot{r} = \dot{y} = \begin{pmatrix} 1 & -l_2 c_2 \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with the Jacobian being always full rank. The closed-form solutions to the two problems of dynamic redundancy optimization are obtained from the general LQ formulation as

$$\boldsymbol{\tau}_A = \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\right)^{\#}\left(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})\right)\right)$$

and

$$\boldsymbol{\tau}_B = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{\#}(\boldsymbol{q})\left(\ddot{\boldsymbol{r}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})\right)\right).$$

Since the robot is at rest, the velocity terms $\boldsymbol{c}$ and $\dot{\boldsymbol{J}}\dot{\boldsymbol{q}}$ are zero. Evaluating the inertia matrix and the task Jacobian in the configuration $\bar{\boldsymbol{q}} = \begin{pmatrix} 1 & \pi/2 \end{pmatrix}^T$,

$$\boldsymbol{M}(\bar{\boldsymbol{q}}) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \qquad \boldsymbol{J}(\bar{\boldsymbol{q}}) = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

we compute

$$\begin{aligned} \boldsymbol{\tau}_A &= \left(\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1/a & 0 \\ 0 & 1/c \end{pmatrix} \right)^{\#} \left(\ddot{y}_d + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1/a & 0 \\ 0 & 1/c \end{pmatrix} \boldsymbol{g}(\boldsymbol{q}) \right) \\ &= \begin{pmatrix} 1/a & 0 \end{pmatrix}^{\#} \left(\ddot{y}_d + \begin{pmatrix} 1/a & 0 \end{pmatrix} \boldsymbol{g}(\boldsymbol{q}) \right) = \begin{pmatrix} a \\ 0 \end{pmatrix} \left(\ddot{y}_d + (1/a)\, g_1(\bar{\boldsymbol{q}}) \right) = \begin{pmatrix} a\, \ddot{y}_d + g_1(\bar{\boldsymbol{q}}) \\ 0 \end{pmatrix}. \end{aligned} \tag{6}$$

Similarly,

$$\begin{aligned} \boldsymbol{\tau}_B &= \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}^{\#} \left(\ddot{y}_d + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1/a & 0 \\ 0 & 1/c \end{pmatrix} \boldsymbol{g}(\boldsymbol{q}) \right) \\ &= \begin{pmatrix} a \\ 0 \end{pmatrix} \left(\ddot{y}_d + (1/a)\, g_1(\bar{\boldsymbol{q}}) \right) = \begin{pmatrix} a\, \ddot{y}_d + g_1(\bar{\boldsymbol{q}}) \\ 0 \end{pmatrix} = \boldsymbol{\tau}_A. \end{aligned} \tag{7}$$

As a result, the two solutions (6) and (7) are identical in this very particular case (in fact, it is here $\left(\boldsymbol{J}\boldsymbol{M}^{-1}\right)^{\#} = \boldsymbol{M}\boldsymbol{J}^{\#}$, an identity which is not true in general). Note that there is no need to derive the expression of the model term $\boldsymbol{g}(\boldsymbol{q})$ for this comparison.

A final remark is in order. The torque commands $\boldsymbol{\tau}_A$ and $\boldsymbol{\tau}_B$, which have been obtained above from the general solution of the associated constrained minimization problems, could have been found in this specific case by inspection. In the configuration $\bar{\boldsymbol{q}}$, the PR robot is fully stretched along the vertical y-axis. In addition, being the robot at rest, any torque applied at the second joint would give *no* contribution to the desired task acceleration $\ddot{y}_d$. Since we pursue in both cases a (weighted) minimum torque norm solution, the second joint torque τ_2 should simply be zero; the entire task (task acceleration $\ddot{y}_d$ in the vertical direction plus gravity compensation) is executed in a unique way by the first joint only.

Exercise #6

The gravity term of the PR robot in Fig. 2 is obtained as the gradient of the sum of the potential energies of each link

$$U_i(\boldsymbol{q}) = -m_i\, \boldsymbol{g}^T \boldsymbol{r}_{0,ci} = -m_i \begin{pmatrix} 0 & -g_0 & 0 \end{pmatrix} \boldsymbol{r}_{0,ci} = m_i g_0\, r_{0,ci_y}, \quad i = 1, 2.$$

Thus (neglecting an arbitrary constant), we have

$$U(\boldsymbol{q}) = U_1(q_1) + U_2(q_1, q_2) = m_1 g_0\, q_1 + m_2 g_0\, (q_1 + d_{c2} s_2)$$

that gives

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T = \begin{pmatrix} (m_1 + m_2)\, g_0 \\ m_2 g_0 d_{c2} c_2 \end{pmatrix}.$$

The gradient of $\boldsymbol{g}(\boldsymbol{q})$ w.r.t. $\boldsymbol{q}$ is the symmetric (here, negative semi-definite) Hessian matrix

$$\frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \frac{\partial^2 U(\boldsymbol{q})}{\partial \boldsymbol{q}^2} = \begin{pmatrix} 0 & 0 \\ 0 & -m_2 g_0 d_{c2} s_2 \end{pmatrix}.$$

Its norm (associated to the standard Euclidean norm of vectors) is given by

$$\left\| \frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} \right\| = \sqrt{\lambda_{max} \left\{ \frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} \left(\frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T \right\}} = \sqrt{\lambda_{max} \left\{ \begin{pmatrix} 0 & 0 \\ 0 & m_2^2 g_0^2 d_{c2}^2 s_2^2 \end{pmatrix} \right\}} = m_2 g_0 d_{c2} \left| s_2 \right| .$$

Thus, an upper bound for this norm is

$$\left\| \frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} \right\| \leq \alpha = m_2 g_0 d_{c2}, \qquad \forall \boldsymbol{q}. \tag{8}$$

This upper bound is tight, being attained at $q_2 = \pm\pi/2$.

* * * * *

Robotics 2

Midterm Test – April 19, 2023

Exercise #1

The end-effector of a 3R planar robot, having equal link lengths $l = 0.5$ [m], is executing a positional trajectory $\boldsymbol{p}_d(t) \in \mathbb{R}^2$ in the plane, commanded by joint accelerations $\ddot{\boldsymbol{q}}(t) \in \mathbb{R}^3$ that are updated every $T_c = 100$ ms. The robot is subject to the following hard bounds on joint velocities and accelerations:

$$|\dot{q}_i| \leq V_{max,i}, \qquad |\ddot{q}_i| \leq A_{max,i}, \qquad i = 1, 2, 3. \tag{1}$$

When the limits in (1) are

$$\boldsymbol{V}_{max} = \begin{pmatrix} 1.5 \\ 1.5 \\ 1 \end{pmatrix} \text{[rad/s]}, \qquad \boldsymbol{A}_{max} = \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix} \text{[rad/s}^2\text{]},$$

and the robot is in the configuration $\boldsymbol{q} = \begin{pmatrix} 0 & 0 & \pi/2 \end{pmatrix}^T$ [rad] with velocity $\dot{\boldsymbol{q}} = \begin{pmatrix} 0.8 & 0 & -0.8 \end{pmatrix}^T$ [rad/s], compute the acceleration command $\ddot{\boldsymbol{q}}$ of minimum norm that realizes the desired end-effector acceleration $\ddot{\boldsymbol{p}}_d = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$ [m/s^2] while complying with the bounds imposed on robot motion.

Exercise #2

A 3R robot with Denavit-Hartenberg (DH) parameters $\alpha_i = 0$, $d_i = 0$, and $a_i = l_i > 0$, for $i = 1, 2, 3$, moves in a vertical plane. The i-th link has mass $m_i > 0$ and position of the center of mass (CoM) ${}^i\boldsymbol{r}_{c,i} = (r_{cx,i}, r_{cy,i}, 0)$ when expressed in the i-th DH frame, for $i = 1, 2, 3$. Define suitable relations between the link masses, lengths, and CoM positions of this robot such that the gravity term in the dynamic model takes the following expression:

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} m_1 g_0 \, r_{cy,1} \cos q_1 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{with } g_0 = 9.81 \text{ [m/s}^2\text{]}. \tag{2}$$

Sketch a figure of a robot having the positions of the link CoMs consistent with (2).

Exercise #3

Compute the 4×4 inertia matrix of the 4P planar robot in Fig. 1. With the robot in a generic configuration $\boldsymbol{q}$, determine the joint velocity command $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ that realizes a desired end-effector velocity $\boldsymbol{v}_d = \begin{pmatrix} v_{xd} & v_{yd} \end{pmatrix}^T$ while minimizing the kinetic energy T of the robot. Which would be the solution instead if the norm of the joint velocity $\|\dot{\boldsymbol{q}}\|$ is minimized?

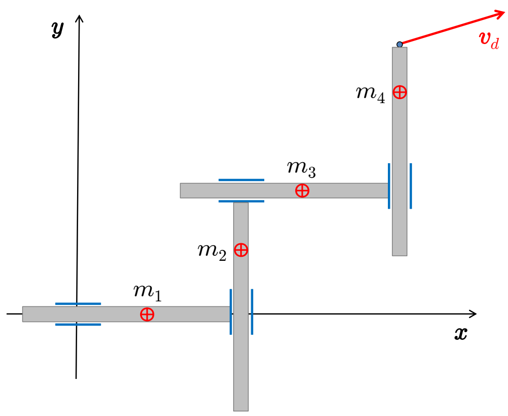


Figure 1: A 4P planar robot in a generic configuration.

Exercise #4

Consider the RPR spatial robot in Fig. 2. Based on the DH frames and joint variables defined therein, provide the expression of the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. Assume that the three links have their center of mass, respectively along the $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, and $\boldsymbol{x}_3$ axes, and that the barycentric inertia matrix of the third link is diagonal and isotropic, i.e., ${}^3\boldsymbol{I}_{c3} = \text{diag}\{I_3, I_3, I_3\}$.

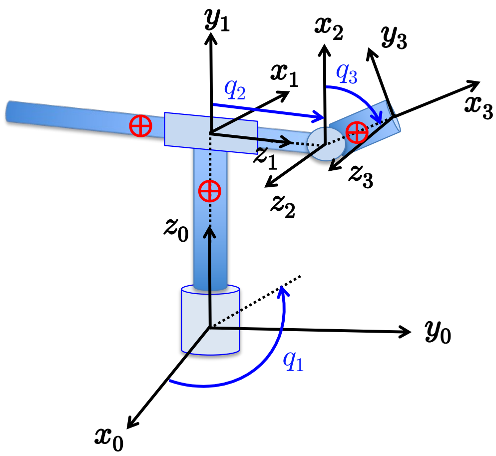


Figure 2: A spatial RPR robot, with DH frames assigned to each link.

Exercise #5

The inertia matrix of a 3-dof robot with coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ is given by

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2q_2 + a_3q_2^2 + 2a_4q_2 \sin q_3 + a_5 \sin^2 q_3 & 0 & 0 \\ 0 & a_3 & a_4 \cos q_3 \\ 0 & a_4 \cos q_3 & a_6 \end{pmatrix}, \tag{3}$$

where $\boldsymbol{a} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{pmatrix}^T$ is the vector of dynamic coefficients. Using (3), compute: *i)* the Coriolis and centrifugal term $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ in the robot dynamic model; *ii)* three **different** factorizations $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{S}'(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{S}''(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$, such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ and $\dot{\boldsymbol{M}} - 2\boldsymbol{S}'$ are skew-symmetric matrices while $\dot{\boldsymbol{M}} - 2\boldsymbol{S}''$ is not; *iii)* the unique 3×6 regressor matrix $\boldsymbol{Y}$ such that $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\,\boldsymbol{a}$.

Exercise #6

A 2-dof robot has the axes of the first prismatic joint and of the second revolute joint coincident and vertical (i.e., aligned with the acceleration of gravity). The two joints should perform a displacement $\boldsymbol{\Delta q} = (\Delta q_1, \Delta q_2)$, by tracing a rest-to-rest cubic trajectory in the same motion time T. The input commands u_1 and u_2 at the joints (respectively, a force and a torque) are bounded as $|u_i| \leq U_{max,i}$, for $i = 1, 2$. Provide the minimum feasible motion time T^* to execute the task, as a function of the problem data and of the robot dynamics. Without loss of generality, assume that the actuators are at least strong enough to sustain statically the weight of the robot links.

[270 minutes (4.5 hours); open books]

Solution

April 19, 2023

Exercise #1

The task kinematics for the given 3R planar robot is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = l \begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{pmatrix},$$

with associated Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = l \begin{pmatrix} -s_1 - s_{12} - s_{123} & -s_{12} - s_{123} & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix}.$$

The end-effector acceleration is then computed as

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}),$$

with

$$\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{J}}(\boldsymbol{q})\,\dot{\boldsymbol{q}} = -l \begin{pmatrix} \dot{q}_1^2\, c_1 + (\dot{q}_1 + \dot{q}_2)^2\, c_{12} + (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2\, c_{123} \\ \dot{q}_1^2\, s_1 + (\dot{q}_1 + \dot{q}_2)^2\, s_{12} + (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2\, s_{123} \end{pmatrix}.$$

When evaluating the terms at the given state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ we obtain

$$\boldsymbol{J} = \begin{pmatrix} -0.5 & -0.5 & -0.5 \\ 1 & 0.5 & 0 \end{pmatrix}, \qquad \dot{\boldsymbol{J}} = \begin{pmatrix} -0.8 & -0.4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{n} = \begin{pmatrix} -0.64 \\ 0 \end{pmatrix}.$$

Therefore, the minimum norm joint acceleration realizing the desired task acceleration $\ddot{\boldsymbol{p}}_d$ in the absence of hard bounds on robot motion is

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\,(\ddot{\boldsymbol{p}}_d - \boldsymbol{n}) = \begin{pmatrix} 0.3333 & 1 \\ -0.6667 & 0 \\ -1.6667 & -1 \end{pmatrix} \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -0.64 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1.88 \\ -1.76 \\ -5.40 \end{pmatrix} \ [\text{rad/s}^2]. \tag{4}$$

In order to verify if this command is feasible, we have to check both the direct limits on joint acceleration (i.e., whether $|\ddot{q}_i| \leq A_{max,i}$ is satisfied for all joints) and the indirect limits induced by the presence of joint velocity bounds. Since the acceleration command $\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}\,(kT_c)$ at $t = kT_c$ is kept constant for an interval T_c, the joint velocity at the next sampling instant will be

$$\dot{\boldsymbol{q}}\,((k+1)T_c) = \dot{\boldsymbol{q}}\,(kT_c) + \ddot{\boldsymbol{q}}\,(kT_c)\,T_c.$$

Thus, the current joint acceleration should also satisfy the bounds (in vector format)

$$-\frac{\boldsymbol{V}_{max} + \dot{\boldsymbol{q}}\,(kT_c)}{T_c} \leq \ddot{\boldsymbol{q}}\,(kT_c) \leq \frac{\boldsymbol{V}_{max} - \dot{\boldsymbol{q}}\,(kT_c)}{T_c}.$$

As a result, we need to check componentwise (at the current instant) if

$$\ddot{Q}_{min,i} = \max\left\{-A_{max,i}, -\frac{V_{max,i} + \dot{q}_i}{T_c}\right\} \leq \ddot{q}_i \leq \min\left\{A_{max,i}, \frac{V_{max,i} - \dot{q}_i}{T_c}\right\} = \ddot{Q}_{max,i}, \qquad i = 1, 2, 3.$$

Plugging in the problem data, we have

$$\ddot{\boldsymbol{Q}}_{min} = \begin{pmatrix} -10 \\ -10 \\ -2 \end{pmatrix}, \qquad \ddot{\boldsymbol{Q}}_{max} = \begin{pmatrix} 7 \\ 10 \\ 10 \end{pmatrix}. \tag{5}$$

While the acceleration (4) is feasible at the first two joints, the third acceleration component $\ddot{q}_3 = -5.4$ exceeds the lower limit $\ddot{Q}_{min,3} = -2$. We apply thus a step of the SNS algorithm, as translated to the acceleration level.

Set first the third joint acceleration to its lower limit, $\ddot{q}_{SNS,3} = -2$. Then, recompute the solution for the other two joints by using the reduced 2×2 Jacobian $\boldsymbol{J}_{-3}$, obtained by removing the third column $\boldsymbol{J}_3$ from the task Jacobian $\boldsymbol{J}$; the desired task acceleration $\ddot{\boldsymbol{p}}_d$ should be modified accordingly to account for the saturated contribution of the third joint. We have

$$\ddot{\boldsymbol{p}}_{SNS,d} = \ddot{\boldsymbol{p}}_d - \boldsymbol{J}_3\, \ddot{q}_{SNS,3} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -0.5 \\ 0 \end{pmatrix} \cdot (-2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and thus, as unique possible solution, we obtain

$$\begin{pmatrix} \ddot{q}_{SNS,1} \\ \ddot{q}_{SNS,2} \end{pmatrix} = \left(\boldsymbol{J}_{-3}\right)^{-1} \left(\ddot{\boldsymbol{p}}_{SNS,d} - \boldsymbol{n}\right) = \begin{pmatrix} -0.5 & -0.5 \\ 1 & 0.5 \end{pmatrix}^{-1} \begin{pmatrix} 1.64 \\ 1 \end{pmatrix} = \begin{pmatrix} 5.28 \\ -8.56 \end{pmatrix}.$$

The solution

$$\ddot{\boldsymbol{q}}_{SNS} = \begin{pmatrix} 5.28 \\ -8.56 \\ -2 \end{pmatrix} \ [\mathrm{rad/s^2}].$$

is now feasible, i.e., it stays within the limits (5) and, by the property of the SNS algorithm, its has also the minimum norm property among all feasible acceleration solutions.

Exercise #2

Since the DH twist angles α_i are all zero, the 3R robot is planar. Choose the axis $\boldsymbol{x}_0$ pointing downward in the vertical plane[1], so that the gravity acceleration vector is $\boldsymbol{g}_0 = \begin{pmatrix} g_0 & 0 & 0 \end{pmatrix}^T$ (with $g_0 = 9.81$ [m/s^2]). The potential energy of each link is given by

$$U_i = -m_i\, \boldsymbol{g}_0^T \boldsymbol{r}_{ci}, \qquad i = 1, 2, 3. \tag{6}$$

In order to use the constant expressions of the CoMs in the local frames, we have

$$\boldsymbol{r}_{c,i}^{hom} = \begin{pmatrix} \boldsymbol{r}_{c,i} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_i(q_1, ..., q_i) \begin{pmatrix} {}^i\boldsymbol{r}_{c,i} \\ 1 \end{pmatrix} = {}^0\boldsymbol{A}_1(q_1) \dots {}^{i-1}\boldsymbol{A}_i(q_i) \begin{pmatrix} r_{cx,i} \\ r_{cy,i} \\ 0 \\ 1 \end{pmatrix}, \qquad i = 1, 2, 3.$$

where the homogeneous transformation matrices are computed from the DH parameters as

$${}^{i-1}\boldsymbol{A}_i(q_i) = \begin{pmatrix} c_i & -s_i & 0 & l_i c_i \\ s_i & c_i & 0 & l_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad i = 1, 2, 3.$$

Performing the computations in (6), one obtains

$$\begin{aligned} U_1 &= m_1 g_0 \left(r_{cy,1}\, s_1 - (l_1 + r_{cx,1})\, c_1\right) \\ U_2 &= m_2 g_0 \left(r_{cy,2}\, s_{12} - (l_2 + r_{cx,2})\, c_{12} - l_1\, c_1\right) \\ U_3 &= m_3 g_0 \left(r_{cy,3}\, s_{123} - (l_3 + r_{cx,3})\, c_{123} - l_2\, c_{12} - l_1\, c_1\right). \end{aligned}$$

[1] A different choice for the direction of $\boldsymbol{x}_0$ (e.g., horizontal or upward) would not affect the conditions that impose $g_2 = g_3 = 0$ in the gravity term of the dynamic model, but only the actual trigonometric function appearing in $g_1(\boldsymbol{q})$, i.e., $\pm \sin q_1$ or $\pm \cos q_1$.

From $U = U_1 + U_2 + U_3$, we get

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = g_0 \begin{pmatrix} m_1 r_{cy,1}\, c_1 + (m_1(l_1 + r_{cx,1}) + (m_2 + m_3)l_1)\, s_1 \\ + m_2 r_{cy,2}\, c_{12} + (m_2(l_2 + r_{cx,2}) + m_3 l_2)\, s_{12} \\ + m_3\, (r_{cy,3}\, c_{123} + (l_3 + r_{cx,3})\, s_{123}) \\ m_2 r_{cy,2}\, c_{12} + (m_2(l_2 + r_{cx,2}) + m_3 l_2)\, s_{12} \\ + m_3\, (r_{cy,3}\, c_{123} + (l_3 + r_{cx,3})\, s_{123}) \\ m_3\, (r_{cy,3}\, c_{123} + (l_3 + r_{cx,3})\, s_{123}) \end{pmatrix} = \begin{pmatrix} g_1(\boldsymbol{q}) \\ g_2(\boldsymbol{q}) \\ g_3(\boldsymbol{q}) \end{pmatrix}.$$

Proceeding backward from the last component, in order to obtain the desired structure (2) of the gravity term, we have to set first

$$r_{cx,3} = -l_3, \quad r_{cy,3} = 0 \qquad \Rightarrow \qquad g_3 \equiv 0,$$

and then also

$$m_2\, r_{cx,2} = -\,(m_2 + m_3)\, l_2, \quad r_{cy,2} = 0 \qquad \Rightarrow \qquad g_2 \equiv 0,$$

and finally

$$m_1 r_{cx,1} = -\,(m_1 + m_2 + m_3)\, l_1 \qquad \Rightarrow \qquad g_1(q_1) = m_1 g_0\, r_{cy,1}\, c_1.$$

Figure 3 shows a sketch of a possible 3R planar robot satisfying the conditions for having the desired gravity term (2) in its dynamic model. We have chosen here $l_1 = l_2 = l_3 = l$ [m], $m_1 = 4\, m_2 = 16\, m_3 = 10$ [kg], and $r_{cy,1} = 0.2\, l$ [m].

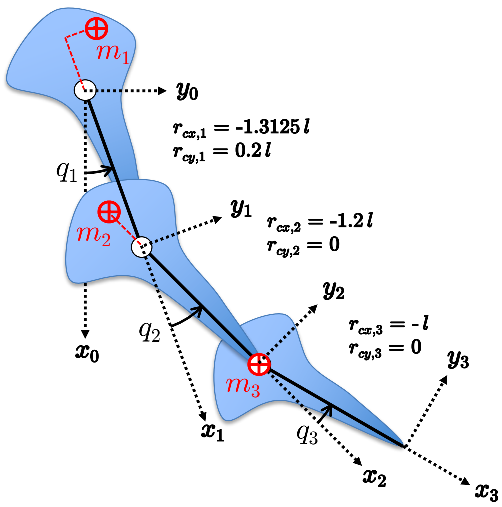


Figure 3: Localization of the CoMs of a 3R planar robot having the dynamic term $\boldsymbol{g}(\boldsymbol{q})$ as in eq. (2).

Exercise #3

Since there is no angular motion, the kinetic energy of the 4P planar robot is simply computed as

$$T = \sum_{i=1}^{4} T_i = \frac{1}{2} \sum_{i=1}^{4} m_i \|\boldsymbol{v}_{ci}\|^2 = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}},$$

with velocity vectors (conveniently written in $\mathbb{R}^2$)

$$\boldsymbol{v}_{c1} = \begin{pmatrix} \dot{q}_1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_{c2} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}, \quad \boldsymbol{v}_{c3} = \begin{pmatrix} \dot{q}_1 + \dot{q}_3 \\ \dot{q}_2 \end{pmatrix}, \quad \boldsymbol{v}_{c4} = \begin{pmatrix} \dot{q}_1 + \dot{q}_3 \\ \dot{q}_2 + \dot{q}_4 \end{pmatrix}.$$

As a result, the robot inertia matrix is constant and is given by

$$\boldsymbol{M} = \begin{pmatrix} m_1+m_2+m_3+m_4 & 0 & m_3+m_4 & 0 \\ 0 & m_2+m_3+m_4 & 0 & m_4 \\ m_3+m_4 & 0 & m_3+m_4 & 0 \\ 0 & m_4 & 0 & m_4 \end{pmatrix}.$$

The end-effector Jacobian for the linear velocity $\boldsymbol{v}$ in the plane (x, y) is also constant:

$$\boldsymbol{J} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

The joint velocity that produces the desired end-effector velocity $\boldsymbol{v}_d$ while minimizing T is obtained by using the inertia-weighted pseudoinverse of $\boldsymbol{J}$:

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_{\boldsymbol{M}}^{\#}\,\boldsymbol{v}_d = \boldsymbol{M}^{-1}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T\right)^{-1}\boldsymbol{v}_d = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_{xd} \\ v_{yd} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v_{xd} \\ v_{yd} \end{pmatrix}. \tag{7}$$

This result is rather intuitive: moving only the last two joints, each by the corresponding component of the end-effector desired velocity, involves the displacement of the minimum amount of mass, and is thus the minimum kinetic energy solution. By this observation, the use of the following intermediate matrix computations is really unnecessary:

$$\boldsymbol{M}^{-1} = \begin{pmatrix} \frac{1}{m_1+m_2} & 0 & -\frac{1}{m_1+m_2} & 0 \\ 0 & \frac{1}{m_2+m_3} & 0 & -\frac{1}{m_2+m_3} \\ -\frac{1}{m_1+m_2} & 0 & \frac{m_1+m_2+m_3+m_4}{(m_1+m_2)(m_3+m_4)} & 0 \\ 0 & -\frac{1}{m_2+m_3} & 0 & \frac{m_2+m_3+m_4}{m_4(m_2+m_3)} \end{pmatrix}, \qquad \boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{J}^T = \begin{pmatrix} \frac{1}{m_3+m_4} & 0 \\ 0 & \frac{1}{m_4} \end{pmatrix}.$$

In comparison with (7), the minimum velocity norm solution

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\#}\boldsymbol{v}_d = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} v_{xd} \\ v_{yd} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} v_{xd} \\ v_{yd} \\ v_{xd} \\ v_{yd} \end{pmatrix}$$

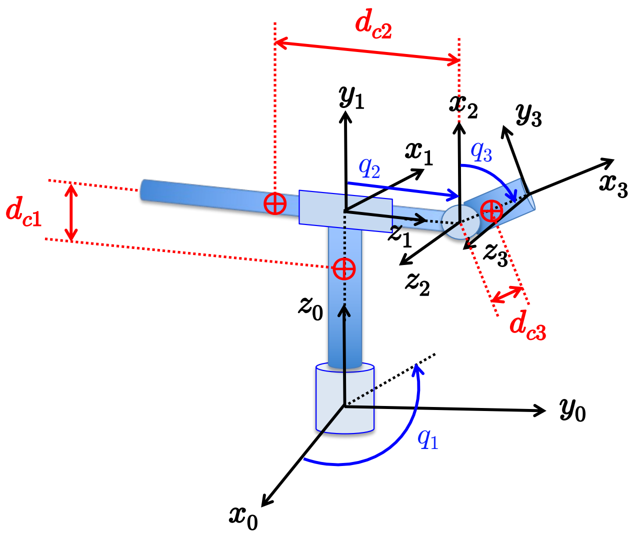
equally distributes the desired Cartesian velocity between the pairs of robot joints that move, respectively, along the x and the y directions.

Exercise #4

We need to compute the kinetic energy of the three links. For $i = 1, 2, 3$, denote by m_i the mass of link i, by l_i its kinematic length (i.e., the parameter d_i or a_i of the DH convention), and by ${}^i\boldsymbol{I}_{ci}$ and ${}^i\boldsymbol{r}_{ci} \in \mathbb{R}^3$, respectively its 3×3 barycentric inertia matrix (for the third short link, this matrix is assumed to be diagonal and uniform) and the constant position vector of its center of mass (CoM), both expressed in the local DH frame. Because of the assumptions on the location of the CoMs of the links, only one component of each ${}^i\boldsymbol{r}_{ci}$ will be different from zero. With reference to Fig. 4, we have

$${}^1\boldsymbol{r}_{c1} = \begin{pmatrix} 0 \\ -d_{c1} \\ 0 \end{pmatrix}, \qquad {}^2\boldsymbol{r}_{c2} = \begin{pmatrix} 0 \\ d_{c2} \\ 0 \end{pmatrix}, \qquad {}^3\boldsymbol{r}_{c3} = \begin{pmatrix} -l_3 + d_{c3} \\ 0 \\ 0 \end{pmatrix}.$$

where $d_{ci} > 0$, for $i = 1, 2, 3$. For the first two links, computation of the kinetic energy is rather straightforward. For the third link, it is convenient to use the moving frame algorithm mainly to obtain ${}^3\omega_3$.



i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	l_1	q_1
2	$-\pi/2$	0	q_2	$\pi/2$
3	0	l_3	0	q_3

Figure 4: Localization of the link CoMs and DH table for the spatial RPR robot.

Link 1

$$T_1 = \frac{1}{2}\, I_{c1,yy}\, \dot{q}_1^2 = \frac{1}{2}\, I_1 \dot{q}_1^2,$$

where we set $I_1 = I_{c1,yy}$ for compactness. Note that the actual position of the CoM of link 1 along the axis $\boldsymbol{z}_0 = \boldsymbol{y}_1$ is irrelevant.

Link 2

$$T_2 = \frac{1}{2}\, m_2 \left((r_{c2,y} - q_2)^2 \, \dot{q}_1^2 + \dot{q}_2^2 \right) + \frac{1}{2}\, I_{c2,xx}\, \dot{q}_1^2 = \frac{1}{2} \left(m_2 \, (q_2 - d_{c2})^2 + I_2 \right) \dot{q}_1^2 + \frac{1}{2}\, m_2 \, \dot{q}_2^2,$$

where $r_{c2,y} = d_{c2} > 0$ is the distance of the CoM of link 2 from the axis of joint 3 and we set $I_{c2,xx} = I_2$ for compactness.

Link 3

Since $d_{c3} > 0$ is the distance of the CoM of link 3 from the axis of joint 3, we have

$$\boldsymbol{p}_{c3} = \begin{pmatrix} (q_2 - d_{c3}s_3)\, s_1 \\ -\,(q_2 - d_{c3}s_3)\, c_1 \\ l_1 + d_{c3}c_3 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} (q_2 - d_{c3}s_3)\, c_1 \dot{q}_1 + (\dot{q}_2 - d_{c3}c_3\, \dot{q}_3)\, s_1 \\ (q_2 - d_{c3}s_3)\, s_1 \dot{q}_1 - (\dot{q}_2 - d_{c3}c_3\, \dot{q}_3)\, c_1 \\ -d_{c3}s_3\, \dot{q}_3 \end{pmatrix}.$$

Moreover,

$${}^1\boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad {}^2\boldsymbol{\omega}_2 = {}^1\boldsymbol{R}_2^T \, {}^1\boldsymbol{\omega}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{q}_1 \\ 0 \\ 0 \end{pmatrix},$$

and so

$${}^3\boldsymbol{\omega}_3 = {}^2\boldsymbol{R}_3^T(q_3) \left({}^2\boldsymbol{\omega}_2 + \begin{pmatrix} 0 \\ 0 \\ \dot{q}_3 \end{pmatrix} \right) = \begin{pmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ 0 \\ \dot{q}_3 \end{pmatrix} = \begin{pmatrix} c_3\, \dot{q}_1 \\ -s_3\, \dot{q}_1 \\ \dot{q}_3 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} T_3 &= \frac{1}{2}\, m_3 \, \boldsymbol{v}_{c3}^T \boldsymbol{v}_{c3} + \frac{1}{2}\, {}^3\boldsymbol{\omega}_3^T \, {}^3\boldsymbol{I}_{c3} \, {}^3\boldsymbol{\omega}_3 \\ &= \frac{1}{2}\, m_3 \left((q_2 - d_{c3}s_3)^2 \, \dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2 \dot{q}_3^2 - 2 d_{c3} c_3 \dot{q}_2 \dot{q}_3 \right) + \frac{1}{2}\, I_3 \left(\dot{q}_1^2 + \dot{q}_3^2 \right). \end{aligned}$$

As a result, the total kinetic energy is

$$T = T_1 + T_2 + T_3 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with the robot inertia matrix given by

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_{11}(q_2,q_3) & 0 & 0 \\ 0 & m_2+m_3 & -m_3 d_{c3}\, c_3 \\ 0 & -m_3 d_{c3}\, c_3 & I_3 + m_3 d_{c3}^2 \end{pmatrix}, \tag{8}$$

where

$$m_{11}(q_2,q_3) = I_1 + I_2 + m_2 d_{c2}^2 + I_3 - 2m_2 d_{c2} q_2 + (m_2+m_3)q_2^2 - 2m_3 d_{c3} q_2 s_3 + m_3 d_{c3}^2 s_3^2.$$

Note finally that by defining the six dynamic coefficients

$$\begin{aligned} a_1 &= I_1 + I_2 + m_2 d_{c2}^2 + I_3 \\ a_2 &= -m_2 d_{c2} \\ a_3 &= m_2 + m_3 \\ a_4 &= -m_3 d_{c3} \\ a_5 &= m_3 d_{c3}^2 \\ a_6 &= I_3 + m_3 d_{c3}^2, \end{aligned}$$

the inertia matrix (8) is exactly the same input matrix (3) of the next exercise.

Exercise #5

This exercise is solved by the following symbolic code of MATLAB.

```
syms q1 q2 q3 dq1 dq2 dq3 ddq1 ddq2 ddq3 a1 a2 a3 a4 a5 a6 real

disp('the given robot inertia matrix')
M=[a1+2*a2*q2+a3*q2^2+2*a4*q2*sin(q3)+a5*(sin(q3))^2  0  0;
   0    a3           a4*cos(q3);
   0    a4*cos(q3)   a6]

disp('Christoffel matrices')
q=[q1;q2;q3];
M1=M(:,1);
C1=(1/2)*(jacobian(M1,q)+jacobian(M1,q)'-diff(M,q1))
M2=M(:,2);
C2=(1/2)*(jacobian(M2,q)+jacobian(M2,q)'-diff(M,q2))
M3=M(:,3);
C3=(1/2)*(jacobian(M3,q)+jacobian(M3,q)'-diff(M,q3))

disp('robot centrifugal and Coriolis terms')
dq=[dq1;dq2;dq3];
c1=dq'*C1*dq;
c2=dq'*C2*dq;
c3=dq'*C3*dq;
c=[c1;c2;c3]

disp('time derivative of the inertia matrix')
dM=diff(M,q1)*dq1+diff(M,q2)*dq2+diff(M,q3)*dq3
```

```
disp('skew-symmetric factorization of velocity terms')
S1=dq'*C1;
S2=dq'*C2;
S3=dq'*C3;
S=[S1;S2;S3]

disp('check skew-symmetry of N=dM-2*S')
N=simplify(dM-2*S)
N_plus_NT=simplify(N+N')

disp('a second, different factorization of velocity terms (yet with skew-symmetry)')
SS=[0 -dq3 dq2;dq3 0 -dq1;-dq2 dq1 0]
Sprime=S+SS
%namely, obtained by adding to S a skew symmetric matrix SS such that SS*dq=0

disp('check validity of Sprime and skew-symmetry of N=dM-2*Sprime')
checkzero=simplify(c-Sprime*dq)
Nprime=simplify(dM-2*Sprime)
Nprime_plus_NprimeT=simplify(Nprime+Nprime')

disp('a third factorization of velocity terms (without skew-symmetry)')
S2prime=S+[0 -dq3 dq2;dq3 0 -dq1;0 0 0]

disp('check validity of S2prime and absence of skew-symmetry of N=dM-2*S2prime')
checkzero=simplify(c-S2prime*dq)
N2prime=simplify(dM-2*S2prime)
N2prime_plus_N2primeT=simplify(N2prime+N2prime')

disp('regressor Y in linear parametrization Y(q,dq,ddq)*a=tau')
ddq=[ddq1;ddq2;ddq3];
tau=M*ddq+c;
a=[a1;a2;a3;a4;a5;a6];
Y=simplify(jacobian(tau,a))
```

The output of this code yields the matrices of Christoffel symbols

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \begin{pmatrix} 0 & a_2 + a_3 q_2 + a_4 s_3 & (a_4 q_2 + a_5 s_3)\, c_3 \\ a_2 + a_3 q_2 + a_4 s_3 & 0 & 0 \\ (a_4 q_2 + a_5 s_3)\, c_3 & 0 & 0 \end{pmatrix} \\
\boldsymbol{C}_2(\boldsymbol{q}) &= \begin{pmatrix} -(a_2 + a_3 q_2 + a_4 s_3) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a_4 s_3 \end{pmatrix} \\
\boldsymbol{C}_3(\boldsymbol{q}) &= \begin{pmatrix} -(a_4 q_2 + a_5 s_3)\, c_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{9}$$

from which the Coriolis and centrifugal terms are obtained (with $c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}$, for $i = 1, 2, 3$):

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} 2\,(a_2 + a_3 q_2 + a_4 s_3)\, \dot{q}_1 \dot{q}_2 + 2\,(a_4 q_2 + a_5 s_3)\, c_3\, \dot{q}_1 \dot{q}_3 \\ -(a_2 + a_3 q_2 + a_4 s_3)\, \dot{q}_1^2 - a_4 s_3\, \dot{q}_3^2 \\ -(a_4 q_2 + a_5 s_3)\, c_3\, \dot{q}_1^2 \end{pmatrix}. \tag{10}$$

The time derivative of the inertia matrix is

$$\dot{\boldsymbol{M}} = \begin{pmatrix} 2\left(a_2 + a_3 q_2 + a_4 s_3\right)\dot{q}_2 + 2\left(a_4 q_2 + a_5 s_3\right)c_3\,\dot{q}_3 & 0 & 0 \\ 0 & 0 & -a_4 s_3\,\dot{q}_3 \\ 0 & -a_4 s_3\,\dot{q}_3 & 0 \end{pmatrix}.$$

The standard factorization of (10) yielding the skew-symmetric property is given by the matrix having its rows $\boldsymbol{S}_i^T$ built with the Christoffel matrices ($\boldsymbol{S}_i^T(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T\boldsymbol{C}_i(\boldsymbol{q})$, for $i = 1,2,3$):

$$\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} \left(a_2 + a_3 q_2 + a_4 s_3\right)\dot{q}_2 + \left(a_4 q_2 + a_5 s_3\right)c_3\,\dot{q}_3 & \left(a_2 + a_3 q_2 + a_4 s_3\right)\dot{q}_1 & \left(a_4 q_2 + a_5 s_3\right)c_3\,\dot{q}_1 \\ -\left(a_2 + a_3 q_2 + a_4 s_3\right)\dot{q}_1 & 0 & -a_4 s_3\,\dot{q}_3 \\ -\left(a_4 q_2 + a_5 s_3\right)c_3\,\dot{q}_1 & 0 & 0 \end{pmatrix}.$$

A different factorization yielding again the skew-symmetric property is obtained by adding a skew-symmetric matrix $\mathrm{Skew}(\dot{\boldsymbol{q}})$ built with the components of $\dot{\boldsymbol{q}}$,

$$\boldsymbol{S}'(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \mathrm{Skew}(\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \begin{pmatrix} 0 & -\dot{q}_3 & \dot{q}_2 \\ \dot{q}_3 & 0 & -\dot{q}_1 \\ -\dot{q}_2 & \dot{q}_1 & 0 \end{pmatrix},$$

which is certainly another valid factorization of $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})$, being $\mathrm{Skew}(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}} \times \dot{\boldsymbol{q}} = \boldsymbol{0}$. Both choices lead in fact to the skew-symmetry, respectively of

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S} = \begin{pmatrix} 0 & -2\left(a_2 + a_3 q_2 + a_4 s_3\right)\dot{q}_1 & -2\left(a_4 q_2 + a_5 s_3\right)c_3\,\dot{q}_1 \\ 2\left(a_2 + a_3 q_2 + a_4 s_3\right)\dot{q}_1 & 0 & a_4 s_3\,\dot{q}_3 \\ 2\left(a_4 q_2 + a_5 s_3\right)c_3\,\dot{q}_1 & -a_4 s_3\,\dot{q}_3 & 0 \end{pmatrix}$$

and of

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}' = \begin{pmatrix} 0 & -2\left(a_2 + a_3 q_2 + a_4 s_3\right)\dot{q}_1 + 2\dot{q}_3 & -2\left(a_4 q_2 + a_5 s_3\right)c_3\,\dot{q}_1 - 2\dot{q}_2 \\ 2\left(a_2 + a_3 q_2 + a_4 s_3\right)\dot{q}_1 - 2\dot{q}_3 & 0 & 2\dot{q}_1 + a_4 s_3\,\dot{q}_3 \\ 2\left(a_4 q_2 + a_5 s_3\right)c_3\,\dot{q}_1 + 2\dot{q}_2 & -2\dot{q}_1 - a_4 s_3\,\dot{q}_3 & 0 \end{pmatrix}.$$

On the other hand, the choice

$$\boldsymbol{S}''(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \begin{pmatrix} 0 & -\dot{q}_3 & \dot{q}_2 \\ \dot{q}_3 & 0 & -\dot{q}_1 \\ 0 & 0 & 0 \end{pmatrix}$$

is still a feasible factorization, being $\boldsymbol{S}''(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}})$, but leads to a matrix

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}'' = \begin{pmatrix} 0 & -2\left(a_2 + a_3 q_2 + a_4 s_3\right)\dot{q}_1 + 2\dot{q}_3 & -2\left(a_4 q_2 + a_5 s_3\right)c_3\,\dot{q}_1 - 2\dot{q}_2 \\ 2\left(a_2 + a_3 q_2 + a_4 s_3\right)\dot{q}_1 - 2\dot{q}_3 & 0 & 2\dot{q}_1 + a_4 s_3\,\dot{q}_3 \\ 2\left(a_4 q_2 + a_5 s_3\right)c_3\,\dot{q}_1 & -a_4 s_3\,\dot{q}_3 & 0 \end{pmatrix}$$

which is not skew-symmetric.

Finally, the regressor matrix $\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ that linearly parametrizes the robot dynamics (in the absence of gravity), i.e.,

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a},$$

is obtained from (3) and (10) as

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 & 2\left(q_2\,\ddot{q}_1 + \dot{q}_1\dot{q}_2\right) & q_2^2\,\ddot{q}_1 + 2q_2\,\dot{q}_1\dot{q}_2 & 2\left(q_2 s_3\,\ddot{q}_1 + \left(s_3\,\dot{q}_2 + q_2 c_3\,\dot{q}_3\right)\dot{q}_1\right) & s_3^2\,\ddot{q}_1 + 2s_3 c_3\,\dot{q}_1\dot{q}_3 & 0 \\ 0 & -\dot{q}_1^2 & \ddot{q}_2 - q_2\,\dot{q}_1^2 & c_3\,\ddot{q}_3 - s_3\left(\dot{q}_1^2 + \dot{q}_3^2\right) & 0 & 0 \\ 0 & 0 & 0 & c_3\left(\ddot{q}_2 - q_2\,\dot{q}_1^2\right) & -s_3 c_3\,\dot{q}_1^2 & \ddot{q}_3 \end{pmatrix}.$$

Exercise #6

The 2-dof system under consideration is a PR robot, as sketched in Fig. 5 together with its relevant dynamic parameters. The kinetic energy of this robot is

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2} m_2 \left(\dot{q}_1^2 + d_{c2}^2 \, \dot{q}_2^2 \right) + \frac{1}{2} I_{c2} \dot{q}_2^2 \qquad \Rightarrow \qquad T = T_1 + T_2 = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}},$$

with

$$\boldsymbol{M} = \begin{pmatrix} m_1 + m_2 & 0 \\ 0 & I_{c2} + m_2 d_{c2}^2 \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix},$$

while the potential energy due to gravity and the corresponding gravity term are

$$U_1 = m_1 g_0 q_1, \quad U_2 = m_2 g_0 q_1 \quad \Rightarrow \quad U = U_1 + U_2 \quad \Rightarrow \qquad \boldsymbol{g} = \frac{\partial U}{\partial \boldsymbol{q}} = \begin{pmatrix} (m_1 + m_2)\, g_0 \\ 0 \end{pmatrix} = \begin{pmatrix} g_1 \\ 0 \end{pmatrix}$$

with $g_0 = 9.81$ [m/s^2]. As a result, the dynamic model of this PR robot is given by two linear and decoupled differential equations:

$$\begin{aligned} M_1 \ddot{q}_1 + g_1 &= u_1 \\ M_2 \ddot{q}_2 &= u_2. \end{aligned} \tag{11}$$

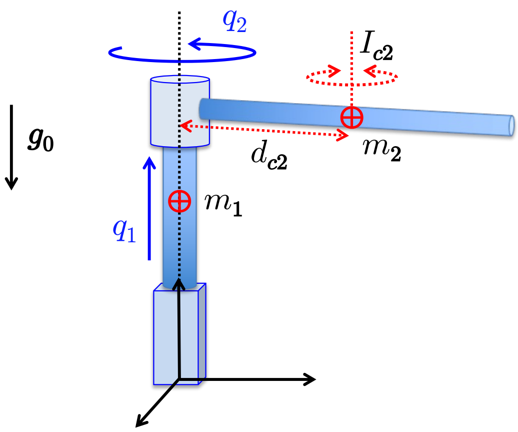


Figure 5: The PR robot with first and second axis coincident and vertical.

The desired rest-to-rest joint trajectory is the cubic polynomial

$$\boldsymbol{q}_d(t) = \boldsymbol{\Delta q} \left(3 \left(\frac{t}{T} \right)^2 - 2 \left(\frac{t}{T} \right)^3 \right), \qquad t \in [0, T],$$

where T is the (coordinated) motion time. The associated acceleration has a linear profile in time

$$\ddot{\boldsymbol{q}}_d(t) = \frac{6 \boldsymbol{\Delta q}}{T^2} \left(1 - 2 \left(\frac{t}{T} \right) \right).$$

From the bounds $|u_i| \leq U_{max,i}$, $i = 1, 2$, and from eqs. (11) it follows that the maximum absolute value of the acceleration, which is reached at $t = 0$ and $t = T$,

$$|\ddot{\boldsymbol{q}}_d(0)| = |\ddot{\boldsymbol{q}}_d(T)| = \frac{6\, |\boldsymbol{\Delta q}|}{T^2},$$

should satisfy componentwise

$$M_1 \frac{6\, |\Delta q_1|}{T^2} \leq U_{max,1} - g_1, \qquad M_2 \frac{6\, |\Delta q_2|}{T^2} \leq U_{max,2}.$$

Therefore, the minimum feasible motion time is given by

$$T^* = \max\left\{\sqrt{\frac{6|\Delta q_1|M_1}{U_{max,1} - g_1}}, \sqrt{\frac{6|\Delta q_2|M_2}{U_{max,2}}}\right\},$$

which is well defined since $U_{max,1} - g_1 > 0$ by assumption.

*****

Robotics 2

June 12, 2023

Exercise 1

Consider the 4R planar robot in Fig. 1, with generic lengths, masses and inertias of the links but with the center of mass of each link placed on its kinematic axis. As shown in the figure, the absolute angles of the links with respect to the axis $\boldsymbol{x}_0$ must be used as generalized coordinates $\boldsymbol{q}$.

$\theta_1 = q_1$

$\theta_2 = -q_1 + q_2$

$\theta_3 = -q_2 + q_3$

$\theta_4 = q_4 - q_3$

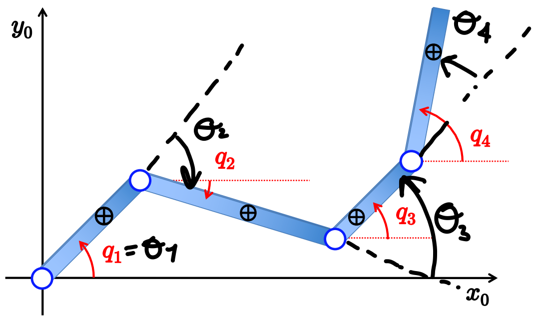


Figure 1: A 4R planar robot.

- Compute the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of this robot.
- From the elements of $\boldsymbol{M}(\boldsymbol{q})$, derive the expression of the robot inertia matrix when using instead the Denavit-Hartenberg joint angles $\boldsymbol{\theta}$ as generalized coordinates.
- With the experience gained for the case $n = 4$, provide the general expression of the kinetic energy $T_i(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of link i in a nR planar robot using the generalized coordinates $\boldsymbol{q}$ and under similar assumptions.

Exercise 2

Let the robot of Fig. 1 have all four links of unitary length, and suppose that we can command the robot using the joint velocities $\dot{\boldsymbol{q}} \in \mathbb{R}^4$. With the robot in the configuration

$$\boldsymbol{q}_0 = \begin{pmatrix} 0 & \frac{\pi}{6} & -\frac{\pi}{3} & -\frac{\pi}{3} \end{pmatrix}^T,$$

consider two (alternative or simultaneous) tasks: *(i)* the end-effector should move with a velocity $\boldsymbol{v}_e \in \mathbb{R}^2$; and *(ii)* the tip of the second link should move with a velocity $\boldsymbol{v}_t \in \mathbb{R}^2$. Determine the joint velocity commands $\dot{\boldsymbol{q}}$ for the following problems:

a. execute at best the end-effector task $\boldsymbol{v}_e = (0.4330, -0.75)$, while minimizing the norm of $\dot{\boldsymbol{q}}$;

b. execute at best the second link tip task $\boldsymbol{v}_t = (-0.5, 0.8660)$, while minimizing the norm of $\dot{\boldsymbol{q}}$;

c. execute at best both tasks $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$ simultaneously;

d. execute at best both tasks $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$, with priority to the end-effector task $\boldsymbol{v}_e$;

e. execute at best both tasks $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$, with priority to the second link tip task $\boldsymbol{v}_t$.

For each case, provide also the obtained velocity errors $\boldsymbol{e}_e$ and $\boldsymbol{e}_t$ on both tasks (whether assigned or not) and their norm.

Exercise 3

For regulating the PRR planar robot shown in Fig. 2 to a desired configuration $\boldsymbol{q}_d$, the PD+gravity compensation torque $\boldsymbol{\tau} = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d)$ is being used, with diagonal gain matrices $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$. For this control law, provide the symbolic expression of the feedforward term $\boldsymbol{g}(\boldsymbol{q}_d)$ and of the minimum constant value for the elements of $\boldsymbol{K}_P$ that guarantees global asymptotic stabilization of any desired equilibrium configuration $\boldsymbol{q}_d$.

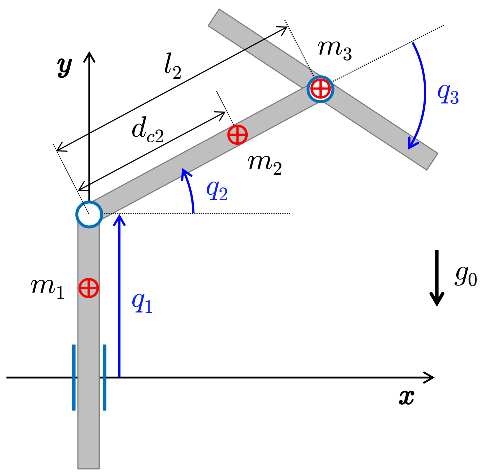


Figure 2: A PRR planar robot under gravity.

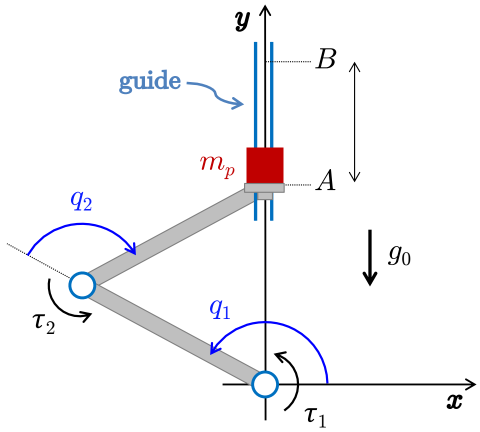


Figure 3: The mechanism with a constrained 2R robot for elevating payloads.

Exercise 4

The mechanism in Fig. 3 elevates payloads by means of a 2R robot, which is constrained at its end effector by a vertical guide. The robot has unitary link lengths. The extension of the vertical motion is limited between points $A = (0, 0.95)$ and $B = (0, 1.45)$ —the robot is never in a singularity. When including also the payload m_p, the inertia matrix of the unconstrained 2R robot is parametrized as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2\, c_2 & a_3 + a_2\, c_2 \\ a_3 + a_2\, c_2 & a_3 \end{pmatrix}.$$

Derive all the individual elements of the reduced dynamic model of this constrained robotic system. In particular, provide:

- the remaining elements $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\boldsymbol{g}(\boldsymbol{q})$ of the 2R robot dynamic model;
- the 1×2 Jacobian of the constraint $\boldsymbol{A}(\boldsymbol{q})$ and a 1×2 completion matrix $\boldsymbol{D}(\boldsymbol{q})$ that guarantees non-singularity in the operating region, together with their time derivatives $\dot{\boldsymbol{A}}(\boldsymbol{q})$ and $\dot{\boldsymbol{D}}(\boldsymbol{q})$;
- a physical interpretation of the pseudo-velocity $v \in \mathbb{R}$;
- the (scalar) reduced inertia $\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})$.

Design then a suitable motion control law for the torque $\boldsymbol{\tau} \in \mathbb{R}^2$ that should impose a desired cyclic motion from A to B and vice versa in a total motion time T. The joint motion of the 2R robot should have a continuous acceleration profile at all times. Moreover, no reaction force $\lambda \in \mathbb{R}$ should be exerted on the end-effector by the constraining guide.

[240 minutes; open books]

Solution

June 12, 2023

Exercise 1

We compute the kinetic energy of this 4R planar robot, taking advantage of the absolute coordinates q_i, $i = 1, \dots, 4$, as shown in Fig. 1. Let m_i be the mass and l_i the kinematic length of link i, d_{ci} the distance along the link axis of the center of mass (CoM) of link i from the previous joint, and I_{ci} the barycentric inertia of link i around the axis normal to the plane of motion. The position $\boldsymbol{p}_{ci}$ and the velocity $\boldsymbol{v}_{ci}$ of the CoM of link i are two-dimensional vectors in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$.

For the first link, we have

$$T_1 = \frac{1}{2}\left(I_{c1} + m_1 d_{c1}^2\right)\dot{q}_1^2.$$

For the second link, being

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \frac{d}{dt}\begin{pmatrix} l_1 c_1 + d_{c2} c_2 \\ l_1 s_1 + d_{c2} s_2 \end{pmatrix} = \begin{pmatrix} -\left(l_1 s_1 \dot{q}_1 + d_{c2} s_2 \dot{q}_2\right) \\ l_1 c_1 \dot{q}_1 + d_{c2} c_2 \dot{q}_2 \end{pmatrix},$$

it is

$$T_2 = \frac{1}{2}\, I_{c2}\dot{q}_2^2 + \frac{1}{2}\, m_2 \|\boldsymbol{v}_{c2}\|^2 = \frac{1}{2} m_2 l_1^2 \dot{q}_1^2 + \frac{1}{2}\left(I_{c2} + m_2 d_{c2}^2\right)\dot{q}_2^2 + m_2 l_1 d_{c2} c_{2-1} \dot{q}_1 \dot{q}_2,$$

where $c_{2-1} = \cos(q_2 - q_1)$.

For the third link, being

$$\boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \frac{d}{dt}\begin{pmatrix} l_1 c_1 + l_2 c_2 + d_{c3} c_3 \\ l_1 s_1 + l_2 s_2 + d_{c3} s_3 \end{pmatrix} = \begin{pmatrix} -\left(l_1 s_1 \dot{q}_1 + l_2 s_2 \dot{q}_2 + d_{c3} s_3 \dot{q}_3\right) \\ l_1 c_1 \dot{q}_1 + l_2 c_2 \dot{q}_2 + d_{c3} c_3 \dot{q}_3 \end{pmatrix},$$

it is

$$\begin{aligned} T_3 &= \frac{1}{2}\, I_{c3}\dot{q}_3^2 + \frac{1}{2}\, m_3 \|\boldsymbol{v}_{c3}\|^2 \\ &= \frac{1}{2} m_3 \left(l_1^2 \dot{q}_1^2 + l_2^2 \dot{q}_2^2\right) + \frac{1}{2}\left(I_{c3} + m_3 d_{c3}^2\right)\dot{q}_3^2 \\ &\quad + m_3 \left(l_1 l_2 c_{2-1} \dot{q}_1 \dot{q}_2 + l_1 d_{c3} c_{3-1} \dot{q}_1 \dot{q}_3 + l_2 d_{c3} c_{3-2} \dot{q}_2 \dot{q}_3\right), \end{aligned}$$

where $c_{3-1} = \cos(q_3 - q_1)$ and $c_{3-2} = \cos(q_3 - q_2)$.

For the fourth and last link, we follow the same pattern and obtain

$$\begin{aligned} T_4 &= \frac{1}{2}\, I_{c4}\dot{q}_4^2 + \frac{1}{2}\, m_4 \|\boldsymbol{v}_{c4}\|^2 \\ &= \frac{1}{2} m_4 \left(l_1^2 \dot{q}_1^2 + l_2^2 \dot{q}_2^2 + l_3^2 \dot{q}_3^2\right) + \frac{1}{2}\left(I_{c4} + m_4 d_{c4}^2\right)\dot{q}_4^2 \\ &\quad + m_4 \left(l_1 l_2 c_{2-1} \dot{q}_1 \dot{q}_2 + l_1 l_3 c_{3-1} \dot{q}_1 \dot{q}_3 + l_2 l_3 c_{3-2} \dot{q}_2 \dot{q}_3 \right. \\ &\qquad\quad \left. + \left(l_1 c_{4-1} \dot{q}_1 + l_2 c_{4-2} \dot{q}_2 + l_3 c_{4-3} \dot{q}_3\right) d_{c4} \dot{q}_4\right), \end{aligned}$$

where $c_{4-1} = \cos(q_4 - q_1)$, $c_{4-2} = \cos(q_4 - q_2)$ and $c_{4-3} = \cos(q_4 - q_4)$.

Finally,

$$T = T_1 + T_2 + T_3 + T_4 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

and the robot inertia matrix is given by

$$
\boldsymbol{M}(\boldsymbol{q}) =
\begin{pmatrix}
I_{c1}+m_1 d_{c1}^2+(m_2+m_3+m_4)\, l_1^2 & & symm & \\
(m_2 d_{c2}+(m_3+m_4)\, l_2)\, l_1 c_{2-1} & I_{c2}+m_2 d_{c2}^2+(m_3+m_4)\, l_2^2 & & \\
(m_3 d_{c3}+m_4 l_3)\, l_1 c_{3-1} & (m_3 d_{c3}+m_4 l_3)\, l_2 c_{3-2} & I_{c3}+m_3 d_{c3}^2+m_4 l_3^2 & \\
m_4 d_{c4} l_1 c_{4-1} & m_4 d_{c4} l_2 c_{4-2} & m_4 d_{c4} l_3 c_{4-3} & I_{c4}+m_4 d_{c4}^2
\end{pmatrix}.
$$

The coordinate transformation between the $\boldsymbol{\theta}$ variables of Denavit-Hartenberg and the generalized coordinates $\boldsymbol{q}$ is linear and is given by

$$
\boldsymbol{q} = \boldsymbol{T}\,\boldsymbol{\theta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \boldsymbol{\theta} \qquad \Longleftrightarrow \qquad \boldsymbol{\theta} = \boldsymbol{T}^{-1}\boldsymbol{q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \boldsymbol{q}.
$$

To obtain the inertia matrix in the new coordinates $\boldsymbol{\theta}$, we first replace the arguments of the cosine functions inside the elements $m_{ij}(q)$ of $\boldsymbol{M}(\boldsymbol{q})$ as follows:

$$
c_{2-1} = \cos\theta_2, \qquad c_{3-1} = \cos(\theta_2+\theta_3), \qquad c_{4-1} = \cos(\theta_2+\theta_3+\theta_4),
$$
$$
c_{3-2} = \cos\theta_3 \qquad c_{4-2} = \cos(\theta_3+\theta_4), \qquad c_{4-3} = \cos\theta_4.
$$

Applying then the transformation rule, one obtains

$$
\widetilde{\boldsymbol{M}}(\boldsymbol{\theta}) = \boldsymbol{T}^T \boldsymbol{M}(\boldsymbol{q})|_{\boldsymbol{q}=\boldsymbol{T}\boldsymbol{\theta}}\, \boldsymbol{T}
$$
$$
= \left.\begin{pmatrix}
\begin{matrix} m_{11}+2m_{12}+2m_{13}+2m_{14} \\ +m_{22}+2m_{23}+2m_{24} \\ +m_{33}+2m_{34}+m_{44} \end{matrix} & & symm & \\
\begin{matrix} m_{12}+m_{13}+m_{14} \\ +m_{22}+2m_{23}+2m_{24} \\ +m_{33}+2m_{34}+m_{44} \end{matrix} & \begin{matrix} m_{22}+2m_{23}+2m_{24} \\ +m_{33}+2m_{34}+m_{44} \end{matrix} & & \\
\begin{matrix} m_{13}+m_{14}+m_{23}+m_{24} \\ +m_{33}+2m_{34}+m_{44} \end{matrix} & \begin{matrix} m_{23}+m_{24} \\ +m_{33}+2m_{34}+m_{44} \end{matrix} & m_{33}+2m_{34}+m_{44} & \\
m_{14}+m_{24}+m_{34}+m_{44} & m_{24}+m_{34}+m_{44} & m_{34}+m_{44} & m_{44}
\end{pmatrix}\right|_{\boldsymbol{q}=\boldsymbol{T}\boldsymbol{\theta}},
$$

which clearly shows how more cumbersome would be the explicit expression of the robot inertia matrix for this robot when using the DH (relative) angles $\boldsymbol{\theta}$.

For the nR planar robot, based on the previous derivations and under the same assumptions, it is easy to find the general expression for the kinetic energy of link i, for $i = 1, \ldots, n$:

$$
T_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\, I_{ci}\dot{q}_i^2 + \frac{1}{2}\, m_i \|\boldsymbol{v}_{ci}\|^2,
$$

with

$$
\|\boldsymbol{v}_{ci}\|^2 = \sum_{j=1}^{i-1} l_j^2 \dot{q}_j^2 + d_{ci}^2 \dot{q}_i^2 + 2\sum_{j=1}^{i-1} \left(\sum_{k=j+1}^{i-1} l_k c_{k-j} \dot{q}_k \right) l_j \dot{q}_j + 2 \left(\sum_{j=1}^{i-1} l_j c_{i-j} \dot{q}_j \right) d_{ci}\dot{q}_i.
$$

Exercise 2

With all links of the 4R planar robot being of unitary length and using again the absolute joint variables of Fig. 1, the Jacobians for the two considered tasks are

$$\boldsymbol{J}_e(\boldsymbol{q}) = \begin{pmatrix} -s_1 & -s_2 & -s_3 & -s_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} \qquad \boldsymbol{v}_e = \boldsymbol{J}_e(\boldsymbol{q})\dot{\boldsymbol{q}}$$

and

$$\boldsymbol{J}_t(\boldsymbol{q}) = \begin{pmatrix} -s_1 & -s_2 & 0 & 0 \\ c_1 & c_2 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{v}_t = \boldsymbol{J}_t(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

In the configuration $\boldsymbol{q}_0$ (see Fig. 4), we have

$$\boldsymbol{J}_e(\boldsymbol{q}_0) = \begin{pmatrix} 0 & -0.5 & 0.8660 & 0.8660 \\ 1 & 0.8660 & 0.5 & 0.5 \end{pmatrix} \quad \Rightarrow \quad \operatorname{rank} \boldsymbol{J}_e(\boldsymbol{q}_0) = 2$$

and

$$\boldsymbol{J}_t(\boldsymbol{q}_0) = \begin{pmatrix} 0 & -0.5 & 0 & 0 \\ 1 & 0.8660 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \operatorname{rank} \boldsymbol{J}_t(\boldsymbol{q}_0) = 2,$$

showing that both tasks can certainly be executed separately, no matter which are the values of $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$. However, the complete Jacobian for the two simultaneous tasks is singular,

$$\operatorname{rank} \boldsymbol{J}(\boldsymbol{q}_0) = \operatorname{rank} \begin{pmatrix} \boldsymbol{J}_e(\boldsymbol{q}_0) \\ \boldsymbol{J}_t(\boldsymbol{q}_0) \end{pmatrix} = 3 < 4,$$

and thus the robot is in an algorithmic singularity.

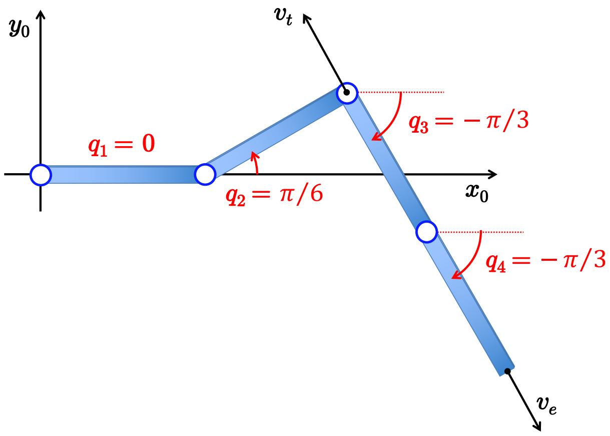


Figure 4: The 4R planar robot in the configuration $\boldsymbol{q}_0$, with the two assigned tasks.

With the above in mind, we set for $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$ the given numerical values (see again Fig. 4), and solve the stated problems as follows.

a. Execute at best the end-effector task $\boldsymbol{v}_e = (0.4330, -0.75)$, while minimizing the norm of $\dot{\boldsymbol{q}}$:

$$\dot{\boldsymbol{q}}_{\mathrm{a}} = \boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_e = \begin{pmatrix} -0.4 \\ -0.5196 \\ 0.1 \\ 0.1 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} \boldsymbol{e}_e &= \boldsymbol{v}_e - \boldsymbol{J}_e(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\mathrm{a}} = \boldsymbol{0} \\ \boldsymbol{e}_t &= \boldsymbol{v}_t - \boldsymbol{J}_t(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\mathrm{a}} = \begin{pmatrix} -0.7598 \\ 1.7160 \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{e}_t\| = 1.8767. \end{aligned}$$

b. Execute at best the second link tip task $\boldsymbol{v}_t = (-0.5, 0.8660)$, while minimizing the norm of $\dot{\boldsymbol{q}}$:

$$\dot{\boldsymbol{q}}_{\rm b} = \boldsymbol{J}_t^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_t = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{l} \boldsymbol{e}_t = \boldsymbol{v}_t - \boldsymbol{J}_t(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\rm b} = \boldsymbol{0} \\ \boldsymbol{e}_e = \boldsymbol{v}_e - \boldsymbol{J}_t(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\rm b} = \begin{pmatrix} 0.9330 \\ -1.6160 \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{e}_e\| = 1.8660. \end{array}$$

c. Execute at best both tasks $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$ simultaneously:

$$\dot{\boldsymbol{q}}_{\rm c} = \boldsymbol{J}^{\#}(\boldsymbol{q}_0)\begin{pmatrix} \boldsymbol{v}_e \\ \boldsymbol{v}_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0.0670 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{e} = \begin{pmatrix} \boldsymbol{v}_e \\ \boldsymbol{v}_t \end{pmatrix} - \boldsymbol{J}(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\rm c} = \begin{pmatrix} 0.4665 \\ -0.8080 \\ -0.4665 \\ 0.8080 \end{pmatrix}$$

$$\Rightarrow \quad \|\boldsymbol{e}\| = 1.3195.$$

d. Execute at best both tasks $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$, with priority to the end-effector task $\boldsymbol{v}_e$:

$$\dot{\boldsymbol{q}}_{\rm d} = \boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_e + (\boldsymbol{J}_t(\boldsymbol{q}_0)\boldsymbol{P}_e(\boldsymbol{q}_0))^{\#}\left(\boldsymbol{v}_t - \boldsymbol{J}_t(\boldsymbol{q}_0)\boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_e\right) = \begin{pmatrix} 0 \\ -0.8660 \\ 0 \\ 0 \end{pmatrix}$$

with $\boldsymbol{P}_e(\boldsymbol{q}_0) = \boldsymbol{I} - \boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{J}_e(\boldsymbol{q}_0)$.

$$\Rightarrow \quad \begin{array}{l} \boldsymbol{e}_e = \boldsymbol{v}_e - \boldsymbol{J}_e(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\rm d} = \boldsymbol{0} \\ \boldsymbol{e}_t = \boldsymbol{v}_t - \boldsymbol{J}_t(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\rm d} = \begin{pmatrix} -0.9330 \\ 1.6160 \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{e}\| = \|\boldsymbol{e}_t\| = 1.8660. \end{array}$$

e. Execute at best both tasks $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$, with priority to the second link tip task $\boldsymbol{v}_t$:

$$\dot{\boldsymbol{q}}_{\rm e} = \boldsymbol{J}_t^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_t + (\boldsymbol{J}_e(\boldsymbol{q}_0)\boldsymbol{P}_t(\boldsymbol{q}_0))^{\#}\left(\boldsymbol{v}_e - \boldsymbol{J}_e(\boldsymbol{q}_0)\boldsymbol{J}_t^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_t\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

with $\boldsymbol{P}_t(\boldsymbol{q}_0) = \boldsymbol{I} - \boldsymbol{J}_t^{\#}(\boldsymbol{q}_0)\boldsymbol{J}_t(\boldsymbol{q}_0)$.

$$\Rightarrow \quad \begin{array}{l} \boldsymbol{e}_t = \boldsymbol{v}_t - \boldsymbol{J}_t(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\rm e} = \boldsymbol{0} \\ \boldsymbol{e}_e = \boldsymbol{v}_e - \boldsymbol{J}_e(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{\rm e} = \begin{pmatrix} 0.9330 \\ -1.6160 \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{e}\| = \|\boldsymbol{e}_e\| = 1.8660. \end{array}$$

Summarizing, one can observe that:

- As expected, the attempt to execute both tasks simultaneously without the use of any priority (case c.) produces errors on all tasks components, due to the algorithmic singularity. On the other hand, the introduction of priority preserves the correct execution of one of the two tasks. Nonetheless, the norm of the error on the complete set of tasks in the first case is smaller ($\|\boldsymbol{e}\| = 1.3195$) than when using priorities ($\|\boldsymbol{e}\| = 1.8660$ in both cases d. and e.).
- In the specific situation considered, the independent execution of either of the two tasks without care of the other (cases a. and b.) and their simultaneous execution with priority given to either of the two tasks (cases d. and e., respectively) produce exactly the same error in norm for the discarded or the lower priority task.

- The two chosen velocity tasks for the tip of the second link and for the end-effector are highly conflicting (as apparent also from the geometry in Fig. 4). In fact, the two velocity vectors $\boldsymbol{v}_e$ and $\boldsymbol{v}_t$ have a common direction, but opposite orientations. Moreover, they lie along the Cartesian direction where the third and fourth link of the robot are stretched: as a consequence, joints 3 and 4 cannot contribute to their simultaneous execution. Note also that these two task velocities are slightly different in norm ($\|\boldsymbol{v}_e\| = 0.8660$, $\|\boldsymbol{v}_t\| = 1$): if we had chosen still the same common direction, but exactly opposite values (i.e., $\boldsymbol{v}_e = -\boldsymbol{v}_t$), the best solution in case c. would have been $\dot{\boldsymbol{q}}_{\rm c} = \boldsymbol{0}$ (the robot does not move!).

Exercise 3

With reference to Fig. 2, the potential energy due to gravity is computed (up to constants) for each link as

$$U_1(q_1) = m_1 g_0 q_1 \qquad U_2(q_1, q_2) = m_2 g_0 \left(q_1 + d_{c2} \sin q_2\right) \qquad U_3(q_1, q_2) = m_3 g_0 \left(q_1 + l_2 \sin q_2\right),$$

where $g_0 = 9.81$. Thus, from $U = U_1 + U_2 + U_3 = U(\boldsymbol{q})$, one has

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0 \left(m_1 + m_2 + m_3\right) \\ g_0 \cos q_2 \left(m_2 d_{c2} + m_3 l_2\right) \\ 0 \end{pmatrix}, \tag{1}$$

from which the feedforward term $\boldsymbol{g}(\boldsymbol{q}_d)$ in the control law follows by direct substitution of $q_2 = q_{2,d}$.

A well known sufficient condition for the global asymptotic stability of a desired equilibrium configuration $\boldsymbol{q}_d$ in a robot controlled by PD + gravity compensation is that $\boldsymbol{K}_{P,m} > \alpha$. For a diagonal gain matrix $\boldsymbol{K}_P$, $\boldsymbol{K}_{P,m}$ is the smallest diagonal element of the matrix. The constant α is defined as a value that bounds the norm of the Hessian matrix of the gravitational potential energy in all configurations, or

$$\left\|\frac{\partial^2 U}{\partial \boldsymbol{q}^2}\right\| = \left\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right\| = \sqrt{\lambda_{max}\left\{\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right)^T \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right)\right\}} \le \alpha, \qquad \forall \boldsymbol{q}. \tag{2}$$

Therefore, from

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -g_0 \sin q_2 \left(m_2 d_{c2} + m_3 l_2\right) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the only positive eigenvalue of the symmetric, positive semi-definite matrix $\left\{\left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right)^T \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right)\right\}$ is

$$\lambda_{max}(q_2) = g_0^2 \left(m_2 d_{c2} + m_3 l_2\right)^2 \sin^2 q_2$$

and so the minimum value for α that globally satisfies (2) is

$$\alpha = g_0 \left(m_2 d_{c2} + m_3 l_2\right) > 0.$$

Indeed, due to the structure of the gravity vector (1) for this robot, with a first constant component and a zero third component, it is easy to see that the sufficient condition is simplified to $\boldsymbol{K}_{P,2} > \alpha$ while the other two diagonal gains $\boldsymbol{K}_{P,1}$ and $\boldsymbol{K}_{P,3}$ only need to be positive.

Exercise 4

Given the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of the unconstrained 2R planar robot, the Coriolis and centrifugal terms are derived using the Christoffel symbols (without the need of knowing the actual expressions of the dynamic coefficients). One obtains

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} -a_2 s_2 \left(2\dot{q}_1 + \dot{q}_2\right)\dot{q}_2 \\ a_2 s_2 \dot{q}_1^2 \end{pmatrix}.$$

As for the gravity term, the potential energy should include also the payload. Therefore, being the link lengths $l_1 = l_2 = 1$, we have

$$U_1(q_1) = m_1 g_0 d_{c1} s_1 \qquad U_2(q_1,q_2) = m_2 g_0 \left(s_1 + d_{c2} s_{12}\right) \qquad U_p(q_1,q_2) = m_p g_0 \left(s_1 + s_{12}\right),$$

where $g_0 = 9.81$. From $U = U_1 + U_2 + U_p = U(\boldsymbol{q})$, one has

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0\left(m_1 d_{c1} + m_2 + m_p\right)c_1 + g_0\left(m_2 d_{c2} + m_p\right)c_{12} \\ g_0\left(m_2 d_{c2} + m_p\right)c_{12} \end{pmatrix} = \begin{pmatrix} a_4 c_1 + a_5 c_{12} \\ a_5 c_{12} \end{pmatrix}.$$

The guide constrains the motion of the robot end-effector to $h(\boldsymbol{q}) = p_x(\boldsymbol{q}) = 0$. Thus, from the direct kinematics, we have

$$h(\boldsymbol{q}) = c_1 + c_{12} = 0 \qquad \Rightarrow \qquad \boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(s_1 + s_{12}\right) & -s_{12} \end{pmatrix}.$$

A convenient choice for completing a square nonsingular matrix is

$$\boldsymbol{D}(\boldsymbol{q}) = \begin{pmatrix} c_1 + c_{12} & c_{12} \end{pmatrix}. \tag{3}$$

In fact, the resulting matrix is nothing else than the robot Jacobian

$$\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -\left(s_1 + s_{12}\right) & -s_{12} \\ c_1 + c_{12} & c_{12} \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q}),$$

whose determinant $\det \boldsymbol{J}(\boldsymbol{q}) = s_2$ never vanishes in the operating region of the constrained mechanism. Therefore, we can safely invert this matrix and obtain

$$\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix}^{-1} = \frac{1}{s_2}\begin{pmatrix} c_{12} & s_{12} \\ -\left(c_1 + c_{12}\right) & -\left(s_1 + s_{12}\right) \end{pmatrix} = \begin{pmatrix} \boldsymbol{E}(\boldsymbol{q}) & \boldsymbol{F}(\boldsymbol{q}) \end{pmatrix}.$$

Moreover, the following time derivatives are needed:

$$\dot{\boldsymbol{A}}(\boldsymbol{q}) = \begin{pmatrix} -c_1\dot{q}_1 - c_{12}\left(\dot{q}_1 + \dot{q}_2\right) & -c_{12}\left(\dot{q}_1 + \dot{q}_2\right) \end{pmatrix} \qquad \dot{\boldsymbol{D}}(\boldsymbol{q}) = \begin{pmatrix} -s_1\dot{q}_1 - s_{12}\left(\dot{q}_1 + \dot{q}_2\right) & -s_{12}\left(\dot{q}_1 + \dot{q}_2\right) \end{pmatrix}.$$

The choice (3) leads also to a simple physical interpretation of the pseudo-velocity

$$v = \boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}} = c_1\dot{q}_1 + c_{12}\left(\dot{q}_1 + \dot{q}_2\right)$$

as the end-effector velocity component along the $\boldsymbol{y}$ direction, i.e., $v = v_y = \dot{p}_y(\boldsymbol{q})$. Finally, the reduced inertia of the constrained robot is evaluated as

$$\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = \frac{1}{s_2^2}\left(a_3 s_1^2 + \left(a_1 - a_3\right)s_{12}^2 - 2a_2 s_1 s_{12} c_2\right) > 0.$$

The motion control law is computed by inverse dynamics and is given by

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q}) \left(\boldsymbol{F}(\boldsymbol{q}) \dot{v}_d - \left(\boldsymbol{E}(\boldsymbol{q}) \dot{\boldsymbol{A}}(\boldsymbol{q}) + \boldsymbol{F}(\boldsymbol{q}) \dot{\boldsymbol{D}}(\boldsymbol{q}) \right) \dot{\boldsymbol{q}} \right) + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}), \tag{4}$$

where each individual term has already been defined, except for the desired pseudo-acceleration $\dot{v}_d$. It can be shown that the control law (4) provides

$$\dot{v} = \dot{v}_d, \qquad \lambda = 0,$$

as requested.

Multiple possibilities are available for the definition of a smooth cyclic motion with period T between points A and B. For instance, one can use two specular quintic polynomials, one for the elevation from A to B in the time interval $t \in [0, T/2]$, the other for returning from B to A with $t \in [T/2, T]$. By imposing zero boundary conditions on the first and second time derivatives at $A_y = 0.95$ and $B_y = 1.45$, we obtain

$$p_{y,d}(t) = \begin{cases} A_y + (B_y - A_y) \left(10\sigma^3 - 15\sigma^4 + 6\sigma^5\right), & \sigma = \dfrac{t}{T/2}, \quad t \in [0, T/2], \\ B_y + (A_y - B_y) \left(10\sigma^3 - 15\sigma^4 + 6\sigma^5\right), & \sigma = \dfrac{t - T/2}{T/2}, \quad t \in [T/2, T]. \end{cases}$$

From this, we get the pseudo-acceleration command

$$\dot{v}_d(t) = \ddot{p}_{y,d}(t) = \begin{cases} \dfrac{60\,(B_y - A_y)}{(T/2)^2} \left(\sigma - 3\sigma^2 + 2\sigma^3\right), & \sigma = \dfrac{t}{T/2}, \quad t \in [0, T/2], \\ \dfrac{60\,(A_y - B_y)}{(T/2)^2} \left(\sigma - 3\sigma^2 + 2\sigma^3\right), & \sigma = \dfrac{t - T/2}{T/2}, \quad t \in [T/2, T]. \end{cases}$$

The desired position profile $p_{y,d}(t)$ and the pseudo-acceleration command $\dot{v}_d(t)$ are shown in Fig. 5, for a chosen motion period of $T = 1$ s.

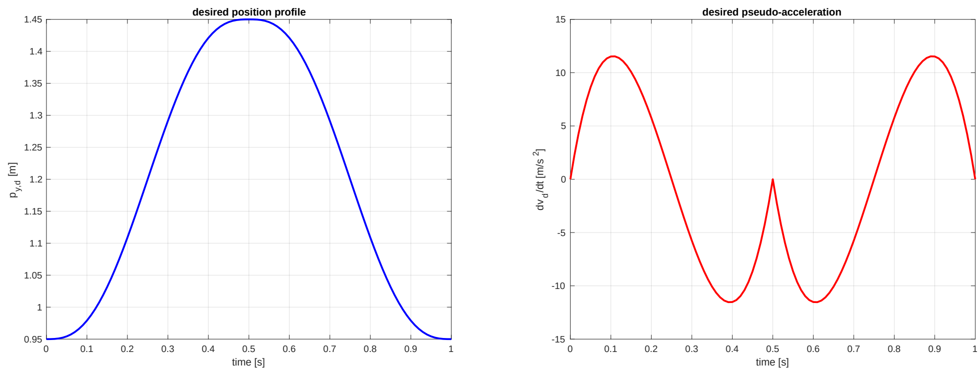


Figure 5: Periodic motion of the constrained robot end-effector: $p_{y,d}(t)$ [left] and $\dot{v}_d(t)$ [right].

* * * * *

Robotics II

October 27, 2014

Exercise 1

The collision detection and isolation method based on the use of residuals that monitor the robot generalized momentum has been presented only for open chain manipulators with *revolute* joints. Consider the planar PRR robot in Fig. 1, moving on a *horizontal* plane. Assuming that the robot at rest at time $t = 0$, provide the explicit expressions of the contributions to the residual vector $\boldsymbol{r} \in \mathbb{R}^3$ in terms of the robot dynamic model components[1]. Comment on analogies or differences that may result due to the presence of *prismatic* joints in the chain, only one in the present case, or one or more in the general case.

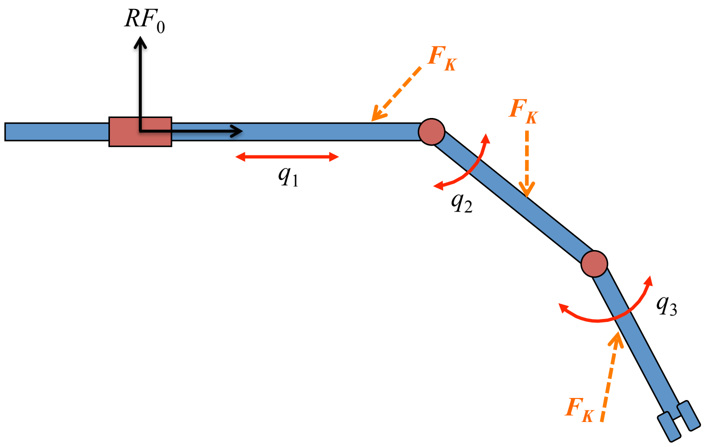


Figure 1: A planar PRR robot that may undergo collisions at any point along its structure (only non-simultaneous collision forces $\boldsymbol{F}_K$ in the plane of motion are considered)

Exercise 2

For the same robot of the previous exercise, assume that second and third links have unitary lengths. Moreover, the force (for joint 1) and torques (for joints 2 and 3) that can be delivered by the actuators at the joints are bounded as follows:

$$|\tau_1| \leq 5 \ \ [\text{N}], \qquad |\tau_2| \leq 1 \ \ [\text{Nm}], \qquad |\tau_3| \leq 2 \ \ [\text{Nm}].$$

The robot should be able to sustain in static conditions contact forces $\boldsymbol{F}$ that are applied to its end-effector in various planar directions. Determine the maximum norm of a contact force that can be applied in *any* planar direction and sustained by the robot when kept always in a fixed configuration. Provide at least one *non-singular* configuration in which the robot achieves this optimal result.

[180 minutes; open books]

[1] For dynamic analysis, you may use whatever generalized coordinates you find most convenient.

Solution

October 27, 2014

Exercise 1

When considering the Lagrangian dynamics of a robot with $\boldsymbol{q} \in \mathbb{R}^n$ that is possibly subject to collision forces, we have

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{\tau}_K, \qquad \boldsymbol{\tau}_K = \boldsymbol{J}_K^T(\boldsymbol{q})\boldsymbol{F}_K,$$

where the left-hand side of the dynamic equation contains the usual inertia terms, Coriolis and centrifugal terms (with a factorization such that $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is a skew-symmetric matrix), and gravity terms, while the non-conservative terms on the right-hand side are the motor torque $\boldsymbol{\tau}$ and the joint torque $\boldsymbol{\tau}_K \in \mathbb{R}^n$ resulting from a collision/contact on the robot structure (i.e., on one of the links). Moreover, $\boldsymbol{F}_K \in \mathbb{R}^m$ is the generalized force at the contact and $\boldsymbol{J}_K^T(\boldsymbol{q})$ is the transpose of the Jacobian of the contact point. Both these quantities are unknown.

To detect the presence of a joint torque $\boldsymbol{\tau}_K$ due to a collision, we use a residual vector based on the robot generalized momentum $\boldsymbol{p} = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$. With the robot starting at rest at time $t = 0$, the residual $\boldsymbol{r} \in \mathbb{R}^n$ is defined as

$$\boldsymbol{r} = \boldsymbol{K}_I \left[\boldsymbol{p} - \int_0^t \left(\boldsymbol{\tau} + \boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{r} \right) ds \right], \qquad \boldsymbol{r}(0) = \boldsymbol{0},$$

with $\boldsymbol{K}_I > 0$ and diagonal. Therefore, for its computation we need the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$, the factorization matrix $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of the Coriolis and centrifugal terms (obtained using Christoffel's coefficients), and the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ of the specific robot.

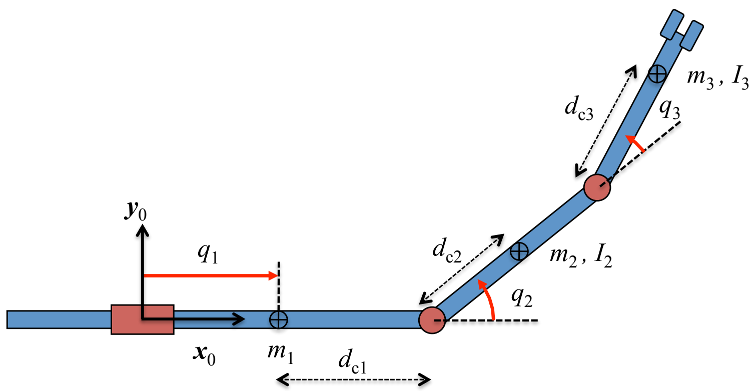


Figure 2: The definition of generalized coordinates and dynamic parameters for the planar PRR robot (the link length are l_2 and l_3)

We define the reference frame axes and the generalized coordinates $\boldsymbol{q} \in \mathbb{R}^3$ for the planar PRR robot as in Fig. 2 —note that these do not follow the DH convention. Since the robot is moving on a horizontal plane, $\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}$.

The kinetic energy of the robot

$$T = \sum_{i=1}^{3} T_i = \sum_{i=1}^{3} \frac{1}{2} \left[m_i \|\boldsymbol{v}_{c_i}\|^2 + \boldsymbol{\omega}_i^T \boldsymbol{I}_i \boldsymbol{\omega}_i \right] = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

is computed as follows[2]:

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2$$

$$\boldsymbol{p}_{c2} = \begin{pmatrix} q_1 + d_{c1} + d_{c2} c_2 \\ d_{c2} s_2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \begin{pmatrix} \dot{q}_1 - d_{c2} s_2 \dot{q}_2 \\ d_{c2} c_2 \dot{q}_2 \end{pmatrix}$$

$$\Rightarrow \quad T_2 = \frac{1}{2} m_2 \left(\dot{q}_1^2 + d_{c2}^2 \dot{q}_2^2 - 2 d_{c2} s_2 \dot{q}_1 \dot{q}_2 \right) + \frac{1}{2} I_2 \dot{q}_2^2$$

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_1 + d_{c1} + l_2 c_2 + d_{c3} c_{23} \\ l_2 s_2 + d_{c3} s_{23} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} \dot{q}_1 - l_2 s_2 \dot{q}_2 - d_{c3} s_{23} (\dot{q}_2 + \dot{q}_3) \\ l_2 c_2 \dot{q}_2 + d_{c3} c_{23} (\dot{q}_2 + \dot{q}_3) \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \quad T_3 \;&=\; \tfrac{1}{2} m_3 \left(\dot{q}_1^2 + l_2^2 \dot{q}_2^2 + d_{c3}^2 (\dot{q}_2 + \dot{q}_3)^2 - 2 l_2 s_2 \dot{q}_1 \dot{q}_2 - 2 d_{c3} s_{23} \dot{q}_1 (\dot{q}_2 + \dot{q}_3) + 2 l_2 d_{c3} c_{23} \dot{q}_2 (\dot{q}_2 + \dot{q}_3) \right) \\ &\quad + \tfrac{1}{2} I_3 (\dot{q}_2 + \dot{q}_3)^2 . \end{aligned}$$

As a result, the inertia matrix is

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1 + m_2 + m_3 & -(m_2 d_{c2} + m_3 l_2) s_2 - m_3 d_{c3} s_{23} & -m_3 d_{c3} s_{23} \\ symm & I_2 + m_2 d_{c2}^2 + I_3 + m_3 d_{c3}^2 + m_3 l_2^2 + 2 l_2 m_3 d_{c3} c_3 & I_3 + m_3 d_{c3}^2 + l_2 m_3 d_{c3} c_3 \\ symm & symm & I_3 + m_3 d_{c3}^2 \end{pmatrix}.$$

Using the following parametrization[3]

$$\begin{aligned} a_1 &= m_1 + m_2 + m_3 \\ a_2 &= m_2 d_{c2} + m_3 l_2 \\ a_3 &= m_3 d_{c3} \\ a_4 &= I_2 + m_2 d_{c2}^2 + I_3 + m_3 d_{c3}^2 + m_3 l_2^2 \\ a_5 &= I_3 + m_3 d_{c3}^2, \end{aligned}$$

the inertia matrix can be more compactly rewritten as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 & -a_2 s_2 - a_3 s_{23} & -a_3 s_{23} \\ -a_2 s_2 - a_3 s_{23} & a_4 + 2 l_2 a_3 c_3 & a_5 + l_2 a_3 c_3 \\ -a_3 s_{23} & a_5 + l_2 a_3 c_3 & a_5 \end{pmatrix} = \begin{pmatrix} \boldsymbol{m}_1(\boldsymbol{q}) & \boldsymbol{m}_2(\boldsymbol{q}) & \boldsymbol{m}_3(\boldsymbol{q}) \end{pmatrix}.$$

For the term $\boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$, we use the formula

$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_3(\boldsymbol{q}) \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{C}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{C}_1^T(\boldsymbol{q})\dot{\boldsymbol{q}} & \boldsymbol{C}_2^T(\boldsymbol{q})\dot{\boldsymbol{q}} & \boldsymbol{C}_3^T(\boldsymbol{q})\dot{\boldsymbol{q}} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix},$$

with matrices $\boldsymbol{C}_i(\boldsymbol{q})$ defined as

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left[\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right) + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \left(\frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i} \right) \right], \qquad \text{for } i = 1, 2, 3.$$

[2] Taking into account the planar nature of the problem, we work with two-dimensional vectors for linear quantities (in the plane $(\boldsymbol{x}_0, \boldsymbol{y}_0)$) and with scalars for angular quantities (components along $\boldsymbol{z}_0$).

[3] Here, the kinematic parameter l_2 is assumed to be known. This allows writing the product $m_3 l_2 d_{c3}$ as $a_3 l_2$, without the need of introducing a sixth dynamic coefficient.

Performing computations yields finally

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(a_2c_2 + a_3c_{23}) & -a_3c_{23} \\ 0 & -a_3c_{23} & -a_3c_{23} \end{pmatrix},$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -l_2a_3s_3 \\ 0 & -l_2a_3s_3 & -l_2a_3s_3 \end{pmatrix},$$

$$\boldsymbol{C}_3(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -l_2a_3s_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The presence of prismatic joints in the robot does not change the overall residual approach to collision detection. Indeed, there are slight changes in the detectability of contact forces. For instance, a force $\boldsymbol{F}_K$ applied to the first link of the PRR robot along a direction that is *normal* to the first joint axis will *not* be detected. For a robot with a first revolute joint, a force applied to the first link along a line *crossing* the first joint axis would *not* be detected. In both cases, note that such forces produce no motion anyway. When a prismatic joint is placed along the structure, as in a planar RPR robot, the same reasoning applies, although detectability becomes in any event easier, thanks to the role of the multiple joints preceding the link being hit.

In general, contact forces $\boldsymbol{F}_K \in \mathcal{N}\left\{\boldsymbol{J}_K^T(\boldsymbol{q})\right\}$ will never be recorded by the residual $\boldsymbol{r}$. In a dual fashion, the residual method will fully detect contact forces that are completely orthogonal to the null space of the contact Jacobian, no matter which is the prismatic or revolute nature of the robot joints.

Exercise 2

We use the same coordinates as in Exercise 1. From the direct kinematics

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_1 + l_2c_2 + l_3c_{23} \\ l_2s_2 + l_3s_{23} \end{pmatrix},$$

the end-effector Jacobian of interest is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 1 & -(l_2s_2 + l_3s_{23}) & -l_3s_{23} \\ 0 & l_2c_2 + l_3c_{23} & l_3c_{23} \end{pmatrix}.$$

Note that the Jacobian is singular (loses rank) when $c_{23} = c_2 = 0$, namely when the second and third links lie both (folded or stretched) along a line orthogonal to the first (prismatic) joint axis. In singular configurations, there are Cartesian directions along which arbitrary forces can be applied without the need of motor torques to keep the structure in static balance. While such configurations are indeed good candidates for the solution of the problem at hand, we should also take into account that even in singular configurations contact forces could be applied along arbitrary planar directions, and thus too large forces in norm may not be sustainable in view of the given actuator limits.

A generic contact force can be parametrized as

$$\boldsymbol{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \|\boldsymbol{F}\| \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \qquad \alpha \in (-\pi.\pi], \tag{1}$$

where α is the angle of the force w.r.t. the $\boldsymbol{x}_0$ axis. For a given robot configuration $\boldsymbol{q}$ and a given value of $\|\boldsymbol{F}\|$, we should check if the actuator limits are still satisfied or not in correspondence to the 'worst case' angle α.

Setting then $l_2 = l_3 = 1$ [m], and using the given actuator bounds, the following inequalities need to be satisfied

$$\begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix} \le \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} = \begin{pmatrix} F_x \\ -(s_2 + s_{23})\, F_x + (c_2 + c_{23})\, F_y \\ -s_{23}F_x + c_{23}F_y \end{pmatrix} \le \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}.$$

Or, using also eq. (1),

$$\begin{pmatrix} |F_x| \\ \|F\| \cdot |s_{2-\alpha} + s_{23-\alpha}| \\ \|F\| \cdot |s_{23-\alpha}| \end{pmatrix} \le \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}. \tag{2}$$

This form is more convenient for analysis. From the first inequality in (2), the norm of $\boldsymbol{F}$ should certainly not exceed 5 N. From the last inequality, no matter which value takes the sum $q_2 + q_3$, we can always select a direction (i.e., a suitable value of α) such that $|s_{23-\alpha}| = 1$, yielding then the worst case. Thus, the norm of $\boldsymbol{F}$ cannot exceed 2 N (which dominates the former condition). Finally, consider the second inequality. If we had a limit on the second torque larger than or equal to 4 Nm, since $|s_{2-\alpha} + s_{23-\alpha}| \le 2$ holds for any combination of q_2, q_3 and α, then this bound would anyway be dominated by the stricter condition on the third torque. Instead, with a maximum value of 1 Nm, we still need to carry the analysis a bit further.

It is easy to see that $c_3 = -1$ (namely, $q_3 = \pm\pi$) implies $s_{2-\alpha} + s_{23-\alpha} \equiv 0$, and so any force can be sustained by the actuator at joint 2 when folding the third link over the second, and independently of the value of q_2. This should not come unexpected, as the application line of any force applied to the end-effector would then always cross the axis of joint 2, producing thus no torque. Note that things would be quite different in the case $l_2 \ne l_3$.

Summarizing, the maximum sustainable contact force applied at the robot end-effector has norm $\|\boldsymbol{F}\| = 2$ N. A non-singular configuration where such a force can be sustained (with at least one torque in saturation) is given by $\boldsymbol{q}^* = (\text{any}, 3\pi/4, -\pi)$ —q_1 is irrelevant. When applying a force of 2 N in the direction normal to link 3 ($\alpha = \pi/4$), we would have as balancing joint torque

$$\boldsymbol{\tau}^* = -\boldsymbol{J}^T(\boldsymbol{q}^*)\boldsymbol{F} = -\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 0 \\ -2 \end{pmatrix}.$$

Note that the Jacobian matrix at $\boldsymbol{q}^*$ has full rank, so this is *not* a singular configuration.

* * * * *

Robotics 2

Remote Exam – June 5, 2020

Exercise #1

A 3R planar robot is subject to hard joint velocity limits $|\dot{q}_i| \leq V_i$, for $i = 1, 2, 3$, with $V_1 = 1$, $V_2 = 1.5$, and $V_2 = 2$ [rad/s]. At the current configuration, its task Jacobian is given by

$$\boldsymbol{J} = \begin{pmatrix} -1 & -1 & -0.5 \\ -0.366 & -0.866 & -0.866 \end{pmatrix}$$

and the task requires a Cartesian velocity $\boldsymbol{v} = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$ [m/s]. Apply the SNS algorithm to find a feasible solution $\dot{\boldsymbol{q}} \in \mathbb{R}^3$ with the least possible norm, including task scaling if needed.

Exercise #2

Determine all the conditions on the constant parameters a, b, c and d, under which the following (linear) equations can be considered the dynamic model of an actual 2-dof robot:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{u}, \qquad \text{with} \quad \boldsymbol{q} \in \mathbb{R}^2, \ \boldsymbol{u} \in \mathbb{R}^2,$$

where

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad \boldsymbol{g} = \begin{pmatrix} 0 \\ d \end{pmatrix}.$$

Any guess about which type of robot this could be? Suppose now that only one generalized coordinate is actuated by a command $u_a \in \mathbb{R}$, while the other is passive. Which component of $\boldsymbol{u}$ should be actuated in order to guarantee the existence of an equilibrium? If we choose a value $u_a > 0$, what would be the instantaneous acceleration of the other (passive) coordinate?

Exercise #3

Consider the RP robot in Fig. 1, moving in a vertical plane. The prismatic joint has a limited range $d \leq q_2 \leq L$. Derive the gravity term $\boldsymbol{g}(\boldsymbol{q})$ in the dynamic model and find the expression (in symbolic form) of a constant $\alpha > 0$ that bounds $\|\partial \boldsymbol{g}(\boldsymbol{q})/\partial \boldsymbol{q}\|$ for all $\boldsymbol{q}$ within the robot workspace.

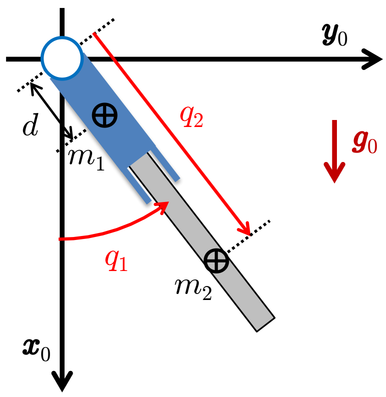


Figure 1: A RP robot, with associated coordinates $\boldsymbol{q}$ and relevant dynamic data.

Exercise #4

The Cartesian robot in Fig. 2 has the two links respectively of mass m_1 and m_2, and carries a payload m_p. It moves under gravity and has relevant viscous friction at both joints. In the absence of a priori information on the dynamic parameters, design an adaptive control law yielding global asymptotic stabilization of the tracking error for a desired trajectory $\boldsymbol{q}_d(t)$.

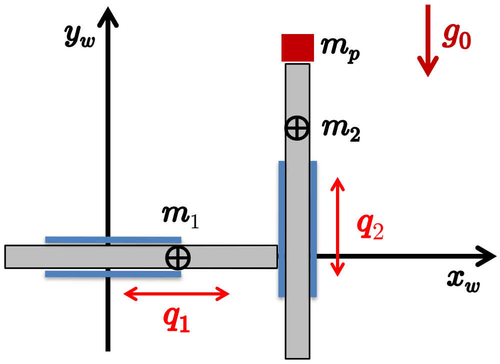


Figure 2: A Cartesian robot moving under gravity.

Exercise #5

A robot should close a door by firmly grasping its handle, pushing the door to the final position, and then turning the handle to lock the door. This interaction task is sketched in Fig. 3. Write down the natural and artificial constraints, defining a suitable task frame. Propose a time behavior for the hybrid references so as to cover the pushing phase and the final locking of the door. As usual in planning of hybrid force-velocity tasks, neglect any friction or environment compliance.

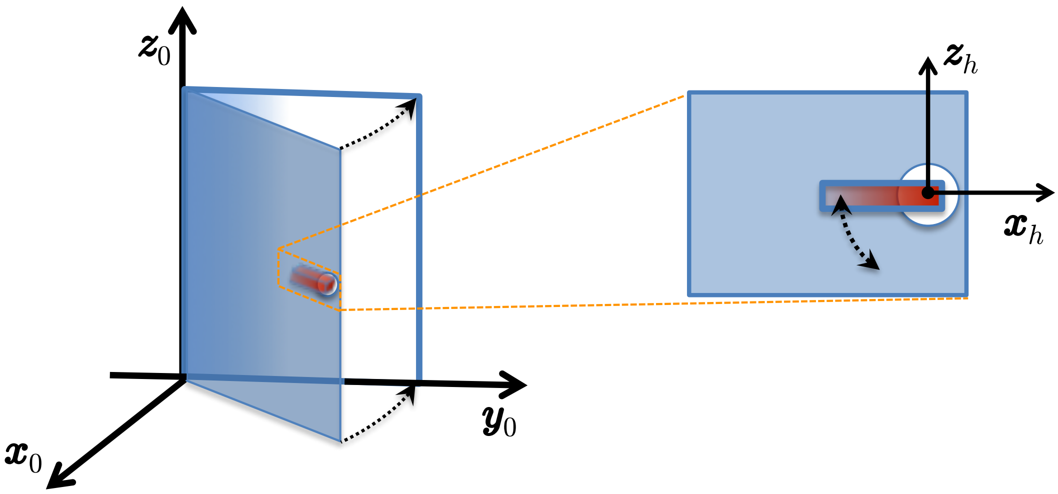


Figure 3: The task of closing a door [left], with handle motion for the final door locking [right].

Exercise #6

With reference to Fig. 4, a planar 2R robot with links of unitary length (in [m]) is in the configuration $\boldsymbol{q} = \begin{pmatrix} \pi/2 & -\pi/2 \end{pmatrix}^T$ [rad] and with a velocity $\dot{\boldsymbol{q}} = \begin{pmatrix} 0 & \pi/4 \end{pmatrix}^T$ [rad/s] when a collision occurs. Detail and comment the detection and isolation properties of the energy-based and momentum-based methods for the three cases of collision with the shown impact forces: *i)* $\boldsymbol{F}_{c1} = \begin{pmatrix} -1 & 0 \end{pmatrix}^T$ [N] at the end-effector; *ii)* $\boldsymbol{F}_{c2} = \begin{pmatrix} 0 & -1 \end{pmatrix}^T$ [N] at the elbow; *iii)* $\boldsymbol{F}_{c3} = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ [N] at the midpoint of the second link.

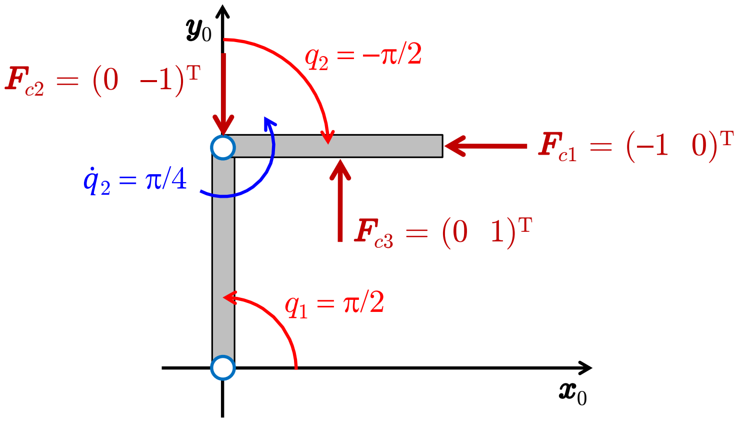


Figure 4: Three cases of collision for a 2R planar robot.

[180 minutes (3 hours); open books]

Solution

June 5, 2020

Exercise #1

The SNS algorithm at the velocity level starts with checking whether the minimum norm solution is feasible with respect to the joint velocity bounds. Using the pseudoinverse of the Jacobian $\boldsymbol{J}$ and the Cartesian velocity $\boldsymbol{v}$, we compute

$$\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{q}}_{PS} = \boldsymbol{J}^{\#}\boldsymbol{v} = \begin{pmatrix} -1.1333 & 0.9309 \\ -0.2124 & -0.3136 \\ 0.6914 & -1.2345 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1.3358 \\ -0.7383 \\ 0.1482 \end{pmatrix} \Leftarrow \begin{matrix} < -V_1 = -1\,!! \\ \in [-V_2, V_2] = [-1.5, 1.5] \\ \in [-V_3, V_3] = [-2, 2] \end{matrix}$$

Therefore, we saturate the first joint velocity at its overdriven limit, $\dot{q}_1 = -V_1 = -1$ [rad/s], and recompute the task velocity $\boldsymbol{v}_1$ that needs to be executed with the remaining two joints,

$$\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{J}_1\dot{q}_{0,1} = \boldsymbol{v} - \boldsymbol{J}_1(-V_1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -0.366 \end{pmatrix}(-1) = \begin{pmatrix} 1 \\ 0.634 \end{pmatrix},$$

where by $\boldsymbol{J}_i$, $i = 1, 2, 3$, we denote the columns of the Jacobian $\boldsymbol{J}$. We rewrite then the reduced problem as

$$\boldsymbol{J}_{(-1)} := \begin{pmatrix} \boldsymbol{J}_2 & \boldsymbol{J}_3 \end{pmatrix} = \begin{pmatrix} -1 & -0.5 \\ -0.8660 & -0.8660 \end{pmatrix}, \quad \dot{\boldsymbol{q}}_{(-1)} := \begin{pmatrix} \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{J}_{(-1)}\dot{\boldsymbol{q}}_{(-1)} = \boldsymbol{v}_1,$$

At this stage, the reduced solution is uniquely determined as

$$\dot{\boldsymbol{q}}_1 = \begin{pmatrix} \dot{q}_{1,1} \\ \dot{q}_{1,2} \end{pmatrix} = \boldsymbol{J}_{(-1)}^{-1}\boldsymbol{v}_1 = \begin{pmatrix} -2 & 1.1547 \\ 2 & -2.3094 \end{pmatrix} \begin{pmatrix} 1 \\ 0.634 \end{pmatrix} = \begin{pmatrix} -1.2679 \\ 0.5359 \end{pmatrix} \Leftarrow \text{ both feasible!}$$

Recombining the joint velocity vector, we have the (minimum norm) feasible solution

$$\dot{\boldsymbol{q}}^* = \begin{pmatrix} \dot{q}_{0,1} \\ \dot{q}_{1,1} \\ \dot{q}_{1,2} \end{pmatrix} = \begin{pmatrix} -1 \\ -1.2679 \\ 0.5359 \end{pmatrix} \text{ [rad/s]},$$

and we can check that is satisfies indeed $\boldsymbol{J}\dot{\boldsymbol{q}}^* = \boldsymbol{v}$. Thus, task scaling is not needed here.

Exercise #2

In order to be the dynamic model of an actual 2-dof robot, the only condition is on the parameters a and c, which need to guarantee the positive definiteness of the inertia matrix $\boldsymbol{M}$. Thus

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} > 0 \qquad \Longleftrightarrow \qquad a > 0, \quad \det \boldsymbol{M} = ac - b^2 > 0 \quad \Rightarrow \quad c > 0.$$

Being the inertia constant, there are no Coriolis and centrifugal effects. Also, friction is neglected. The 2-dof robot could be a planar arm with two prismatic joints (2P), whose axes are twisted by an angle $\alpha_1 \neq 0$. Moreover, in order to include a constant gravity term only on the second joint, the plane of motion is vertical and the first prismatic joint horizontal. We would have then for the kinetic and potential energy

$$T = T_1 + T_2 = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1^2 + \dot{q}_2^2 + 2\cos\alpha_1\,\dot{q}_1\dot{q}_2), \qquad U = U_2 = m_2 g_0\, q_2 \sin\alpha_1,$$

where $m_1 > 0$ and $m_2 > 0$ are the masses of the two links, so that the inertia matrix and the gravity vector for this 2P robot are

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} m_1 + m_2 & m_2 \cos\alpha_1 \\ m_2 \cos\alpha_1 & m_2 \end{pmatrix}, \qquad \boldsymbol{g} = \begin{pmatrix} 0 \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ m_2 g_0 \sin\alpha_1 \end{pmatrix},$$

with $\det \boldsymbol{M} = m_1 m_2 + m_2^2(1 - \cos\alpha_1) > 0$. Note that the coefficient $b = m_2 \cos\alpha_1$ could be positive or negative (depending on the twist angle $|\alpha_1| < \pi/2$ or, respectively, $> \pi/2$). Same for $d = m_2 g_0 \sin\alpha_1$ (depending on the sign of α_1).

Looking now at the equilibrium condition (i.e., $\ddot{\boldsymbol{q}} = \boldsymbol{0}$) in the underactuated case, it is evident that the second joint has to be the actuated one,

$$\boldsymbol{g} = \boldsymbol{u} \qquad \Rightarrow \qquad \begin{pmatrix} 0 \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ u_a \end{pmatrix}.$$

The equilibrium force at the second joint is $u_a = d$ $(= m_2 g_0 \sin\alpha_1)$, the same for all equilibrium configurations $\boldsymbol{q}_e \in \mathbb{R}^2$. When choosing a value $u_a > 0$, the instantaneous acceleration $\ddot{q}_1$ of the first (passive) joint is obtained from

$$\ddot{\boldsymbol{q}} = \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \boldsymbol{M}^{-1} (\boldsymbol{u} - \boldsymbol{g}) = \frac{1}{\det \boldsymbol{M}} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} 0 \\ u_a - d \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} -b \\ a \end{pmatrix} (u_a - d),$$

and thus $\ddot{q}_1 = b(d - u_a)/(ac - b^2)$. Being the determinant of $\boldsymbol{M}$ positive, the sign of $\ddot{q}_1$ will depend on the sign of the product $b(d - u_a)$. For instance, when $b > 0$ and for a large $u_a > |d|$, the acceleration of the first (passive) joint will be negative. On the other hand, the acceleration of the second (actuated) joint will always be positive, as soon as the control force overcomes the gravity term ($u_a > |d|$).

Exercise #3

From Fig. 1, we have for the potential energy due to gravity

$$U(\boldsymbol{q}) = U_1(q_1) + U_2(\boldsymbol{q}) = -m_1 g_0 d \cos q_1 - m_2 g_0 q_2 \cos q_1.$$

Therefore, using the compact notation for trigonometric quantities, we have

$$\boldsymbol{g}(\boldsymbol{q}) = \frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} (m_1 d + m_2 q_2) g_0 s_1 \\ -m_2 g_0 c_1 \end{pmatrix},$$

and then the symmetric matrix (representing the Hessian of $U(\boldsymbol{q})$)

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial \boldsymbol{g}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} (m_1 d + m_2 q_2) g_0 c_1 & m_2 g_0 s_1 \\ m_2 g_0 s_1 & 0 \end{pmatrix}.$$

This matrix is not definite in sign. Therefore, in order to evaluate its norm, we have to use the general form

$$\|\boldsymbol{A}(\boldsymbol{q})\| = \sqrt{\lambda_{\max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right)},$$

and compute the real eigenvalues of the positive semi-definite, symmetric matrix[1]

$$\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q}) = \begin{pmatrix} \left((m_1 d + m_2 q_2) g_0 c_1\right)^2 + \left(m_2 g_0 s_1\right)^2 & m_2 (m_1 d + m_2 q_2) g_0^2 s_1 c_1 \\ m_2 (m_1 d + m_2 q_2) g_0^2 s_1 c_1 & \left(m_2 g_0 s_1\right)^2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}.$$

[1]We have $a_1 > 0$, $a_3 \geq 0$, and thus $a_1 + a_3 > 0$. Also, $a_1 a_3 - a_2^2 \geq 0$.

From

$$\det\left(\lambda \boldsymbol{I} - \boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right) = \det\begin{pmatrix} \lambda - a_1 & -a_2 \\ -a_2 & \lambda - a_3 \end{pmatrix} = \lambda^2 - (a_1 + a_3)\lambda + (a_1 a_3 - a_2^2),$$

we obtain the maximum (real and positive) eigenvalue as

$$\lambda_{\max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right) = \frac{a_1 + a_3}{2} + \frac{\sqrt{(a_1 + a_3)^2 - 4(a_1 a_3 - a_2^2)}}{2} = \frac{a_1 + a_3 + \sqrt{(a_1 - a_3)^2 + 4a_2^2}}{2} > 0.$$

Substituting the expressions of the a_i's, we get

$$\begin{aligned}
&\lambda_{\max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right) \\
&= \frac{1}{2}\left(\left((m_1 d + m_2 q_2) g_0 c_1\right)^2 + 2\left(m_2 g_0 s_1\right)^2 + \sqrt{\left((m_1 d + m_2 q_2) g_0 c_1\right)^4 + 4\left(m_2 (m_1 d + m_2 q_2) g_0^2 s_1 c_1\right)^2}\right) \\
&= \frac{1}{2}\left(\left((m_1 d + m_2 q_2) g_0 c_1\right)^2 + 2\left(m_2 g_0 s_1\right)^2 + (m_1 d + m_2 q_2)\, g_0^2 c_1 \sqrt{(m_1 d + m_2 q_2)^2 c_1^2 + 4\left(m_2 s_1\right)^2}\right).
\end{aligned}$$

This expression can be upper bounded in different ways, using also the upper limit for the prismatic joint $q_2 \le L$. Replacing for instance $c_1 \to 1$, $s_1 \to 1$, and $q_2 = L$, we finally obtain the upper bound

$$\|\boldsymbol{A}(\boldsymbol{q})\| = \sqrt{\lambda_{\max}\left(\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{A}(\boldsymbol{q})\right)} \le \alpha$$

with[2]

$$\alpha = \frac{g_0}{\sqrt{2}}\sqrt{\left(m_1 d + m_2 L\right)^2 + 2\, m_2^2 + (m_1 d + m_2 L)\sqrt{(m_1 d + m_2 L)^2 + 4\, m_2^2}} > 0.$$

This constant is used, e.g., in the proof of the global asymptotic stability of a PD control law with gravity compensation $\boldsymbol{g}(\boldsymbol{q}_d)$, in which the minimum (positive) value $\boldsymbol{K}_{P,m}$ of the diagonal matrix of proportional gains $\boldsymbol{K}_P$ should satisfy $\boldsymbol{K}_{P,m} > \alpha$.

Exercise #4

The dynamic model of the Cartesian robot in Fig. 2 is very simple (in fact, this robot has a linear and decoupled dynamics). We have

$$T = T_1 + T_2 + T_p = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2}(m_2 + m_p)\left(\dot{q}_1^2 + \dot{q}_2^2\right), \qquad U = U_2 + U_p = (m_2 + m_p) g_0 q_2,$$

and thus, from the Euler-Lagrange equations, considering also the presence of viscous friction

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} + \boldsymbol{F}_v \dot{\boldsymbol{q}} = \boldsymbol{u}, \tag{1}$$

with

$$\boldsymbol{M} = \begin{pmatrix} m_1 + m_2 + m_p & 0 \\ 0 & m_2 + m_p \end{pmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} 0 \\ (m_2 + m_p) g_0 \end{pmatrix}, \quad \boldsymbol{F}_v = \begin{pmatrix} f_{v1} & 0 \\ 0 & f_{v2} \end{pmatrix}.$$

[2] One may notice that there is a unit inconsistency among the terms in the expression of α. In fact, we are taking the norm of a matrix $\boldsymbol{A}(\boldsymbol{q})$ that has elements expressed in different units. This happens because the robot has joints of different nature (revolute and prismatic): the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ has the first component expressed in [Nm] (a torque) and the second in [N] (a force).

The dynamic model (1) can be linearly re-parametrized as

$$\boldsymbol{Y}(\dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{u} \tag{2}$$

with

$$\boldsymbol{Y}(\dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \begin{pmatrix} \ddot{q}_1 & 0 & \dot{q}_1 & 0 \\ 0 & \ddot{q}_2 + g_0 & 0 & \dot{q}_2 \end{pmatrix}, \qquad \boldsymbol{a} = \begin{pmatrix} m_1 + m_2 + m_p \\ m_2 + m_p \\ f_{v1} \\ f_{v2} \end{pmatrix}.$$

The adaptive control law for tracking a desired trajectory $\boldsymbol{q}_d(t)$ (at least twice differentiable w.r.t. time) is

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{Y}(\dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, \hat{\boldsymbol{a}} + \boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_D \dot{\boldsymbol{e}}, \qquad \boldsymbol{K}_P, \boldsymbol{K}_D > 0, \\ \dot{\hat{\boldsymbol{a}}} &= \boldsymbol{\Gamma} \boldsymbol{Y}^T(\dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, (\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}), \quad \boldsymbol{\Gamma} > 0, \end{aligned} \tag{3}$$

with $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$, $\dot{\boldsymbol{e}} = \dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}$, $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda} \boldsymbol{e}$, and $\ddot{\boldsymbol{q}}_r = \ddot{\boldsymbol{q}}_d + \boldsymbol{\Lambda} \dot{\boldsymbol{e}}$ ($\boldsymbol{\Lambda} = \boldsymbol{K}_P \boldsymbol{K}_D^{-1} > 0$), and where

$$\boldsymbol{Y}(\dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r) = \begin{pmatrix} \ddot{q}_{r1} & 0 & \dot{q}_{r1} & 0 \\ 0 & \ddot{q}_{r2} + g_0 & 0 & \dot{q}_{r2} \end{pmatrix}.$$

The gain matrices $\boldsymbol{K}_P$, $\boldsymbol{K}_D$, and $\boldsymbol{\Gamma}$ are taken diagonal. We additionally remark the following.

- The problem is fully decoupled into parallel subproblems for each of the two prismatic joints. Consider for instance the first joint. From eqs. (1), (2), and (3), we have for the closed-loop system
$$\begin{aligned} (m_1 + m_2 + m_p)\ddot{q}_1 + f_{v1}\dot{q}_1 &= u_1, \\ u_1 &= \hat{a}_1 \ddot{q}_{r1} + \hat{a}_3 \dot{q}_{r1} + k_{p1} e_1 + k_{d1} \dot{e}_1, \\ \dot{\hat{a}}_1 &= \gamma_1\, \ddot{q}_{r1} (\dot{q}_{r1} - \dot{q}_1), \\ \dot{\hat{a}}_3 &= \gamma_3\, \dot{q}_{r1} (\dot{q}_{r1} - \dot{q}_1), \end{aligned}$$
with $e_1 = q_{d1} - q_1$, $\dot{e}_1 = \dot{q}_{d1} - \dot{q}_1$, $\dot{q}_{r1} = \dot{q}_{d1} + \frac{k_{P1}}{k_{D1}}\, e_1$, and $\ddot{q}_{r1} = \ddot{q}_{d1} + \frac{k_{P1}}{k_{D1}}\, \dot{e}_1$.
- Despite the linear dynamics of the considered robot, the closed-loop system is indeed still nonlinear because of the interplay between the robot state $\boldsymbol{x} = (\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathbb{R}^4$ and the state $\hat{\boldsymbol{a}} \in \mathbb{R}^4$ of the adaptive controller.

Exercise #5

With reference to Fig. 5, where a possible task frame is defined, the $\boldsymbol{x}_t$ and $\boldsymbol{z}_t$ axes of the task frame have been chosen to coincide with the homologous ones at the door handle. The natural constraints are the following:

$$v_x = 0, \qquad v_z = 0, \qquad \omega_x = 0, \qquad \omega_z = 0, \qquad f_y = 0, \qquad \mu_y = 0.$$

In these constraints, we neglect any friction effect and mass/inertia or compliance of the environment (which is thus assumed to be purely geometric). The complementary artificial constraints are chosen then as:

$$f_x = 0, \qquad f_z = 0, \qquad \mu_x = 0, \qquad \mu_z = 0, \qquad v_y = v_{d,y}(t), \qquad \omega_y = \omega_{d,y}(t).$$

The first four (zero) values for forces and moments that are assigned as references to the hybrid task controller reflect the desire to limit the mechanical stress on the door handle grasped by the robot end-effector. The last two time-varying references are used instead to specify the way feasible motions are handled during the task of closing a door. While moving the door, we set $\omega_{d,y} = 0$ (no motion around the rotation axis $\boldsymbol{y}_h = \boldsymbol{y}_t$ of the door handle). On the other hand, the time law $v_{d,y}(t)$ will define the way the door closing should be performed, e.g., with a trapezoidal profile for a rest-to-rest motion from the initial position (door open) to the desired approach position (door nearly closed), using a fast or slow cruise speed. When the approach position is reached, the linear motion is ended ($v_{d,y} = 0$) and $\omega_{d,y}(t)$ is used to turn the handle, preparing it for the final phase of door locking. Note finally that when the door touches the door frame on the wall, the contact situation changes and, accordingly, also the definition of natural and artificial constraints.

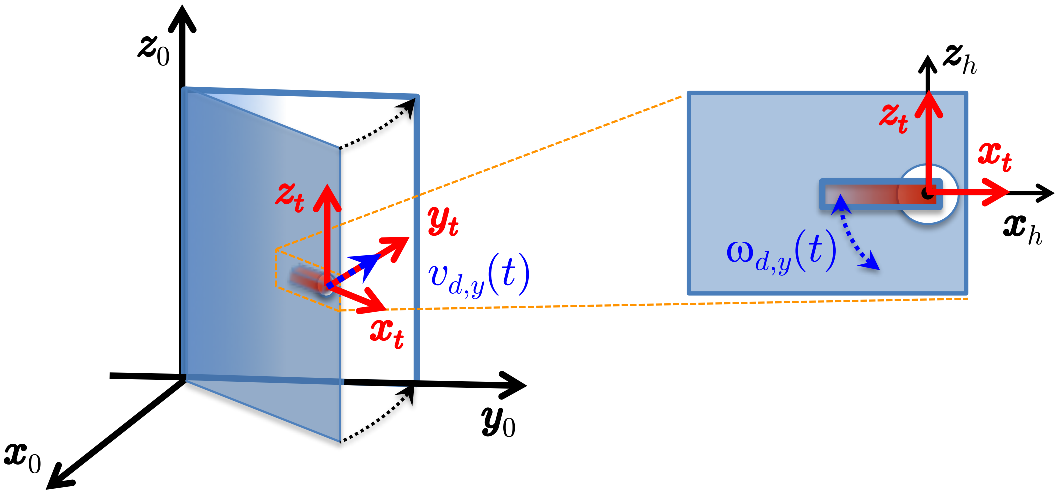


Figure 5: Two views of the task frame assignment for closing a door, with associated time-varying specification of two artificial constraints.

Exercise #6

The energy-based method fails to detect a collision when $\dot{\boldsymbol{q}} = \boldsymbol{0}$ or, more in general, when the colliding force $\boldsymbol{F}_c$ is orthogonal to the velocity $\boldsymbol{v}_c$ of the contact point. In fact, in this case we have $\boldsymbol{v}_c^T \boldsymbol{F}_c = \left(\boldsymbol{J}_c(\boldsymbol{q})\dot{\boldsymbol{q}}\right)^T \boldsymbol{F}_c = \dot{\boldsymbol{q}}^T \left(\boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c\right) = \dot{\boldsymbol{q}}^T \boldsymbol{\tau}_c = 0$, thus not exciting the scalar residual σ. In the given configuration $\boldsymbol{q} = \begin{pmatrix} \pi/2 & -\pi/2 \end{pmatrix}^T$ of the planar 2R robot, we verify this condition for the three considered cases:

$$\boldsymbol{v}_{c1} = \boldsymbol{J}_{c1}\dot{\boldsymbol{q}} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \pi/4 \end{pmatrix} = \begin{pmatrix} 0 \\ \pi/4 \end{pmatrix} \text{ [m/s]} \quad \Rightarrow \quad \boldsymbol{v}_{c1}^T \boldsymbol{F}_{c1} = \begin{pmatrix} 0 & \pi/4 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0;$$

$$\boldsymbol{v}_{c2} = \boldsymbol{J}_{c2}\dot{\boldsymbol{q}} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \pi/4 \end{pmatrix} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{v}_{c2}^T \boldsymbol{F}_{c2} = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0;$$

$$\boldsymbol{v}_{c3} = \boldsymbol{J}_{c3}\dot{\boldsymbol{q}} = \begin{pmatrix} -1 & 0 \\ 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0 \\ \pi/4 \end{pmatrix} = \begin{pmatrix} 0 \\ \pi/8 \end{pmatrix} \text{ [m/s]} \Rightarrow \boldsymbol{v}_{c3}^T \boldsymbol{F}_{c3} = \begin{pmatrix} 0 & \pi/8 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \pi/8 \text{ [Nm]}.$$

Therefore, only in the third case we are able to detect the occurrence of a collision with the energy-based method.

As for the momentum-based method, a collision is detected (and possibly isolated) provided $\boldsymbol{F}_c$ is not in the null space of the transpose of the contact Jacobian $\boldsymbol{J}_c$. In fact, if $\boldsymbol{F}_c \in \mathcal{N}\left\{\boldsymbol{J}_c^T(\boldsymbol{q})\right\}$ the contact force is balanced by the reaction of the rigid robot structure, yielding $\boldsymbol{\tau}_c = \boldsymbol{J}_c^T(\boldsymbol{q})\boldsymbol{F}_c = \mathbf{0}$ and thus not exciting the residual vector $\boldsymbol{r}$. We verify next if the detection condition holds for the three considered cases, drawing also conclusions on the isolation property. For the tip contact on link 2:

$$\mathcal{N}\left\{\boldsymbol{J}_{c1}^T\right\} = \mathcal{N}\left\{\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\right\} = \emptyset \;\Rightarrow\; \boldsymbol{F}_{c1} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \notin \mathcal{N}\left\{\boldsymbol{J}_{c1}^T\right\},\; \boldsymbol{\tau}_{c1} = \boldsymbol{J}_{c1}^T\boldsymbol{F}_{c1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ [Nm]}.$$

Therefore, this collision will be detected with the momentum-based method (contrary to what happens with the energy-based method). However, the second component r_2 of the residual vector will be unaffected (being $\tau_{c1,2} = 0$), leading to the wrong conclusion that the contact occurred on link 1. Indeed, this is a singular situation for the isolation property (the contact force vector $\boldsymbol{F}_{c1}$ passes through the axis of joint 2). For the contact at the robot elbow:

$$\mathcal{N}\left\{\boldsymbol{J}_{c2}^T\right\} = \mathcal{N}\left\{\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\} = \alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad\Rightarrow\quad \boldsymbol{F}_{c2} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \mathcal{N}\left\{\boldsymbol{J}_{c2}^T\right\},\; \boldsymbol{\tau}_{c2} = \boldsymbol{J}_{c2}^T\boldsymbol{F}_{c2} = \mathbf{0}.$$

Therefore, also the momentum-based method is not able to detect this collision. In fact, the contact force vector $\boldsymbol{F}_{c2}$ passes through both joint axes and is balanced entirely by the internal reaction force of the robot structure. Finally, for the contact at the midpoint of link 2:

$$\mathcal{N}\left\{\boldsymbol{J}_{c3}^T\right\} = \mathcal{N}\left\{\begin{pmatrix} -1 & 0.5 \\ 0 & 0.5 \end{pmatrix}\right\} = \emptyset \;\Rightarrow\; \boldsymbol{F}_{c3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin \mathcal{N}\left\{\boldsymbol{J}_{c3}^T\right\},\; \boldsymbol{\tau}_{c3} = \boldsymbol{J}_{c3}^T\boldsymbol{F}_{c3} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \text{ [Nm]}.$$

Therefore, the momentum-based method detects the collision and also correctly isolates the contact as occurring on the second link.

* * * * *

Robotics 2

Remote Exam – July 15, 2020

Exercise #1

Consider the 4-dof robot in Fig. 1, made by a 3R planar arm mounted on a rail. The robot has the last three links of equal length ℓ. The generalized coordinates $\boldsymbol{q} \in \mathbb{R}^4$ to be used are also shown. Determine the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of the dynamic model of this robot (if needed, define symbolically any missing parameters).

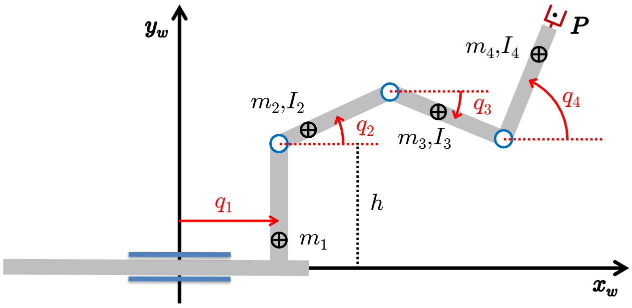


Figure 1: A 4-dof planar robot with generalized coordinates $\boldsymbol{q}$ and relevant parameters.

Exercise #2

For the same robot in Fig. 1, assume $\ell = h = 1$ [m] and consider the following tasks, to be executed using a kinematic control scheme with joint velocity commands $\dot{\boldsymbol{q}} \in \mathbb{R}^4$.

Task 1. Trace with the end-effector counterclockwise a circle of radius $R = 3$ [m], centered at $\boldsymbol{C} = (7, 0)$, starting from point $\boldsymbol{P}_0 = (10, 0)$ and with constant speed $v = 1$ [m/s].

Task 2. Keep the second link always horizontal ($q_2(t) = 0$).

Define the augmented Jacobian $\boldsymbol{J}_A(\boldsymbol{q})$ for both tasks 1 and 2. Choose a suitable initial robot configuration so as to stay at time $t = 0$ in $\boldsymbol{P}_0$ and compute there the minimum joint velocity norm solution that realizes both tasks simultaneously. Determine the first point $\boldsymbol{P}_s$ on the circular path where an algorithmic singularity of $\boldsymbol{J}_A(\boldsymbol{q})$ necessarily occurs. In that situation, compute the minimum joint velocity norm solution that realizes the first task only. Will the execution of the second task be relaxed or not?

Exercise #3

A 6-dof robot should hold firmly with its three-fingered gripper a cylindric payload, and move it along a desired path on a frictionless plane with one of its bases in full contact with the plane, as shown in Fig. 2. Define an associated task frame where the natural (geometric) constraints and the artificial (control) constraints of this hybrid task can be defined and realized in a decoupled way. Where reasonable, provide also values for the control references.

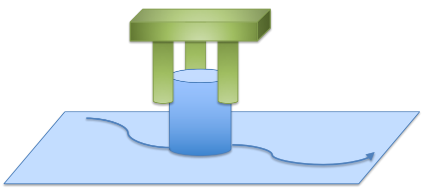


Figure 2: The hybrid task of moving along a path a cylinder in contact with a planar surface.

Exercise #4

During the accurate execution of a smooth joint trajectory $\boldsymbol{q}_d(t)$ lasting $T = 3$ [s] with the 2R planar robot shown in Fig. 3 moving under gravity, the maximum torques of the two joints exceed at some instants their bounds, as given by $|\tau_i| \le \tau_{max,i}$, $i = 1, 2$. We have in particular

$$\tau_1(t_1) = \max_{t\in[0,T]} \tau_1(t) = 140 > 100 = \tau_{max,1}, \qquad \tau_2(t_2) = \max_{t\in[0,T]} \tau_2(t) = 25 > 20 = \tau_{max,2} \qquad \text{[Nm]}.$$

The robot links have equal length $\ell = 0.5$ [m] and equal, uniformly distributed mass $m = 5$ [kg]. The robot configurations at the time instants $t = t_1$ and $t = t_2$ are

$$\boldsymbol{q}(t_1) = \begin{pmatrix} \pi/4 & 0 \end{pmatrix}^T, \qquad \boldsymbol{q}(t_2) = \begin{pmatrix} -\pi/4 & 3\pi/4 \end{pmatrix}^T \qquad \text{[rad]}.$$

In order to recover motion feasibility, a uniform trajectory scaling is used. What will be the minimum feasible motion time $T' = kT$ thus obtained? What are the values of the new joint torques $\tau_1(kt_1)$ and $\tau_2(kt_2)$?

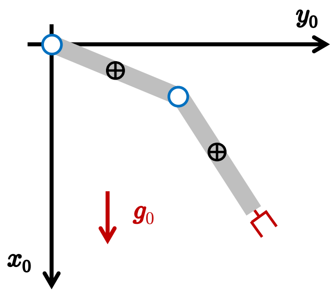


Figure 3: A 2R robot moving under gravity. Joint variables are defined by the D-H convention.

Exercise #5

Consider again the 2R robot in Fig. 3, with the same definition of joint variables and using the same kinematic and dynamic parameters. The robot is initially at rest at $t = 0$ in $\boldsymbol{q}(0) = \boldsymbol{q}_0$. Provide the explicit expressions of all terms in the following three feedback control laws, each achieving its own objective.

a. Global exponential stabilization of the state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$, with decoupled transient evolutions of the position errors $e_i(t) = q_{d,i} - q_i(t)$, $i = 1, 2$, of the form $e_1(t) = e_1(0)\,(1 + 5t)\,\exp(-5t)$ and $e_2(t) = e_2(0)\,(2\exp(-5t) - \exp(-10t))$.

b. Global asymptotic stabilization of the state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$, without knowledge of the robot inertia matrix.

c. Exponential stabilization of the end-effector position $\boldsymbol{p} = \boldsymbol{p}_d \in \mathbb{R}^2$ with zero velocity $\dot{\boldsymbol{p}} = \boldsymbol{0}$, up to kinematic singularities.

[180 minutes (3 hours); open books]

Solution

July 15, 2020

Exercise #1

Note first that we are using the *absolute* angles w.r.t. to the $\boldsymbol{x}_0$-axis for the orientation of the second to the fourth link (thus, not the Denavit-Hartenberg relative angles). Also, denote by $d_{ci} > 0$ the distance of the center of mass of link i from joint i, for $i = 2, 3, 4$. The individual contributions to the kinetic energy of this 4-dof planar robot are computed as follows.

$$T_1 = \tfrac{1}{2} m_1 \dot{q}_1^2$$

$$\boldsymbol{p}_{c2} = \begin{pmatrix} q_1 + d_{c2}\cos q_2 \\ h + d_{c1}\sin q_2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \begin{pmatrix} \dot{q}_1 - d_{c2}\sin q_2\,\dot{q}_2 \\ d_{c2}\cos q_2\,\dot{q}_2 \end{pmatrix}$$

$$T_2 = \tfrac{1}{2} I_2 \dot{q}_2^2 + \tfrac{1}{2} m_2 \left(\dot{q}_1^2 + d_{c2}^2\,\dot{q}_2^2 - 2 d_{c2}\sin q_2\,\dot{q}_1\dot{q}_2\right)$$

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_1 + \ell\cos q_2 + d_{c3}\cos q_3 \\ h + \ell\sin q_2 + d_{c3}\sin q_3 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} \dot{q}_1 - \ell\sin q_2\,\dot{q}_2 - d_{c3}\sin q_3\,\dot{q}_3 \\ \ell\cos q_2\,\dot{q}_2 + d_{c3}\cos q_3\,\dot{q}_3 \end{pmatrix}$$

$$T_3 = \tfrac{1}{2} I_3 \dot{q}_3^2 + \tfrac{1}{2} m_3 \left(\dot{q}_1^2 + \ell^2\,\dot{q}_2^2 + d_{c3}^2\,\dot{q}_3^2 - 2\ell\sin q_2\,\dot{q}_1\dot{q}_2 - 2 d_{c3}\sin q_3\,\dot{q}_1\dot{q}_3 + 2 d_{c3}\ell\cos(q_3 - q_2)\,\dot{q}_2\dot{q}_3\right)$$

$$\boldsymbol{p}_{c4} = \begin{pmatrix} q_1 + \ell(\cos q_2 + \cos q_3) + d_{c4}\cos q_4 \\ h + \ell(\sin q_2 + \sin q_3) + d_{c4}\sin q_4 \end{pmatrix} \Rightarrow \boldsymbol{v}_{c4} = \begin{pmatrix} \dot{q}_1 - \ell(\sin q_2\,\dot{q}_2 + \sin q_3\,\dot{q}_3) - d_{c4}\sin q_4\,\dot{q}_4 \\ \ell(\cos q_2\,\dot{q}_2 + \cos q_3\,\dot{q}_3) + d_{c4}\cos q_4\,\dot{q}_4 \end{pmatrix}$$

$$\begin{aligned} T_4 = {} & \tfrac{1}{2} I_4 \dot{q}_4^2 + \tfrac{1}{2} m_4 \left(\dot{q}_1^2 + \ell^2(\dot{q}_2^2 + \dot{q}_3^2 + 2\cos(q_3 - q_2)\,\dot{q}_2\dot{q}_3) + d_{c4}^2\,\dot{q}_4^2\right. \\ & \left. - 2\ell(\sin q_2\,\dot{q}_2 + \sin q_3\,\dot{q}_3)\dot{q}_1 - 2 d_{c4}\sin q_4\,\dot{q}_1\dot{q}_4 + 2 d_{c4}\ell(\cos(q_4 - q_2)\,\dot{q}_2 + \cos(q_4 - q_3)\,\dot{q}_3)\,\dot{q}_4\right). \end{aligned}$$

From

$$T = \sum_{i=1}^{4} T_i = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} \boldsymbol{M}_{ij}(\boldsymbol{q})\,\dot{q}_i\dot{q}_j,$$

the elements $\boldsymbol{M}_{ij} = \boldsymbol{M}_{ji}$ of the 4×4 symmetric inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of this robot are

$$\begin{aligned} \boldsymbol{M}_{11} &= m_1 + m_2 + m_3 + m_4 \\ \boldsymbol{M}_{12} &= -(m_2 d_{c2} + (m_3 + m_4)\ell)\sin q_2 \\ \boldsymbol{M}_{13} &= -(m_3 d_{c3} + m_4\ell)\sin q_3 \\ \boldsymbol{M}_{14} &= -m_4 d_{c4}\sin q_4 \\ \boldsymbol{M}_{22} &= I_2 + m_2 d_{c2}^2 + (m_3 + m_4)\ell^2 \\ \boldsymbol{M}_{23} &= (m_3 d_{c3} + m_4\ell)\ell\cos(q_3 - q_2) \\ \boldsymbol{M}_{24} &= m_4 d_{c4}\ell\cos(q_4 - q_2) \\ \boldsymbol{M}_{33} &= I_3 + m_3 d_{c3}^2 + m_4\ell^2 \\ \boldsymbol{M}_{34} &= m_4 d_{c4}\ell\cos(q_4 - q_3) \\ \boldsymbol{M}_{44} &= I_4 + m_4 d_{c4}^2. \end{aligned}$$

Exercise #2

The kinematics of the first task of dimension $m_1 = 2$ is given by

$$\boldsymbol{r}_1 = \boldsymbol{p} = \begin{pmatrix} q_1 + \ell(\cos q_2 + \cos q_3 + \cos q_4) \\ h + \ell(\sin q_2 + \sin q_3 + \sin q_4) \end{pmatrix} = \boldsymbol{f}_1(\boldsymbol{q}),$$

with Jacobian

$$\boldsymbol{J}_1(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_1(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 1 & -\ell \sin q_2 & -\ell \sin q_3 & -\ell \sin q_4 \\ 0 & \ell \cos q_2 & \ell \cos q_3 & \ell \cos q_4 \end{pmatrix},$$

while the kinematics of the second task of dimension $m_2 = 1$ is given just by

$$r_2 = q_2 = f_2(\boldsymbol{q}),$$

with Jacobian

$$\boldsymbol{J}_2 = \frac{\partial f_2(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix},$$

The augmented Jacobian is then

$$\boldsymbol{J}_A(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_1(\boldsymbol{q}) \\ \boldsymbol{J}_2 \end{pmatrix} = \begin{pmatrix} 1 & -\ell \sin q_2 & -\ell \sin q_3 & -\ell \sin q_4 \\ 0 & \ell \cos q_2 & \ell \cos q_3 & \ell \cos q_4 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that $\boldsymbol{J}_A(\boldsymbol{q})$ is singular, i.e., $\operatorname{rank}(\boldsymbol{J}_A(\boldsymbol{q})) < 3 = m_A(= m_1 + m_2)$, if and only if $\cos q_3 = \cos q_4 = 0$. This happens when the third and fourth link are aligned (or folded) along the $\boldsymbol{y}_w$ direction.

With reference to Fig. 4, the augmented task requires

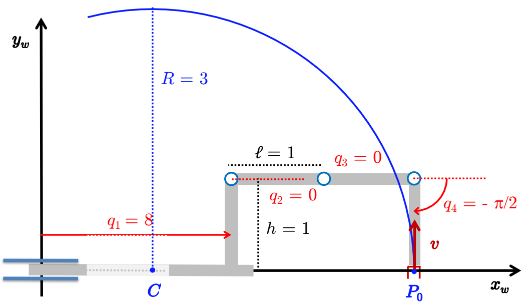
$$\boldsymbol{r}_{1d}(t) = \boldsymbol{C} + R \begin{pmatrix} \cos \dfrac{vt}{R} \\ \sin \dfrac{vt}{R} \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{r}}_{1d}(t) = v \begin{pmatrix} -\sin \dfrac{vt}{R} \\ \cos \dfrac{vt}{R} \end{pmatrix}$$

with $\boldsymbol{r}_{1d}(0) = \boldsymbol{P}_0$, $\dot{\boldsymbol{r}}_{1d}(0) = \begin{pmatrix} 0 & v \end{pmatrix}^T$ and $\|\dot{\boldsymbol{r}}_{1d}(t)\| = v$, as well as

$$r_{2d}(t) = 0 \quad \Rightarrow \quad \dot{r}_{2d} = 0.$$

Setting now $\ell = h = 1$, in order to be consistent with the augmented task at the initial time $t = 0$, there will be multiple robot configurations (q_1, q_3, q_4) that satisfy the desired end-effector positioning with $q_2 = 0$ (second link horizontal and pointing to the right):

$$\begin{pmatrix} q_1 + 1 + \cos q_3 + \cos q_4 \\ \sin q_3 + \sin q_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} = \boldsymbol{P}_0$$



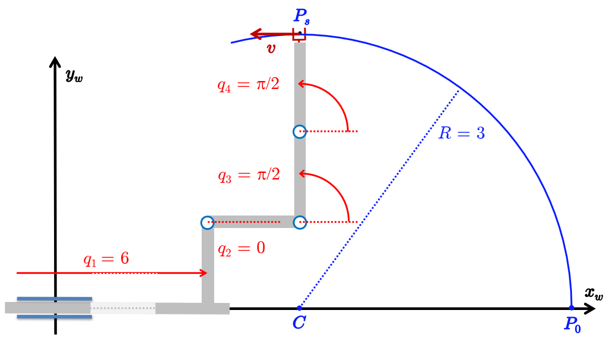


Figure 4: The 4-dof planar robot in the initial configuration $\boldsymbol{q}_0$ [left] and in the singular configuration $\boldsymbol{q}_s$ [right] for the two tasks of tracing the circle with its end effector (task 1) while keeping the second link horizontal (task 2).

A simple choice is to pick $q_1 = 8$ [m], $q_3 = 0$, $q_4 = -\pi/2$ [rad], as in Fig. 4 [left]. The configuration $\boldsymbol{q}_0 = \boldsymbol{q}(0) = \begin{pmatrix} 8 & 0 & 0 & -\pi/2 \end{pmatrix}^T$ is not singular. Accordingly, any augmented task velocity $\dot{\boldsymbol{r}}_d \in \mathbb{R}^3$ can be instantaneously realized (actually in a infinite number of ways). The minimum norm joint velocity solution is obtained using pseudoinversion ($\boldsymbol{J}_A^{\#} = \boldsymbol{J}_A^T(\boldsymbol{J}_A\boldsymbol{J}_A^T)^{-1}$):

$$\dot{\boldsymbol{q}}_0 = \boldsymbol{J}_A^{\#}(\boldsymbol{q}_0)\dot{\boldsymbol{r}}_d(0) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^{\#} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \\ 0.5 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

As a result, only the third joint moves, rotating counterclockwise. As shown in Fig. 4 [right], the first point on the circle where the augmented task necessarily encounters a singularity is at $\boldsymbol{P}_s = (\boldsymbol{C}_x, \boldsymbol{C}_y + R) = (7, 3)$. The robot arrives there at some instant $t = t_s > 0$ and can satisfy the positional/orientation tasks in only one configuration $\boldsymbol{q}_s = \boldsymbol{q}(t_s) = \begin{pmatrix} 6 & 0 & \pi/2 & \pi/2 \end{pmatrix}^T$, which is indeed singular. Note that this is a true *algorithmic singularity*, since both tasks are full rank ($\operatorname{rank}(\boldsymbol{J}_1(\boldsymbol{q}_s)) = 2$, $\operatorname{rank}(\boldsymbol{J}_2) = 1$) but $\operatorname{rank}(\boldsymbol{J}_A(\boldsymbol{q}_s)) = 2 < 3 = m_A$. Indeed, one can still compute the pseudoinverse solution, which provides

$$\dot{\boldsymbol{q}}_s = \boldsymbol{J}_A^{\#}(\boldsymbol{q}_s)\dot{\boldsymbol{r}}_d(t_s) = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}^{\#} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.3333 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ -0.3333 & 0 & 0 \\ -0.3333 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

The prismatic joint retracts, while joints 3 and 4 will rotate counterclockwise. When evaluating the execution of the augmented task with this joint velocity, we find

$$\boldsymbol{J}_A(\boldsymbol{q}_s)\dot{\boldsymbol{q}}_s = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \dot{\boldsymbol{r}}_d(t_s)!$$

Thus, the entire velocity task is still satisfied. In fact, despite the loss of rank of the augmented Jacobian, it is easy to see that $\dot{\boldsymbol{r}}_d(t_s) \in \mathcal{R}\{\boldsymbol{J}_A(\boldsymbol{q}_s)\}$. The pseudoinverse joint velocity returns then the exact solution also in this case. We note finally that the robot will not be able to trace the entire circle, being the lower part outside its workspace.

Exercise #3

With reference to Fig. 5, we define the task frame with axis $\boldsymbol{z}_t$ normal to the plane of motion and passing through the center of the cylinder base, and axis $\boldsymbol{x}_t$ tangential to the path on the plane. The natural constraints are then

$$f_x = 0, \qquad f_y = 0, \qquad v_z = 0, \qquad \omega_x = 0, \qquad \omega_y = 0, \qquad \mu_z = 0,$$

in which we neglected any friction effect at the contact. The complementary artificial constraints are

$$v_x = v_{x,d}(t) > 0, \qquad v_y = 0, \qquad f_z = f_{z,d} \neq 0, \qquad \mu_x = 0, \qquad \mu_y = 0, \qquad \omega_z = \omega_{z,d}.$$

The value of the velocity v_y (normal to the path) is chosen to be zero, signifying that the robot end-effector/payload should strictly follow the path on the plane. A non-zero $f_{z,d}$ can be chosen

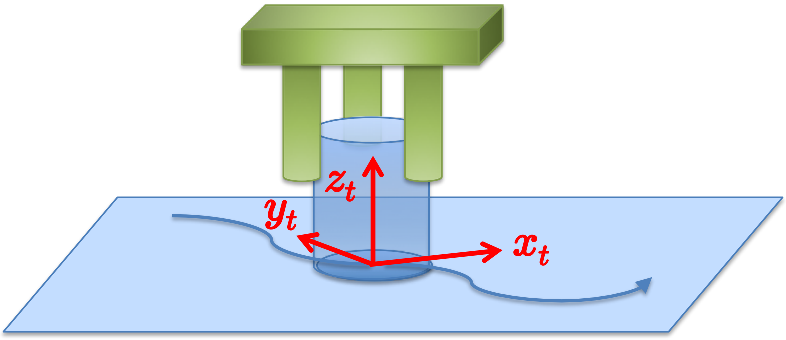


Figure 5: The task frame assignment for the contact motion of a cylinder following a path on a frictionless plane.

so as to enforce full surface contact with the base, despite of the presence of friction (and other disturbances) in the real world. The two reaction torques μ_x and μ_y are set to zero, in order not to stress the object while in contact. Finally, $\omega_{z,d}$ can be set to zero or not, depending on whether the cylinder should keep its orientation or rotate around its major axis while the center of its base is following the path traced on the plane.

Exercise #4

In this exercise, we just need to derive the gravity term in the dynamic model of the 2R planar robot. No information is required in fact on the inertial terms. Using the Denavit-Hartenberg coordinates, the $\boldsymbol{q} = \mathbf{0}$ configuration will correspond to the robot arm being stretched downward along the $\boldsymbol{x}_0$-axis, the configuration of minimum potential energy. Therefore, being $m_1 = m_2 = m$ and $d_{c1} = d_{c2} = \ell/2$, the potential energy due to gravity is

$$\begin{aligned} U = U_1 + U_2 &= -m_1 g_0\, d_{c1} \cos q_1 - m_2 g_0 \left(\ell \cos q_1 + d_{c2} \cos(q_1 + q_2)\right) \\ &= -m g_0 \ell \left(\frac{3}{2} \cos q_1 + \frac{1}{2} \cos(q_1 + q_2)\right), \end{aligned}$$

and so

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} m g_0 \ell \left(\dfrac{3}{2} \sin q_1 + \dfrac{1}{2} \sin(q_1 + q_2)\right) \\ m g_0 \dfrac{\ell}{2} \sin(q_1 + q_2) \end{pmatrix} = \begin{pmatrix} g_1(\boldsymbol{q}) \\ g_2(\boldsymbol{q}) \end{pmatrix}. \tag{1}$$

Setting $m = 5$ [kg], $\ell = 0.5$ [m] and $g_0 = 9.81$ [m/s^2], by evaluating (1) at $\boldsymbol{q}(t_1) = (\pi/4, 0)$ and $\boldsymbol{q}(t_2) = (-\pi/4, 3\pi/4)$ we obtain the gravity torques at the two joints

$$g_1(\boldsymbol{q}(t_1)) = 34.6836 \ \text{[Nm]} \qquad \text{and} \qquad g_2(\boldsymbol{q}(t_2)) = 12.2625 \ \text{[Nm]}.$$

The uniform time scaling factor $k > 1$ needed to recover feasibility of the entire motion is computed from

$$k_1 = \sqrt{\frac{\tau_1(t_1) - g_1(\boldsymbol{q}(t_1))}{\tau_{max,1} - g_1(\boldsymbol{q}(t_1))}} = 1.2698, \qquad k_2 = \sqrt{\frac{\tau_2(t_2) - g_2(\boldsymbol{q}(t_2))}{\tau_{max,2} - g_2(\boldsymbol{q}(t_2))}} = 1.2830,$$

as

$$k = \max\{k_1, k_2\} = 1.2830 \,(= k_2).$$

Thus, the second joint is the one with higher relative violation of the torque limit (once gravity is removed). The motion time is then increased from $T = 3$ [s] to the new value $T' = kT = 3.8491$ [s], which is the minimum feasible one under uniform time scaling. The values of the new joint torques (expressed in [Nm]) at the scaled instants $t_1' = kt_1$ and $t_2' = kt_2$ are computed as

$$\tau_1(t_1') = \frac{\tau_1(t_1) - g_1(\boldsymbol{q}(t_1))}{k^2} + g_1(\boldsymbol{q}(t_1) = 98.6589 < 100 = \tau_{max,1},$$

$$\tau_2(t_2') = \frac{\tau_2(t_2) - g_2(\boldsymbol{q}(t_2))}{k^2} + g_2(\boldsymbol{q}(t_2) = 20 = \tau_{max,2}.$$

As expected, the second joint torque will be in saturation at the scaled instant t_2'.

Exercise #5

The three requested motion tasks are all regulation problems. The dynamic terms needed for the various feedback control laws are listed first. We make reference to the 2R robot in Fig. 3, with the same definition of joint variables and using the same parameters. The inertia matrix is

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2 \cos q_2 & a_3 + a_2 \cos q_2 \\ a_3 + a_2 \cos q_2 & a_3 \end{pmatrix},$$

with $a_1 = I_1 + I_2 + \frac{3}{2}\, m\ell^2$, $a_2 = \frac{1}{2}\, m\ell^2$, $a_3 = I_2 + \frac{1}{4} m\ell^2$. The Coriolis and centrifugal terms are

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -a_2 \sin q_2 (\dot q_2 + 2\dot q_1)\dot q_2 \\ a_2 \sin q_2 \dot q_1^2 \end{pmatrix}.$$

The gravity vector has been already computed in Exercise #4, and is rewritten here as

$$\boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} a_4 \sin q_1 + a_5 \sin(q_1 + q_2) \\ a_5 \sin(q_1 + q_2) \end{pmatrix}$$

with $a_4 = \frac{1}{2}\, m g_0 \ell$, $a_5 = \frac{1}{2}\, m g_0 \ell$. From the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$, the robot Jacobian that maps the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^2$ to the velocity $\dot{\boldsymbol{p}} \in \mathbb{R}^2$ of the end effector is

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\ell(\sin q_1 + \sin(q_1 + q_2)) & -\ell \sin(q_1 + q_2) \\ \ell(\cos q_1 + \cos(q_1 + q_2)) & \ell \cos(q_1 + q_2) \end{pmatrix}.$$

Finally, we shall need also the time derivative of the Jacobian matrix, namely

$$\dot{\boldsymbol{J}}(\boldsymbol{q}) = \begin{pmatrix} -\ell\left(\cos q_1\, \dot q_1 + \cos(q_1 + q_2)\,(\dot q_1 + \dot q_2)\right) & -\ell \cos(q_1 + q_2)\,(\dot q_1 + \dot q_2) \\ -\ell\left(\sin q_1\, \dot q_1 + \sin(q_1 + q_2)\,(\dot q_1 + \dot q_2)\right) & -\ell \sin(q_1 + q_2)\,(\dot q_1 + \dot q_2) \end{pmatrix}.$$

a. Global exponential stabilization of the state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$, with decoupled transient evolutions of the position errors $e_i(t) = q_{d,i} - q_i(t)$, $i = 1, 2$, of the form $e_1(t) = e_1(0)\,(1 + 5t)\, \exp(-5t)$ and $e_2(t) = e_2(0)\,(2 \exp(-5t) - \exp(-10t))$.

This is obtained by feedback linearization control in the joint space:

$$\boldsymbol{u} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}), \quad \text{with } \boldsymbol{a} = -\boldsymbol{K}_D\, \dot{\boldsymbol{q}} + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}),$$

with $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$ and both diagonal. The desired error transients are obtained by choosing suitable gains in the linear and decoupled second-order dynamics

$$\ddot e_i + K_{D,i} \dot e_i + K_{P,i} e_i = 0, \qquad i = 1, 2, \tag{2}$$

for the two position errors $e_i(t) = q_{d,i} - q_i(t)$. For joint 1, substitute $e_1(t) = e_1(0)\,(1+5t)\,\exp(-5t)$ and its first and second time derivatives in (2):

$$e_1(0)\,(-25+125t)\,\exp(-5t) - K_{D,1}\,e_1(0)\,25t\exp(-5t) + K_{P,1}\,e_1(0)\,(1+5t)\,\exp(-5t) = 0.$$

Since $e_1(0)\exp(-5t) \neq 0$ for any finite $t \geq 0$, this common factor can be eliminated so as to obtain

$$(-25+125t) - 25K_{D,1}t + (1+5t)\,K_{P,1} = 0.$$

By the principle of polynomial identity (w.r.t. the powers of t), this implies

$$125 - 25\,K_{D,1} + 5\,K_{P,1} = 0, \quad -25 + K_{P,1} = 0 \qquad \Rightarrow \qquad K_{P,1} = 25,\ K_{D,1} = 10.$$

Moreover, transforming eq. (2) for $i = 1$ in the Laplace domain and using these values leads to

$$\left(s^2 + 10s + 25\right) e_1(s) = (s+5)^2 e_1(s) = 0,$$

namely, the error dynamics at the first joint is characterized by two real and coincident negative eigenvalues in -5.

We proceed similarly for joint 2. Substitute $e_2(t) = e_2(0)\,(2\exp(-5t) - \exp(-10t))$ and its first and second time derivatives in (2):

$$e_2(0)\,(50\exp(-5t) - 100\exp(-10t)) + K_{D,2}\,e_2(0)\,(-10\exp(-5t) + 10\exp(-10t))$$
$$+ K_{P,2}\,e_2(0)\,(2\exp(-5t) - \exp(-10t)) = 0.$$

Being $e_2(0) \neq 0$, in order for this expression to vanish identically at all times $t \geq 0$, we should zero the coefficients multiplying the two different exponentials $\exp(-5t)$ and $\exp(-10t)$. This yields

$$50 - 10\,K_{D,2} + 2\,K_{P,2} = 0, \quad -100 + 10\,K_{D,2} - K_{P,2} = 0 \qquad \Rightarrow \qquad K_{P,2} = 50,\ K_{D,2} = 15.$$

Moreover, transforming eq. (2) for $i = 2$ in the Laplace domain and using these values leads to

$$\left(s^2 + 15s + 50\right) e_2(s) = (s+5)(s+10)\,e_2(s) = 0,$$

namely, the error dynamics at the second joint has two real and distinct negative eigenvalues in -5 and -10.

b. Global asymptotic stabilization of the state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$, without knowledge of the robot inertia matrix.

This can be obtained by multiple choices, the most common being a PD control with gravity cancellation

$$\boldsymbol{u} = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}),$$

with symmetric $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$, typically chosen diagonal. In alternative, one can use gravity compensation

$$\boldsymbol{u} = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\,\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d)$$

further requiring that the minimum eigenvalue of $\boldsymbol{K}_P$ is larger than a finite upper bound $\alpha > 0$ on the norm of the Hessian $\partial^2 U(\boldsymbol{q})/\partial \boldsymbol{q}^2$ of the gravitational potential energy U.

c. Exponential stabilization of the end-effector position $\boldsymbol{p} = \boldsymbol{p}_d \in \mathbb{R}^2$ with zero velocity $\dot{\boldsymbol{p}} = \mathbf{0}$, up to kinematic singularities.

In this case, we require a feedback linearization control in the Cartesian space:

$$\boldsymbol{u} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\boldsymbol{a} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}),$$

with

$$\boldsymbol{a} = -\boldsymbol{K}_D\, \dot{\boldsymbol{p}} + \boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{p}) = -\boldsymbol{K}_D\, \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} + \boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}))$$

and where $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$ are both chosen diagonal.

* * * * *

Robotics 2

Remote Exam – September 11, 2020

Exercise #1

Consider the planar 3R robot with links of equal length L in Fig. 1, driven by joint velocity commands $\dot{\boldsymbol{q}} \in \mathbb{R}^3$. The end effector should trace the nominal linear path from A to B with a constant speed $v > 0$, while the entire robot avoids any collision with a single static obstacle O. The obstacle is of known circular shape but uncertain radius $R \leq L/4$, and is placed at least at a distance $\rho_{min} = L$ from the robot base and not farther than $\rho_{max} = 3L$. Its actual location is unknown a priori, but the obstacle can be detected by an omnidirectional proximity sensor mounted on the robot end effector and having a sensing range $\sigma = 1.5L$.

Define a sensor-based control scheme that makes the robot perform *at best* the assigned task, and describe qualitatively its expected performance.

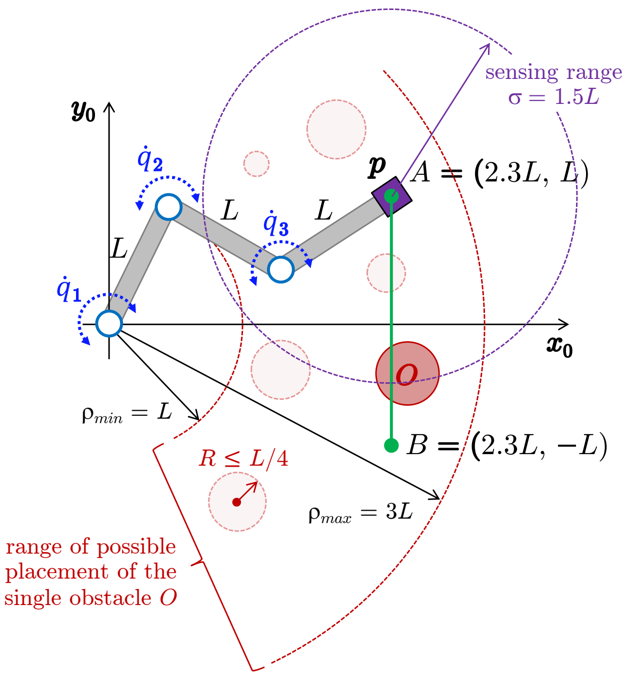


Figure 1: The 3R robot with the nominal Cartesian path AB to be traced and the static circular obstacle O to be avoided (several possible alternative locations of the single obstacle are shown).

Exercise #2

In an image-based visual servoing control scheme, the camera mounted on the robot end effector (eye-in-hand) looks for three specific points in the scene, characterized by their point features in the image plane with coordinates (u_i, v_i), for $i = 1, 2, 3$. Derive the expression of the 2×6 *interaction matrix* $\boldsymbol{J}_b$ associated to the geometric barycenter $\boldsymbol{b} \in \mathbb{R}^2$ of the triangle in the image having these point features as the three vertices, namely

$$\dot{\boldsymbol{b}} = \boldsymbol{J_b} \begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{\Omega} \end{pmatrix},$$

where $\boldsymbol{V} \in \mathbb{R}^3$ and $\boldsymbol{\Omega} \in \mathbb{R}^3$ are respectively the linear and angular velocity of the camera.

Provide then an instantaneous motion of the camera such that $\dot{\boldsymbol{b}} = \boldsymbol{0}$.

Exercise #3

With reference to the planar RP robot in Fig. 2, using the symbolic parameters specified therein, determine the complete expression of the *Cartesian inertia matrix* $\boldsymbol{M}_{\boldsymbol{p}}(\boldsymbol{q})$ of the robot at the tip $\boldsymbol{p} \in \mathbb{R}^2$ as a function of the configuration $\boldsymbol{q}$.

Provide then the numerical value of $\boldsymbol{M}_{\boldsymbol{p}}(\boldsymbol{q}^*)$ at $\boldsymbol{q}^* = \begin{pmatrix} 0 & 3 \end{pmatrix}^T$ [rad, m], using the parameters:

$$l_1 = 1, \quad d_1 = 0.5, \quad m_1 = 3, \quad I_1 = 0.25; \qquad d_2 = 0.5, \quad m_2 = 0.5, \quad I_2 = 0.875.$$

Assuming that this robot is at rest in $\boldsymbol{q}^*$ on a horizontal plane, check that the tip acceleration $\ddot{\boldsymbol{p}} \in \mathbb{R}^2$ has the same direction of any force $\boldsymbol{F} \in \mathbb{R}^2$ in the plane applied to the tip of the robot.

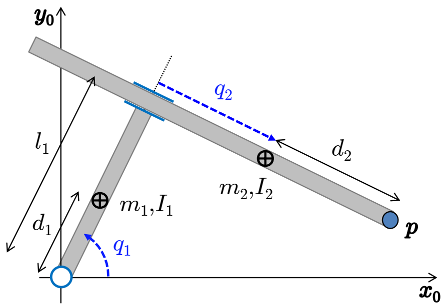


Figure 2: A 2-dof RP robot with its relevant kinematic and dynamic parameters.

Exercise #4

A 3R robot moves on a horizontal plane in the presence of geometric constraints in the Cartesian space, as illustrated in Fig. 3. Assume that the distance k satisfies the inequalities $l_1 < k < l_1 + l_2$.

- Determine the dimension M of the constraints and their possible expression $\boldsymbol{h}(\boldsymbol{q}) = \boldsymbol{0}$, with the associated Jacobian matrix $\boldsymbol{A}(\boldsymbol{q}) = \partial \boldsymbol{h}(\boldsymbol{q})/\partial \boldsymbol{q}$.
- Define the $(3-M) \times 3$ matrix $\boldsymbol{D}(\boldsymbol{q})$ to complete $\boldsymbol{A}(\boldsymbol{q})$ in a nonsingular way, as well as the blocks $\boldsymbol{E}(\boldsymbol{q})$ and $\boldsymbol{F}(\boldsymbol{q})$ of the inverse that appear in the *reduced dynamics* of this constrained robot. The reduced model should hold for any $\boldsymbol{q}$ such that the contact situation remains as in Fig. 3.
- Let m_{ij} be the elements of the (symmetric) inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of this robot in free space. Give the full expression of the $(3-M)\times(3-M)$ *reduced inertia matrix* in the constrained case.

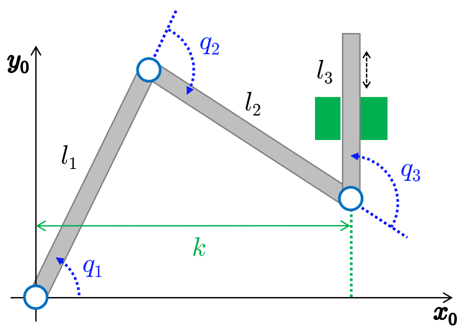


Figure 3: A 3R robot with geometric constraints in the Cartesian space limiting its motion.

Exercise #5

Consider a 2R robot arm of human-like size and weight moving in a vertical plane with negligible friction. Two model-based controllers are being compared in a trajectory tracking problem in the joint space. The first is based on feedback linearization, yielding a control torque $\boldsymbol{u}_{FBL}(t)$, the second is a Lyapunov-based control law with global asymptotic convergence property, yielding a torque $\boldsymbol{u}_{GLB}(t)$.

Assuming that the PD gain matrices are the same in both control laws, provide the explicit expression of the torque difference $\boldsymbol{\Delta u}(t) = \boldsymbol{u}_{FBL}(t) - \boldsymbol{u}_{GLB}(t)$.

Next, when the desired joint trajectory and the PD gains are respectively

$$\boldsymbol{q}_d(t) = \begin{pmatrix} \dfrac{\pi}{2} + 3\sin\dfrac{\pi t}{2} \\ 1 - \cos 2\pi t \end{pmatrix}, \qquad \boldsymbol{K}_P = 100 \cdot I_{2\times 2}, \quad \boldsymbol{K}_D = 20 \cdot I_{2\times 2},$$

assume that at time $t = 2$ s, the current robot configuration is $\boldsymbol{q}(2) = \begin{pmatrix} \pi/2 & -\pi/2 \end{pmatrix}^T$ [rad] and the velocity tracking error is zero. For each robot joint, determine which is the controller that uses the larger instantaneous torque in absolute value.

[240 minutes (4 hours); open books]

Solution

September 11, 2020

Exercise #1

This is an open-ended exercise and many possible schemes could be devised. Here, we present one where the 3R robot uses its redundancy with respect to the two-dimensional motion task assigned to its end effector as a mean to avoid the single obstacle detected online. The end-effector motion task has a higher priority than the collision avoidance task, as long as the minimum distance between the robot and the obstacle O stays above some threshold $\varepsilon > 0$. Otherwise, the controller switches the priority order and moves the robot primarily to avoid collision with the obstacle, trying to keep also the tracking error as small as possible. Indeed, when the obstacle is placed on the end-effector path, this switching will certainly happen. To implement such a control strategy, we use the following items.

- A clearance function defined by

$$H(\boldsymbol{q}) = \min_{\substack{\boldsymbol{a}(\boldsymbol{q})\in \text{robot} \\ \boldsymbol{b}\in \text{obstacle}}} \|\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{b}\|. \tag{1}$$

  In this expression, the detection of points $\boldsymbol{b}$ belonging to the circular obstacle O is made by the proximity sensor mounted on the end-effector. The sensor is able to reconstruct the entire visible surface of the obstacle and thus determine the closest point to the robot. The range of the proximity sensor covers the entire area of interest. The location of every point $\boldsymbol{a}(\boldsymbol{q})$ on the robot body is known from the encoder measures of $\boldsymbol{q}$ and via the direct kinematics of the robot.

  The clearance function in (1) can also be approximated by choosing a number of control points on the robot body for robot-obstacle distance computation, rather than the entire robot skeleton. For instance, one can use the three points $\boldsymbol{P}_{j2}$ = location of joint 2, $\boldsymbol{P}_{j3}$ = location of joint 3, and $\boldsymbol{P}_{ee}$ = end-effector location. Then, (1) would be replaced by the clearance function

$$H(\boldsymbol{q}) = \min_{\boldsymbol{b}\in \text{obstacle}} \left\{\|\boldsymbol{P}_{j2}(q_1) - \boldsymbol{b}\|, \|\boldsymbol{P}_{j3}(q_1,q_2) - \boldsymbol{b}\|, \|\boldsymbol{P}_{ee}(q_1,q_2,q_3) - \boldsymbol{b}\|\right\}. \tag{2}$$

  In both cases, we can use the gradient $\nabla_{\boldsymbol{q}} H(\boldsymbol{q}) = (\partial H(\boldsymbol{q})/\partial \boldsymbol{q})^T$ in the control scheme, together with a stepsize factor $\alpha > 0$. While both clearance functions (1) and (2) are continuous (in space and time), their gradient may have discontinuities.

- The primary task Jacobian associated to the end-effector position $\boldsymbol{p}$, as given by

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -(l_1 s_1 + l_2 s_{12} + l_3 s_{123}) & -(l_2 s_{12} + l_3 s_{123}) & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \end{pmatrix},$$

  with the usual shorthand notation for trigonometric quantities (e.g., $s_{123} = \sin(q_1 + q_2 + q_3)$). We will use the pseudoinverse $\boldsymbol{J}^{\#}(\boldsymbol{q})$ of this matrix. Moreover, the desired task velocity will be $\dot{\boldsymbol{p}}_d = v\,(B - A)\,/\|B - A\|$.

- A switching control law defined as

$$\dot{\boldsymbol{q}} = \begin{cases} \boldsymbol{J}^{\#}(\boldsymbol{q})\,(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_P\,(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q}))) + \Big(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\Big)\,\alpha\nabla_{\boldsymbol{q}} H(\boldsymbol{q}) & \\ \quad = \alpha\nabla_{\boldsymbol{q}} H(\boldsymbol{q}) + \boldsymbol{J}^{\#}(\boldsymbol{q})\Big(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_P\,(\boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q})) - \alpha \boldsymbol{J}(\boldsymbol{q})\nabla_{\boldsymbol{q}} H(\boldsymbol{q})\Big), & \text{if } H(\boldsymbol{q}) > \epsilon, \\ \alpha\nabla_{\boldsymbol{q}} H(\boldsymbol{q}), & \text{if } H(\boldsymbol{q}) \le \epsilon. \end{cases} \tag{3}$$

When the obstacle is sufficiently far, with the control law (3) the robot executes exactly the desired end-effector trajectory $\boldsymbol{p}_d(t)$, while the gradient of the clearance function is projected in the one-dimension null space of the primary task. Instead, when the robot is getting too close to the obstacle, the control law will move away the nearest point of the robot. In the given form of the gradient of $H(\boldsymbol{q})$ in the configuration space, obstacle repulsion is a three-dimensional task which leaves no space for a secondary consideration of the original tracking task. Therefore, during this phase, the end-effector path is typically abandoned[1]. When the minimum clearance ε is recovered, the control will switch back to the previous law, trying to recover the accumulated error with respect to the desired Cartesian trajectory. In order to do so, a position error feedback term with (diagonal) matrix gain $\boldsymbol{K}_P > 0$ has been added in (3).

We note finally that the proposed control scheme is purely reactive (there is no planning for obstacle avoidance) and only local in scope (we cannot exclude that the robot gets stuck before reaching the point B). Further, there is no motion stop explicitly involved with a control switch. As a consequence, switches are typically associated with discontinuities of the joint velocity commands.

Exercise #2

Associated to a point $\boldsymbol{P}_i = (X_i, Y_i, Z_i)$ in the Cartesian space, with its coordinates expressed in the camera frame, there is a point feature $\boldsymbol{f}_{\boldsymbol{p}_i} = (u_i, v_i)$ in the image plane whose interaction matrix $\boldsymbol{J}_{p_i}$ takes the form (see the lecture slides)

$$\boldsymbol{J}_{p_i}(u_i, v_i, Z_i) = \begin{pmatrix} -\dfrac{\lambda}{Z_i} & 0 & \dfrac{u_i}{Z_i} & \dfrac{u_i v_i}{\lambda} & -\dfrac{u_i^2}{\lambda} - \lambda & v_i \\ 0 & -\dfrac{\lambda}{Z_i} & \dfrac{v_i}{Z_i} & \dfrac{v_i^2}{\lambda} + \lambda & -\dfrac{u_i v_i}{\lambda} & -u_i \end{pmatrix},$$

where $\lambda > 0$ is the focal length of the camera and the depth $Z_i > 0$ is limited (by the visual range of the camera). The geometric barycenter $\boldsymbol{b} \in \mathbb{R}^2$ of a triangle in the image plane is simply obtained from its three vertices, namely the coordinates of the point features $\boldsymbol{f}_{\boldsymbol{p}_i}$, $i = 1, 2, 3$, as

$$\boldsymbol{b} = \begin{pmatrix} b_u \\ b_v \end{pmatrix} = \frac{1}{3} \begin{pmatrix} u_1 + u_2 + u_3 \\ v_1 + v_2 + v_3 \end{pmatrix}.$$

Therefore, the interaction matrix $\boldsymbol{J}_{\boldsymbol{b}}$ takes the form

$$\begin{aligned} \boldsymbol{J}_{\boldsymbol{b}} &= \frac{1}{3}\Big(\boldsymbol{J}_{p_1}(u_1, v_1, Z_1) + \boldsymbol{J}_{p_2}(u_2, v_2, Z_2) + \boldsymbol{J}_{p_3}(u_3, v_3, Z_3)\Big) \\ &= \frac{1}{3} \begin{pmatrix} -\lambda \displaystyle\sum_{i=1}^{3} \frac{1}{Z_i} & 0 & \displaystyle\sum_{i=1}^{3} \frac{u_i}{Z_i} & \dfrac{1}{\lambda} \displaystyle\sum_{i=1}^{3} u_i v_i & -\dfrac{1}{\lambda} \displaystyle\sum_{i=1}^{3} u_i^2 - 3\lambda & \displaystyle\sum_{i=1}^{3} v_i \\ 0 & -\lambda \displaystyle\sum_{i=1}^{3} \frac{1}{Z_i} & \displaystyle\sum_{i=1}^{3} \frac{v_i}{Z_i} & \dfrac{1}{\lambda} \displaystyle\sum_{i=1}^{3} v_i^2 + 3\lambda & -\dfrac{1}{\lambda} \displaystyle\sum_{i=1}^{3} u_i v_i & -\displaystyle\sum_{i=1}^{3} u_i \end{pmatrix} \\ &= \boldsymbol{J}_{\boldsymbol{b}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{Z}), \end{aligned}$$

[1]Other collision avoidance schemes can be defined, still using an artificial potential to keep the robot away from the obstacle. Obstacle avoidance can be formulated as a two-dimensional task, when the repulsive action is defined along the gradient of the Cartesian clearance, or even as a one-dimensional task, when only the projection of the repulsive Cartesian alction along the clearance direction is specified. In both cases, it is possible to accommodate the tracking task in the null space of the avoidance task.

with $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{v} = (v_1, v_2, v_3)$, and depths $\boldsymbol{Z} = (Z_1, Z_2, Z_3)$. Note that the obtained interaction matrix $\boldsymbol{J_b}$ is in general *different* from the interaction matrix associated to the single point feature of the Cartesian barycenter $\boldsymbol{P_b} = \frac{1}{3}(\boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{P}_3)$ of the three points in 3D space. Possible camera motions that belong to the null space of $\boldsymbol{J_b}$ are

$$\boldsymbol{V} = \begin{pmatrix} \displaystyle\sum_{i=1}^{3} \frac{u_i}{Z_i} \\ \displaystyle -\sum_{i=1}^{3} \frac{v_i}{Z_i} \\ \displaystyle \lambda \sum_{i=1}^{3} \frac{1}{Z_i} \end{pmatrix}, \quad \boldsymbol{\Omega} = \boldsymbol{0} \qquad \text{(a pure linear motion)},$$

or

$$\boldsymbol{V} = \begin{pmatrix} \displaystyle\sum_{i=1}^{3} v_i \\ \displaystyle\sum_{i=1}^{3} u_i \\ 0 \end{pmatrix}, \quad \boldsymbol{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \displaystyle \lambda \sum_{i=1}^{3} \frac{1}{Z_i} \end{pmatrix} \qquad \begin{array}{c}\text{(linear motion parallel to the image plane,} \\ \text{with angular motion around the optical axis).}\end{array}$$

More independent camera motions with $\dot{\boldsymbol{b}} = \boldsymbol{0}$ exist, since the dimension of $\mathcal{N}\{\boldsymbol{J_b}\}$ is at least 4.

Exercise #3

We compute first the kinetic energy of the RP robot. For the two links, we have

$$T_1 = \frac{1}{2}\left(I_1 + m_1 d_1^2\right)\dot{q}_1^2, \qquad T_2 = \frac{1}{2} I_2 \dot{q}_1^2 + \frac{1}{2} m_2 \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2},$$

with

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \frac{d}{dt}\begin{pmatrix} l_1 \cos q_1 + q_2 \sin q_1 \\ l_1 \sin q_1 - q_2 \cos q_1 \end{pmatrix} = \begin{pmatrix} (q_2 \cos q_1 - l_1 \sin q_1)\,\dot{q}_1 + \sin q_1\,\dot{q}_2 \\ (l_1 \cos q_1 + q_2 \sin q_1)\,\dot{q}_1 - \cos q_1\,\dot{q}_2 \end{pmatrix}.$$

Therefore, from $T = T_1 + T_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$, we obtain the inertia matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_1 + m_1 d_1^2 + I_2 + m_2 l_1^2 + m_2 q_2^2 & -m_2 l_1 \\ -m_2 l_1 & m_2 \end{pmatrix}.$$

The Jacobian associated to the linear velocity $\boldsymbol{v} = \dot{\boldsymbol{p}} \in \mathbb{R}^2$ of the robot tip is computed as

$$\dot{\boldsymbol{p}} = \frac{d}{dt}\begin{pmatrix} l_1 \cos q_1 + (q_2 + d_2)\sin q_1 \\ l_1 \sin q_1 - (q_2 + d_2)\cos q_1 \end{pmatrix} = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\Rightarrow \quad \boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} (q_2 + d_2)\cos q_1 - l_1 \sin q_1 & \sin q_1 \\ l_1 \cos q_1 + (q_2 + d_2)\sin q_1 & -\cos q_1 \end{pmatrix}.$$

A singularity occurs when $\det \boldsymbol{J}(\boldsymbol{q}) = -(q_2+d_2) = 0$. Out of this singularity, and using also a shorthand notation for the trigonometric terms, the Cartesian inertia matrix at the robot tip is

$$
\begin{aligned}
\boldsymbol{M}_{\boldsymbol{p}}(\boldsymbol{q}) &= \boldsymbol{J}^{-T}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q}) \\
&= \frac{1}{(q_2+d_2)^2}\begin{pmatrix} -c_1 & -l_1c_1-(q_2+d_2)s_1 \\ -s_1 & (q_2+d_2)c_1-l_1s_1 \end{pmatrix}\begin{pmatrix} I_1+m_1d_1^2+I_2+m_2l_1^2+m_2q_2^2 & -m_2l_1 \\ -m_2l_1 & m_2 \end{pmatrix} \\
&\qquad \cdot \begin{pmatrix} -c_1 & -s_1 \\ -l_1c_1-(q_2+d_2)s_1 & (q_2+d_2)c_1-l_1s_1 \end{pmatrix},
\end{aligned}
$$

with elements

$$
\begin{aligned}
\boldsymbol{M}_{\boldsymbol{p},11} &= \frac{1}{(q_2+d_2)^2}\Big(I_1+I_2+m_1d_1^2+m_2q_2^2+\left(2m_2d_2q_2+m_2d_2^2-I_1-I_2-m_1d_1^2\right)s_1^2\Big), \\
\boldsymbol{M}_{\boldsymbol{p},12} &= \boldsymbol{M}_{\boldsymbol{p},21} = \frac{1}{(q_2+d_2)^2}\Big(I_1+I_2+m_1d_1^2-m_2d_2^2-2m_2d_2q_2\Big)s_1c_1, \\
\boldsymbol{M}_{\boldsymbol{p},22} &= \frac{1}{(q_2+d_2)^2}\Big(m_2(q_2+d_2)^2+\left(I_1+I_2+m_1d_1^2-m_2d_2^2-2m_2d_2q_2\right)s_1^2\Big).
\end{aligned}
$$

The Cartesian inertia matrix is not diagonal in general. However, when evaluating $\boldsymbol{M}_{\boldsymbol{p}}(\boldsymbol{q}^*)$ for $\boldsymbol{q}^* = \begin{pmatrix} 0 & 3 \end{pmatrix}^T$ [rad, m] and using the numerical parameters given in the text, we obtain

$$
\begin{aligned}
\boldsymbol{M}_{\boldsymbol{p}}(\boldsymbol{q}^*) &= \boldsymbol{J}^{-T}(\boldsymbol{q}^*)\boldsymbol{M}(\boldsymbol{q}^*)\boldsymbol{J}^{-1}(\boldsymbol{q}^*) \\
&= \begin{pmatrix} 0.2857 & 0.2857 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 6.625 & -0.5 \\ -0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 0.2857 & 0 \\ 0.2857 & -1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} = 0.5\cdot \boldsymbol{I}_{2\times 2}
\end{aligned}
$$

As a result, we can immediately verify that the Cartesian acceleration $\ddot{\boldsymbol{p}} \in \mathbb{R}^2$ of the robot end effector in response to an arbitrary force $\boldsymbol{F} \in \mathbb{R}^2$ applied at the tip when the robot is at rest in $\boldsymbol{q}^*$ and in the absence of gravity is

$$
\ddot{\boldsymbol{p}} = \boldsymbol{M}_{\boldsymbol{p}}^{-1}(\boldsymbol{q}^*)\boldsymbol{F} = 2\boldsymbol{F},
$$

namely, $\ddot{\boldsymbol{p}}$ has the same direction of the applied force $\boldsymbol{F}$.

Exercise #4

The 3R robot (having $N = 3$ degrees of freedom) is geometrically constrained in its Cartesian motion in two ways:

- it cannot change the absolute orientation of the last link, which remains always parallel to $\boldsymbol{y}_0$;
- it cannot move the tip of the second link away from the axis $x = k$.

Therefore, this situation can be modeled by $M = 2$ scalar constraints, written in the joint space as

$$
\boldsymbol{h}(\boldsymbol{q}) = \begin{pmatrix} q_1+q_2+q_3-\dfrac{\pi}{2} \\ l_1\cos q_1 + l_2\cos(q_1+q_2) - k \end{pmatrix} = \mathbf{0}.
$$

The Jacobian of these constraints is

$$
\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial \boldsymbol{h}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} 1 & 1 & 1 \\ -\left(l_1\sin q_1 + l_2\sin(q_1+q_2)\right) & -l_2\sin(q_1+q_2) & 0 \end{pmatrix}.
$$

The rank of matrix $\boldsymbol{A}(\boldsymbol{q})$ is 2, except when $\sin(q_1 + q_2) = \sin q_1 = 0$, which occur if and only if $q_1 = \{0, \pi\}$ and $q_2 = \{0, \pi\}$, i.e., when the first two links are stretched or folded along the $\boldsymbol{x}_0$-axis. However, these singular configurations are not allowed by the geometric constraints (thanks to the two inequalities imposed on the parameter k, otherwise arbitrary).

The first step for deriving the reduced dynamics of this constrained robot is to find a matrix $\boldsymbol{D}(\boldsymbol{q})$ of size $(N - M) \times M = 1 \times 3$ such that it completes $\boldsymbol{A}(\boldsymbol{q})$, building a nonsingular square matrix in the operating region. A suitable choice that satisfies this requirement is

$$\boldsymbol{D}(\boldsymbol{q}) = \begin{pmatrix} l_1 \cos q_1 + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) & 0 \end{pmatrix},$$

which leads to

$$\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -\left(l_1 \sin q_1 + l_2 \sin(q_1 + q_2)\right) & -l_2 \sin(q_1 + q_2) & 0 \\ l_1 \cos q_1 + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) & 0 \end{pmatrix}, \quad \det \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = l_1 l_2 \sin q_2. \tag{4}$$

The determinant is never zero for all $\boldsymbol{q}$ such that the contact situation remains the same as in Fig. 3. As a matter of fact, this choice of $\boldsymbol{D}(\boldsymbol{q})$ reconstructs the 2×2 Jacobian of the 2R substructure made by the first two links of the 3R robot. The constrained robot has only one degree of freedom left, which is described by the scalar term

$$v = \boldsymbol{D}(\boldsymbol{q})\dot{\boldsymbol{q}} = \left(l_1 \cos q_1 + l_2 \cos(q_1 + q_2)\right) \dot{q}_1 + l_2 \cos(q_1 + q_2)\, \dot{q}_2.$$

This pseudovelocity represents the motion of the tip of the second link along the direction $\boldsymbol{y}_0$.

The second step of the procedure is to invert the matrix in (4) so as to define the blocks $\boldsymbol{E}(\boldsymbol{q})$ and $\boldsymbol{F}(\boldsymbol{q})$ in the inverse. We obtain

$$\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix}^{-1} = \frac{1}{l_1 l_2 \sin q_2} \begin{pmatrix} 0 & l_2 \cos(q_1 + q_2) & l_2 \sin(q_1 + q_2) \\ 0 & -\left(l_1 \cos q_1 + l_2 \cos(q_1 + q_2)\right) & -\left(l_1 \sin q_1 + l_2 \sin(q_1 + q_2)\right) \\ l_1 l_2 \sin q_2 & l_1 \cos q_1 & l_1 \sin q_1 \end{pmatrix}.$$

Thus, we have the partition into the first $M = 2$ columns

$$\boldsymbol{E}(\boldsymbol{q}) = \frac{1}{l_1 l_2 \sin q_2} \begin{pmatrix} 0 & l_2 \cos(q_1 + q_2) \\ 0 & -\left(l_1 \cos q_1 + l_2 \cos(q_1 + q_2)\right) \\ l_1 l_2 \sin q_2 & l_1 \cos q_1 \end{pmatrix}$$

and the last $N - M = 1$ column

$$\boldsymbol{F}(\boldsymbol{q}) = \frac{1}{l_1 l_2 \sin q_2} \begin{pmatrix} l_2 \sin(q_1 + q_2) \\ -\left(l_1 \sin q_1 + l_2 \sin(q_1 + q_2)\right) \\ l_1 \sin q_1 \end{pmatrix}.$$

Finally, introducing in symbolic form the elements of the robot inertia of the 3R robot in free space

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix} > 0,$$

we have that the reduced inertia matrix is a scalar given by[2]

$$
\begin{aligned}
\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) &= \boldsymbol{F}^T(\boldsymbol{q}) \cdot \frac{1}{l_1 l_2 \sin q_2} \begin{pmatrix} (m_{11}-m_{12})\, l_2 \sin(q_1+q_2) + (m_{13}-m_{12})\, l_1 \sin q_1 \\ (m_{12}-m_{22})\, l_2 \sin(q_1+q_2) + (m_{23}-m_{22})\, l_1 \sin q_1 \\ (m_{13}-m_{23})\, l_2 \sin(q_1+q_2) + (m_{33}-m_{23})\, l_1 \sin q_1 \end{pmatrix}. \\
&= \frac{1}{(l_1 l_2 \sin q_2)^2} \Big((m_{11}+m_{22}-2m_{12})\, l_2^2 \sin^2(q_1+q_2) \\
&\qquad + (m_{22}+m_{33}-2m_{23})\, l_1^2 \sin^2 q_1 \\
&\qquad + 2\,(m_{22}-m_{12}+m_{13}-m_{23})\, l_1 l_2 \sin q_1 \sin(q_1+q_2) \Big).
\end{aligned}
$$

Exercise #5

The trajectory tracking control law based on feedback linearization is

$$\boldsymbol{u}_{FBL} = \boldsymbol{M}(\boldsymbol{q})\Big(\ddot{\boldsymbol{q}}_d + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}) + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q})\Big) + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}). \tag{5}$$

The Lyapunov-based control law with global asymptotic convergence property is

$$\boldsymbol{u}_{GLB} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}_d + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_d + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}) + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}), \tag{6}$$

where $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is a skew symmetric matrix and the PD gains are by hypothesis the same as in (5). Since the Coriolis and centrifugal terms in (5) can always be rewritten as $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ using the same factorization used in (6), the difference between the two control torques can be written in general as

$$\boldsymbol{\Delta u} = \boldsymbol{u}_{FBL} - \boldsymbol{u}_{GLB} = (\boldsymbol{M}(\boldsymbol{q}) - \boldsymbol{I})\,(\boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}) + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q})) + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\,(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_d)\,. \tag{7}$$

From the desired joint trajectory, we obtain

$$\dot{\boldsymbol{q}}_d(t) = \begin{pmatrix} \dfrac{3\pi}{2}\cos\dfrac{\pi t}{2} \\ 2\pi \sin 2\pi t \end{pmatrix}, \qquad \ddot{\boldsymbol{q}}_d(t) = \begin{pmatrix} -\dfrac{3\pi^2}{4}\sin\dfrac{\pi t}{2} \\ 4\pi^2 \cos 2\pi t \end{pmatrix}.$$

At time $t = 2$ s, we have thus

$$\boldsymbol{q}_d(2) = \begin{pmatrix} \dfrac{\pi}{2} \\ 0 \end{pmatrix}, \qquad \dot{\boldsymbol{q}}_d(2) = \begin{pmatrix} -\dfrac{3\pi}{2} \\ 0 \end{pmatrix}, \qquad \ddot{\boldsymbol{q}}_d(2) = \begin{pmatrix} 0 \\ 4\pi^2 \end{pmatrix},$$

while the robot state and the position and velocity errors are

$$\boldsymbol{q}(2) = \begin{pmatrix} \dfrac{\pi}{2} \\ -\dfrac{\pi}{2} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{e}(2) = \begin{pmatrix} 0 \\ \dfrac{\pi}{2} \end{pmatrix}, \qquad \dot{\boldsymbol{q}}(2) = \dot{\boldsymbol{q}}_d(2) \quad \Rightarrow \quad \dot{\boldsymbol{e}}(2) = \boldsymbol{0}.$$

In this case, the only information needed in eq. (7) is the inertia matrix of the 2R robot. From the lecture slides, this matrix has the form

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2\cos q_2 & a_3 + a_2\cos q_2 \\ a_3 + a_2 \cos q_2 & a_3 \end{pmatrix},$$

[2]Performing computations by hand in the given sequence is surprisingly faster than setting up a similarly efficient code using symbolic programming!

with dynamic coefficients $a_i > 0$, $i = 1, 2, 3$. Using the PD gains given in the text, we finally obtain

$$\boldsymbol{\Delta u}(2) = (\boldsymbol{M}(\boldsymbol{q}(2)) - \boldsymbol{I}) \cdot \boldsymbol{K}_P\, \boldsymbol{e}(2) = \begin{pmatrix} a_1 - 1 & a_3 \\ a_3 & a_3 - 1 \end{pmatrix} \cdot 100 \begin{pmatrix} 0 \\ \dfrac{\pi}{2} \end{pmatrix} = 50\pi \cdot \begin{pmatrix} a_3 \\ a_3 - 1 \end{pmatrix}.$$

It is quite reasonably to assume that $a_3 = I_2 + m_2 d_{c2}^2 > 1$, being the robot arm of human-like size and weight. Thus, both components of $\boldsymbol{\Delta u}(2)$ are positive. However, to determine which controller is using the larger torques in absolute value at $t = 2$ s, we need to assess also the signs of the components of at least one of the two torque commands[3]. We evaluate then the Lyapunov-based tracking controller under the assumed conditions, obtaining

$$\boldsymbol{u}_{GLB}(2) = \boldsymbol{M}(\boldsymbol{q}(2))\, \ddot{\boldsymbol{q}}_d(2) + \boldsymbol{S}(\boldsymbol{q}(2), \dot{\boldsymbol{q}}_d(2))\, \dot{\boldsymbol{q}}_d(2) + \boldsymbol{g}(\boldsymbol{q}(2)) + \boldsymbol{K}_P\, \boldsymbol{e}(2).$$

For each term in the expression of $\boldsymbol{u}_{GLB}(2)$, the following can be easily observed:

$$\boldsymbol{M}(\boldsymbol{q}(2))\, \ddot{\boldsymbol{q}}_d(2) = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_3 \end{pmatrix} \begin{pmatrix} 0 \\ 4\pi^2 \end{pmatrix} = 4\pi^2 a_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} > \boldsymbol{0},$$

$$\begin{aligned} \boldsymbol{S}(\boldsymbol{q}(2), \dot{\boldsymbol{q}}_d(2))\, \dot{\boldsymbol{q}}_d(2) = \boldsymbol{c}(\boldsymbol{q}(2), \dot{\boldsymbol{q}}_d(2)) &= \begin{pmatrix} -a_2 \sin q_2(2) \left(\dot{q}_{d2}^2(2) - 2\, \dot{q}_{d1}(2) \dot{q}_{d2}(2)\right) \\ a_2 \sin q_2(2)\, \dot{q}_{d1}^2(2) \end{pmatrix} \\ &= -\frac{9\pi^2}{4}\, a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leq \boldsymbol{0}, \end{aligned}$$

$$\boldsymbol{g}(\boldsymbol{q}(2)) = \begin{pmatrix} a_4 \cos q_1(2) + a_5 \cos(q_1(2) + q_2(2)) \\ a_5 \cos(q_1(2) + q_2(2)) \end{pmatrix} = a_5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} > \boldsymbol{0},$$

$$\boldsymbol{K}_P\, \boldsymbol{e}(2) = 50 \begin{pmatrix} 0 \\ \pi \end{pmatrix} \geq \boldsymbol{0}.$$

Despite of the negative addend in the second component of the velocity term, it can be safely concluded that this single term is compensated by the multiple other positive ones, so that $\boldsymbol{u}_{GLB}(2) > \boldsymbol{0}$ holds componentwise. Thus, it is also $\boldsymbol{u}_{FBL}(2) = \boldsymbol{u}_{GLB}(2) + \boldsymbol{\Delta u}(2) > \boldsymbol{0}$ componentwise. For both two components, the feedback linearization law requires at $t = 2$ s a larger torque (in absolute value, but in fact positive) than the Lyapunov-based control law.

* * * * *

[3]Suppose that $\Delta u = a - b > 0$. If $b > 0$, both a and b will be positive and a is certainly larger than b. If instead $b < 0$, we could have both $|b| > |a|$ or viceversa in absolute value, and thus also the sign of a should be checked.

Robotics 2

January 12, 2021

Exercise #1

The 3R robot in Fig. 1 moves in a vertical plane. Its control architecture consists of an ideal low-level controller that is able to execute any (reasonable) reference joint velocity command $\dot{\boldsymbol{q}}_r \in \mathbb{R}^3$ (including $\dot{\boldsymbol{q}}_r = \mathbf{0}$), as received by a high-level control law. The assigned task requires that the end effector is kept at a desired position P_d in the plane, while the robot minimizes the potential energy $U_g(\boldsymbol{q})$ due to gravity. Design a suitable high-level control law that realizes this task in a robust way (i.e., rejecting also positioning errors for the end effector). Provide the detailed symbolic expression of all terms needed in this law. Compute then the numerical value of $\dot{\boldsymbol{q}}_r$ at the starting instant $t = 0$, assuming that the links have a uniform distribution of mass and using the data:

$$l_1 = 0.5, \quad l_2 = 0.4, \quad l_3 = 0.3 \quad [\text{m}], \qquad m_1 = 5, \quad m_2 = 3, \quad m_3 = 2 \quad [\text{kg}],$$
$$q_1(0) = \pi, \quad q_2(0) = 0, \quad q_3(0) = -\pi/2 \quad [\text{rad}] \qquad \Rightarrow \qquad P_d = (-0.9, 0.3).$$

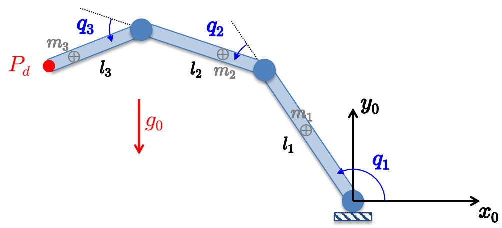


Figure 1: A planar 3R robot with a constant desired position P_d defined for its end effector.

Exercise #2

Consider again the situation of Exercise #1. Assume now that the robot is torque-controlled, namely that the control architecture is able to impose any (reasonable) reference joint torque command $\boldsymbol{\tau}_r \in \mathbb{R}^3$. Design a suitable torque-level control law that realizes the same previous task in a robust way (i.e., rejecting also position and/or velocity errors for the end effector). [*Note: You don't have to detail the expression of the terms in the control law, just define the structure.*]

Exercise #3

Derive the dynamic model of the 2R polar robot in Fig. 2, assuming that $I_{2,xx} \neq I_{2,yy} = I_{2,zz}$. Consider also the presence of viscous friction at the two joints.

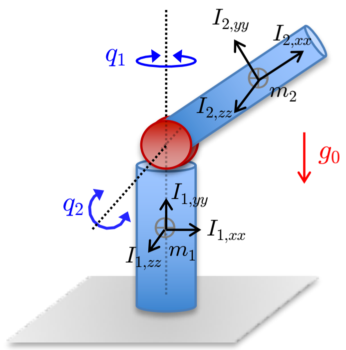


Figure 2: A 2R polar robot. Mass and diagonal barycentric inertia of each link is indicated.

Exercise #4

Consider again the robot of Exercise #3. Answer the following questions on its dynamics.

a) Suppose that the first robot joint is kept at a constant speed $\dot{q}_1 = \Omega > 0$ by a constant torque $\bar{\boldsymbol{\tau}}$ of *minimum* possible norm $\|\bar{\boldsymbol{\tau}}\|$ applied at the joints. Find an associated constant steady-state position $q_2 = \bar{q}_2$ and the expressions of $\bar{\tau}_1$ and $\bar{\tau}_2$. Is the joint angle $\bar{q}_2$ unique for a given $\Omega > 0$? And is the associated minimum norm torque $\bar{\boldsymbol{\tau}}$ unique?

b) Assuming that only the gravity acceleration g_0 and the kinematic parameters are known for this robot, what is the *minimum* number p of uncertain dynamic coefficients $\boldsymbol{a}$ that fully describe the robot dynamics in the linearly parametrized way $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{\tau}$? Give the symbolic expressions for the vector $\boldsymbol{a} \in \mathbb{R}^p$ and the $2 \times p$ matrix $\boldsymbol{Y}$.

Exercise #5

A PPR robot moving on a horizontal plane may collide with some unknown obstacle in the environment at an a priori unknown point P_c along its structure, as illustrated in Fig. 3. Suppose that a single collision occurs and that the interaction can be assumed as pointwise, modeled by an unknown and unmeasurable pure force $\boldsymbol{F}_{c,i} \in \mathbb{R}^2$ applied to the robot, respectively with $i = 1$, 2, or 3 according to which link is involved. Define the complete expression of the dynamic terms in a model-based residual vector $\boldsymbol{r} \in \mathbb{R}^3$ that can be used for collision detection and isolation. For every possible situation, analyze if the collision can be detected or not, if the colliding link can be isolated or not, if the colliding force $\boldsymbol{F}_{c,i}$ can be identified (completely or in part) or not, and if the location of the collision point $P_{c,i}$ along the ith link can be determined or not.

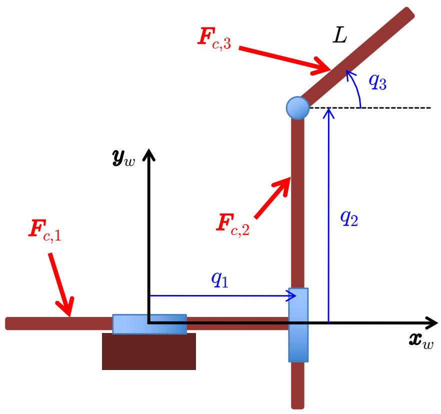


Figure 3: A planar PPR robot undergoing a possible collision during motion.

[240 minutes (4 hours); open books]

Solution

January 12, 2021

Exercise #1

In the absence of position error at the robot end effector level, a suitable high-level control law satisfying the requested self-motion task is obtained by projecting the negative gradient of the gravitational potential energy $U_g(\boldsymbol{q})$ in the null space of the task Jacobian $\boldsymbol{J}(\boldsymbol{q})$, or

r_dot = 0 since r is constant

$$\dot{\boldsymbol{q}}_r = -\alpha \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q}) \right) \nabla_{\boldsymbol{q}} U_g(\boldsymbol{q}) = \alpha \left(\boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q}) - \boldsymbol{I} \right) \boldsymbol{g}(\boldsymbol{q}). \tag{1}$$

The scalar $\alpha > 0$ is a step size in the anti-gradient direction $-\nabla_{\boldsymbol{q}} U_g = -\left(\partial U_g / \partial \boldsymbol{q}\right)^T$ and $\boldsymbol{g}(\boldsymbol{q})$ is the gravity term in the robot dynamic model[1]. To make the control law robust w.r.t. transient Cartesian position errors $\boldsymbol{e}_p = \boldsymbol{p}_d - \boldsymbol{p} = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})$, where $\boldsymbol{f}(\boldsymbol{q})$ is the (positional) task kinematics of the robot, we modify (1) as

$$\dot{\boldsymbol{q}}_r = \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{K}_P\boldsymbol{e}_p + \alpha \left(\boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q}) - \boldsymbol{I} \right) \boldsymbol{g}(\boldsymbol{q}) = -\alpha\, \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{J}^{\#}(\boldsymbol{q}) \left(\boldsymbol{K}_P\boldsymbol{e}_p + \alpha\, \boldsymbol{J}(\boldsymbol{q})\boldsymbol{g}(\boldsymbol{q}) \right), \tag{2}$$

where $\boldsymbol{K}_P > 0$ is a (typically, diagonal) control gain matrix. Out of singularities, this produces an exponentially converging error dynamics since $\dot{\boldsymbol{e}}_p = -\dot{\boldsymbol{p}} = -\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}_r = -\boldsymbol{K}_P\boldsymbol{e}_p$. In order to evaluate (2), we need the following terms (using the usual compact trigonometric notation):

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} l_1c_1 + l_2c_{12} + l_3c_{123} \\ l_1s_1 + l_2s_{12} + l_3s_{123} \end{pmatrix},$$

$$\boldsymbol{J}(\boldsymbol{q}) = \left(\frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} -(l_1s_1 + l_2s_{12} + l_3s_{123}) & -(l_2s_{12} + l_3s_{123}) & -l_3s_{123} \\ l_1c_1 + l_2c_{12} + l_3c_{123} & l_2c_{12} + l_3c_{123} & l_3c_{123} \end{pmatrix},$$

$$U_g(\boldsymbol{q}) = g_0 \left(m_1 d_{c1} s_1 + m_2(l_1s_1 + d_{c2}s_{12}) + m_3(l_1s_1 + l_2s_{12} + d_{c3}s_{123}) \right),$$

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U_g(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = g_0 \begin{pmatrix} (m_1d_{c1} + (m_2+m_3)l_1)\, c_1 + (m_2d_{c2} + m_3l_2)\, c_{12} + m_3d_{c3}\, c_{123} \\ (m_2d_{c2} + m_3l_2)\, c_{12} + m_3d_{c3}\, c_{123} \\ m_3d_{c3}\, c_{123} \end{pmatrix},$$

where $d_{ci} > 0$ is the distance of the center of mass of link i from the joint axis i, for $i = 1, 2, 3$. With the given data, it is $\boldsymbol{f}(\boldsymbol{q}(0)) = \begin{pmatrix} 0.9 & 0.3 \end{pmatrix}^T = \boldsymbol{p}_d$, so that $\boldsymbol{e}_p = \boldsymbol{0}$ and we don't need to know the actual value of $\boldsymbol{K}_p$. For the remaining terms in (2), we compute

$$\boldsymbol{J}(\boldsymbol{q}(0)) = \begin{pmatrix} -0.3 & -0.3 & -0.3 \\ -0.9 & -0.4 & 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{J}^{\#}(\boldsymbol{q}(0)) = \begin{pmatrix} 0.5464 & -1.1475 \\ -1.2295 & 0.0820 \\ -2.6503 & 1.0656 \end{pmatrix}$$

and

$$\boldsymbol{g}(\boldsymbol{q}(0)) = g_0 \begin{pmatrix} -5.15 \\ -1.4 \\ 0 \end{pmatrix} = \begin{pmatrix} -50.522 \\ -13.734 \\ 0 \end{pmatrix} \text{ [Nm]},$$

[1] Note that α converts here a joint torque into a joint velocity. Thus, it has dimensional units [rad$\cdot$(Nm$\cdot$s)$^{-1}$].

where $g_0 = 9.81$ [m/s^{-2}]. Setting for instance $\alpha = 1$, we finally obtain

$$\dot{\boldsymbol{q}}_r(0) = \begin{pmatrix} 2.5731 \\ -5.7895 \\ 3.2164 \end{pmatrix} \text{ [rad/s]}.$$

The first and third link start moving counterclockwise, while the second link will rotate clockwise. Robot motion will continue until the projection of the gravity term $\boldsymbol{g}(\boldsymbol{q})$ in the null space of the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ will vanish (in general, not implying that $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{0}$).

Exercise #2

When the robot is in a torque-controlled mode, its full dynamics

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

has to be taken into account. To address the same task as in Exercise #1 with torque control, we apply first a feedback linearization law in the joint space, i.e.,

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) \qquad \Rightarrow \qquad \ddot{\boldsymbol{q}} = \boldsymbol{a}. \tag{3}$$

HOW TO FIND $\ddot{q} = a$

$\ddot{r} = J\ddot{q} + \dot{J}\dot{q}$

$\ddot{q} = -J^{\#}(\dot{J}\dot{q})$

A joint acceleration command performing a robot self-motion, as driven by the negative gradient of the potential energy due to gravity[2], is then designed as

$$\boldsymbol{a} = -\boldsymbol{J}^{\#}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\left(\alpha\, \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_v\dot{\boldsymbol{q}}\right). \tag{4}$$

In (4), a damping velocity term $-\boldsymbol{K}_v\dot{\boldsymbol{q}}$, with $\boldsymbol{K}_v > 0$ and diagonal, has been added in the null space of the task Jacobian in order to stabilize the joint motion. This is customary (and almost mandatory) when resolving redundancy at the acceleration level.

For rejecting position and/or velocity errors that may occur around the desired constant end-effector position, the command (4) is modified as

$$\boldsymbol{a} = \boldsymbol{J}^{\#}(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{K}_D\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) - \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\left(\alpha\, \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_v\dot{\boldsymbol{q}}\right), \tag{5}$$

including thus a PD action, with (typically diagonal) gain matrices $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$, on the Cartesian position error, and taking into account that $\dot{\boldsymbol{p}}_d = \boldsymbol{0}$. Plugging (5) into (3) yields finally the desired torque control law

$$\begin{aligned} \boldsymbol{\tau}_r = \boldsymbol{M}(\boldsymbol{q}) \Big[& \boldsymbol{J}^{\#}(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{K}_D\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) \\ & - \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\left(\alpha\, \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_v\dot{\boldsymbol{q}}\right)\Big] + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}). \end{aligned} \tag{6}$$

Exercise #3

Kinetic energy

$$T_1 = \frac{1}{2}\, I_{1,yy}\, \dot{q}_1^2 \qquad\qquad T_2 = \frac{1}{2}\, m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2}\, \boldsymbol{\omega}_2^T \boldsymbol{I}_2 \boldsymbol{\omega}_2 = \frac{1}{2}\, m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2}\, {}^2\boldsymbol{\omega}_2^T\, {}^2\boldsymbol{I}_2\, {}^2\boldsymbol{\omega}_2$$

[2]In this case, α converts a joint torque into a joint acceleration. Thus, it has dimensional units [rad·Nm^{-1}·s^{-2}]. Similarly, the units of $\boldsymbol{K}_v$ are [s^{-1}].

$$\boldsymbol{p}_{c2} = \begin{pmatrix} d_{c2}\cos q_2 \cos q_1 \\ d_{c2}\cos q_2 \sin q_1 \\ d_{c2}\sin q_2 \end{pmatrix} = \begin{pmatrix} d_{c2}\, c_1 c_2 \\ d_{c2}\, s_1 c_2 \\ d_{c2}\, s_2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -d_{c2}(s_1 c_2\, \dot{q}_1 + c_1 s_2\, \dot{q}_2) \\ d_{c2}(c_1 c_2\, \dot{q}_1 - s_1 s_2\, \dot{q}_2) \\ d_{c2}\, c_2\, \dot{q}_2 \end{pmatrix}$$

$$\Rightarrow \quad \|\boldsymbol{v}_{c2}\|^2 = d_{c2}^2 \left(\dot{q}_2^2 + c_2^2 \dot{q}_1^2\right)$$

$${}^1\boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} \ \Rightarrow \ {}^1\boldsymbol{\omega}_2 = \begin{pmatrix} 0 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \ \Rightarrow \ {}^2\boldsymbol{\omega}_2 = {}^1\boldsymbol{R}_2^T(q_2)\, {}^1\boldsymbol{\omega}_2 = \begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} s_2\dot{q}_1 \\ c_2\dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

$$\Rightarrow \ T_2 = \frac{1}{2}\, m_2\, d_{c2}^2 \left(\dot{q}_2^2 + c_2^2\dot{q}_1^2\right) + \frac{1}{2} \begin{pmatrix} s_2\dot{q}_1 & c_2\dot{q}_1 & \dot{q}_2 \end{pmatrix} \begin{pmatrix} I_{2,xx} & 0 & 0 \\ 0 & I_{2,yy} & 0 \\ 0 & 0 & I_{2,zz} \end{pmatrix} \begin{pmatrix} s_2\dot{q}_1 \\ c_2\dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

$$= \frac{1}{2}\, m_2\, d_{c2}^2 \left(\dot{q}_2^2 + c_2^2\dot{q}_1^2\right) + \frac{1}{2}\, I_{2,zz}\, \dot{q}_2^2 + \frac{1}{2}\, \left(I_{2,xx} s_2^2 + I_{2,yy} c_2^2\right) \dot{q}_1^2$$

$$T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T_1 + T_2 = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

Inertia matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} I_{1,yy} + I_{2,xx}s_2^2 + \left(I_{2,yy} + m_2 d_{c2}^2\right) c_2^2 & 0 \\ 0 & I_{2,zz} + m_2 d_{c2}^2 \end{pmatrix} \tag{7}$$

$$= \begin{pmatrix} I_{1,yy} + I_{2,xx} + \left(I_{2,yy} + m_2 d_{c2}^2 - I_{2,xx}\right) c_2^2 & 0 \\ 0 & I_{2,zz} + m_2 d_{c2}^2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 + a_2\, c_2^2 & 0 \\ 0 & a_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{m}_1(q_2) & \boldsymbol{m}_2 \end{pmatrix}$$

Potential energy and gravity vector

$$U_1 = 0 \qquad U_2 = g_0 m_2 d_{c2} s_2 \qquad U(\boldsymbol{q}) = U_1 + U_2$$

$$\Rightarrow \ \boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} 0 \\ g_0 m_2 d_{c2} c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ a_4\, c_2 \end{pmatrix}$$

Coriolis and centrifugal vector

$$\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{m}_1}{\partial \boldsymbol{q}}\right) + \left(\frac{\partial \boldsymbol{m}_1}{\partial \boldsymbol{q}}\right)^T - \left(\frac{\partial \boldsymbol{M}}{\partial q_1}\right)\right) = \begin{pmatrix} 0 & -a_2 s_2 c_2 \\ -a_2 s_2 c_2 & 0 \end{pmatrix}$$

$$c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \dot{\boldsymbol{q}} = -2\, a_2 s_2 c_2\, \dot{q}_1 \dot{q}_2 = -a_2 \sin(2q_2) \dot{q}_1 \dot{q}_2$$

$$\boldsymbol{C}_2(\boldsymbol{q}) = \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{m}_2}{\partial \boldsymbol{q}}\right) + \left(\frac{\partial \boldsymbol{m}_2}{\partial \boldsymbol{q}}\right)^T - \left(\frac{\partial \boldsymbol{M}}{\partial q_2}\right)\right) = \begin{pmatrix} a_2 s_2 c_2 & 0 \\ -0 & 0 \end{pmatrix}$$

$$c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \dot{\boldsymbol{q}} = a_2 s_2 c_2\, \dot{q}_1^2$$

$$\Rightarrow \ \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -2\, a_2 s_2 c_2\, \dot{q}_1 \dot{q}_2 \\ a_2 s_2 c_2\, \dot{q}_1^2 \end{pmatrix}$$

Dynamic model (including viscous friction)

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}.\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{F}\dot{\boldsymbol{q}} = \boldsymbol{\tau} \quad \Longleftrightarrow \quad \begin{cases} \left(a_1 + a_2\, c_2^2\right) \ddot{q}_1 - 2\, a_2 s_2 c_2\, \dot{q}_1 \dot{q}_2 + f_1 \dot{q}_1 = \tau_1 \\ a_3 \ddot{q}_2 + a_2 s_2 c_2\, \dot{q}_1^2 + a_4\, c_2 + f_2 \dot{q}_2 = \tau_2 \end{cases} \tag{8}$$

Exercise #4

a) With reference to eqs. (8), set $\dot{q}_1 = \Omega > 0$, $\ddot{q}_1 = 0$, $q_2 = \bar{q}_2$, $\dot{q}_2 = \ddot{q}_2 = 0$ for the desired steady state. We get

$$f_1 \Omega = \tau_1 \tag{9}$$

$$a_2 \sin \bar{q}_2 \cos \bar{q}_2\, \Omega^2 + a_4 \cos \bar{q}_2 = \tau_2 \tag{10}$$

From (9), the torque on joint 1 should necessarily compensate the loss of energy due to viscous friction when the joint is rotating with a speed Ω, so $\bar{\tau}_1 = f_1 \Omega$. Because of the minimum norm requirement for the steady-state torque $\bar{\boldsymbol{\tau}}$, we seek then a solution $\bar{q}_2$ in (10) for $\tau_2 = \bar{\tau}_2 = 0$. This is possible in two cases.

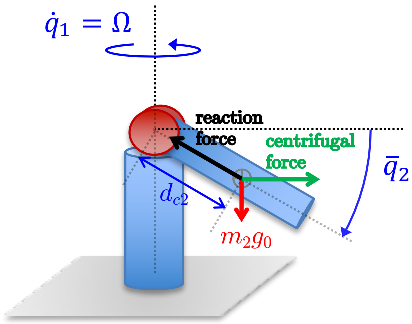


Figure 4: Equilibrium of forces for a polar robot spinning its first joint at a constant $\dot{q}_1 = \Omega > 0$.

- For $\bar{q}_2 = \pm\pi/2$ (second link vertical, up or down along the first joint axis). Indeed, $\cos \bar{q}_2 = 0$ and (10) will be an identity with zero applied torque. However, these equilibria are both unstable: small perturbations to this steady-state condition will let the second robot joint deviate from any of these two configurations.

- When $a_2 \sin \bar{q}_2\, \Omega^2 + a_4 = 0$. This corresponds to a special balancing between the vector sum of the gravity force (pointing vertically and downward) and centrifugal force (pointing horizontally and radially from the first joint axis), both applied to the center of mass of link 2, and the internal reaction force by the rigid robot structure (see Fig. 4). This balance is obtained for

$$\bar{q}_2 = \arcsin \frac{-a_4}{a_2\, \Omega^2} = -\arcsin \frac{g_0 m_2 d_{c2}}{\left(I_{2,yy} + m_2 d_{c2}^2 - I_{2,xx}\right) \Omega^2} \in \left(-\frac{\pi}{2}, 0\right). \tag{11}$$

  The domain of definition for $\bar{q}_2$ follows from $d_{c2} > 0$ and from the fact that $I_{2,yy} + m_2 d_{c2}^2 > I_{2,xx}$ always holds, namely that the inertia of link 2 around an axis belonging to its base is less than the baricentral inertia around an axis stretching along the link length. Such inequality can be proven for any rigid body, no matter what mass it has and how the mass is distributed in the body volume[3]. For low values of Ω, the angle in (11) would be close to $-\pi/2$; for large values of Ω, we have instead $\bar{q}_2 \to 0^-$. It can be shown that this equilibrium is dynamically stable.

[3]Triangular inequalities hold among the elements of a diagonal barycentric inertia matrix of a rigid body (or for the elements on its principal axes), such as $I_{yy} + I_{zz} > I_{xx}$. In addition, the inertia I around any barycentric axis of a body of mass m is smaller than md^2, where d is the distance from the CoM to a parallel axis at the body end. These two physical properties together prove the inequality in the text.

In any case, we note that the (minimum norm) torque for this dynamic equilibrium will be the same:

$$\bar{\boldsymbol{\tau}} = \begin{pmatrix} f_1 \Omega \\ 0 \end{pmatrix}, \qquad \|\bar{\boldsymbol{\tau}}\| = f_1 \, |\Omega| \, .$$

Indeed, when considering also a constant torque $\bar{\tau}_2 \neq 0$ applied at the second joint (and thus, a total torque with $\|\bar{\boldsymbol{\tau}}\| > f_1 \, |\Omega|$), we may find other steady-state solutions for $\bar{q}_2$.

b) The minimum number of dynamic coefficients required is $p = 6$. From eqs. (8), one has

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 & c_2^2 \ddot{q}_1 - 2 s_2 c_2 \, \dot{q}_1 \dot{q}_2 & 0 & 0 & \dot{q}_1 & 0 \\ 0 & s_2 c_2 \, \dot{q}_1^2 & \ddot{q}_2 & c_2 & 0 & \dot{q}_2 \end{pmatrix}, \qquad \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} I_{1,yy} + I_{2,xx} \\ I_{2,yy} + m_2 d_{c2}^2 - I_{2,xx} \\ I_{2,zz} + m_2 d_{c2}^2 \\ g_0 m_2 d_{c2} \\ f_1 \\ f_2 \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}} . \ddot{\boldsymbol{q}}) \, \boldsymbol{a} = \boldsymbol{\tau}.$$

A comment is in place on the alternative definition of coefficients in (7), before introducing the trigonometric substitution $s_2^2 = 1 - c_2^2$. When $I_{2,yy} = I_{2,zz}$, as in the present case, only three independent dynamic coefficients would appear anyway in the inertia matrix, although with the different definitions

$$a_1' = I_{1,yy}, \quad a_2' = I_{2,xx}, \quad a_3' = I_{2,yy} + m_2 d_{c2}^2 = I_{2,zz} + m_2 d_{c2}^2 = a_3,$$

whereas $a_i' = a_i$ for the remaining $i = 4, 5, 6$. Although the associated regressor matrix $\boldsymbol{Y}'$ would look slightly different, both parametrizations are minimal ($p = 6$). On the other hand, this would no longer be true for the parametrization suggested by (7) in case $I_{2,yy} \neq I_{2,zz}$: 4 dynamic coefficients would then be used in $\boldsymbol{M}$, leading to a total of 7 coefficients in the dynamic model.

Exercise #5

We shall derive first the terms in the dynamic model of the planar PPR robot in Fig. 3, following a Lagrangian approach. In the absence of gravity, we have

$$\boldsymbol{M}(\boldsymbol{q}) \ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \dot{\boldsymbol{q}} = \boldsymbol{\tau} + \boldsymbol{\tau}_c, \qquad \boldsymbol{\tau}_c = \boldsymbol{J}_c^T(\boldsymbol{q}) \boldsymbol{F}_c,$$

where $\boldsymbol{S}\dot{\boldsymbol{q}}$ is a factorization of the quadratic velocity terms such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is a skew-symmetric matrix (or, equivalently, $\dot{\boldsymbol{M}} = \boldsymbol{S} + \boldsymbol{S}^T$), $\boldsymbol{\tau}_c$ is the joint torque resulting from a collision with a force $\boldsymbol{F}_c$, and $\boldsymbol{J}_c$ is the Jacobian of the collision point along the structure.

The kinetic energy of the first two links, moved by two prismatic joints with orthogonal axes, is

$$T_1 + T_2 = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \left(\dot{q}_1^2 + \dot{q}_2^2 \right).$$

From

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_1 + d_{c3} c_3 \\ q_2 + d_{c3} s_3 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} \dot{q}_1 - d_{c3} s_3 \dot{q}_3 \\ \dot{q}_2 + d_{c3} c_3 \dot{q}_3 \end{pmatrix}, \qquad \omega_3 = \dot{q}_3,$$

the kinetic energy of the third (rotational) link is computed as

$$T_3 = \frac{1}{2} m_3 \|\boldsymbol{v}_{c3}\|^2 + \frac{1}{2} I_3 \, \omega_3^2 = \frac{1}{2} m_3 \left(\dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2 \dot{q}_3^2 + 2 \, d_{c3} (-s_3 \dot{q}_1 + c_3 \dot{q}_2) \dot{q}_3 \right) + \frac{1}{2} I_3 \dot{q}_3^2.$$

From the kinetic energy of the system,

$$T = T_1 + T_2 + T_3 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

the inertia matrix is extracted as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1 + m_2 + m_3 & 0 & -m_3 d_{c3} s_3 \\ 0 & m_2 + m_3 & m_3 d_{c3} c_3 \\ -m_3 d_{c3} s_3 & m_3 d_{c3} c_3 & I_3 + m_3 d_{c3}^2 \end{pmatrix}.$$

The (purely) centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ are derived using the Christoffel symbols, i.e., for each component

$$c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial \boldsymbol{q}_i}\right), \qquad i = 1, 2, 3$$

being $\boldsymbol{m}_i$ the ith column of the inertia matrix $\boldsymbol{M}$. We obtain

$$\begin{aligned}
\boldsymbol{C}_1(\boldsymbol{q}) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m_3 d_{c3} c_3 \end{pmatrix} & \Rightarrow \qquad & c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 d_{c3} c_3 \dot{q}_3^2 \\
\boldsymbol{C}_2(\boldsymbol{q}) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m_3 d_{c3} s_3 \end{pmatrix} & \Rightarrow \qquad & c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -m_3 d_{c3} s_3 \dot{q}_3^2 \\
\boldsymbol{C}_3(\boldsymbol{q}) &= \boldsymbol{0} & \Rightarrow \qquad & c_3(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0,
\end{aligned}$$

and thus

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -m_3 d_{c3} c_3 \dot{q}_3^2 \\ -m_3 d_{c3} s_3 \dot{q}_3^2 \\ 0 \end{pmatrix}.$$

Using again the Christoffel symbols, a suitable factorization matrix for $\boldsymbol{c}$ is computed as

$$\boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_3(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -m_3 d_{c3} c_3 \dot{q}_3 \\ 0 & 0 & -m_3 d_{c3} s_3 \dot{q}_3 \\ 0 & 0 & 0 \end{pmatrix},$$

The model-based residual for collision detection and isolation can then be evaluated in all its terms, namely

$$\boldsymbol{r}(t) = \boldsymbol{K}_I \left(\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} - \int_0^t \left(\boldsymbol{\tau} + \boldsymbol{S}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{r}\right) ds\right), \qquad \boldsymbol{K}_I > 0, \tag{12}$$

where we have assumed that $\dot{\boldsymbol{q}}(0) = \boldsymbol{0}$ (the robot starts at rest). In particular, we have for the second term in the integral

$$\boldsymbol{S}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{pmatrix} 0 \\ 0 \\ -m_3 d_{c3} \left(s_3 \dot{q}_2 + c_3 \dot{q}_1\right) \dot{q}_3 \end{pmatrix}.$$

The residual $\boldsymbol{r}$ in (12) is affected, through the joint torque $\boldsymbol{\tau}_{ci}$, by a collision force $\boldsymbol{F}_{ci} = \begin{pmatrix} F_x & F_y \end{pmatrix}^T$ acting at the collision point P_{ci} on the robot link i as

$$\dot{\boldsymbol{r}} = \boldsymbol{K}_I\left(\boldsymbol{\tau}_{ci} - \boldsymbol{r}\right) = \boldsymbol{K}_I\left(\boldsymbol{J}_{ci}^T(\boldsymbol{q})\boldsymbol{F}_{ci} - \boldsymbol{r}\right), \qquad i = 1, 2, 3,$$

except for some singular cases. Essentially, there are directions along which the point P_{ci} cannot be given a linear instantaneous velocity in the motion plane by means of a joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$. Next, we shall distinguish between collisions on the first, second, or third link and analyze the various possible situations in terms of collision detection and isolation, as well as collision force identification and localization. Assuming that sufficiently large gains can be chosen in the diagonal matrix $\boldsymbol{K}_I$, we will have

$$\boldsymbol{r} \approx \boldsymbol{\tau}_{ci},$$

and the residual $\boldsymbol{r}$ (in particular, its components r_i) can be used as a proxy for $\boldsymbol{\tau}_{ci}$ when reasoning about the nature of collisions. In the following, all quantities will be expressed in the world frame RF_w of Fig. 3.

- **Collision on link 1**. The position of the collision point along the first link[4] and the associated Jacobian and joint torque are

$$\boldsymbol{p}_{c1} = \begin{pmatrix} q_1 - \rho_1 \\ 0 \end{pmatrix}, \text{ with } \rho_1 \in [0, 2l_{1,max}] \;\Rightarrow\; \boldsymbol{J}_{c1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;\Rightarrow\; \boldsymbol{\tau}_{c1} = \boldsymbol{J}_{c1}^T\boldsymbol{F}_{c1} = \begin{pmatrix} F_x \\ 0 \\ 0 \end{pmatrix}.$$

we don't
have rho
in tau
so we can't
find the
collision point

  Clearly, a collision is not detected at all when

$$\boldsymbol{F}_{c1}^0 = \begin{pmatrix} 0 \\ F_y \end{pmatrix} \quad\Rightarrow\quad \boldsymbol{J}_{c1}^T\boldsymbol{F}_{c1}^0 = \boldsymbol{0}.$$

  Only the intensity F_x of $\boldsymbol{F}_{c1}$ can be identified. The closer is the alignment of $\boldsymbol{F}_{c1}$ to the axis of joint 1, the poorer will be the detection. Moreover, we will never have an information on the localization of the collision point P_{c1}.

- **Collision on link 2**. The position of the contact point along the second link and the associated Jacobian and joint torque are

$$\boldsymbol{p}_{c2} = \begin{pmatrix} q_1 \\ q_2 - \rho_2 \end{pmatrix}, \text{ with } \rho_2 \in [0, 2l_{2,max}] \;\Rightarrow\; \boldsymbol{J}_{c2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \;\Rightarrow\; \boldsymbol{\tau}_{c2} = \boldsymbol{J}_{c2}^T\boldsymbol{F}_{c2} = \begin{pmatrix} F_x \\ F_y \\ 0 \end{pmatrix}.$$

we don't
have rho
in tau
so we can't
find the
collision
point

  Thus, the collision will always be detected and the collision force $\boldsymbol{F}_{c2}$ fully identified. However, once again, no information on the localization of the collision point P_{c2} is provided by the residual. This is also critical for the isolation of the actual link in collision. In fact, when $F_y = 0$ in $\boldsymbol{F}_{c2}$, we obtain $\boldsymbol{\tau}_{c2} = \boldsymbol{\tau}_{c1}$ and there will be no way to understand whether the collision occurred on link 1 (with the force intensity being possibly identified only in part) or on link 2 (with direction and intensity of the collision force being fully identified).

- **Collision on link 3**. The position of the contact point along the third link and the associated Jacobian and joint torque are

$$\boldsymbol{p}_{c3} = \begin{pmatrix} q_1 + \rho_3\, c_3 \\ q_2 + \rho_3\, s_3 \end{pmatrix}, \quad \text{with } \rho_3 \in [0, L] \quad\Rightarrow\quad \boldsymbol{J}_{c3}(\boldsymbol{q}) = \begin{pmatrix} 1 & 0 & -\rho_3\, s_3 \\ 0 & 1 & \rho_3\, c_3 \end{pmatrix}$$

[4] For simplicity, assume that the prismatic joints have limited excursions, i.e., $q_i \in [-l_{i,max}, l_{i,max}]$, for $i = 1, 2$.

$$\Rightarrow \quad \boldsymbol{\tau}_{c3} = \boldsymbol{J}_{c3}^T(\boldsymbol{q})\boldsymbol{F}_{c3} = \begin{pmatrix} F_x \\ F_y \\ \rho_3 \left(c_3 F_y - s_3 F_x\right) \end{pmatrix}.$$

The first two components of $\boldsymbol{\tau}_{c3}$ (in practice, of its proxy $\boldsymbol{r}$) show again that the collision is always detected, and that the collision force $\boldsymbol{F}_{c3}$ is fully identified as well. In this case, localization of the actual collision point P_{c3} (i.e., the value ρ_3) is also possible, provided that the third component $\tau_{c3,3} \neq 0$. In fact, we can estimate then ρ_3 as

$$\rho_3 = \frac{\tau_{c3,3}}{c_3 F_y - s_3 F_x} = \frac{\tau_{c3,3}}{c_3 \tau_{c3,2} - s_3 \tau_{c3,1}} \approx \frac{r_3}{c_3 r_2 - s_3 r_1} = \hat{\rho}_3.$$

Such localization will fail when

$$\boldsymbol{F}_{c3} = \|\boldsymbol{F}_{c3}\| \begin{pmatrix} c_3 \\ s_3 \end{pmatrix} \qquad \Rightarrow \qquad \tau_{c3,3} = 0,$$

namely $\boldsymbol{F}_{c3}$ is aligned with the third link. Moreover, in this situation we obtain $\boldsymbol{\tau}_{c3} = \boldsymbol{\tau}_{c2}$ and also the isolation of the actual link in collision will fail. In fact, we cannot distinguish between a collision occurred on link 2 or 3. The same happens when a force $\boldsymbol{F}_{c3}$ hits the third link at its base (being then ρ_3).

* * * * *

Robotics 2

February 4, 2021

Exercise #1

Consider the RRPR robot in Fig. 1, where all relevant kinematic and dynamic parameters are also shown. The robot moves in a vertical plane. Compute the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ and the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ in the Lagrangian dynamic model of this robot. Provide a linear factorization of each term, $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{Y}_M(\boldsymbol{q},\ddot{\boldsymbol{q}})\boldsymbol{a}_M$ and $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}_g(\boldsymbol{q})\boldsymbol{a}_g$, introducing dynamic coefficients $\boldsymbol{a}_m \in \mathbb{R}^{p_m}$ and $\boldsymbol{a}_g \in \mathbb{R}^{p_g}$. Find also all open-loop equilibrium configurations $\boldsymbol{q}_e$, i.e., such that $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$.

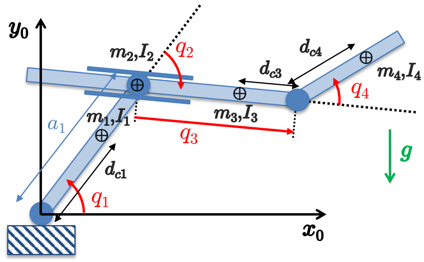


Figure 1: A 4-dof RRPR robot: generalized coordinates, kinematic and dynamic parameters.

Exercise #2

Consider the situation depicted in Fig. 2. The two Cartesian robots A and B are commanded by motors and transmissions that generate linear forces $\boldsymbol{\tau}_A \in \mathbb{R}^2$ and $\boldsymbol{\tau}_B \in \mathbb{R}^2$ along the axes of their prismatic joints. Each control force component is bounded as

$$|\tau_{A,i}| \leq \tau_{A,max}, \quad i = 1,2, \qquad |\tau_{B,i}| \leq \tau_{B,max}, \quad i = 1,2.$$

The two robots hold firmly a payload mass m_p and cooperate in moving the mass in minimum time along a horizontal linear path $\boldsymbol{p} = \boldsymbol{p}(s)$, from point P_{in} to point P_{fin} in a rest-to-rest mode. In addition, it is desired that the sum $H = \|\boldsymbol{\tau}_A\|^2 + \|\boldsymbol{\tau}_B\|^2$ is always minimized instantaneously. Motion occurs in the vertical plane, and we assume that motors are powerful enough to sustain the weight of the respective robot and of the payload. Provide a dynamic model of this cooperating task and determine the optimal time profiles of the four commands $\boldsymbol{\tau}_A$ and $\boldsymbol{\tau}_B$.

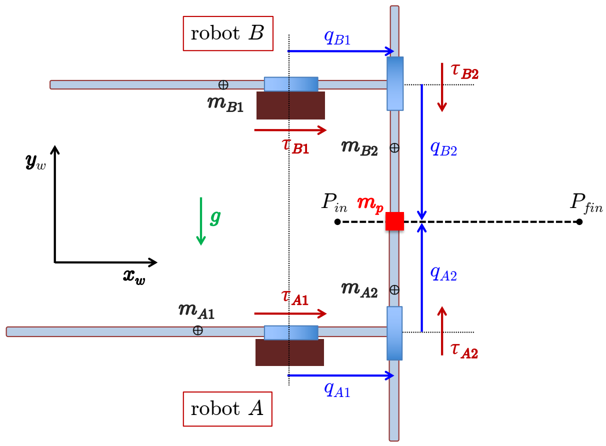


Figure 2: Two Cartesian robots that move a mass m_p under gravity along a linear horizontal path.

Exercise #3

The planar 3R robot shown in Fig. 3 is commanded by joint torques $\boldsymbol{\tau} \in \mathbb{R}^3$ that use feedback from the current state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$. The robot is initially at rest in the configuration $\boldsymbol{q}_{in} = (-\pi/9,\ 11\pi/18,\ -\pi/4)$ and should then perform a self-motion so as to guarantee that the third joint asymptotically reaches the final value $q_{3,fin} = -\pi/2$, while keeping the position of its end-effector always at the same initial point P_{in}. Design a torque control scheme that completes this task in a robust way, i.e., by rejecting transient position and/or velocity errors and without encountering any singular situation for the control law. *Hint: Use an approach based on joint space decomposition.*

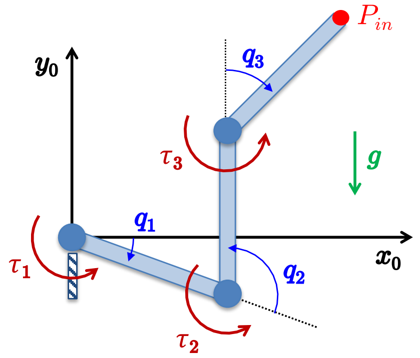


Figure 3: A 3R robot that should perform a self-motion task with constant end-effector position.

Exercise #4

Figure 4 shows a simple 1-dof model of the interaction between a robot of mass $m_r > 0$, commanded by a force F, and a rigid environment, with a force sensor of stiffness $k_s > 0$ measuring the contact force F_c. The two coefficients $b_r > 0$ and $b_s > 0$ represent, respectively, the viscous friction affecting robot motion and the viscous damping of the force sensor. The reference position $x_r = 0$ is when the robot mass is in contact with $F_c = 0$. Provide the dynamic model of this system assuming linearity of all effects. Based only on the measured force, design a control law for F that is able to regulate asymptotically the contact force F_c to a desired constant value $F_d > 0$, despite uncertainty in all model parameters. Provide the associated steady-state values of $F = \bar{F}$ and $x_r = \bar{x}_r$.

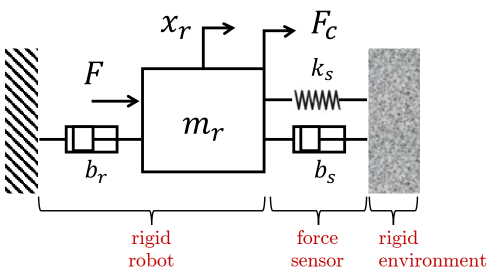


Figure 4: A 1-dof model of interaction between a robot and a rigid environment.

[240 minutes (4 hours); open books]

Solution

February 4, 2021

Exercise #1

The dynamic modeling steps and the requested linear factorizations of terms are quite standard procedures.[1] They are sketched hereafter for the given RRPR planar robot without further comments. We assume that the kinematic parameter a_1 ([m]) and the gravity acceleration $g_0 = 9.81$ [m/s^2] are accurately known. We finally evaluate the open-loop equilibria of the robot under gravity.

Kinetic energy

$$T_1 = \frac{1}{2}\left(I_1 + m_1 d_{c1}^2\right)\dot{q}_1^2 \qquad T_2 = \frac{1}{2}\, m_2 a_1^2 \dot{q}_1^2 + \frac{1}{2} I_2 \left(\dot{q}_1 + \dot{q}_2\right)^2$$

$$T_3 = \frac{1}{2}\, m_3 \|\boldsymbol{v}_{c3}\|^2 + \frac{1}{2}\, I_3 \left(\dot{q}_1 + \dot{q}_2\right)^2 \qquad T_4 = \frac{1}{2}\, m_4 \|\boldsymbol{v}_{c4}\|^2 + \frac{1}{2}\, I_4 \left(\dot{q}_1 + \dot{q}_2 + \dot{q}_4\right)^2$$

$$\boldsymbol{p}_{c3} = \begin{pmatrix} a_1 \cos q_1 + \cos\left(q_1 + q_2\right)\left(q_3 - d_{c3}\right) \\ a_1 \sin q_1 + \sin\left(q_1 + q_2\right)\left(q_3 - d_{c3}\right) \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} -\left(a_1 s_1 + s_{12}(q_3 - d_{c3})\right)\dot{q}_1 - s_{12}(q_3 - d_{c3})\dot{q}_2 + c_{12}\dot{q}_3 \\ \left(a_1 c_1 + c_{12}(q_3 - d_{c3})\right)\dot{q}_1 + c_{12}(q_3 - d_{c3})\dot{q}_2 + s_{12}\dot{q}_3 \end{pmatrix}$$

$$\boldsymbol{p}_{c4} = \begin{pmatrix} a_1 \cos q_1 + q_3 \cos\left(q_1 + q_2\right) + d_{c4} \cos\left(q_1 + q_2 + q_4\right) \\ a_1 \sin q_1 + q_3 \sin\left(q_1 + q_2\right) + d_{c4} \sin\left(q_1 + q_2 + q_4\right) \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{v}_{c4} = \dot{\boldsymbol{p}}_{c4} = \begin{pmatrix} -\left(a_1 s_1 + q_3 s_{12} + d_{c4} s_{124}\right)\dot{q}_1 - \left(q_3 s_{12} + d_{c4} s_{124}\right)\dot{q}_2 + c_{12}\,\dot{q}_3 - d_{c4} s_{124}\,\dot{q}_4 \\ \left(a_1 c_1 + q_3 c_{12} + d_{c4} c_{124}\right)\dot{q}_1 + \left(q_3 c_{12} + d_{c4} c_{124}\right)\dot{q}_2 + s_{12}\,\dot{q}_3 + d_{c4} c_{124}\,\dot{q}_4 \end{pmatrix}$$

$$T(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T_1 + T_2 + T_3 + T_4 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Inertia matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ \vdots & m_{22} & m_{23} & m_{24} \\ \vdots & \ddots & m_{33} & m_{34} \\ \text{symm} & \cdots & \cdots & m_{44} \end{pmatrix}$$

$$\begin{aligned}
m_{11} &= I_1 + m_1 d_{c1}^2 + I_2 + I_3 + m_3 d_{c3}^2 + I_4 + m_4 d_{c4}^2 + \left(m_2 + m_3 + m_4\right) a_1^2 + \left(m_3 + m_4\right) q_3^2 - 2\, m_3 d_3\, q_3 \\
&\quad - 2\, m_3 d_{c3}\, a_1 \cos q_2 + 2\left(m_3 + m_4\right) a_1 q_3 \cos q_2 + 2\, m_4 d_{c4}\left(a_1 \cos(q_2 + q_4) + q_3 \cos q_4\right) \\
m_{12} &= I_2 + I_3 + m_3 d_{c3}^2 + I_4 + m_4 d_{c4}^2 + \left(m_3 + m_4\right) q_3^2 - 2\, m_3 d_{c3}\, q_3 \\
&\quad - m_3 d_{c3}\, a_1 \cos q_2 + \left(m_3 + m_4\right) a_1 q_3 \cos q_2 + m_4 d_{c4}\left(a_1 \cos(q_2 + q_4) + 2\, q_3 \cos q_4\right) \\
m_{13} &= \left(m_3 + m_4\right) a_1 \sin q_2 - m_4 d_{c4} \sin q_4 \\
m_{14} &= I_4 + m_4 d_{c4}^2 + m_4 d_{d4}\left(a_1 \cos(q_2 + q_4) + q_3 \cos q_4\right) \\
m_{22} &= I_2 + I_3 + m_3 d_{c3}^2 + I_4 + m_4 d_{c4}^2 + \left(m_3 + m_4\right) q_3^2 - 2\, m_3 d_{c3}\, q_3 + 2\, m_4 d_{c4}\, q_3 \cos q_4 \\
m_{23} &= -m_3 d_{c3} \sin q_4 \\
m_{24} &= I_4 + m_4 d_{c4}^2 + m_4 d_{c4}\, q_3 \cos q_4 \\
m_{33} &= m_3 + m_4 \\
m_{34} &= -m_4 d_{c4} \sin q_4 \\
m_{44} &= I_4 + m_4 d_{c4}^2
\end{aligned}$$

[1] Note that not all the generalized coordinates in Fig. 1 correspond to the joint variables in the DH convention.

Linear parametrization of the inertia term

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{Y}_M(\boldsymbol{q},\ddot{\boldsymbol{q}})\,\boldsymbol{a}_M, \qquad \boldsymbol{a}_M \in \mathbb{R}^6$$

$$\begin{aligned}
m_{11} &= a_{M1} + a_{M4}\, q_3^2 - 2\, a_{M3}\, q_3 + 2\,(a_{M4}\, q_3 - a_{M3})\, a_1\, c_2 + 2\, a_{M5}\,(a_1 c_{24} + q_3 c_4)\\
m_{12} &= a_{M2} + a_{M4}\, q_3^2 - 2\, a_{M3}\, q_3 + (a_{M4}\, q_3 - a_{M3})\, a_1\, c_2 + a_{M5}\,(a_1 c_{24} + 2\, q_3\, c_4)\\
m_{13} &= a_{M4}\, a_1\, s_2 - a_{M5}\, s_4\\
m_{14} &= a_{M6} + a_{M5}\,(a_1\, c_{24} + q_3\, c_4)\\
m_{22} &= a_{M2} + a_{M4}\, q_3^2 - 2\, a_{M3}\, q_3 + 2\, a_{M5}\, q_3\, c_4\\
m_{23} &= -a_{M3}\, s_4\\
m_{24} &= a_{M6} + a_{M5}\, q_3\, c_4\\
m_{33} &= a_{M4}\\
m_{34} &= -a_{M5}\, s_4\\
m_{44} &= a_{M6}
\end{aligned}$$

$$\boldsymbol{a}_M = \begin{pmatrix} a_{M1}\\ a_{M2}\\ a_{M3}\\ a_{M4}\\ a_{M5}\\ a_{M6}\end{pmatrix} = \begin{pmatrix} I_1 + m_1 d_{c1}^2 + I_2 + I_3 + m_3 d_{c3}^2 + I_4 + m_4 d_{c4}^2 + (m_2 + m_3 + m_4)\, a_1^2\\ I_2 + I_3 + m_3 d_{c3}^2 + I_4 + m_4 d_{c4}^2\\ m_3 d_{c3}\\ m_3 + m_4\\ m_4 d_{c4}\\ I_4 + m_4 d_{c4}^2 \end{pmatrix}$$

$$\boldsymbol{Y}_M(\boldsymbol{q},\ddot{\boldsymbol{q}}) =$$

$$\begin{pmatrix}
\ddot{q}_1 & \ddot{q}_2 & \begin{matrix}-2\,(q_3 + a_1 s_2)\,\ddot{q}_1\\ -(2\, q_3 + a_1 s_2)\,\ddot{q}_2\end{matrix} & \begin{matrix}\left(q_3^2 + 2\, q_3\, a_1 c_2\right)\ddot{q}_1\\ +\left(q_3^2 + q_3\, a_1 c_2\right)\ddot{q}_2\\ +a_1 s_2\, \ddot{q}_3\end{matrix} & \begin{matrix}2\,(a_1 c_{24} + q_3 c_4)\,\ddot{q}_1\\ +(a_1 c_{24} + 2\, q_3\, c_4)\,\ddot{q}_2\\ -s_4\,\ddot{q}_3\,(a_1\, c_{24} + q_3\, c_4)\,\ddot{q}_4\end{matrix} & \ddot{q}_4\\[2ex]
0 & \ddot{q}_1 + \ddot{q}_2 & \begin{matrix}\left(q_3^2 + q_3\, a_1 c_2\right)\ddot{q}_1\\ -2\, q_3\, \ddot{q}_2\end{matrix} & \begin{matrix}\left(q_3^2 + q_3\, a_1 c_2\right)\ddot{q}_1\\ +q_3^2\, \ddot{q}_2\end{matrix} & \begin{matrix}(a_1 c_{24} + 2\, q_3\, c_4)\,\ddot{q}_1\\ +q_3\, c_4\,(2\,\ddot{q}_2 + \ddot{q}_4)\end{matrix} & \ddot{q}_4\\[2ex]
0 & 0 & -s_3 \ddot{q}_2 & a_1 s_2\, \ddot{q}_1 + \ddot{q}_3 & -s_4\,(\ddot{q}_1 + \ddot{q}_4) & 0\\[2ex]
0 & 0 & 0 & 0 & \begin{matrix}(a_1\, c_{24} + q_3\, c_4)\,\ddot{q}_1\\ +q_3\, c_4\, \ddot{q}_2 - s_4\, \ddot{q}_3\end{matrix} & \ddot{q}_1 + \ddot{q}_2 + \ddot{q}_4
\end{pmatrix}$$

Potential energy

$$U_1 = m_1 d_{c1} g_0 \sin q_1 \qquad\qquad U_2 = m_2 a_1 g_0 \sin q_1$$

$$U_3 = m_3 g_0\,(a_1 \sin q_1 + (q_3 - d_{c3})\sin(q_1 + q_2))$$

$$U_4 = m_4 g_0\,(a_1 \sin q_1 + q_3 \sin(q_1 + q_2) + d_{c4}\sin(q_1 + q_2 + q_4))$$

$$U(\boldsymbol{q}) = U_1 + U_2 + U_3 + U_4$$

Gravity vector

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_1\\ g_2\\ g_3\\ g_4\end{pmatrix}$$

$$
\begin{aligned}
g_1 &= (m_1 d_{c1} + (m_2+m_3+m_4)\,a_1)\,g_0\cos q_1 - m_3 d_{c3}\,g_0\,c_{12}\\
&\quad + (m_3+m_4)\,g_0\,q_3\,c_{12} + m_4 d_{c4}\,g_0\,c_{124}\\
g_2 &= -m_3 d_{c3}\,g_0\,c_{12} + (m_3+m_4)\,g_0\,q_3\,c_{12} + m_4 d_{c4}\,g_0\,c_{124}\\
g_3 &= (m_3+m_4)\,g_0\,s_{12}\\
g_4 &= m_4 d_{c4}\,g_0\,c_{124}
\end{aligned}
$$

Linear parametrization of the gravity vector

$$
\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}_g(\boldsymbol{q})\,\boldsymbol{a}_g, \qquad \boldsymbol{a}_g \in \mathbb{R}^4
$$

$$
\boldsymbol{a}_g = \begin{pmatrix} a_{g1}\\ a_{g2}\\ a_{g3}\\ a_{g4}\end{pmatrix} = \begin{pmatrix} m_1 d_{c1} + (m_2+m_3+m_4)\,a_1\\ m_3 d_{c3}\\ m_3+m_4\\ m_4 d_{c4}\end{pmatrix}
$$

$$
\boldsymbol{Y}_g(\boldsymbol{q}) = \begin{pmatrix} g_0\,c_1 & -g_0\,c_{12} & g_0\,q_3\,c_{12} & g_0\,c_{124}\\ 0 & -g_0\,c_{12} & g_0\,q_3\,c_{12} & g_0\,c_{124}\\ 0 & 0 & g_0\,s_{12} & 0\\ 0 & 0 & 0 & g_0\,c_{124}\end{pmatrix}
$$

Note. We kept separated the two linear parametrizations, as requested. Indeed, $a_{g2} = a_{M3}$, $a_{g3} = a_{M4}$ and $a_{g4} = a_{M5}$, so that only $6+1 = 7$ different dynamic coefficients would be needed in total.

Open-loop equilibria

$$
\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0} \quad\Longleftrightarrow\quad \begin{cases} q_{e1} = \pm\dfrac{\pi}{2}\\ (m_3+m_4)\,q_{e3} = m_3 d_{c3}\\ q_{e1}+q_{e2} = \{0,\pi\}\\ q_{e1}+q_{e2}+q_{e4} = \pm\dfrac{\pi}{2}\end{cases} \quad\Longleftrightarrow\quad \begin{cases} q_{e1} = \pm\dfrac{\pi}{2}\\ q_{e2} = \pm\dfrac{\pi}{2}\\ q_{e3} = \dfrac{m_3 d_{c3}}{m_3+m_4} < d_{c3}\\ q_{e4} = \pm\dfrac{\pi}{2}\end{cases}
$$

One of the equilibria is shown in Fig. 5.

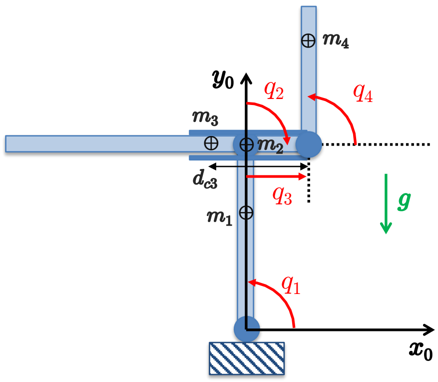


Figure 5: An equilibrium configuration of the RRPR robot: $\boldsymbol{q}_e = \left(\dfrac{\pi}{2}, -\dfrac{\pi}{2}, \dfrac{m_3 d_{c3}}{m_3+m_4}, \dfrac{\pi}{2}\right)$.

Exercise #2

We define first a path parametrization (for the payload motion) and see how the direct kinematics of each robot is related to the common cooperative task. The problem is then addressed by considering the dynamics of each of the two robots and of the payload separately. However, the three subsystems will interact among each other via exchanged forces. For these, the principle of action and reaction holds: a force applied from a robot to the payload is equal to the same force applied by the payload to that robot.

Once we put everything together, the original optimal control problem is naturally decomposed in *i)* a minimum-time motion problem for the total mass of the system moving along the path, and *ii)* a minimum internal force problem along the normal to the path (while compensating for gravity). The first problem is solved by a common bang-bang profile for the command forces acting on the first (horizontal) prismatic joints of the two robots. The second problem is solved by equally distributing the total gravity load between the two command forces on the second (vertical) prismatic joints of the two robots. The resulting commands provide the minimum value of the objective function H (sum of the squared norms of the robot inputs) among all force commands that produce the same motion in minimum time along the given path.

Path parametrization

The simplest parametrization of the desired path is a linear one. Expressing it in the world reference frame RF_w, we have

$$\boldsymbol{p}(s) = \boldsymbol{p}_{in} + \frac{\boldsymbol{p}_{fin} - \boldsymbol{p}_{in}}{L}\, s = \begin{pmatrix} p_{in,x} + s \\ p_{in,y} \end{pmatrix}, \qquad s \in [0, L], \qquad L = \left\| \boldsymbol{p}_{fin} - \boldsymbol{p}_{in} \right\|.$$

The acceleration along the path is then given by

$$\ddot{\boldsymbol{p}} = \begin{pmatrix} \ddot{p}_x \\ \ddot{p}_y \end{pmatrix} = \begin{pmatrix} \ddot{s} \\ 0 \end{pmatrix}. \tag{1}$$

Task kinematics

Indeed, the payload position $\boldsymbol{p}$ coincides with the end effector position of both robots. As seen from each robot side, the direct kinematics of each robot is related to the task by $\boldsymbol{p} = \boldsymbol{f}_A(\boldsymbol{q}_A) = \boldsymbol{f}_B(\boldsymbol{q}_B)$, with

$$\boldsymbol{f}_A(\boldsymbol{q}_A) = \begin{pmatrix} q_{A1} + q_{A1,0} \\ q_{A2,0} \end{pmatrix} = \begin{pmatrix} q_{B1} + q_{B1,0} \\ q_{B2,0} \end{pmatrix} = \boldsymbol{f}_B(\boldsymbol{q}_B),$$

for some constant (but irrelevant hereafter) values $q_{A1,0}$, $q_{A2,0}$, $q_{B1,0}$, and $q_{B2,0}$. Differentiating twice w.r.t. time and using (1), yields

$$\ddot{q}_{A1} = \ddot{q}_{B1} = \ddot{p}_x = \ddot{s}, \qquad \ddot{q}_{A2} = \ddot{q}_{B2} = \ddot{p}_y = 0. \tag{2}$$

Payload dynamics

With reference to the free-body diagram in Fig. 6, we shall consider the forces applied by the two robots A and B to the payload as decomposed in those that contribute to motion (along the path tangent) and those that may generate internal forces on the rigid payload (along the normal to the path) —and will produce in any event no motion. In this simple cooperative task, such decomposition occurs along two fixed directions, i.e., those of the world frame RF_w. Thus, we have

$$\begin{aligned} m_p\, \ddot{p}_x &= F_{A,motion} + F_{B,motion} \\ m_p\, \ddot{p}_y + m_p\, g_0 &= F_{A,internal} - F_{B,internal}. \end{aligned}$$

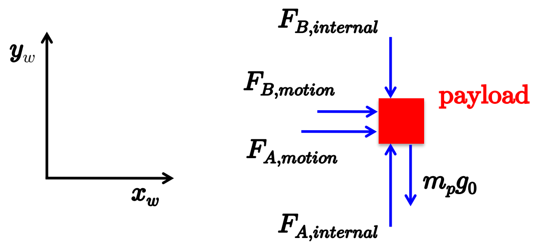


Figure 6: Free-body diagram of the forces applied to the payload.

Using (1), these simplify to

$$m_p\, \ddot{s} = F_{A,motion} + F_{B,motion} \tag{3}$$

$$m_p\, g_0 = F_{A,internal} - F_{B,internal} \tag{4}$$

Dynamics of the cooperating robots

For the two robots A and B, one has

$$\begin{aligned} (m_{A1} + m_{A2})\, \ddot{q}_{A1} &= \tau_{A1} - F_{A,motion} \\ m_{A2}\, \ddot{q}_{A2} + m_{A2}\, g_0 &= \tau_{A2} - F_{A,internal}, \end{aligned} \tag{5}$$

and, respectively,

$$\begin{aligned} (m_{B1} + m_{B2})\, \ddot{q}_{B1} &= \tau_{B1} - F_{B,motion} \\ m_{B2}\, \ddot{q}_{B2} - m_{B2}\, g_0 &= \tau_{B2} - F_{B,internal}. \end{aligned} \tag{6}$$

As already mentioned, the force components F_i appearing on the right-hand sides of these equations are those applied to the robots by the payload dynamics.

Minimum time motion

Putting together (3) with the first components of (5) and (6), and using (2), we obtain

$$m_p\, \ddot{s} = \tau_{A1} - (m_{A1} + m_{A2})\, \ddot{s} + \tau_{B1} - (m_{B1} + m_{B2})\, \ddot{s}$$

or

$$m_{tot}\, \ddot{s} = \tau_{A1} + \tau_{B1} = \tau_{motion}, \qquad \text{with} \quad m_{tot} = m_p + m_{A1} + m_{A2} + m_{B1} + m_{B2}. \tag{7}$$

As a result, the rest-to-rest minimum time solution for the total mass m_{tot} moving under an equivalent force command τ_{motion}, bounded as

$$|\tau_{motion}| \le \tau_{A,max} + \tau_{B,max} = \tau_{max},$$

will be given by the bang-bang profiles τ^*_{A1}, τ^*_{B1}, and τ^*_{motion}, as illustrated in Fig. 7. Accordingly, the minimum motion time is

$$T^* = \sqrt{\frac{4L}{\ddot{s}_{max}}} = 2\sqrt{\frac{L\, m_{tot}}{\tau_{max}}}.$$

Minimum internal force

Consider now (4) together with the second components of (5) and (6). Using again (2), we obtain

$$m_p\, g_0 = \tau_{A2} - m_{A2} g_0 - \tau_{B2} - m_{B2} g_0,$$

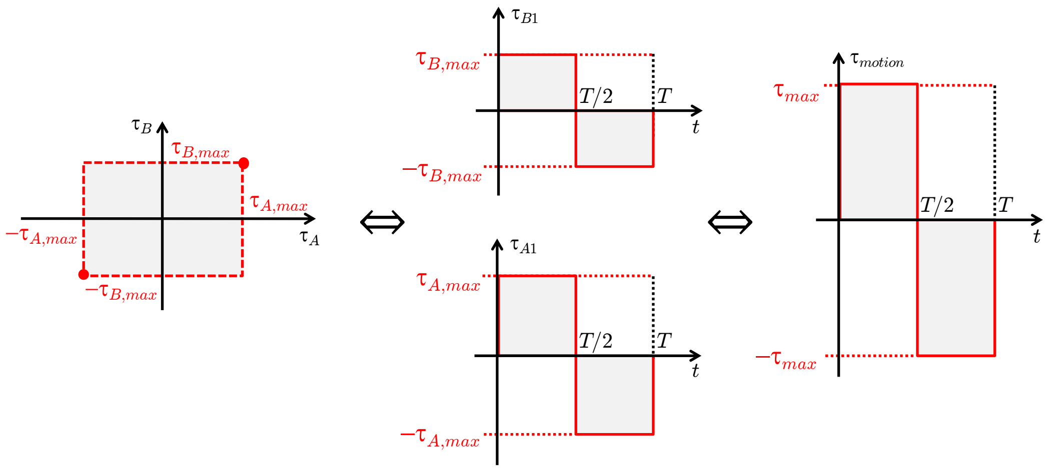


Figure 7: Bang-bang profiles of the individual robot force commands τ_{A1}^* and τ_{B1^*} and of the equivalent total force τ_{motion}^* in the minimum motion time solution.

or

$$(m_p + m_{A2} + m_{B2})\, g_0 = \tau_{A2} - \tau_{B2} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \tau_{A2} \\ \tau_{B2} \end{pmatrix} = \boldsymbol{J}_{internal} \begin{pmatrix} \tau_{A2} \\ \tau_{B2} \end{pmatrix}. \tag{8}$$

The requirement of minimizing the objective function

$$H = \|\boldsymbol{\tau}_A\|^2 + \|\boldsymbol{\tau}_B\|^2 = \tau_{A1}^2 + \tau_{A2}^2 + \tau_{B1}^2 + \tau_{B2}^2,$$

in view of the (unique) minimum time solution already found for $\tau_{A1} = \tau_{A1}^*$ and $\tau_{B1} = \tau_{B1}^*$, reduces to the minimization of the quadratic sub-function

$$H' = \tau_{A2}^2 + \tau_{B2}^2,$$

subject to the linear constraint (8). It is easy to see that the (unique) solution to this simple LQ problem for the remaining robot commands is

$$\tau_{A2}^* = \frac{1}{2}\,(m_p + m_{A2} + m_{B2})\, g_0, \qquad \tau_{B2}^* = -\frac{1}{2}\,(m_p + m_{A2} + m_{B2})\, g_0 = -\tau_{A2}^*. \tag{9}$$

Indeed, any other force pair of the perturbed form

$$\tau_{A2} = \tau_{A2}^* + \Delta, \qquad \tau_{B2} = \tau_{B2}^* + \Delta, \qquad \forall \Delta \in \mathbb{R},$$

will still satisfy the linear constraint (8) of the reduced problem. These force perturbations are in fact in the null space of the Jacobian of the constraint, or

$$\Delta \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathcal{N}\left\{\boldsymbol{J}_{internal}\right\},$$

and therefore produce solutions that are larger in norm (thus, with a higher H'). Such perturbations have the meaning of internal forces that may arise when the payload is rigidly held by the two robots. Thus, the optimal solution (9) has the nice physical interpretation of minimizing the internal forces ($\Delta = 0$). We finally note that, if gravity were not present (e.g., for motions occurring on a horizontal plane), the solution that minimizes the internal forces would be $\tau_{A2}^* = \tau_{B2}^* = 0$.

Exercise #3

With the dynamics of this planar 3R robot given by

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

we apply first a feedback linearization law in the joint space, i.e.,

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) \qquad \Rightarrow \qquad \ddot{\boldsymbol{q}} = \boldsymbol{a}, \tag{10}$$

to convert the self-motion task into a purely kinematic one. The robot should always keep the position of its end effector

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} l_1 \cos q_1 + l_2 \cos(q_1+q_2) + l_3 \cos(q_1+q_2+q_3) \\ l_1 \sin q_1 + l_2 \sin(q_1+q_2) + l_3 \sin(q_1+q_2+q_3) \end{pmatrix}$$

at

$$\boldsymbol{p}_{in} = \boldsymbol{f}(\boldsymbol{q}_{in}) = \begin{pmatrix} l_1 \cos\frac{\pi}{9} + \frac{\sqrt{2}}{2} l_3 \\ -l_1 \cos\frac{\pi}{9} + l_2 + \frac{\sqrt{2}}{2} l_3 \end{pmatrix}.$$

Indeed, the 3R robot has $n - m = 3 - 2 = 1$ degree of redundancy for the positioning task in the plane. A joint acceleration command performing a robot self-motion, as driven by the target position $q_{3,fin}$ for the third joint, can then be designed as

$$\boldsymbol{a} = -\boldsymbol{J}^{\#}(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\left(\begin{pmatrix} 0 \\ 0 \\ \alpha\,(q_{3,fin} - q_3) \end{pmatrix} - \boldsymbol{K}_v\dot{\boldsymbol{q}}\right), \tag{11}$$

for some $\alpha > 0$. In (11), a damping velocity term $-\boldsymbol{K}_v\dot{\boldsymbol{q}}$, with diagonal gain matrix $\boldsymbol{K}_v > 0$, has been added in the null space of the task Jacobian so as to stabilize the joint motion[2]. In order to reject also position and/or velocity errors that may occur around the desired constant end-effector position $\boldsymbol{p}_{in}$, a more robust version of the command (11) is given by

$$\boldsymbol{a} = \boldsymbol{J}^{\#}(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{K}_D\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\begin{pmatrix} -k_{v,1}\,\dot{q}_1 \\ -k_{v,2}\,\dot{q}_2 \\ \alpha\,(q_{3,fin} - q_3) - k_{v,3}\,\dot{q}_3 \end{pmatrix}, \tag{12}$$

including thus a PD action, with (typically diagonal) gain matrices $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$, on the Cartesian position error, and taking into account that $\dot{\boldsymbol{p}}_{in} = \boldsymbol{0}$. Plugging (12) into (10) yields the torque control law

$$\begin{aligned} \boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\Big[&\boldsymbol{J}^{\#}(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{K}_D\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) \\ &+ \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\begin{pmatrix} -k_{v,1}\,\dot{q}_1 \\ -k_{v,2}\,\dot{q}_2 \\ \alpha\,(q_{3,fin} - q_3) - k_{v,3}\,\dot{q}_3 \end{pmatrix}\Big] + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}). \end{aligned} \tag{13}$$

It should be noted that the control law (13), or simply the acceleration law (11) and its robust version (12), may or may not guarantee the reaching of the final desired value for the third joint.

[2]This is rather customary (and almost mandatory) when resolving redundancy at the acceleration level.

In principle, at a steady-state with $\dot{\boldsymbol{q}} = \boldsymbol{0}$, the projection operator $\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{J}^{\#}\boldsymbol{J}$ (with its columns $\boldsymbol{P}_i$, $i = 1, 2, 3$) may still mask the presence of a residual configuration error. Thus, one should also show that there exists no configuration $\boldsymbol{q}^*$ such that $\boldsymbol{f}(\boldsymbol{q}^*) = \boldsymbol{p}_{in}$ (i.e., belonging to the self-motion manifold in the joint space associated to the initial point P_{in}) for which $\boldsymbol{P}_3(\boldsymbol{q}^*)\, e_3 = \boldsymbol{0}$ while $e_3 = q_{3,fin} - q_3^* \neq 0$. Such statement about non-existence is in fact true, but its formal proof is not trivial. Therefore, an alternative approach that directly guarantees also the convergence of q_3 to $q_{3,fin}$ may be more attractive.

In the following, we will illustrate the use of a joint space decomposition approach at the acceleration level (i.e., to be applied, after feedback linearization, to $\ddot{\boldsymbol{q}} = \boldsymbol{a}$). In this case, one focuses on the command to be given to the third joint, the one that has a special target assigned, leaving to the other two joints the task of keeping the end effector at the desired position P_{in}. First, for the third joint we choose the control law

$$\ddot{q}_3 = a_3 = k_{p,3}\,(q_{3,fin} - q_3) - k_{d,3}\,\dot{q}_3, \qquad k_{p,3} > 0,\; k_{d,3} > 0. \tag{14}$$

As a result, the error e_3 will satisfy the linear differential equation

$$\ddot{e}_3 + k_{d,3}\dot{e}_3 + k_{p,3}e_3 = 0,$$

which guarantees that q_3 will converge exponentially from any initial state to the desired $q_{3,fin}$, with $\dot{q}_3 = \dot{q}_{3,fin} = 0$. Moreover, by a suitable choice of the gains $k_{p,3}$ and $k_{d,3}$, the natural motion of q_3 will always remain in the interval $[q_{3,fin}, q_{3,in}] = [-\pi/2, -\pi/4]$ (i.e., without overshooting or wandering). Next, decompose the second-order differential kinematics as follows:

$$\begin{aligned}\ddot{\boldsymbol{p}} &= \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}_{12}(\boldsymbol{q})\ddot{\boldsymbol{q}}_{12} + \boldsymbol{J}_3(\boldsymbol{q})\,\ddot{q}_3 + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \\ &= \begin{pmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -l_3 s_{123} \\ l_3 c_{123} \end{pmatrix} \ddot{q}_3 + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}.\end{aligned} \tag{15}$$

The square sub-Jacobian $\boldsymbol{J}_{12}$ made by the first two columns of $\boldsymbol{J}$ has

$$\det \boldsymbol{J}_{12}(\boldsymbol{q}) = l_1\,(l_2 \sin q_2 + l_3 \sin(q_2 + q_3)).$$

As long as this determinant is different from zero, we can set $\ddot{\boldsymbol{p}} = \boldsymbol{0}$ in (15) and solve for $\ddot{\boldsymbol{q}}_{12}$ so as to realize our self-motion task by

$$\ddot{\boldsymbol{q}}_{12} = -\boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\left(\boldsymbol{J}_3(\boldsymbol{q})\,\ddot{q}_3 + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right), \tag{16}$$

for any motion $\ddot{q}_3$, in particular that given by (14). To introduce more robustness in the task of keeping the end-effector position at $\boldsymbol{p}_{in}$, we replace

$$\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_{in} = \boldsymbol{0} \qquad \Rightarrow \qquad \ddot{\boldsymbol{p}} = \boldsymbol{K}_P\,(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{K}_D\,\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{17}$$

with (diagonal) 2×2 gain matrices $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$ weighting, respectively, the position error $\boldsymbol{e}_P = \boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})$ and the velocity error $\boldsymbol{e}_D = \dot{\boldsymbol{p}}_{in} - \dot{\boldsymbol{p}} = -\dot{\boldsymbol{p}} = -\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$. Using (14) and (17) in eq. (15) and solving again for $\ddot{\boldsymbol{q}}_{12}$ yields

$$\ddot{\boldsymbol{q}}_{12} = \boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\left(\boldsymbol{K}_P\,(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{K}_D\,\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{J}_3(\boldsymbol{q})\,(k_{P,3}\,(q_{3,fin} - q_3) - k_{d,3}\,\dot{q}_3) - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right). \tag{18}$$

We can also combine (14) and (18) in a single formula as

$$\boldsymbol{a} = \begin{pmatrix} \ddot{\boldsymbol{q}}_{12} \\ \ddot{q}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{12}^{-1}(\boldsymbol{q}) & -\boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\,\boldsymbol{J}_3(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{K}_P\,(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})) - \left(\boldsymbol{K}_D\,\boldsymbol{J}(\boldsymbol{q}) - \dot{\boldsymbol{J}}(\boldsymbol{q})\right)\dot{\boldsymbol{q}} \\ k_{P,3}\,(q_{3,fin} - q_3) - k_{d,3}\,\dot{q}_3 \end{pmatrix}. \tag{19}$$

Finally, plugging (19) into (10) yields the torque control law

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{J}_{12}^{-1}(\boldsymbol{q}) & -\boldsymbol{J}_{12}^{-1}(\boldsymbol{q})\, \boldsymbol{J}_3(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{K}_P \left(\boldsymbol{p}_{in} - \boldsymbol{f}(\boldsymbol{q})\right) - \left(\boldsymbol{K}_D\, \boldsymbol{J}(\boldsymbol{q}) - \dot{\boldsymbol{J}}(\boldsymbol{q})\right) \dot{\boldsymbol{q}} \\ k_{P,3} \left(q_{3,fin} - q_3\right) - k_{d,3}\, \dot{q}_3 \end{pmatrix} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}). \tag{20}$$

The last thing to check is the absence of singularities for $\boldsymbol{J}_{12}(\boldsymbol{q})$ during the self-motion under the control law (20), or simply the acceleration law (19). It can be shown that $\det \boldsymbol{J}_{12}(\boldsymbol{q}) = 0$ if and only if the end-effector of the 3R robot finds itself aligned with the first link of the structure. From the illustration in Fig. 8, it is rather evident that such condition is not encountered in this task.

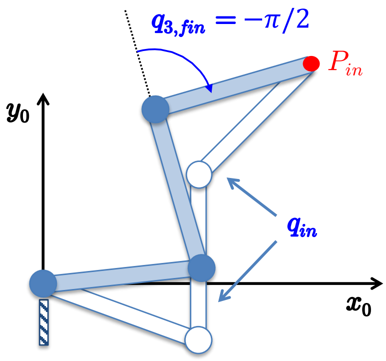


Figure 8: Configuration reached by the 3R robot at the end of the controlled self-motion task.

Exercise #4

The dynamics of the system represented in Fig. 4 is

$$m_r \ddot{x}_r + (b_r + b_s)\, \dot{x}_r + k_s x_r = F, \tag{21}$$

with the contact force measured by the sensor given by

$$F_c = k_s x_r.$$

Since we deal with a linear dynamics, one can also transform (21) in the Laplace domain and represent the system by its transfer function from the control input F to the controlled output F_c as

$$P(s) = \frac{F_c(s)}{F(s)} = \frac{k_s x_r(s)}{F(s)} = \frac{k_s}{m_r s^2 + (b_r + b_s)\, s + k_s}. \tag{22}$$

Since the physical parameters m_r, b_r, b_s and k_s are all positive, $P(s)$ has two poles with negative real part, and the open-loop system is thus asymptotically stable (with a unitary steady-state gain, $P(0) = 1$). The simplest feedback controller $C(s)$ that tries to regulate the contact force to a (constant, but arbitrary) desired value F_d is a proportional law to the force error $F_e = F_d - F_c$,

$$F = K_P\, (F_d - F_c) = K_P\, F_e \qquad \Longleftrightarrow \qquad C(s) = \frac{F(s)}{F_e(s)} = K_P > 0.$$

The input-output transfer function of the closed-loop system would then be

$$W(s) = \frac{F(s)}{F_d(s)} = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{k_s K_P}{m_r s^2 + (b_r + b_s)\, s + k_s\, (1 + K_P)},$$

which is still asymptotically stable, but with a non-unitary gain $W(0) = K_P/(1 + K_P) \neq 1$. This means that the steady-state output response to a desired step input F_d would have an error (unless $K_P \to \infty$, which is impossible). The value of this force error can also be found from the input-error transfer function,

$$W_e(s) = \frac{F_e(s)}{F_d(s)} = \frac{F_d(s) - F(s)}{F_d(s)} = 1 - W(s) = \frac{1}{1 + P(s)C(s)} = \frac{m_r s^2 + (b_r + b_s)\, s + k_s}{m_r s^2 + (b_r + b_s)\, s + k_s\,(1 + K_P)}.$$

In fact, from the final value theorem, the steady-state error for a constant F_d is computed as

$$F_{e,\infty} = \lim_{t\to\infty} F_e(t) = \lim_{s\to 0} F_e(s) = \lim_{s\to 0} W_e(s) F_d(s) = W_e(0)\, F_d = \frac{1}{1 + K_P}\, F_d \neq 0.$$

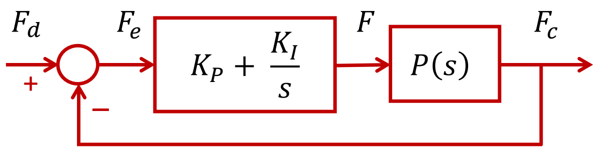


Figure 9: The closed-loop scheme of the robot-environment system under PI force control.

In order to eliminate this steady-state error in a robust way (i.e., using feedback), we need an integral action[3] (a pole in $s = 0$) in the controller $C(s)$. With reference to Fig. 9, we consider then a proportional-integral (PI) controller on the force error $F_e = F_d - F_c$, or

$$F(t) = K_P\, F_e(t) + K_I \int_0^t F_e(\tau) d\tau \qquad \Longleftrightarrow \qquad C(s) = K_P + \frac{K_I}{s} = \frac{K_P s + K_I}{s} \tag{23}$$

Combining (23) with (22) gives the closed-loop system

$$W(s) = \frac{F(s)}{F_d(s)} = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{k_s\,(K_P s + K_I)}{m_r s^3 + (b_r + b_s)\, s^2 + k_s\,(1 + K_P)\, s + k_s K_I}, \tag{24}$$

with gain $W(0) = 1$. To check the conditions under which the three poles of $W(s)$ will all have negative real part, we apply the Routh criterion. From the Routh table built for the polynomial denominator of $W(s)$ in (24)

$$\begin{array}{c|cc}
3 & m_r & k_s\,(1 + K_P) \\
2 & b_r + b_s & k_s K_I \\
1 & k_s\,(1 + K_P) - \dfrac{m_r k_s K_I}{b_r + b_s} & \\
0 & k_s K_I &
\end{array}$$

we see that the elements in the first column have the same (here, positive) sign iff

$$K_I > 0, \qquad (1 + K_P) - \frac{m_r K_I}{b_r + b_s} > 0.$$

[3]From the elementary feedback theory, in order to guarantee zero error at steady state in the step response, the control system in Fig. 9 should be asymptotically stable and have (at least) a pole in $s = 0$ in the forward path (type I). If the process $P(s)$ does not have already such a pole, it should be introduced in the controller.

Therefore, choosing the two gains K_P and K_I in the ranges

$$K_I \geq \frac{b_r + b_s}{m_r} > 0, \qquad K_P > \frac{m_r K_I - (b_r + b_s)}{b_r + b_s} \geq 0$$

will ensure asymptotic stability of the closed-loop system (in a robust way with respect to uncertainties in system parameters —there is only a need to enforce inequalities that are simple to overbound). Moreover, the force error at steady state will be zero as expected, since the input-error transfer function

$$W_e(s) = 1 - W(s) = \frac{\left(m_r s^2 + (b_r + b_s)\, s + k_s\right) s}{m_r s^3 + (b_r + b_s)\, s^2 + k_s\,(1 + K_P)\, s + k_s K_I},$$

has a zero at $s = 0$, and thus

$$F_{e,\infty} = W_e(0)\, F_d = 0 \cdot F_d = 0.$$

As a result, at steady state

$$\bar{F}_c = F_d = k_s \bar{x}_r, \qquad \bar{x}_r = \frac{F_d}{k_s}, \qquad \bar{F} = K_I \int_0^\infty F_e(\tau) d\tau = F_d.$$

* * * * *

Robotics 2

June 11, 2021

Exercise #1

Suppose that a routine is available that computes numerically the pseudoinverse of a matrix $\boldsymbol{A}$, e.g., the `pinv` function in MATLAB, $\boldsymbol{A}^{\#} = \texttt{pinv}(\boldsymbol{A})$. Given a $m \times n$, **full rank** matrix $\boldsymbol{J}$, with $m < n$, and a $n \times n$, positive definite, symmetric weighting matrix $\boldsymbol{W}$, prove formally that the weighted pseudoinverse $\boldsymbol{J}^{\#}_{\boldsymbol{W}}$ can be computed as $\boldsymbol{J}^{\#}_{\boldsymbol{W}} = \boldsymbol{W}^{-1/2}\,\texttt{pinv}(\boldsymbol{J}\boldsymbol{W}^{-1/2})$. As a verification, provide a simple numerical example with $m = 2$, $n = 3$.

Exercise #2

A single link mounted on a passive elastic support is moved on a horizontal plane by a torque τ applied by a motor to the revolute joint at its base, as sketched in Fig. 1. The generalized coordinates q_1 and q_2 are defined therein, together with the relevant dynamic parameters: mass m, distance $d > 0$ of the CoM from the joint, and barycentric inertia I_L of the link; stiffness $k > 0$ of the linear spring in the support. The spring is undeformed when $q_1 = 0$. Derive first the Lagrangian dynamic model of this simple robotic system. Next, address the following two dynamic problems.

a) An input torque $\tau_0 > 0$ is applied at $t = 0$, with the system at $\boldsymbol{q}(0) = \dot{\boldsymbol{q}}(0) = \boldsymbol{0}$ (zero initial state). Determine if the spring will initially be compressed or extended and if the link will start moving clockwise or counterclockwise. Provide the expression of the initial accelerations $\ddot{q}_1(0)$ and $\ddot{q}_2(0)$ of the two coordinates .

b) Starting at $t = 0$ in a generic non-zero initial state $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0))$, define a control law $\tau = \tau(\boldsymbol{q}, \dot{\boldsymbol{q}})$ such that $\boldsymbol{q}_2(t)$ will exponentially converge to zero. At steady state, determine the residual dynamics of the other coordinate $q_1(t)$ and provide a physical interpretation of this result.

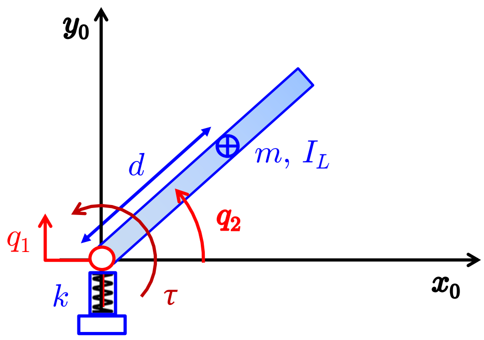


Figure 1: A single-link robotic system mounted on a flexible base.

Exercise #3

Draw a 3-dof robot whose dynamic model is given by

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{u}, \qquad \boldsymbol{M} = \begin{pmatrix} m_1 + m_2 + m_3 & 0 & 0 \\ 0 & m_2 + m_3 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \qquad \boldsymbol{g} = \begin{pmatrix} 0 \\ 0 \\ -m_3 g_0 \end{pmatrix},$$

with masses $m_i > 0$, $i = 1, 2, 3$, and $g_0 = 9.81$ [m/s^2].

Exercise #4

Consider the planar 3R robot with equal link lengths in Fig. 2 and absolute joint coordinates $\boldsymbol{q} = (q_1, q_2, q_3)$ defined therein. The robot is equipped with three motors producing torques $\boldsymbol{u} = (u_1, u_2, u_3)$ that perform work on the Denavit-Hartenberg angles $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$, respectively. Provide the expression of the torque vector $\boldsymbol{u_q} = (u_{q1}, u_{q2}, u_{q3})$ that should appear on the right-hand side of the robot dynamic model written in terms of the absolute coordinates $\boldsymbol{q}$ (i.e., in $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u_q}$), as function of the components of $\boldsymbol{u}$.

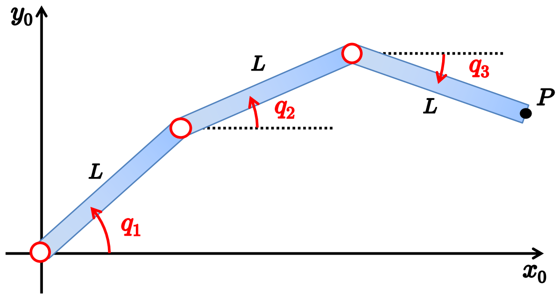


Figure 2: A planar 3R robot, with absolute joint coordinates $\boldsymbol{q}$ and equal links of length L.

Exercise #5

With reference to Fig. 3, the end-point P of a planar 2R robot should execute a desired trajectory $\boldsymbol{p}_d(t) \in \mathbb{R}^2$, specified by a circular path in the Cartesian plane with a time-varying desired tangential velocity. Motion occurs on a horizontal plane and the circle should be traced counterclockwise. An initial position and/or velocity error between the robot end-point and the desired trajectory is present at $t = 0$. Later on, external disturbances may also affect occasionally the execution of the desired motion task. Design a torque control law for the robot such that the trajectory tracking error dynamics is exponentially stable, linear, and decoupled along the normal and tangential directions to the path. The error behaviors in these two directions (represented, respectively, by the $\boldsymbol{x}_t$ and $\boldsymbol{y}_t$ axes of a time-varying frame that moves with the desired task) should be critically damped, with a dominant reaction time to position errors in the normal direction which is about five times faster than in the tangential one. Assume that no kinematic singularities are encountered (as suggested also by the placement of the circle in Fig. 3).

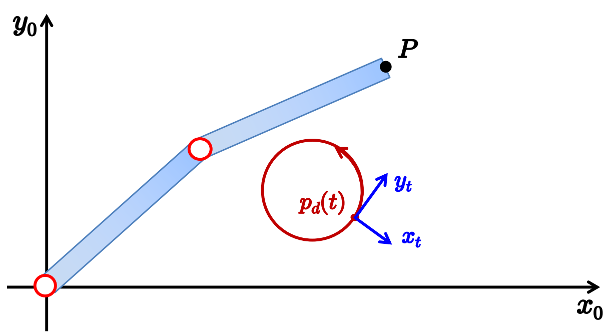


Figure 3: Tracking a circular Cartesian trajectory with a planar 2R robot.

[240 minutes (4 hours); open books]

Solution

June 11, 2021

Exercise #1

The weighted pseudoinverse $\boldsymbol{J}_{\boldsymbol{W}}^{\#}$ of a $m \times n$ matrix $\boldsymbol{J}$, with $\text{rank}\,\boldsymbol{J} = m < n$ and a symmetric matrix $\boldsymbol{W} > 0$, is given by

$$\boldsymbol{J}_{\boldsymbol{W}}^{\#} = \boldsymbol{W}^{-1}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T\right)^{-1},$$

with the three relations holding

$$\boldsymbol{J}\boldsymbol{J}_{\boldsymbol{W}}^{\#}\boldsymbol{J} = \boldsymbol{J}, \qquad \boldsymbol{J}_{\boldsymbol{W}}^{\#}\boldsymbol{J}\boldsymbol{J}_{\boldsymbol{W}}^{\#} = \boldsymbol{J}_{\boldsymbol{W}}^{\#}, \qquad \left(\boldsymbol{J}\boldsymbol{J}_{\boldsymbol{W}}^{\#}\right)^T = \boldsymbol{J}\boldsymbol{J}_{\boldsymbol{W}}^{\#}, \tag{1}$$

but not the fourth one (i.e., in general $(\boldsymbol{J}_{\boldsymbol{W}}^{\#}\boldsymbol{J})^T \neq \boldsymbol{J}_{\boldsymbol{W}}^{\#}\boldsymbol{J}$). On the other hand, for a full row rank matrix $\boldsymbol{A}$ the `pinv` routine provides as output

$$\boldsymbol{A}^{\#} = \texttt{pinv}(\boldsymbol{A}) = \boldsymbol{A}^T\left(\boldsymbol{A}\boldsymbol{A}^T\right)^{-1}.$$

Therefore,

$$\begin{aligned}
\boldsymbol{W}^{-1/2}\,\texttt{pinv}(\boldsymbol{J}\boldsymbol{W}^{-1/2}) &= \boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^T\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\right)^T\right)^{-1} \\
&= \boldsymbol{W}^{-1/2}\,\boldsymbol{W}^{-1/2}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{W}^{-1/2}\,\boldsymbol{W}^{-1/2}\boldsymbol{J}^T\right)^{-1} \\
&= \boldsymbol{W}^{-1}\boldsymbol{J}^T\left(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T\right)^{-1} = \boldsymbol{J}_{\boldsymbol{W}}^{\#}.
\end{aligned}$$

As a numerical example with $m = 2$, $n = 3$, for

$$\boldsymbol{J} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \qquad \boldsymbol{W} = \begin{pmatrix} 7 & & \\ & 8 & \\ & & 9 \end{pmatrix},$$

we obtain

$$\boldsymbol{W}^{-1/2}\,\texttt{pinv}(\boldsymbol{J}\boldsymbol{W}^{-1/2}) = \begin{pmatrix} -0.9792 & 0.4583 \\ -0.0417 & 0.0833 \\ 0.6875 & -0.2083 \end{pmatrix} = \boldsymbol{J}_{\boldsymbol{W}}^{\#},$$

which satisfies indeed the defining identities (1).

Note that the same formula holds also in the singular case ($\text{rank}\,\boldsymbol{J} < m$), being then

$$\boldsymbol{J}_{\boldsymbol{W}}^{\#} = \boldsymbol{W}^{-1}\boldsymbol{J}^T\texttt{pinv}\left(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T\right) = \boldsymbol{W}^{-1/2}\,\texttt{pinv}(\boldsymbol{J}\boldsymbol{W}^{-1/2}).$$

For instance, with

$$\boldsymbol{J} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \end{pmatrix}, \qquad \text{rank}\,\boldsymbol{J} = 1,$$

and the same previous diagonal $\boldsymbol{W} > 0$, we obtain

$$\boldsymbol{W}^{-1/2}\,\texttt{pinv}(\boldsymbol{J}\boldsymbol{W}^{-1/2}) = \begin{pmatrix} 0.0051 & 0.0205 \\ 0.0090 & 0.0358 \\ 0.0119 & 0.0477 \end{pmatrix} = \boldsymbol{J}_{\boldsymbol{W}}^{\#}.$$

Exercise #2

We compute the kinetic energy T of the link and the potential energy U_e due to the elastic spring. The position and velocity of the center of mass of the link are

$$\boldsymbol{p}_c = \begin{pmatrix} d\cos q_2 \\ q_1 + d\sin q_2 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_c = \dot{\boldsymbol{p}}_c = \begin{pmatrix} -d\sin q_2\, \dot{q}_2 \\ \dot{q}_1 + d\cos q_2\, \dot{q}_2 \end{pmatrix},$$

while the angular velocity of the link has only the z-component $\omega_z = \dot{q}_2$. Therefore, the kinetic energy is

$$T = \frac{1}{2} m \left(\dot{q}_1^2 + d^2\dot{q}_2^2 + 2d\cos q_2\, \dot{q}_1\dot{q}_2\right) + \frac{1}{2} I_L\, \dot{q}_2^2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with the inertia matrix given by

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m & md\cos q_2 \\ md\cos q_2 & I_L + md^2 \end{pmatrix}.$$

The velocity terms in the dynamic model are computed through the standard Christoffel's symbols. We obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\left\{ \begin{pmatrix} 0 & 0 \\ 0 & -md\sin q_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -md\sin q_2 \end{pmatrix}^T - \boldsymbol{0} \right\} = \begin{pmatrix} 0 & 0 \\ 0 & -md\sin q_2 \end{pmatrix},$$

yielding

$$c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q})\dot{\boldsymbol{q}} = -md\sin q_2\, \dot{q}_2^2.$$

On the other hand, it is easy to see that

$$\boldsymbol{C}_2(\boldsymbol{q}) = \boldsymbol{0} \qquad \Rightarrow \qquad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0.$$

The elastic potential U_e and its gradient are

$$U_e = \frac{1}{2} k\, q_1^2 \qquad \Rightarrow \qquad \nabla_{\boldsymbol{q}} U_e = \begin{pmatrix} k\, q_1 \\ 0 \end{pmatrix}$$

As a result, the dynamic model of the robotic system is

$$\begin{pmatrix} m & md\cos q_2 \\ md\cos q_2 & I_L + md^2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -md\sin q_2\, \dot{q}_2^2 \\ 0 \end{pmatrix} + \begin{pmatrix} k\, q_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \tau \end{pmatrix}. \tag{2}$$

For case a), at the initial state $\boldsymbol{q}(0) = \dot{\boldsymbol{q}}(0) = \boldsymbol{0}$ and with an input $\tau = \tau_0 > 0$, we solve from (2) for the acceleration $\ddot{\boldsymbol{q}}(0)$ as

$$\begin{pmatrix} \ddot{q}_1(0) \\ \ddot{q}_2(0) \end{pmatrix} = \boldsymbol{M}^{-1}(\boldsymbol{q}(0)) \begin{pmatrix} 0 \\ \tau_0 \end{pmatrix} = \frac{1}{\det \boldsymbol{M}(\boldsymbol{q}(0))} \begin{pmatrix} -md\cos q_2(0) \\ m \end{pmatrix} \tau_0 = \frac{\tau_0}{I_L} \begin{pmatrix} -d \\ 1 \end{pmatrix}. \tag{3}$$

As a consequence, the spring will be initially compressed ($\ddot{q}_1(0) < 0$) and the link will start moving counterclockwise ($\ddot{q}_2(0) > 0$).

For case b), we isolate $\ddot{q}_1$ from the first equation in (2),

$$\ddot{q}_1 = \frac{1}{m}\left(md\left(\sin q_2\, \dot{q}_2^2 - \cos q_2\, \ddot{q}_2\right) - k\, q_1\right),$$

and replace it in the second, obtaining thus

$$\left(I_L + md^2(1-\cos^2 q_2)\right)\ddot q_2 + md^2 \sin q_2 \cos q_2\, \dot q_2^2 - dk\, q_1 \cos q_2 = \tau. \tag{4}$$

Consider now the nonlinear feedback control law

$$\tau = \left(I_L + md^2(1-\cos^2 q_2)\right)\left(-k_d\, \dot q_2 + k_p\,(q_{2d} - q_2)\right) + md^2 \sin q_2 \cos q_2\, \dot q_2^2 - dk\, q_1 \cos q_2, \tag{5}$$

with $k_p > 0$ and $k_d > 0$, and any constant value for q_{2d}. Note that the (inertial) factor multiplying $\ddot q_2$ in this control law is always positive. Then, plugging the torque (5) into (4) will exactly linearize and stabilize the dynamics of the coordinate q_2 in a global sense, yielding

$$\ddot q_2 + k_d \dot q_2 + k_p q_2 = k_p q_{2,d} \qquad \Rightarrow \qquad q_2(t) \to q_{2d} \quad \text{exponentially.}$$

At steady state, one has $\dot q_2 = \ddot q_2 = 0$. Thus, the first dynamic equation in (2) provides

$$m\, \ddot q_1 + k\, q_1 = 0.$$

This is the dynamics of an undamped mass m suspended on a spring of stiffness k. If initially excited at some $t = \bar t > 0$, i.e., for $q_1(\bar t) \neq 0$ and/or $\dot q_1(\bar t) \neq 0$, the mass will oscillate forever as

$$q_1(t) = q_1(\bar t) \cos \omega (t - \bar t) + \frac{\dot q_1(\bar t)}{\omega} \sin \omega (t - \bar t), \qquad \omega = \sqrt{\frac{k}{m}} > 0, \quad \forall t \geq \bar t.$$

Accordingly, the control law (5) boils down at steady state to the oscillatory command

$$\tau_{ss}(t) = -\left(dk \cos q_{2d}\right) q_1(t),$$

which will prevent the link from rotating.

Exercise #3

It is easy to recognize that the dynamic model refers to a Cartesian 3P robot with orthogonal axes, see Fig. 4. Only the third (vertical) prismatic joint is subject to gravity. The structure is also called a portal robot. It supports and moves heavy payloads and is usually equipped with an additional 3R spherical wrist mounted on the end effector.

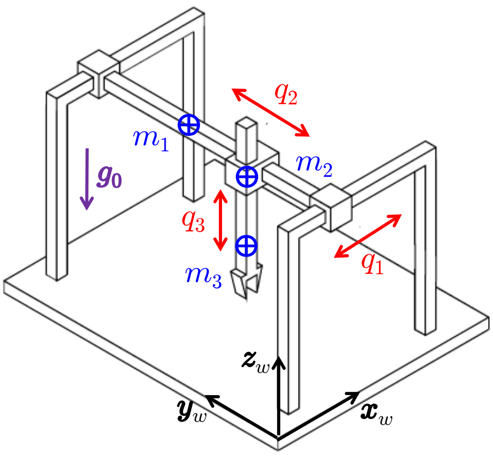


Figure 4: A Cartesian 3P robot with a portal structure.

Exercise #4

This is a straightforward application of the principle of virtual work. The (absolute) coordinates q_i, $i = 1, 2, 3$, in Fig. 2 are related to the (relative) joint variables θ_i, $i = 1, 2, 3$, of the Denavit-Hartenberg (DH) notation by the linear transformation

$$\begin{aligned} q_1 &= \theta_1 \\ q_2 &= \theta_1 + \theta_2 \\ q_3 &= \theta_1 + \theta_2 + \theta_3 \end{aligned} \qquad \Rightarrow \qquad \boldsymbol{q} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \boldsymbol{\theta} = \boldsymbol{T}\boldsymbol{\theta}.$$

The joint torques $\boldsymbol{u}$ produced by the three motors and performing work on the DH angles $\boldsymbol{\theta}$ and the torques $\boldsymbol{u_q}$ performing work on the $\boldsymbol{q}$ coordinates that appear in the dynamic model

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u_q},$$

are related by the identity

$$\boldsymbol{u}_q^T \dot{\boldsymbol{q}} = \boldsymbol{u}^T \dot{\boldsymbol{\theta}} \qquad \Rightarrow \qquad \boldsymbol{u}_q^T \dot{\boldsymbol{q}} = \boldsymbol{u}_{\boldsymbol{q}}^T \boldsymbol{T}\dot{\boldsymbol{\theta}} = \boldsymbol{u}^T \dot{\boldsymbol{\theta}}, \quad \forall \dot{\boldsymbol{\theta}}$$

or

$$\boldsymbol{u} = \boldsymbol{T}^T \boldsymbol{u_q} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{u_q} \qquad \Rightarrow \qquad \begin{aligned} u_1 &= u_{q1} + u_{q2} + u_{q3} \\ u_2 &= u_{q2} + u_{q3} \\ u_3 &= u_{q3} \end{aligned}$$

and its inverse

$$\boldsymbol{u_q} = \boldsymbol{T}^{-T} \boldsymbol{u} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \boldsymbol{u} \qquad \Rightarrow \qquad \begin{aligned} u_{q1} &= u_1 - u_2 \\ u_{q2} &= u_2 - u_3 \\ u_{q3} &= u_3. \end{aligned}$$

Exercise #5

The solution is provided by a feedback linearization control law designed to stabilize in a decoupled way the normal and tangential trajectory errors in the time-varying reference frame $(\boldsymbol{x}_t, \boldsymbol{y}_t)$ associated to the task. For this, having defined the position error $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p}$, the error vector of interest in the rotated task frame is

$${}^t\boldsymbol{e} = {}^0\boldsymbol{R}_t^T(t)\, \boldsymbol{e} = {}^0\boldsymbol{R}_t^T(t)\, (\boldsymbol{p}_d - \boldsymbol{p})\,,$$

where

$${}^0\boldsymbol{R}_t(t) = \begin{pmatrix} \cos\alpha_d(t) & -\sin\alpha_d(t) \\ \sin\alpha_d(t) & \cos\alpha_d(t) \end{pmatrix}$$

is the 2×2 (planar) rotation matrix characterizing the current orientation (by the angle $\alpha_d(t)$) of the task frame $(\boldsymbol{x}_t, \boldsymbol{y}_t)$ moving with $\boldsymbol{p}_d(t)$ w.r.t. the absolute frame $(\boldsymbol{x}_0, \boldsymbol{y}_0)$. Accordingly, the time derivative of ${}^t\boldsymbol{e}$ is[1]

$${}^t\dot{\boldsymbol{e}} = {}^0\boldsymbol{R}_t^T(t)\, \dot{\boldsymbol{e}} + {}^0\dot{\boldsymbol{R}}_t^T(t)\, \boldsymbol{e} = {}^0\boldsymbol{R}_t^T(t)\, \dot{\boldsymbol{e}} + {}^0\boldsymbol{R}_t^T(t)\boldsymbol{S}^T(\dot{\alpha}_d(t))\, \boldsymbol{e} = {}^0\boldsymbol{R}_t^T(t)\, \left(\dot{\boldsymbol{e}} + \boldsymbol{S}^T(\dot{\alpha}_d(t))\, \boldsymbol{e}\right), \qquad (6)$$

[1] In the following, the two formats of the expression of ${}^t\dot{\boldsymbol{e}}$ in (6), and later of ${}^t\ddot{\boldsymbol{e}}$ in (7), can be used equivalently: either the one containing time derivatives of the rotation matrix, or the one where these are substituted by their explicit computation.

with

$$\boldsymbol{S}(\dot{\alpha}_d(t)) = \begin{pmatrix} 0 & -\dot{\alpha}_d(t) \\ \dot{\alpha}_d(t) & 0 \end{pmatrix}.$$

Similarly, its second time derivative is

$$\begin{aligned} {}^t\ddot{\boldsymbol{e}} &= {}^0\boldsymbol{R}_t^T(t)\,\ddot{\boldsymbol{e}} + 2\,{}^0\dot{\boldsymbol{R}}_t^T(t)\,\dot{\boldsymbol{e}} + {}^0\ddot{\boldsymbol{R}}_t^T(t)\,\boldsymbol{e} \\ &= {}^0\boldsymbol{R}_t^T(t)\,\ddot{\boldsymbol{e}} + 2\,{}^0\boldsymbol{R}_t^T(t)\boldsymbol{S}^T(\dot{\alpha}_d(t))\,\dot{\boldsymbol{e}} + {}^0\boldsymbol{R}_t^T(t)\boldsymbol{S}^T(\ddot{\alpha}_d(t))\,\boldsymbol{e} - {}^0\boldsymbol{R}_t^T(t)\boldsymbol{D}(\dot{\alpha}_d(t))\,\boldsymbol{e} \\ &= {}^0\boldsymbol{R}_t^T(t)\,\left(\ddot{\boldsymbol{e}} + 2\,\boldsymbol{S}^T(\dot{\alpha}_d(t))\,\dot{\boldsymbol{e}} + \left(\boldsymbol{S}^T(\ddot{\alpha}_d(t)) - \boldsymbol{D}(\dot{\alpha}_d(t))\right)\boldsymbol{e}\right), \end{aligned} \tag{7}$$

being

$$\boldsymbol{D}(\dot{\alpha}_d^2(t)) = -\left(\boldsymbol{S}^T(\dot{\alpha}_d(t))\right)^2 = \begin{pmatrix} \dot{\alpha}_d^2(t) & 0 \\ 0 & \dot{\alpha}_d^2(t) \end{pmatrix}.$$

Therefore, to satisfy the problem specifications, we should impose to the controlled robot the following linear and decoupled error dynamics

$${}^t\ddot{\boldsymbol{e}} + {}^t\boldsymbol{K}_D\,{}^t\dot{\boldsymbol{e}} + {}^t\boldsymbol{K}_P\,{}^t\boldsymbol{e} = \boldsymbol{0}, \tag{8}$$

with diagonal, positive definite gain matrices (task gains)

$${}^t\boldsymbol{K}_P = \begin{pmatrix} k_{P,norm} & 0 \\ 0 & k_{P,tang} \end{pmatrix} > 0, \qquad {}^t\boldsymbol{K}_D = \begin{pmatrix} k_{D,norm} & 0 \\ 0 & k_{D,tang} \end{pmatrix} > 0,$$

where the subscripts *norm* and *tang* are used to denote, respectively, the normal direction $\boldsymbol{x}_t$ and the tangential direction $\boldsymbol{y}_t$ of the current task frame.

Consider the dynamic model of the planar 2R robot (without gravity term, being this on a horizontal plane)

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{\tau}$$

and the direct (and first- and second-order differential) kinematics of the robot

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}), \qquad \dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}.$$

Assuming no singularities of the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ are encountered, we apply the control law

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})\left(\boldsymbol{a} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{9}$$

with

$$\boldsymbol{a} = \ddot{\boldsymbol{p}}_d + {}^0\boldsymbol{R}_t(t)\left({}^t\boldsymbol{K}_D\,{}^t\dot{\boldsymbol{e}} + {}^t\boldsymbol{K}_P\,{}^t\boldsymbol{e}\right) + 2\,\boldsymbol{S}^T(\dot{\alpha}_d(t))\,\dot{\boldsymbol{e}} + \left(\boldsymbol{S}^T(\ddot{\alpha}_d(t)) - \boldsymbol{D}(\dot{\alpha}_d(t))\right)\boldsymbol{e}. \tag{10}$$

This gives

$$\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_d + {}^0\boldsymbol{R}_t(t)\left({}^t\boldsymbol{K}_D\,{}^t\dot{\boldsymbol{e}} + {}^t\boldsymbol{K}_P\,{}^t\boldsymbol{e}\right) + 2\,\boldsymbol{S}^T(\dot{\alpha}_d(t))\,\dot{\boldsymbol{e}} + \left(\boldsymbol{S}^T(\ddot{\alpha}_d(t)) - \boldsymbol{D}(\dot{\alpha}_d(t))\right)\boldsymbol{e}$$

or

$${}^0\boldsymbol{R}_t^T(t)\left(\ddot{\boldsymbol{e}} + \boldsymbol{S}^T(\dot{\alpha}_d(t))\,2\,\dot{\boldsymbol{e}} + \left(\boldsymbol{S}^T(\ddot{\alpha}_d(t)) - \boldsymbol{D}(\dot{\alpha}_d(t))\right)\boldsymbol{e}\right) + {}^t\boldsymbol{K}_D\,{}^t\dot{\boldsymbol{e}} + {}^t\boldsymbol{K}_P\,{}^t\boldsymbol{e} = \boldsymbol{0},$$

which is equivalent to the expression (8) of the desired behavior for the task error ${}^t\boldsymbol{e}(t)$.

Note that the commanded acceleration $\boldsymbol{a}$ in (10) can be equivalently written using the time derivatives of the task rotation matrix ${}^0\boldsymbol{R}_t(t)$ within the expressions of ${}^t\dot{\boldsymbol{e}}$ in (6) and ${}^t\ddot{\boldsymbol{e}}$ in (7). This would lead to

$$\boldsymbol{a} = \ddot{\boldsymbol{p}}_d + {}^0\boldsymbol{R}_t(t)\left({}^t\boldsymbol{K}_D\,{}^t\dot{\boldsymbol{e}} + {}^t\boldsymbol{K}_P\,{}^t\boldsymbol{e} + 2\,{}^0\dot{\boldsymbol{R}}_t^T(t)\,\dot{\boldsymbol{e}} + {}^0\ddot{\boldsymbol{R}}_t^T(t)\,\boldsymbol{e}\right), \tag{11}$$

which produces indeed the same target result (8). Moreover, equation (10) can be rewritten only in terms of the position and velocity errors $\boldsymbol{e}$ and $\dot{\boldsymbol{e}}$ expressed in the base frame as

$$\boldsymbol{a} = \ddot{\boldsymbol{p}}_d + \boldsymbol{K}_D(t)\,\dot{\boldsymbol{e}} + \boldsymbol{K}_P(t)\,\boldsymbol{e} = \ddot{\boldsymbol{p}}_d + \boldsymbol{K}_D(t)\,(\dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}}) + \boldsymbol{K}_P(t)\,(\boldsymbol{p}_d - \boldsymbol{p}), \tag{12}$$

where we defined the two time-varying gain matrices associated to the task

$$\boldsymbol{K}_P(t) = {}^0\boldsymbol{R}_t(t)\,{}^t\boldsymbol{K}_P\,{}^0\boldsymbol{R}_t^T(t) + {}^0\boldsymbol{R}_t(t)\,{}^t\boldsymbol{K}_D\,{}^0\boldsymbol{R}_t^T(t)\,\boldsymbol{S}^T(\dot{\alpha}_d(t)) + \boldsymbol{S}^T(\ddot{\alpha}_d(t)) - \boldsymbol{D}(\dot{\alpha}_d(t))$$

and

$$\boldsymbol{K}_D(t) = {}^0\boldsymbol{R}_t(t)\,{}^t\boldsymbol{K}_D\,{}^0\boldsymbol{R}_t^T(t) + 2\,\boldsymbol{S}^T(\dot{\alpha}_d(t)).$$

We note that the desired task acceleration $\ddot{\boldsymbol{p}}_d$ needs not to be rotated in the expression (12), and that extra terms appear in these time-varying gain matrices, related to the changing orientation of the task frame. If the same trajectory were assigned along a linear path, then $\dot{\alpha}_d = \ddot{\alpha}_d = 0$ and these gain matrices would become constant

$${}^t\boldsymbol{K}_P = {}^0\boldsymbol{R}_t\,\boldsymbol{K}_P\,{}^0\boldsymbol{R}_t^T, \qquad {}^t\boldsymbol{K}_D = {}^0\boldsymbol{R}_t\,\boldsymbol{K}_D\,{}^0\boldsymbol{R}_t^T.$$

Finally, in order to assign a critical damping and the desired time scale separation between the normal and tangential error components, we use standard results on linear second-order dynamic systems. In order to impose two asymptotically stable eigenvalues/poles in

$$-a \pm i\,b = -\omega\left(\zeta \pm i\sqrt{1-\zeta^2}\right) \qquad \text{with } \omega > 0,\ \zeta > 0,\ a = \zeta\omega > 0,\ b = \sqrt{1-\zeta^2}\omega \geq 0,$$

to the characteristic equation $s^2 + k_D s + k_P = 0$ that governs the dynamics of each component of the tracking error in the task frame, one needs to choose

$$k_D = 2\zeta\omega, \qquad k_P = \omega^2.$$

The coefficient ζ affects the way oscillations at the natural frequency ω are damped. Moreover, the error response over time is enveloped by a decaying exponential $e^{-\zeta\omega t}$. Therefore, to achieve our design target for the transient errors, we choose first a common critical damping $\zeta = 0.7$ (or larger) for both the tangential and normal directions. Then, once a sufficiently high frequency $\omega_{tang} > 0$ has been selected for the tangential direction, we set

$$k_{P,tang} = \omega_{tang}^2, \qquad k_{D,tang} = 2\zeta\omega_{tang},$$

and

$$k_{P,norm} = \omega_{norm}^2 = (5\,\omega_{tang})^2 = 25\,\omega_{tang}^2, \qquad k_{D,norm} = 2\zeta\omega_{norm} = 10\zeta\omega_{tang}.$$

* * * * *

Robotics 2

July 12, 2021

Exercise #1

Consider the planar 3R robot in Fig. 1. The three links have all equal length L. The robot is controlled by a joint acceleration command $\boldsymbol{u} = \ddot{\boldsymbol{q}} \in \mathbb{R}^3$. The input commands are bounded componentwise as $|u_i| \leq U_{max,i}$, for $i = 1, 2, 3$. Moreover, let $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) \in \mathbb{R}^2$ be the end-effector position. At a given instant $t = t_0$, the robot is in a generic state $(\boldsymbol{q}(t_0), \dot{\boldsymbol{q}}(t_0)) = (\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0) \in \mathbb{R}^6$. Assume in the following that it is always $\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$. For this robot, provide (if it exists) a *feasible* solution $\boldsymbol{u}_0 = \boldsymbol{u}(t_0)$ to each of the following problems. If there are multiple feasible solutions, provide the one having *minimum norm*.

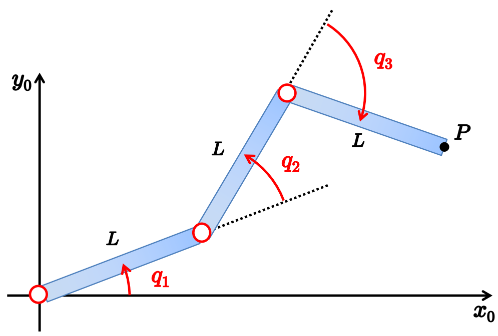


Figure 1: A planar 3R robot.

Is it possible to define $\boldsymbol{u}_0$ so that the end-effector acceleration instantaneously vanishes, i.e., $\ddot{\boldsymbol{p}}_0 = \ddot{\boldsymbol{p}}(t_0) = \boldsymbol{0}$? If so, under which conditions on the current robot state $(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0)$? Provide a specific example (or counterexample) illustrating the situation, giving the numerical values of $\boldsymbol{q}_0$, $\dot{\boldsymbol{q}}_0$, and $\boldsymbol{u}_0$ (and of the resulting $\ddot{\boldsymbol{p}}_0$, if different from zero).

Exercise #2

The dynamics of a robot arm with n joints that may be subject to actuator faults can be written in a standard form as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} - \boldsymbol{\tau}_f, \tag{1}$$

where $\boldsymbol{\tau}_f \in \mathbb{R}^n$ is an additional torque that models a generic actuator fault when present. Consider now as fault an *incipient block* of the joint/motor i, with $i \in \{1, \ldots, n\}$. This situation is represented by an acceleration of the faulted joint that behaves as $\ddot{q}_i = -\lambda_i \dot{q}_i$, with $\lambda_i \gg 1$, until the joint eventually stops. Show that this fault can be described as in (1), by providing the model-based expression of the fault $\tau_{f,i}$ (with $\tau_{f,j} \equiv 0$, for $j \neq i$). Find also the expression of the accelerations of the other joints, $j \neq i$, at the instant when this type of fault occurs.

Exercise #3

The 3-dof RPR robot in Fig. 2 moves on a horizontal plane. The robot should asymptotically track a smooth joint trajectory $\boldsymbol{q}_d(t)$ in the presence of a partly unknown dynamic model. In fact, only the dynamic parameters m_3, d_{c3}, and I_3 are assumed to be known. Derive first the dynamic model of this robot, neglecting any dissipative frictional effect.

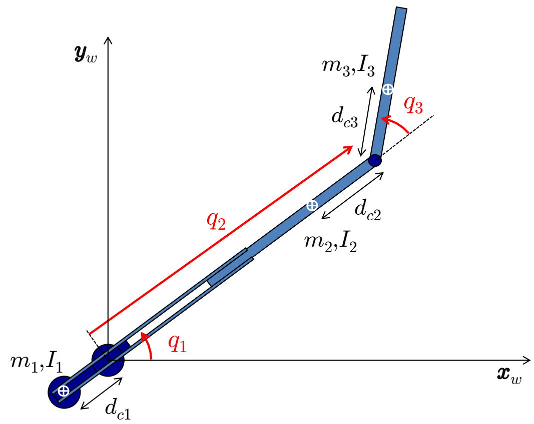


Figure 2: A planar RPR robot with its generalized coordinates and dynamic parameters.

Consider next the following adaptive trajectory tracking control law that takes advantage of the partly known dynamics

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{Y}_U(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, \hat{\boldsymbol{a}}_U + \boldsymbol{Y}_K(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, \boldsymbol{a}_K + \boldsymbol{K}_P\,(\boldsymbol{q}_d - \boldsymbol{q}) + \boldsymbol{K}_D\,(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}) \\ \dot{\hat{\boldsymbol{a}}}_U &= \boldsymbol{\Gamma}\, \boldsymbol{Y}_U^T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\,(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}), \end{aligned} \tag{2}$$

where $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}\,(\boldsymbol{q}_d - \boldsymbol{q}) = \dot{\boldsymbol{q}}_d + \boldsymbol{K}_D^{-1}\boldsymbol{K}_P\,(\boldsymbol{q}_d - \boldsymbol{q})$, $\boldsymbol{K}_P > 0$, $\boldsymbol{K}_D > 0$, and $\boldsymbol{\Gamma} > 0$ are diagonal gain matrices, $\boldsymbol{a}_U \in \mathbb{R}^{p_u}$ and $\boldsymbol{a}_K \in \mathbb{R}^{p_k}$ are vectors containing, respectively, the *unknown* and the *known* dynamic coefficients of the robot model. The $3 \times p_u$ matrix $\boldsymbol{Y}_U$ and $3 \times p_k$ matrix $\boldsymbol{Y}_K$ are the associated regressors in the linear parametrization of the dynamic model. Provide the explicit expressions of all terms in the adaptive control law (2). If time remains, sketch a proof of the asymptotic stability of the trajectory tracking error with this modified adaptive law.

[180 minutes (3 hours); open books]

Solution

July 12, 2021

Exercise #1

The second-order differential kinematics of a robot with n joints performing a m-dimensional task (with $m \leq n$) is

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{u} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{3}$$

with vector $\boldsymbol{h} \in \mathbb{R}^m$ being quadratic in $\dot{\boldsymbol{q}}$. The joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^n$ is taken here as input command $\boldsymbol{u}$. For the planar 3R robot of Fig. 1, we have $n = 3$, $m = 2$, and the terms in (3) are the 2×3 task Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = L \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix},$$

its time derivative

$$\dot{\boldsymbol{J}}(\boldsymbol{q}) = -L \cdot$$
$$\begin{pmatrix} c_1\dot{q}_1 + c_{12}(\dot{q}_1 + \dot{q}_2) + c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & c_{12}(\dot{q}_1 + \dot{q}_2) + c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ s_1\dot{q}_1 + s_{12}(\dot{q}_1 + \dot{q}_2) + s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & s_{12}(\dot{q}_1 + \dot{q}_2) + s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \end{pmatrix},$$

and the product of matrix $\dot{\boldsymbol{J}}$ by the joint velocity $\dot{\boldsymbol{q}}$

$$\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -L \cdot$$
$$\begin{pmatrix} (c_1 + c_{12} + c_{123})\,\dot{q}_1^2 + 2\,(c_{12} + c_{123})\,\dot{q}_1\dot{q}_2 + 2\,c_{123}\,\dot{q}_1\dot{q}_3 + (c_{12} + c_{123})\,\dot{q}_2^2 + 2\,c_{123}\,\dot{q}_2\dot{q}_3 + c_{123}\,\dot{q}_3^2 \\ (s_1 + s_{12} + s_{123})\,\dot{q}_1^2 + 2\,(s_{12} + s_{123})\,\dot{q}_1\dot{q}_2 + 2\,s_{123}\,\dot{q}_1\dot{q}_3 + (s_{12} + s_{123})\,\dot{q}_2^2 + 2\,s_{123}\,\dot{q}_2\dot{q}_3 + s_{123}\,\dot{q}_3^2 \end{pmatrix},$$

having used the shorthand notation for trigonometric functions (e.g., $c_{123} = \cos(q_1 + q_2 + q_3)$).

The minimum norm solution of (3) for $\ddot{\boldsymbol{p}} = \boldsymbol{0}$ at a generic state $(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0)$ is

$$\boldsymbol{u}_0 = -\boldsymbol{J}^{\#}(\boldsymbol{q}_0)\boldsymbol{h}(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0). \tag{4}$$

In the absence of bounds on the command $\boldsymbol{u}_0$, this acceleration would return $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$ if and only if $\boldsymbol{h}(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0) \in \mathcal{R}\{\boldsymbol{J}(\boldsymbol{q}_0)\}$. In particular, when rank $\{\boldsymbol{J}(\boldsymbol{q}_0)\} = 2$ (regular case), this condition is always satisfied and we only need to check whether $\boldsymbol{u}_0$ is feasible, i.e., if $|\boldsymbol{u}_{0,i}| \leq U_{max,i}$, for all $i = 1, 2, 3$. If so, then we stop. The same would happen in the singular case (rank $\{\boldsymbol{J}(\boldsymbol{q}_0)\} < 2$).

If instead the acceleration command (4) is unfeasible, we should attempt the use of the general solution to (3) for $\ddot{\boldsymbol{p}} = \boldsymbol{0}$, namely

$$\boldsymbol{u}_0 = -\boldsymbol{J}^{\#}(\boldsymbol{q}_0)\boldsymbol{h}(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0) + \Big(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q}_0)\boldsymbol{J}(\boldsymbol{q}_0)\Big)\,\ddot{\boldsymbol{q}}_0, \tag{5}$$

with an extra joint acceleration $\ddot{\boldsymbol{q}}_0$ projected in the null space of $\boldsymbol{J}$. This term may possibly help in recovering feasibility when the preferred minimum norm solution (4) is unfeasible. The definition of the actual command (5) is obtained by a simple variant of the SNS (Saturation in the Null Space) method for redundant robots.

Consider for simplicity only the regular case for the Jacobian, rank $\{\boldsymbol{J}(\boldsymbol{q}_0)\} = 2$. Taking into account that there is only $n - m = 1$ degree of redundancy in the problem, if two (or all) of the scalar components of (4) are out of bounds, then no feasible solution will exist in any case. Otherwise, Algorithm 1 (written in pseudo-code) will provide a feasible solution, if one exists. The bounds on the commands are organized in vector form as $\boldsymbol{U}_{max} = \begin{pmatrix} U_{max,1} & U_{max,2} & U_{max,3} \end{pmatrix}^T \in \mathbb{R}^3$. We set also $\boldsymbol{J}_0 = \boldsymbol{J}(\boldsymbol{q}_0)$, $\dot{\boldsymbol{J}}_0 = \dot{\boldsymbol{J}}(\boldsymbol{q}_0)$, and $\boldsymbol{h}_0 = \boldsymbol{h}(\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0) = \dot{\boldsymbol{J}}_0\dot{\boldsymbol{q}}_0$.

Algorithm 1 SNS method for finding a feasible solution, if it exists (case $n=3$, $m=2$)

if the command $\boldsymbol{u}_0$ in (4) satisfies $|\boldsymbol{u}_0| \leq \boldsymbol{U}_{max}$ (componentwise) **then**
  STOP % *the minimum norm solution (4) is feasible and returns already* $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$
else
  $j^* = \arg\max_{i=1,2,3} \left| \left(\boldsymbol{J}_0^{\#} \boldsymbol{h}_0 \right)_i \right|$ % *it is then* $|u_{0,j^*}| > U_{max,j^*}$*; by assumption,*
  % u_{0,j^*} *will be the only command out of bounds*
  set $u_{0,j^*} = \operatorname{sign}\left(-\boldsymbol{J}_0^{\#} \boldsymbol{h}_0\right)_{j^*} U_{max,j^*}$ % *the overdriven command is now saturated*
  define the 2×2 matrix $\bar{\boldsymbol{J}}_{\{-j^*\}}$ by deleting the j^*-th column $\boldsymbol{J}_{j^*}$ from $\boldsymbol{J}_0$,
   $\boldsymbol{u}_{\{-j^*\}} \in \mathbb{R}^2$ by deleting u_{j^*} from $\boldsymbol{u}$, and $\boldsymbol{U}_{max,\{-j^*\}} \in \mathbb{R}^2$ deleting U_{max,j^*} from $\boldsymbol{U}_{max}$
  set $\boldsymbol{a} = \boldsymbol{J}_{j^*} u_{0,j^*} + \boldsymbol{h}_0$ % *... one needs to solve* $\bar{\boldsymbol{J}}_{\{-j^*\}} \boldsymbol{u}_{\{-j^*\}} + \boldsymbol{a} = \boldsymbol{0}$
  compute $\boldsymbol{u}_{0,\{-j^*\}} = -\bar{\boldsymbol{J}}_{\{-j^*\}}^{\#} \boldsymbol{a}$ % *... a unique solution if rank* $\left\{ \bar{\boldsymbol{J}}_{\{-j^*\}} \right\} = 2$!!
  if $\left| \boldsymbol{u}_{0,\{-j^*\}} \right| \leq \boldsymbol{U}_{max,\{-j^*\}}$ (componentwise) **then**
   recompose $\boldsymbol{u}_{0,\mathrm{SNS}}$ from u_{0,j^*} and $\boldsymbol{u}_{0,\{-j^*\}}$
   STOP % *a new feasible solution* $\boldsymbol{u}_{0,\mathrm{SNS}}$ *has been found, returning* $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$
  else
   EXIT % *there is no feasible solution providing* $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$
  end if
end if

We provide next several numerical examples illustrating different situations[1]. In all cases, we have set $L = 1$ [m] for the link lengths and chosen

$$\boldsymbol{U}_{max} = \begin{pmatrix} 15\,\pi \\ 10\,\pi \\ 10\,\pi \end{pmatrix} = \begin{pmatrix} 47.1239 \\ 31.4159 \\ 31.4159 \end{pmatrix} \ [\mathrm{rad/s^2}]$$

as bounds for the acceleration commands. The same data will be used also for Problem #1b of this Exercise. Computations are performed in MATLAB.

1. *Regular configuration* $\boldsymbol{q}_0 = (0, \pi/2, \pi/2)$ [rad], *joint velocity* $\dot{\boldsymbol{q}}_0 = (\pi, \pi, 0)$ [rad/s]

   We evaluate the terms in (3)

$$\boldsymbol{J}_0 = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \quad \dot{\boldsymbol{J}}_0 = \begin{pmatrix} \pi & 2\pi & 2\pi \\ -2\pi & -2\pi & 0 \end{pmatrix}, \quad \boldsymbol{h}_0 = \begin{pmatrix} 3\,\pi^2 \\ -4\,\pi^2 \end{pmatrix} = \begin{pmatrix} 29.6088 \\ -39.4784 \end{pmatrix},$$

   and compute the pseudoinverse as

$$\boldsymbol{J}_0^{\#} = \boldsymbol{J}_0^T \left(\boldsymbol{J}_0 \, \boldsymbol{J}_0^T \right)^{-1} = \begin{pmatrix} -2/3 & 1/3 \\ -1/3 & -1/3 \\ 1/3 & -2/3 \end{pmatrix}.$$

   Note that the end-effector velocity $\dot{\boldsymbol{p}}_0 = \boldsymbol{J}_0 \dot{\boldsymbol{q}}_0 = \begin{pmatrix} -2\pi & -\pi \end{pmatrix}^T$ [m/s] is different from zero. Applying the minimum norm solution (4), we obtain the feasible command

$$\boldsymbol{u}_0 = -\boldsymbol{J}_0^{\#} \, \boldsymbol{h}_0 = \begin{pmatrix} 10\,\pi^2/3 \\ -\pi^2/3 \\ -11\,\pi^2/3 \end{pmatrix} = \begin{pmatrix} 32.8987 \\ -3.2899 \\ -36.1885 \end{pmatrix} \ [\mathrm{rad/s^2}],$$

   which returns $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$.

[1] Indeed, only one example was requested in the text.

2. *Same regular configuration* $\boldsymbol{q}_0 = (0, \pi/2, \pi/2)$ [rad], *new velocity* $\dot{\boldsymbol{q}}_0 = (\pi, \pi, -\pi/4)$ [rad/s]

   The Jacobian $\boldsymbol{J}_0$ and its pseudoinverse $\boldsymbol{J}_0^{\#}$ are the same as before. The different terms to be computed are

$$\dot{\boldsymbol{J}}_0 = \begin{pmatrix} 3\pi/4 & 7\pi/4 & 7\pi/4 \\ -2\pi & -2\pi & 0 \end{pmatrix} = \begin{pmatrix} 2.3562 & 5.4978 & 5.4978 \\ -6.2832 & -6.2832 & 0 \end{pmatrix}$$

   and

$$\boldsymbol{h}_0 = \begin{pmatrix} 33\,\pi^2/16 \\ -4\,\pi^2 \end{pmatrix} = \begin{pmatrix} 20.3561 \\ -39.4784 \end{pmatrix}.$$

   The end-effector velocity is $\dot{\boldsymbol{p}}_0 = \begin{pmatrix} -2\pi & -3\pi/4 \end{pmatrix}^T \neq \boldsymbol{0}$. The minimum norm solution (4) is now

$$\boldsymbol{u}_0 = -\boldsymbol{J}_0^{\#}\,\boldsymbol{h}_0 = \begin{pmatrix} 65\,\pi^2/24 \\ -31\,\pi^2/48 \\ -161\,\pi^2/48 \end{pmatrix} = \begin{pmatrix} 26.7302 \\ -6.3741 \\ -33.1043 \end{pmatrix} \text{ [rad/s}^2\text{]},$$

   which is unfeasible, being the third joint acceleration $u_{0,3} = -33.1043$ larger (in module) than its bound $U_{max,3} = 31.4159$. The other two acceleration commands remain instead within their bounds, so that we can apply Algorithm 1 to check if a feasible solution can be found by the SNS method. Being $j^* = 3$, we set $u_{0,3} = -U_{max,3} = -10\,\pi = -31.4159$ (saturation to the closest limit, the negative one) and compute

$$\bar{\boldsymbol{J}}_{\{-3\}} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \; \boldsymbol{a} = -\boldsymbol{J}_3\,U_{max,3} + \boldsymbol{h}_0 = -\begin{pmatrix} 0 \\ -1 \end{pmatrix} 10\,\pi + \begin{pmatrix} 33\,\pi^2/16 \\ -4\,\pi^2 \end{pmatrix} = \begin{pmatrix} 20.3561 \\ -8.0625 \end{pmatrix}.$$

   Thus

$$\boldsymbol{u}_{0,\{-3\}} = -\bar{\boldsymbol{J}}_{\{-3\}}^{\#}\,\boldsymbol{a} = -\bar{\boldsymbol{J}}_{\{-3\}}^{-1}\,\boldsymbol{a} = \begin{pmatrix} 28.4186 \\ -8.0625 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{u}_{0,\text{SNS}} = \begin{pmatrix} 28.4186 \\ -8.0625 \\ -31.4159 \end{pmatrix}.$$

   The SNS result $\boldsymbol{u}_{0,\text{SNS}}$ is now a feasible solution and returns again $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$.

3. *Singular configuration* $\boldsymbol{q}_0 = (0, 0, \pi)$ [rad], *joint velocity* $\dot{\boldsymbol{q}}_0 = (\pi/2, -\pi, \pi/2)$ [rad/s]

   The robot has the second link stretched and the third folded. This is a singular configuration since

$$\boldsymbol{J}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \qquad \Rightarrow \qquad \text{rank}\,\{\boldsymbol{J}_0\} = 1, \qquad \mathcal{R}\{\boldsymbol{J}_0\} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

   The end-effector is placed at $\boldsymbol{p}_0 = \boldsymbol{f}(\boldsymbol{q}_0) = (1, 0)$. Note also that the end-effector velocity is now

$$\dot{\boldsymbol{p}}_0 = \boldsymbol{J}_0 \dot{\boldsymbol{q}}_0 = \boldsymbol{0} \qquad \Longleftrightarrow \qquad \dot{\boldsymbol{q}}_0 = \begin{pmatrix} \pi/2 \\ -\pi \\ \pi/2 \end{pmatrix} \in \mathcal{N}\{\boldsymbol{J}_0\}.$$

   We evaluate the other terms in (3):

$$\dot{\boldsymbol{J}}_0 = \begin{pmatrix} 0 & \pi/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{h}_0 = \begin{pmatrix} -\pi^2/2 \\ 0 \end{pmatrix} = \begin{pmatrix} -4.9348 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{h}_0 \notin \mathcal{R}\{\boldsymbol{J}_0\}.$$

Computing the pseudoinverse of the singular task Jacobian,

$$\boldsymbol{J}_0^{\#} = \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \\ 0 & -0.5 \end{pmatrix},$$

we obtain that the minimum norm solution (4) is simply $\boldsymbol{u}_0 = -\boldsymbol{J}_0^{\#}\,\boldsymbol{h}_0 = \boldsymbol{0}$. Therefore, the end-effector acceleration cannot be modified in any case, remaining equal to

$$\ddot{\boldsymbol{p}}_0 = \boldsymbol{h}_0 = \begin{pmatrix} -4.9348 \\ 0 \end{pmatrix} \neq \boldsymbol{0}.$$

4. *Another singular case* $\boldsymbol{q}_0 = (0, \pi, -\pi)$ [rad], *same joint velocity* $\dot{\boldsymbol{q}}_0 = (\pi/2, -\pi, \pi/2)$ [rad/s]

   In this last example, the robot has the second and third links both folded. We are again in a singularity, being

$$\boldsymbol{J}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \qquad \text{rank}\,\{\boldsymbol{J}_0\} = 1, \qquad \mathcal{R}\{\boldsymbol{J}_0\} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

   The end-effector position $\boldsymbol{p}_0$ is the same as in the previous example. On the other hand, the end-effector velocity is now

$$\dot{\boldsymbol{p}}_0 = \boldsymbol{J}_0\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 0 \\ \pi \end{pmatrix} \neq \boldsymbol{0} \qquad \Longleftrightarrow \qquad \dot{\boldsymbol{p}}_0 \in \mathcal{R}\{\boldsymbol{J}_0\}.$$

   Moreover, being

$$\dot{\boldsymbol{J}}_0 = \begin{pmatrix} -\pi & -\pi/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{h}_0 = \dot{\boldsymbol{J}}_0\dot{\boldsymbol{q}}_0 = \boldsymbol{0} \quad (!!),$$

   the minimum norm solution (4) will be $\boldsymbol{u}_0 = -\boldsymbol{J}_0^{\#}\,\boldsymbol{h}_0 = \boldsymbol{0}$ as in the previous example. However, the (feasible) solution $\boldsymbol{u}_0$ will produce now $\ddot{\boldsymbol{p}}_0 = \boldsymbol{h}_0 = \boldsymbol{0}$, keeping thus the end-effector at the same instantaneous velocity $\dot{\boldsymbol{p}}_0$.

Exercise #2

Denote the inverse of the (symmetric) robot inertia matrix as

$$\boldsymbol{H}(\boldsymbol{q}) = \boldsymbol{M}^{-1}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{h}_1^T(\boldsymbol{q}) \\ \boldsymbol{h}_2^T(\boldsymbol{q}) \\ \vdots \\ \boldsymbol{h}_n^T(\boldsymbol{q}) \end{pmatrix},$$

with $\boldsymbol{h}_i(\boldsymbol{q})$ being the i-th column of $\boldsymbol{H}(\boldsymbol{q})$, for $i = 1, 2, \ldots, n$. Also, denote by $h_{ii}(\boldsymbol{q})$ the i-th element on the diagonal of $\boldsymbol{H}(\boldsymbol{q})$, for $i = 1, 2, \ldots, n$.

With reference to the dynamic model (1), the incipient blocking fault of the motor at the joint i is modeled by a vector $\boldsymbol{\tau}_f \in \mathbb{R}^n$ with the single nonzero i-th component having the expression

$$\tau_{f,i} = \frac{1}{h_{ii}(\boldsymbol{q})}\left(\boldsymbol{h}_i^T(\boldsymbol{q})\,(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})) + \lambda_i \dot{q}_i\right), \qquad \lambda_i \gg 1, \tag{6}$$

whereas $\tau_{f,j} = 0$, for all $j \neq i$.

In fact, since the acceleration vector $\ddot{\boldsymbol{q}} \in \mathbb{R}^n$ is given by

$$\ddot{\boldsymbol{q}} = \boldsymbol{H}(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{\tau}_f\right),$$

using (6), the scalar component of the acceleration at joint i will be

$$\begin{aligned}
\ddot{q}_i &= \boldsymbol{h}_i^T(\boldsymbol{q}) \left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q}) - \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \tau_{f,i} \right) \\
&= \boldsymbol{h}_i^T(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right) - h_{ii}(\boldsymbol{q}) \cdot \frac{1}{h_{ii}(\boldsymbol{q})} \left(\boldsymbol{h}_i^T(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right) + \lambda_i \dot{q}_i \right) \\
&= -\lambda_i \dot{q}_i,
\end{aligned}$$

as requested. Moreover, the acceleration at any other joint $j \neq i$ is

$$\ddot{q}_j = \boldsymbol{h}_j^T(\boldsymbol{q})\left(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right) - h_{ji}(\boldsymbol{q})\, \tau_{f,i},$$

showing that the fault of motor i will affect also the acceleration of the other joints, because of the inertial couplings of the inverse of the robot inertia matrix (the off-diagonal terms $h_{ji}(\boldsymbol{q})$, $j \neq i$).

Exercise #3

The robot in Fig. 2 has $n = 3$ joints and moves on the horizontal plane. Neglecting friction effects, its dynamic model is

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau}.$$

In computing the kinetic energy T of the robot, we take into account that the motion is planar; thus, linear velocities will be 2D vectors in the plane $(\boldsymbol{x}_w, \boldsymbol{y}_w)$, while angular velocities will be just scalars (in the $\boldsymbol{z}_w$-direction). For the first link, it is

$$T_1 = \frac{1}{2}\left(I_1 + m_1 d_{c1}^2\right)\dot{q}_1^2.$$

For the second link, since

$$\boldsymbol{p}_{c2} = (q_2 - d_{c2}) \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} &= \dot{q}_2 \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} + (q_2 - d_{c2})\,\dot{q}_1 \begin{pmatrix} -\sin q_1 \\ \cos q_1 \end{pmatrix} \\ &= \begin{pmatrix} \cos q_1 & -\sin q_1 \\ \sin q_1 & \cos q_1 \end{pmatrix} \begin{pmatrix} \dot{q}_2 \\ (q_2 - d_{c2})\,\dot{q}_1 \end{pmatrix} = \boldsymbol{R}(q_1) \begin{pmatrix} -\sin q_1 \\ \cos q_1 \end{pmatrix}, \end{aligned}$$

it follows

$$T_2 = \frac{1}{2} I_2\, \omega_2^2 + \frac{1}{2} m_2\, \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2} = \frac{1}{2}\left(I_2\, \dot{q}_1^2 + m_2\left((q_2 - d_{c2})^2\, \dot{q}_1^2 + \dot{q}_2^2\right)\right).$$

For the third link, from

$$\begin{aligned}
\boldsymbol{p}_{c3} &= \begin{pmatrix} q_2 \cos q_1 + d_{c3} \cos(q_1 + q_3) \\ q_2 \sin q_1 + d_{c3} \sin(q_1 + q_3) \end{pmatrix} \\
\Rightarrow \quad \boldsymbol{v}_{c3} &= \begin{pmatrix} \cos q_1 \dot{q}_2 - q_2 \sin q_1 \dot{q}_1 - d_{c3} \sin(q_1 + q_3)(\dot{q}_1 + \dot{q}_3) \\ \sin q_1 \dot{q}_2 + q_2 \cos q_1 \dot{q}_1 + d_{c3} \cos(q_1 + q_3)(\dot{q}_1 + \dot{q}_3) \end{pmatrix},
\end{aligned}$$

we obtain

$$
\begin{aligned}
T_3 &= \frac{1}{2} I_3\, \omega_3^2 + \frac{1}{2} m_3\, \boldsymbol{v}_{c3}^T \boldsymbol{v}_{c3} \\
&= \frac{1}{2} I_3 \left(\dot{q}_1 + \dot{q}_3\right)^2 + \frac{1}{2}\, m_3 \left(q_2^2\, \dot{q}_1^2 + \dot{q}_2^2 + d_{c3}^2 \left(\dot{q}_1 + \dot{q}_3\right)^2 + 2\, d_{c3} \left(q_2 \cos q_3\, \dot{q}_1 - \sin q_3\, \dot{q}_2\right)\left(\dot{q}_1 + \dot{q}_3\right)\right).
\end{aligned}
$$

Thus, being

$$
T = \sum_{i=1}^{3} T_i = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad \text{with} \quad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix},
$$

we can write the single elements of the symmetric inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ as follows:

$$
\begin{aligned}
m_{11} &= I_1 + m_1 d_{c1}^2 + I_2 + m_2 d_{c2}^2 + I_3 + m_3 d_{c3}^2 \\
&\quad - 2\, m_2 d_{c2}\, q_2 + (m_2 + m_3)\, q_2^2 + 2\, m_3 d_{c3}\, q_2 \cos q_3 \\
m_{12} &= -m_3 d_{c3} \sin q_3 \\
m_{13} &= I_3 + m_3 d_{c3}^2 + m_3 d_{c3}\, q_2 \cos q_3 \\
m_{22} &= m_2 + m_3 \\
m_{23} &= -m_3 d_{c3} \sin q_3 \\
m_{33} &= I_3 + m_3 d_{c3}^2.
\end{aligned} \tag{7}
$$

The inertial term in the dynamic model can be rewritten in terms of $p = 5$ coefficients that collect the dynamic parameters of the robot,

$$
\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \begin{pmatrix} a_1 + 2a_2 q_2 + a_3 q_2^2 + 2a_4 q_2 \cos q_3 & -a_4 \sin q_3 & a_5 + a_4 q_2 \cos q_3 \\ -a_4 \sin q_3 & a_3 & -a_4 \sin q_3 \\ a_5 + a_4 q_2 \cos q_3 & -a_4 \sin q_3 & a_5 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} = \boldsymbol{Y}_M(\boldsymbol{q}, \ddot{\boldsymbol{q}})\boldsymbol{a},
$$

with the vector of dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^5$ defined by

$$
\begin{aligned}
a_1 &= I_1 + m_1 d_{c1}^2 + I_2 + m_2 d_{c2}^2 + I_3 + m_3 d_{c3}^2 \\
a_2 &= -m_2 d_{c2} \\
a_3 &= m_2 + m_3 \\
a_4 &= m_3 d_{c3} \\
a_5 &= I_3 + m_3 d_{c3}^2.
\end{aligned} \tag{8}
$$

Similarly, the entire dynamic model (because of the absence of gravity) is linearly parametrized in terms of the same dynamic coefficients $\boldsymbol{a}$, with a suitable $n \times p = 3 \times 5$ regressor matrix $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$. However, the parametrization (8) does not separate the unknown from the known dynamic coefficients in the inertia matrix (and thus in the dynamic model). On the other hand, the proposed adaptive law (2) does not need updating coefficients that involve only known dynamic parameters, i.e., m_3, d_{c3}, and I_3.

Therefore, looking at the expressions of the elements m_{ij} in (7) and at the definitions of the coefficients a_i in (8), we can organize and separate terms in the inertia matrix as

$$\begin{aligned}
m_{11} &= \left(I_1 + m_1 d_{c1}^2 + I_2 + m_2 d_{c2}^2\right) + I_3 + m_3 d_{c3}^2 - 2\, m_2 d_{c2}\, q_2 + (m_2 + m_3)\, q_2^2 + 2\, m_3 d_{c3}\, q_2 \cos q_3 \\
&= a_{U1} + a_{K1} + 2\, a_{U2}\, q_2 + (a_{U3} + a_{K2})\, q_2^2 + 2\, a_{K3}\, q_2 \cos q_3 \\
m_{12} &= -m_3 d_{c3} \sin q_3 = -a_{K3} \sin q_3 \\
m_{13} &= I_3 + m_3 d_{c3}^2 + m_3 d_{c3}\, q_2 \cos q_3 = a_{K1} + a_{K3}\, q_2 \cos q_3 \\
m_{22} &= m_2 + m_3 = a_{U3} + a_{K2} \\
m_{23} &= -m_3 d_{c3} \sin q_3 = -a_{K3} \sin q_3 \\
m_{33} &= I_3 + m_3 d_{c3}^2 = a_{K1},
\end{aligned}$$

with $p_u = 3$ *unknown* dynamic coefficients

$$\begin{aligned}
a_{U1} &= I_1 + m_1 d_{c1}^2 + I_2 + m_2 d_{c2}^2 \\
a_{U2} &= -m_2 d_{c2} \\
a_{U3} &= m_2,
\end{aligned} \tag{9}$$

organized as components of a vector $\boldsymbol{a}_U \in \mathbb{R}^3$, and, respectively, $p_k = 3$ *known* dynamic coefficients

$$\begin{aligned}
a_{K1} &= I_3 + m_3 d_{c3}^2 \\
a_{K2} &= m_3 \\
a_{K3} &= m_3 d_{c3},
\end{aligned} \tag{10}$$

organized as components of a vector $\boldsymbol{a}_K \in \mathbb{R}^3$. Despite the total number of dynamic coefficients is now higher than before ($p_u + p_k = 6 > 5 = p$), the number of those to be updated in the adaptive law is actually lower ($p_u = 3$).

To complete the dynamic modeling, we have to derive also the Coriolis and centrifugal terms. This will be done by referring directly to the double parametrization by $\boldsymbol{a}_U$ and $\boldsymbol{a}_K$, in view of their final use in the adaptive control law (2). Rewrite the robot inertia matrix as

$$\begin{aligned}
&\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{m}_1(\boldsymbol{q}) & \boldsymbol{m}_2(\boldsymbol{q}) & \boldsymbol{m}_3(\boldsymbol{q}) \end{pmatrix} = \\
&\begin{pmatrix} a_{U1} + a_{K1} + 2\, a_{U2}\, q_2 + (a_{U3} + a_{K2})\, q_2^2 + 2\, a_{K3}\, q_2 \cos q_3 & -a_{K3} \sin q_3 & a_{K1} + a_{K3}\, q_2 \cos q_3 \\ -a_{K3} \sin q_3 & a_{U3} + a_{K2} & -a_{K3} \sin q_3 \\ a_{K1} + a_{K3}\, q_2 \cos q_3 & -a_{K3} \sin q_3 & a_{K1} \end{pmatrix}.
\end{aligned} \tag{11}$$

Using the Christoffel's symbols for computing the components $c_i(\boldsymbol{q}, \dot{\boldsymbol{q}})$ of the Coriolis and centrifugal vector $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$,

$$\boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i} \right), \qquad c_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q}) \dot{\boldsymbol{q}}, \qquad i = 1, 2, 3,$$

we obtain

$$\boldsymbol{C}_1(\boldsymbol{q}) \;=\; \frac{1}{2}\left(\begin{pmatrix} 0 & 2\,a_{U2}+2\,(a_{U3}+a_{K2})\,q_2+2\,a_{K3}\cos q_3 & -2\,a_{K3}\,q_2\sin q_3 \\ 0 & 0 & -a_{K3}\cos q_3 \\ 0 & a_{K3}\cos q_3 & -a_{K3}\,q_2\sin q_3 \end{pmatrix} + \begin{pmatrix} & \dots & \end{pmatrix}^T - \boldsymbol{O}\right)$$

$$= \begin{pmatrix} 0 & a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3 & -a_{K3}\,q_2\sin q_3 \\ a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3 & 0 & 0 \\ -a_{K3}\,q_2\sin q_3 & 0 & -a_{K3}\,q_2\sin q_3 \end{pmatrix}$$

$$\Rightarrow \qquad c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) = 2\,(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot q_1\dot q_2 - a_{K3}\,q_2\sin q_3\,(2\,\dot q_1+\dot q_3)\,\dot q_3,$$

$$\boldsymbol{C}_2(\boldsymbol{q}) \;=\; \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & -a_{K3}\cos q_3 \\ 0 & 0 & 0 \\ 0 & 0 & -a_{K3}\cos q_3 \end{pmatrix} + \begin{pmatrix} \dots \end{pmatrix}^T - \begin{pmatrix} 2\,a_{U2}+2\,(a_{U3}+a_{K2})\,q_2+2\,a_{K3}\cos q_3 & 0 & a_{K3}\cos q_3 \\ 0 & 0 & 0 \\ a_{K3}\cos q_3 & 0 & 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -\,(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3) & 0 & -a_{K3}\cos q_3 \\ 0 & 0 & 0 \\ -a_{K3}\cos q_3 & 0 & -a_{K3}\cos q_3 \end{pmatrix}$$

$$\Rightarrow \qquad c_2(\boldsymbol{q},\dot{\boldsymbol{q}}) = -\,(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot q_1^2 - a_{K3}\cos q_3\,(2\,\dot q_1+\dot q_3)\,\dot q_3,$$

$$\boldsymbol{C}_3(\boldsymbol{q}) \;=\; \frac{1}{2}\left(\begin{pmatrix} 0 & a_{K3}\cos q_3 & -a_{K3}\,q_2\sin q_3 \\ 0 & 0 & -a_{K3}\cos q_3 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} & \dots & \end{pmatrix}^T \right.$$

$$\left. - \begin{pmatrix} -2a_{K3}\,q_2\sin q_3 & -a_{K3}\cos q_3 & -a_{K3}\,q_2\sin q_3 \\ -a_{K3}\cos q_3 & 0 & -a_{K3}\cos q_3 \\ -a_{K3}\,q_2\sin q_3 & -a_{K3}\cos q_3 & 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} a_{K3}\,q_2\sin q_3 & a_{K3}\cos q_3 & 0 \\ a_{K3}\cos q_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \qquad c_3(\boldsymbol{q},\dot{\boldsymbol{q}}) = a_{K3}\,q_2\sin q_3\,\dot q_1^2 + 2\,a_{K3}\cos q_3\,\dot q_1\dot q_2,$$

yielding

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} 2\,(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot q_1\dot q_2 - a_{K3}\,q_2\sin q_3\,(2\,\dot q_1+\dot q_3)\,\dot q_3 \\ -\,(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot q_1^2 - a_{K3}\cos q_3\,(2\,\dot q_1+\dot q_3)\,\dot q_3 \\ a_{K3}\,q_2\sin q_3\,\dot q_1^2 + 2\,a_{K3}\cos q_3\,\dot q_1\dot q_2 \end{pmatrix} = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}},$$

with the factorizing matrix

$$\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \mathrm{col}\left\{\dot{\boldsymbol{q}}^T\boldsymbol{C}_i(\boldsymbol{q},\dot{\boldsymbol{q}})\right\} = \tag{12}$$

$$\begin{pmatrix} \begin{matrix}(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot q_2 \\ -a_{K3}\,q_2\sin q_3\,\dot q_3\end{matrix} & (a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot q_1 & -a_{K3}\,q_2\sin q_3\,(\dot q_1+\dot q_3) \\ \begin{matrix}-\,(a_{U2}+(a_{U3}+a_{K2})\,q_2+a_{K3}\cos q_3)\,\dot q_1 \\ -a_{K3}\cos q_3\,\dot q_3\end{matrix} & 0 & -a_{K3}\cos q_3\,(\dot q_1+\dot q_3) \\ a_{K3}\,q_2\sin q_3\,\dot q_1 + a_{K3}\cos q_3\,\dot q_2 & a_{K3}\cos q_3\,\dot q_1 & 0 \end{pmatrix}$$

being such that $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is skew-symmetric (check this!).

As a result, the complete dynamic model can be linearly parametrized as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a}_U + \boldsymbol{Y}_K(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a}_K \tag{13}$$

with the $n \times p_u = 3 \times 3$ regressor matrix $\boldsymbol{Y}_U$ for the unknown coefficients (conveniently separating the contributions by the inertial and by the velocity terms)

$$\begin{aligned}\boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) &= \boldsymbol{Y}_{U,M}(\boldsymbol{q},\ddot{\boldsymbol{q}}) + \boldsymbol{Y}_{U,c}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ &= \begin{pmatrix} \ddot{q}_1 & 2\,q_2\,\ddot{q}_1 & q_2^2\,\ddot{q}_1 \\ 0 & 0 & \ddot{q}_2 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2\,\dot{q}_1\dot{q}_2 & q_2\,\dot{q}_1\dot{q}_2 \\ 0 & -\dot{q}_1^2 & -q_2\,\dot{q}_1^2 \\ 0 & 0 & 0 \end{pmatrix}\end{aligned} \tag{14}$$

and, similarly, with the $n \times p_k = 3 \times 3$ regressor matrix $\boldsymbol{Y}_K$ for the known coefficients

$$\begin{aligned}\boldsymbol{Y}_K(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) &= \boldsymbol{Y}_{K,M}(\boldsymbol{q},\ddot{\boldsymbol{q}}) + \boldsymbol{Y}_{K,c}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ &= \begin{pmatrix} \ddot{q}_1 + \ddot{q}_3 & q_2^2\,\ddot{q}_1 & q_2\cos q_3\,(2\,\ddot{q}_1 + \ddot{q}_3) - \sin q_3\,\ddot{q}_2 \\ 0 & \ddot{q}_2 & -\sin q_3\,(\ddot{q}_1 + \ddot{q}_3) \\ \ddot{q}_1 + \ddot{q}_3 & 0 & q_2\cos q_3\,\ddot{q}_1 - \sin q_3\,\ddot{q}_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 2\,q_2\,\dot{q}_1\dot{q}_2 & 2\,\cos q_3\,\dot{q}_1\dot{q}_2 - q_2\sin q_3\,(2\,\dot{q}_1 + \dot{q}_3)\,\dot{q}_3 \\ 0 & -q_2\,\dot{q}_1^2 & -\cos q_3\,(\dot{q}_1 + \dot{q}_3)^2 \\ 0 & 0 & q_2\sin q_3\,\dot{q}_1^2 + 2\cos q_3\,\dot{q}_1\dot{q}_2 \end{pmatrix}.\end{aligned} \tag{15}$$

To complete the design of the (partly) adaptive control law (2), we need to evaluate the two above regressors using suitable arguments. In particular, for the regressor $\boldsymbol{Y}_U$ we have

$$\boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r) = \boldsymbol{Y}_{U,M}(\boldsymbol{q},\ddot{\boldsymbol{q}}_r) + \boldsymbol{Y}_{U,c}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r)\,. \tag{16}$$

Inside the second (velocity-dependent) addend, we have to split the quadratic velocity terms by exploiting the factorization given by matrix $\boldsymbol{S}$ in (12). For this, let

$$\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}_U(\boldsymbol{q}) + \boldsymbol{M}_K(\boldsymbol{q})$$

be a decomposition of the inertia matrix in elements that depends (linearly) only on $\boldsymbol{a}_U$ and, respectively, only on $\boldsymbol{a}_K$. Accordingly, one can decompose also

$$\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}_U(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{S}_K(\boldsymbol{q},\dot{\boldsymbol{q}}).$$

It can be easily shown that the three matrices

$$\dot{\boldsymbol{M}} - 2\boldsymbol{S}, \qquad \dot{\boldsymbol{M}}_U - 2\boldsymbol{S}_U, \qquad \dot{\boldsymbol{M}}_K - 2\boldsymbol{S}_K,$$

satisfy all the skew-symmetry property (as requested by the adaptive control law —see below). In particular, for this robot we have

$$\boldsymbol{M}_U(\boldsymbol{q}) = \begin{pmatrix} a_{U1} + 2\,a_{U2}\,q_2 + a_{U3}\,q_2^2 & 0 & 0 \\ 0 & a_{U3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\boldsymbol{S}_U(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} (a_{U2}+a_{U3}\,q_2)\,\dot{q}_2 & (a_{U2}+a_{U3}\,q_2)\,\dot{q}_1 & 0 \\ -\,(a_{U2}+a_{U3}\,q_2)\,\dot{q}_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify the skew-symmetry of $\dot{\boldsymbol{M}}_U - 2\boldsymbol{S}_U$. As a result, the second term $\boldsymbol{Y}_{U,c}$ in (16) is obtained by using the identity

$$\boldsymbol{S}_U(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}_r = \boldsymbol{Y}_{U,c}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r)\,\boldsymbol{a}_U = \begin{pmatrix} 0 & \dot{q}_1\dot{q}_{2r}+\dot{q}_2\dot{q}_{1r} & q_2\,(\dot{q}_1\dot{q}_{2r}+\dot{q}_2\dot{q}_{1r}) \\ 0 & -\dot{q}_1\dot{q}_{1r} & -q_2\,\dot{q}_1\dot{q}_{1r} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_{U1} \\ a_{U2} \\ a_{U3} \end{pmatrix}. \tag{17}$$

Summarizing, the regressor matrix $\boldsymbol{Y}_U$ needed in control (2), together with its transpose for the adaptation law, is computed by (16) using $\boldsymbol{Y}_{U,M}(\boldsymbol{q},\ddot{\boldsymbol{q}}_r)$ from (14) and $\boldsymbol{Y}_{U,c}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r)$ from (17):

$$\boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r) = \begin{pmatrix} \ddot{q}_{1r} & 2\,q_2\,\ddot{q}_{1r}+\dot{q}_1\dot{q}_{2r}+\dot{q}_2\dot{q}_{1r} & q_2^2\,\ddot{q}_{2r}+q_2\,(\dot{q}_1\dot{q}_{2r}+\dot{q}_2\dot{q}_{1r}) \\ 0 & -\dot{q}_1\dot{q}_{1r} & \ddot{q}_{2r}-q_2\,\dot{q}_1\dot{q}_{1r} \\ 0 & 0 & 0 \end{pmatrix}.$$

The same considerations can be repeated for $\boldsymbol{M}_K$, $\boldsymbol{S}_K$, and $\boldsymbol{Y}_{K,c}$, leading to $\boldsymbol{Y}_K(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)$ (these computations are left to the reader).

The modified adaptive control law (2) provides (global) asymptotic stability of the trajectory tracking error. The proof follows the same argument as in the complete adaptive case (simpler in the absence of friction terms), once the above partitioned notation for unknown and known terms has been introduced. Consider in fact the same positive definite Lyapunov candidate

$$V = \frac{1}{2}\boldsymbol{s}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{s} + \boldsymbol{e}^T\boldsymbol{K}_P\,\boldsymbol{e} + \frac{1}{2}\tilde{\boldsymbol{a}}_U^T\,\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{a}}_U \geq 0, \tag{18}$$

where $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$, $\boldsymbol{s} = \dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}} = \dot{\boldsymbol{e}} + \boldsymbol{\Lambda}\boldsymbol{e}$, $\boldsymbol{\Lambda} = \boldsymbol{K}_D^{-1}\boldsymbol{K}_P > 0$, $\tilde{\boldsymbol{a}}_U = \boldsymbol{a}_U - \hat{\boldsymbol{a}}_U$, and the gain matrices $\boldsymbol{K}_P > 0$, $\boldsymbol{K}_D > 0$ and $\boldsymbol{\Gamma} > 0$ have been chosen as diagonal. Note that only the unknown dynamic coefficients, those that need to be updated online, and their estimates appear in the Lyapunov candidate. The time derivative of (18) is

$$\dot{V} = \frac{1}{2}\boldsymbol{s}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\boldsymbol{s} + \boldsymbol{s}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}} + 2\,\boldsymbol{e}^T\boldsymbol{K}_P\dot{\boldsymbol{e}} - \tilde{\boldsymbol{a}}_U^T\,\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{a}}}_U \tag{19}$$

The closed-loop dynamics, i.e., eq. (13) with the control (2), is

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} &= \boldsymbol{Y}_U\,(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\hat{\boldsymbol{a}}_U + \boldsymbol{Y}_K\,(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\boldsymbol{a}_K + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}} \\ &= \hat{\boldsymbol{M}}_U(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \hat{\boldsymbol{S}}_U(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \boldsymbol{M}_K(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \boldsymbol{S}_K(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}}. \end{aligned} \tag{20}$$

Subtracting both sides of eq. (20) from the identity

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r = \boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\boldsymbol{a}_U + \boldsymbol{Y}_K(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\boldsymbol{a}_K$$

yields

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}} + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{s} &= \boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,(\boldsymbol{a}_U - \hat{\boldsymbol{a}}_U) - \boldsymbol{K}_P\boldsymbol{e} - \boldsymbol{K}_D\dot{\boldsymbol{e}} \\ &= \boldsymbol{Y}_U(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\,\tilde{\boldsymbol{a}}_U - (\boldsymbol{K}_P\boldsymbol{e} + \boldsymbol{K}_D\dot{\boldsymbol{e}}) \end{aligned} \tag{21}$$

where the known term $\boldsymbol{Y}_K \boldsymbol{a}_K$ has been cancelled. Substituting $\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{s}}$ from (21) into (19) and using the update law $\dot{\hat{\boldsymbol{a}}}_U = \boldsymbol{\Gamma}\, \boldsymbol{Y}_U^T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, \boldsymbol{s}$ gives

$$
\begin{aligned}
\dot{V} &= \frac{1}{2}\, \boldsymbol{s}^T \left(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\, \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \right) \boldsymbol{s} + \boldsymbol{s}^T \boldsymbol{Y}_U(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, \tilde{\boldsymbol{a}}_U - \boldsymbol{s}^T \left(\boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_D \dot{\boldsymbol{e}} \right) \\
&\quad + 2\, \boldsymbol{e}^T \boldsymbol{K}_P \dot{\boldsymbol{e}} - \tilde{\boldsymbol{a}}_U^T\, \boldsymbol{\Gamma}^{-1} \cdot \boldsymbol{\Gamma}\, \boldsymbol{Y}_U^T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\, \boldsymbol{s} \\
&= -\left(\dot{\boldsymbol{e}} + \boldsymbol{K}_D^{-1} \boldsymbol{K}_P\, \boldsymbol{e} \right)^T \left(\boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_D \dot{\boldsymbol{e}} \right) + 2\, \boldsymbol{e}^T \boldsymbol{K}_P \dot{\boldsymbol{e}} \\
&= -\dot{\boldsymbol{e}}^T \boldsymbol{K}_D\, \dot{\boldsymbol{e}} - \boldsymbol{e}^T \boldsymbol{K}_P \boldsymbol{K}_D^{-1} \boldsymbol{K}_P\, \boldsymbol{e} \leq 0,
\end{aligned}
$$

where we used the skew-symmetry of $\dot{\boldsymbol{M}} - 2\, \boldsymbol{S}$ and the diagonality of the gain matrices. The proof is completed by Barbalat lemma and LaSalle theorem.

* * * * *

Robotics 2

September 10, 2021

Exercise #1

Consider the planar 3R robot in Fig. 1. The three links have all equal length L. The robot is controlled by a joint acceleration command $\boldsymbol{u} = \ddot{\boldsymbol{q}} \in \mathbb{R}^3$. The input commands are bounded componentwise as $|u_i| \leq U_{max,i}$, for $i = 1, 2, 3$. Moreover, let $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) \in \mathbb{R}^2$ be the end-effector position. At a given instant $t = t_0$, the robot is in a generic state $(\boldsymbol{q}(t_0), \dot{\boldsymbol{q}}(t_0)) = (\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0) \in \mathbb{R}^6$, with $\dot{\boldsymbol{q}}_0 \notin \mathcal{N}\{\boldsymbol{J}(\boldsymbol{q}_0)\}$. Which feasible command $\boldsymbol{u}_0 = \boldsymbol{u}(t_0)$ would you apply to *stop as fast as possible* the Cartesian motion of the end-effector, while *keeping its velocity aligned* with the direction of $\dot{\boldsymbol{p}}_0 = \dot{\boldsymbol{p}}(t_0) \neq \boldsymbol{0}$? If there are multiple feasible solutions, provide the one having minimum norm. Illustrate your findings with a numerical example, providing the values of $\boldsymbol{q}_0$, $\dot{\boldsymbol{q}}_0$, $\boldsymbol{u}_0$, and of the resulting acceleration $\ddot{\boldsymbol{p}}_0 = \ddot{\boldsymbol{p}}(t_0)$.

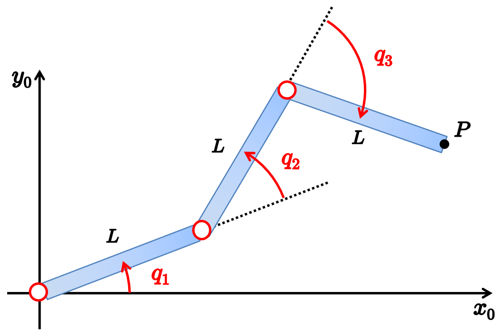


Figure 1: A planar 3R robot.

Exercise #2

For the same robot in Fig. 1, assume that the three links of equal length $L = 0.5$ [m] are all modeled as thin rods with a uniformly distributed mass of $m = 5$ [kg]. Provide the eigenvalues of the 2×2 Cartesian inertia matrix $\boldsymbol{M}_p$, when the robot is in the regular configuration $\boldsymbol{q}^* = (\pi/2, \pi/2, 0)$.

Hint: Use an equivalent expression for the Cartesian inertia matrix $\boldsymbol{M}_p = \boldsymbol{J}^{-T}\boldsymbol{M}\boldsymbol{J}^{-1}$ *that applies both to square and non-square Jacobians under the same full rank assumption.*

Exercise #3

The end-effector of a 2-dof Cartesian robot with different link masses m_1 and m_2 moves in a vertical plane $(\boldsymbol{x}, \boldsymbol{y})$ making contact with an environment. There is no force/torque sensor mounted on the robot. Design an impedance control law that shapes the response between interacting forces and tracking errors by assigning the same two real, negative and coincident eigenvalues (i.e., in $-\lambda < 0$) to the closed-loop linear dynamics along the two decoupled directions $\boldsymbol{x}$ and $\boldsymbol{y}$.

[180 minutes (3 hours); open books]

Solution

September 10, 2021

Exercise #1

The second-order differential kinematics of a robot with n joints performing a m-dimensional task (with $m \leq n$) is

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{u} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{1}$$

with vector $\boldsymbol{h} \in \mathbb{R}^m$ being quadratic in $\dot{\boldsymbol{q}}$. The joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^n$ is taken as the input command $\boldsymbol{u}$.

At time $t = t_0$, the task velocity is $\dot{\boldsymbol{p}}_0 = \dot{\boldsymbol{p}}(t_0) = \boldsymbol{J}(\boldsymbol{q}(t_0))\,\dot{\boldsymbol{q}}(t_0) = \boldsymbol{J}_0\dot{\boldsymbol{q}}_0$, which is necessarily different from zero since one should choose a $\dot{\boldsymbol{q}}_0 \notin \mathcal{N}\{\boldsymbol{J}_0\}$. We impose to the end-effector an acceleration $\ddot{\boldsymbol{p}}_0$ (actually, a deceleration) that is aligned with $\dot{\boldsymbol{p}}_0$ and whose components are opposite in sign to the associated velocity components. Therefore, we set

$$\ddot{\boldsymbol{p}}_0 = -\lambda\,\dot{\boldsymbol{p}}_0 = -\lambda\,\boldsymbol{J}_0\dot{\boldsymbol{q}}_0, \qquad \text{with } \lambda \geq 0,$$

and choose the largest possible (non-negative) value for the scalar λ such that the minimum norm joint acceleration solution $\boldsymbol{u}_0$ to (1) is feasible. It is then

$$\boldsymbol{u}_0 = \boldsymbol{J}_0^{\#}\left(\ddot{\boldsymbol{p}}_0 - \boldsymbol{h}_0\right) = -\lambda\,\boldsymbol{J}_0^{\#}\boldsymbol{J}_0\,\dot{\boldsymbol{q}}_0 - \boldsymbol{J}_0^{\#}\dot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0. \tag{2}$$

Define now the two n-dimensional vectors[1]

$$\boldsymbol{a} = -\boldsymbol{J}_0^{\#}\boldsymbol{J}_0\,\dot{\boldsymbol{q}}_0, \qquad \boldsymbol{b} = -\boldsymbol{J}_0^{\#}\dot{\boldsymbol{J}}_0\,\dot{\boldsymbol{q}}_0, \tag{3}$$

and organize the bounds on the commands in vector form as

$$\boldsymbol{U}_{max} = \begin{pmatrix} U_{max,1} \\ U_{max,2} \\ \vdots \\ U_{max,n} \end{pmatrix}.$$

The problem is formulated as a simple linear program (LP) as follows:

$$\max \lambda \qquad \text{s.t.} \quad -\boldsymbol{U}_{max} \leq \boldsymbol{a}\,\lambda + \boldsymbol{b} \leq \boldsymbol{U}_{max}, \quad \lambda \geq 0, \tag{4}$$

where vector inequalities are to be considered component-wise. Note first that $\boldsymbol{a} \neq \boldsymbol{0}$ (although some of its components may possibly vanish). In fact, $\dot{\boldsymbol{p}}_0 = \boldsymbol{J}_0\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$ is a realizable velocity, as generated by $\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$; thus, the pseudoinverse of such task velocity cannot produce a zero joint velocity. The feasible set may be empty, in which case no instantaneous acceleration solution exists. Moreover, if the optimal value of problem (4) is $\lambda = 0$, the end-effector will not be able to instantaneously decelerate; the problem has again no actual solution at $t = t_0$. Nonetheless, it is convenient to keep the value $\lambda = 0$ in the feasible set, so as to guarantee the existence of a solution to problem (4) whenever its (closed) feasible set is non-empty.

[1] One can also define the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with a positive sign in front. Being the bounds on the command $\boldsymbol{u}$ symmetric, the linear inequalities in (4) would remain the same.

The optimal solution λ^* to (4) is easily found. For $i = 1, \ldots, n$, let

$$\lambda_i = \begin{cases} -\infty & \text{if } b_i < -U_{max,i} \text{ and } a_i \le 0, \\ \dfrac{U_{max,i} - b_i}{a_i} & \text{if } b_i < -U_{max,i} \text{ and } a_i > 0, \\ \max\left\{-\dfrac{U_{max,i} + b_i}{a_i}, \dfrac{U_{max,i} - b_i}{a_i}\right\} & \text{if } U_{max,i} \le b_i \le U_{max,i} \text{ and } a_i \ne 0, \\ +\infty & \text{if } U_{max,i} \le b_i \le U_{max,i} \text{ and } a_i = 0, \\ -\dfrac{U_{max,i} + b_i}{a_i} & \text{if } b_i > U_{max,i} \text{ and } a_i < 0, \\ -\infty & \text{if } b_i > U_{max,i} \text{ and } a_i \ge 0. \end{cases} \tag{5}$$

We compute then

$$\lambda^* = \min_{i=1,\ldots,n} \lambda_i, \tag{6}$$

with the following conclusions:

$\lambda^* > 0 \quad \Rightarrow \quad \lambda^*$ is the optimal solution, with a feasible acceleration $\boldsymbol{u}_0^* = \boldsymbol{a}\lambda^* + \boldsymbol{b}$;

$\lambda^* = 0 \quad \Rightarrow \quad$ the resulting joint acceleration is $\boldsymbol{u}_0 = -\boldsymbol{J}_0^{\#} \dot{\boldsymbol{J}}_0 \dot{\boldsymbol{q}}_0$, yielding $\ddot{\boldsymbol{p}}_0 = \boldsymbol{0}$;

$\lambda^* = -\infty \quad \Rightarrow \quad$ there is no solution to the problem (the feasible set is empty).

In the optimal solution $\boldsymbol{u}_0^*$, at least one joint acceleration will saturate one of its bounds. When $\lambda^* = 0$, the end-effector will keep instantaneously the same velocity $\dot{\boldsymbol{p}}_0$, with no deceleration. When there is no solution to the problem, the end-effector will no longer be able to move exactly along the direction of $\dot{\boldsymbol{p}}_0$ (in either way). Some of the various possible situations for a generic single component λ_i are illustrated in Fig. 2. Figure 3 shows geometrically some resulting cases for $\boldsymbol{u}_0^*$ with $n = 2$ components (thus, when $m = 1$).

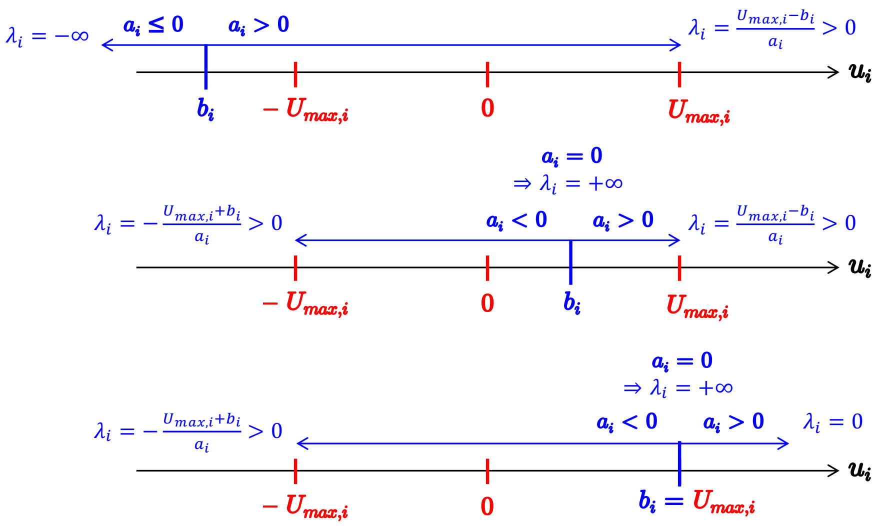


Figure 2: Examples of evaluation of λ_i for a generic single component.

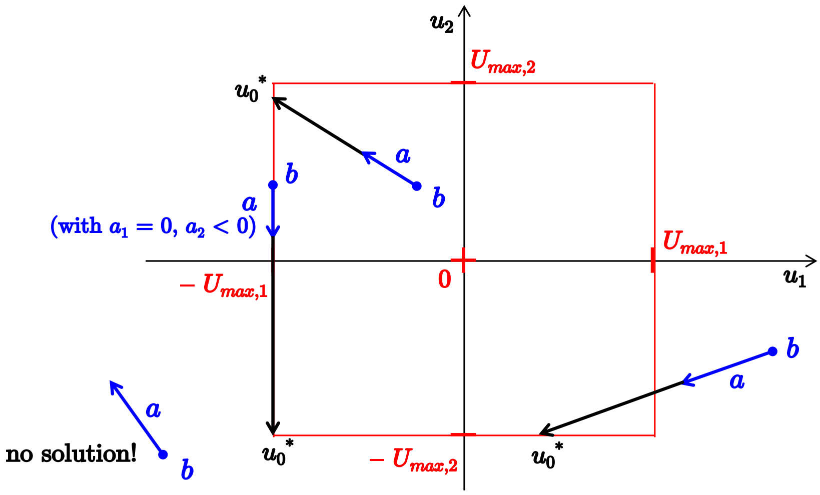


Figure 3: Examples of existence or not of a solution $\boldsymbol{u}_0^*$ and its geometrical evaluation when $n = 2$.

For the planar 3R robot of Fig. 1, we have $n = 3$, $m = 2$, and the terms in (1) are the 2×3 task Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = L \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix}, \tag{7}$$

its time derivative

$$\dot{\boldsymbol{J}}(\boldsymbol{q}) = -L \cdot$$
$$\begin{pmatrix} c_1\dot{q}_1 + c_{12}(\dot{q}_1 + \dot{q}_2) + c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & c_{12}(\dot{q}_1 + \dot{q}_2) + c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & c_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \\ s_1\dot{q}_1 + s_{12}(\dot{q}_1 + \dot{q}_2) + s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & s_{12}(\dot{q}_1 + \dot{q}_2) + s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) & s_{123}(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \end{pmatrix},$$

and the product of matrix $\dot{\boldsymbol{J}}$ by the joint velocity $\dot{\boldsymbol{q}}$

$$\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -L \begin{pmatrix} (c_1 + c_{12} + c_{123})\,\dot{q}_1^2 + 2\,(c_{12} + c_{123})\,\dot{q}_1\dot{q}_2 + 2\,c_{123}\,\dot{q}_1\dot{q}_3 + (c_{12} + c_{123})\,\dot{q}_2^2 + 2\,c_{123}\,\dot{q}_2\dot{q}_3 + c_{123}\,\dot{q}_3^2 \\ (s_1 + s_{12} + s_{123})\,\dot{q}_1^2 + 2\,(s_{12} + s_{123})\,\dot{q}_1\dot{q}_2 + 2\,s_{123}\,\dot{q}_1\dot{q}_3 + (s_{12} + s_{123})\,\dot{q}_2^2 + 2\,s_{123}\,\dot{q}_2\dot{q}_3 + s_{123}\,\dot{q}_3^2 \end{pmatrix},$$

having used the shorthand notation for trigonometric functions (e.g., $c_{123} = \cos(q_1 + q_2 + q_3)$).

As a first numerical example, set $L = 1$ [m] for the link lengths and choose

$$\boldsymbol{U}_{max} = \begin{pmatrix} 15\,\pi \\ 10\,\pi \\ 10\,\pi \end{pmatrix} = \begin{pmatrix} 47.1239 \\ 31.4159 \\ 31.4159 \end{pmatrix} \ [\text{rad/s}^2]$$

as values for the (symmetric) bounds for the acceleration commands[2]. At time $t = t_0$, consider the robot state

$$\boldsymbol{q}_0 = \begin{pmatrix} 0 \\ \pi/2 \\ \pi/2 \end{pmatrix} \ [\text{rad}], \qquad \dot{\boldsymbol{q}}_0 = \begin{pmatrix} \pi/2 \\ \pi/2 \\ 0 \end{pmatrix} \ [\text{rad/s}].$$

[2]These bounds are the same used in Exercise 1 of the exam of July 12, 2021.

We compute then from the previous formulas

$$\boldsymbol{J}_0 = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \qquad \dot{\boldsymbol{p}}_0 = \boldsymbol{J}_0 \dot{\boldsymbol{q}}_0 = \begin{pmatrix} -\pi \\ -\pi/2 \end{pmatrix} \text{ [m/s]}$$

and

$$\dot{\boldsymbol{J}}_0 = \begin{pmatrix} \pi/2 & \pi & \pi \\ -\pi & -\pi & 0 \end{pmatrix}, \qquad \boldsymbol{h}_0 = \dot{\boldsymbol{J}}_0 \dot{\boldsymbol{q}}_0 = \begin{pmatrix} 3\,\pi^2/4 \\ -\pi^2 \end{pmatrix} = \begin{pmatrix} 7.4022 \\ -9.8696 \end{pmatrix} \text{ [m/s}^2\text{]}.$$

The pseudoinverse of the task Jacobian is

$$\boldsymbol{J}_0^{\#} = \boldsymbol{J}_0^T \left(\boldsymbol{J}_0 \boldsymbol{J}_0^T \right)^{-1} = \begin{pmatrix} -2/3 & 1/3 \\ -1/3 & -1/3 \\ 1/3 & -2/3 \end{pmatrix}.$$

Therefore, from (3) we obtain

$$\boldsymbol{a} = -\boldsymbol{J}_0^{\#} \dot{\boldsymbol{p}}_0 = -\begin{pmatrix} \pi/2 \\ \pi/2 \\ 0 \end{pmatrix}, \qquad \boldsymbol{b} = -\boldsymbol{J}_0^{\#} \boldsymbol{h}_0 = \begin{pmatrix} 5\,\pi^2/6 \\ -\pi^2/12 \\ -11\,\pi^2/12 \end{pmatrix} = \begin{pmatrix} 8.2247 \\ -0.8225 \\ -9.0471 \end{pmatrix}.$$

None of the components of vector $\boldsymbol{b}$ (related to the Cartesian drift acceleration $\boldsymbol{h}_0$) is outside the acceleration bounds specified by $\boldsymbol{U}_{max}$. As a result, according to the law (5–6), an optimal solution certainly exists and is given by

$$\lambda^* = 19.4764 \text{ [s}^{-1}\text{]} \quad \Rightarrow \quad \boldsymbol{u}_0^* = \begin{pmatrix} -22.3688 \\ -31.4159 \\ -9.0471 \end{pmatrix} \text{ [rad/s}^2\text{]} \quad \Rightarrow \quad \ddot{\boldsymbol{p}}_0 = \begin{pmatrix} 61.1869 \\ 30.5935 \end{pmatrix} \text{ [m/s}^2\text{]}.$$

As expected, there is at least a component of $\boldsymbol{u}_0^*$ that is saturated (only the second one, at its negative lower bound). The obtained task acceleration is $\ddot{\boldsymbol{p}}_0 = -\lambda^* \dot{\boldsymbol{p}}_0$, as expected.

To verify further the method, consider now the following joint velocity at time $t = t_0$,

$$\dot{\boldsymbol{q}}_0 = \begin{pmatrix} 2\pi \\ \pi/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6.2832 \\ 1.5708 \\ 0 \end{pmatrix} \text{ [rad/s]},$$

with the first component four times higher than before, all the rest being the same. The changed terms are

$$\dot{\boldsymbol{p}}_0 = \begin{pmatrix} -7.8540 \\ -1.5708 \end{pmatrix} \text{ [m/s]}, \quad \dot{\boldsymbol{J}}_0 = \begin{pmatrix} 1.5708 & 7.8540 & 7.8540 \\ -7.8540 & -7.8540 & 0 \end{pmatrix}, \quad \boldsymbol{h}_0 = \begin{pmatrix} 22.2066 \\ -61.6850 \end{pmatrix} \text{ [m/s}^2\text{]},$$

and thus

$$\boldsymbol{a} = \begin{pmatrix} -4.7124 \\ -3.1416 \\ 1.5708 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 35.3661 \\ -13.1595 \\ -48.5256 \end{pmatrix}.$$

The component b_3 of vector $\boldsymbol{b}$ exceeds now the lover bound $-U_{max,3} = -31.4159$ [rad/s^2]. However, $a_3 > 0$ and thus an optimal solution exists. According to the law (5–6), we obtain

$$\lambda^* = 5.8112 \text{ [s}^{-1}\text{]} \quad \Rightarrow \quad \boldsymbol{u}_0^* = \begin{pmatrix} 7.9814 \\ -31.4159 \\ -39.3973 \end{pmatrix} \text{ [rad/s}^2\text{]} \quad \Rightarrow \quad \ddot{\boldsymbol{p}}_0 = \begin{pmatrix} 45.6411 \\ 9.1282 \end{pmatrix} \text{ [m/s}^2\text{]}.$$

As before, the second component of $\boldsymbol{u}_0^*$ is saturated at its negative lower bound. The rate of decrease of the Cartesian velocity is now slower[3], because the optimal λ^* is also smaller and the rate of decrease of $\ddot{\boldsymbol{p}}_0$ depends on λ^* only. In fact, it is immediate to see that

$$\ddot{\boldsymbol{p}} = -\lambda\,\dot{\boldsymbol{p}} \qquad \Rightarrow \qquad \dot{\boldsymbol{p}}(t) = e^{-\lambda(t-t_0)}\,\dot{\boldsymbol{p}}(t_0) \simeq (1-\lambda\,dt)\,\dot{\boldsymbol{p}}_0 \qquad \Rightarrow \qquad \frac{\dot{\boldsymbol{p}}(t)-\dot{\boldsymbol{p}}_0}{dt} \simeq -\lambda\,\dot{\boldsymbol{p}}_0,$$

for a sufficiently small $dt = t - t_0 > 0$.

As a last example, we double the joint velocity at time $t = t_0$ with respect to the first case,

$$\dot{\boldsymbol{q}}_0 = \begin{pmatrix} \pi \\ \pi \\ 0 \end{pmatrix} = \begin{pmatrix} 3.1416 \\ 3.1416 \\ 0 \end{pmatrix} \ \text{[rad/s]},$$

all the rest being again the same[4]. The changed terms are now

$$\dot{\boldsymbol{p}}_0 = \begin{pmatrix} -6.2832 \\ -3.1416 \end{pmatrix} \ \text{[m/s]}, \quad \dot{\boldsymbol{J}}_0 = \begin{pmatrix} 3.1416 & 6.2832 & 6.2832 \\ -6.2832 & -6.2832 & 0 \end{pmatrix}, \quad \boldsymbol{h}_0 = \begin{pmatrix} 29.6088 \\ -39.4784 \end{pmatrix} \ \text{[m/s}^2\text{]},$$

and

$$\boldsymbol{a} = \begin{pmatrix} -3.1416 \\ -3.1416 \\ 0 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 32.8987 \\ -3.2899 \\ -36.1885 \end{pmatrix}.$$

Again, the third component of $\boldsymbol{b}$ exceeds its lower bound $-U_{max,3} = -31.4159$ [rad/s^2]. However, since $a_3 = 0$, no solution exists in this case. In fact, according to the law (5–6), it is

$$\lambda_1 = 25.4720, \quad \lambda_2 = 8.9528, \quad \text{but} \quad \lambda_3 = -\infty \qquad \Rightarrow \qquad \lambda^* = -\infty.$$

Exercise #2

Note first that the $m \times m$ Cartesian inertia matrix of a robot with a $m \times n$ Jacobian $\boldsymbol{J}(\boldsymbol{q})$ that has full rank m can always be written as[5]

$$\boldsymbol{M}_p(\boldsymbol{q}) = \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}, \tag{8}$$

where $\boldsymbol{M}(\boldsymbol{q}) > 0$ is the $n \times n$ inertia matrix in the configuration space. The derivation of (8) for the case $m < n$ (redundant robot) with a full rank Jacobian is simple. Let the robot dynamics in joint space be

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}, \tag{9}$$

and the second-order differential kinematics to the Cartesian space be

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{10}$$

Extracting $\ddot{\boldsymbol{q}}$ from (9), using the transformation of generalized forces $\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F}$, and substituting in (10) yields

$$\begin{aligned} \ddot{\boldsymbol{p}} &= \boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right) + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \\ &= \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)\boldsymbol{F} - \boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\left(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})\right) + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} \end{aligned}$$

[3]This happens independently from the value of $\|\ddot{\boldsymbol{p}}_0\|$, which is smaller here than in the first case.
[4]This case coincides with the first one considered in Exercise 1 of the exam of July 12, 2021.
[5]The expression (8) appears also in the lecture slides on robot redundancy (block 2, part 2, p. 10) and on collision detection and reaction (block 19, p. 40).

or

$$\boldsymbol{M}_p(\boldsymbol{q})\ddot{\boldsymbol{p}} + \boldsymbol{c}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}_p(\boldsymbol{q}) = \boldsymbol{F}, \tag{11}$$

with $\boldsymbol{M}_p(\boldsymbol{q})$ as in (8) and

$$\boldsymbol{c}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{M}_p(\boldsymbol{q}) \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\, \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right), \qquad \boldsymbol{g}_p(\boldsymbol{q}) = \boldsymbol{M}_p(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\, \boldsymbol{g}(\boldsymbol{q}).$$

Indeed, the dynamic description (11) is incomplete when $m < n$ and should be complemented by additional $n - m$ dynamic equations (e.g., judiciously extracted from the original complete dynamics (9) in the joint space). On the other hand, when the Jacobian is square ($m = n$) and nonsingular, the terms in the (now complete) Cartesian dynamic model (11) simplify to

$$\boldsymbol{M}_p(\boldsymbol{q}) = \boldsymbol{J}^{-T}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q})$$

and

$$\boldsymbol{c}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{J}^{-T}(\boldsymbol{q})\, \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{M}_p(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{g}_p(\boldsymbol{q}) = \boldsymbol{J}^{-T}(\boldsymbol{q})\, \boldsymbol{g}(\boldsymbol{q}).$$

The Jacobian of the planar 3R robot in Fig. 1 is given by (7). The inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ is extracted from the kinetic energy of the three links of the robot. Under the assumption of equal uniform thin rods, the center of mass of each link is at $d_{ci} = L/2$ on its kinematic axis and the barycentral inertia (around the axis normal to the plane) equals $I_i = (1/12)\, mL^2$. For the first link, we have then

$$T_1 = \frac{1}{2}\left(md_{c1}^2 + I_1\right)\dot{q}_1^2 = \frac{1}{2}\left(m\left(\frac{L}{2}\right)^2 + \frac{1}{12}\, mL^2\right)\dot{q}_1^2 = \frac{1}{2}\,\frac{mL^2}{3}\,\dot{q}_1^2.$$

For the second link, we have

$$\boldsymbol{p}_{c2} = L\begin{pmatrix} c_1 + \frac{1}{2}\, c_{12} \\ s_1 + \frac{1}{2}\, s_{12} \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = L\begin{pmatrix} -s_1\dot{q}_1 - \frac{1}{2}\, s_{12}\left(\dot{q}_1 + \dot{q}_2\right) \\ c_1\dot{q}_1 + \frac{1}{2}\, c_{12}\left(\dot{q}_1 + \dot{q}_2\right) \end{pmatrix},$$

and thus

$$\begin{aligned} T_2 &= \frac{1}{2}\left(m\left\|\boldsymbol{v}_{c2}\right\|^2 + I_2\left(\dot{q}_1 + \dot{q}_2\right)^2\right) \\ &= \frac{1}{2}\left(mL^2\left(\dot{q}_1^2 + \frac{1}{4}\left(\dot{q}_1 + \dot{q}_2\right)^2 + c_2\,\dot{q}_1\left(\dot{q}_1 + \dot{q}_2\right)\right) + \frac{1}{12}\, mL^2\left(\dot{q}_1 + \dot{q}_2\right)^2\right) \\ &= \frac{1}{2}\, mL^2\left(\left(\frac{4}{3} + c_2\right)\dot{q}_1^2 + \left(\frac{2}{3} + c_2\right)\dot{q}_1\dot{q}_2 + \frac{1}{3}\,\dot{q}_2^2\right). \end{aligned}$$

Finally, for the third link

$$\boldsymbol{p}_{c3} = L\begin{pmatrix} c_1 + c_{12} + \frac{1}{2}\, c_{123} \\ s_1 + s_{12} + \frac{1}{2}\, s_{123} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = L\begin{pmatrix} -\left(s_1\dot{q}_1 + s_{12}\left(\dot{q}_1 + \dot{q}_2\right) + \frac{1}{2}\, s_{123}\left(\dot{q}_1 + \dot{q}_2 + \dot{q}_3\right)\right) \\ c_1\dot{q}_1 + c_{12}\left(\dot{q}_1 + \dot{q}_2\right) + \frac{1}{2}\, c_{123}\left(\dot{q}_1 + \dot{q}_2 + \dot{q}_3\right) \end{pmatrix},$$

and so

$$
\begin{aligned}
T_3 &= \frac{1}{2}\left(m\left\|\boldsymbol{v}_{c3}\right\|^2 + I_3\left(\dot{q}_1+\dot{q}_2+\dot{q}_3\right)^2\right)\\
&= \frac{1}{2}\left(mL^2\left(\dot{q}_1^2 + \left(\dot{q}_1+\dot{q}_2\right)^2 + \frac{1}{4}\left(\dot{q}_1+\dot{q}_2+\dot{q}_3\right)^2 + 2\,c_2\,\dot{q}_1\left(\dot{q}_1+\dot{q}_2\right)\right.\right.\\
&\qquad\qquad \left.\left. +\; c_{23}\,\dot{q}_1\left(\dot{q}_1+\dot{q}_2+\dot{q}_3\right) + c_3\left(\dot{q}_1+\dot{q}_2\right)\left(\dot{q}_1+\dot{q}_2+\dot{q}_3\right)\right) + \frac{1}{12}\,mL^2\left(\dot{q}_1+\dot{q}_2+\dot{q}_3\right)^2\right) \;.\\
&= \frac{1}{2}\,mL^2\left(\left(\frac{7}{3}+2\,c_2+c_{23}+c_3\right)\dot{q}_1^2 + \left(\frac{8}{3}+2\,c_2+c_{23}+2c_3\right)\dot{q}_1\dot{q}_2 + \left(\frac{2}{3}+c_{23}+c_3\right)\dot{q}_1\dot{q}_3\right.\\
&\qquad\qquad \left. + \left(\frac{4}{3}+c_3\right)\dot{q}_2^2 + \left(\frac{2}{3}+c_3\right)\dot{q}_2\dot{q}_3 + \frac{1}{3}\,\dot{q}_3^2\right).
\end{aligned}
$$

Therefore, from

$$
T = T_1 + T_2 + T_3 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}
$$

we obtain

$$
\boldsymbol{M}(\boldsymbol{q}) = mL^2\begin{pmatrix} 4+3\,c_2+c_{23}+c_3 & \frac{5}{3}+\frac{3}{2}\,c_2+\frac{1}{2}\,c_{23}+c_3 & \frac{1}{3}+\frac{1}{2}\,c_{23}+\frac{1}{2}\,c_3 \\ \vdots & \frac{5}{3}+c_3 & \frac{1}{3}+\frac{1}{2}\,c_3 \\ \text{symm} & \cdots & \frac{1}{3}\end{pmatrix}. \tag{12}
$$

Since $L = 0.5$ [m] and $mL^2 = 5{\cdot}0.5^2 = 1.25$ [kgm^2], evaluating eqs. (7) and (12) at the configuration $\boldsymbol{q}^* = (\pi/2, \pi/2, 0)$ gives:

$$
\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{q}^*) = \begin{pmatrix} -0.5 & 0 & 0 \\ -1 & -1 & -0.5 \end{pmatrix},
$$

$$
\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{q}^*) = \begin{pmatrix} \frac{25}{4} & \frac{10}{3} & \frac{25}{24} \\ \frac{10}{3} & \frac{10}{3} & \frac{25}{24} \\ \frac{25}{24} & \frac{25}{24} & \frac{5}{12}\end{pmatrix} \simeq \begin{pmatrix} 6.25 & 3.3333 & 1.0417 \\ 3.3333 & 3.3333 & 1.0417 \\ 1.0417 & 1.0417 & 0.4167 \end{pmatrix}.
$$

The Cartesian inertia matrix in (8) is

$$
\boldsymbol{M}_p = \boldsymbol{M}_p(\boldsymbol{q}^*) = \begin{pmatrix} 35/3 & 0 \\ 0 & 35/24 \end{pmatrix} \simeq \begin{pmatrix} 11.6667 & 0 \\ 0 & 1.4583 \end{pmatrix}.
$$

Its two eigenvalues are then

$$
\lambda_1 = 11.6667, \qquad \lambda_2 = 1.4583.
$$

Note that the Cartesian inertia is fully decoupled in the configuration $\boldsymbol{q}^*$. This is by no means the generic case, although the matrix $\boldsymbol{M}_p(\boldsymbol{q})$ is always symmetric, and also positive definite as long as the Jacobian is full rank (its eigenvalues are always real, and strictly positive outside singularities).

Exercise #3

With reference to Fig. 4, the dynamic model of the 2-dof Cartesian robot in contact with a generic environment is

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g} = \boldsymbol{\tau} + \boldsymbol{F}, \tag{13}$$

with

$$\boldsymbol{M} = \begin{pmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{pmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} 0 \\ m_2 g_0 \end{pmatrix}, \quad \boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}, \quad \boldsymbol{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix},$$

where $\boldsymbol{\tau} \in \mathbb{R}^2$ is the control input force at the prismatic joints and $\boldsymbol{F} \in \mathbb{R}^2$ is the contact force exerted from the environment on the robot end effector.

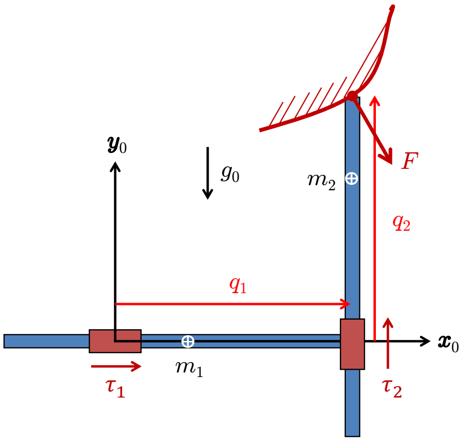


Figure 4: A 2-dof Cartesian robot making contact with an environment.

The desired linear and decoupled impedance model is

$$\boldsymbol{M}_d\, \ddot{\boldsymbol{e}} + \boldsymbol{D}_d\, \dot{\boldsymbol{e}} + \boldsymbol{K}_d\, \boldsymbol{e} = \boldsymbol{F}, \tag{14}$$

where

$$\boldsymbol{e} = \boldsymbol{p} - \boldsymbol{p}_d, \qquad \boldsymbol{p} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

with $\boldsymbol{p}_d \in \mathbb{R}^2$ being the desired position of the robot end effector, and

$$\boldsymbol{M}_d = \begin{pmatrix} M_{dx} & 0 \\ 0 & M_{dy} \end{pmatrix} > 0, \qquad \boldsymbol{D}_d = \begin{pmatrix} D_{dx} & 0 \\ 0 & D_{dy} \end{pmatrix} > 0, \qquad \boldsymbol{K}_d = \begin{pmatrix} K_{dx} & 0 \\ 0 & K_{dy} \end{pmatrix} > 0.$$

Since there is no force/torque sensor available, we have to choose necessarily the actual (Cartesian) inertia as desired (apparent) inertia:

$$\boldsymbol{M}_d = \boldsymbol{M} \qquad \Longleftrightarrow \qquad M_{dx} = m_1 + m_2, \quad M_{dy} = m_2. \tag{15}$$

From (13) and (14), with (15), the required control law is

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{M}\ddot{\boldsymbol{p}}_d + \boldsymbol{g} - \boldsymbol{D}_d\, \dot{\boldsymbol{e}} - \boldsymbol{K}_d\, \boldsymbol{e} \\ &= \begin{pmatrix} (m_1 + m_2)\, \ddot{x}_d \\ m_2\, \ddot{y}_d \end{pmatrix} + \begin{pmatrix} 0 \\ m_2 g_0 \end{pmatrix} + \begin{pmatrix} D_{dx}\,(\dot{x}_d - \dot{x}) + K_{dx}\,(x_d - x) \\ D_{dy}\,(\dot{y}_d - \dot{y}) + K_{dy}\,(y_d - y) \end{pmatrix}, \end{aligned} \tag{16}$$

which has the standard form of a PD action with a feedforward acceleration term (if $\ddot{\boldsymbol{p}}_d \neq \boldsymbol{0}$) and gravity cancellation. For the choice of the gains, we rewrite the impedance model (14), using again (15), in the Laplace domain,

$$\left(\boldsymbol{M}s^2 + \boldsymbol{D}_d s + \boldsymbol{K}_d\right)\boldsymbol{e}(s) = \boldsymbol{F}(s),$$

and impose the desired dynamic characteristics between $\boldsymbol{F}(s)$ and $\boldsymbol{e}(s)$ in each Cartesian direction:

$$\left(\boldsymbol{I}s^2 + \boldsymbol{M}^{-1}\boldsymbol{D}_d\ s + \boldsymbol{M}^{-1}\boldsymbol{K}_d\right) = \begin{pmatrix} (s+\lambda)^2 & 0 \\ 0 & (s+\lambda)^2 \end{pmatrix} = \begin{pmatrix} s^2 + 2\lambda s + \lambda^2 & 0 \\ 0 & s^2 + 2\lambda s + \lambda^2 \end{pmatrix}.$$

As a result,

$$D_{dx} = 2\,(m_1 + m_2)\,\lambda, \qquad K_{dx} = (m_1 + m_2)\,\lambda^2,$$

and

$$D_{dy} = 2m_2\lambda, \qquad K_{dy} = m_2\lambda^2.$$

* * * * *

Robotics 2

January 11, 2022

Exercise #1

The RPR robot in Fig. 1 moves on a horizontal plane, carrying at its end effector a payload of mass m_p and inertia I_p. The links of the robot can be considered as uniform rods of length l_i and mass m_i, $i = 1, 2, 3$. The robot control architecture receives as reference input a high-level joint velocity command $\dot{\boldsymbol{q}}_r \in \mathbb{R}^3$.

[i] The first task requires to move the end-effector point P along a desired trajectory $\boldsymbol{p}_d(t) \in \mathbb{R}^2$, while locally minimizing the robot kinetic energy $T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$. Design a high-level control law $\dot{\boldsymbol{q}}_r^{[i]}$ realizing this task and provide the detailed symbolic expression of all its terms.

[ii] Consider next the presence of a circular obstacle $\mathcal{O}_{obs}$ of radius r that is placed in a known position P_{obs} in the robot workspace. Modify the previous control law into a $\dot{\boldsymbol{q}}_r^{[ii]}$ so as to try also avoiding collisions between the full robot body and the obstacle. Provide the symbolic expression of the additional terms in this law.

[iii] Using the following kinematic and dynamic data

$$l_1 = 0.45,\ l_2 = 0.7,\ l_3 = 0.35\ [\text{m}], \quad m_1 = m_2 = 10,\ m_3 = 4,\ m_p = 2\ [\text{kg}], \quad I_p = 0.01\ [\text{kgm}^2],$$

compute the numerical values of the command $\dot{\boldsymbol{q}}_r^{[i]}$ when the robot is in the configuration $\boldsymbol{q} = \bar{\boldsymbol{q}} = (\pi/4, 0.25, -\pi/4)$ [rad,m,rad] and the desired end-effector velocity is $\dot{\boldsymbol{p}}_d = (1, -1)$. For case [ii], repeat the same evaluation for $\dot{\boldsymbol{q}}_r^{[ii]}$, when the circular obstacle is placed at the point $P_{obs} = (0.1 + l_1\sqrt{2}, -0.1) \simeq (0.736, -0.1)$ [m] and has radius $r = 0.05$ [m].

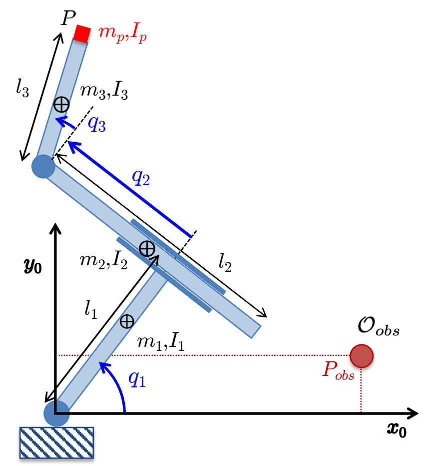


Figure 1: A planar RPR robot with a payload on the end effector. A circular obstacle $\mathcal{O}_{obs}$ may also be present in a generic position P_{obs} in the robot workspace.

Exercise #2

Consider a robot manipulator with n revolute joints and with dynamic model given by

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}. \tag{1}$$

At time $t = 0$, the robot is in a state $\boldsymbol{x}(0) = \begin{pmatrix} \boldsymbol{q}^T(0) & \dot{\boldsymbol{q}}^T(0) \end{pmatrix}^T = \begin{pmatrix} \boldsymbol{q}_0^T & \dot{\boldsymbol{q}}_0^T \end{pmatrix}^T \in \mathbb{R}^{2n}$, with $\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$.

[i] Define a joint torque law $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{x}, t)$ that is continuous w.r.t. time and that will bring the robot with a coordinated joint motion in *exactly* T seconds to an equilibrium state, i.e., to an arbitrary configuration $\boldsymbol{q}(T)$ with $\dot{\boldsymbol{q}}(0) = \mathbf{0}$, where the robot will remain for all $t \geq T$.

[ii] According to your choice, you should be able to provide in closed form both the reached joint configuration $\boldsymbol{q}_f = \boldsymbol{q}(T)$ and the resulting initial acceleration $\ddot{\boldsymbol{q}}_0 = \ddot{\boldsymbol{q}}(0)$.

[iii] Consider the case of a robot with $\boldsymbol{g}(\boldsymbol{q}) \equiv \mathbf{0}$. Assume that acceleration bounds $|\ddot{q}_i| \leq A_{max,i}$, $i = 1, \ldots, n$, are imposed on the already defined robot motion and that at least one of the joints exceeds its bound at some time instant $\bar{t} \in [0, T]$. Provide the expression of the minimum factor $k > 0$ such that the robot trajectory resulting from an uniform scaling of the motion time to $T' = kT$ will satisfy all the given bounds.

Exercise #3

The end effector of the PR robot in Fig. 2 is constrained to move along a line segment, between points A and B. Assume that all dissipative effects are negligible and that the robot dynamic model in free space has the form (1).

[i] Derive the (one-dimensional) *reduced* dynamic model of the constrained robot and the explicit expression of the force multiplier $\lambda \in \mathbb{R}$.

[ii] If the end-effector has to execute a rest-to-rest motion from A to B with a cubic profile in a total time T without generating any constraint force at the contact during the motion, what would be the explicit expression of the control law?

[iii] Using the following data

$$L = 1,\ d_2 = 0.6 \text{ [m]}, \quad m_1 = 15,\ m_2 = 8 \text{ [kg]}, \quad I_2 = 1.2 \text{ [kgm}^2\text{]},$$
$$A = (0.7, 2), \quad B = (0.5, 1) \text{ [m]}, \quad T = 2 \text{ [s]},$$

compute at the initial time $t = 0$ the numerical value $\boldsymbol{\tau}(0) \in \mathbb{R}^2$ of the force/torque commands at the joints that execute the task specified in [ii].

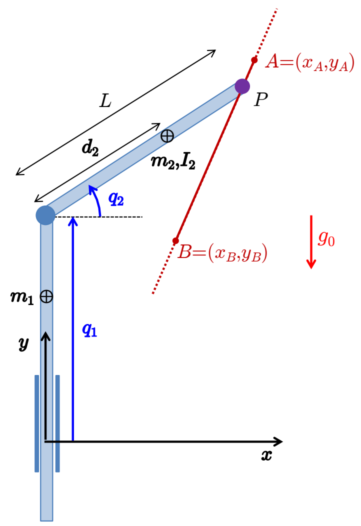


Figure 2: A planar PR robot with its end effector constrained on a segment AB of a line.

[210 minutes (3.5 hours); open books]

Solution

January 11, 2022

Exercise #1

There is a unique solution to the problem of finding a joint velocity command $\dot{\boldsymbol{q}}_r \in \mathbb{R}^N$ (here, $N = 3$) that realizes a given task velocity $\dot{\boldsymbol{p}}_d \in \mathbb{R}^M$, with $M < N$ (here, $M = 2$), by minimizing the kinetic energy of the robot, i.e., for problem [i],

$$\min_{\dot{\boldsymbol{q}}} \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \text{s.t.} \quad \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} = \dot{\boldsymbol{p}}_d,$$

where $\boldsymbol{M}(\boldsymbol{q}) > 0$ is the robot inertia matrix and $\boldsymbol{J}(\boldsymbol{q})$ is the task Jacobian, both evaluated at the current configuration $\boldsymbol{q}$. The solution is obtained by the weighted pseudoinverse

$$\dot{\boldsymbol{q}}_r^{[i]} = \boldsymbol{J}^{\#}_{\boldsymbol{M}}(\boldsymbol{q})\,\dot{\boldsymbol{p}}_d = \boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\dot{\boldsymbol{p}}_d. \tag{2}$$

For the planar RPR robot in Fig. 1, the Jacobian matrix is computed from the direct kinematics[1]

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} l_1 c_1 - q_2 s_1 + l_3 c_{13} \\ l_1 s_1 + q_2 c_1 + l_3 s_{13} \end{pmatrix}$$

as

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(l_1 s_1 + q_2 c_1 + l_3 s_{13}\right) & -s_2 & -l_3 s_{13} \\ l_1 c_1 - q_2 s_1 + l_3 c_{13} & c_1 & l_3 c_{13} \end{pmatrix}. \tag{3}$$

Next, we compute the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ by extracting its elements from the total robot kinetic energy T. In doing so, we also use the fact that the links are uniform thin rods of length l_i, $i = 1, 2, 3$. The center of mass is then located at the link midpoint (at a distance $d_i = l_i/2$ to the end of the rod), while the barycentric inertia (around an axis normal to the plane) is $I_i = (1/12)\, m_i l_i^2$. The payload will be included in the kinetic energy of the robot.

Kinetic energy and inertia matrix

Link 1

$$T_1 = \frac{1}{2}\left(I_1 + m_1 d_1^2\right)\dot{q}_1^2 = \frac{1}{2}\,\frac{m_1 l_1^2}{3}\,\dot{q}_1^2$$

Link 2

$$T_2 = \frac{1}{2}\, m_2 \left\|\boldsymbol{v}_{c2}\right\|^2 + \frac{1}{2}\, I_2\, \dot{q}_1^2 \quad \Rightarrow$$

$$\boldsymbol{p}_{c2} = \begin{pmatrix} l_1 c_1 - (q_2 - d_2)\, s_1 \\ l_1 s_1 + (q_2 - d_2)\, c_1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -\left(l_1 s_1 + (q_2 - d_2)\, c_1\right)\dot{q}_1 - s_1\,\dot{q}_2 \\ \left(l_1 c_1 - (q_2 - d_2)\, s_1\right)\dot{q}_1 + c_1\,\dot{q}_2 \end{pmatrix}$$

$$\Rightarrow \quad \left\|\boldsymbol{v}_{c2}\right\|^2 = \left(l_1^2 + (q_2 - d_2)^2\right)\dot{q}_1^2 + \dot{q}_2^2 + 2 l_1\,\dot{q}_1\dot{q}_2 \quad \Rightarrow$$

$$\begin{aligned} T_2 &= \frac{1}{2}\left(\left(I_2 + m_2\left(l_1^2 + (q_2 - d_2)^2\right)\right)\dot{q}_1^2 + m_2\,\dot{q}_2^2 + 2 m_2 l_1\,\dot{q}_1\dot{q}_2\right) \\ &= \frac{1}{2}\left(m_2\left(l_1^2 + \frac{l_2^2}{3} + q_2^2 - l_2 q_2\right)\dot{q}_1^2 + m_2\,\dot{q}_2^2 + 2 m_2 l_1\,\dot{q}_1\dot{q}_2\right) \end{aligned}$$

[1] In the following, the shorthand notation for trigonometric quantities is used (e.g., $s_{13} = \sin(q_1 + q_3)$).

Link 3

$$T_3 = \frac{1}{2}\, m_3 \left\|\boldsymbol{v}_{c3}\right\|^2 + \frac{1}{2}\, I_3 \left(\dot q_1 + \dot q_3\right)^2 \quad \Rightarrow$$

$$\boldsymbol{p}_{c3} = \begin{pmatrix} l_1 c_1 - q_2 s_1 + d_3 c_{13} \\ l_1 s_1 + q_2 c_1 + d_3 s_{13} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c3} = \begin{pmatrix} -\left(l_1 s_1 + q_2 c_1 + d_3 s_{13}\right) \dot q_1 - s_1\, \dot q_2 - d_3 s_{13}\, \dot q_3 \\ \left(l_1 c_1 - q_2 s_1 + d_3 c_{13}\right) \dot q_1 + c_1\, \dot q_2 + d_3 c_{13}\, \dot q_3 \end{pmatrix}$$

$$\Rightarrow \quad \left\|\boldsymbol{v}_{c3}\right\|^2 = \left(l_1^2 + q_2^2 + d_3^2 + 2 l_1 d_3 c_3 + 2 q_2 d_3 s_3\right) \dot q_1^2 + \dot q_2^2 + d_3^2\, \dot q_3^2$$
$$+ 2\left(l_1 + d_3 c_3\right) \dot q_1 \dot q_2 + 2 d_3 \left(d_3 + l_1 c_3 + q_2 s_3\right) \dot q_1 \dot q_3 + 2 d_3 c_3\, \dot q_2 \dot q_3 \quad \Rightarrow$$

$$T_3 = \frac{1}{2}\left(\left(I_3 + m_3\left(l_1^2 + q_2^2 + d_3^2 + 2 l_1 d_3 c_3 + 2 q_2 d_3 s_3\right)\right) \dot q_1^2 + m_3 \dot q_2^2 + \left(I_3 + m_3 d_3^2\right) \dot q_3^2\right.$$
$$\left. + 2 m_3 \left(l_1 + d_3 c_3\right) \dot q_1 \dot q_2 + 2\left(I_3 + m_3 d_3\left(d_3 + l_1 c_3 + q_2 s_3\right)\right) \dot q_1 \dot q_3 + 2 m_3 d_3 c_3\, \dot q_2 \dot q_3\right)$$

$$= \frac{1}{2}\left(m_3\left(l_1^2 + \frac{l_3^2}{3} + q_2^2 + l_1 l_3 c_3 + q_2 l_3 s_3\right) \dot q_1^2 + m_3\, \dot q_2^2 + m_3 \frac{l_3^2}{3}\, \dot q_3^2\right.$$
$$\left. + 2 m_3 \left(l_1 + \frac{l_3}{2}\right) \dot q_1 \dot q_2 + 2 m_3 \left(\frac{l_3^2}{3} + l_1 \frac{l_3}{2} c_3 + q_2 \frac{l_3}{2} s_3\right) \dot q_1 \dot q_3 + 2 m_3 \frac{l_3}{2}\, \dot q_2 \dot q_3\right)$$

Payload

$$T_p = \frac{1}{2}\, m_p \left\|\boldsymbol{v}_p\right\|^2 + \frac{1}{2}\, I_p \left(\dot q_1 + \dot q_3\right)^2 \quad \Rightarrow$$

$$\boldsymbol{p}_p = \begin{pmatrix} l_1 c_1 - q_2 s_1 + l_3 c_{13} \\ l_1 s_1 + q_2 c_1 + l_3 s_{13} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_p = \begin{pmatrix} -\left(l_1 s_1 + q_2 c_1 + l_3 s_{13}\right) \dot q_1 - s_1\, \dot q_2 - l_3 s_{13}\, \dot q_3 \\ \left(l_1 c_1 - q_2 s_1 + l_3 c_{13}\right) \dot q_1 + c_1\, \dot q_2 + l_3 c_{13}\, \dot q_3 \end{pmatrix}$$

$$\Rightarrow \quad \left\|\boldsymbol{v}_p\right\|^2 = \left(l_1^2 + q_2^2 + l_3^2 + 2 l_1 l_3 c_3 + 2 q_2 l_3 s_3\right) \dot q_1^2 + \dot q_2^2 + l_3^2\, \dot q_3^2$$
$$+ 2\left(l_1 + l_3 c_3\right) \dot q_1 \dot q_2 + 2 l_3 \left(l_3 + l_2 c_3 + q_3 s_3\right) \dot q_1 \dot q_3 + 2 l_3 c_3\, \dot q_2 \dot q_3 \quad \Rightarrow$$

$$T_p = \frac{1}{2}\left(\left(I_p + m_p\left(l_1^2 + l_3^2 + q_2^2 + 2 l_1 l_3 c_3 + 2 q_2 l_3 s_3\right)\right) \dot q_1^2 + m_p \dot q_2^2 + \left(I_p + m_p l_3^2\right) \dot q_3^2\right.$$
$$\left. + 2 m_p \left(l_1 + l_3 c_3\right) \dot q_1 \dot q_2 + 2\left(I_p + m_p\left(l_3^2 + l_1 l_3 c_3 + q_2 l_3 s_3\right)\right) \dot q_1 \dot q_3 + 2 m_p l_3 c_3\, \dot q_2 \dot q_3\right)$$

Therefore, from

$$T = T_1 + T_2 + T_3 + T_p = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \frac{1}{2} \sum_{i,j=1}^{3} m_{ij}(\boldsymbol{q})\, \dot q_i \dot q_j$$

we obtain by extraction the coefficients of $\dot q_i \dot q_j$ the elements $m_{ij} = m_{ji}$ of the symmetric inertia matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix}, \tag{4}$$

with

$$
\begin{aligned}
m_{11} &= \frac{m_1 l_1^2}{3} + m_2\left(l_1^2 + \frac{l_2^2}{3} + q_2^2 - l_2 q_2\right) + m_3\left(l_1^2 + \frac{l_3^2}{3} + q_2^2 + l_1 l_3 c_3 + q_2 l_3 s_3\right) \\
&\quad + I_p + m_p\left(l_1^2 + l_3^2 + q_2^2 + 2 l_1 l_3 c_3 + 2 q_2 l_3 s_3\right) \\
m_{12} &= \left(m_2 + m_3 + m_p\right) l_1 + \left(m_3 \frac{l_3}{2} + m_p l_3\right) c_3 \\
m_{13} &= I_p + m_p l_3^2 + m_3 \frac{l_3^2}{3} + \left(m_3 \frac{l_3}{2} + m_p l_3\right)\left(l_1 c_3 + q_2 s_3\right) \\
m_{22} &= m_2 + m_3 + m_p \\
m_{23} &= \left(m_3 \frac{l_3}{2} + m_p l_3\right) c_3 \\
m_{33} &= m_3 \frac{l_3^2}{3} + I_p + m_p l_3^2.
\end{aligned}
$$

Using the given data, we have from (3)

$$
\boldsymbol{J}(\bar{\boldsymbol{q}}) = \begin{pmatrix} -0.495 & -0.7071 & 0 \\ 0.4914 & 0.7071 & 0.35 \end{pmatrix}
$$

and from (4)

$$
\boldsymbol{M}(\bar{\boldsymbol{q}}) = \begin{pmatrix} 5.6126 & 8.1899 & 0.6163 \\ 8.1899 & 16 & 0.9899 \\ 0.6163 & 0.9899 & 0.4183 \end{pmatrix}.
$$

Thus, we compute numerically the solution (2) obtaining

$$
\dot{\boldsymbol{q}}_r^{[i]} = \boldsymbol{J}_{\boldsymbol{M}}^{\#}(\bar{\boldsymbol{q}})\,\dot{\boldsymbol{p}}_d = \begin{pmatrix} -2.0329 & 0.1042 \\ 0.0088 & -0.0729 \\ 2.8365 & 2.8582 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2.1371 \\ 0.0818 \\ -0.0217 \end{pmatrix} \quad \text{[rad/s,m/s,rad/s]}. \tag{5}
$$

For case [ii], the solution includes a null space term with the gradient of the distance function $H_{obs}(\boldsymbol{q})$ between the fixed obstacle $\mathcal{O}_{obs}$ and the entire robot body $\mathcal{R} = \mathcal{R}(\boldsymbol{q})$ —the robot occupies a region in the Cartesian space that depends indeed on the current configuration $\boldsymbol{q}$. This results in

$$
\begin{aligned}
\dot{\boldsymbol{q}}_r^{[ii]} &= \boldsymbol{J}_{\boldsymbol{M}}^{\#}(\boldsymbol{q})\,\dot{\boldsymbol{p}}_d + \alpha\left(\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{M}}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\right)\nabla_q H_{obs}(\boldsymbol{q}) \\
&= \alpha \nabla_q H_{obs}(\boldsymbol{q}) + \boldsymbol{J}_{\boldsymbol{M}}^{\#}(\boldsymbol{q})\big(\dot{\boldsymbol{p}}_d - \alpha \boldsymbol{J}(\boldsymbol{q})\nabla_q H_{obs}(\boldsymbol{q})\big),
\end{aligned} \tag{6}
$$

for a sufficiently small step size $\alpha > 0$. The distance function from the obstacle (also known as the *clearance* of the robot) is defined as

$$
H_{obs}(\boldsymbol{q}) = \min_{\substack{\boldsymbol{a}(\boldsymbol{q}) \in \mathcal{R} \\ \boldsymbol{b} \in \mathcal{O}_{obs}}} \|\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{b}\|, \tag{7}
$$

with the Euclidean norm

$$
\|\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{b}\| = \sqrt{(\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{b})^T (\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{b})}.
$$

Accordingly, the gradient of $H_{obs}(\boldsymbol{q})$ is evaluated[2] as

$$\nabla_q H_{obs}(\boldsymbol{q}) = \left(\frac{\partial H_{obs}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \frac{1}{2}\,\frac{1}{\|\boldsymbol{a}(\boldsymbol{q})-\boldsymbol{b}\|}\left(\frac{\partial \boldsymbol{a}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T(\boldsymbol{a}(\boldsymbol{q})-\boldsymbol{b}). \tag{8}$$

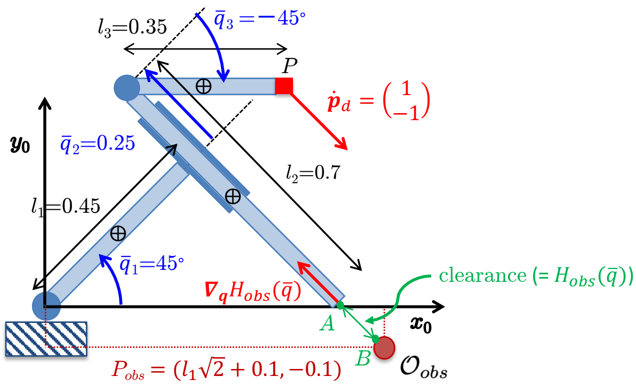


Figure 3: The clearance between the RPR robot and the circular obstacle $\mathcal{O}_{obs}$ when the robot is in the configuration $\bar{\boldsymbol{q}} = (\pi/4, 0.25, -\pi/4)$.

We use now the given data and illustrate the situation in Fig. 3. At $\boldsymbol{q} = \bar{\boldsymbol{q}}$, the two points on the robot $\mathcal{R}$ and on the obstacle $\mathcal{O}_{obs}$ that are giving the clearance of the robot are point A at the lower end of the second link and point B on the boundary of the circular obstacle (at a distance $r = 0.05$ [m] from its center $P_{obs} = (0.736, -0.1)$ [m]), along the line having the *same* orientation of the second link. For varying $\boldsymbol{q}$, the position of the point A on the robot and its Jacobian are

$$\boldsymbol{a}(\boldsymbol{q}) = \begin{pmatrix} l_1c_1 - (q_2-l_2)s_1 \\ l_1s_1 + (q_2-l_2)c_1 \end{pmatrix} \quad\Rightarrow\quad \frac{\partial \boldsymbol{a}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -l_1s_1 - (q_2-l_2)c_1 & -s_1 & 0 \\ l_1c_1 - (q_2-l_2)s_1 & c_1 & 0 \end{pmatrix}.$$

Moreover the position of the point B on the obstacle is given by

$$\boldsymbol{b} = \boldsymbol{p}_{obs} + r\,\frac{\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{p}_{obs}}{\|\boldsymbol{a}(\boldsymbol{q}) - \boldsymbol{p}_{obs}\|}.$$

Thus, we have from (7)

$$\boldsymbol{a}(\bar{\boldsymbol{q}}) = \begin{pmatrix} 0.6364 \\ 0 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0.7007 \\ -0.0646 \end{pmatrix} \quad\Rightarrow\quad H_{obs}(\bar{\boldsymbol{q}}) = \|\boldsymbol{a}(\bar{\boldsymbol{q}}) - \boldsymbol{b}\| = 0.0911 \text{ [m]},$$

with the gradient in (8) being

$$\nabla_q H_{obs}(\bar{\boldsymbol{q}}) = \begin{pmatrix} 0.4509 \\ 1 \\ 0 \end{pmatrix}.$$

[2]The expression in (8) holds in situations when the function $H_{obs}(\boldsymbol{q})$ is differentiable. This occurs when the two points on the robot $\mathcal{R}$ and on the obstacle $\mathcal{O}_{obs}$ that determine the minimum distance at the current configuration $\boldsymbol{q}$ do not jump form one location to another. In practice, the gradient $\nabla_q H_{obs}(\boldsymbol{q})$ is often computed numerically by finite differences of $H_{obs}(\boldsymbol{q})$ between two successive configuration $\boldsymbol{q}(t_k)$ and $\boldsymbol{q}(t_{k-1})$ attained during motion (with some safeguarding rule to obtain bounded variations).

Setting for instance $\alpha = 1$, the evaluation of (6) yields the solution

$$\dot{\boldsymbol{q}}_r^{[ii]} = \begin{pmatrix} -3.6742 \\ 1.1577 \\ -0.0373 \end{pmatrix} \text{ [rad/s,m/s,rad/s]}. \tag{9}$$

Compare now the two joint velocity commands in (5) and (9). It is evident that the presence of the obstacle will modify the inertia-weighted minimum norm solution by pushing away the second link through the sliding of the second (prismatic) joint in the positive direction. In order to still achieve the desired end-effector velocity $\dot{\boldsymbol{p}}_d$ while compensating for this extra joint motion, the first robot joint in $\dot{\boldsymbol{q}}_r^{[ii]}$ will rotate in the clockwise direction by a larger amount than in $\dot{\boldsymbol{q}}_r^{[i]}$.

Exercise #2

To address the general problem of a robot motion that has to be completed in a prescribed *finite* time, it is convenient to apply first to (1) the nonlinear feedback law

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{u}(t) + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}), \tag{10}$$

obtaining the linear and decoupled system $\ddot{\boldsymbol{q}} = \boldsymbol{u}$. Then, we use the acceleration command $\boldsymbol{u}(t)$, which has to be continuous w.r.t. time for all $t > 0$, to plan a state-to-rest trajectory $\boldsymbol{q}_d(t)$ in exactly T seconds. Since the final configuration is not specified a priori, but the state to be reached at time $t = T$ should be an equilibrium, we will impose the following minimal set of (asymmetric) boundary conditions to the interpolating trajectory:

$$\boldsymbol{q}_d(0) = \boldsymbol{q}_0, \qquad \dot{\boldsymbol{q}}_d(0) = \dot{\boldsymbol{q}}_0 \neq 0, \qquad \dot{\boldsymbol{q}}_d(T) = \boldsymbol{0}, \qquad \ddot{\boldsymbol{q}}_d(T) = \boldsymbol{0}. \tag{11}$$

One can satisfy these conditions by choosing a cubic trajectory in normalized time $\sigma = t/T \in [0, 1]$,

$$\boldsymbol{q}_d(\sigma) = \boldsymbol{a}\,\sigma^3 + \boldsymbol{b}\,\sigma^2 + \boldsymbol{c}\,\sigma + \boldsymbol{d}, \tag{12}$$

with $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^n$, $\boldsymbol{c} \in \mathbb{R}^n$, and $\boldsymbol{d} \in \mathbb{R}^n$. The joint motion will automatically be coordinated since the total time T is the same for all joints. Imposing the boundary conditions (11) on the vector function (12) leads to

$$\boldsymbol{q}_d(0) = \boldsymbol{q}_0 \quad \Rightarrow \quad \boldsymbol{d} = \boldsymbol{q}_0, \qquad\qquad \dot{\boldsymbol{q}}_d(0) = \dot{\boldsymbol{q}}_0 \quad \Rightarrow \quad \boldsymbol{c} = \dot{\boldsymbol{q}}_0,$$

and

$$\begin{cases} \dot{\boldsymbol{q}}_d(1) = \boldsymbol{0} \\ \ddot{\boldsymbol{q}}_d(1) = \boldsymbol{0} \end{cases} \quad \Rightarrow \quad \begin{cases} \dfrac{1}{T}\,(3\boldsymbol{a} + 2\boldsymbol{b} + \dot{\boldsymbol{q}}_0) = \boldsymbol{0} \\ \dfrac{1}{T^2}\,(6\boldsymbol{a} + 2\boldsymbol{b}) = \boldsymbol{0} \end{cases} \quad \Rightarrow \quad \begin{cases} \boldsymbol{a} = \dfrac{\dot{\boldsymbol{q}}_0}{3} \\ \boldsymbol{b} = -\dot{\boldsymbol{q}}_0. \end{cases}$$

As a result, we obtain

$$\boldsymbol{q}_d(t) = \frac{\dot{\boldsymbol{q}}_0}{3}\left(\frac{t}{T}\right)^3 - \dot{\boldsymbol{q}}_0\left(\frac{t}{T}\right)^2 + \dot{\boldsymbol{q}}_0\left(\frac{t}{T}\right) + \boldsymbol{q}_0, \tag{13}$$

and thus the requested values

$$\boldsymbol{q}_f = \boldsymbol{q}_d(T) = \boldsymbol{q}_0 + \frac{\dot{\boldsymbol{q}}_0}{3}, \qquad\qquad \ddot{\boldsymbol{q}}_0 = \ddot{\boldsymbol{q}}_d(0) = -\frac{2\dot{\boldsymbol{q}}_0}{T^2}. \tag{14}$$

The actual torque law realizing the task will be given by (10) with

$$\boldsymbol{u}(t) = \begin{cases} \ddot{\boldsymbol{q}}_d(t) = \dfrac{2\dot{\boldsymbol{q}}_0}{T^2}\left(\dfrac{t}{T} - 1\right), & t \in [0, T] \\ \boldsymbol{0}, & t \geq T. \end{cases} \tag{15}$$

We note that, after stopping the robot at time $t = T$ in $\boldsymbol{q} = \boldsymbol{q}_f$, the applied torque (10) with (15) will become equal to $\boldsymbol{\tau}(t) = \boldsymbol{g}(\boldsymbol{q}_f)$, keeping the robot in equilibrium for all times $t \geq T$ as requested.

The choice of a cubic function (12) is also convenient for addressing the presence of bounds on the joint accelerations. In fact, the accelerations in (15) are linear functions of time and their maximum (absolute) value is always attained at $t = 0$, as given by the second equation in (14). We have

$$|\ddot{q}_i(t)| \leq A_{max,i}, \quad \forall t \in [0, +\infty) \qquad \Longleftrightarrow \qquad \max_{t\in[0,T]} |\ddot{q}_{d,i}(t)| = \frac{2|\dot{q}_{0,i}|}{T^2} \leq A_{max,i}, \qquad i = 1, \ldots, n.$$

Assuming that at least one of the joints exceeds its bound during the planned motion implies that this will happen at $t = 0$. Define the index i^* of the maximum violating joint as

$$i^* = \arg\left\{\max_{i=1,\ldots,n} \frac{|\dot{q}_{0,i}|}{A_{max,i}}\right\}, \qquad \text{with } |\ddot{q}_{d,i^*}(0)| = \frac{2|\dot{q}_{0,i^*}|}{T^2} > A_{max,i^*}.$$

Then, the uniform scaling of motion time with minimum $k > 0$ that guarantees satisfaction of all acceleration bounds is given by

$$\frac{2|\dot{q}_{0,i^*}|}{(kT)^2} = A_{max,i^*} \qquad \Rightarrow \qquad k = \frac{1}{T}\sqrt{\frac{2|\dot{q}_{0,i^*}|}{A_{max,i^*}}} > 1 \qquad \Rightarrow \qquad T' = kT. \tag{16}$$

The absence of the gravity term $\boldsymbol{g}(\boldsymbol{q})$ plays no role in the solution. In that case, when the robot comes to rest at the end of the motion, the control law will simply vanish ($\boldsymbol{\tau}(t) = \boldsymbol{0}$, for $t \geq T$).

Exercise #3

We derive first the dynamic model of the PR robot in Fig. 2 when moving in an unconstrained way in the vertical plane (under gravity).

Kinetic energy and inertia matrix

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} I_2 \dot{q}_2^2,$$

$$\boldsymbol{p}_{c2} = \begin{pmatrix} d_2 \cos q_2 \\ q_1 + d_2 \sin q_2 \end{pmatrix} \Rightarrow \boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \begin{pmatrix} -d_2 \sin q_2\, \dot{q}_2 \\ \dot{q}_1 + d_2 \cos q_2\, \dot{q}_2 \end{pmatrix} \Rightarrow \|\boldsymbol{v}_{c2}\|^2 = \dot{q}_1^2 + d_2^2 \dot{q}_2^2 + 2 d_2 \cos q_2\, \dot{q}_1 \dot{q}_2$$

$$\Rightarrow \quad T = T_1 + T_2 = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \text{with} \quad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1 + m_2 & m_2 d_2 \cos q_2 \\ m_2 d_2 \cos q_2 & I_2 + m_2 d_{c2}^2 \end{pmatrix}.$$

Potential energy and gravity vector

$$U_1 = m_1 g_0\, q_1, \qquad U_2 = m_2 g_0\, (q_1 + \sin q_2) \quad \Rightarrow \quad U = U_1 + U_2$$

$$\Rightarrow \quad \boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0\,(m_1 + m_2) \\ g_0 m_2 d_2 \cos q_2 \end{pmatrix}.$$

Coriolis and centrifugal vector

$$
\begin{aligned}
&\boldsymbol{C}_1(\boldsymbol{q}) = \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{m}_1}{\partial \boldsymbol{q}}\right) + \left(\frac{\partial \boldsymbol{m}_1}{\partial \boldsymbol{q}}\right)^T - \left(\frac{\partial \boldsymbol{M}}{\partial q_1}\right)\right) = \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_2 \sin q_2 \end{pmatrix} \\
\Rightarrow \quad & c_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \dot{\boldsymbol{q}} = -m_2 d_2 \sin q_2\, \dot{q}_2^2 \\
&\boldsymbol{C}_2(\boldsymbol{q}) = \frac{1}{2}\left(\left(\frac{\partial \boldsymbol{m}_2}{\partial \boldsymbol{q}}\right) + \left(\frac{\partial \boldsymbol{m}_2}{\partial \boldsymbol{q}}\right)^T - \left(\frac{\partial \boldsymbol{M}}{\partial q_2}\right)\right) = \boldsymbol{0} \quad \Rightarrow \quad c_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \\
\Rightarrow \quad & \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -m_2 d_2 \sin q_2\, \dot{q}_2^2 \\ 0 \end{pmatrix}.
\end{aligned}
$$

Dynamic model

$$
\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \qquad \Longleftrightarrow
$$

$$
\begin{cases} (m_1 + m_2)\, \ddot{q}_1 + m_2 d_2 \cos q_2\, \ddot{q}_2 - m_2 d_2 \sin q_2\, \dot{q}_2^2 + g_0\, (m_1 + m_2) = \tau_1 \\ m_2 d_2 \cos q_2\, \ddot{q}_1 + \left(I_2 + m_2 d_2^2\right) \ddot{q}_2 + g_0 m_2 d_2 \cos q_2 = \tau_2. \end{cases} \tag{17}
$$

We write next the Cartesian constraint on the end-effector position $\boldsymbol{p} = (p_x, p_y)$: point P should to belong to the line $\mathcal{L}$ passing through the two points A and B. Assuming that both $x_A \neq x_B$ and $y_A \neq y_B$ hold true, we can use the parametric expression of the line

$$
\mathcal{L}: \quad \frac{x - x_B}{x_A - x_B} = \frac{y - y_B}{y_A - y_B} \quad \Rightarrow \quad P \in \mathcal{L}: \quad k(\boldsymbol{p}) = \frac{p_y - y_B}{y_A - y_B} - \frac{p_x - x_B}{x_A - x_B} = 0. \tag{18}
$$

Substituting the direct kinematics $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ for the point P in (18) yields

$$
\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} L \cos q_2 \\ q_1 + L \sin q_2 \end{pmatrix} \quad \Rightarrow \quad h(\boldsymbol{q}) = k\left(\boldsymbol{f}(\boldsymbol{q})\right) = \frac{q_1 + L \sin q_2 - y_B}{y_A - y_B} - \frac{L \cos q_2 - x_B}{x_A - x_B} = 0, \tag{19}
$$

with the Jacobian of the scalar constraint given by

$$
\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} \dfrac{1}{y_A - y_B} & \dfrac{L \cos q_2}{y_A - y_B} + \dfrac{L \sin q_2}{x_A - x_B} \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{A}(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}. \tag{20}
$$

In the assumed hypothesis on the relative location of the two points A and B, the matrix $\boldsymbol{A}(\boldsymbol{q})$ is always well defined and has full rank.

In order to obtain the reduced dynamic model of the constrained PR robot, the basic step is to define a 1×2 (row) vector $\boldsymbol{D}(\boldsymbol{q})$ that is linearly independent from $\boldsymbol{A}(\boldsymbol{q})$. A simple choice is the following constant matrix with rank one:

$$
\boldsymbol{D} = \begin{pmatrix} 0 & y_A - y_B \end{pmatrix} \quad \Rightarrow \quad \det \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D} \end{pmatrix} = 1. \tag{21}
$$

Thus,

$$
\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D} \end{pmatrix}^{-1} = \begin{pmatrix} y_A - y_B & -\left(\dfrac{L \cos q_2}{y_A - y_B} + \dfrac{L \sin q_2}{x_A - x_B}\right) \\ 0 & \dfrac{1}{y_A - y_B} \end{pmatrix} \triangleq \begin{pmatrix} \boldsymbol{E} & \boldsymbol{F}(\boldsymbol{q}) \end{pmatrix}.
$$

We define then the pseudo-velocity $v \in \mathbb{R}$ on the Cartesian line and the inverse mapping to $\dot{\boldsymbol{q}}$ as

$$v = \boldsymbol{D}\,\dot{\boldsymbol{q}} = (y_A - y_B)\,\dot{q}_2\,, \qquad \dot{\boldsymbol{q}} = \boldsymbol{F}(\boldsymbol{q})v = \begin{pmatrix} -\left(\dfrac{L\cos q_2}{y_A - y_B} + \dfrac{L\sin q_2}{x_A - x_B}\right) \\ \dfrac{1}{y_A - y_B} \end{pmatrix} v. \tag{22}$$

Being

$$\dot{\boldsymbol{D}} = \boldsymbol{0} \qquad \text{and} \qquad \dot{\boldsymbol{A}}(\boldsymbol{q}) = \begin{pmatrix} 0 & \left(\dfrac{L\cos q_2}{x_A - x_B} - \dfrac{L\sin q_2}{y_A - y_B}\right)\dot{q}_2 \end{pmatrix},$$

the reduced dynamic model of the constrained PR robot is given by the single differential equation

$$\left(\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\right)\dot{v} = \boldsymbol{F}^T(\boldsymbol{q})\big(\boldsymbol{\tau} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\big) + \boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{E}\,\dot{\boldsymbol{A}}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\,v, \tag{23}$$

with the scalars

$$\begin{aligned}\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = {} & (m_1 + m_2)\left(\frac{L\cos q_2}{y_A - y_B} + \frac{L\sin q_2}{x_A - x_B}\right)^2 + \frac{I_2 + m_2 d_{c2}^2}{(y_A - y_B)^2} \\ & - 2\,\frac{m_2 d_2 \cos q_2}{y_A - y_B}\left(\frac{L\cos q_2}{y_A - y_B} + \frac{L\sin q_2}{x_A - x_B}\right)\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{E}\,\dot{\boldsymbol{A}}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = {} & -(m_1 + m_2)\left(\frac{L\cos q_2}{x_A - x_B} - \frac{L\sin q_2}{y_A - y_B}\right)^2 \dot{q}_2 \\ & \frac{m_2 d_2 \cos q_2}{y_A - y_B}\left(\frac{L\cos q_2}{x_A - x_B} - \frac{L\sin q_2}{y_A - y_B}\right)\dot{q}_2\,.\end{aligned}$$

Similarly, the multiplier $\lambda \in \mathbb{R}$ that produces the normal force when attempting to violate the constraint is

$$\lambda = \left(\boldsymbol{E}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\right)\dot{v} - \left(\boldsymbol{E}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{E}\,\dot{\boldsymbol{A}}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\right)v + \boldsymbol{E}^T\big(\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{\tau}\big), \tag{24}$$

with the scalars

$$\boldsymbol{E}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = (y_A - y_B)\left(\frac{m_2 d_2 \cos q_2}{y_A - y_B} - (m_1 + m_2)\left(\frac{L\cos q_2}{y_A - y_B} + \frac{L\sin q_2}{x_A - x_B}\right)\right)$$

and

$$\boldsymbol{E}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{E}\,\dot{\boldsymbol{A}}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = (m_1 + m_2)(y_A - y_B)\left(\frac{L\cos q_2}{x_A - x_B} - \frac{L\sin q_2}{y_A - y_B}\right)\dot{q}_2\,.$$

The desired rest-to-rest motion from A to B on the line $\mathcal{L}$ in time T is planned in a parametric way by defining the path as

$$\boldsymbol{p}_d(s) = \boldsymbol{p}_A + s\,\frac{\boldsymbol{p}_B - \boldsymbol{p}_A}{\Delta}, \qquad s \in [0,\Delta]\,, \qquad \Delta = \|\boldsymbol{p}_B - \boldsymbol{p}_A\| = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2},$$

where $\boldsymbol{p}_A \in \mathbb{R}^2$ and $\boldsymbol{p}_B \in \mathbb{R}^2$ are, respectively, the position vectors of point A and point B, and the timing law with a cubic profile as

$$s = s_d(t) = \Delta\left(3\left(\frac{t}{T}\right)^2 - 2\left(\frac{t}{T}\right)^3\right), \qquad t \in [0,T]\,.$$

Note that the parameter s is here the actual length of the path traced during motion. The desired pseudo-velocity and pseudo-acceleration are then

$$v_d(t) = \dot{s}_d(t) = 6\Delta\,\frac{t}{T}\left(1-\frac{t}{T}\right), \qquad \dot{v}_d(t) = \frac{6\Delta}{T^2}\left(1-2\,\frac{t}{T}\right), \qquad t \in [0,T]\,.$$

At time $t=0$, the position $\boldsymbol{p}$ of the robot end effector should be matched with the position $\boldsymbol{p}_A$ of the point A (on the linear constraint $h(\boldsymbol{q})=0$). The initial configuration $\boldsymbol{q}(0)$ is obtained by solving the inverse kinematics problem for the PR robot

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} L\cos q_2 \\ q_1 + L\sin q_2 \end{pmatrix} = \begin{pmatrix} x_A \\ y_A \end{pmatrix} = \boldsymbol{p}_A.$$

This yields[3]

$$\boldsymbol{q}_0 = \boldsymbol{q}(0) = \boldsymbol{f}^{-1}(\boldsymbol{p}_A) = \begin{pmatrix} y_A - \sin\left(\arccos\left(\dfrac{x_A}{L}\right)\right) \\ \arccos\left(\dfrac{x_A}{L}\right) \end{pmatrix} = \begin{pmatrix} y_A - \sqrt{1-\left(\dfrac{x_A}{L}\right)^2} \\ \arccos\left(\dfrac{x_A}{L}\right) \end{pmatrix}.$$

Moreover, since the robot starts at rest, it is $\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{q}}(0) = \boldsymbol{0}$ (consistently with $v_d(0)=0$).

To execute the desired constrained motion with $\dot{v} = \dot{v}_d(t)$ and $\lambda = \lambda_d = 0$ (no forces are generated normal to the line $\mathcal{L}$), we apply the inverse constrained dynamics control law

q_ddot is from slide #14 n. 33

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\,\dot{v}_d + \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{M}(\boldsymbol{q})\boldsymbol{E}\,\dot{\boldsymbol{A}}(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{25}$$

with

$$\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = \begin{pmatrix} -(m_1+m_2)\left(\dfrac{L\cos q_2}{x_A-x_B} - \dfrac{L\sin q_2}{y_A-y_B}\right) + \dfrac{m_2 d_2\cos q_2}{y_A-y_B} \\ -m_2 d_2\cos q_2\left(\dfrac{L\cos q_2}{x_A-x_B} - \dfrac{L\sin q_2}{y_A-y_B}\right) + \dfrac{I_2+m_2 d_2^2}{y_A-y_B} \end{pmatrix}$$

and

$$\boldsymbol{M}(\boldsymbol{q})\boldsymbol{E}\,\dot{\boldsymbol{A}}(\boldsymbol{q}) = \begin{pmatrix} 0 & (m_1+m_2)(y_A-y_B)\left(\dfrac{L\cos q_2}{x_A-x_B} - \dfrac{L\sin q_2}{y_A-y_B}\right)\dot{q}_2 \\ 0 & m_2 d_2\cos q_2\,(y_A-y_B)\left(\dfrac{L\cos q_2}{x_A-x_B} - \dfrac{L\sin q_2}{y_A-y_B}\right)\dot{q}_2 \end{pmatrix}.$$

At time $t=0$, we have

$$\boldsymbol{\tau}_0 = \boldsymbol{\tau}(0) = \boldsymbol{M}(\boldsymbol{q}_0)\boldsymbol{F}(\boldsymbol{q}_0)\dot{v}_d(0) + \boldsymbol{g}(\boldsymbol{q}_0) \tag{26}$$

With the given numerical data, we compute the following relevant quantities: the path length and the initial pseudo-acceleration

$$\Delta = 1.0198 \text{ [m]}, \qquad \dot{v}_d(0) = 1.5297 \text{ [m/s}^2\text{]};$$

the inverse kinematics solution at the point A

$$\boldsymbol{q}_0 = \begin{pmatrix} 1.2859 \\ 0.7954 \end{pmatrix} \text{ [m,rad]};$$

[3] We chose arbitrarily only one of the two inverse solutions. The other solution has a sign $-$ in front of arccos and a sign $+$ in front of the square root.

the inertia matrix, the gravity vector, and the $\boldsymbol{F}$ term evaluated at the initial configuration $\boldsymbol{q}_0$

$$\boldsymbol{M}(\boldsymbol{q}_0) = \begin{pmatrix} 23 & 3.36 \\ 3.36 & 4.08 \end{pmatrix}, \qquad \boldsymbol{g}(\boldsymbol{q}_0) = \begin{pmatrix} 225.63 \\ 32.96 \end{pmatrix}, \qquad \boldsymbol{F}(\boldsymbol{q}_0) = \begin{pmatrix} 4.27 \\ 1 \end{pmatrix}.$$

Finally, plugging in (26) the above values, we obtain

$$\boldsymbol{\tau}_0 = \begin{pmatrix} 381.02 \\ 61.15 \end{pmatrix} \text{ [N,Nm]}.$$

* * * * *

Robotics 2

February 3, 2022

Exercise #1

The RPR robot in Fig. 1 moves in a vertical plane and is controlled by the joint torque $\boldsymbol{\tau} \in \mathbb{R}^3$.

a) Provide a linear parametrization of the gravity term $\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{a}_G$ in the robot dynamic model, where the matrix $\boldsymbol{G}(\boldsymbol{q})$ contains only known kinematic quantities (including the gravity acceleration g_0). Introduce kinematic and dynamic parameters as needed.

b) Design a control law $\boldsymbol{\tau} = \boldsymbol{\tau}_r(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{e}_p)$, driven by the Cartesian error $\boldsymbol{e}_p = \boldsymbol{p}_d - \boldsymbol{p}$, that achieves regulation of the end-effector position to a desired constant value $\boldsymbol{p}_d \in \mathbb{R}^2$, up to singularities. Give the explicit expression of all terms in this control law.

c) Find a robot configuration $\boldsymbol{q}_s$ and an associated desired position $\boldsymbol{p}_d$, with $\boldsymbol{e}_p = \boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q}_s) \neq \boldsymbol{0}$, such that the robot will *not* move when at rest in $\boldsymbol{q}_s$ under the action of the previous control law $\boldsymbol{\tau} = \boldsymbol{\tau}_r(\boldsymbol{q}_s, \boldsymbol{0}, \boldsymbol{e}_p)$.

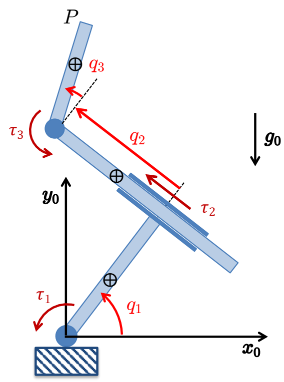


Figure 1: A planar RPR robot.

Exercise #2

Consider the planar 4R robot in Fig. 2, having equal links of unitary length. The robot is commanded by the joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^4$.

a) The end effector of the robot should follow a desired smooth position trajectory $\boldsymbol{p}_d(t) \in \mathbb{R}^2$. Provide the general form of the command $\ddot{\boldsymbol{q}}_a$ that executes the task in nominal conditions, while minimizing instantaneously the objective function

$$H = \frac{1}{2} \| \ddot{\boldsymbol{q}} + \boldsymbol{K}_d \dot{\boldsymbol{q}} \|^2, \qquad \boldsymbol{K}_d > 0. \tag{1}$$

Moreover, study the singularities that may be encountered during the execution of this task.

b) Consider again the problem in item a), but now with the desired task augmented in order to keep the end-effector angular speed at some constant value $\omega_{z,d} \in \mathbb{R}$. Provide the general form of the command $\ddot{\boldsymbol{q}}_b$ that executes the extended task.

c) Compute the numerical value of $\ddot{\boldsymbol{q}}_a$ when the robot is in the nominal state $\boldsymbol{x}_d = (\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d) \in \mathbb{R}^8$ and for a desired $\ddot{\boldsymbol{p}}_d \in \mathbb{R}^2$ given by

$$\boldsymbol{q}_d = \begin{pmatrix} \pi/4 \\ \pi/3 \\ -\pi/2 \\ 0 \end{pmatrix} \text{ [rad]}, \qquad \dot{\boldsymbol{q}}_d = \begin{pmatrix} -0.8 \\ 1 \\ 0.2 \\ 0 \end{pmatrix} \text{ [rad/s]}, \qquad \ddot{\boldsymbol{p}}_d = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ [m/s}^2\text{]},$$

having set $\boldsymbol{K}_d = \boldsymbol{I}_{4\times 4}$ in (1).

d) Compute the numerical value of $\ddot{\boldsymbol{q}}_b$ in the same conditions of item c). What are the values of $\boldsymbol{p}_d$, $\dot{\boldsymbol{p}}_d$, and $\omega_{z,d}$ in this case?

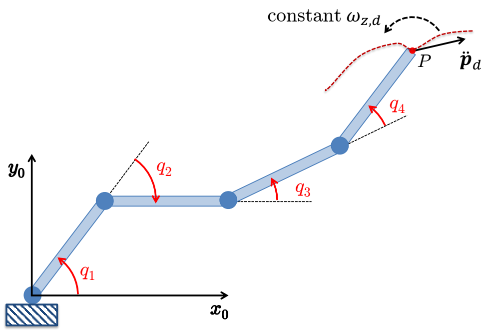


Figure 2: A planar 4R robot, with a sketch of its Cartesian tasks.

Exercise #3

Figure 3 shows a 2R robot with unitary link lengths whose end effector is mechanically constrained to move only along the vertical segment between points $A = (0, 1)$ and $B = (0, \sqrt{2})$, under the action of the single available motor torque τ at the first joint. The second joint is passive and all dissipative effects can be neglected.

a) Derive a (one-dimensional) *reduced* dynamic model of the constrained robot.

b) If the robot is in an equilibrium state with its end effector in A, what is the applied torque τ_0?

c) Suppose that the end effector should execute a rest-to-rest motion from A and B with a sinusoidal acceleration profile at an angular frequency $\omega = 0.5$ [rad/s]. What is the explicit expression of the needed torque command $\tau_d(t)$?

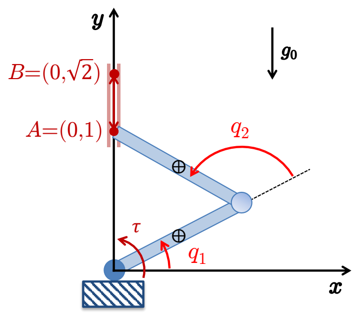


Figure 3: A 2R robot in constrained motion, with one actuator only at the first joint.

[210 minutes (3.5 hours); open books]

Solution

February 3, 2022

Exercise #1

We compute first the gravity term $\boldsymbol{g}(\boldsymbol{q})$ needed in the regulation law and provide a linear parametrization for it. The potential energy of each link is computed from the general expression (with vectors in $\mathbb{R}^2$, being the problem planar)

$$U_i = -m_i\,\boldsymbol{g}_0^T\boldsymbol{r}_{i,c_i}, \qquad \boldsymbol{g}_0 = \begin{pmatrix} 0 \\ -g_0 \end{pmatrix} \qquad \Rightarrow \qquad U_i = m_i g_0\, r_{i,c_{i_y}}$$

with $g_0 = 9.81$ [m/s^2]. Thus,

$$\begin{aligned} U_1 &= m_1 g_0\, d_1 \sin q_1, \\ U_2 &= m_2 g_0 \left(l_1 \sin q_1 + (q_2 - d_2)\cos q_1\right), \\ U_3 &= m_3 g_0 \left(l_1 \sin q_1 + q_2 \cos q_1 + d_3 \sin(q_1+q_3)\right), \end{aligned}$$

where l_1 is the length of link 1, d_1 and d_3 are the positions of the center mass of link 1 and link 3, respectively, computed from their proximal base, and d_2 is the position of the center mass of link 2 as computed from its distal base (i.e., the axis of joint 3). As a result, from $U = U_1 + U_2 + U_3$, we obtain

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T = g_0 \begin{pmatrix} (m_1 d_1 + (m_2+m_3) l_1)\cos q_1 - (m_2(q_2-d_2) + m_3 q_2)\sin q_1 \\ + m_3 d_3 \cos(q_1+q_3) \\ (m_2+m_3)\cos q_1 \\ m_3 d_3 \cos(q_1+q_3) \end{pmatrix}. \tag{2}$$

This can be linearly parametrized by a (3×4) regressor matrix $\boldsymbol{G}(\boldsymbol{q})$ and a vector of dynamic coefficients $\boldsymbol{a}_G \in \mathbb{R}^4$ as follows:

$$\boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{a}_G = \begin{pmatrix} g_0\left(l_1 \cos q_1 - q_2 \sin q_1\right) & g_0\,\cos q_1 & g_0\,\sin q_1 & g_0\,\cos(q_1+q_3) \\ g_0\,\cos q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_0\,\cos(q_1+q_3) \end{pmatrix} \begin{pmatrix} m_2+m_3 \\ m_1 d_1 \\ m_2 d_2 \\ m_3 d_3 \end{pmatrix}.$$

Indeed, also other parametrizations that are still minimal (i.e., of dimension 4) can be found.

The required Cartesian regulator is given by[1]

$$\boldsymbol{\tau}_r = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{K}_P\left(\boldsymbol{p}_d - \boldsymbol{f}(\boldsymbol{q})\right) - \boldsymbol{K}_D\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}), \qquad \text{with } \boldsymbol{K}_P > 0,\ \ \boldsymbol{K}_D > 0. \tag{3}$$

Having already given the gravity term in (2), the missing expressions in (3) are the direct kinematics

$$\boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} l_1 \cos q_1 - q_2 \sin q_1 + l_3 \cos(q_1+q_3) \\ l_1 \sin q_1 + q_2 \cos q_1 + l_3 \sin(q_1+q_3) \end{pmatrix}$$

and its associated Jacobian

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(l_1 \sin q_1 + q_2 \cos q_1 + l_3 \sin(q_1+q_3)\right) & -\sin q_1 & -l_3 \sin(q_1+q_3) \\ l_1 \cos q_1 - q_2 \sin q_1 + l_3 \cos(q_1+q_3) & \cos q_1 & l_3 \cos(q_1+q_3) \end{pmatrix}. \tag{4}$$

[1] Alternatively, one can replace the joint damping term $-\boldsymbol{K}_D\dot{\boldsymbol{q}}$ by a Cartesian damping $-\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{K}_{D,c}\,\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$, with $\boldsymbol{K}_{D,c} > 0$. However, there is no actual advantage in doing so.

To answer to item c), we need to find the singular configurations $\boldsymbol{q}_s$ of $\boldsymbol{J}(\boldsymbol{q})$. In fact, the controller (3) gets stuck with the robot at rest in an end-effector position $\boldsymbol{p}_s = \boldsymbol{f}(\boldsymbol{q}_s)$ different from the desired $\boldsymbol{p}_d$ if and only if the (gain scaled) error $\boldsymbol{K}_P \boldsymbol{e}_p$, with $\boldsymbol{e}_p = \boldsymbol{p}_d - \boldsymbol{p}_s \neq \boldsymbol{0}$, lies in the null space of the transpose of the Jacobian $\boldsymbol{J}_s = \boldsymbol{J}(\boldsymbol{q}_s)$. And this can happen only when the Jacobian in (4) loses rank. Analyzing the three minors (obtained by deleting one column of $\boldsymbol{J}$ at a time), we have

$$\det \boldsymbol{J}_{-1} = l_3 \sin q_3, \qquad \det \boldsymbol{J}_{-2} = l_3 \left(l_1 \sin q_3 - q_2 \cos q_3 \right), \qquad \det \boldsymbol{J}_{-3} = -(q_2 + l_3 \sin q_3).$$

These are simultaneously zero if and only if $q_2 = 0$ and $q_3 = 0$ or π (with arbitrary q_1). Take for example $\boldsymbol{q}_s = (q_1, 0, 0)$. We have

$$\boldsymbol{p}_s = (l_1 + l_3) \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \qquad \boldsymbol{J}_s = \begin{pmatrix} -(l_1 + l_3)\sin q_1 & -\sin q_1 & -l_3 \sin q_1 \\ (l_1 + l_3)\cos q_1 & \cos q_1 & l_3 \cos q_1 \end{pmatrix}$$

and so

$$\mathcal{N}\left\{ \boldsymbol{J}_s^T \right\} = \alpha \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}.$$

Consider a gain $\boldsymbol{K}_P = k_P \boldsymbol{I}_{2\times 2}$, $k_P > 0$. By choosing

$$\boldsymbol{p}_d = \Delta \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix}, \quad \Delta < l_1 + l_3 \qquad \Rightarrow \qquad \boldsymbol{K}_P \boldsymbol{e}_p = k_P \left(\Delta - (l_1 + l_3) \right) \begin{pmatrix} \cos q_1 \\ \sin q_1 \end{pmatrix} \neq \boldsymbol{0},$$

with the robot at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$), the control law (3) becomes simply $\boldsymbol{\tau}_r = \boldsymbol{\tau}_r(\boldsymbol{q}_s, \boldsymbol{0}, \boldsymbol{e}_p) = \boldsymbol{g}(\boldsymbol{q}_s)$. In the closed-loop system, with the robot dynamics evaluated at the state $\boldsymbol{x}_s = (\boldsymbol{q}_s, \boldsymbol{0})$, it is

$$\boldsymbol{M}(\boldsymbol{q}_s)\,\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}_s, \boldsymbol{0}) + \boldsymbol{g}(\boldsymbol{q}_s) = \boldsymbol{g}(\boldsymbol{q}_s) \qquad \Rightarrow \qquad \boldsymbol{M}(\boldsymbol{q}_s)\,\ddot{\boldsymbol{q}} = \boldsymbol{0} \qquad \Leftrightarrow \qquad \ddot{\boldsymbol{q}} = \boldsymbol{0},$$

so that the state $\boldsymbol{x}_s$ is an equilibrium and the robot will not move under the action of (3), despite of the residual Cartesian position error.

Exercise #2

We compute the direct and the differential kinematics up to the second order of the planar 4R robot for the positional task of its end effector. Being all links of unitary length, we have

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} c_1 + c_{12} + c_{123} + c_{1234} \\ s_1 + s_{12} + s_{123} + s_{1234} \end{pmatrix},$$

with the shorthand notation for trigonometric quantities (e.g., $c_{123} = \cos(q_1 + q_2 + q_3)$). Differentiating once w.r.t. time, we obtain

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

with the (2×4) Jacobian matrix

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(s_1 + s_{12} + s_{123} + s_{1234}) & -(s_{12} + s_{123} + s_{1234}) & -(s_{123} + s_{1234}) & -s_{1234} \\ c_1 + c_{12} + c_{123} + c_{1234} & c_{12} + c_{123} + c_{1234} & c_{123} + c_{1234} & c_{1234} \end{pmatrix}. \tag{5}$$

Differentiating a second time, we get

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{6}$$

with the quadratic term in the joint velocities

$$\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}} = -\begin{pmatrix} c_1\,\dot{q}_1^2 + c_{12}\,(\dot{q}_1+\dot{q}_2)^2 + c_{123}\,(\dot{q}_1+\dot{q}_2+\dot{q}_3)^2 + c_{1234}\,(\dot{q}_1+\dot{q}_2+\dot{q}_3+\dot{q}_4)^2 \\ s_1\,\dot{q}_1^2 + s_{12}\,(\dot{q}_1+\dot{q}_2)^2 + s_{123}\,(\dot{q}_1+\dot{q}_2+\dot{q}_3)^2 + s_{1234}\,(\dot{q}_1+\dot{q}_2+\dot{q}_3+\dot{q}_4)^2 \end{pmatrix}.$$

At the current robot state $(\boldsymbol{q},\dot{\boldsymbol{q}})$ and for a given $\ddot{\boldsymbol{p}} = \ddot{\boldsymbol{p}}_d$, finding a solution $\ddot{\boldsymbol{q}}$ to (6) that minimizes instantaneously the objective function (1) is a standard LQ problem with the unique solution

$$\begin{aligned} \ddot{\boldsymbol{q}}_a &= \boldsymbol{J}^{\#}(\boldsymbol{q})\,(\ddot{\boldsymbol{p}}_d - \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})) - \Big(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\Big)\,\boldsymbol{K}_d\,\dot{\boldsymbol{q}} \\ &= -\boldsymbol{K}_d\,\dot{\boldsymbol{q}} + \boldsymbol{J}^{\#}(\boldsymbol{q})\Big(\,\ddot{\boldsymbol{p}}_d - \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{J}(\boldsymbol{q})\boldsymbol{K}_d\,\dot{\boldsymbol{q}}\Big). \end{aligned} \tag{7}$$

In fact, the *preferred* acceleration in the objective function (1), i.e., the one that would minimize H in the unconstrained case, is $\ddot{\boldsymbol{q}}_0 = -\boldsymbol{K}_d\dot{\boldsymbol{q}}$. This achieves damping of joint velocities in the null space of the task, as apparent from the first expression in (7). The second equivalent expression is however more efficient to evaluate.

The singularities of $\boldsymbol{J}(\boldsymbol{q})$ may affect the execution of the task and need to be analyzed in advance. Even if not strictly necessary, it is convenient for this purpose to simplify the Jacobian by the following factorization[2]

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -s_1 & -s_{12} & -s_{123} & -s_{1234} \\ c_1 & c_{12} & c_{123} & c_{1234} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \boldsymbol{J}'(\boldsymbol{q})\,\boldsymbol{T}, \tag{8}$$

where $\boldsymbol{T}$ is a nonsingular matrix (with $\det \boldsymbol{T} = 1$). Thus, the configurations at which $\boldsymbol{J}'(\boldsymbol{q})$ loses rank coincide with those of $\boldsymbol{J}(\boldsymbol{q})$. By inspecting the six (2×2) minors of $\boldsymbol{J}'$, it is easy to see that this happens in the singular configurations

$$\boldsymbol{q}_s : \{\, q_1 \text{ is arbitrary}, \;\; q_2 = 0 \text{ or } \pi, \;\; q_3 = 0 \text{ or } \pi, \;\; q_4 = 0 \text{ or } \pi \,\}.$$

In all these 8 types of singular configurations, the links of the robot are either stretched or folded along the single direction specified by q_1.

We extend now the task with the angular component, concerning the orientation of the end-effector. We proceed incrementally, using the results obtained so far for the two-dimensional task. We have

$$\boldsymbol{p}_e = \begin{pmatrix} \boldsymbol{f}(\boldsymbol{q}) \\ q_1 + q_2 + q_3 + q_4 \end{pmatrix},$$

and thus

$$\dot{\boldsymbol{p}}_e = \boldsymbol{J}_e(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{J}(\boldsymbol{q}) \\ 1 \quad 1 \quad 1 \quad 1 \end{pmatrix}\dot{\boldsymbol{q}}$$

and

$$\ddot{\boldsymbol{p}}_e = \boldsymbol{J}_e(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_e(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \boldsymbol{J}_e(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{n}_e(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{J}_e(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \begin{pmatrix} \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) \\ 0 \end{pmatrix}.$$

[2]This is a common trick for planar nR robot structures.

The solution of the LQ problem takes exactly the same form as in (7), once the augmented quantities are used, including the augmented desired acceleration $\ddot{\boldsymbol{p}}_{e,d}$, which is given by $\ddot{\boldsymbol{p}}_d \in \mathbb{R}^2$ with an extra third component equal to 0, or

$$\ddot{\boldsymbol{p}}_{e,d} = \begin{pmatrix} \ddot{\boldsymbol{p}}_d \\ \dot{\omega}_{z,d} \end{pmatrix} = \begin{pmatrix} \ddot{\boldsymbol{p}}_d \\ 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned} \ddot{\boldsymbol{q}}_b &= \boldsymbol{J}_e^{\#}(\boldsymbol{q}) \left(\ddot{\boldsymbol{p}}_{e,d} - \boldsymbol{n}_e(\boldsymbol{q}, \dot{\boldsymbol{q}})\right) - \left(\boldsymbol{I} - \boldsymbol{J}_e^{\#}(\boldsymbol{q})\boldsymbol{J}_e(\boldsymbol{q})\right) \boldsymbol{K}_d \, \dot{\boldsymbol{q}} \\ &= -\boldsymbol{K}_d \, \dot{\boldsymbol{q}} + \boldsymbol{J}_e^{\#}(\boldsymbol{q})\Big(\ddot{\boldsymbol{p}}_{e,d} - \boldsymbol{n}_e(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{J}_e(\boldsymbol{q})\boldsymbol{K}_d \, \dot{\boldsymbol{q}} \Big). \end{aligned} \tag{9}$$

The only difference will be in the analysis of the singularities of the (3×4) Jacobian $\boldsymbol{J}_e(\boldsymbol{q})$. Proceeding as before with the same transformation matrix $\boldsymbol{T}$, we have

$$\boldsymbol{J}_e(\boldsymbol{q}) = \boldsymbol{J}'_e(\boldsymbol{q})\,\boldsymbol{T}, \qquad \text{with} \quad \boldsymbol{J}'_e(\boldsymbol{q}) = \begin{pmatrix} -s_1 & -s_{12} & -s_{123} & -s_{1234} \\ c_1 & c_{12} & c_{123} & c_{1234} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

Accordingly, the singular configurations of $\boldsymbol{J}_e(\boldsymbol{q})$ are namely

$$\boldsymbol{q}_s : \{\, q_1 \text{ is arbitrary}, \;\; q_2 = 0 \text{ or } \pi, \;\; q_3 = 0 \text{ or } \pi, \;\; q_4 \text{ is arbitrary} \,\},$$

namely the same as those of the (2×3) Jacobian of a planar 3R robot in a positional task. In these 4 types of singular configurations, the first three links of the robot are either stretched or folded along the single direction specified again by q_1.

Using the values $\boldsymbol{q}_d$, $\dot{\boldsymbol{q}}_d$, and $\ddot{\boldsymbol{p}}_d$ specified in item c) of the text of this Exercise, we evaluate (7) and (9) as

$$\ddot{\boldsymbol{q}}_a = \begin{pmatrix} -0.3074 \\ -1.6378 \\ 2.0496 \\ 1.1248 \end{pmatrix} \; [\text{rad/s}^2], \qquad \ddot{\boldsymbol{q}}_b = \begin{pmatrix} 0.3163 \\ -2.5735 \\ 3.0503 \\ -0.7932 \end{pmatrix} \; [\text{rad/s}^2].$$

The extended position and orientation task vector and the end-effector linear and angular velocity at the current state $\boldsymbol{x}_d = (\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d)$ are

$$\boldsymbol{p}_e = \begin{pmatrix} \boldsymbol{p}_d \\ \phi_d \end{pmatrix} = \begin{pmatrix} 2.3801 \\ 2.1907 \\ 0.2618 \end{pmatrix} \; [\text{m, m, rad}], \qquad \dot{\boldsymbol{p}}_e = \begin{pmatrix} \dot{\boldsymbol{p}}_d \\ \omega_{z,d} \end{pmatrix} = \begin{pmatrix} 0.1654 \\ 0.1553 \\ 0.4 \end{pmatrix} \; [\text{m/s, m/s, rad/s}].$$

Exercise #3

Apart from the knowledge of unitary link lengths, no dynamic information is provided about the planar 2R robot in Fig. 3, Therefore, in its dynamic model,

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u} + \boldsymbol{A}^T(\boldsymbol{q})\lambda = \begin{pmatrix} \tau \\ 0 \end{pmatrix} + \boldsymbol{A}^T(\boldsymbol{q})\lambda, \tag{11}$$

we will use a parametric form for the inertia, Coriolis/centrifugal, and gravity terms

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + 2a_2c_2 & a_3 + a_2c_2 \\ a_3 + a_2c_2 & a_3 \end{pmatrix}, \;\; \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -a_2s_2\,(\dot{q}_2 + 2\dot{q}_1)\,\dot{q}_2 \\ a_2s_2\dot{q}_1^2 \end{pmatrix}, \;\; \boldsymbol{g}(\boldsymbol{q}) = \begin{pmatrix} a_4c_1 + a_5c_{12} \\ a_5c_{12} \end{pmatrix},$$

with dynamic coefficients a_i, $i = 1\ldots,5$, and using the shorthand notation for trigonometric quantities. On the right-hand side of (11), we have taken into account that there is no motor at the second joint (the joint is passive) and that the robot is subject to a holonomic constraint, providing a reaction force $\lambda \in \mathbb{R}$. The scalar constraint on the robot end effector is written as

$$k(\boldsymbol{p}) = p_x = 0 \qquad \Rightarrow \qquad h(\boldsymbol{q}) = k\left(\boldsymbol{f}(\boldsymbol{q})\right) = c_1 + c_{12} = 0.$$

The Jacobian of this constraint is

$$\boldsymbol{A}(\boldsymbol{q}) = \frac{\partial h(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -\left(s_1 + s_{12}\right) & -s_{12} \end{pmatrix}. \tag{12}$$

In the assumed hypothesis on the location of the two points A and B, the matrix $\boldsymbol{A}(\boldsymbol{q})$ is always well defined and with full rank ($= 1$).

In order to obtain the reduced dynamic model of the constrained 2R robot, the basic step is to define a 1×2 (row) vector $\boldsymbol{D}(\boldsymbol{q})$ that is linearly independent from $\boldsymbol{A}(\boldsymbol{q})$, possibly everywhere in the region of interest. A useful choice is given by the following matrix (also with rank one in the constrained space)

$$\boldsymbol{D}(\boldsymbol{q}) = \begin{pmatrix} c_1 + c_{12} & c_{12} \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix} = \boldsymbol{J}(\boldsymbol{q}), \tag{13}$$

generating in this way the (2×2) robot Jacobian. This matrix is always nonsingular in the constrained region of operation of the 2R robot. Thus,

$$\begin{pmatrix} \boldsymbol{A}(\boldsymbol{q}) \\ \boldsymbol{D}(\boldsymbol{q}) \end{pmatrix}^{-1} = \boldsymbol{J}^{-1}(\boldsymbol{q}) = \frac{\operatorname{adj}\left\{\boldsymbol{J}(\boldsymbol{q})\right\}}{\det \boldsymbol{J}(\boldsymbol{q})} = \frac{1}{s_2}\begin{pmatrix} c_{12} & s_{12} \\ -\left(c_1 + c_{12}\right) & -\left(s_1 + s_{12}\right) \end{pmatrix} \triangleq \begin{pmatrix} \boldsymbol{E}(\boldsymbol{q}) & \boldsymbol{F}(\boldsymbol{q}) \end{pmatrix}.$$

We define then the pseudo-velocity $v_y \in \mathbb{R}$ on the Cartesian line and the inverse mapping to $\dot{\boldsymbol{q}}$ as

$$v_y = \boldsymbol{D}(\boldsymbol{q})\,\dot{\boldsymbol{q}} = c_1\,\dot{q}_1 + c_{12}\left(\dot{q}_1 + \dot{q}_2\right), \qquad \dot{\boldsymbol{q}} = \boldsymbol{F}(\boldsymbol{q})\,v_y = \begin{pmatrix} s_{12} \\ -\left(s_1 + s_{12}\right) \end{pmatrix} v_y. \tag{14}$$

Note that a subscript y has been added to v since this is exactly the y-component of the end-effector velocity, the only allowed by the constraint. Since the elements of the matrix $\boldsymbol{F}(\boldsymbol{q})$ are available analytically, we can obtain its time derivative in closed form[3]

$$\dot{\boldsymbol{F}}(\boldsymbol{q}) = \frac{1}{s_2}\begin{pmatrix} c_{12}\left(\dot{q}_1 + \dot{q}_2\right) \\ -c_1\dot{q}_1 - c_{12}\left(\dot{q}_1 + \dot{q}_2\right) \end{pmatrix} - \frac{c_2\,\dot{q}_2}{s_2^2}\begin{pmatrix} s_{12} \\ -\left(s_1 + s_{12}\right) \end{pmatrix}$$

Thus, the reduced dynamic model of the constrained 2R robot is given by the single differential equation

$$\left(\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q})\right)\dot{v}_y = \boldsymbol{F}^T(\boldsymbol{q})\left(\boldsymbol{u} - \boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{F}}(\boldsymbol{q})\,v_y\right), \tag{15}$$

with the (always positive) reduced inertia given by the scalar

$$\boldsymbol{F}^T(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}(\boldsymbol{q}) = \frac{1}{s_2^2}\left(a_3 s_1^2 + (a_1 - a_3)s_{12}^2 - 2a_2 s_1 c_2 s_{12}\right).$$

[3]This means that we don't need to replace the term $\dot{\boldsymbol{F}}(\boldsymbol{q})\dot{\boldsymbol{q}}$ by the longer expression $-\left(\dot{\boldsymbol{E}}(\boldsymbol{q})\dot{\boldsymbol{A}}(\boldsymbol{q}) + \boldsymbol{F}(\boldsymbol{q})\dot{\boldsymbol{D}}(\boldsymbol{q})\right)\dot{\boldsymbol{q}}$ within the derivations of the constrained dynamics. This is reflected in the form of the resulting equation (15).

Being only the single torque τ available as input, we have

$$\boldsymbol{F}^T(\boldsymbol{q})\,\boldsymbol{u} = \frac{1}{s_2}\begin{pmatrix} s_{12} & -(s_1+s_{12}) \end{pmatrix}\begin{pmatrix} \tau \\ 0 \end{pmatrix} = \frac{s_{12}}{s_2}\,\tau, \tag{16}$$

with $s_2 \neq 0$ in the domain of interest. For the following developments, the derivation of the expression of the force multiplier λ is not needed.

For item b), we need to find the robot configuration associated with point A. The inverse kinematic solution for this point can be found by geometric inspection: $\boldsymbol{q}_A = (\pi/6,\, 2\pi/3)$. Setting the robot at rest, we have that

$$\dot{\boldsymbol{q}} = \boldsymbol{0} \qquad \Rightarrow \qquad \boldsymbol{c}(\boldsymbol{q}_A, \boldsymbol{0}) = \boldsymbol{0}, \quad v_y = \boldsymbol{D}(\boldsymbol{q}_A)\dot{\boldsymbol{q}} = 0,$$

and from (15) it follows

$$\left(\boldsymbol{F}^T(\boldsymbol{q}_A)\boldsymbol{M}(\boldsymbol{q}_A)\boldsymbol{F}(\boldsymbol{q}_A)\right)\dot{v}_y = \boldsymbol{F}^T(\boldsymbol{q}_A)\,\boldsymbol{u} - \boldsymbol{F}^T(\boldsymbol{q}_A)\,\boldsymbol{g}(\boldsymbol{q}_A) = \left.\frac{s_{12}}{s_2}\right|_{\boldsymbol{q}=\boldsymbol{q}_A}\tau - \left.\frac{s_{12}a_4c_1 - s_1a_5c_{12}}{s_2}\right|_{\boldsymbol{q}=\boldsymbol{q}_A}.$$

In order to have $\dot{v}_y = 0$, i.e., an equilibrium, we need to apply the torque

$$\tau_0 = \left.\left(a_4c_1 - \frac{s_1}{s_{12}}\,a_5c_{12}\right)\right|_{\boldsymbol{q}=\boldsymbol{q}_A} = \frac{\sqrt{3}}{2}\,(a_4 + a_5)\,. \tag{17}$$

The value in (17) implicitly takes into account the reaction force (i.e., λ) imposed by the constraint at the end effector, which helps sustaining the robot against gravity in the configuration $\boldsymbol{q}_A$, even in the absence of a motor torque at joint 2. In fact, note that this is *not* the first component of the gravity torque $\boldsymbol{g}(\boldsymbol{q}_A)$, which is

$$g_1(\boldsymbol{q}_A) = \left.(a_4c_1 + a_5c_{12})\right|_{\boldsymbol{q}=\boldsymbol{q}_A} = \frac{\sqrt{3}}{2}\,(a_4 - a_5)\,.$$

For item c), we design first the required motion trajectory for the end-effector. We start from the desired acceleration profile

$$\dot{v}_{y,d}(t) = \Delta\,\sin\omega t, \qquad t \in [0,T], \quad T = \frac{2\pi}{\omega}, \tag{18}$$

which is zero at start (for $t = 0$) and end (for $t = T$). The amplitude $\Delta > 0$ needs yet to be defined. Integrating this acceleration profile, and imposing zero speed at the motion start and end, we obtain

$$v_{y,d}(t) = \frac{\Delta}{\omega}\,(1 - \cos\omega t)\,, \qquad t \in [0,T]. \tag{19}$$

The positional trajectory starting from $A = (0,1)$ at time $t = 0$ is then

$$p_{y,d}(t) = y_A + \frac{\Delta}{\omega}\left(t - \frac{1}{\omega}\sin\omega t\right).$$

We impose that the motion ends in point $B = (0, \sqrt{2})$ at time $t = T = 2\pi/\omega$, obtaining eventually the value of Δ

$$p_{y,d}(T) = y_A + \frac{\Delta}{\omega}\frac{2\pi}{\omega} = y_B \qquad \Rightarrow \qquad \Delta = \frac{\omega^2}{2\pi}\,(y_B - y_A) = \frac{\omega^2}{2\pi}\left(\sqrt{2} - 1\right) > 0.$$

To compute the torque $\tau_d(t)$ that realizes the motion, we use (18), (19), and again (15), evaluated along the nominal state trajectory $\boldsymbol{x}_d(t) = (\boldsymbol{q}_d(t), \dot{\boldsymbol{q}}_d(t))$:

$$
\begin{aligned}
\tau_d(t) = \left.\frac{s_2}{s_{12}}\right|_{\boldsymbol{q}=\boldsymbol{q}_d(t)} \cdot \boldsymbol{F}^T(\boldsymbol{q}_d(t)) \Big(& \boldsymbol{M}(\boldsymbol{q}_d(t)) \boldsymbol{F}(\boldsymbol{q}_d(t))\, \dot{v}_{y,d}(t) + \boldsymbol{c}(\boldsymbol{q}_d(t), \dot{\boldsymbol{q}}_d(t)) + \boldsymbol{g}(\boldsymbol{q}_d(t)) \\
& + \boldsymbol{M}(\boldsymbol{q}_d(t)) \dot{\boldsymbol{F}}(\boldsymbol{q}_d(t))\, v_{y,d}(t) \Big) .
\end{aligned}
\tag{20}
$$

To determine the state trajectory $\boldsymbol{x}_d(t)$, $t \in [0, T]$, to be used in (20), one starts with

$$
\boldsymbol{q}_d(0) = \boldsymbol{q}_A, \qquad \dot{\boldsymbol{q}}_d(0) = \mathbf{0},
$$

and integrates forward in time the joint acceleration

$$
\ddot{\boldsymbol{q}}_d = \boldsymbol{F}(\boldsymbol{q}_d)\, \dot{v}_{y,d} + \dot{\boldsymbol{F}}(\boldsymbol{q}_d) \boldsymbol{F}(\boldsymbol{q}_d) \dot{\boldsymbol{q}}_d,
$$

as driven by the desired (reduced) acceleration profile $\dot{v}_{y,d}(t)$ in (18).

* * * * *

Robotics 2

June 10, 2022

Exercise #1

The PR robot in Fig. 1 moves on a *horizontal* plane. Its inertia matrix has the form

$$M(q) = \begin{pmatrix} A & B\cos q_2 \\ B\cos q_2 & C \end{pmatrix} > 0. \tag{1}$$

Using only the symbolic coefficients A, B and C in (1), provide the expression of the regressor matrix $\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)$ and of the complete *adaptive* control law that guarantees global asymptotic tracking of a smooth joint trajectory $\boldsymbol{q}_d(t)$, without a priori information on the values of the dynamic coefficients. Neglect all dissipative effects.

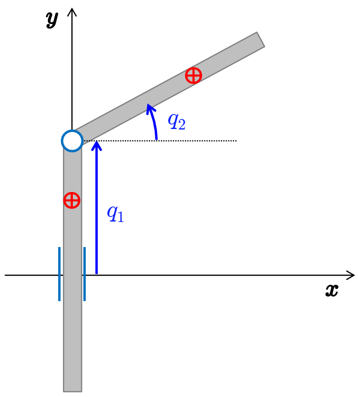


Figure 1: A planar PR robot with the definition of the joint variables q_1 and q_2.

Exercise #2

A macro-micro planar 4R robot is commanded by kinematic control at the joint velocity level. The first two links have equal length, $L_1 = L_2 = L$, and the last two links are also equal in length but four times shorter, $L_3 = L_4 = L/4$ (micro-manipulator). A trajectory $\boldsymbol{p}_{d,1}(t) \in \mathbb{R}^2$ is assigned to the robot end-effector, which is the highest priority task. A second trajectory $\boldsymbol{p}_{d,2}(t) \in \mathbb{R}^2$ is assigned to the tip of the second link, which is the end of the supporting macro part of the robot. Determine the arm configurations $\boldsymbol{q}_s$ at which *algorithmic* singularities occur for the extended Jacobian of the two simultaneous motion tasks. Specify the additional conditions needed in such algorithmic singularities under which a task priority approach would enable perfect execution of the primary task (with some least-squares error on the secondary task).

Exercise #3

The dynamic model of a robot with n elastic joints interacting with the environment can be expressed by two second-order differential equations, each of dimension n,

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}_J + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F} \tag{2}$$

$$\boldsymbol{B}\ddot{\boldsymbol{\theta}} + \boldsymbol{\tau}_J = \boldsymbol{\tau}, \tag{3}$$

named respectively, the *link dynamics* and the *motor dynamics* of the elastic joint robot. In these equations, $\boldsymbol{\theta} \in \mathbb{R}^n$ are the motor variables (before joint elasticity), $\boldsymbol{q} \in \mathbb{R}^n$ are the link variables (after joint elasticity), and $\boldsymbol{\tau}_J = \boldsymbol{K}_J\,(\boldsymbol{\theta} - \boldsymbol{q}) \in \mathbb{R}^n$ is the elastic joint torque measured by the joint torque sensors, with a diagonal joint stiffness matrix $\boldsymbol{K}_J > 0$. Moreover, the robot is equipped with two encoders per joint, measuring both $\boldsymbol{\theta}$ and $\boldsymbol{q}$. The dynamic terms on the left-hand side of the top equation (2) are the same as in a rigid robot; on the right-hand side, there is also the end-effector robot Jacobian $\boldsymbol{J}(\boldsymbol{q}) = (\partial \boldsymbol{f}(\boldsymbol{q})/\partial \boldsymbol{q})$ associated to the Cartesian task vector $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{q}) \in \mathbb{R}^m$ and the related generalized Cartesian forces $\boldsymbol{F} \in \mathbb{R}^m$. On the other hand, in the bottom equation (3), the diagonal matrix $\boldsymbol{B} > 0$ collects the reflected motor inertias, while $\boldsymbol{\tau} \in \mathbb{R}^n$ are the input torques available for control. Suppose also that $m = n$.

- Using feedback from joint torque sensors and motor velocities, design first a control law for $\boldsymbol{\tau}$ such that the motor dynamics becomes

  $$\boldsymbol{B}_0\,\ddot{\boldsymbol{\theta}} + \boldsymbol{D}_0\dot{\boldsymbol{\theta}} + \boldsymbol{\tau}_J = \boldsymbol{u}, \qquad \text{for a diagonal, } \textit{arbitrary small } \boldsymbol{B}_0 > 0 \text{ and a suitable } \boldsymbol{D}_0 > 0.$$

- Thanks to this inertial reduction, the motor dynamics is made arbitrarily fast so that we can assume $\boldsymbol{\tau}_J \simeq \boldsymbol{u}$ (this fast dynamics can be seen as if it were always at steady-state). Complete then the control design on the robot link dynamics by imposing to the Cartesian task vector $\boldsymbol{x}$, the following impedance model

  $$\boldsymbol{M}_x(\boldsymbol{q})\,\ddot{\boldsymbol{x}} + \left(\boldsymbol{S}_x(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{D}_m\right)\dot{\boldsymbol{x}} + \boldsymbol{K}_m(\boldsymbol{x} - \boldsymbol{x}_d) = \boldsymbol{F} \tag{4}$$

  where $\boldsymbol{x}_d$ is constant, $\boldsymbol{K}_m > 0$ and $\boldsymbol{D}_m > 0$ are desired, diagonal stiffness and damping matrices, and, assuming to work out of kinematic singularities,

  $$\boldsymbol{M}_x(\boldsymbol{q}) = \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}, \quad \boldsymbol{S}_x(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{J}^{-T}(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q},\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q}) - \boldsymbol{M}_x(\boldsymbol{q})\dot{\boldsymbol{J}}(\boldsymbol{q})\boldsymbol{J}^{-1}(\boldsymbol{q}).$$

Write the final control law for the input torque $\boldsymbol{\tau} \in \mathbb{R}^n$ in explicit form only in terms of the original state variables $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\boldsymbol{\theta}$, and $\dot{\boldsymbol{\theta}}$. Moreover, if an external constant force $\boldsymbol{F} = \bar{\boldsymbol{F}}$ is being applied from the environment to the robot, which will the equilibrium $\boldsymbol{x} = \boldsymbol{x}_E$ at steady state and what will be the value $\boldsymbol{\tau}_E$ of the control torque $\boldsymbol{\tau}$ at this equilibrium?

Exercise #4

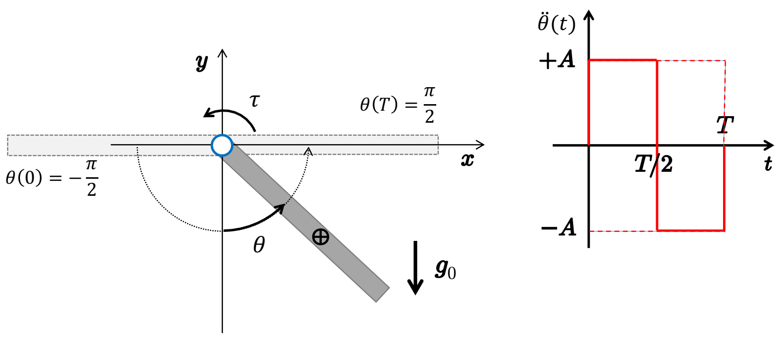


Figure 2: A one-link arm moving under gravity and the desired joint acceleration profile.

Consider the actuated link under gravity in Fig. 2, with the input torque bounded as $|\tau| \le \tau_{max}$. The link should perform a rest-to-rest motion from $\theta(0) = -\pi/2$ to $\theta(T) = \pi/2$ with the bang-bang acceleration profile $\ddot{\theta}(t)$ shown in the same figure. Determine analytically the minimum feasible time T to execute the desired motion with this type of trajectory.

[210 minutes (3.5 hours); open books]

Solution

June 10, 2022

Exercise #1

We compute the Coriolis and centrifugal terms using the Christoffel symbols:

$$c_i(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T \boldsymbol{C}_i(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i}\right), \qquad i = 1,2,$$

where $\boldsymbol{m}_i(\boldsymbol{q})$ is the i-th column of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$. This leads to

$$\begin{aligned} \boldsymbol{C}_1(\boldsymbol{q}) &= \begin{pmatrix} 0 & 0 \\ 0 & -B\sin q_2 \end{pmatrix} & \Rightarrow \qquad & c_1(\boldsymbol{q},\dot{\boldsymbol{q}}) = -B\sin q_2\,\dot{q}_2^2, \\ \boldsymbol{C}_2(\boldsymbol{q}) &= \boldsymbol{O} & \Rightarrow \qquad & c_2(\boldsymbol{q},\dot{\boldsymbol{q}}) \equiv 0. \end{aligned}$$

Accordingly, the factorization of these terms with the skew-symmetric property is trivial (being only a centrifugal term present):

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \begin{pmatrix} 0 & -B\sin q_2 \dot{q}_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} -B\sin q_2\,\dot{q}_2^2 \\ 0 \end{pmatrix}.$$

Having defined $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}(\boldsymbol{q}_d - \boldsymbol{q}) = \dot{\boldsymbol{q}}_d + \boldsymbol{K}_D^{-1}\boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q})$, the adaptive control law is

$$\begin{aligned} \boldsymbol{\tau} &= \hat{\boldsymbol{M}}(\boldsymbol{q})\ddot{\boldsymbol{q}}_r + \hat{\boldsymbol{S}}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_r + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}) \\ &= \boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\hat{\boldsymbol{a}} + \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) + \boldsymbol{K}_D(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}), \qquad \text{(diagonal) } \boldsymbol{K}_P > 0, \boldsymbol{K}_D > 0), \end{aligned}$$

with

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r) = \begin{pmatrix} \ddot{q}_{r1} & \ddot{q}_{r2}\cos q_2 - \dot{q}_2\dot{q}_{2r}\sin q_2 & 0 \\ 0 & \ddot{q}_{r1}\cos q_2 & \ddot{q}_{r2} \end{pmatrix}, \qquad \hat{\boldsymbol{a}} = \begin{pmatrix} \hat{A} \\ \hat{B} \\ \hat{C} \end{pmatrix},$$

and adaptation law

$$\dot{\hat{\boldsymbol{a}}} = \boldsymbol{\Gamma}\boldsymbol{Y}^T(\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}}_r,\ddot{\boldsymbol{q}}_r)\left(\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}\right), \qquad \boldsymbol{\Gamma} > 0.$$

Exercise #2

The two tasks of dimension $m_1 = m_2 = 2$ are defined by the following kinematics:

$$\begin{aligned} \boldsymbol{p}_1 &= \boldsymbol{f}_1(\boldsymbol{q}) = L\begin{pmatrix} c_1 + c_{12} \\ s_1 + s_{12} \end{pmatrix} + \frac{L}{4}\begin{pmatrix} c_{123} + c_{1234} \\ s_{123} + s_{1234} \end{pmatrix} = \boldsymbol{p}_2 + \frac{L}{4}\begin{pmatrix} c_{123} + c_{1234} \\ s_{123} + s_{1234} \end{pmatrix}, \\ \boldsymbol{p}_2 &= \boldsymbol{f}_2(\boldsymbol{q}) = L\begin{pmatrix} c_1 + c_{12} \\ s_1 + s_{12} \end{pmatrix}. \end{aligned}$$

The associated Jacobians $\boldsymbol{J}_i(\boldsymbol{q}) = \partial \boldsymbol{f}_i(\boldsymbol{q})/\partial \boldsymbol{q}$, for $i = 1,2$, are

$$\boldsymbol{J}_1(\boldsymbol{q}) = L\begin{pmatrix} -\left(s_1 + s_{12} + \dfrac{s_{123}+s_{1234}}{4}\right) & -\left(s_{12} + \dfrac{s_{123}+s_{1234}}{4}\right) & -\dfrac{s_{123}+s_{1234}}{4} & -\dfrac{s_{1234}}{4} \\ c_1 + c_{12} + \dfrac{c_{123}+c_{1234}}{4} & c_{12} + \dfrac{c_{123}+c_{1234}}{4} & \dfrac{c_{123}+c_{1234}}{4} & \dfrac{c_{1234}}{4} \end{pmatrix}$$

$$\boldsymbol{J}_2(\boldsymbol{q}) = L\begin{pmatrix} -\left(s_1 + s_{12}\right) & -s_{12} & 0 & 0 \\ c_1 + c_{12} & c_{12} & 0 & 0 \end{pmatrix}.$$

Since $n = m_1 + m_2 = 4$, the simultaneous execution of both task leads to the square *extended Jacobian*

$$\boldsymbol{J}_E(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_1(\boldsymbol{q}) \\ \boldsymbol{J}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_{11}(\boldsymbol{q}_M, \boldsymbol{q}_m) & \boldsymbol{J}_{12}(\boldsymbol{q}_M, \boldsymbol{q}_m) \\ \boldsymbol{J}_{21}(\boldsymbol{q}_M) & \boldsymbol{O} \end{pmatrix}$$

with $\boldsymbol{q}_M = (q_1, q_2)$ and $\boldsymbol{q}_m = (q_3, q_4)$ being respectively the variables of the macro- and micro-manipulator.

The determinant of $\boldsymbol{J}_E(\boldsymbol{q})$ is easily computed as

$$\det \boldsymbol{J}_E(\boldsymbol{q}) = \det \boldsymbol{J}_{12}(\boldsymbol{q}_M, \boldsymbol{q}_m) \cdot \det \boldsymbol{J}_{21}(\boldsymbol{q}_M) = \frac{L^4}{16} \sin q_2 \sin q_4.$$

In particular, $\sin q_2 = 0$ corresponds certainly to a rank loss of the secondary task Jacobian $\boldsymbol{J}_2$ (namely, a singularity of the square $\boldsymbol{J}_{21}$ matrix), whereas $\sin q_4 = 0$ corresponds to a singularity of the square block $\boldsymbol{J}_{12}$. However, it could be that the 2×4 primary task Jacobian $\boldsymbol{J}_2$ has still full rank even if $\sin q_4 = 0$. In this case, the primary task may still be generically realized by a task priority strategy.

The analysis of the rank deficiencies of the extended Jacobian $\boldsymbol{J}_E$ and of its sub-parts can be simplified by using the following two invertible transformations (acting on the columns and on the rows of the matrix).

- Redefining the joint velocity as

$$\dot{\boldsymbol{q}} = \boldsymbol{T}\,\dot{\boldsymbol{q}}' \qquad \text{with } \boldsymbol{T} = \frac{1}{L}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -4 & 4 \end{pmatrix},$$

  leads to

$$\boldsymbol{J}_E(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}_E(\boldsymbol{q})\boldsymbol{T}\,\dot{\boldsymbol{q}}' = \boldsymbol{J}'_E(\boldsymbol{q})\dot{\boldsymbol{q}}'$$

  with the simpler form

$$\boldsymbol{J}'_E(\boldsymbol{q}) = \begin{pmatrix} -s_1 & -s_{12} & -s_{123} & -s_{1234} \\ c_1 & c_{12} & c_{123} & c_{1234} \\ -s_1 & -s_{12} & 0 & 0 \\ c_1 & c_{12} & 0 & 0 \end{pmatrix}.$$

- Expressing the task velocities in the frame rotated with joint 1,

$$\begin{aligned} \begin{pmatrix} {}^1\dot{\boldsymbol{p}}_1 \\ {}^1\dot{\boldsymbol{p}}_2 \end{pmatrix} &= \begin{pmatrix} {}^0\boldsymbol{R}_1^T(\boldsymbol{q}_1) & \boldsymbol{O} \\ \boldsymbol{O} & {}^0\boldsymbol{R}_1^T(\boldsymbol{q}_1) \end{pmatrix} \begin{pmatrix} {}^0\dot{\boldsymbol{p}}_1 \\ {}^0\dot{\boldsymbol{p}}_2 \end{pmatrix} \\ &= \begin{pmatrix} {}^0\boldsymbol{R}_1^T(\boldsymbol{q}_1) & \boldsymbol{O} \\ \boldsymbol{O} & {}^0\boldsymbol{R}_1^T(\boldsymbol{q}_1) \end{pmatrix} \boldsymbol{J}'_E(\boldsymbol{q})\dot{\boldsymbol{q}}' = {}^1\boldsymbol{J}'_E(\boldsymbol{q})\dot{\boldsymbol{q}}', \end{aligned}$$

  further simplifies the extended Jacobian to

$${}^1\boldsymbol{J}'_E(\boldsymbol{q}) = \begin{pmatrix} 0 & -s_2 & -s_{23} & -s_{234} \\ 1 & c_2 & c_{23} & c_{234} \\ 0 & -s_2 & 0 & 0 \\ 1 & c_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} {}^1\boldsymbol{J}'_1(\boldsymbol{q}) \\ {}^1\boldsymbol{J}'_2(\boldsymbol{q}) \end{pmatrix}.$$

With the above in mind, a true *algorithmic* singularity occurs when $\sin q_2 \neq 0$, i.e., the second task Jacobian $\boldsymbol{J}_2$ is full rank, the first task Jacobian $\boldsymbol{J}_1$ is also of full rank, but the extended Jacobian $\boldsymbol{J}_E$ is singular, which implies then necessarily $\sin q_4 = 0$. To check if this situation is at all possible, consider the simplified Jacobian ${}^1\boldsymbol{J}_1'$ evaluated in particular for $q_4 = 0$ (but with $\sin q_2 \neq 0$). We have —this is labeled below as case a:

$$ {}^1\boldsymbol{J}_1'(\boldsymbol{q})\big|_{q_4=0} = \begin{pmatrix} 0 & -s_2 & -s_{23} & -s_{23} \\ 1 & c_2 & c_{23} & c_{23} \end{pmatrix} \qquad \Rightarrow \qquad \text{has rank} = 2 \text{ (just as } \boldsymbol{J}_1\text{)}. $$

This means that adopting a task priority solution will allow the highest priority task to be exactly executed (in this case, together with the second one being also rank $\boldsymbol{J}_2 = 2$), whereas inversion of the square extended Jacobian $\boldsymbol{J}_E$ would be impossible.

A further interesting situation is when the secondary task is singular ($\sin q_2 = 0$), together with the sub-Jacobian $\boldsymbol{J}_{12}$ of the first task ($\sin q_4 = 0$). For instance, when $q_2 = q_4 = 0$ one has

$$ {}^1\boldsymbol{J}_1'(\boldsymbol{q})\big|_{q_2=q_4=0} = \begin{pmatrix} 0 & 0 & -s_3 & -s_3 \\ 1 & 1 & c_3 & c_3 \end{pmatrix} \qquad \Rightarrow \qquad \text{has rank} = 2 \text{ iff } \sin q_3 \neq 0. $$

Also in this situation, labeled below as case b, the task priority solution would allow a correct execution of the highest priority task, whereas the secondary task would report an error (minimized in a least square sense).

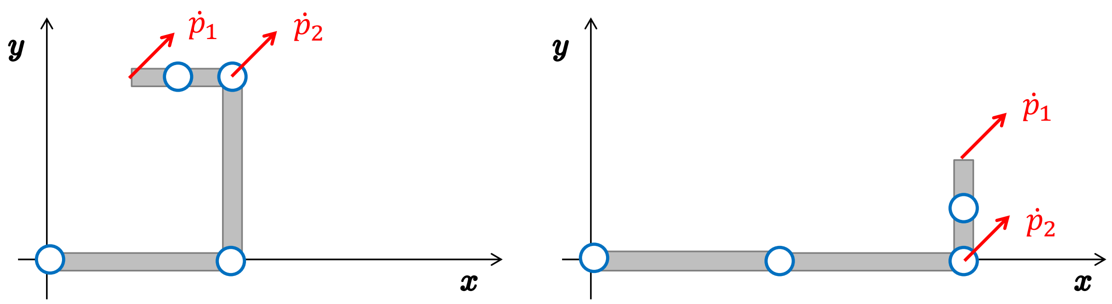


Figure 3: Cases a (left) and b (right) of singularities for the extended Jacobian $\boldsymbol{J}_E$ of the 4R macro-micro robot.

The above two situations are depicted in Fig. 3. The task priority solutions are computed as

$$ \dot{\boldsymbol{q}}_{TP} = \boldsymbol{J}_1^{\#}(\boldsymbol{q})\dot{\boldsymbol{p}}_1 + (\boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{P}_1(\boldsymbol{q}))^{\#}\left(\dot{\boldsymbol{p}}_2 - \boldsymbol{J}_2(\boldsymbol{q})\boldsymbol{J}_1^{\#}(\boldsymbol{q})\dot{\boldsymbol{p}}_1\right), \tag{5} $$

where $\boldsymbol{P}_1(\boldsymbol{q}) = \boldsymbol{I} - \boldsymbol{J}_1^{\#}(\boldsymbol{q})\boldsymbol{J}_1(\boldsymbol{q})$. In both cases, we have set the link length parameter and the task velocities to

$$ L = 1 \text{ [m]}, \qquad \dot{\boldsymbol{p}}_1 = \dot{\boldsymbol{p}}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ [m/s]}. $$

When $\boldsymbol{q}_a = (0, \pi/2, \pi/2, 0)$ (algorithmic singularity, case a), the joint velocity (5) provides the correct solution for both tasks:

$$ \dot{\boldsymbol{q}}_{TP,a} = \begin{pmatrix} 1 \\ -2 \\ 0.8 \\ 0.4 \end{pmatrix} \text{ [rad/s]} \qquad \Rightarrow \qquad \boldsymbol{J}_E(\boldsymbol{q}_a)\dot{\boldsymbol{q}}_{TP,a} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ [m/s]}. $$

When $\boldsymbol{q}_b = (0, 0, \pi/2, 0)$ (case b), the Jacobian $\boldsymbol{J}_2$ of the second task is not of full rank, and thus $\boldsymbol{J}_E$ is singular), the joint velocity (5) still provides the correct solution for the high-priority task

$$\dot{\boldsymbol{q}}_{TP,b} = \begin{pmatrix} 0.7586 \\ -0.5172 \\ -1.793 \\ -0.8966 \end{pmatrix} \text{[rad/s]} \qquad \Rightarrow \qquad \boldsymbol{J}_E(\boldsymbol{q}_b)\dot{\boldsymbol{q}}_{TP,b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{[m/s]}.$$

while the x-component of the second task velocity $\dot{\boldsymbol{p}}_2$ is clearly not realized.

Exercise #3

The first step in the control design is achieved simply by choosing

this is just the inverse formula of theta_ddot in the first request, plugged in the motor equations

$$\boldsymbol{\tau} = \left(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{B}_0^{-1}\right)\boldsymbol{\tau}_J + \boldsymbol{B}\boldsymbol{B}_0^{-1}\left(\boldsymbol{u} - \boldsymbol{D}_0\dot{\boldsymbol{\theta}}\right). \tag{6}$$

Assuming that $\boldsymbol{\tau}_J = \boldsymbol{u}$, then the rest of the procedure follows a standard impedance control design for a task that involves a constant reference $\boldsymbol{x}_d$ and when the apparent inertia is chosen as the natural Cartesian inertia of the robot, with consistent Coriolis/centrifugal terms included in the model to be matched. Thus, no force feedback will be required. The Cartesian model of the link dynamics (viz., that of a rigid robot) is

$$\boldsymbol{M}_x(\boldsymbol{q})\ddot{\boldsymbol{x}} + \boldsymbol{S}_x(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{x}} + \boldsymbol{g}_x(\boldsymbol{q}) = \boldsymbol{F} + \boldsymbol{J}^{-T}(\boldsymbol{q})\boldsymbol{u}. \tag{7}$$

Comparing (7) with the target model (4), the equivalence is obtained by choosing

this is obtained isolating u from the classical cartesian dynamic model (7) plugging then it in the desired impedance model given in the text

$$\boldsymbol{u} = \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{K}_m(\boldsymbol{x}_d - \boldsymbol{x}) - \boldsymbol{D}_m\dot{\boldsymbol{x}} + \boldsymbol{g}_x(\boldsymbol{q})\right) = \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{K}_m(\boldsymbol{x}_d - \boldsymbol{x}) - \boldsymbol{D}_m\dot{\boldsymbol{x}}\right) + \boldsymbol{g}(\boldsymbol{q}), \tag{8}$$

being $\boldsymbol{g}_x(\boldsymbol{q}) = \boldsymbol{J}^{-T}(\boldsymbol{q})\boldsymbol{g}(\boldsymbol{q})$. Putting together eqs. (6) and (8) gives finally

$$\boldsymbol{\tau} = \left(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{B}_0^{-1}\right)\boldsymbol{K}_J(\boldsymbol{\theta} - \boldsymbol{q}) + \boldsymbol{B}\boldsymbol{B}_0^{-1}\left(\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{K}_m(\boldsymbol{x}_d - \boldsymbol{f}(\boldsymbol{q})) - \boldsymbol{D}_m\boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) + \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{D}_0\dot{\boldsymbol{\theta}}\right).$$

If an external constant force $\boldsymbol{F} = \bar{\boldsymbol{F}}$ is applied at the robot end effector, the system will find the steady-state equilibrium

$$\boldsymbol{x}_E = \boldsymbol{x}_d + \boldsymbol{K}_m^{-1}\bar{\boldsymbol{F}}.$$

The value of the control signal $\boldsymbol{u}$ at this equilibrium is

$$\boldsymbol{u}_E = \boldsymbol{J}^T(\boldsymbol{q}_E)\bar{\boldsymbol{F}} + \boldsymbol{g}(\boldsymbol{q}_E),$$

where the equilibrium configuration $\boldsymbol{q}_E$ is such that $\boldsymbol{f}(\boldsymbol{q}_E) = \boldsymbol{x}_E$. Moreover, since the elastic torque at steady state will be $\boldsymbol{\tau}_{J,E} = \boldsymbol{u}_E$, it follows from (6) that the control torque at the equilibrium is $\boldsymbol{\tau}_E = \boldsymbol{u}_E$.

Exercise #4

The dynamics of the actuated pendulum under gravity is given by

$$I\ddot{\theta} + mg_0 d\sin\theta = \tau,$$

where $m > 0$ is the link mass, $d > 0$ is the distance of its center of mass from the joint, and $I = I_c + md^2 > 0$ is the link inertia around the joint axis at the base. With the bang-bang acceleration profile $\ddot{\theta}(t)$ given in Fig. 2, when starting at rest in $\theta(0) = \theta_0$ one has by integration

$$\dot{\theta}(t) = \begin{cases} At, & t \in [0, T/2] \\ A(T-t), & t \in [T/2, T] \end{cases} \qquad \theta(t) = \begin{cases} \theta_0 + \frac{1}{2}At^2, & t \in [0, T/2] \\ \theta_0 + \frac{1}{4}AT^2 - \frac{1}{2}A(T-t)^2, & t \in [T/2, T] \end{cases}$$

yielding $\theta(T) = \theta_0 + \frac{1}{4}AT^2$. Thus, for a rest-to-rest angular motion $\Delta = \theta(T) - \theta_0$ to be executed in T seconds, the bang-bang value A of the acceleration will have to be

$$A = \frac{4\Delta}{T^2}.$$

From the inverse dynamics for the desired motion, one should note that the maximum of the gravity torque $\tau_g = mg_0 d \sin\theta$ in *absolute value* occurs at $\theta = \pm\pi/2$. However, the gravity torque has always opposite sign of the inertial torque $I\ddot{\theta} = \pm IA$ (see the dotted blue line in the right of Fig. 4), subtracting from the total torque that the actuator needs to deliver for the desired motion. Stated differently, it helps in the acceleration phase ($\tau_g(t)$ has negative sign when $t \in [0, T/2)$) as well as in the deceleration phase ($\tau_g(t)$ has positive sign when $t \in (T/2, T]$). Therefore, we easily see that

$$\max_{t\in[0,T]} |\tau(t)| = |\tau\,(T/2)| = IA = I\,\frac{4\Delta}{T^2} \leq \tau_{max},$$

i.e., at $t = T/2$, where $\tau_g(T/2) = \tau_g(\theta = 0) = 0$. From this, the minimum motion time for the required $\Delta = \pi$ is

$$T_{min} = \sqrt{\frac{4\Delta I}{\tau_{max}}} = \sqrt{\frac{4\pi I}{\tau_{max}}}.$$

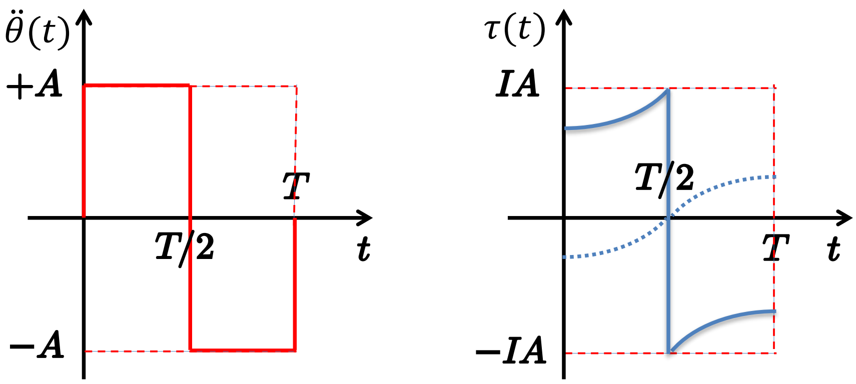


Figure 4: The joint acceleration profile and the associated joint torque profile.

* * * * *

Robotics 2

July 8, 2022

Exercise #1

A generic 3R spatial manipulator, which is self-balanced with respect to gravity, is driven by three actuators that deliver the torques $\boldsymbol{\tau} = \begin{pmatrix} \tau_1 & \tau_2 & \tau_3 \end{pmatrix}^T$. When using the generalized coordinates $\boldsymbol{q} \in \mathbb{R}^3$, the robot dynamic model is expressed in compact form as

$$\boldsymbol{M_q}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c_q}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau_q}, \tag{1}$$

where

$$\boldsymbol{M_q} = \begin{pmatrix} m_{11}(q_2, q_3) & 0 & 0 \\ 0 & m_{22}(q_3) & m_{23}(q_3) \\ 0 & m_{23}(q_3) & m_{33} \end{pmatrix}, \quad \boldsymbol{c_q} = \begin{pmatrix} c_1(q_2, q_3, \dot{q}_2, \dot{q}_3) \\ c_2(q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) \\ c_3(q_2, q_3, \dot{q}_1, \dot{q}_2) \end{pmatrix}, \quad \boldsymbol{\tau_q} = \begin{pmatrix} \tau_1 \\ \tau_2 + \tau_3 \\ \tau_3 \end{pmatrix}.$$

- Find the set of coordinates $\boldsymbol{p} \in \mathbb{R}^3$ on which the torque vector $\boldsymbol{\tau} \in \mathbb{R}^3$ produces work component-wise, and give the coordinate transformation between $\boldsymbol{q}$ and $\boldsymbol{p}$.
- Write the dynamic model in the coordinates $\boldsymbol{p}$, expressing the elements of the inertia matrix $\boldsymbol{M_p}$ and of the Coriolis and centrifugal vector $\boldsymbol{c_p}$ in terms of the elements m_{ij} and c_i of model (1). For compactness, there is no need to replace the dependences on $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ by those on $(\boldsymbol{p}, \dot{\boldsymbol{p}})$ within these terms.

Exercise #2

The dynamic model of a serial manipulator with n revolute joints can always be written as

$$\begin{pmatrix} m_{11}(\boldsymbol{q}) & \boldsymbol{m}_{12}^T(\boldsymbol{q}) \\ \boldsymbol{m}_{12}(\boldsymbol{q}) & \boldsymbol{M}_{22}(\boldsymbol{q}) \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{\boldsymbol{q}}_2 \end{pmatrix} + \begin{pmatrix} n_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ \boldsymbol{n}_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \boldsymbol{\tau}_2 \end{pmatrix}, \tag{2}$$

where the joint variables $\boldsymbol{q} \in \mathbb{R}^n$ are partitioned in $q_1 \in \mathbb{R}$ and $\boldsymbol{q}_2 \in \mathbb{R}^{n-1}$ and, similarly, the joint torques $\boldsymbol{\tau} \in \mathbb{R}^n$ in $\tau_1 \in \mathbb{R}$ and $\boldsymbol{\tau}_2 \in \mathbb{R}^{n-1}$. The inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ and the dynamic terms $\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})$ in (2) have been partitioned accordingly. Suppose that a constraint is imposed on the first joint, so that $q_1(t) = k$ (an arbitrary constant value).

- Derive the explicit form of the $(n-1)$-dimensional *reduced dynamics* of the constrained robot.
- Provide the corresponding expression of the force multiplier $\lambda \in \mathbb{R}$ that arises when attempting to violate the constraint during a generic robot motion.
- Define control laws for τ_1 and for $\boldsymbol{\tau}_2$ that regulate the robot to a desired configuration $\boldsymbol{q}_d$, which is feasible (i.e., such that $q_{1d} = k$), while keeping $\lambda(t) = 0$ at all times.

Exercise #3

With reference to Fig. 1, consider a Cartesian (PP) robot with links of mass m_1 and m_2, moving in a vertical plane. The end-effector should transfer from rest to rest between two generic points P_s and P_g in *minimum time*, with the two input force commands being bounded as

$$|u_i| \leq U_{i,max}, \qquad i = x, y.$$

The robot starts at an equilibrium and should remain in equilibrium when the motion ends.

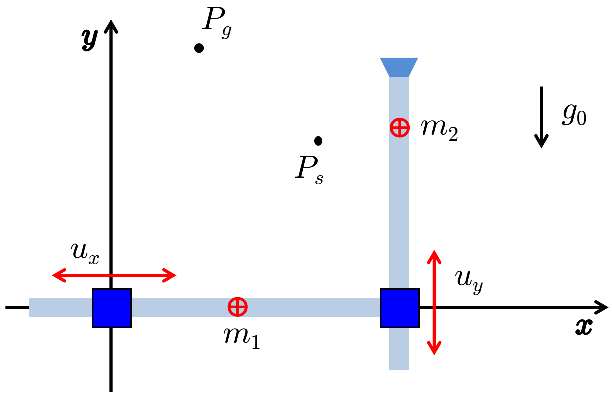


Figure 1: A Cartesian robot in a point-to-point task in the vertical plane.

- Determine the minimum feasible value T^* of the transfer time in a parametric form with respect to the problem data.
- For the numerical values

$$P_s = (1, 0.3), \quad P_g = (0.6, 0.7) \ \text{[m]}, \quad m_1 = 5, \ \ m_2 = 3 \ \text{[kg]}, \quad U_{x,max} = U_{y,max} = 40 \ \text{[N]},$$

  evaluate time T^* and sketch the optimal profiles of force, acceleration, velocity, and position of the two robot joints.
- Is the end-effector path associated to this time-optimal trajectory a straight line segment between P_s and P_g? (Support your answer with an argument: a simple 'yes' or 'no' does not count!).

[180 minutes; open books]

Solution

July 8, 2022

Exercise #1

The objective is to obtain the robot dynamic equations in the transformed coordinates $\boldsymbol{p} = \boldsymbol{t}(\boldsymbol{q})$ such that

$$\boldsymbol{M_p}(\boldsymbol{p})\ddot{\boldsymbol{p}} + \boldsymbol{c_p}(\boldsymbol{p}, \dot{\boldsymbol{p}}) = \boldsymbol{\tau_p} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}, \tag{3}$$

i.e., in the right-hand side of eq. (3) the three available actuators torques $\boldsymbol{\tau}$ are those performing work of the coordinates $\boldsymbol{p}$.
Since the following holds by duality

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{t}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \boldsymbol{J_t}(\boldsymbol{q})\dot{\boldsymbol{q}} \qquad \Longleftrightarrow \qquad \boldsymbol{\tau_q} = \boldsymbol{J}_t^T(\boldsymbol{q})\boldsymbol{\tau_p},$$

we extract from the right-hand side of (1) the required Jacobian of the transformation,

$$\boldsymbol{\tau_q} = \begin{pmatrix} \tau_1 \\ \tau_2 + \tau_3 \\ \tau_3 \end{pmatrix} = \boldsymbol{J_t^T}\boldsymbol{\tau_p} \qquad \Rightarrow \qquad \boldsymbol{J_t^T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

which turns out to be constant. Therefore, the change of coordinates is linear

$$\boldsymbol{p} = \boldsymbol{J_t}\,\boldsymbol{q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_2 + q_3 \end{pmatrix}$$

and its inverse is

$$\boldsymbol{q} = \boldsymbol{J_t^{-1}}\boldsymbol{p} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 - p_2 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}} = \boldsymbol{J_t^{-1}}\dot{\boldsymbol{p}}, \quad \ddot{\boldsymbol{q}} = \boldsymbol{J_t^{-1}}\ddot{\boldsymbol{p}}.$$

Plugging these into (1) yields finally (3) with

$$\begin{aligned} \boldsymbol{M_p}(\boldsymbol{p}) &= \boldsymbol{J_t^{-T}}\,\boldsymbol{M_q}(\boldsymbol{q})\,\boldsymbol{J_t^{-1}} \\ &= \begin{pmatrix} m_{11}(p_2, p_3 - p_2) & 0 & 0 \\ 0 & m_{22}(p_3 - p_2) + m_{33} - 2m_{23}(p_3 - p_2) & m_{23}(p_3 - p_2) - m_{33} \\ 0 & m_{23}(p_3 - p_2) - m_{33} & m_{33} \end{pmatrix} \end{aligned} \tag{4}$$

$$\boldsymbol{c}_p(\boldsymbol{p}, \dot{\boldsymbol{p}}) = \boldsymbol{J}_t^{-T}\,\boldsymbol{c}_q(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} c_1(p_2, p_3 - p_2, \dot{p}_2, \dot{p}_3 - \dot{p}_2) \\ c_2(p_2, p_3 - p_2, \dot{p}_1, \dot{p}_2, \dot{p}_3 - \dot{p}_2) - c_3(p_2, p_3 - p_2, \dot{p}_1, \dot{p}_2) \\ c_3(p_2, p_3 - p_2, \dot{p}_1, \dot{p}_2) \end{pmatrix}, \tag{5}$$

where the arguments in the functions on the right-hand sides of (4) and (5) have been substituted with the inverse mappings $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \left(\boldsymbol{J_t^{-1}}\boldsymbol{p}, \boldsymbol{J_t^{-1}}\dot{\boldsymbol{p}}\right)$. This is not strictly needed in general (nor required by the text), but is particularly simple here because of the linearity of the transformation.

Exercise #2

We apply the standard procedure for obtaining the reduced dynamic model, which is particularly simple in this case.

The Jacobian of the scalar constraint $h(\boldsymbol{q}) = q_1(t) - k = 0$ is $\boldsymbol{A} = \partial h(\boldsymbol{q})/\partial \boldsymbol{q} = \begin{pmatrix} 1 & \boldsymbol{0}_{1\times(n-1)} \end{pmatrix}$. Therefore, the obvious completion of $\boldsymbol{A}$ with a matrix $\boldsymbol{D}$ to obtain a square nonsingular matrix is

$$\begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix} = \begin{pmatrix} 1 & \boldsymbol{0}_{1\times(n-1)} \\ \boldsymbol{0}_{(n-1)\times 1} & \boldsymbol{I}_{(n-1)\times(n-1)} \end{pmatrix} = \boldsymbol{I}_{n\times n},$$

and thus

$$\begin{pmatrix} \boldsymbol{E} & \boldsymbol{F} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{D} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \boldsymbol{0}_{1\times(n-1)} \\ \boldsymbol{0}_{(n-1)\times 1} & \boldsymbol{I}_{(n-1)\times(n-1)} \end{pmatrix}.$$

As a result, the pseudo-velocity vector is

$$\boldsymbol{v} = \boldsymbol{D}\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_2 \in \mathbb{R}^{n-1}.$$

Being all the defined transformation matrices constant, the reduced dynamics becomes

$$\left(\boldsymbol{F}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}\right)\dot{\boldsymbol{v}} = \boldsymbol{F}^T\left(\boldsymbol{\tau} - \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})\right),$$

or

$$\boldsymbol{M}_{22}(\boldsymbol{q})\,\ddot{\boldsymbol{q}}_2 = \boldsymbol{\tau}_2 - \bar{\boldsymbol{n}}_2(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{\tau}_2 - \bar{\boldsymbol{c}}_2(\boldsymbol{q},\dot{\boldsymbol{q}}) - \bar{\boldsymbol{g}}_2(\boldsymbol{q}), \tag{6}$$

where a 'bar' on a dynamic term means that:

- $\boldsymbol{M}_{22}$ is identical to the same block in (2), because q_1 does not appear in the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of any robot (a so-called *cyclic variable*);
- $\bar{\boldsymbol{c}}_2$ is evaluated at $\dot{q}_1 = 0$ while, as a result of the previous property, is also independent from q_1;
- $\bar{\boldsymbol{g}}_2$ is evaluated at $q_1 = q_{1d} = k$.

Similarly, the expression of the (scalar) force multiplier λ becomes

$$\begin{aligned} \lambda = \boldsymbol{E}^T\left(\boldsymbol{M}(\boldsymbol{q})\boldsymbol{F}\dot{\boldsymbol{v}} + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{\tau}\right) &= \boldsymbol{m}_{12}^T(\boldsymbol{q})\ddot{\boldsymbol{q}}_2 + \bar{n}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) - \tau_1 \\ &= \boldsymbol{m}_{12}^T(\boldsymbol{q})\boldsymbol{M}_{22}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_2 - \bar{\boldsymbol{n}}_2(\boldsymbol{q},\dot{\boldsymbol{q}})\right) + \bar{n}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) - \tau_1, \end{aligned} \tag{7}$$

where eq. (6) has been used.

For any arbitrary choice of the torque $\boldsymbol{\tau}_2$, the control law applied at joint 1 to make sure that $\lambda(t) \equiv 0$ is then

$$\tau_1 = \bar{n}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{m}_{12}^T(\boldsymbol{q})\boldsymbol{M}_{22}^{-1}(\boldsymbol{q})\left(\boldsymbol{\tau}_2 - \bar{\boldsymbol{n}}_2(\boldsymbol{q},\dot{\boldsymbol{q}})\right). \tag{8}$$

Note that gravity effects acting on joint 1 are also cancelled at rest by the $\bar{g}_1$ torque within $\bar{n}_1$. Furthermore, in order to achieve regulation to a desired $\boldsymbol{q}_{2d}$, one can use a feedback linearization approach yielding

$$\boldsymbol{\tau}_2 = \boldsymbol{M}_{22}(\boldsymbol{q})\left(\boldsymbol{K}_P(\boldsymbol{q}_{2d} - \boldsymbol{q}_2) - \boldsymbol{K}_D\dot{\boldsymbol{q}}_2\right) + \bar{\boldsymbol{n}}_2(\boldsymbol{q},\dot{\boldsymbol{q}}), \qquad \text{with } \boldsymbol{K}_P > 0,\ \boldsymbol{K}_D > 0, \tag{9}$$

which provides exponential and decoupled stabilization of the error $\boldsymbol{e}_2 = \boldsymbol{q}_{d2} - \boldsymbol{q}_2$ to zero. In alternative, one can design a simpler PD regulator with gravity cancellation

$$\boldsymbol{\tau}_2 = \boldsymbol{K}_P(\boldsymbol{q}_{2d} - \boldsymbol{q}_2) - \boldsymbol{K}_D\dot{\boldsymbol{q}}_2 + \bar{\boldsymbol{g}}_2(\boldsymbol{q},\dot{\boldsymbol{q}}), \qquad \text{with } \boldsymbol{K}_P > 0,\ \boldsymbol{K}_D > 0. \tag{10}$$

It is straightforward to prove asymptotic stability of the closed-loop system with (10), using a Lyapunov/LaSalle argument on the reduced dynamics (6).

Exercise #3

The task does not require any coordination between the two joints, nor there is a velocity bound. Thus, each joint will move as fast as possible with a bang-bang force profile. The minimum transfer time will be given by the slowest joint completing its motion (while the fastest joint remains at rest for some interval).

The two scalar problems are however different because of the presence of gravity on the vertical (second) joint, which offsets its feasible acceleration range. With $\boldsymbol{q} = (x, y)$, the dynamic model of this PP robot is

$$\begin{aligned} (m_1 + m_2)\, \ddot{x} &= u_x \\ m_2\, \ddot{y} + m_2\, g_0 &= u_y, \end{aligned} \tag{11}$$

being $g_0 = 9.81$ [m/s^2]. As a result

$$\begin{aligned} |u_x| \leq U_{x,max} \qquad &\Rightarrow \qquad |\ddot{x}| \leq \frac{U_{x,max}}{m_1 + m_2} \\ |u_y| \leq U_{y,max} \qquad &\Rightarrow \qquad -\left(\frac{U_{y,max}}{m_2} + g_0\right) \leq \ddot{y} \leq \frac{U_{y,max}}{m_2} - g_0, \end{aligned}$$

with an asymmetric feasible range for $\ddot{y}$. Moreover, let $P_g - P_s = (\Delta x, \Delta y)$ be the required displacement.

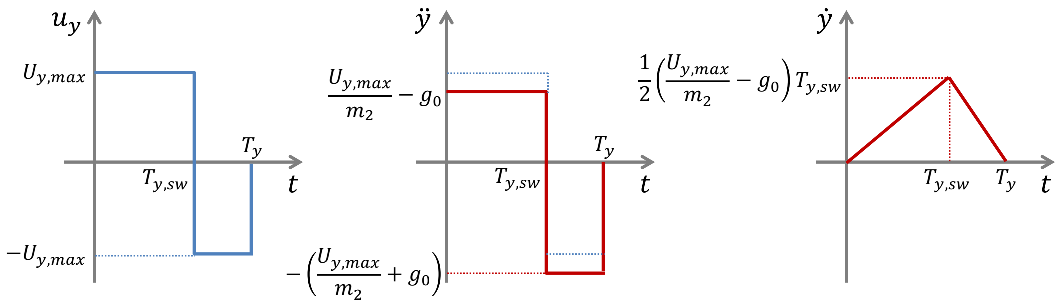


Figure 2: Force, acceleration and velocity profiles for the joint moving under gravity by $\Delta y > 0$.

Consider first the time-optimal motion for the y-axis under gravity. With reference to Fig. 2, which shows only the case of $\Delta y > 0$, the acceleration is bang-bang and the velocity is triangular, both switching at $T_{y,sw}$ and being typically asymmetric in time with respect to the total interval T_y. Two relations are then obtained from these behaviors: *i*) the area with sign covered by the acceleration profile (i.e., its integral) should be zero in order to obtain a rest-to-rest motion, i.e.,

$$\left(\operatorname{sign}(\Delta y)\, \frac{U_{y,max}}{m_2} - g_0\right) T_{y,sw} - \left(\operatorname{sign}(\Delta y)\, \frac{U_{y,max}}{m_2} + g_0\right) (T_y - T_{y,sw}) = 0;$$

ii) the area with sign covered by the velocity profile should be equal to the required displacement Δy of the joint, i.e.,

$$\frac{1}{2}\left(\operatorname{sign}(\Delta y)\, \frac{U_{y,max}}{m_2} - g_0\right) T_{y,sw}^2 + \frac{1}{2}\left(\operatorname{sign}(\Delta y)\, \frac{U_{y,max}}{m_2} - g_0\right) T_{y,sw}\, (T_y - T_{y,sw}) = \Delta y.$$

Solving these two equations for T_y and $T_{y,sw}$ gives

$$T_y = 2\sqrt{\frac{m_2\,|\Delta y|\,U_{y,max}}{U^2_{y,max} - (m_2 g_0)^2}} \tag{12}$$

and

$$T_{y,sw} = \frac{T_y}{2}\left(1 + \text{sign}(\Delta y)\,\frac{m_2 g_0}{U_{y,max}}\right) \neq \frac{T_y}{2}. \tag{13}$$

If $\Delta y > 0$ then $T_{y,sw} > T_y/2$ and, viceversa, if $\Delta y < 0$ then $T_{y,sw} < T_y/2$.

The time-optimal motion for the x-axis without gravity is a sub-case of the formulas (12) and (13), obtained by setting $g_0 = 0$ and replacing m_2 with the total mass $m_1 + m_2$ driven by this joint. Thus,

$$T_x = 2\sqrt{\frac{(m_1 + m_2)\,|\Delta x|}{U_{x,max}}} \qquad \text{and} \qquad T_{x,sw} = \frac{T_x}{2}. \tag{14}$$

Therefore,

$$T^* = \max\{T_x, T_y\}\,. \tag{15}$$

Note that the joint that arrives first should remain then at rest, waiting for the slower joint to reach its goal. Indeed, the faster joint could also remain at rest at the beginning and then start moving at an instant such that task completion occurs simultaneously at T^* for both joints. In any event, to stay at rest at steady state, the horizontal joint does not require any force ($u_{x,ss} = 0$), whereas the vertical joint should sustain gravity ($u_{y,ss} = m_2 g_0$). Except for very special combinations of problem data, the above minimum-time control strategy will not result in a coordinated robot motion (i.e., all joints start and end their motion at the same instant, without intermediate stops).

With the problem data, it is $\Delta x = -0.4$, $\Delta y = 0.4$ [m] and we obtain the following motion times (in seconds):

$$T_x = 0.5657, \quad T_{x,sw} = 0.2828, \quad T_y = 0.5115, \quad T_{y,sw} = 0.4439 \qquad \Rightarrow \qquad T^* = 0.5657.$$

The results are reported in Fig. 3. We note that, since in this case the y-axis is faster, when this joint reaches its goal (at $t = T_y < T^*$), the control input switches to the steady-state equilibrium force $u_{y,ss} = 20.43$ [N].

The resulting Cartesian path of the robot end-effector is not a straight line segment between P_s and P_g —see Fig. 4. Rather, the initial part of the path, between $t = 0$ and $t = T_{x,sw}$, is linear since

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\dfrac{U_{y,max}}{m_2}\,t}{\dfrac{U_{x,max}}{m_1 + m_2}\,t} = \frac{U_{y,max}}{U_{x,max}}\,\frac{m_2}{m_1 + m_2} = k;$$

it is followed then by two different curvilinear parts[1], between $t = T_{x,sw}$ and $t = T_{y,sw}$ and between $t = T_{y,sw}$ and $t = T_y$; the last part, between $t = T_y$ and $t = T_x = T^*$, is again a (very short) linear segment.

It can be shown[2] that even if the problem data were such that the two joints complete their task at the same instant (i.e., $T^* = T_x = T_y$), the resulting path would still not be a linear segment between P_s and P_g.

[1]These parts have no easy geometric expressions. In fact, the tangent to each curve is a rational function given by the ratio of linear polynomials in time t.

[2]If you write a code for this problem, try out $P_g = (0.673, 0.7)$ with all the rest being the same. This will result in a coordinated joint motion, but still without a resulting linear Cartesian path. Try out also a motion task in favor of gravity ($\Delta y < 0$), in order to better understand the need of the absolute value and sign of Δy in eqs. (12) and (13).

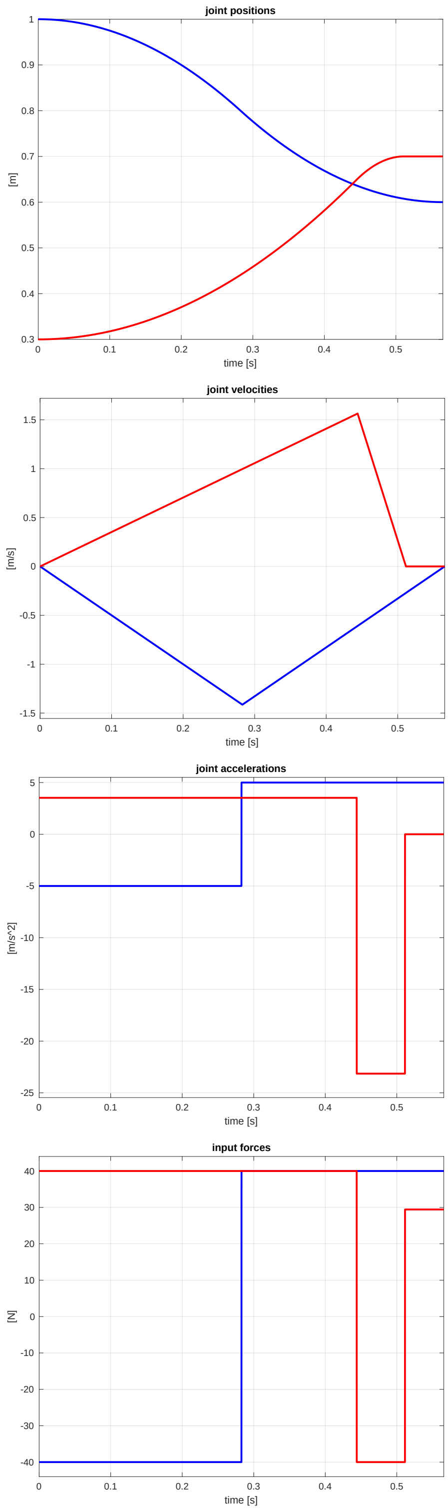


Figure 3: From top to bottom: Joint position, velocity, acceleration, and input force in the time-optimal solution (blue = x-axis, red = y-axis).

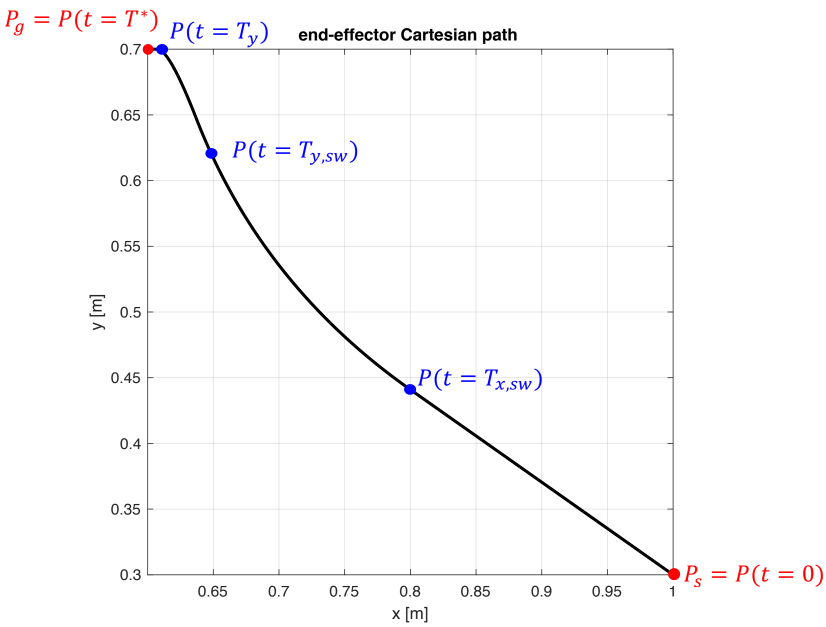


Figure 4: Cartesian path of the robot end-effector.

* * * * *

Robotics 2

September 9, 2022

Exercise 1

Consider a planar 3R robot with equal links of length $L > 0$ that is commanded with joint velocities $\dot{\boldsymbol{q}} \in \mathbb{R}^3$. Each joint has the same available motion range

$$q_i \in [q_{i,min}, q_{i,max}] = [-\Delta, +\Delta], \qquad \text{with } \Delta = \frac{3\pi}{4} \text{ [rad]}, \qquad \text{for } i = 1, 2, 3.$$

The joint range function is defined in general as

$$H(\boldsymbol{q}) = \frac{1}{2N} \sum_{i=1}^{N} \left(\frac{q_i - \bar{q}_i}{q_{i,max} - q_{i,min}} \right)^2, \qquad \text{with } \bar{q}_i = \frac{q_{i,min} + q_{i,max}}{2},$$

where N is the number of robot joints. The end-effector position $\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) \in \mathbb{R}^2$ of this robot is constrained to be at $\boldsymbol{p} = \boldsymbol{0}$. Define a kinematic control law for $\dot{\boldsymbol{q}}$ that will always satisfy this task constraint and the joint range limits, when starting from an initial feasible configuration $\boldsymbol{q}(0)$ and while attempting to minimize the function $H(\boldsymbol{q})$. Show that the robot converges to a unique configuration $\bar{\boldsymbol{q}}$ such that $\nabla H(\bar{\boldsymbol{q}}) \neq 0$ but $\dot{\boldsymbol{q}} = \boldsymbol{0}$. Provide the values of $\bar{\boldsymbol{q}}$, $H(\bar{\boldsymbol{q}})$ and $\nabla H(\bar{\boldsymbol{q}})$.

Exercise 2

The large RP robot in Fig. 1, with coordinates $\boldsymbol{q} = (q_1, q_2)$ and dynamic parameters m_2, d_{c2}, I_1 and I_2 defined therein, moves on a horizontal plane. The robot is controlled by the generalized force $\boldsymbol{u} = (\tau, F)$ [Nm, N] at the joints, while dissipative effects can be neglected. The robot end-effector point P should execute a linear rest-to-rest trajectory from $P_i = (2, 3)$ [m] to $P_f = (-2, 0)$ [m] in $T = 2$ [s], with bang-coast-bang symmetric acceleration profile and cruising speed $V = 3$ [m/s]. The robot is initially at rest in $\boldsymbol{q}(0) = (0, 2)$ [rad,m].

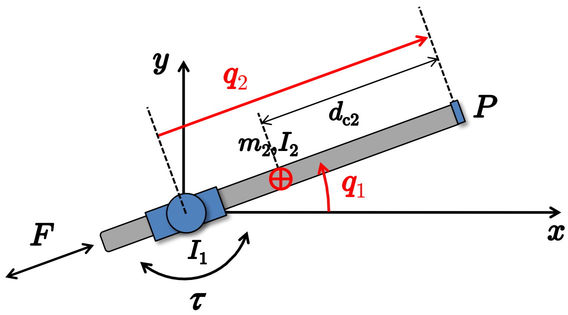


Figure 1: A RP planar robot with the relevant parameters and variables.

Define a control law for $\boldsymbol{u}$ such that the following performance is obtained:

- the Cartesian trajectory error $\boldsymbol{e} = \boldsymbol{p}_d - \boldsymbol{p}$ goes to zero exponentially;
- the trajectory error components e_t and e_n, respectively along the tangent and the normal directions to the path, have a decoupled dynamics, governed by the differential equations

$$\ddot{e}_t + 4\,\dot{e}_t + 4\,e_t = 0, \qquad \ddot{e}_n + 8\,\dot{e}_n + 16\,e_n = 0.$$

Provide the explicit expression of all needed terms in the resulting control law, and in particular the analytic expression of generalized force $\boldsymbol{u}(t)$ at the initial time $t = 0$. Compute then its numerical value $\boldsymbol{u}(0) = (\tau(0), F(0))$ when the dynamic parameters are

$$m_2 = 10 \text{ [kg]}, \qquad d_{c2} = 2.5 \text{ [m]}, \qquad I_1 = I_2 = 20 \text{ [kg}\cdot\text{m}^2\text{]}.$$

Compute also the numerical value of the actual position $\boldsymbol{p}(t)$ of the end-effector at the half-time $t = T/2 = 1$ [s] of the robot motion.

Exercise 3

Consider the simple mechanical system with one degree of freedom in Fig. 2(a). The actuated mass $B > 0$ moves on a frictionless horizontal plane and is driven by an input force F. Its position is θ. The mass is connected through a pulley and a rigid cable to a second mass M that is suspended vertically under gravity.

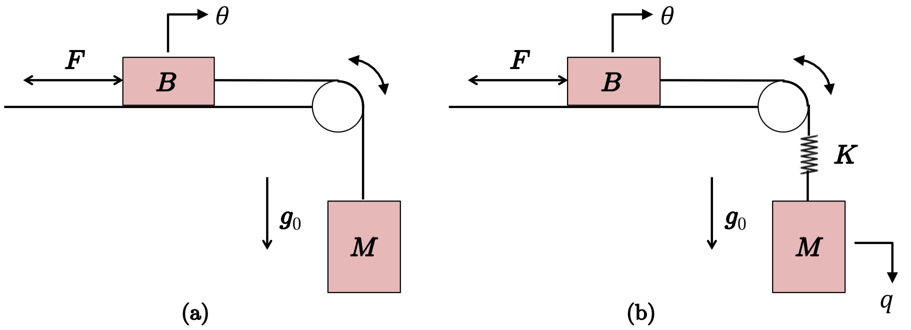


Figure 2: A mass B driven by an input force F and connected to a second mass M: the cable is rigid in (a) and flexible with stiffness K in (b).

- Derive the dynamic model of this system.
- Prove that the PID control law

$$F = K_P\,(\theta_d - \theta) - K_D\dot{\theta} + K_I \int (\theta_d - \theta(\tau))\,d\tau,$$

  with suitable choices of the gains K_P, K_D and K_I, will asymptotically stabilize the desired equilibrium state $(\theta, \dot{\theta}) = (\theta_d, 0)$ of the closed-loop system. Will also exponential stabilization be achieved in this case?
- Consider the PD control law with iterative feedforward

$$F = K_P\,(\theta_d - \theta) - K_D\dot{\theta} + v_{i-1},$$

  where the feedforward is updated at successive steady states $(\theta, \dot{\theta}) = (\theta_i, 0)$, $i = 1, 2, \dots,$ as

$$v_i = v_{i-1} + K_P\,(\theta_d - \theta_i)\,, \qquad \text{(with } v_0 = 0\text{)}.$$

  Prove that, with suitable gains K_P and K_D, this iterative learning control will globally, asymptotically stabilize the desired state $(\theta, \dot{\theta}) = (\theta_d, 0)$ of the closed-loop system. In particular, show that the convergence of the sequence $\{\theta_i\}$ exactly to θ_d occurs in a finite number of iterations.
- Assuming now, as in Fig. 2(b), that the cable is elastic with finite stiffness $K > 0$, derive the new dynamic model using the additional coordinate q for the position of the mass M.

[210 minutes; open books]

Solution

September 9, 2022

Exercise 1

The end-effector position $\boldsymbol{p} \in \mathbb{R}^2$ of the considered planar 3R robot is

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = L \begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123} \end{pmatrix} = \boldsymbol{f}(\boldsymbol{q}),$$

with the usual shorthand notation for trigonometric functions, e.g., $c_{123} = \cos(q_1 + q_2 + q_3)$. The associated 2×3 Jacobian matrix in $\dot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$ is given by

$$\boldsymbol{J}(\boldsymbol{q}) = L \begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123} \end{pmatrix} = \begin{pmatrix} -p_y & -p_y + L\, s_1 & -p_y + L\,(s_1 + s_{12}) \\ p_x & p_x - L\, c_1 & p_x - L\,(c_1 + c_{12}) \end{pmatrix},$$

where the last equivalent expression will be convenient for what follows. Further, from the given joint limits, it is $\bar{q}_i = 0$, for $i = 1, 2, 3$, and the specific joint range function of this robot is

$$H(\boldsymbol{q}) = \frac{1}{6} \sum_{i=1}^{3} \left(\frac{q_i}{2\Delta} \right)^2 = \frac{1}{24\,\Delta^2} \left(q_1^2 + q_2^2 + q_3^2 \right), \qquad \Delta = \frac{3\pi}{4} \text{ [rad]},$$

with gradient

$$\nabla_{\boldsymbol{q}} H(\boldsymbol{q}) = \frac{1}{12\,\Delta^2} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}.$$

To minimize this function without moving the end-effector ($\dot{\boldsymbol{p}} = \boldsymbol{0}$), the joint velocity should be chosen as

$$\dot{\boldsymbol{q}} = -\alpha \left(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q}) \right) \nabla_{\boldsymbol{q}} H(\boldsymbol{q}) = -\alpha\, \boldsymbol{P}(\boldsymbol{q}) \nabla_{\boldsymbol{q}} H(\boldsymbol{q}), \tag{1}$$

where $\boldsymbol{J}^{\#}$ is the pseudoinverse of matrix $\boldsymbol{J}$, $\boldsymbol{P}$ is the projection matrix in its null space, and $\alpha > 0$ is a suitable scalar step. When starting from an initial feasible configuration $\boldsymbol{q}(0)$ such that $\boldsymbol{p}(0) = \boldsymbol{f}(\boldsymbol{q}(0)) = \boldsymbol{0}$, the command (1) will keep satisfying the task constraint on the position of the robot end-effector. However, one should also check that the configuration $\boldsymbol{q}$ remains always within the hard limits of the joint ranges.

The problem is largely simplified by the requirement that the end-effector should be kept in particular at the origin. This implies that the three links of the robot will always form an equilateral triangle (with sides L), whose orientation is parametrized by the first joint angle q_1. Thus, only one of the two classes of configurations are feasible:

$$\boldsymbol{q}_{right} = (q_1, 2\pi/3, 2\pi/3) \qquad \text{or} \qquad \boldsymbol{q}_{left} = (q_1, -2\pi/3, -2\pi/3).$$

Consider then the right-arm class (the following treatment is analogous for the left-arm case). With $p_x = p_y = 0$ and $q_2 (= q_3) = 2\pi/3$, the Jacobian becomes after trigonometric simplifications

$$\boldsymbol{J}(q_1) = \begin{pmatrix} 0 & L \sin q_1 & L \sin(q_1 + \pi/3) \\ 0 & -L \cos q_1 & -L \cos(q_1 + \pi/3) \end{pmatrix}.$$

Any configuration in the $\boldsymbol{q}_{right}$ class is clearly regular: in fact, $\det(\boldsymbol{J}(q_1)\boldsymbol{J}^T(q_1)) = 0.75L^4 \neq 0$. Then, the pseudoinverse $\boldsymbol{J}^{\#}(q_1)$ can be computed as

$$\boldsymbol{J}^{\#}(q_1) = \boldsymbol{J}^T(q_1)\left(\boldsymbol{J}(q_1)\boldsymbol{J}^T(q_1)\right)^{-1} = \frac{2\sqrt{3}}{3L}\begin{pmatrix} 0 & 0 \\ \cos(q_1+\pi/3) & \sin(q_1+\pi/3) \\ \cos q_1 & \sin q_1 \end{pmatrix},$$

and the projection matrix $\boldsymbol{P}$ becomes the constant matrix

$$\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This result should not come unexpected: no non-zero velocity of joints 2 or 3 will be admissible if the end-effector should keep the position $\boldsymbol{p} = \boldsymbol{0}$. Conversely, any velocity of joint 1 does move the end-effector and in fact is left unchanged when multiplied by the projector $\boldsymbol{P}$.

Therefore, starting from $\boldsymbol{q}(0) = (q_1(0), 2\pi/3, 2\pi/3)$, with $q_1(0) \in [-\Delta, +\Delta]$, eq. (1) becomes

$$\dot{\boldsymbol{q}} = -\alpha\,\boldsymbol{P}(\boldsymbol{q})\nabla_{\boldsymbol{q}}H(\boldsymbol{q}) = -\frac{\alpha}{12\,\Delta^2}\begin{pmatrix} q_1 \\ 0 \\ 0 \end{pmatrix}.$$

Accordingly, if $q_1(0) > 0$ then $\dot{q}_1 < 0$ and, vice versa, if $q_1(0) < 0$ then $\dot{q}_1 > 0$. This guarantees that the robot motion will execute correctly the Cartesian task, while remaining always within the joint limits. The motion will stop when q_1 reaches zero. Thus, the final reached configuration is in any case

$$\bar{\boldsymbol{q}} = \begin{pmatrix} 0 \\ 2\pi/3 \\ 2\pi/3 \end{pmatrix},$$

with

$$H(\bar{\boldsymbol{q}}) = \frac{1}{24\,\Delta^2}\left(\left(\frac{2\pi}{3}\right)^2 + \left(\frac{2\pi}{3}\right)^2\right) = 0.0658,$$

and

$$\nabla H(\bar{\boldsymbol{q}}) = \frac{1}{12\,\Delta^2}\begin{pmatrix} 0 \\ 2\pi/3 \\ 2\pi/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.0314 \\ 0.0314 \end{pmatrix} \neq 0.$$

Exercise 2

The problem is solved by a feedback linearization control law that imposes a prescribed linear and decoupled dynamics to the trajectory errors in the task space. The task space for this planar robot is a two-dimensional Cartesian space for the end-effector, which is rotated so as to align its axes with the tangent and normal directions to the desired linear path.

We define first the desired task trajectory for the robot end-effector. The path is a segment from $P_i = (2,3)$ to $P_f = (-2,0)$, having length $L = \|P_f - P_i\| = 5$ [m] and being expressed in terms of its arc length σ as

$$\boldsymbol{p}(\sigma) = P_i + \frac{P_f - P_i}{L}\,\sigma = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}\sigma, \qquad \sigma \in [0, L] = [0, 5].$$

One can associate to the path (see Fig. 3) a rotated right-handed planar frame $\Sigma_R = (\boldsymbol{t}, \boldsymbol{n})$, having axis $\boldsymbol{t}$ tangent to the path and axis $\boldsymbol{n}$ normal to it. The orientation of this frame is given by the 2×2 matrix

$$\boldsymbol{R}(\alpha) = \begin{pmatrix} \boldsymbol{t} & \boldsymbol{n} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad \text{with } \alpha = \text{atan2}\left\{-\frac{3}{5}, -\frac{4}{5}\right\} = -2.4981 \text{ [rad]} = -143.13°,$$

and thus

$$\boldsymbol{R}(\alpha) = \begin{pmatrix} -0.8 & 0.6 \\ -0.6 & -0.8 \end{pmatrix}. \tag{2}$$

The task frame Σ_R will be used to conveniently express the dynamically decoupled error components $e_t(t)$ and $e_n(t)$ along the trajectory.

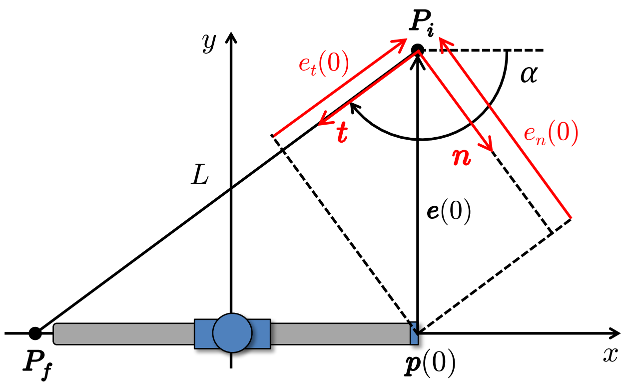


Figure 3: The desired linear path, with the definition of the task frame $\Sigma_R = (\boldsymbol{t}, \boldsymbol{n})$ whose constant orientation is given by matrix $\boldsymbol{R}(\alpha)$. The initial position of the robot end-effector $\boldsymbol{p}(0)$ is also shown, with the Cartesian initial error $\boldsymbol{e}(0) \in \mathbb{R}^2$ and the (rotated) initial position error components $e_t(0)$ and $e_n(0)$.

The timing law $\sigma = \sigma_d(t)$ on the reference path is a rest-to-rest motion with a symmetric bang-coast-bang acceleration profile. For such a profile, the maximum acceleration A along the path is computed from the total motion time T, the path length L, and the cruise speed V as

$$T = \frac{LA + V^2}{AV} \qquad \Rightarrow \qquad A = \frac{V^2}{TV - L} = \frac{9}{2 \cdot 3 - 5} = 9 \text{ [m/s}^2\text{]},$$

whereas the acceleration and deceleration phases last each $T_s = V/A = 0.333$ [s]. The assumed existence of a cruising velocity at $V = 3$ [m/s] is confirmed by the obtained value $T_s < T/2 = 1$. Thus, the initial acceleration of the timing law is $\ddot{\sigma}_d(0) = A = 9$ [m/s^2]. Accordingly, the initial desired end-effector velocity and acceleration in the base frame are

$$\dot{\boldsymbol{p}}_d(0) = \boldsymbol{0}, \qquad \ddot{\boldsymbol{p}}_d(0) = \frac{P_f - P_i}{L}\,\ddot{\sigma}_d(0) = -\begin{pmatrix} 7.2 \\ 5.4 \end{pmatrix} \text{ [m/s}^2\text{]},$$

while the same quantities expressed in the rotated task frame are indeed

$${}^R\dot{\boldsymbol{p}}_d(0) = \boldsymbol{0}, \qquad {}^R\ddot{\boldsymbol{p}}_d(0) = \boldsymbol{R}^T(\alpha)\,\ddot{\boldsymbol{p}}_d(0) = \begin{pmatrix} 9 \\ 0 \end{pmatrix} \text{ [m/s}^2\text{]}.$$

Next, define the initial position and velocity error of the robot end-effector with respect to the desired trajectory. The direct kinematics of the RP robot is

$$\boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q}) = \begin{pmatrix} q_2 \cos q_1 \\ q_2 \sin q_1 \end{pmatrix}. \tag{3}$$

The robot is initially at rest in $\boldsymbol{q}(0) = (0, 2)$. Thus,

$$\boldsymbol{p}(0) = \boldsymbol{f}(\boldsymbol{q}(0)) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \text{ [m]}, \qquad \dot{\boldsymbol{q}}(0) = \boldsymbol{0} \quad \Rightarrow \quad \dot{\boldsymbol{p}}(0) = \boldsymbol{0},$$

and so

$$\boldsymbol{e}(0) = \boldsymbol{p}_d(0) - \boldsymbol{p}(0) = \begin{pmatrix} e_x(0) \\ e_y(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \qquad \dot{\boldsymbol{e}}(0) = \dot{\boldsymbol{p}}_d(0) - \dot{\boldsymbol{p}}(0) = \boldsymbol{0}.$$

In the rotated frame Σ_R, we have

$${}^R\boldsymbol{e}(0) = \boldsymbol{R}^T(\alpha)\,\boldsymbol{e}(0) = \begin{pmatrix} e_t(0) \\ e_n(0) \end{pmatrix} = \begin{pmatrix} -1.8 \\ -2.4 \end{pmatrix}, \qquad {}^R\dot{\boldsymbol{e}}(0) = \boldsymbol{R}^T(\alpha)\,\dot{\boldsymbol{e}}(0) = \boldsymbol{0}.$$

Consider now the robot dynamics. Since the RP robot moves on the horizontal plane, its dynamic model has $\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}$:

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{u}.$$

Applying the feedback linearizing control

$$\boldsymbol{u} = \boldsymbol{M}(\boldsymbol{q})\boldsymbol{a}_{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \qquad \text{with} \quad \boldsymbol{a}_{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q}) \left(\boldsymbol{R}(\alpha)\boldsymbol{a} - \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}\right), \tag{4}$$

leads to the Cartesian acceleration of the end-effector in the task frame

$$\ddot{\boldsymbol{p}} = \boldsymbol{R}(\alpha)\,\boldsymbol{a} \qquad \Rightarrow \qquad {}^R\ddot{\boldsymbol{p}} = \boldsymbol{R}^T(\alpha)\,\ddot{\boldsymbol{p}} = \boldsymbol{a}.$$

Choosing then

$$\boldsymbol{a} = {}^R\ddot{\boldsymbol{p}}_d + {}^R\boldsymbol{K}_D\left({}^R\dot{\boldsymbol{p}}_d - {}^R\dot{\boldsymbol{p}}\right) + {}^R\boldsymbol{K}_P\left({}^R\boldsymbol{p}_d - {}^R\boldsymbol{p}\right) \tag{5}$$

with

$${}^R\boldsymbol{K}_P = \begin{pmatrix} k_{P,t} & 0 \\ 0 & k_{P,n} \end{pmatrix} \qquad \text{and} \qquad {}^R\boldsymbol{K}_D = \begin{pmatrix} k_{D,t} & 0 \\ 0 & k_{D,n} \end{pmatrix},$$

yields for the dynamics of the task trajectory error ${}^R\boldsymbol{e} = {}^R\boldsymbol{p}_d - {}^R\boldsymbol{p}$

$${}^R\ddot{\boldsymbol{e}} + {}^R\boldsymbol{K}_D{}^R\dot{\boldsymbol{e}} + {}^R\boldsymbol{K}_P{}^R\boldsymbol{e} = \boldsymbol{0},$$

or, componentwise,

$$\ddot{e}_t + k_{D,t}\,\dot{e}_t + k_{P,t}\,e_t = 0, \qquad \ddot{e}_n + k_{D,n}\,\dot{e}_n + k_{P,n}\,e_n = 0. \tag{6}$$

Accordingly, by choosing the scalar control gains

$$k_{P,t} = 4, \qquad k_{D,t} = 4, \qquad k_{P,n} = 16, \qquad k_{D,n} = 8, \tag{7}$$

we obtain the prescribed dynamic behavior for the transient errors along the tangent and normal directions to the desired trajectory.

Putting together (4) and (5), the final control law is thus

$$\boldsymbol{u} = \boldsymbol{M}(\boldsymbol{q}) \left(\boldsymbol{J}^{-1}(\boldsymbol{q}) \boldsymbol{R}(\alpha) \left({}^R\ddot{\boldsymbol{p}}_d + {}^R\boldsymbol{K}_D {}^R\dot{\boldsymbol{e}} + {}^R\boldsymbol{K}_P {}^R\boldsymbol{e} \right) - \boldsymbol{J}^{-1}(\boldsymbol{q}) \dot{\boldsymbol{J}}(\boldsymbol{q}) \dot{\boldsymbol{q}} \right) + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{8}$$

Beside (2) and (3), the other kinematic terms needed in the control law (8) are the Jacobian matrix and its inverse

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{J}^{-1}(\boldsymbol{q}) = -\frac{1}{q_2} \begin{pmatrix} \sin q_1 & -\cos q_1 \\ -q_2 \cos q_1 & -q_2 \sin q_1 \end{pmatrix},$$

and the time derivative of the Jacobian

$$\dot{\boldsymbol{J}}(\boldsymbol{q}) = \begin{pmatrix} -\dot{q}_1 q_2 \cos q_1 - \dot{q}_2 \sin q_1 & -\dot{q}_1 \sin q_1 \\ -\dot{q}_1 q_2 \sin q_1 + \dot{q}_2 \cos q_1 & \dot{q}_1 \cos q_1 \end{pmatrix}.$$

From these, it follows also

$$\dot{\boldsymbol{J}}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \begin{pmatrix} -\dot{q}_1^2 q_2 \cos q_1 - 2 \dot{q}_1 \dot{q}_2 \sin q_1 \\ -\dot{q}_1^2 q_2 \sin q_1 + 2 \dot{q}_1 \dot{q}_2 \cos q_1 \end{pmatrix} \quad \Rightarrow \quad -\boldsymbol{J}^{-1}(\boldsymbol{q}) \dot{\boldsymbol{J}}(\boldsymbol{q}) \dot{\boldsymbol{q}} = \frac{1}{q_2} \begin{pmatrix} -2 \dot{q}_1 \dot{q}_2 \\ \dot{q}_1^2 q_2^2 \end{pmatrix}.$$

Moreover, the dynamic terms used in (8) are the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ and the Coriolis and centrifugal terms $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$. These are obtained as usual from the following steps.

Kinetic energy

$$T = T_1 + T_2$$

with

$$T_1 = \frac{1}{2} I_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2} I_2 \dot{q}_1^2 + \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2,$$

where

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \frac{d}{dt} \left((q_2 - d_{c2}) \begin{pmatrix} c_1 \\ s_1 \end{pmatrix} \right) = \dot{q}_2 \begin{pmatrix} c_1 \\ s_1 \end{pmatrix} + (q_2 - d_{c2}) \dot{q}_1 \begin{pmatrix} -s_1 \\ c_1 \end{pmatrix} = \boldsymbol{R}(q_1) \begin{pmatrix} \dot{q}_2 \\ (q_2 - d_{c2}) \dot{q}_1 \end{pmatrix},$$

and so

$$T = \frac{1}{2} (I_1 + I_2) \dot{q}_1^2 + \frac{1}{2} m_2 \left(\dot{q}_2^2 + (q_2 - d_{c2})^2 \dot{q}_1^2 \right).$$

Inertia matrix

$$T = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}} \qquad \Rightarrow \qquad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{m}_1(\boldsymbol{q}) & \boldsymbol{m}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} I_1 + I_2 + m_2 (q_2 - d_{c2})^2 & 0 \\ 0 & m_2 \end{pmatrix}.$$

Coriolis and centrifugal terms

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q}) \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q}) \dot{\boldsymbol{q}} \end{pmatrix}, \quad \text{with } \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2} \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i} \right), \quad i = 1, 2.$$

Since

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & m_2 (q_2 - d_{c2}) \\ m_2 (q_2 - d_{c2}) & 0 \end{pmatrix}, \qquad \boldsymbol{C}_2(\boldsymbol{q}) = -\begin{pmatrix} m_2 (q_2 - d_{c2}) & 0 \\ 0 & 0 \end{pmatrix},$$

we have

$$\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{pmatrix} 2m_2\,(q_2-d_{c2})\,\dot{q}_1\dot{q}_2 \\ -m_2\,(q_2-d_{c2})\,\dot{q}_1^2 \end{pmatrix}.$$

Moreover, the analytic expression of the generalized force $\boldsymbol{u}(t)$ at the initial time $t=0$ is obtained by particularizing (8) at the initial robot state $(\boldsymbol{q},\dot{\boldsymbol{q}}) = (\boldsymbol{q}(0),\boldsymbol{0})$ and with the initial task position error ${}^R\boldsymbol{e}(0)$ and task velocity error ${}^R\dot{\boldsymbol{e}}(0)=\boldsymbol{0}$. We have

$$\boldsymbol{u}(0) = \boldsymbol{M}(\boldsymbol{q}(0))\boldsymbol{J}^{-1}(\boldsymbol{q}(0))\boldsymbol{R}(\alpha)\left({}^R\ddot{\boldsymbol{p}}_d(0) + {}^R\boldsymbol{K}_P\,{}^R\boldsymbol{e}(0)\right). \tag{9}$$

With the given dynamic data, the numerical value of the initial control (9) is

$$\boldsymbol{u}(0) = \begin{pmatrix} \tau(0) \\ F(0) \end{pmatrix} = \begin{pmatrix} 629.8 \\ -244.8 \end{pmatrix} \text{ [Nm, N]}.$$

Finally, the numerical value of the position $\boldsymbol{p}(t)$ of the end-effector at the half-time $t=T/2=1$ of the motion is obtained as $\boldsymbol{p}(T/2) = \boldsymbol{p}_d(T/2) - \boldsymbol{e}(T/2)$, from the knowledge of the desired trajectory $\boldsymbol{p}_d(t)$ and using the position error $\boldsymbol{e}(t)$ resulting from the feedback linearization control law (8) —thus, analytically and without the need of a numerical simulation of the robot dynamics!

For the desired trajectory position, one has

$$\boldsymbol{p}_d(1) = P_i + \frac{P_f - P_i}{2} = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \text{ [m]},$$

namely, the midpoint on the linear path. On the other hand, the closed-form solutions of the linear differential equations in (6) for the initial values $e_t(0)=1.8$, $e_n(0)=2.4$, $\dot{e}_t(0)=\dot{e}_n(0)=0$, and for the specific numerical gains in (7) are[1]

$$e_t(t) = -1.8\,(1+2t)\exp(-2t), \qquad e_n(t) = -2.4\,(1+4t)\exp(-4t). \tag{10}$$

Thus

$$\boldsymbol{e}(1) = \boldsymbol{R}(\alpha)\,{}^R\boldsymbol{e}(1) = \boldsymbol{R}(\alpha)\begin{pmatrix} -5.4\,\exp(-2) \\ -12\,\exp(-4) \end{pmatrix} = \begin{pmatrix} -0.8 & 0.6 \\ -0.6 & -0.8 \end{pmatrix}\begin{pmatrix} -0.7308 \\ -0.2198 \end{pmatrix} = \begin{pmatrix} 0.4528 \\ 0.6143 \end{pmatrix},$$

and therefore

$$\boldsymbol{p}(1) = \boldsymbol{p}_d(1) - \boldsymbol{e}(1) = \begin{pmatrix} -0.4528 \\ 0.8857 \end{pmatrix} \text{ [m]}.$$

Figure 4 shows the evolution in time of the relevant variables concerning the robot end-effector: on the left, the tracking errors $e_t(t)$ and $e_n(t)$ along the tangential and normal directions to the path; at the center, the components $e_x(t)$ and $e_y(t)$ of the tracking error in the base frame; on the right, the coordinates $p_x(t)$ and $p_y(t)$ of the actual position of the end-effector. One can see that the chosen PD gains of the linear part of the control design are not sufficient large to fully recover the tracking error before the end of the trajectory. Indeed, these residual errors will approach exponentially zero after $T=5$ [s], when the desired reference motion has ended and the trajectory tracking problem has become a regulation problem.

[1] The two differential equations (6) are written in the Laplace domain as

$$\left(s^2+4s+4\right)e_t(s) = (s+2)^2\,e_t(s) = 0, \qquad \left(s^2+4s+16\right)e_n(s) = (s+4)^2\,e_n(s) = 0,$$

i.e., they have two real and coincident roots, respectively in -2 and -4. Accordingly, the time solutions for arbitrary initial conditions are

$$e_t(t) = e_t(0)\exp(-2t) + (\dot{e}_t(0) + 2e_t(0))\,t\exp(-2t), \qquad e_n(t) = e_n(0)\exp(-4t) + (\dot{e}_n(0) + 4e_n(0))\,t\exp(-4t).$$

The results in (10) follow for the considered case with $\dot{e}_t(0) = \dot{e}_n(0) = 0$.

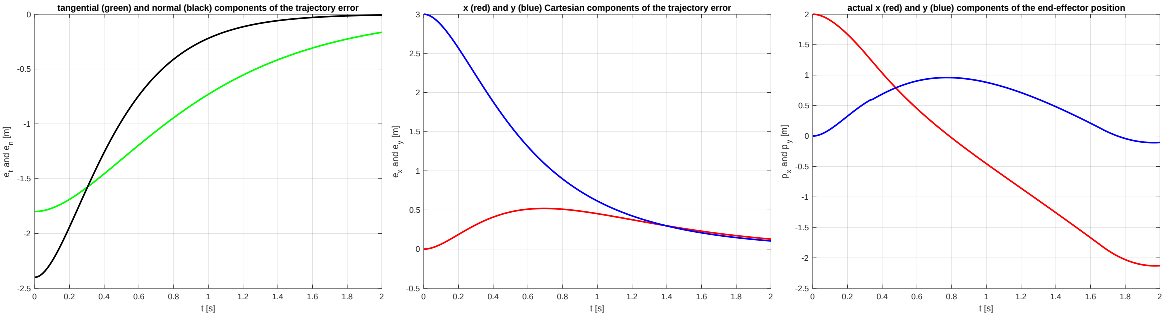


Figure 4: Time evolution of the tracking errors $e_t(t)$ and $e_n(t)$ in the task frame [left] and of the tracking errors $e_x(t)$ and $e_y(t)$ in the base frame [center]; coordinates $p_x(t)$ and $p_y(t)$ of the actual end-effector position [right].

Extra comments to the solution of Exercise 2

The evolutions of the Cartesian errors and positions, respectively in the center and right plots of Fig. 4, deserve a special comment. While the initial errors $e_t(0)$ and $e_n(0)$ are both non-zero, the position component $p_x(0)$ is matched with its desired value $p_{d,x}(0)$ and the initial Cartesian position error limited to the y-direction. However, the reaction to trajectory errors is designed to be decoupled in the $\boldsymbol{t}$ and $\boldsymbol{n}$ (task) directions, not in the $\boldsymbol{x}$ and $\boldsymbol{y}$ (Cartesian) directions. Therefore, the presence of an initial error along $\boldsymbol{y}$ will induce later on also an error along $\boldsymbol{x}$, because the control action is designed to reduce separately the two error components in the task directions (see Fig. 4 [left]). After both these task errors have been sufficiently reduced, also the two Cartesian errors will eventually be reduced in a monotonic way. Note also that the y-position of the end-effector (the blue plot in Fig. 4 [right]) increases first in order to approach the planned path, but then reduces because the desired trajectory has reduced as well its y-component (see the path in Fig. 3).

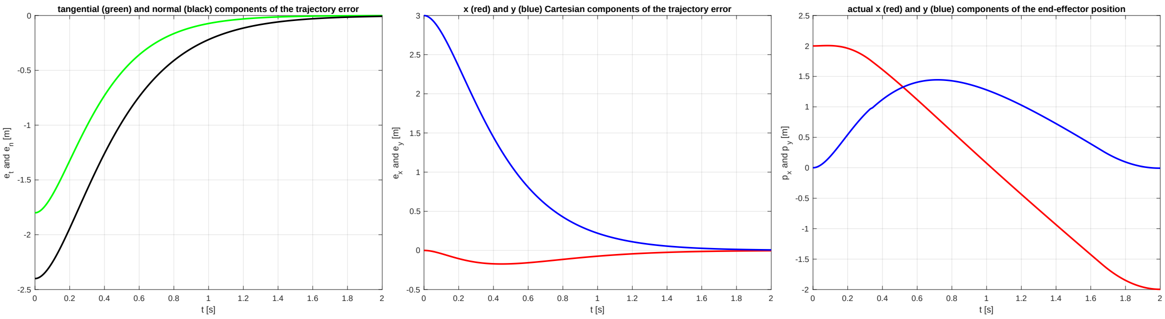


Figure 5: Improved evolution of the tracking errors in task frame [left] and in base frame [center] and actual position of the end-effector [right] when the double pole of the linear controller in the tangential direction is moved from -2 to -5.

A better, and perhaps clearer, behavior could be observed if the control gains in the tangential direction were properly increased. Choosing for instance

$$k_{P,t} = 25, \qquad k_{D,t} = 10$$

would lead in the Laplace domain to a double pole in $s = -5$ (rather than in $s = -2$) along the tangential direction, with the associated error decreasing at a faster exponential rate as

$$e_t(t) = -1.8\,(1 + 5t)\exp(-5t),$$

which is to be compared with $e_t(t)$ in (10). The control gains in the normal direction have remained unchanged. The obtained results are shown in Fig. 5, where the green profile in the left plot is the new $e_t(t)$. The faster reaction to this tangential error has a counter-effect on $e_x(t)$, which now becomes negative for some time during the transient, although with a much smaller magnitude than before. In this case, all errors have vanished by the end of the desired motion ($T = 5$ [s]).

Exercise 3

The dynamic model of the two-mass system with a rigid cable of Fig. 2(a) is given by

$$(B + M)\,\ddot{\theta} - M g_0 = F.$$

This can be obtained by simple balance of inertial, gravity and external forces or from a Lagrangian approach, with $L = T - U_g$ and where $T = \frac{1}{2}B\dot{\theta}^2 + \frac{1}{2}M\dot{\theta}^2$ and $U_g = -M g_0 \theta + U_0$. Note also that, being the gravity term $g = \partial U_g/\partial\theta = -M g_0$ constant, its gradient $\partial g/\partial\theta = 0$ is upper bounded just by $\alpha = 0$; thus, to have a unique closed-loop equilibrium, we expect no positive lower bound larger than zero for the proportional gain K_P in order to contrast gravity.

As for the control design, both proposed control laws do not use the knowledge of the masses B and M. Thus, we shall assume in the following that their value is unknown, while only an upper bound is available on each, i.e., and $0 < B \le B_{max}$ and $0 < M \le M_{max}$.

Consider first the PID control law

$$F = K_P\,(\theta_d - \theta) - K_D\dot{\theta} + K_I \int (\theta_d - \theta)\, d\tau, \tag{11}$$

with θ_d being constant. This leads to the following closed-loop equation for the error $e = \theta_d - \theta$

$$(B + M)\,\ddot{e} + K_D\dot{e} + K_P e + K_I \int_0^t e\, d\tau + M g_0 = 0.$$

In order to have the desired closed-loop equilibrium for $t \to \infty$, the error e should vanish and thus the integral term should balance at steady state the gravity term:

$$F_\infty = K_I \int_0^\infty e(\tau)\, d\tau = -M g_0.$$

To verify the asymptotic stability of the desired closed-loop equilibrium, we use the fact that the system dynamics is linear and transform the closed-loop equation in the Laplace domain. The gravity term $d = M g_0$ is considered here as a constant external disturbance. We have

$$\left((B + M)\, s^2 + K_D s + K_P + \frac{K_I}{s} \right) \theta(s) = \left(K_P + \frac{K_I}{s} \right) \theta_d(s) + d(s),$$

or, multiplying by s and re-organizing terms,

$$\theta(s) = \frac{K_P s + K_I}{(B + M)\, s^3 + K_D s^2 + K_P s + K_I}\, \theta_d(s) + \frac{s}{(B + M)\, s^3 + K_D s^2 + K_P s + K_I}\, d(s).$$

The input-output and disturbance-output transfer functions, respectively $W(s) = \theta(s)/\theta_d(s)$ and $W_d(s) = \theta(s)/d(s)$, have a common polynomial at the denominator. The asymptotic stability of the closed-loop system depends on the localization of the three closed-loop poles, namely the three roots of the characteristic equation

$$(B + M)\, s^3 + K_D s^2 + K_P s + K_I = 0.$$

Using the Routh criterion, we build the Routh table

$$\begin{array}{c|cc} 3 & B+M & K_P \\ 2 & K_D & K_I \\ 1 & K_P - \dfrac{(B+M)K_I}{K_D} & \\ 0 & K_I & \end{array}$$

From this, we find that all three roots will be asymptotically stable (i.e., will have negative real parts) if and only if there is no change of sign in the first column of the table, namely

$$K_I > 0, \qquad K_D > 0, \qquad K_P > \frac{(B+M)K_I}{K_D} > 0.$$

Under this condition, the steady-state value of θ for the input-output relation will be

$$\theta_{ss} = \lim_{t\to\infty} \theta(t) = \lim_{s\to 0} W(s)\theta_d = \theta_d,$$

i.e., the desired one, while for the disturbance-output relation due to gravity it will be

$$\theta_{ss} = \lim_{t\to\infty} \theta(t) = \lim_{s\to 0} W_d(s)d = 0.$$

i.e., the effect of gravity is completely rejected. By superposition of the two effects, it is $\theta_{ss} = \theta_d$.

Summarizing, the PID control law (11) will achieve the desired objective, provided that positive gains are used for the integral and derivative action, and a positive and sufficiently large gain is used for the proportional action. In particular, choosing

$$K_P > \frac{(B_{max} + M_{max})K_I}{K_D} > 0.$$

will guarantee a robust performance in spite of uncertainties on the two masses. Finally, since the system has a linear dynamics, the obtained asymptotic stabilization will be both global and exponential.

Consider next the PD control law with feedforward

$$F = K_P\,(\theta_d - \theta) - K_D\dot{\theta} + v_{i-1}, \tag{12}$$

at a generic iteration of the method. The following closed-loop equation holds for the error $e = \theta_d - \theta$

$$(B+M)\,\ddot{e} + K_D\dot{e} + K_P e + v_{i-1} + Mg_0 = 0. \tag{13}$$

In order for the error e to vanish at steady state, the feedforward should balance the gravity term:

$$v_{i-1} = -Mg_0.$$

If this is not the case, the steady-state error $e_i = \theta_d - \theta_i \neq 0$ at the end of iteration i will satisfy the equilibrium equation

$$K_P e_i + v_{i-1} = -Mg_0 \qquad \Rightarrow \qquad \theta_i = \theta_d + \frac{1}{K_P}\,(v_{i-1} + Mg_0) \neq \theta_d.$$

Analyzing eq. (13) as before, this (wrong) equilibrium will be asymptotically reached if and only if $K_P > 0$ and $K_D > 0$. The uniqueness of such equilibrium is again guaranteed without any further condition on the proportional gain, thanks to the linearity of the system. When the feedforward is updated at successive steady states $(\theta, \dot{\theta}) = (\theta_i, 0)$, $i = 1, 2, \dots$, as

$$v_i = v_{i-1} + K_P \left(\theta_d - \theta_i\right),$$

the desired closed-loop equilibrium state $(q_d, 0)$ will be reached by iteration. Remarkably, when initializing the feedforward term with $v_0 = 0$ (i.e., a pure PD is applied at the first iteration), convergence occurs in just two iterations. In fact, at the end of the first iteration, we will have

$$K_P e_1 = K_P(\theta_d - \theta_1) = -M g_0$$

and the update is

$$v_1 = v_0 + K_P(\theta_d - \theta_1) = 0 - M g_0 = -M g_0.$$

Then, at the end of the second iteration, we will have

$$K_P e_2 + v_1 = K_P(\theta_d - \theta_2) - M g_0 = -M g_0 \qquad \Rightarrow \qquad \theta_2 = \theta_d.$$

Summarizing, the PD + iterative feedforward control (12) will achieve the desired objective, provided that positive gains are used for the proportional and derivative action; the convergence is achieved in two iterations without any further condition on the proportional gain.

Finally, if the cable is flexible (in the domain of linear elasticity) with a finite stiffness $K > 0$ as in Fig. 2(b), the dynamic model consists of two differential equations for the generalized coordinates θ and q. We have

$$\begin{aligned} B\ddot{\theta} + K\left(\theta - q\right) &= F \\ M\ddot{q} + K\left(q - \theta\right) - M g_0 &= 0. \end{aligned}$$

In a Lagrangian framework, these equations are obtained with the vector of generalized coordinates $Q = (\theta, q)$, having $L = T - (U_g + U_e)$, and where $T = \frac{1}{2}B\dot{\theta}^2 + \frac{1}{2}M\dot{q}^2$, $U_g = -M g_0 q + U_0$, and the added elastic potential is $U_e = \frac{1}{2}K(\theta - q)^2$.

* * * * *

Robotics 2

October 21, 2022

Exercise

Consider the PR robot in Fig. 1, moving in a vertical plane.

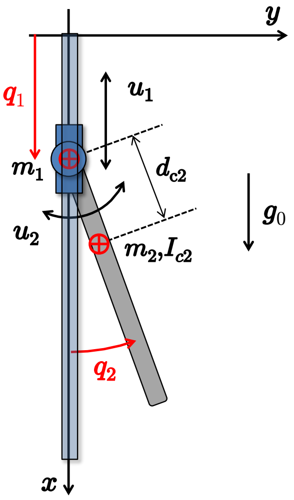


Figure 1: A PR planar robot with the relevant dynamic parameters and variables.

1. Derive the dynamic model of the robot in the form
$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u}.$$

2. Find a linear parametrization of the dynamic model in the form
$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{u},$$
where $\boldsymbol{a} \in \mathbb{R}^p$ has the minimal possible dimension p (the gravity acceleration g_0 is known).

3. Design a control law $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{q}_d)$ that globally asymptotically stabilizes the robot to the desired configuration $\boldsymbol{q}_d = (0, \pi)$ [m,rad], when *only* the total mass $m = m_1 + m_2$ of the robot, the acceleration g_0, and the length ℓ_2 of the second link are known.

4. Suppose that the robot is initially in equilibrium at $\boldsymbol{q}_{in} = \boldsymbol{q}(0) = (0, 0)$. Under the action of the control law designed in step 3, determine the sign of the initial acceleration $\ddot{q}_1(0)$ of the first joint, in case this is different from zero.

5. Assume now that all robot dynamic parameters are known. Show how it is possible to design a model-based command $\boldsymbol{u} = \boldsymbol{u}(t)$ that will transfer the robot from the state $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)) = (\boldsymbol{q}_{in}, \boldsymbol{0})$ to the final state $(\boldsymbol{q}(T), \dot{\boldsymbol{q}}(T)) = (\boldsymbol{q}_d, \boldsymbol{0})$ in a given time $T > 0$ and *without* moving the first joint.

6. The robot input commands are now limited as $|u_i(t)| \le U_i$, $i = 1, 2$. Consider the rest-to-rest task of moving in minimum time T from $\boldsymbol{q}(0) = (q_1(0), \bar{q}_2)$ to $\boldsymbol{q}(T) = (q_1(0) - \Delta, \bar{q}_2)$, with $\Delta > 0$, while keeping the second joint *constantly* at $q_2 = \bar{q}_2 > 0$. Determine the optimal solution in an analytic way and sketch the time-optimal profiles of $\ddot{q}_1(t)$, $\ddot{q}_2(t)$, $u_1(t)$ and $u_2(t)$ for $t \in [0, T]$.

[180 minutes; open books]

Solution

October 21, 2022

1. Dynamic model

Kinetic energy

$$T = T_1 + T_2,$$

with

$$T_1 = \frac{1}{2} m_1 \dot{q}_1^2, \qquad T_2 = \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} I_{c2}\, \dot{q}_2^2,$$

where

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \frac{d}{dt}\begin{pmatrix} q_1 + d_{c2}c_2 \\ d_{c2}s_2 \end{pmatrix} = \begin{pmatrix} \dot{q}_1 - d_{c2}s_2\dot{q}_2 \\ d_{c2}c_2\dot{q}_2 \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{v}_{c2}\|^2 = \dot{q}_1^2 + d_{c2}^2\dot{q}_2^2 - 2d_{c2}s_2\dot{q}_1\dot{q}_2.$$

Thus

$$T = \frac{1}{2}\left(m_1 + m_2\right)\dot{q}_1^2 + \frac{1}{2}\left(I_{c2} + m_2 d_{c2}^2\right)\dot{q}_2^2 - m_2 d_{c2} s_2 \dot{q}_1 \dot{q}_2.$$

Inertia matrix

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} \quad \Rightarrow \quad \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{m}_1(\boldsymbol{q}) & \boldsymbol{m}_2(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} m_1 + m_2 & -m_2 d_{c2} s_2 \\ -m_2 d_{c2} s_2 & I_{c2} + m_2 d_{c2}^2 \end{pmatrix}.$$

Coriolis and centrifugal terms

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} \dot{\boldsymbol{q}}^T \boldsymbol{C}_1(\boldsymbol{q})\,\dot{\boldsymbol{q}} \\ \dot{\boldsymbol{q}}^T \boldsymbol{C}_2(\boldsymbol{q})\,\dot{\boldsymbol{q}} \end{pmatrix}, \quad \text{with } \boldsymbol{C}_i(\boldsymbol{q}) = \frac{1}{2}\left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}} + \left(\frac{\partial \boldsymbol{m}_i(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T - \frac{\partial \boldsymbol{M}(\boldsymbol{q})}{\partial q_i}\right), \quad i = 1, 2.$$

Since

$$\boldsymbol{C}_1(\boldsymbol{q}) = \begin{pmatrix} 0 & 0 \\ 0 & -m_2 d_{c2} c_2 \end{pmatrix}, \qquad \boldsymbol{C}_2(\boldsymbol{q}) = \boldsymbol{O},$$

we have

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -m_2 d_{c2} c_2 \dot{q}_2^2 \\ 0 \end{pmatrix}.$$

Potential energy and gravity terms

$$U = U_1 + U_2 = -m_1 g_0 q_1 - m_2 g_0 \left(q_1 + d_{c2} c_2\right),$$

and so

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} -\left(m_1 + m_2\right) g_0 \\ m_2 d_{c2}\, g_0 s_2 \end{pmatrix}.$$

Robot equations

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u}$$
$$\Downarrow$$
$$\begin{pmatrix} m_1 + m_2 & -m_2 d_{c2} s_2 \\ -m_2 d_{c2} s_2 & I_{c2} + m_2 d_{c2}^2 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -m_2 d_{c2} c_2 \dot{q}_2^2 \\ 0 \end{pmatrix} + \begin{pmatrix} -\left(m_1 + m_2\right) g_0 \\ m_2 d_{c2}\, g_0 s_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{1}$$

2. Linear parametrization

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a} = \begin{pmatrix} \ddot{q}_1 - g_0 & -s_2\,\ddot{q}_2 - c_2\,\dot{q}_2^2 & 0 \\ 0 & -s_2\,(\ddot{q}_1 - g_0) & \ddot{q}_2 \end{pmatrix} \begin{pmatrix} m_1 + m_2 \\ m_2 d_{c2} \\ I_{c2} + m_2 d_{c2}^2 \end{pmatrix},$$

with a minimal number $p = 3$ of dynamic coefficients a_i, $i = 1, 2, 3$.

3. Regulation control

Under the given assumptions, we can design a PD plus constant gravity compensation law as

$$\boldsymbol{u} = \boldsymbol{K}_P(\boldsymbol{q}_d - \boldsymbol{q}) - \boldsymbol{K}_D\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d) \tag{2}$$

with diagonal gain matrices $\boldsymbol{K}_P > 0$, $\boldsymbol{K}_D > 0$ and with $K_{Pm} > \alpha$, where $\|\partial \boldsymbol{g}/\partial \boldsymbol{q}\| \leq \alpha$. In fact, for $\boldsymbol{q}_d = (0, \pi)$, the control law (2) becomes simply

$$\boldsymbol{u} = \begin{pmatrix} -K_{P1}q_1 - K_{D1}\dot{q}_1 - mg_0 \\ K_{P2}(\pi - q_2) - K_{D2}\dot{q}_2 \end{pmatrix}, \tag{3}$$

where $m\,(= m_1 + m_2)$ and g_0 are known. Moreover,

$$\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & 0 \\ 0 & m_2 d_{c2}\,g_0 c_2 \end{pmatrix} \quad \Rightarrow \quad \left\| \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} \right\| = \sqrt{\lambda_{\max}\left\{ \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} \right)^T \right\}} = m_2 d_{c2}\,g_0 |c_2| < m\ell_2 g_0 = \alpha,$$

being ℓ_2 also known. Thus, to guarantee global asymptotic stabilization to $\boldsymbol{q}_d$ with the control law (3) we choose

$$K_{Pm} = \min\{K_{P1}, K_{P2}\} \geq m\ell_2 g_0.$$

4. Initial acceleration

Isolating the acceleration $\ddot{\boldsymbol{q}}$ from the dynamic model (1) and evaluating it at $t = 0$, with initial state $(\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)) = (\boldsymbol{0}, \boldsymbol{0})$ and when using the control law (3), gives

$$\ddot{\boldsymbol{q}}(0) = \boldsymbol{M}^{-1}(\boldsymbol{q}(0))\,(\boldsymbol{u}(0) - \boldsymbol{g}(\boldsymbol{q}(0))) = \frac{1}{\det \boldsymbol{M}(\boldsymbol{q}(0))} \begin{pmatrix} I_{c2} + m_2 d_{c2}^2 & 0 \\ 0 & m_1 + m_2 \end{pmatrix} \begin{pmatrix} 0 \\ K_{P2}\,\pi \end{pmatrix},$$

since in particular $q_{d1} - q_1(0) = 0$ and $u_1(0) - g_1(0) = -mg_0 + (m_1 + m_2)\,g_0 = 0$. Thus,

$$\ddot{q}_1(0) = 0.$$

This should not be unexpected, being the dynamics of the two joints fully decoupled in the initial state and joint 1 still at an equilibrium under the control law (3). Note that a simple PD control law without the gravity compensation term $\boldsymbol{g}(\boldsymbol{q}_d)$ in (3) would result in an initial acceleration $\ddot{q}_1(0) = g_0 = 9.81 > 0$, i.e., the first (prismatic) joint would initially slide downwards.

5. Inverse dynamics command

The desired motion task is obtained by using inverse dynamics on a suitable rest-to-rest trajectory interpolating the initial and final configuration in a given time T. Consider for instance the cubic polynomial trajectory

$$\boldsymbol{q}_d(t) = \boldsymbol{q}_{in} + (\boldsymbol{q}_d - \boldsymbol{q}_{in}) \left(-2\left(\frac{t}{T}\right)^3 + 3\left(\frac{t}{T}\right)^2 \right), \quad t \in [0, T],$$

or, componentwise,

$$q_{d1}(t) = 0 \quad \Rightarrow \quad \dot{q}_{d1}(t) = \ddot{q}_{d1}(t) = 0,$$

$$q_{d2}(t) = \pi\left(-2\left(\frac{t}{T}\right)^3 + 3\left(\frac{t}{T}\right)^2\right) \Rightarrow \dot{q}_{d2}(t) = \frac{6\pi}{T}\left(-\left(\frac{t}{T}\right)^2 + \frac{t}{T}\right) \Rightarrow \ddot{q}_{d2}(t) = \frac{6\pi}{T^2}\left(1 - 2\,\frac{t}{T}\right).$$

Accordingly, the required command is computed as

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{u}_d(t) &= \boldsymbol{M}(\boldsymbol{q}_d(t))\ddot{\boldsymbol{q}}_d(t) + \boldsymbol{c}(\boldsymbol{q}_d(t), \dot{\boldsymbol{q}}_d(t)) + \boldsymbol{g}(\boldsymbol{q}_d(t)) \\ &= \begin{pmatrix} -m_2 d_{c2}\left(\sin q_{d2}(t)\,\ddot{q}_{d2}(t) - \cos q_{d2}(t)\,\dot{q}_{d2}^2(t)\right) - (m_1 + m_2)\,g_0 \\ \left(I_{c2} + m_2 d_{c2}^2\right)\ddot{q}_{d2}(t) + m_2 d_{c2}\,g_0 \sin q_{d2}(t)\end{pmatrix}, \quad t \in [0, T].\end{aligned}$$

6. Minimum time motion

The problem can be formulated as a minimum-time motion on a prescribed path in the joint space, where q_1 needs to move between $q_1(0)$ and $q_1(0) - \Delta$ while q_2 is kept always at the constant value $\bar{q}_2$ (hence, $\dot{q}_2 = \ddot{q}_2 = 0$). Since the velocity term $\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{0}$ for $\dot{q}_2 = 0$, from eq. (1) the dynamics along this path is then

$$\boldsymbol{M}(\bar{q}_2)\begin{pmatrix}\ddot{q}_1 \\ 0\end{pmatrix} + \boldsymbol{g}(\bar{q}_2) = \begin{pmatrix} m_1 + m_2 \\ -m_2 d_{c2}\sin\bar{q}_2\end{pmatrix}(\ddot{q}_1 - g_0) = \boldsymbol{u}, \tag{4}$$

which is parametrized by the single acceleration $\ddot{q}_1(t)$, with $s(t) = q_1(t)$ acting as the (scalar) timing law. Note that the two differential equations in (4) are linear and not independent. This means that one equation should be used in forward dynamics (with the optimal input command) to determine the acceleration $\ddot{q}_1$, while the other will be used in inverse dynamics to find the other input command.

For compactness, define the two constants

$$a = m_1 + m_2 > 0, \qquad b = m_2 d_{c2}\sin\bar{q}_2 > 0.$$

being $\bar{q}_2 \in (0, \pi)$ and thus $\sin\bar{q}_2 > 0$. Then, from (4) and using the bounds on the input commands, we have

$$-U_1 \le a\,(\ddot{q}_1 - g_0) \le U_1, \qquad -U_2 \le -b\,(\ddot{q}_1 - g_0) \le U_2.$$

Manipulating the inequalities, we obtain that $\ddot{q}_1(t)$, for $t \in [0, T]$, is bounded by

$$\max\left\{-\frac{U_1}{a} + g_0, -\frac{U_2}{b} + g_0\right\} = \ddot{q}_1^- \le \ddot{q}_1(t) \le \ddot{q}_1^+ = \min\left\{\frac{U_1}{a} + g_0, \frac{U_2}{b} + g_0\right\}. \tag{5}$$

While $\ddot{q}_1^+ > 0$ clearly holds, it is necessary to enforce $\ddot{q}_1^- < 0$ in order to be able to perform any desired (rest-to-rest) motion transfer by $\Delta \lesseqgtr 0$ with a suitable sequence of positive and negative acceleration of the first joint. However, this condition is trivially obtained once we realize that the minimum requirement for the actuation torques at each joint is that they should be able to sustain (at least) the robot gravity load in any configuration. As a result, we can safely assume that

$$U_1 > (m_1 + m_2)\,g_0 = a\,g_0, \qquad U_2 > m_2 d_{c2}\,g_0 > m_2 d_{c2}\sin\bar{q}_2\,g_0 = b\,g_0 \qquad \Rightarrow \qquad \ddot{q}_1^- < 0.$$

Note that the two possible saturation levels for the acceleration of the first joint (at its positive or negative value) correspond to two different physical situations: either because the first actuator pushes/pulls the robot as fast as possible (U_1 saturates), or because the second actuator reaches its limit capability in order to keep the second link at the fixed configuration $\bar{q}_2$ (U_2 saturates). Furthermore, it follows from the expressions of the bounds in (5) that the same actuator will saturate during the acceleration and deceleration phases, while equation (4) shows that the two input commands will always have opposite signs.

With such asymmetric bounds on the feasible acceleration of joint 1, the requested rest-to-rest motion task for a displacement $-\Delta < 0$ (thus, moving against gravity) will be executed in minimum time T by the following acceleration command:

$$\ddot{q}_1(t) = \begin{cases} \ddot{q}_1^-, & t \in [0, T_s), \\ \ddot{q}_1^+, & t \in [T_s, T). \end{cases}$$

The values of T_s and T are obtained from the two relationships:

velocity at time T $$\ddot{q}_1^- \, T_s + \ddot{q}_1^+ \, (T - T_s) = 0 \qquad \text{(rest-to-rest motion enforced)}$$

Acceleration at time T $$\frac{1}{2}\, \ddot{q}_1^- \, T_s^2 - \frac{1}{2}\, \ddot{q}_1^+ \, (T - T_s)^2 = -\Delta \qquad \text{(net displacement to be achieved).}$$

As a result,

$$T = \sqrt{\frac{2\Delta \left(\ddot{q}_1^+ - \ddot{q}_1^-\right)}{|\ddot{q}_1^-| \, \ddot{q}_1^+}}, \qquad T_s = \frac{\ddot{q}_1^+}{\ddot{q}_1^+ - \ddot{q}_1^-}\, T.$$

Note that in case of opposite values of the positive and negative acceleration ($\ddot{q}_1^+ = -\ddot{q}_1^- = A$), these formulas return the usual symmetric bang-bang profile with $T = \sqrt{4\Delta/A}$ and $T_s = T/2$.

Qualitative optimal profiles of $\ddot{q}_1(t)$, $u_1(t)$ and $u_2(t)$ for $t \in [0, T]$ are sketched in Fig. 2, assuming here that $U_1/a < U_2/b$. Indeed, $\ddot{q}_2(t) = 0$ at any time t (thus, it is not shown).

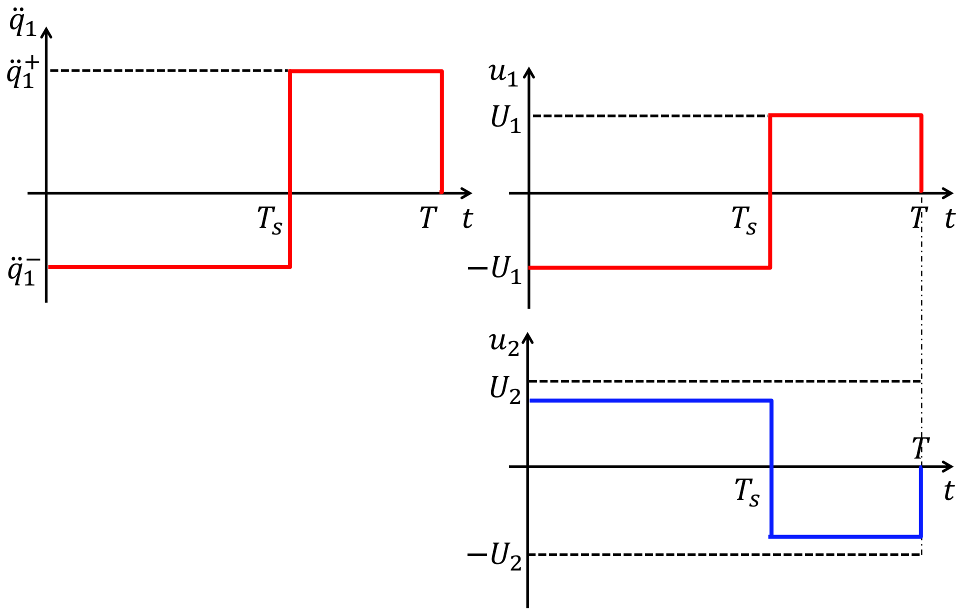


Figure 2: Time-optimal profiles of $\ddot{q}_1(t)$, $u_1(t)$ and $u_2(t)$ for a displacement $-\Delta < 0$ of q_1.

On the other hand, Figure 3 shows the numerical results obtained with the data

$$m_1 = 8, \;\; m_2 = 5 \text{ [kg]}, \quad d_{c2} = 1 \text{ [m]}, \quad \bar{q}_2 = \frac{\pi}{4} \text{ [rad]}, \quad U_1 = 260 \text{ [N]}, \;\; U_2 = 100 \text{ [Nm]}, \tag{6}$$

yielding for a displacement of $-\Delta = -1$ [m] of q_1:

$$\ddot{q}_1^- = -10.19, \quad \ddot{q}_1^+ = 29.81 \text{ [rad/s}^2\text{]}, \qquad T = 0.5132, \quad T_s = 0.3825 \text{ [s]}.$$

The first joint saturates its command ($|u_1| = U_1 = 260$ [N]), whereas the command to the second joint is set to a maximum (absolute) value of $|u_2| = -b\left(\ddot{q}_1^- - g_0\right) = 70.7107 < 100 = U_2$ [Nm].

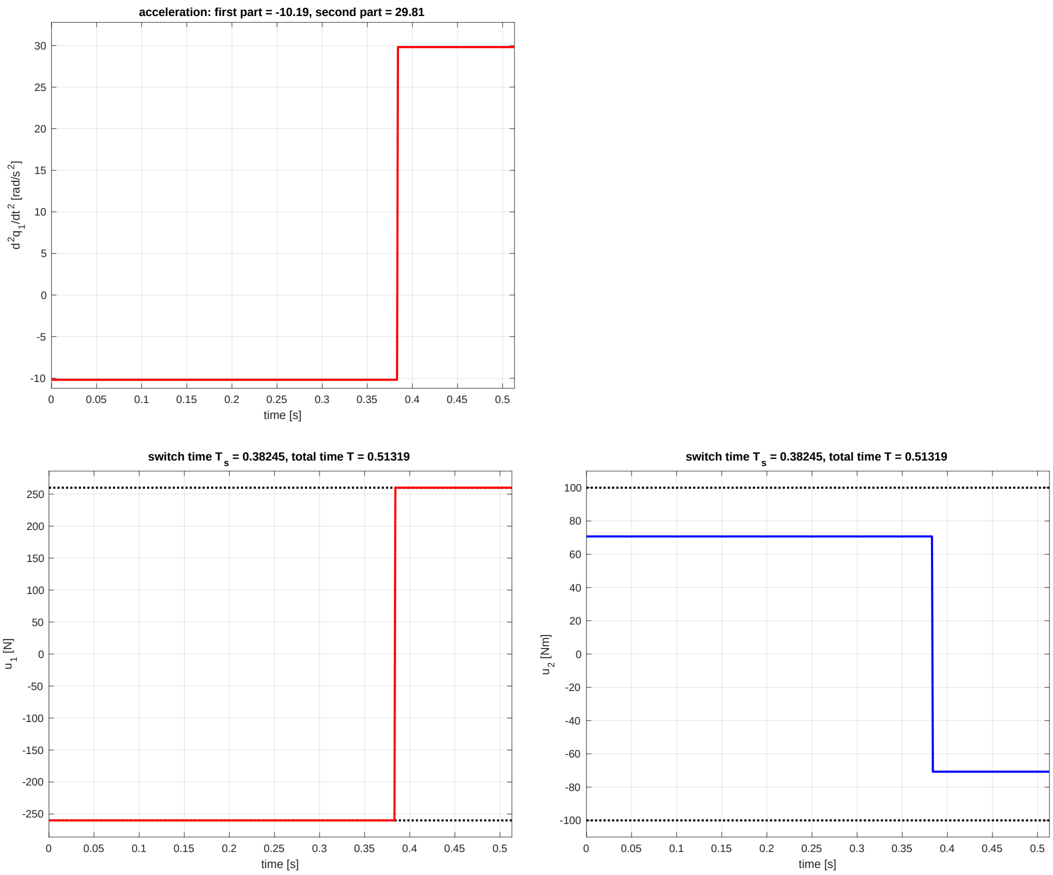


Figure 3: Time-optimal profiles of $\ddot{q}_1(t)$, $u_1(t)$ and $u_2(t)$ for the data in (6) and $-\Delta = -1$.

* * * * *

Robotics 2

January 25, 2023

Exercise 1

The 2R planar robot in Fig. 1 moves in a vertical plane. The second link has its center of mass on the axis of the second joint. Viscous friction is present at both joints.

a. Derive the dynamic model of this robot in Lagrangian form. Find then a linear parametrization of the model as

$$\boldsymbol{Y}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\,\boldsymbol{a}=\boldsymbol{\tau},$$

where the vector of dynamic coefficients $\boldsymbol{a}\in\mathbb{R}^p$ has the least dimension p (the gravity acceleration g_0 and the link lengths are assumed to be known).

b. Consider the control law $\boldsymbol{\tau}=\boldsymbol{K}_P(\boldsymbol{q}_d-\boldsymbol{q})-\boldsymbol{K}_D\dot{\boldsymbol{q}}+\boldsymbol{g}(\boldsymbol{q}_d)$, with diagonal gain matrices $\boldsymbol{K}_P>0$ and $\boldsymbol{K}_D\geq 0$ and with constant gravity compensation at $\boldsymbol{q}_d$. Which are the *minimum* values of the four control gains K_{P1}, K_{P2}, K_{D1} and K_{D2} that guarantee global asymptotic stabilization of *any* generic desired equilibrium state $(\boldsymbol{q},\dot{\boldsymbol{q}})=(\boldsymbol{q}_d,\boldsymbol{0})$?

c. With the robot dynamic parameters being unknown (except for the position of the center of mass of the second link), design an adaptive control law that is able to obtain global asymptotic tracking of a desired smooth trajectory $\boldsymbol{q}_d(t)$.

d. Suppose now that: *i)* the robot moves on a horizontal plane, *ii)* friction at the joints is negligible, and *iii)* the motor torques are bounded as $|\tau_i(t)|\leq\tau_{max,i}$, $i=1,2$. Consider the rest-to-rest task of moving in minimum time the first joint by $\Delta>0$, while keeping the second joint *constantly* at its initial value $q_2(0)$. Determine the optimal solution and the minimum time T in analytic form. Sketch the time-optimal profiles of $\dot{q}_1(t)$, $\ddot{q}_1(t)$, $\tau_1(t)$ and $\tau_2(t)$, for $t\in[0,T]$.

Exercise 2

A 2P Cartesian robot on a horizontal plane is equipped with a F/T sensor at the end-effector. The robot should keep contact with a linear surface, which makes an angle $\alpha\in(0,\pi)$ with the x-axis, while moving at a constant tangential speed $v_d>0$ and applying a constant normal force $f_d>0$ (see Fig. 2). The environment is compliant with stiffness K_n and frictionless, so that it can provide only normal reaction forces. Design an hybrid force-velocity control law that realizes exponential stabilization of the velocity and force errors in a decoupled way along the two task directions. *Hint: Because of the surface compliance, one can consider in the analysis also a small deformation $\delta_n(t)$ at the contact in the normal task direction, and relate $f_n(t)$ and $v_n(t)$ to it.*

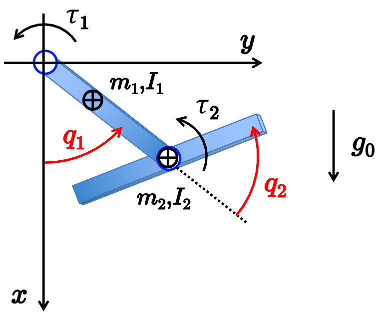


Figure 1: A 2R planar robot with a balanced second link.

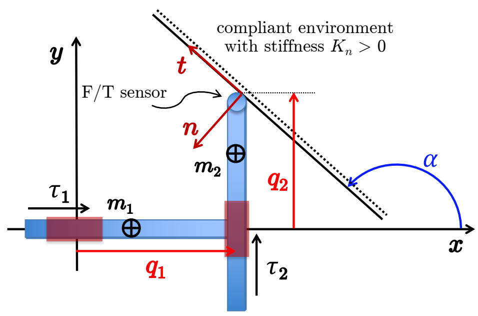


Figure 2: A hybrid force-velocity task to be executed by a 2P Cartesian robot.

[180 minutes; open books]

Solution

January 25, 2023

Exercise 1

a. Dynamic model

Kinetic energy

$$T = T_1 + T_2 = \frac{1}{2}\left(I_1 + m_1 d_{c1}^2\right)\dot{q}_1^2 + \frac{1}{2} m_2 \|\boldsymbol{v}_{c2}\|^2 + \frac{1}{2} I_2 \left(\dot{q}_1 + \dot{q}_2\right)^2$$

where

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \frac{d}{dt}\begin{pmatrix} l_1 c_1 \\ l_1 s_1 \end{pmatrix} = \begin{pmatrix} -l_1 s_1 \dot{q}_1 \\ l_1 c_1 \dot{q}_1 \end{pmatrix} \quad \Rightarrow \quad \|\boldsymbol{v}_{c2}\|^2 = l_1^2 \dot{q}_1^2.$$

Inertia matrix

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}} \quad \Rightarrow \quad \boldsymbol{M} = \begin{pmatrix} I_{tot} & I_2 \\ I_2 & I_2 \end{pmatrix} > 0 \qquad \text{(constant)},$$

with $I_{tot} = I_1 + m_1 d_{c1}^2 + m_2 l_1^2 + I_2 = I_0 + I_2 > I_2$. Coriolis and centrifugal terms are zero.

Potential energy and gravity terms

$$U = U_1 + U_2 = -m_1 g_0 d_{c1} c_1 - m_2 g_0 l_1 c_1,$$

and so

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} \left(m_1 d_{c1} + m_2 l_1\right) g_0 s_1 \\ 0 \end{pmatrix}.$$

Robot equations

$$\begin{aligned} \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{F}_v \dot{\boldsymbol{q}} &= \boldsymbol{\tau} \\ &\Downarrow \\ I_{tot}\,\ddot{q}_1 + I_2\,\ddot{q}_2 + \left(m_1 d_{c1} + m_2 l_1\right) g_0 s_1 + F_{v1}\,\dot{q}_1 &= \tau_1 \\ I_2\,\ddot{q}_1 + I_2\,\ddot{q}_2 + F_{v2}\,\dot{q}_2 &= \tau_2. \end{aligned} \tag{1}$$

Linear parametrization

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\,\boldsymbol{a} = \begin{pmatrix} \ddot{q}_1 & \ddot{q}_2 & g_0 s_1 & \dot{q}_1 & 0 \\ 0 & \ddot{q}_1 + \ddot{q}_2 & 0 & 0 & \dot{q}_2 \end{pmatrix} \begin{pmatrix} I_{tot} \\ I_2 \\ m_1 d_{c1} + m_2 l_1 \\ F_{v1} \\ F_{v2} \end{pmatrix},$$

with $p = 5$ dynamic coefficients a_i, $i = 1, \ldots, 5$. This is obviously a factorization with the least possible number of dynamic coefficients, although not the only one with 5 coefficients; we may, e.g., replace I_{tot} by I_0 in $\boldsymbol{a}$, obtaining a new regressor matrix $\boldsymbol{Y}$ having $Y_{12} = \ddot{q}_1 + \ddot{q}_2$ as the only changed element.

b. Regulation control law

Under the given assumptions, for the PD plus constant gravity compensation law

$$\boldsymbol{\tau} = \boldsymbol{K}_P \left(\boldsymbol{q}_d - \boldsymbol{q}\right) - \boldsymbol{K}_D \dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}_d) \tag{2}$$

the following minimal values for the four control gains are sufficient for the global asymptotic stability of the closed-loop system:

$$K_{P1} > \alpha = a_3 g_0 = \left(m_1 d_{c1} + m_2 l_1\right) g_0 > 0, \qquad K_{P2} > 0, \qquad K_{D1} = K_{D2} = 0, \tag{3}$$

where $\alpha \geq \|\partial \boldsymbol{g}/\partial \boldsymbol{q}\|$, for all $\boldsymbol{q}$. In fact, the closed-loop system (1),(2) can be rewritten as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \left(\boldsymbol{F}_v + \boldsymbol{K}_D\right)\dot{\boldsymbol{q}} + \left(\boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{g}(\boldsymbol{q}_d)\right) = \boldsymbol{K}_P \left(\boldsymbol{q}_d - \boldsymbol{q}\right).$$

Thus, being $\boldsymbol{F}_v > 0$, the presence of viscous friction allows to set to zero the derivative gains $\boldsymbol{K}_D$ in the control law, without prejudice for the asymptotic stability. Moreover, at any equilibrium ($\dot{\boldsymbol{q}} = \ddot{\boldsymbol{q}} = \boldsymbol{0}$), we have

$$\left(\boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{g}(\boldsymbol{q}_d)\right) = \boldsymbol{K}_P \left(\boldsymbol{q}_d - \boldsymbol{q}\right) \quad \Leftrightarrow \quad \begin{aligned} a_3 g_0 \left(\sin q_1 - \sin q_{d1}\right) &= K_{P1}\left(q_{d1} - q_1\right) \\ 0 &= K_{P2}\left(q_{d2} - q_2\right). \end{aligned}$$

It is clear that these two equilibrium conditions are decoupled each to other. In the first equation, it is sufficient to have $K_{P1} > a_3$ in order to have a *unique* equilibrium solution at $q_1 = q_{d1}$. Instead, in the second equation $K_{P2} > 0$ is already sufficient to guarantee that $q_2 = q_{d2}$ is the unique equilibrium.

It is easy to see that the original Lyapunov proof showing global asymptotic stability of the desired state $(\boldsymbol{q}_d, \boldsymbol{0})$ with the PD+ control (2) works as well in the present case under the gain assumptions (3). As a result, the standard sufficient condition

$$K_{Pm} = \min\left\{K_{P1}, K_{P2}\right\} \geq \alpha \quad \Rightarrow \quad K_{P1} \geq \alpha, \quad K_{P2} \geq \alpha > 0$$

is relaxed: it is sufficient to have just a positive proportional gain $K_{P2} > 0$ at joint 2, without any strictly positive lower bound. Moreover, the conditions (3) on the positional gains become also *necessary* for global asymptotic stabilization if we consider that the same control law (2) should work for *any* chosen $\boldsymbol{q}_d$. In particular, the necessity of $K_{P1} > \alpha = a_3 g_0$ follows from the local analysis of the behavior of the closed-loop system linearized around $q_{d1} = \pi$.

c. Adaptive control law for trajectory tracking

Based on the previous results, an adaptive control law for tracking a desired smooth trajectory $\boldsymbol{q}_d(t)$, with global asympotic stability of the tracking error $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$, takes the following expression:

$$\begin{aligned} \boldsymbol{\tau} &= \hat{\boldsymbol{M}}\ddot{\boldsymbol{q}}_r + \hat{\boldsymbol{g}}(\boldsymbol{q}) + \hat{\boldsymbol{F}}_v \dot{\boldsymbol{q}}_r + \boldsymbol{K}_P \boldsymbol{e} + \boldsymbol{K}_D \dot{\boldsymbol{e}} \\ \dot{\hat{\boldsymbol{a}}} &= \begin{pmatrix} \dot{\hat{I}}_{tot} \\ \dot{\hat{I}}_2 \\ \dot{\widehat{m_1 d_{c1} + m_2 l_1}} \\ \dot{\hat{F}}_{v1} \\ \dot{\hat{F}}_{v2} \end{pmatrix} = \boldsymbol{\Gamma} \boldsymbol{Y}^T(\boldsymbol{q}, \dot{\boldsymbol{q}}_r, \ddot{\boldsymbol{q}}_r)\boldsymbol{s} = \boldsymbol{\Gamma} \begin{pmatrix} \ddot{q}_{r1} & 0 \\ \ddot{q}_{r2} & \ddot{q}_{r1} + \ddot{q}_{r2} \\ g_0 s_1 & 0 \\ \dot{q}_{r1} & 0 \\ 0 & \dot{q}_{r2} \end{pmatrix} (\dot{\boldsymbol{q}}_r - \dot{\boldsymbol{q}}), \end{aligned}$$

with (diagonal) $\boldsymbol{K}_P > 0$, $\boldsymbol{K}_D > 0$ and $\boldsymbol{\Gamma} > 0$, a modified reference velocity $\dot{\boldsymbol{q}}_r = \dot{\boldsymbol{q}}_d + \boldsymbol{\Lambda}\boldsymbol{e}$, and the choice $\boldsymbol{\Lambda} = \boldsymbol{K}_D^{-1}\boldsymbol{K}_P$.

d. Rest-to-rest motion in minimum time

On the horizontal plane and without dissipative terms, the dynamic model (1) reduces to

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{\tau} \quad \Rightarrow \quad \begin{aligned} I_{tot}\,\ddot{q}_1 + I_2\,\ddot{q}_2 &= \tau_1 \\ I_2\,\ddot{q}_1 + I_2\,\ddot{q}_2 &= \tau_2. \end{aligned} \tag{4}$$

The specified motion task requires the second joint to remain at rest in the same initial configuration[1], thus imposing $\dot{q}_2 = \ddot{q}_2 = 0$. The two equations (4) are then rewritten in a direct/inverse dynamic form as

$$\ddot{q}_1 = \frac{1}{I_{tot}}\,\tau_1, \qquad \tau_2 = I_2\,\ddot{q}_1 = \frac{I_2}{I_{tot}}\,\tau_1 = \frac{I_2}{I_2 + I_0}\,\tau_1 < \tau_1. \tag{5}$$

The rest-to-rest motion in minimum time for the first joint will be a bang-bang profile in acceleration (and torque). Similarly, because of the coupling between the two commands in (5), also the second torque that keeps joint 2 at rest will have a bang-bang profile. However, only one between the two commanded torques is allowed to reach its bound, depending on the relative values of $\tau_{max,1}$ and $\tau_{max,2}$ and on the robot inertias. In fact, introducing a scalar parameter α to possibly scale the maximum torque at joint 1, we have

$$\tau_1 = \alpha\,\tau_{max,1}, \;\; \alpha \in (0,1] \quad \Rightarrow \quad \tau_2 = \frac{I_2}{I_2 + I_0}\,\alpha\,\tau_{max,1} \le \tau_{max,2} \quad \Rightarrow \quad \alpha \le \frac{I_2 + I_0}{I_2}\,\frac{\tau_{max,2}}{\tau_{max,1}}. \tag{6}$$

Therefore, the maximum torque that can be applied at joint 1 (complying with both bounds) is

$$\bar{\tau}_1 = \min\left\{1, \frac{I_2 + I_0}{I_2}\,\frac{\tau_{max,2}}{\tau_{max,1}}\right\}\tau_{max,1} = \min\left\{\tau_{max,1}, \frac{I_2 + I_0}{I_2}\,\tau_{max,2}\right\}. \tag{7}$$

Accordingly, the torque needed at joint 2 to keep it at rest will be

$$\bar{\tau}_2 = \frac{I_2}{I_2 + I_0}\,\bar{\tau}_1 = \min\left\{\frac{I_2}{I_2 + I_0}\,\tau_{max,1}, \tau_{max,2}\right\}. \tag{8}$$

In order to perform in minimum time the desired rest-to-rest displacement $\Delta > 0$ of the first joint without moving the second, we apply the torques (symmetric in time, so $T_s = T/2$)

$$\tau_1(t) = \begin{cases} \bar{\tau}_1, & t \in [0, T/2] \\ -\bar{\tau}_1, & t \in [T/2, T], \end{cases} \qquad \tau_2(t) = \begin{cases} \bar{\tau}_2, & t \in [0, T/2] \\ -\bar{\tau}_2, & t \in [T/2, T], \end{cases}$$

where T is the minimum motion time, yet to be determined. The acceleration and the velocity of joint 1 will have, respectively, a bang-bang and a triangular time profile:

$$\ddot{q}_1(t) = \begin{cases} \dfrac{\bar{\tau}_1}{I_{tot}}, & t \in [0, T/2] \\ -\dfrac{\bar{\tau}_1}{I_{tot}}, & t \in [T/2, T], \end{cases} \qquad \dot{q}_1(t) = \begin{cases} \dfrac{\bar{\tau}_1}{I_{tot}}\,t, & t \in [0, T/2] \\ \dfrac{\bar{\tau}_1}{I_{tot}}\,T - \dfrac{\bar{\tau}_1}{I_{tot}}\,t, & t \in [T/2, T]. \end{cases}$$

Finally, the minimum time T is obtained by equating the area below the (positive) velocity profile to the displacement $\Delta > 0$. We obtain

$$\frac{\bar{\tau}_1}{I_{tot}}\,\frac{T}{2}\cdot\frac{T}{2} = \Delta \qquad \Rightarrow \qquad T = \sqrt{\frac{4\Delta I_{tot}}{\bar{\tau}_1}}.$$

[1] The actual value of $q_2(0)$ is irrelevant for what follows.

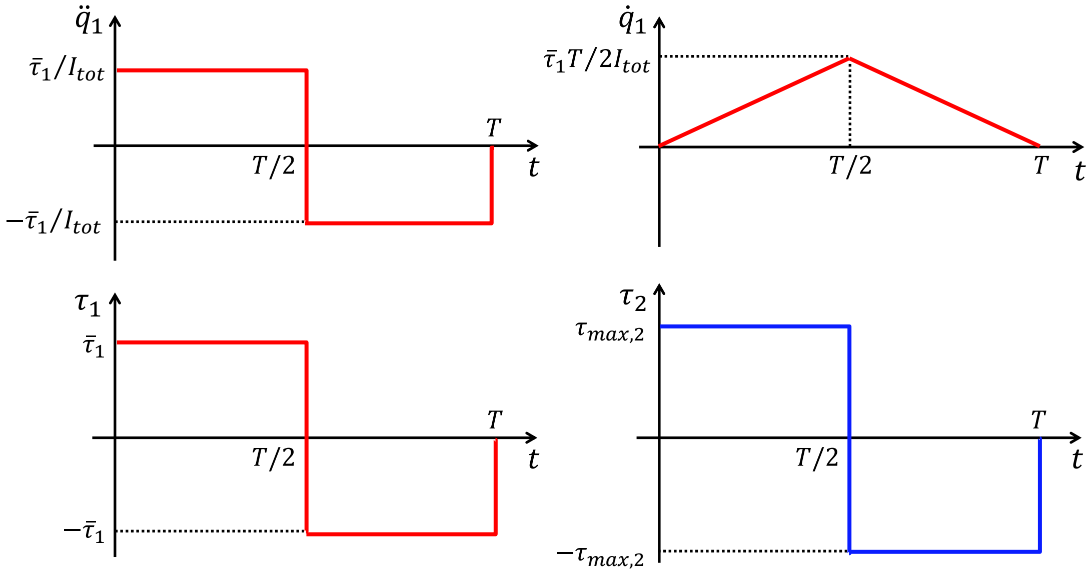


Figure 3: Minimum-time profiles for a displacement $\Delta > 0$ of q_1, when $\tau_2(t) = \pm\tau_{max,2}$: $\ddot{q}_1(t)$ and $\dot{q}_1(t)$ [top]; $\tau_1(t)$ and $\tau_2(t)$ [bottom].

Figure 3 shows the minimum-time profiles of $\ddot{q}_1(t)$ and $\dot{q}_1(t)$, and of the torques $\tau_1(t)$ and $\tau_2(t)$, assuming that the torque at joint 2 is the one that saturates its bound in (7) and (8), i.e., $\alpha < 1$ in (6). Thus, $\bar{\tau}_1 = (I_2 + I_0)\,\tau_{max,2}/I_2 < \tau_{max,1}$.

Exercise 2

The dynamic model of the Cartesian robot in contact with the environment is

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{\tau} + \boldsymbol{\tau}_f \quad \Rightarrow \quad \begin{pmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = \begin{pmatrix} \tau_1 + \tau_{f1} \\ \tau_2 + \tau_{f2} \end{pmatrix}, \tag{9}$$

where $\boldsymbol{\tau}_f$ are the joint torques resulting from the forces exerted by the environment on the robot (and performing work on $\boldsymbol{q}$).

The orientation of the task frame shown in Fig. 2 is given by a 2×2 constant rotation matrix $\boldsymbol{R}$ in the plane. Accordingly, the following relationships between task velocities $\boldsymbol{v}$ and forces $\boldsymbol{f}$ and joint velocities $\dot{\boldsymbol{q}}$ and torques $\boldsymbol{\tau}_f$ hold, all vectors being in $\mathbb{R}^2$:

$$\boldsymbol{R} = \begin{pmatrix} c_\alpha & -s_\alpha \\ s_\alpha & c_\alpha \end{pmatrix}, \qquad \boldsymbol{v} = \boldsymbol{R}^T\dot{\boldsymbol{q}}, \qquad \boldsymbol{v} = \begin{pmatrix} v_t \\ v_n \end{pmatrix}, \qquad \boldsymbol{\tau}_f = \boldsymbol{R}\boldsymbol{f}, \qquad \boldsymbol{f} = \begin{pmatrix} 0 \\ f_n \end{pmatrix}.$$

Since the environment is frictionless, we have set $f_t = 0$: applied contact forces can be balanced by reaction forces f_n only along the normal to the environment. On the other hand, being the environment compliant, a non-zero (though small) normal velocity v_n may also be present at the contact. Let δ_n be the deformation at the contact point along the environment normal. Then

$$v_n = \dot{\delta}_n, \qquad f_n = K_n \delta_n. \tag{10}$$

as it was a spring: stiffness * deformation

With the above notations, we rewrite the dynamic model (9) in the task space as

$$\boldsymbol{R}^T\boldsymbol{M}\boldsymbol{R}\dot{\boldsymbol{v}} = \boldsymbol{R}^T\boldsymbol{\tau} + \boldsymbol{R}^T\boldsymbol{\tau}_f \qquad \Rightarrow \qquad \bar{\boldsymbol{M}}\dot{\boldsymbol{v}} = \bar{\boldsymbol{\tau}} + \boldsymbol{f}, \tag{11}$$

with

$$\bar{\boldsymbol{M}} = \begin{pmatrix} m_1 c_\alpha^2 + m_2 & -m_1 c_\alpha s_\alpha \\ -m_1 c_\alpha s_\alpha & m_1 s_\alpha^2 + m_2 \end{pmatrix}, \qquad \bar{\boldsymbol{\tau}} = \begin{pmatrix} c_\alpha \tau_1 + s_\alpha \tau_2 \\ -s_\alpha \tau_1 + c_\alpha \tau_2 \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} 0 \\ f_n \end{pmatrix}.$$

The dynamics (11) in the task space is still linear but coupled between the t and n axes. We proceed then with the following decoupling control law:

$$\bar{\boldsymbol{\tau}} = \bar{\boldsymbol{M}} \begin{pmatrix} K_v\,(v_d - v_t) \\ K_f\,((f_d - f_n) - K_d v_n) \end{pmatrix} - \begin{pmatrix} 0 \\ f_n \end{pmatrix}, \qquad K_v > 0,\ K_f > 0,\ K_d > 0. \tag{12}$$

The closed-loop system (11),(12) becomes

$$\begin{aligned} \dot{v}_t &= K_v\,(v_d - v_t) \\ \dot{v}_n &= K_f\,((f_d - f_n) - K_d v_n). \end{aligned} \tag{13}$$

Along the tangential direction of the task space, the control action is proportional to the velocity error $e_v = v_d - v_t$. Since v_d is constant, $\dot{e}_v = -\dot{v}_t$ and the first equation in (13) is rewritten as

$$\dot{e}_v = -K_v e_v \quad \Rightarrow \quad e_v(t) = e_v(0)\exp\left(-K_v\,t\right),$$

which shows exponential stabilization to zero of the tangential velocity error.

Along the normal direction of the task space, the control law cancels any (measured) contact force f_n, adds a proportional action on the force error $e_f = f_d - f_n$, and includes a velocity damping $-K_d v_n$. For analysis, using the relationships (10) of the compliant environment, the second equation in (13) can be expressed in terms of the deformation δ_n as

$$M_f \ddot{\delta}_n + K_d \dot{\delta}_n + K_n \delta_n = f_d, \qquad \text{with } M_f = \frac{1}{K_f} > 0. \tag{14}$$

Thus, an impedance-like behavior has been obtained, where the apparent mass M_f and damping K_d can be chosen freely, while the stiffness K_n is the one of the environment. Moreover, the forcing term on the right-hand side is the desired contact force f_d, rather than the actual one f_n as in a standard impedance design. This setting is indeed appropriate. In fact, the second-order dynamics (14) is stable and converges exponentially[2], as $t \to \infty$, to the constant equilibrium deformation $\bar{\delta}_n = f_d/K_n$. In turn, this implies that the normal force at steady state is the desired one:

$$\lim_{t\to\infty} f_n(t) = K_n \bar{\delta}_n = f_d.$$

Finally, the control torque in the joint space is obtained from (12) as

$$\boldsymbol{\tau} = \boldsymbol{R}\,\bar{\boldsymbol{\tau}} = \boldsymbol{M}\boldsymbol{R} \begin{pmatrix} K_v\,(v_d - v_t) \\ K_f\,((f_d - f_n) - K_d v_n) \end{pmatrix} - \boldsymbol{\tau}_f \qquad \left(\text{being } \boldsymbol{\tau}_f = \begin{pmatrix} -s_\alpha \\ c_\alpha \end{pmatrix} f_n \right).$$

* * * * *

[2] Other than by setting $\dot{\delta}_n = \ddot{\delta}_n = 0$ in (14), the steady-state deformation $\bar{\delta}_n$ can also be computed by analyzing in the Laplace domain the system response to a step input f_d. It is

$$W(s) = \frac{\delta_n(s)}{f_d(s)} = \frac{1}{M_f s^2 + F_d s + K_n} \quad \Rightarrow \quad \bar{\delta}_n = \lim_{t\to\infty} \delta_n(t) = \lim_{s\to 0} s\,\delta_n(s) = \lim_{s\to 0} s\,W(s)\cdot\frac{f_d}{s} = W(0)\,f_d = \frac{f_d}{K_n}.$$

Robotics 2

February 13, 2023

Exercise 1

The torque controlled 3R planar robot in Fig. 1 moves on a horizontal plane, performing a two-dimensional trajectory task with its end-effector. The links have equal length l and equal uniformly distributed mass m, with barycentric inertia $I_c = ml^2/12$. While at rest in the configuration $\bar{\boldsymbol{q}} = (\pi/4, -\pi/2, \pi/2)$ [rad], the end-effector should accelerate with $\ddot{\boldsymbol{p}}_d = (1, 0)$ [m/s^2] (as in figure).

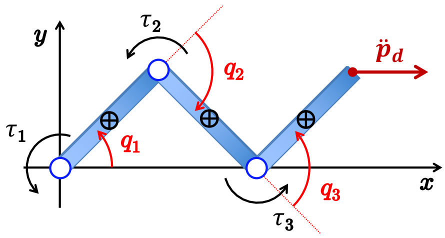


Figure 1: A 3R planar robot with equal links.

Determine, in a parametric way with respect to l and m, the torques $\boldsymbol{\tau}_i \in \mathbb{R}^3$, for $i = A, B, C$, that realize instantaneously the following objectives:

- $\boldsymbol{\tau}_A$ minimizes the squared norm of the joint accelerations $H_A = \frac{1}{2}\left\|\ddot{\boldsymbol{q}}\right\|^2$;
- $\boldsymbol{\tau}_B$ minimizes the squared norm of the absolute joint accelerations $H_B = \frac{1}{2}\left\|\ddot{\boldsymbol{q}}_a\right\|^2$, where

$$\ddot{q}_{a,i} = \sum_{j=1}^{i} \ddot{q}_j, \qquad i = 1, 2, 3;$$

- $\boldsymbol{\tau}_C$ minimizes the squared norm of the inertia-weighted joint accelerations $H_C = \frac{1}{2}\ddot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}$.

Comment on the obtained results in terms of the control efforts at the joint level.

Exercise 2

Consider the single link under gravity in Fig. 2, with all dynamic parameters specified therein. The link should perform a rest-to-rest motion from $\theta(0) = 0$ to $\theta(T) = \pi$ (a swing-up maneuver, counterclockwise), by following a cubic polynomial interpolating trajectory $\theta_d(t)$ under the torque bound $|u| \leq u_{max}$. Suppose that the maximum available torque is large enough to sustain gravity in any configuration, typically with some extra torque left for dynamic motion.

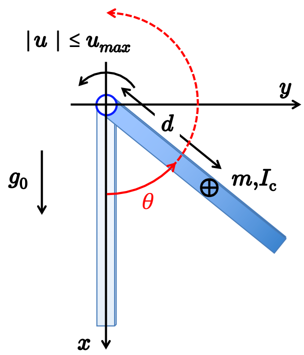


Figure 2: Swing-up maneuver of a single link under maximum torque bound.

Determine the expression of the minimum time $T = T_{min}$ for performing the task, discussing any assumption that you may introduce. Furthermore, suppose that, with a time $T = 1$ s, the motion is unfeasible for a given set of data. What will be the minimum uniform time scaling factor k of the original trajectory that allows to execute the task in a feasible way?

Exercise 3

Figure 3 shows a simplified one-dimensional model of two robots permanently interacting in a compliant mode at the level of their end effectors[1]. Compliance at the contact is modeled by a spring with stiffness $K > 0$. The two robots have equivalent masses m_1 and m_2 and are subject to control forces F_1 and F_2. Their positions are given by q_1 and q_2, with the zero reference for both variables corresponding to when the spring has no deformation (as shown in the figure).

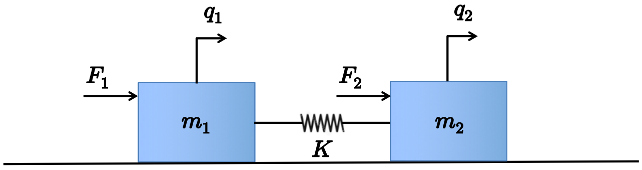


Figure 3: Two masses interacting through a spring.

Define the two control laws

$$F_1 = K_{P1}(q_{1d} - q_1) - K_{D1}\dot{q}_1, \qquad F_2 = K_{P2}(q_{2d} - q_2) - K_{D2}\dot{q}_2, \tag{1}$$

with all gains strictly positive, and where the target positions q_{1d} and q_{2d} for the two masses are generic but different (i.e., $q_{1d} \neq q_{2d}$). These control laws have a decentralized structure, since they both use feedback information only local to the controlled mass, i.e., F_i is function only of $(q_i, \dot{q}_i)$, for $i = 1, 2$.

- Find the unique equilibrium state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\bar{\boldsymbol{q}}, \boldsymbol{0})$ for the closed-loop system under the control (1).
- Prove the global asymptotic stability of this equilibrium state by a Lyapunov/LaSalle argument.
- Is the equilibrium configuration $\bar{\boldsymbol{q}}$ such that $\bar{q}_1 = q_{1d}$ and $\bar{q}_2 = q_{2d}$? If not, how would you modify the controllers (1), possibly keeping the decentralized structure, for the same previous target positions so that $\bar{\boldsymbol{q}} = \boldsymbol{q}_d$ becomes the unique asymptotically stable equilibrium configuration?

[180 minutes; open books]

[1] This ideal situation is not unrealistic. In fact, it can be obtained by applying a preliminary feedback linearizing and decoupling control law in the Cartesian space to two articulated robot manipulators.

Solution

February 13, 2023

Exercise 1

For the considered instantaneous situation, we need to compute only the 2×3 Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ and the 3×3 inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ of the robot. In fact, since the robot moves on a horizontal plane ($\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}$) and is currently at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$), its dynamic model simplifies to

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{\tau}, \tag{2}$$

while the second-order differential kinematics for the positional task becomes

$$\ddot{\boldsymbol{p}} = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}}. \tag{3}$$

Moreover, the absolute joint coordinates $\boldsymbol{q}_a$ are related to $\boldsymbol{q}$ by a constant matrix:

$$\boldsymbol{q}_a = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \boldsymbol{q} = \boldsymbol{T}\boldsymbol{q} \qquad \Rightarrow \qquad \ddot{\boldsymbol{q}}_a = \boldsymbol{T}\ddot{\boldsymbol{q}}. \tag{4}$$

For a desired end-effector acceleration $\ddot{\boldsymbol{p}}_d$ at the current state $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\bar{\boldsymbol{q}}, \boldsymbol{0})$, being $\bar{\boldsymbol{q}}$ a nonsingular configuration for the Jacobian, the three schemes that are locally using the robot redundancy are obtained as particular solutions of the general LQ (Linear Quadratic) optimization problem by the following torques:

- minimization of the squared norm of the joint accelerations

  $$H_A = \frac{1}{2}\left\|\ddot{\boldsymbol{q}}\right\|^2 = \frac{1}{2}\,\ddot{\boldsymbol{q}}^T\ddot{\boldsymbol{q}}$$

  gives

  $$\boldsymbol{\tau}_A = \boldsymbol{M}(\bar{\boldsymbol{q}})\boldsymbol{J}^\dagger(\bar{\boldsymbol{q}})\,\ddot{\boldsymbol{p}}_d, \qquad \text{with } \boldsymbol{J}^\dagger(\boldsymbol{q}) = \boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}; \tag{5}$$

- minimization of the squared norm of the absolute joint accelerations

  $$H_B = \frac{1}{2}\left\|\ddot{\boldsymbol{q}}_a\right\|^2 = \frac{1}{2}\left\|\boldsymbol{T}\ddot{\boldsymbol{q}}\right\|^2 = \frac{1}{2}\,\ddot{\boldsymbol{q}}^T\boldsymbol{W}\ddot{\boldsymbol{q}}$$

  gives

  $$\boldsymbol{\tau}_B = \boldsymbol{M}(\bar{\boldsymbol{q}})\boldsymbol{J}^\dagger_{\boldsymbol{W}}(\bar{\boldsymbol{q}})\,\ddot{\boldsymbol{p}}_d, \qquad \text{with } \boldsymbol{J}^\dagger_{\boldsymbol{W}}(\boldsymbol{q}) = \boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}, \tag{6}$$

  where, from (4), $\boldsymbol{W}$ is the symmetric, positive definite weighting matrix

  $$\boldsymbol{W} = \boldsymbol{T}^T\boldsymbol{T} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} > 0\,;$$

- minimization of the squared norm of the inertia-weighted joint accelerations

  $$H_C = \frac{1}{2}\,\ddot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}$$

gives

$$\boldsymbol{\tau}_C = \boldsymbol{M}(\bar{\boldsymbol{q}})\boldsymbol{J}^\dagger_{\boldsymbol{M}}(\bar{\boldsymbol{q}})\,\ddot{\boldsymbol{p}}_d = \boldsymbol{J}^T(\bar{\boldsymbol{q}})\left(\boldsymbol{J}(\bar{\boldsymbol{q}})\boldsymbol{M}^{-1}(\bar{\boldsymbol{q}})\boldsymbol{J}^T(\bar{\boldsymbol{q}})\right)^{-1}\ddot{\boldsymbol{p}}_d. \tag{7}$$

Note that, in view of (2), this case is equivalent to the minimization of the squared norm of the inverse inertia-weighted torques:

$$H_C = \frac{1}{2}\,\ddot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \frac{1}{2}\,\left(\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{\tau}\right)^T\boldsymbol{M}(\boldsymbol{q})\left(\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{\tau}\right) = \frac{1}{2}\,\boldsymbol{\tau}^T\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{\tau}.$$

We proceed then by computing the required matrices. Note that, because of the uniform nature of the links, the symbolic factors l and $m\,l^2$ can be isolated in the computations of kinematic and, respectively, dynamic terms.

Jacobian

$$\boldsymbol{p} = l\begin{pmatrix} c_1 + c_{12} + c_{123} \\ s_1 + s_{12} + s_{123}\end{pmatrix} \quad\Rightarrow\quad \boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{q}} = l\begin{pmatrix} -(s_1 + s_{12} + s_{123}) & -(s_{12} + s_{123}) & -s_{123} \\ c_1 + c_{12} + c_{123} & c_{12} + c_{123} & c_{123}\end{pmatrix}.$$

Therefore, at $\bar{\boldsymbol{q}} = (\pi/4, -\pi/2, \pi/2)$ we have

$$\boldsymbol{J}(\bar{\boldsymbol{q}}) = l\begin{pmatrix} -0.7071 & 0 & -0.7071 \\ 2.1213 & 1.4142 & 0.7071\end{pmatrix}.$$

Jacobian pseudoinverse

$$\boldsymbol{J}^\dagger(\bar{\boldsymbol{q}}) = \frac{1}{l}\begin{pmatrix} -0.2357 & 0.2357 \\ 0.9428 & 0.4714 \\ -1.1785 & -0.2357\end{pmatrix}.$$

Jacobian weighted pseudoinverse

Being

$$\boldsymbol{W}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2\end{pmatrix},$$

we obtain

$$\boldsymbol{J}^\dagger_{\boldsymbol{W}}(\bar{\boldsymbol{q}}) = \frac{1}{l}\begin{pmatrix} -0.3536 & 0.3536 \\ 1.0607 & 0.3536 \\ -1.0607 & -0.3536\end{pmatrix}.$$

Kinetic energy

By the uniform mass distribution of the link, which is also considered as a thin rod, we have

$$T_1 = \frac{1}{2}\,m\left(\frac{l}{2}\right)^2\dot{q}_1^2 + \frac{1}{2}\left(\frac{1}{12}\,m\,l^2\right)\dot{q}_1^2 = \frac{1}{6}\,m\,l^2\dot{q}_1^2.$$

For the second link, being

$$\boldsymbol{v}_{c2} = \dot{\boldsymbol{p}}_{c2} = \frac{d}{dt}\left(l\begin{pmatrix} c_1 \\ s_1\end{pmatrix} + \frac{l}{2}\begin{pmatrix} c_{12} \\ s_{12}\end{pmatrix}\right) = l\begin{pmatrix} -s_1\dot{q}_1 - \frac{1}{2}\,s_{12}\,(\dot{q}_1 + \dot{q}_2) \\ c_1\dot{q}_1 + \frac{1}{2}\,c_{12}\,(\dot{q}_1 + \dot{q}_2)\end{pmatrix},$$

it is

$$\left\|\boldsymbol{v}_{c2}\right\|^2 = l^2 \left(\dot{q}_1^2 + \frac{1}{4} \left(\dot{q}_1 + \dot{q}_2\right)^2 + c_2\, \dot{q}_1(\dot{q}_1 + \dot{q}_2) \right),$$

and thus

$$\begin{aligned} T_2 &= \frac{1}{2}\, m \left\|\boldsymbol{v}_{c2}\right\|^2 + \frac{1}{2} \left(\frac{1}{12}\, m\, l^2 \right) \left(\dot{q}_1 + \dot{q}_2\right)^2 \\ &= \frac{1}{6}\, m\, l^2 \left(4\, \dot{q}_1^2 + \dot{q}_2^2 + 2\, \dot{q}_1 \dot{q}_2 + 3\, c_2 \dot{q}_1(\dot{q}_1 + \dot{q}_2)\right). \end{aligned}$$

Similarly, for the third link it is

$$\boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \frac{d}{dt} \left(l \begin{pmatrix} c_1 + c_{12} \\ s_1 + s_{12} \end{pmatrix} + \frac{l}{2} \begin{pmatrix} c_{123} \\ s_{123} \end{pmatrix} \right) = l \begin{pmatrix} -s_1 \dot{q}_1 - s_{12}\left(\dot{q}_1 + \dot{q}_2\right) - \frac{1}{2}\, s_{123}\left(\dot{q}_1 + \dot{q}_2 + \dot{q}_3\right) \\ c_1 \dot{q}_1 + c_{12}\left(\dot{q}_1 + \dot{q}_2\right) + \frac{1}{2}\, c_{123}\left(\dot{q}_1 + \dot{q}_2 + \dot{q}_3\right) \end{pmatrix},$$

so that

$$\begin{aligned} \left\|\boldsymbol{v}_{c3}\right\|^2 = l^2 \Big(& \dot{q}_1^2 + (\dot{q}_1 + \dot{q}_2)^2 + \frac{1}{2} \left(\dot{q}_1 + \dot{q}_2 + \dot{q}_3\right)^2 \\ & + 2\, c_2\, \dot{q}_1(\dot{q}_1 + \dot{q}_2) + c_{23}\, \dot{q}_1(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) + c_3 \left(\dot{q}_1 + \dot{q}_2\right)(\dot{q}_1 + \dot{q}_2 + \dot{q}_3) \Big), \end{aligned}$$

and thus

$$\begin{aligned} T_3 &= \frac{1}{2}\, m \left\|\boldsymbol{v}_{c3}\right\|^2 + \frac{1}{2} \left(\frac{1}{12}\, m\, l^2 \right) \left(\dot{q}_1 + \dot{q}_2 + \dot{q}_3\right)^2 \\ &= \frac{1}{6}\, m\, l^2 \big(7\, \dot{q}_1^2 + 4\, \dot{q}_2^2 + \dot{q}_3^2 + 8\, \dot{q}_1 \dot{q}_2 + 2\, \dot{q}_1 \dot{q}_2 + 2\, \dot{q}_2 \dot{q}_3 \\ &\qquad + 6\, c_2\, \dot{q}_1(\dot{q}_1 + \dot{q}_2) + 3\, c_3 \left((\dot{q}_1 + \dot{q}_2)^2 + \dot{q}_3(\dot{q}_1 + \dot{q}_2)\right) + 3\, c_{23}\, \dot{q}_1(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)\big). \end{aligned}$$

Finally,

$$T = T_1 + T_2 + T_3 = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q}) \dot{\boldsymbol{q}}.$$

Inertia matrix

Rather than rewriting the lengthy terms contained in the contributions T_i of the kinetic energy, we directly evaluate numerically inertia matrix at $\bar{\boldsymbol{q}}$. Factoring out the common symbolic factor $m\, l^2$, one has

$$\boldsymbol{M}(\bar{\boldsymbol{q}}) = m\, l^2 \begin{pmatrix} 5.0000 & 2.1667 & 0.8333 \\ 2.1667 & 1.1667 & 0.3333 \\ 0.8333 & 0.3333 & 0.3333 \end{pmatrix},$$

with inverse

$$\boldsymbol{M}^{-1}(\bar{\boldsymbol{q}}) = \frac{1}{m\, l^2} \begin{pmatrix} 0.6316 & -0.6316 & -0.9474 \\ -0.6316 & 1.3816 & 0.1974 \\ -0.9474 & 0.1974 & 5.1711 \end{pmatrix}.$$

At this stage, we can evaluate the three solutions (5)–(7) obtaining

$$\boldsymbol{\tau}_A = ml \begin{pmatrix} -0.1179 \\ 0.6678 \\ -0.2750 \end{pmatrix}, \qquad \boldsymbol{\tau}_B = ml \begin{pmatrix} -0.3536 \\ 0.6482 \\ -0.2946 \end{pmatrix}, \qquad \boldsymbol{\tau}_C = ml \begin{pmatrix} 0.4950 \\ 0.7189 \\ -0.2239 \end{pmatrix}. \tag{8}$$

The associated accelerations, computed as $\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\bar{\boldsymbol{q}})\boldsymbol{\tau}$, are

$$\ddot{\boldsymbol{q}}_A = \frac{1}{l}\begin{pmatrix} -0.2357 \\ 0.9428 \\ -1.1785 \end{pmatrix}, \qquad \ddot{\boldsymbol{q}}_B = \frac{1}{l}\begin{pmatrix} -0.3536 \\ 1.0607 \\ -1.0607 \end{pmatrix}, \qquad \ddot{\boldsymbol{q}}_C = \frac{1}{l}\begin{pmatrix} 0.0707 \\ 0.6364 \\ -1.4849 \end{pmatrix}. \tag{9}$$

There are no major differences between the three results in (8) and (9), except for the fact that the torque and the acceleration of the first joint in the inertia-weighted case C have an opposite sign with respect to the other two cases. Moreover, solution C has lower acceleration at joint 2 and (even less) at joint 1, due to the fact that the inertia of the robot has been taken into account. This result is consistent with the intuitive idea that in a serial manipulator it is more convenient, in terms of torque/acceleration efforts, to move distal joints in the chain rather than proximal ones.

For comparison, consider also a fourth case in which the solution $\boldsymbol{\tau}$ minimizes the squared norm of the torques:

$$\min H_D = \frac{1}{2}\|\boldsymbol{\tau}\|^2 \quad \Rightarrow \quad \boldsymbol{\tau}_D = \left(\boldsymbol{J}(\bar{\boldsymbol{q}})\boldsymbol{M}^{-1}(\bar{\boldsymbol{q}})\right)^{\dagger}\ddot{\boldsymbol{p}}_d = ml\begin{pmatrix} -0.0339 \\ 0.6748 \\ -0.2680 \end{pmatrix} \quad \Rightarrow \quad \ddot{\boldsymbol{q}}_D = \frac{1}{l}\begin{pmatrix} -0.1937 \\ 0.9008 \\ -1.2205 \end{pmatrix}.$$

This solution has, by construction, the minimum norm of the torque and, compared to the other control torques, also by far the lowest value of torque at the first joint.

Exercise 2

The dynamic model of the actuated pendulum in Fig. 2 is

$$\left(I_c + md^2\right)\ddot{\theta} + mg_0 d \sin\theta = u. \tag{10}$$

In the following, let $I = I_c + md^2$. The assigned cubic trajectory for performing the swing-up maneuver in time T can be written (in normalized time) as

$$\theta_d(t) = \pi\left(-2\tau^3 + 3\tau^2\right), \qquad \tau = \frac{t}{T} \in [0, 1],$$

with acceleration

$$\ddot{\theta}_d(t) = \frac{6\pi}{T^2}\left(1 - 2\tau\right).$$

By inverse dynamics, the torque needed to execute this trajectory is then

$$u_d(t) = \frac{6\pi I}{T^2}\left(1 - 2\tau\right) + mg_0 d \sin\left(\pi\left(-2\tau^3 + 3\tau^2\right)\right), \qquad \tau \in [0, 1]. \tag{11}$$

The torque (11) is the sum of two terms: a linear contribution $u_a(t)$ due to acceleration, which is maximum in absolute value at the start and end of the trajectory, with $u_a(0) = -u_a(T) = 6\pi I/T^2$, and zero at the midpoint $t = T/2$; and a sinusoidal contribution $u_g(t)$ due to gravity, which is zero at the start and end of the trajectory, always positive otherwise, and maximum at the midpoint, with $u_g(T/2) = mg_0 d$. It is easy to show that the superposition of the two torques will have a maximum value which occurs certainly in the first half of the motion (where both terms are positive), but not necessarily at $t = 0$ or $t = T/2$ (i.e., $\tau = 0.5$). Moreover, it is also clear that the faster will be the assigned trajectory (i.e., the smaller the total motion time T), the more will the acceleration term grow and dominate the gravitatonal term, which does not change in fact its profile being dependent only on the configuration θ.

In Fig. 4, using the numerical data

$$I = 1.5 \text{ [Nm s}^2\text{]}, \qquad mg_0 d = 14.715 \text{ [Nm]}, \tag{12}$$

we report for illustration two typical situations, the first for a slow trajectory having $T_s = 1.5$ s, the second for a fast trajectory with $T_f = 0.8$ s. The exchanged roles of the two contributions in assessing the maximum absolute value of the torque is clear.

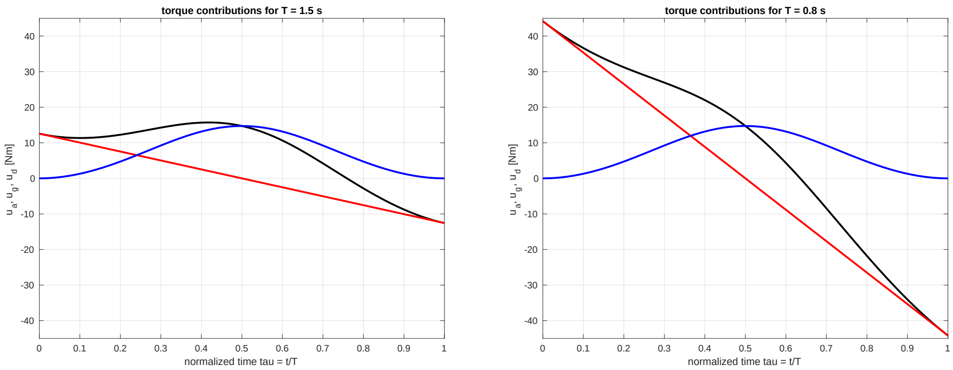


Figure 4: The two contributions $u_a(t)$ (in red) and $u_g(t)$ (in blue) to the total driving torque $u_d(t)$ (in black): slow trajectory with $T_s = 1.5$ s [left]; fast trajectory with $T_f = 0.8$ s [right].

In practice, the available torque u_{max} will not only be larger that mg_0d (the maximum gravity load on the link), but also capable of providing a sufficient acceleration at the time instants $t = 0$ and $t = T$ (where $u_d(t) = u_a(t)$), so as to quickly start and stop motion. Therefore, given a sufficiently large maximum torque u_{max}, the minimum time T_{min} will be specified by the value of the acceleration component at $t = 0$. Thus,

$$u_d(0) = u_a(0) = \frac{6\pi I}{T^2} = u_{max} \qquad \Rightarrow \qquad T_{min} = \sqrt{\frac{6\pi I}{u_{max}}}. \tag{13}$$

Using the same data as in (12) and setting $u_{max} = 20$ [Nm] leads to the optimal solution of Fig. 5, with minimum time $T_{min} = 1.1890$ s, as evaluated from (13).

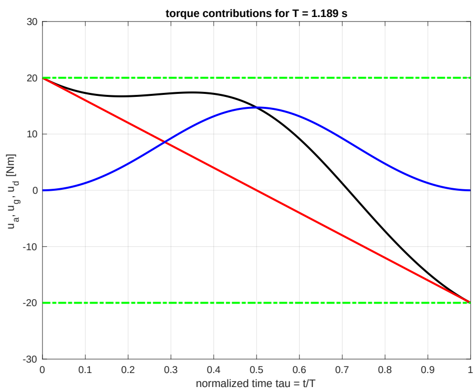


Figure 5: The two contributions $u_a(t)$ (in red) and $u_g(t)$ (in blue) in the time-optimal driving torque $u_d(t)$ (in black).

Finally, consider again the same pendulum with $u_{max} = 18$ [Nm] and set the motion time to

$T = 1$ s. The corresponding torque profile will be unfeasible since

$$u_d(0) = \frac{6I\pi}{T^2} = 28.27 > 18 = u_{max}.$$

Therefore, the minimum uniform time scaling factor to recover feasibility is computed as

$$k = \sqrt{\frac{6\pi I}{u_{max}}} = 1.2533 > 1. \tag{14}$$

The scaled trajectory is slower, with a longer duration $T_s = kT = 1.2533$ s. Feasibility is automatically recovered, with the bound being saturated only at instants with the largest unfeasible torque (in absolute value). Note that, when computing the scaling factor, gravity needs not to be removed because the maximum violating torque already occurs at an instant with zero gravity contribution. The effect of uniform time scaling on the unfeasible trajectory is illustrated in Fig. 6.

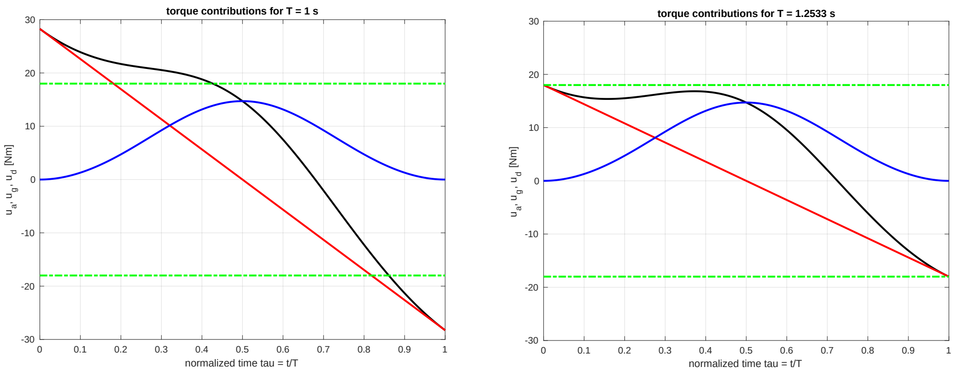


Figure 6: The contributing and total torques for an unfeasible trajectory with $T = 1$ s [left] and after uniform time scaling by $k = 1.2533$ [right].

Exercise 3

The dynamic model of the system of two masses with a spring in between is

$$\begin{aligned} m_1\ddot{q}_1 + K(q_1 - q_2) &= F_1 \\ m_2\ddot{q}_2 + K(q_2 - q_1) &= F_2. \end{aligned} \tag{15}$$

With the control (1), the closed-loop system becomes

$$\begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} K_{D1} & \\ & K_{D2} \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} K + K_{P1} & -K \\ -K & K + K_{P2} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} K_{P1}q_{1d} \\ K_{P2}q_{2d} \end{pmatrix}. \tag{16}$$

At an equilibrium ($\dot{\boldsymbol{q}} = \ddot{\boldsymbol{q}} = \boldsymbol{0}$), it is then

$$\bar{\boldsymbol{K}}\boldsymbol{q} = \begin{pmatrix} K + K_{P1} & -K \\ -K & K + K_{P2} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} K_{P1}q_{1d} \\ K_{P2}q_{2d} \end{pmatrix} = \boldsymbol{K}_P\boldsymbol{q}_d = \bar{\boldsymbol{q}}_d, \tag{17}$$

where $\boldsymbol{K}_P = \text{diag}\,\{K_{P1}, K_{P2}\}$ Since the stiffness/control matrix $\bar{\boldsymbol{K}}$ is nonsingular ($\det \bar{\boldsymbol{K}} = K(K_{P1} + K_{P2}) + K_{P1}K_{P2} > 0$), equation (17) can be solved for the unique equilibrium configuration

$$\bar{\boldsymbol{q}} = \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix} = \bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{q}}_d = \frac{1}{K(K_{P1} + K_{P2}) + K_{P1}K_{P2}}\begin{pmatrix} K(K_{P1}q_{1d} + K_{P2}q_{2d}) + K_{P1}K_{P2}q_{1d} \\ K(K_{P1}q_{1d} + K_{P2}q_{2d}) + K_{P1}K_{P2}q_{2d} \end{pmatrix}. \tag{18}$$

The equilibrium position $\bar{q}_i$ of each mass is in general different from its target value q_{id}, for $i = 1, 2$. From eq. (18), it follows also that only when $q_{1d} = q_{2d} = q_d$ (a case excluded here), it is then $\bar{q}_1 = \bar{q}_2 = q_d$.

To show that the unique equilibrium state $(\bar{\boldsymbol{q}}, \boldsymbol{0})$ is globally asymptotically stable (in fact, exponentially stable since the system is linear) one can follow in principle two ways.

The first is to leverage the linearity of the closed-loop system dynamics (16). It can be recognized that the three matrices $\boldsymbol{M} = \text{diag}\,\{m_1, m_2\}$, $\boldsymbol{D} = \text{diag}\,\{K_{D1}, K_{D2}\}$ and $\bar{\boldsymbol{K}}$ are all positive definite: this is a sufficient condition for concluding on asymptotic stability of mechanical systems in this form. Along the same lines, one can apply tools from linear systems theory to draw the same conclusion: e.g., by computing the four eigenvalues of system (16), once put into a state-space format, and verifying that their real parts are on the left-hand side of the complex plane.

The second way is to follow, as requested, a Lyapunov/LaSalle analysis. Consider the following function as natural Lyapunov candidate:

$$V = \frac{1}{2}\, m_1 \dot{q}_1^2 + \frac{1}{2}\, m_2 \dot{q}_2^2 + \frac{1}{2}\, K\left(q_1 - q_2\right)^2 + \frac{1}{2}\, K_{P1}\left(q_{1d} - q_1\right)^2 + \frac{1}{2}\, K_{P2}\left(q_{2d} - q_2\right)^2 \geq 0. \tag{19}$$

This function is composed by the total energy of the system (kinetic and elastic potential in the first three terms) and by the equivalent elastic potential energy introduced by the control laws (1). Indeed, V is always non-negative and is zero only at the equilibrium $\boldsymbol{q} = \bar{\boldsymbol{q}}$, with $\dot{\boldsymbol{q}} = \boldsymbol{0}$. In fact, setting to zero the gradient of (19) with respect to $\boldsymbol{q}$, as a necessary condition for a minimum, one has

$$\nabla_{\boldsymbol{q}} V = \left(\frac{\partial V}{\partial \boldsymbol{q}}\right)^T = \bar{\boldsymbol{K}}\boldsymbol{q} - \bar{\boldsymbol{q}}_d = \boldsymbol{0},$$

which is exactly (17) and thus uniquely solved by $\bar{\boldsymbol{q}}$. Moreover, since $\partial^2 V / \partial \boldsymbol{q}^2 = \bar{\boldsymbol{K}} > 0$, this will be a minimum of V. By taking the time derivative of V and evaluating it along the trajectories of the closed-loop system (16), we obtain after few simplifications:

$$\dot{V} = \ldots = -\bar{\boldsymbol{D}}\dot{\boldsymbol{q}} \leq 0 \qquad \Rightarrow \qquad \dot{V} = 0 \iff \dot{\boldsymbol{q}} = \boldsymbol{0}.$$

When $\dot{\boldsymbol{q}} = \boldsymbol{0}$, the closed-loop system (16) simplifies to

$$\bar{\boldsymbol{M}}\ddot{\boldsymbol{q}} + \bar{\boldsymbol{K}}\boldsymbol{q} = \bar{\boldsymbol{q}}_d \qquad \Rightarrow \qquad \ddot{\boldsymbol{q}} = \bar{\boldsymbol{M}}^{-1}\left(\bar{\boldsymbol{q}}_d - \bar{\boldsymbol{K}}\boldsymbol{q}\right) \qquad \Rightarrow \qquad \ddot{\boldsymbol{q}} = \boldsymbol{0} \iff \boldsymbol{q} = \bar{\boldsymbol{q}},$$

thanks to (17). Therefore, by LaSalle theorem, the system trajectories will globally converge to the unique equilibrium state $(\bar{\boldsymbol{q}}, \boldsymbol{0})$ (i.e., the single element in the largest invariant set contained in the set of states corresponding to $\dot{V} = 0$), which is then asymptotically stable. Again, being the considered system linear, asymptotic stability is equivalent here to exponential stability.

Finally, a modification of the control laws (1) is needed in order to enforce $\bar{\boldsymbol{q}} = \boldsymbol{q}_d$, i.e., to eliminate the constant final position errors at steady state. A straightforward solution would be to cancel the effect of elasticity on both masses, namely defining the new control laws as

$$F_1 = K_{P1}\left(q_{1d} - q_1\right) - K_{D1}\dot{q}_1 + K\left(q_1 - q_2\right), \qquad F_2 = K_{P2}\left(q_{2d} - q_2\right) - K_{D2}\dot{q}_2 + K\left(q_2 - q_1\right). \tag{20}$$

This would fully decouple the behavior of the two controlled masses, being the closed-loop system

$$\begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} K_{D1} & \\ & K_{D2} \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} K_{P1} & \\ & K_{P2} \end{pmatrix}\begin{pmatrix} q_1 - q_{1d} \\ q_2 - q_{2d} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{21}$$

It is easy to verify that mass m_i would independently reach its target position q_{id} at the equilibrium, for $i = 1, 2$.

However, the control modification (20) destroys the decentralized structure of (1), being the input force to each mass also a function of the position of the other mass. If the original decentralized structure has to be kept, one can include integral terms in the two controllers, i.e.,

this is probably an error of the professor: is q_i without the dot in the integral

$$F_i = K_{Pi}\left(q_{id} - \dot{q}_i\right) - K_{Di}\dot{q}_i + K_{Ii}\int \left(q_{id} - \dot{q}_i\right) dt, \qquad i = 1, 2,$$

and then study the conditions for the control gains K_{Pi}, K_{Di} and K_{Ii} ($i = 1, 2$) that guarantee asymptotic stability of the closed-loop system. This choice has the advantage of requiring no information at all about the parameters of the dynamic system (except for some bounds). A simpler solution is to include feedforward terms in the control laws (1), i.e.,

$$F_i = K_{Pi}\left(q_{id} - \dot{q}_i\right) - K_{Di}\dot{q}_i + F_{i,ffw}, \qquad i = 1, 2, \tag{22}$$

with

$$\begin{aligned} \boldsymbol{F}_{ffw} = \begin{pmatrix} F_{1,ffw} \\ F_{2,ffw} \end{pmatrix} &= \bar{\boldsymbol{K}}\boldsymbol{q}_d - \bar{\boldsymbol{q}}_d \\ &= \begin{pmatrix} K + K_{P1} & -K \\ -K & K + K_{P2} \end{pmatrix} \begin{pmatrix} q_{1d} \\ q_{2d} \end{pmatrix} - \begin{pmatrix} K_{P1}q_{1d} \\ K_{P2}q_{2d} \end{pmatrix} \\ &= \begin{pmatrix} K & -K \\ -K & K \end{pmatrix} \begin{pmatrix} q_{1d} \\ q_{2d} \end{pmatrix} = \begin{pmatrix} K\left(q_{1d} - q_{2d}\right) \\ K\left(q_{2d} - q_{1d}\right) \end{pmatrix}. \end{aligned} \tag{23}$$

The new equilibrium conditions are obtained by modifying accordingly (17) as

$$\bar{\boldsymbol{K}}\boldsymbol{q} = \bar{\boldsymbol{q}}_d + \boldsymbol{F}_{ffw} = \bar{\boldsymbol{q}}_d + \bar{\boldsymbol{K}}\boldsymbol{q}_d - \bar{\boldsymbol{q}}_d = \bar{\boldsymbol{K}}\boldsymbol{q}_d, \qquad \text{with } \bar{\boldsymbol{K}} > 0,$$

which has the unique solution $\boldsymbol{q} = \boldsymbol{q}_d$ as desired. The Lyapunov/LaSalle analysis for the controller (22),(23) follows then in a similar way by using

$$V' = V + \left(\boldsymbol{a}_d - \boldsymbol{q}\right)^T \boldsymbol{F}_{ffw}, \qquad \text{with } \boldsymbol{a}_d = \begin{pmatrix} \dfrac{3q_{1d} + q_{2d}}{4} \\ \dfrac{q_{1d} + 3q_{2d}}{4} \end{pmatrix},$$

which can be shown to be a suitable Lyapunov candidate, i.e., $V' \geq 0$ and $V' = 0$ if and only if $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{q}_d, \boldsymbol{0})$, obtaining eventually global asymptotic (exponential) stability of the unique closed-loop equilibrium state $(\boldsymbol{q}_d, \boldsymbol{0})$. The actual verification is left as an exercise to the reader.

* * * * *

Robotics 2
Midterm Test – April 24, 2024

Exercise #1

Consider the 3R planar robot in the configuration $\boldsymbol{q}$ shown in Fig. 1, controlled by the joint velocity $\dot{\boldsymbol{q}} \in \mathbb{R}^3$.

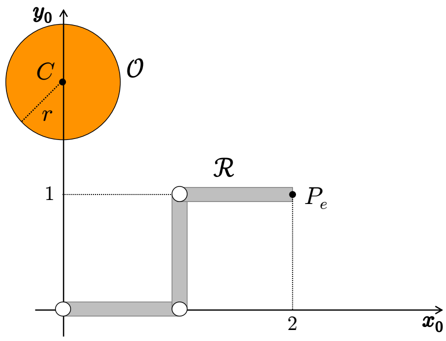


Figure 1: The 3R planar robot $\mathcal{R}$ with a single circular obstacle $\mathcal{O}$ in its workspace.

i) A desired Cartesian velocity $\boldsymbol{v}_e \in \mathbb{R}^2$ should be imposed to the end-effector point P_e of the robot $\mathcal{R}$, while maximizing the distance function $H_{\text{dist}}(\boldsymbol{q})$ between the robot body (i.e., its kinematic skeleton) and a circular obstacle $\mathcal{O}$ centered at $C = (0, 2)$ and of radius $r = 0.5$ m, as expressed by

$$H_{\text{dist}}(\boldsymbol{q}) = \min_{\boldsymbol{p}\in\mathcal{R}, \boldsymbol{o}\in\mathcal{O}} \|\boldsymbol{p}(\boldsymbol{q}) - \boldsymbol{o}\| = \|P_m(\boldsymbol{q}) - O_m\|,$$

where $P_m(\boldsymbol{q})$ and O_m are the closest points, respectively on the robot and on the obstacle, when the robot is in the given configuration $\boldsymbol{q}$. Determine the symbolic expression and the numerical value of the velocity command $\dot{\boldsymbol{q}}_{PG}$ according to the Projected Gradient (PG) method when $\boldsymbol{v}_e = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ m/s. Verify your result and compute also the resulting velocity $\boldsymbol{v}_m \in \mathbb{R}^2$ of the point P_m.

ii) Suppose now that the robot $\mathcal{R}$ has two tasks assigned, ordered by priority: the first task is to impose the same previous velocity $\boldsymbol{v}_e$ to the point P_e; the second task is to impose the velocity $\boldsymbol{v}_m$ obtained in the previous item to the point P_m. Determine the symbolic expression and the numerical value of the velocity command $\dot{\boldsymbol{q}}_{TP}$ according to the Task Priority (TP) method. Do the joint velocities $\dot{\boldsymbol{q}}_{TP}$ and $\dot{\boldsymbol{q}}_{PG}$ have different directions in the joint space or not? Explain why. What if we choose $\boldsymbol{v}_m = \alpha\,(P_m - O_m)$, for some $\alpha > 0$, as desired velocity for P_m?

Exercise #2

Figure 2 shows a 2R robot moving under gravity in the 3D space, with the associated Denavit-Hartenberg (D-H) frames. The following assumptions are made on the position of the center of mass and on the barycentric inertia of the two links, when these quantities are expressed in the frame attached to the respective link[1]:

$$^1\boldsymbol{r}_{c1} = \begin{pmatrix} r_{c1,x} \\ r_{c1,y} \\ r_{c1,z} \end{pmatrix}, \; ^2\boldsymbol{r}_{c2} = \begin{pmatrix} r_{c2,x} \\ 0 \\ 0 \end{pmatrix}, \; ^1\boldsymbol{I}_{c1} = \begin{pmatrix} I_{c1,xx} & I_{c1,xy} & I_{c1,xz} \\ I_{c1,xy} & I_{c1,yy} & I_{c1,yz} \\ I_{c1,xz} & I_{c1,yz} & I_{c1,zz} \end{pmatrix}, \; ^2\boldsymbol{I}_{c2} = \text{diag}\{I_{c2,xx}, I_{c2,yy}, I_{c2,zz}\}.$$

Note in particular that the center of mass of the first link is not on the axis of joint 1.

i) Derive the dynamic model in the form

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau},$$

and a factorization matrix $\boldsymbol{S}$ such that $\boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} = \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and matrix $\dot{\boldsymbol{M}} - 2\boldsymbol{S}$ is skew-symmetric.

[1] All given symbolic quantities are assumed to be generically non-zero.

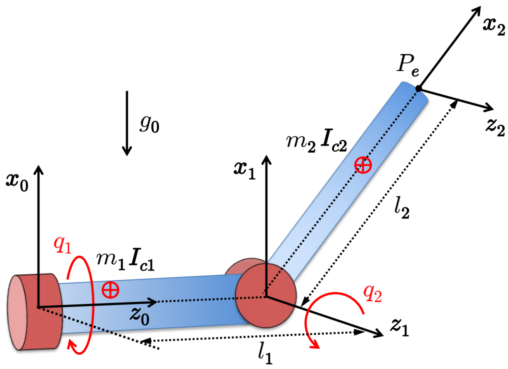


Figure 2: A 2R spatial robot with its D-H frames.

ii) Provide a minimal linear parametrization of the model

$$\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{\tau},$$

giving the symbolic expression of the dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^p$ and of the $2 \times p$ regressor matrix $\boldsymbol{Y}$. Assume that the acceleration of gravity g_0 is known.

iii) What is the expression of the torque $\boldsymbol{\tau}_d(t) = (\tau_{d1}(t), \tau_{d2}(t))$ needed to execute the desired motion $\boldsymbol{q}_d(t) = (q_{d1}(t), q_{d2}(t)) = (2\,t, \pi/4)$, with $t \in [0, \pi]$?

iv) When $\boldsymbol{\tau} = \boldsymbol{0}$, find as many as possible (unforced) equilibrium states $\boldsymbol{x}_e = (\boldsymbol{q}_e, \boldsymbol{0})$ of the robot —specify also the assumptions under which such equilibrium configurations $\boldsymbol{q}_e$ exist.

v) Choose mechanical parameters for the links such that the gravity term $\boldsymbol{g}(\boldsymbol{q})$ vanishes for all $\boldsymbol{q}$.

vi) Under the conditions found in item *v)*, with the robot at rest and being $\boldsymbol{\tau} = \boldsymbol{0}$, a force $\boldsymbol{F}_e \in \mathbb{R}^2$ is applied to the tip of the second link along a direction lying in the horizontal plane parallel to the plane $(\boldsymbol{y}_0, \boldsymbol{z}_0)$. Will the resulting tip acceleration $\ddot{\boldsymbol{p}}_e$ be in the same direction of the applied force $\boldsymbol{F}_e$ or not? Elaborate your answer.

Exercise #3

With reference to the standard recursive Newton-Euler algorithm for inverse dynamics, consider the calling instruction

$$\boldsymbol{u} = NE_0(\boldsymbol{q}, \boldsymbol{0}, \boldsymbol{0})$$

for a 6R robot in a nonsingular configuration $\boldsymbol{q}$. What will be the output $\boldsymbol{u}$ when the robot end-effector is *i)* in free space, or *ii)* subject to a known active wrench (force and moment) $\boldsymbol{F}_e \in \mathbb{R}^6$?

Exercise #4

Consider a rigid robot manipulator with n revolute joints, without external contact forces/moments and dissipative effects. The components of the motor torque $\boldsymbol{\tau} \in \mathbb{R}^n$ are bounded as

$$|\tau_i| \leq T_i, \qquad \text{with } T_i \geq 5 \cdot \max_{\boldsymbol{q}} |g_i(\boldsymbol{q})|\,, \qquad i = 1, \ldots, n,$$

namely with bounds that are large enough to guarantee that the robot always sustains at least its own weight under gravity (here, with a conservative margin factor of 5). Let the robot state at time $t = t_0$ be $\boldsymbol{x}(t_0) = (\boldsymbol{q}(t_0), \dot{\boldsymbol{q}}(t_0))$, with a total robot energy $E(t_0) = E_0$.

i) If the robot is in a generic state $\boldsymbol{x}(t_0) = (\boldsymbol{q}_0, \dot{\boldsymbol{q}}_0)$ with $\dot{\boldsymbol{q}}_0 \neq \boldsymbol{0}$, choose the torque $\boldsymbol{\tau}_0 = \boldsymbol{\tau}(t_0)$ to be delivered by the motors so that the total energy E will instantaneously decrease as much as possible.

ii) Suppose now that the robot is in a state $\boldsymbol{x}(t_0) = (\boldsymbol{q}_0, \boldsymbol{0})$, namely **at rest** and in a configuration $\boldsymbol{q}_0$ such that $\boldsymbol{g}(\boldsymbol{q}_0) \neq \boldsymbol{0}$. Choose again the motor torque $\boldsymbol{\tau}_0$ so that the total energy E will decrease. *Hint:* If this is not possible, choose $\boldsymbol{\tau}_0$ so that $\dot{E}$ will instantaneously decrease.

[210 minutes (3.5 hours); open books]

Solution

April 24, 2024

Exercise #1

The needed formulas to be evaluated at $\boldsymbol{q}_0 = (0, \pi/2, -\pi/2)$ are the following.

i) For the Projected Gradient (PG) method:

$$\begin{aligned}\dot{\boldsymbol{q}}_{PG} &= \boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_e + \boldsymbol{P}_e(\boldsymbol{q}_0)\nabla_{\boldsymbol{q}} H_{\text{dist}}(\boldsymbol{q}_0) = \boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_e + \left(\boldsymbol{I} - \boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{J}_e(\boldsymbol{q}_0)\right)\nabla_{\boldsymbol{q}} H_{\text{dist}}(\boldsymbol{q}_0)\\ &= \nabla_{\boldsymbol{q}} H_{\text{dist}}(\boldsymbol{q}_0) + \boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\left(\boldsymbol{v}_e - \boldsymbol{J}_e(\boldsymbol{q}_0)\nabla_{\boldsymbol{q}} H_{\text{dist}}(\boldsymbol{q}_0)\right),\end{aligned}$$

with

$$H_{\text{dist}}(\boldsymbol{q}_0) = \|P_m(\boldsymbol{q}_0) - C\| - r, \qquad \nabla_{\boldsymbol{q}} H_{\text{dist}}(\boldsymbol{q}_0) = \frac{1}{2}\,\boldsymbol{J}_m(\boldsymbol{q}_0)\,\frac{P_m(\boldsymbol{q}_0) - C}{\|P_m(\boldsymbol{q}_0) - C\|},$$

where $P_m(\boldsymbol{q}_0) = (1, 1)$. Moreover,

$$\boldsymbol{v}_m = \boldsymbol{J}_m(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{PG}.$$

ii) For the Task Priority (TP) method:

$$\dot{\boldsymbol{q}}_{TP} = \boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_e + \left(\boldsymbol{J}_m(\boldsymbol{q}_0)\boldsymbol{P}_e(\boldsymbol{q}_0)\right)^{\#}\left(\boldsymbol{v}_2 - \boldsymbol{J}_m(\boldsymbol{q}_0)\boldsymbol{J}_e^{\#}(\boldsymbol{q}_0)\boldsymbol{v}_e\right),$$

where in the first case (A) $\boldsymbol{v}_2 = \boldsymbol{v}_m$ and in the second case (B) $\boldsymbol{v}_2 = \left(1 - \frac{r}{\|P_m - C\|}\right)(P_m - C)$ (namely, a velocity along the direction of the gradient of the Cartesian clearance between robot and obstacle, with norm equal to $H_{\text{dist}}(\boldsymbol{q}_0)$).

The MATLAB code `Ex1_Midterm.m`, whose source text is given below, has been used for both items *i)* and *ii)*. The output is attached at the end of this pdf file.

The result is that $\dot{\boldsymbol{q}}_{TP,A} = \dot{\boldsymbol{q}}_{PG}$: the TP method returns in this case exactly the requested velocity $\boldsymbol{v}_m$ for the robot point P_m. This is because $\boldsymbol{v}_m$ is certainly realizable together with the velocity $\boldsymbol{v}_E$ of the end-effector, having been defined as the natural outcome of the PG method for point P_m. On the other hand, $\dot{\boldsymbol{q}}_{TP,B} \neq \dot{\boldsymbol{q}}_{PG}$, both in direction and intensity in the joint space. In this case, the requested velocity for the second task, one that would move the closest robot point P_m in the farthest away direction from the obstacle, is not compatible with the velocity $\boldsymbol{v}_E$ of the end-effector.[2] As a result, $\boldsymbol{J}_m(\boldsymbol{q}_0)\dot{\boldsymbol{q}}_{TP,B}$ will not be equal to the desired $\boldsymbol{v}_2$ in case B.

```
% Robotics 2 Midterm
% 24 April 2024
% Ex #1
%
% Projected Gradient (PG) method (with obstacle avoidance)
% and Task Priority (TP) method (with two variants)
% for a 3R planar robot

clear all, clc

disp('*** Projected Gradient and Task Priority (TP) methods for a 3R planar robot ***')
disp(' ')

syms q1 q2 q3 real

q=[q1 q2 q3]';

disp('kinematics of the 3R planar robot (links of unitary length)')
```

[2]This should not be unexpected for generic directions $\boldsymbol{v}_m$ and $\boldsymbol{v}_e$, as the sum of the dimensions of these two tasks ($m = m_1 + m_2 = 4$) is larger than the dimension of the joint space ($n = 3$).

```
pe_simb=[cos(q1)+cos(q1+q2)+cos(q1+q2+q3);sin(q1)+sin(q1+q2)+sin(q1+q2+q3)]
pm_simb=[cos(q1)+cos(q1+q2);sin(q1)+sin(q1+q2)]

Je_simb=jacobian(pe_simb,q)
Jm_simb=jacobian(pm_simb,q)

disp('at the given configuration (as in the text figure)')

q0=[0;pi/2;-pi/2]

disp('numerical kinematics')

pe=subs(pe_simb,q,q0)
pm=subs(pm_simb,q,q0)

Je=subs(Je_simb,q,q0)
Jm=subs(Jm_simb,q,q0)

disp('clearance')

C=[0;2];r=0.5;
H=norm(pm-C)-r

disp('desired end-effector velocity')

ve=[0 1]'

disp('projected gradient (PG) method')

nablaH=0.5*Jm'*(pm-C)/norm(pm-C)
nablaH=eval(nablaH)
alfa1=1 % works also for a different alfa1>0

Je_pinv=pinv(Je)
dq_PG=alfa1*nablaH+Je_pinv*(ve-alfa1*Je*nablaH)
dq_PG=eval(dq_PG)

disp('check PG solution')

ve_check=eval(Je*dq_PG)
vm=eval(Jm*dq_PG)

% TP with two cases:  A & B

disp('Task Priority (TP) - case A')

Pe=eye(3)-Je_pinv*Je
vm_A=vm

dq_TP_A=Je_pinv*ve+pinv(Jm*Pe)*(vm_A-Jm*Je_pinv*ve)
dq_TP_A=eval(dq_TP_A)

disp('check TP_A solution')

ve_check=eval(Je*dq_TP_A)
vm_check=eval(Jm*dq_TP_A)

disp('Task Priority (TP) - case B')

alfa2=1   % minimum error = vm-Jm*dq_TP_B is obtained with alfa2=0.9
vm_B=alfa2*(1-(r/norm(pm-C)))*(pm-C)
vm_B=eval(vm_B)

dq_TP_B=Je_pinv*ve+pinv(Jm*Pe)*(vm_B-Jm*Je_pinv*ve)
```

```
dq_TP_B=eval(dq_TP_B)

disp('check TP_B solution')

ve_check=eval(Je*dq_TP_B)
vm_check=eval(Jm*dq_TP_B)
vm_check_error=eval(vm_B-Jm*dq_TP_B)

% end
```

Exercise #2

It is useful (though not strictly necessary) to use the moving frame algorithm for deriving the kinetic energy of the links, as well as the Denavit-Hartenberg (D-H) homogeneous transformation matrices for the potential energy due to gravity. For this, one should start from the table of D-H parameters given in Tab. 1.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	l_1	q_1
2	0	l_2	0	q_2

Table 1: D-H parameters corresponding to the frames in Fig. 2.

i) The dynamic model computations provide the following terms in symbolic form:

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 + a_2 \cos^2 q_2 & 0 \\ 0 & a_3 \end{pmatrix} =$$

$$\begin{pmatrix} I_{c1,yy} + I_{c2,xx} + m_1(r_{c1,x}^2 + r_{c1.z}^2) + \left(I_{c2,yy} - I_{c2,xx} + m_2(l_2 + r_{c2.x})^2\right)\cos^2 q_2 & 0 \\ 0 & I_{c2,zz} + m_2(l_2 + r_{c2,x})^2 \end{pmatrix}$$

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} -2a_2 \dot{q}_1 \dot{q}_2 \cos q_2 \sin q_2 \\ a_2 \dot{q}_1^2 \cos q_2 \sin q_2 \end{pmatrix} = \begin{pmatrix} -a_2 \dot{q}_2 \cos q_2 \sin q_2 & -a_2 \dot{q}_1 \cos q_2 \sin q_2 \\ a_2 \dot{q}_1 \cos q_2 \sin q_2 & 0 \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{S}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$$

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{q}) &= \begin{pmatrix} a_4\, g_0 \cos q_1 \sin q_2 + a_5\, g_0 \sin q_1 + a_6\, g_0 \cos q_1 \\ a_4\, g_0 \cos q_1 \sin q_2 \end{pmatrix} \\ &= \begin{pmatrix} -m_2\,(l_2 + r_{c2,x})\, g_0 \cos q_1 \sin q_2 - m_1 r_{c1,x}\, g_0 \sin q_1 - m_1 r_{c1,z}\, g_0 \cos q_1 \\ -m_2\,(l_2 + r_{c2,x})\, g_0 \cos q_1 \sin q_2 \end{pmatrix}. \end{aligned}$$

ii) A linear parametrization of the dynamic model, $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\boldsymbol{a} = \boldsymbol{\tau}$, of dimension $p = 6$ uses the following dynamic coefficients

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} \begin{pmatrix} I_{c1,yy} + I_{c2,xx} + m_1\left(r_{c1,x}^2 + r_{c1.z}^2\right) \\ I_{c2,yy} - I_{c2,xx} + m_2\,(l_2 + r_{c2,x})^2 \\ I_{c2,zz} + m_2\,(l_2 + r_{c2,x})^2 \\ -m_2\,(l_2 + r_{c2,x}) \\ -m_1 r_{c1,x} \\ -m_1 r_{c1,z} \end{pmatrix}$$

and the 2×6 regressor matrix

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 & \cos^2 q_2\, \ddot{q}_1 - 2\sin q_2 \cos q_2\, \dot{q}_1 \dot{q}_2 & 0 & g_0 \sin q_1 \cos q_2 & g_0 \sin q_1 & g_0 \cos q_1 \\ 0 & \sin q_2 \cos q_2\, \dot{q}_1 \dot{q}_2 & \ddot{q}_2 & g_0 \cos q_1 \sin q_2 & 0 & 0 \end{pmatrix}.$$

iii) The inverse dynamics torque associated to the desired motion $\boldsymbol{q}_d(t) = (2\,t, \pi/4)$ is given by

$$\boldsymbol{\tau}_d(t) = \begin{pmatrix} a_5 g_0 + a_4 \frac{\sqrt{2}}{2}\, g_0 \sin 2t + a_6 g_0 \cos 2t \\ 2a_2 + a_4 \frac{\sqrt{2}}{2}\, g_0 \cos 2t \end{pmatrix}$$

iv) The (unforced) equilibrium configurations $\boldsymbol{q}_e$ are found by solving the condition $\boldsymbol{g}(\boldsymbol{q}_e) = \boldsymbol{0}$. In particular, zeroing the second component of $\boldsymbol{g}$ leads to the two following cases:

$$\cos q_{1,e} = 0 \quad \Rightarrow \quad q_{1,e} = \{\pm\frac{\pi}{2}\} \quad \Rightarrow \quad q_{2,e} = \pi \pm \arccos \frac{a_5}{a_4} \qquad \Leftarrow \quad \text{valid only if } |a_5| \leq |a_4|;$$

$$\sin q_{2,e} = 0 \quad \Rightarrow \quad q_{2,e} = \left\{ \begin{array}{lll} 0 & \Rightarrow & q_{1,e} = 2 \arctan \dfrac{a_4 + a_5 \pm \sqrt{(a_4 + a_5)^2 + a_6^2}}{a_6} \\ & & \\ \pi & \Rightarrow & q_{1,e} = \pm 2 \arctan \dfrac{a_4 - a_5 + \sqrt{(a_4 + a_5)^2 + a_6^2}}{a_6} \end{array} \right\} \quad \Leftarrow \quad \text{always valid.}$$

v) The mechanical conditions for having $\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}$ for all $\boldsymbol{q}$ are:

$$r_{c2,x} = -l_2 \quad \Rightarrow \quad a_4 = 0; \qquad r_{c1,x} = 0 \quad \Rightarrow \quad a_5 = 0; \qquad r_{c1,z} = 0 \quad \Rightarrow \quad a_6 = 0.$$

vi) With the robot at rest ($\dot{\boldsymbol{q}} = \boldsymbol{0}$), with $\boldsymbol{g}(\boldsymbol{q}) \equiv \boldsymbol{0}$ from the previous item, and with $\boldsymbol{\tau} = \boldsymbol{0}$, the dynamics in reaction to a force $\boldsymbol{F}_e \in \mathbb{R}^2$ applied to the end-effector only in the plane $(\boldsymbol{y}_0, \boldsymbol{z}_0)$ is

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{J}_{e,yz}^T(\boldsymbol{q})\boldsymbol{F}_e, \qquad \text{with } \boldsymbol{J}_{e,yz}(\boldsymbol{q}) = \frac{\partial \begin{pmatrix} p_{e,y} \\ p_{e,z} \end{pmatrix}}{\partial \boldsymbol{q}} = \begin{pmatrix} l_2 \cos q_1 \cos q_2 & -l_2 \sin q_1 \sin q_2 \\ 0 & -l_2 \cos q_2 \end{pmatrix}.$$

Therefore, the end-effector acceleration is computed as

$$\begin{aligned} \ddot{\boldsymbol{p}}_e &= \left(\boldsymbol{J}_{e,yz}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{J}_{e,yz}^T(\boldsymbol{q})\right)\boldsymbol{F}_e = \boldsymbol{M}_p^{-1}(\boldsymbol{q})\boldsymbol{F}_e \\ &= \begin{pmatrix} l_2^2\left(F_{e,y}\left((a_1 + a_2\cos^2 q_2)\cos^2 q_1 \cos^2 q_2 + a_3 \sin^2 q_1 \sin^2 q_2\right) + F_{e,z} a_3 \sin q_1 \sin q_2 \cos q_2\right) \\ a_3 l_2^2 \cos q_2 \left(F_{e,y} \sin q_1 \sin q_2 + F_{e,z} \cos q_2\right) \end{pmatrix}. \end{aligned}$$

where the matrix premultiplying $\boldsymbol{F}_e$ is the inverse of the robot inertia matrix at the end-effector level, restricted to the plane of interest in the Cartesian space. Being this matrix non-diagonal, the direction of the acceleration $\ddot{\boldsymbol{p}}_e$ will differ from the direction of the applied force $\boldsymbol{F}_e$ (although the scalar product $\boldsymbol{F}_e^T\ddot{\boldsymbol{p}}_e > 0$, out of singularities of the Jacobian $\boldsymbol{J}_{e,yz}$).

All the above computations have been done first by hand. The MATLAB code `Ex1_Midterm.m`, whose source text is given below, has been used to verify the correctness of all items from *i)* to *vi)*. In the source, some extra steps useful for symbolic manipulation and further comments are preceded by '`% ADL:`'). The output is attached at the end of this pdf file.

```
% Robotics 2 Midterm
% 24 April 2024
% Ex #2
%
% Dynamic model and analysis of a 2-dof RR spatial robot

clear all, clc

disp('*** Dynamic model and analysis of a 2-dof RR spatial robot ***')
disp(' ')

disp('kinematic quantities, possibly used later in dynamics')
```

```
syms alpha d a theta real
syms q1 q2 L1 L2 real

% number of joints
N=2;

% DH table of parameters
DHTABLE = [ -pi/2      0        sym(‘L1’)  sym(‘q1’);
              0      sym(‘L2’)       0      sym(‘q2’)];

% DH transformation matrix
TDH = [ cos(theta) -sin(theta)*cos(alpha)  sin(theta)*sin(alpha) a*cos(theta);
        sin(theta)  cos(theta)*cos(alpha) -cos(theta)*sin(alpha) a*sin(theta);
          0             sin(alpha)             cos(alpha)           d;
          0                  0                      0               1];

A = cell(1,N);
for i = 1:N
alpha = DHTABLE(i,1);
a = DHTABLE(i,2);
d = DHTABLE(i,3);
theta = DHTABLE(i,4);
Ai = subs(TDH);
end

A01=A1;
R01=A01(1:3,1:3)
p01=A01(1:3,4)
r01=simplify(R01’*p01)
A12=A2;
R12=A12(1:3,1:3)
p12=A12(1:3,4)
r12=simplify(R12’*p12)

disp(‘end-effector position and associated Jacobian (in base frame)’)

p_ee_hom=A01*A12*[0 0 0 1]’;
p_ee=p_ee_hom(1:3)

q=[q1 q2]’;
J_ee=jacobian(p_ee,q)

disp(‘dynamic parameters’)

syms dq1 dq2 ddq1 ddq2 g0 real
syms m1 m2 rc1x rc1y rc1z rc2x real
syms Ic1xx Ic1xy Ic1xz Ic1yy Ic1yz Ic1zz real
syms Ic2xx Ic2yy Ic2zz real
syms a1 a2 a3 a4 a5 a6 real

rc1=[rc1x rc1y rc1z]’
rc2=[rc2x  0  0]’

Ic1=[Ic1xx Ic1xy Ic1xz;
     Ic1xy Ic1yy Ic1yz;
     Ic1xz Ic1yz Ic1zz]
Ic2=diag([Ic2xx Ic2yy Ic2zz])
```

```
gv0=[-g0 0 0]'

disp('moving frames algorithm for computing link kinetic energies')

om0=zeros(3,1);
v0=zeros(3,1);
z0=[0 0 1]';

om1=R01'*(om0+dq1*z0)
v1=R01'*v0+cross(om1,r01)
vc1=simplify(v1+cross(om1,rc1))
T1=simplify(0.5*m1*norm(vc1)^2+0.5*om1'*Ic1*om1);
% ADL: extra step!
T1=collect(T1,m1)

om2=R12'*(om1+dq2*z0)
v2=R12'*v1+cross(om2,r12)
vc2=simplify(v2+cross(om2,rc2))
T2=simplify(0.5*m2*norm(vc2)^2+0.5*om2'*Ic2*om2);
% ADL: extra steps!
T2=subs(T2,sin(q2)^2,1-cos(q2)^2);
T2=collect(T2,cos(q2))

disp('robot kinetic energy')

T=simplify(T1+T2)

disp('robot inertia matrix')

dq=[dq1 dq2]';
M=simplify(hessian(T,dq))

disp('linear parametrization of inertia matrix')

% ADL: by inspection!
% a1=Ic1yy+m1*(rc1x^2+rc1z^2)+Ic2xx
% a2=Ic2yy-Ic2xx+m2*(L2+rc2x)^2
% a3=Ic2zz+m2*(L2+rc2x)^2

M=subs(M,Ic1yy+m1*(rc1x^2+rc1z^2)+Ic2xx,Ic2yy-Ic2xx+m2*(L2+rc2x)^2,Ic2zz+m2*(L2+rc2x)^2,a1,a2,a3)

disp('Christoffel matrices')

M1=M(:,1);
C1=(1/2)*(jacobian(M1,q)+jacobian(M1,q)'-diff(M,q1))
M2=M(:,2);
C2=(1/2)*(jacobian(M2,q)+jacobian(M2,q)'-diff(M,q2))

disp('robot centrifugal and Coriolis terms')

c1=dq'*C1*dq
c2=dq'*C2*dq
c=[c1;c2]

disp('skew-symmetric factorization of velocity terms')

S1=dq'*C1;
S2=dq'*C2;
S=[S1;S2]
```

```
disp('check skew-symmetry of N=dM-2*S')

dM=diff(M,q2)*dq2 % ADL: +diff(M,q1)*dq1 is useless here ...
N=simplify(dM-2*S)

disp('computing link potential energies')

rc01_hom=A01*[rc1;1];
rc01=rc01_hom(1:3)
U1=-m1*gv0'*rc01

rc02_hom=A01*A12*[rc2;1];
rc02=rc02_hom(1:3)
U2=-m2*gv0'*rc02;
% ADL: extra steps!
U2=collect(U2,cos(q2));
U2temp=collect(U2/cos(q2),cos(q1));
U2=U2temp*cos(q2)

disp('robot potential energy')

U=simplify(U1+U2)

disp('gravity terms')

g=jacobian(U,q)'

disp('linear parametrization of gravity terms')

% ADL: by inspection!  (g0 is known)
% a4=-m2*(L2+rc2x)
% a5=-m1*rc1x
% a6=-m1*rc1z

% ADL: proceed separately!
g=subs(g,-m2*(L2+rc2x),a4);
% ADL: extra step!
g=expand(g);
g=subs(g,-m1*rc1x,-m1*rc1z,a5,a6)

disp(['regressor matrix Y and vector a of dynamic coefficients ' ...
'in a linear parametrization Y(q,dq,ddq)*a=tau'])

ddq=[ddq1 ddq2]';
tau=M*ddq+c+g;

a_symb=[a1 a2 a3 a4 a5 a6]';
Y=simplify(jacobian(tau,a_symb))

a=subs(a_symb,a1,a2,a3,a4,a5,a6,Ic1yy+m1*(rc1x^2+rc1z^2)+Ic2xx,...
Ic2yy-Ic2xx+m2*(L2+rc2x)^2,Ic2zz+m2*(L2+rc2x)^2,-m2*(L2+rc2x),-m1*rc1x,-m1*rc1z)

disp('inverse dynamics computation for q1d=2t, q2d=pi/2')

syms t real

q1d=2*t;
q2d=pi/4;
dq1d=2;
dq2d=0;
```

```
ddq1d=0;
ddq2d=0;

taud=subs(tau,q1,q2,dq1,dq2,ddq1,ddq2,q1d,q2d,dq1d,dq2d,ddq1d,ddq2d);
% ADL: extra step!
taud=collect(taud,sin(2*t))

disp('search of equilibrium configurations qe:  g(qe)=0')

% cos(q1)=0 --> g(2)=0 % ADL: requires that a5<a4

q1_e1=pi/2
g_q1_0=subs(g,q1,q1_e1);q2_e1=solve(g_q1_0(1)==0,q2,'Real',true)

% with q1_e1=pi/2
% q2_e1=pi - acos(a5/a4)
% q2_e1=pi + acos(a5/a4)
% ADL: with q1_e1=-pi/2, it is the same pair!

% sin(q2)=0 --> g(2)=0 % ADL: always valid!

q2_e1=0
g_q2_0=subs(g,q2,q2_e1);q1_e1=solve(g_q2_0(1)==0,q1,'Real',true)

% with q2_e1=0
% q1_e1=2*atan((a4 + a5 + (a4^2 + 2*a4*a5 + a5^2 + a6^2)^(1/2))/a6)
% q1_e1=2*atan((a4 + a5 - (a4^2 + 2*a4*a5 + a5^2 + a6^2)^(1/2))/a6)
% ADL: with q2_e1=pi (not shown)
% q1_e2=-2*atan((a4 - a5 + (a4^2 - 2*a4*a5 + a5^2 + a6^2)^(1/2))/a6)
% q1_e2= 2*atan((a5 - a4 + (a4^2 - 2*a4*a5 + a5^2 + a6^2)^(1/2))/a6)

disp('mechanical conditions for balancing gravity:  g(q)=0, for all q')

rc2x=-L2  %a4=0
rc1x=0    %a5=0
rc1z=0    %a6=0
a_0=eval(subs(a,rc1x,rc1z,rc2x,0,0,-L2))
g_0=eval(subs(g,a4,a5,a6,a_0(4),a_0(5),a_0(6)))

disp(['tip acceleration ddp for a tip force F in horizontal plane (y0,z0), ' ...
'when the robot is at rest (dq=0) and tau=g=0'])

syms Fy Fz real

J_ee_yz=J_ee(2:3,:)
M_ee_inv=J_ee_yz*M*J_ee_yz'
F=[Fy Fz]';
ddp=simplify(M_ee_inv*F)

% end
```

Exercise #3

This is quite simple. When the robot end-effector is in free space, the backward recursion of the Newton-Euler algortihm is initialized from the last link with zero forces and moments. The gravity term is removed from the picture (the subscript $\alpha = 0$ implies $g_0 = 0$, and thus $\boldsymbol{g}(\boldsymbol{q}) = \mathbf{0}$), while the velocity and the inertial terms are zero because $\dot{\boldsymbol{q}} = \mathbf{0}$ and $\ddot{\boldsymbol{q}} = \mathbf{0}$. Therefore, one has

$$\boldsymbol{u}^{(i)} = NE_0(\boldsymbol{q}, \mathbf{0}, \mathbf{0}) = \mathbf{0}.$$

On the other hand, when a force $\boldsymbol{f}_e \in \mathbb{R}^3$ and/or a moment $\boldsymbol{\tau}_e \in \mathbb{R}^3$ is applied to the robot end-effector, the components of this wrench are used to initialize the backward recursion from the last link, so that one obtains in this case

$$\boldsymbol{u}^{(ii)} = NE_0(\boldsymbol{q}, \boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{F}_e, \qquad \text{with } \boldsymbol{F}_e = \begin{pmatrix} \boldsymbol{f}_e \\ \boldsymbol{\tau}_e \end{pmatrix},$$

where $\boldsymbol{J}(\boldsymbol{q})$ is the 6×6 geometric Jacobian of the robot.

Exercise #4

In the absence of dissipative effects and assuming that no external contact forces/moments are present, the time evolution of the total energy $E = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) + U(\boldsymbol{q})$ is given by

$$\dot{E} = \dot{\boldsymbol{q}}^T \boldsymbol{\tau},$$

where $\boldsymbol{\tau} \in \mathbb{R}^n$ is the joint torque delivered to the robot by the motors. When $\dot{\boldsymbol{q}} \neq \boldsymbol{0}$, the maximum instantaneous decrease of E under the given torque bounds is obtained by choosing a command $\boldsymbol{\tau}_0^{(i)}$ with components given by

$$\tau_{0,j}^{(i)} = \begin{cases} -T_j \operatorname{sign} \dot{q}_j, & \text{if } \dot{q}_j \neq 0 \\ \text{any}, & \text{if } \dot{q}_j = 0, \end{cases} \qquad j = 1, \ldots, n.$$

The maximum decrease of E will be

$$\dot{E} = - \sum_{\substack{j=1,\ldots,n \\ \dot{q}_j \neq 0}} T_j \, |\dot{q}_j| < 0.$$

Only the joints that are currently in motion will contribute to the instantaneous reduction of E. For the others, one can simply set $\tau_{0,j}^{(i)} = 0$.

When $\dot{\boldsymbol{q}} = \boldsymbol{0}$, then $\dot{E} = 0$ no matter how $\boldsymbol{\tau}$ is chosen. Thus, in order to induce a decrease in the immediate future evolution of E, one should look at the second time derivative of E, namely

$$\ddot{E} = \ddot{\boldsymbol{q}}^T \boldsymbol{\tau} + \dot{\boldsymbol{q}}^T \dot{\boldsymbol{\tau}} = \ddot{\boldsymbol{q}}^T \boldsymbol{\tau},$$

where the second term vanishes being $\dot{\boldsymbol{q}} = \boldsymbol{0}$. The joint acceleration $\ddot{\boldsymbol{q}}$ will depend only on the current robot configuration $\boldsymbol{q}_0$ and on the input torque $\boldsymbol{\tau}$, being now $\dot{\boldsymbol{q}} = \boldsymbol{0}$ (and thus $\boldsymbol{c}(\boldsymbol{q}_0, \boldsymbol{0}) = \boldsymbol{0}$):

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{q}_0) \, (\boldsymbol{\tau} - \boldsymbol{g}(\boldsymbol{q}_0)) \, .$$

As a result, the second time derivative of E becomes

$$\ddot{E} = \boldsymbol{\tau}^T \boldsymbol{M}^{-1}(\boldsymbol{q}_0) \, \boldsymbol{\tau} - \boldsymbol{g}^T(\boldsymbol{q}_0) \boldsymbol{M}^{-1}(\boldsymbol{q}_0) \, \boldsymbol{\tau},$$

namely a complete quadratic function of the input torque $\boldsymbol{\tau}$ to be chosen, with a positive definite quadratic term ($\boldsymbol{\tau}^T \boldsymbol{M}^{-1}(\boldsymbol{q}_0) \, \boldsymbol{\tau} > 0$, for all $\boldsymbol{\tau} \neq \boldsymbol{0}$, from the property of the robot inertia matrix and of its inverse). The maximum decrease of E in the immediate future will be obtained by choosing $\boldsymbol{\tau}$ so as to minimize $\ddot{E} < 0$ (i.e., imposing the largest possible absolute value of $\ddot{E}$, with a leading minus sign). To find stationary points for the function $\ddot{E}(\boldsymbol{\tau})$, we set

$$\nabla_{\boldsymbol{\tau}} \ddot{E} = \left(\frac{\partial \ddot{E}}{\partial \boldsymbol{\tau}} \right)^T = 2\boldsymbol{M}^{-1}(\boldsymbol{q}_0) \, \boldsymbol{\tau} - \boldsymbol{M}^{-1}(\boldsymbol{q}_0) \, \boldsymbol{g}(\boldsymbol{q}_0) = \boldsymbol{0}.$$

This linear equation has the unique solution

$$\boldsymbol{\tau}_0^{(ii)} = \frac{\boldsymbol{g}(\boldsymbol{q}_0)}{2} \neq \boldsymbol{0},$$

having assumed that $\boldsymbol{q}_0$ is not an (unforced) equilibrium configuration for the robot. Being the Hessian matrix $\nabla^2_{\boldsymbol{\tau}}\ddot{E} = 2\boldsymbol{M}^{-1}(\boldsymbol{q}_0) > 0$, $\boldsymbol{\tau}_0^{(ii)}$ is a minimum for $\ddot{E}$ with respect to $\boldsymbol{\tau}$. Moreover, this torque is certainly feasible for the given bounds — by at least one order of magnitude! The attained value of $\ddot{E}$ is

$$\ddot{E}(\boldsymbol{\tau}_0^{(ii)}) = -\frac{1}{4}\,\boldsymbol{g}^T(\boldsymbol{q}_0)\boldsymbol{M}^{-1}(\boldsymbol{q}_0)\,\boldsymbol{g}(\boldsymbol{q}_0) < 0.$$

It is easy to verify that the following expressions hold for the individual contributions to the total energy (kinetic and potential energy) at the given instant of time t_0, when $\boldsymbol{x}(t_0) = (\boldsymbol{q}_0, \boldsymbol{0})$ and $\boldsymbol{\tau}(t_0) = \boldsymbol{\tau}_0^{(ii)}$:

$$T = 0, \qquad \dot{T} = 0, \qquad \ddot{T} = \frac{1}{4}\,\boldsymbol{g}^T(\boldsymbol{q}_0)\boldsymbol{M}^{-1}(\boldsymbol{q}_0)\,\boldsymbol{g}(\boldsymbol{q}_0) > 0,$$

and

$$U = U(\boldsymbol{q}_0) \geq 0, \qquad \dot{U} = 0, \qquad \ddot{U} = -\frac{1}{2}\,\boldsymbol{g}^T(\boldsymbol{q}_0)\boldsymbol{M}^{-1}(\boldsymbol{q}_0)\,\boldsymbol{g}(\boldsymbol{q}_0) < 0.$$

Note finally that if we had $\boldsymbol{g}(\boldsymbol{q}_0) = \boldsymbol{0}$, then also $\ddot{E} = 0$ and two situations may arise. Either $U(\boldsymbol{q}_0) > 0$, and one should then investigate the sign of higher order time derivatives ($\dddot{E}$ and more) in order to assess the evolution of the total energy; or $U(\boldsymbol{q}_0) = 0$, and then the robot is at $E(t_0) = 0$, a global minimum of the total energy, which cannot be decreased in any case.

* * * * *

*** Projected Gradient and Task Priority (TP) methods for a 3R planar robot ***

kinematics of the 3R planar robot (links of unitary length)

pe_simb =

cos(q1 + q2 + q3) + cos(q1 + q2) + cos(q1)
sin(q1 + q2 + q3) + sin(q1 + q2) + sin(q1)

pm_simb =

cos(q1 + q2) + cos(q1)
sin(q1 + q2) + sin(q1)

Je_simb =

[- sin(q1 + q2 + q3) - sin(q1 + q2) - sin(q1), - sin(q1 + q2 + q3) - sin(q1 + q2), -sin(q1 + q2 + q3)]
[cos(q1 + q2 + q3) + cos(q1 + q2) + cos(q1),   cos(q1 + q2 + q3) + cos(q1 + q2), cos(q1 + q2 + q3)]

Jm_simb =

[- sin(q1 + q2) - sin(q1), -sin(q1 + q2), 0]
[cos(q1 + q2) + cos(q1), cos(q1 + q2), 0]

at the given configuration (as in the text figure)

q0 =

```
        0
   1.5708
  -1.5708
```

numerical kinematics

pe =

2
1

pm =

1
1

Je =

[-1, -1, 0]
[2, 1, 1]

Jm =

[-1, -1, 0]
[1, 0, 0]

clearance

H =

```
2^(1/2) - 1/2

desired end-effector velocity

ve =

     0
     1

projected gradient (PG) method

nablaH =

-2^(1/2)/2
-2^(1/2)/4
         0

nablaH =

   -0.7071
   -0.3536
         0

alfa1 =

     1

Je_pinv =

[ 0,  1/3]
[-1, -1/3]
[ 1,  2/3]

dq_PG =

 1/3 - 2^(1/2)/12
 2^(1/2)/12 - 1/3
 2^(1/2)/12 + 2/3

dq_PG =

    0.2155
   -0.2155
    0.7845

check PG solution

ve_check =

     0
     1

vm =

         0
    0.2155
```

Task Priority (TP) - case A

Pe =

[1/3, -1/3, -1/3]
[-1/3, 1/3, 1/3]
[-1/3, 1/3, 1/3]

vm_A =

0
0.2155

dq_TP_A =

3881782278985403/18014398509481984
-3881782278985403/18014398509481984
14132616230496581/18014398509481984

dq_TP_A =

0.2155
-0.2155
0.7845

check TP_A solution

ve_check =

0
1

vm_check =

0
0.2155

Task Priority (TP) - case B

alfa2 =

1

vm_B =

1 - 2^(1/2)/4
2^(1/2)/4 - 1

vm_B =

0.6464
-0.6464

dq_TP_B =

-2911336709239053/4503599627370496

```
 2911336709239053/4503599627370496
 7414936336609549/4503599627370496

dq_TP_B =

  -0.6464
   0.6464
   1.6464

check TP_B solution

ve_check =

     0
     1

vm_check =

        0
   -0.6464

vm_check_error =

   0.6464
        0

>>
```

```
*** Dynamic model and analysis of a 2-dof RR spatial robot ***

kinematic quantities, possibly used later in dynamics

R01 =

[cos(q1),  0, -sin(q1)]
[sin(q1),  0,  cos(q1)]
[      0, -1,        0]

p01 =

 0
 0
L1

r01 =

  0
-L1
  0

R12 =

[cos(q2), -sin(q2), 0]
[sin(q2),  cos(q2), 0]
[      0,        0, 1]

p12 =

L2*cos(q2)
L2*sin(q2)
         0

r12 =

L2
 0
 0

end-effector position and associated Jacobian (in base frame)

p_ee =

L2*cos(q1)*cos(q2)
L2*cos(q2)*sin(q1)
   L1 - L2*sin(q2)

J_ee =

[-L2*cos(q2)*sin(q1), -L2*cos(q1)*sin(q2)]
[ L2*cos(q1)*cos(q2), -L2*sin(q1)*sin(q2)]
[                  0,         -L2*cos(q2)]

dynamic parameters

rc1 =
```

```
rc1x
rc1y
rc1z

rc2 =

rc2x
  0
  0

Ic1 =

[Ic1xx, Ic1xy, Ic1xz]
[Ic1xy, Ic1yy, Ic1yz]
[Ic1xz, Ic1yz, Ic1zz]

Ic2 =

[Ic2xx,     0,     0]
[    0, Ic2yy,     0]
[    0,     0, Ic2zz]

gv0 =

-g0
  0
  0

moving frames algorithm for computing link kinetic energies

om1 =

  0
-dq1
  0

v1 =

0
0
0

vc1 =

-dq1*rc1z
       0
 dq1*rc1x

T1 =

((dq1^2*(rc1x^2 + rc1z^2))/2)*m1 + (Ic1yy*dq1^2)/2

om2 =

-dq1*sin(q2)
-dq1*cos(q2)
```

dq2

v2 =

0
L2*dq2
L2*dq1*cos(q2)

vc2 =

0
dq2*(L2 + rc2x)
dq1*cos(q2)*(L2 + rc2x)

T2 =

((Ic2yy*dq1^2)/2 - (Ic2xx*dq1^2)/2 + (dq1^2*m2*(L2 + rc2x)^2)/2)*cos(q2)^2 + (Ic2xx*dq1^2)/2 + (Ic2zz*dq2^2)/2 + (dq2^2*m2*(L2 + rc2x)^2)/2

robot kinetic energy

T =

(Ic1yy*dq1^2)/2 + ((Ic2yy*dq1^2)/2 - (Ic2xx*dq1^2)/2 + (dq1^2*m2*(L2 + rc2x)^2)/2)*cos(q2)^2 + ((dq1^2*(rc1x^2 + rc1z^2))/2)*m1 + (Ic2xx*dq1^2)/2 + (Ic2zz*dq2^2)/2 + (dq2^2*m2*(L2 + rc2x)^2)/2

robot inertia matrix

M =

```
[Ic2xx + Ic1yy + cos(q2)^2*(Ic2yy - Ic2xx + m2*(L2 + rc2x)^2) + m1*(rc1x^2 + rc1z^2),                       0]
[                                                                    0, Ic2zz + m2*(L2 + rc2x)^2]
```

linear parametrization of inertia matrix

M =

```
[a2*cos(q2)^2 + a1,  0]
[                0, a3]
```

Christoffel matrices

C1 =

```
[                0, -a2*cos(q2)*sin(q2)]
[-a2*cos(q2)*sin(q2),                  0]
```

C2 =

```
[a2*cos(q2)*sin(q2), 0]
[                0, 0]
```

robot centrifugal and Coriolis terms

c1 =

-2*a2*dq1*dq2*cos(q2)*sin(q2)

c2 =

a2*dq1^2*cos(q2)*sin(q2)

c =

```
-2*a2*dq1*dq2*cos(q2)*sin(q2)
    a2*dq1^2*cos(q2)*sin(q2)
```

skew-symmetric factorization of velocity terms

S =

```
[-a2*dq2*cos(q2)*sin(q2), -a2*dq1*cos(q2)*sin(q2)]
[ a2*dq1*cos(q2)*sin(q2),                       0]
```

check skew-symmetry of N=dM-2*S

dM =

```
[-2*a2*dq2*cos(q2)*sin(q2), 0]
[                        0, 0]
```

N =

```
[                  0, a2*dq1*sin(2*q2)]
[-a2*dq1*sin(2*q2),                 0]
```

computing link potential energies

rc01 =

```
rc1x*cos(q1) - rc1z*sin(q1)
rc1z*cos(q1) + rc1x*sin(q1)
                L1 - rc1y
```

U1 =

g0*m1*(rc1x*cos(q1) - rc1z*sin(q1))

rc02 =

```
L2*cos(q1)*cos(q2) + rc2x*cos(q1)*cos(q2)
L2*cos(q2)*sin(q1) + rc2x*cos(q2)*sin(q1)
          L1 - L2*sin(q2) - rc2x*sin(q2)
```

U2 =

cos(q1)*cos(q2)*g0*m2*(L2 + rc2x)

robot potential energy

U =

g0*m1*(rc1x*cos(q1) - rc1z*sin(q1)) + g0*m2*cos(q1)*cos(q2)*(L2 + rc2x) gravity terms

g =

```
- g0*m1*(rc1z*cos(q1) + rc1x*sin(q1)) - g0*m2*cos(q2)*sin(q1)*(L2 + rc2x)
                         -g0*m2*cos(q1)*sin(q2)*(L2 + rc2x)
```

linear parametrization of gravity terms

g =

```
a5*g0*sin(q1) + a6*g0*cos(q1) + a4*g0*cos(q2)*sin(q1)
                     a4*g0*cos(q1)*sin(q2)
```

regressor matrix Y and vector a of dynamic coefficients in a linear parametrization Y(q,dq,ddq)*a=tau

Y =

```
[ddq1, ddq1*cos(q2)^2 - 2*dq1*dq2*cos(q2)*sin(q2),    0, g0*cos(q2)*sin(q1), g0*sin(q1), g0*cos(q1)]
[   0,                   (dq1^2*sin(2*q2))/2, ddq2, g0*cos(q1)*sin(q2),          0,          0]
```

a =

```
Ic2xx + Ic1yy + m1*(rc1x^2 + rc1z^2)
   Ic2yy - Ic2xx + m2*(L2 + rc2x)^2
         Ic2zz + m2*(L2 + rc2x)^2
                 - m2*(L2 + rc2x)
                     - m1*rc1x
                     - m1*rc1z
```

inverse dynamics computation for q1d=2t, q2d=pi/2

taud =

```
(a5*g0 + (2^(1/2)*a4*g0)/2)*sin(2*t) + a6*g0*cos(2*t)
                2*a2 + (2^(1/2)*a4*g0*cos(2*t))/2
```

search of equilibrium configurations qe: g(qe)=0

q1_e1 =

   1.5708

Warning: Solutions are only valid under certain conditions. To include parameters and conditions in the solution, specify the 'ReturnConditions' value as 'true'.
> In sym/solve>warnIfParams (line 478)
In sym/solve (line 357)
In Ex2_Midterm (line 189)

q2_e1 =

pi - acos(a5/a4)
pi + acos(a5/a4)

q2_e1 =

    0

```
q1_e1 =

2*atan((a4 + a5 + (a4^2 + 2*a4*a5 + a5^2 + a6^2)^(1/2))/a6)
2*atan((a4 + a5 - (a4^2 + 2*a4*a5 + a5^2 + a6^2)^(1/2))/a6)

mechanical conditions for balancing gravity: g(q)=0, for all q

rc2x =

-L2

rc1x =

   0

rc1z =

   0

a_0 =

Ic2xx + Ic1yy
Ic2yy - Ic2xx
      Ic2zz
          0
          0
          0

g_0 =

   0
   0

tip acceleration ddp for a tip force F in horizontal plane (y0,z0), when the robot is at rest (dq=0) and tau=g=0

J_ee_yz =

[L2*cos(q1)*cos(q2), -L2*sin(q1)*sin(q2)]
[                 0,        -L2*cos(q2)]

M_ee_inv =

[L2^2*cos(q1)^2*cos(q2)^2*(a2*cos(q2)^2 + a1) + L2^2*a3*sin(q1)^2*sin(q2)^2,
L2^2*a3*cos(q2)*sin(q1)*sin(q2)]
[                                   L2^2*a3*cos(q2)*sin(q1)*sin(q2),            L2^2*a3*cos(q2)^2]

ddp =

Fy*(L2^2*cos(q1)^2*cos(q2)^2*(a2*cos(q2)^2 + a1) + L2^2*a3*sin(q1)^2*sin(q2)^2) +
Fz*L2^2*a3*cos(q2)*sin(q1)*sin(q2)
                                                  L2^2*a3*cos(q2)*(Fz*cos(q2) + Fy*sin(q1)*sin(q2))


>>
```

Robotics 2

June 12, 2024

Exercise 1

Table 1 contains the Denavit-Hartenberg parameters of a robot with three revolute joints.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	θ_1
2	0	$a_2 > 0$	0	θ_2
3	0	$a_3 > 0$	0	θ_3

Table 1: Denavit-Hartenberg parameters of a 3R robot

Use the recursive algorithm based on moving frames to compute the kinetic energy of this robot, making reasonable assumptions to simplify the inertial properties of the links, e.g., assume that each link has a cylindric body with uniformly distributed mass and center of mass on the kinematic axis of the link. Provide at the end the robot inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ and determine a minimal linear parametrization of the inertial term $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}$.

Exercise 2

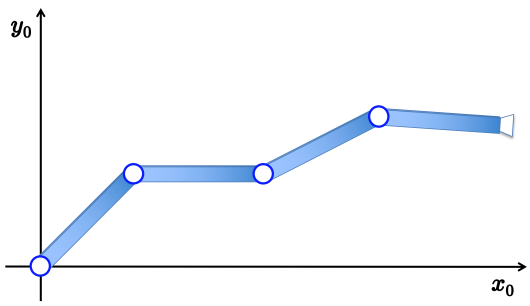


Figure 1: A 4R planar robot.

Consider the 4R planar robot in Fig. 1, having links with unit length. The primary task for the end-effector is to point at a moving target in the plane (x_0, y_0). The available extra degrees of freedom of the robot are used to keep the joints close to the middle of their ranges, which are defined, using D-H variables, as

$$q_1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad q_2 \in \left[0, \frac{\pi}{2}\right] \qquad q_i \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right], \quad i = 3, 4. \tag{1}$$

Define the primary task Jacobian $\boldsymbol{J}(\boldsymbol{q})$ and determine the joint velocity command $\dot{\boldsymbol{q}} \in \mathbb{R}^4$ that realizes at best the desired robot behavior. Provide the numerical value of $\dot{\boldsymbol{q}}$ when the robot is at $\boldsymbol{q} = (0, \pi/2, 0, -\pi/4)$, while the target is being correctly pointed at and has an instantaneous velocity $\dot{\boldsymbol{p}}_t = (-1, -1)$ [m/s]. How would you modify the joint velocity command if the target was placed at $\boldsymbol{p}_t = (0, 2.5)$ [m] and had the same previous velocity $\dot{\boldsymbol{p}}_t$? Provide the new numerical value of the command $\dot{\boldsymbol{q}}$ with your modified control strategy.

Exercise 3

The dynamics of a robot with n elastic joints moving in the absence of gravity is described by the $2n$ second-order differential equations

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{K}(\boldsymbol{q} - \boldsymbol{\theta}) = \boldsymbol{0} \tag{2}$$

$$\boldsymbol{B}_m \ddot{\boldsymbol{\theta}} + \boldsymbol{K}(\boldsymbol{\theta} - \boldsymbol{q}) = \boldsymbol{u}, \tag{3}$$

where (2) are the n link equations and (3) are the n motor equations, with link positions $\boldsymbol{q} \in \mathbb{R}^n$ and motor positions $\boldsymbol{\theta} \in \mathbb{R}^n$ as generalized coordinates. $\boldsymbol{K}$ is the diagonal, positive definite stiffness matrix of the joints, while $\boldsymbol{B}_m$ is the diagonal, positive definite inertia matrix of the motors.

For the input torque $\boldsymbol{u} \in \mathbb{R}^n$, consider the PD control law on the motor variables

$$\boldsymbol{u} = \boldsymbol{K}_P(\boldsymbol{\theta}_d - \boldsymbol{\theta}) - \boldsymbol{K}_D \dot{\boldsymbol{\theta}}, \tag{4}$$

where $\boldsymbol{K}_P$ and $\boldsymbol{K}_D$ are diagonal and positive definite gain matrices, and $\boldsymbol{\theta}_d \in \mathbb{R}^n$ is a desired constant motor position. Prove that

$$(\boldsymbol{q}, \boldsymbol{\theta}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{\theta}}) = (\boldsymbol{\theta}_d, \boldsymbol{\theta}_d, \boldsymbol{0}, \boldsymbol{0})$$

is the unique, globally asymptotically stable equilibrium state for the closed-loop system made by eqs. (2) and (3) under the control law (4).

Exercise 4

The PR robot in Fig. 2 may be subject to a generic unknown fault u_{f1} on the force produced by first actuator. Based on the symbolic terms of the dynamic model of this robot, design a scalar residual function $r_1(t)$ such that $r_1(t) \equiv 0$ in the absence of a fault of this actuator, while it evolves otherwise as $\dot{r}_1(t) = k_1(u_{f1}(t) - r_1(t))$, for a given $k_1 > 0$. If the unknown fault consists in a constant force $u_{f1} = 2$ N being subtracted to the commanded force u_1 at the first joint starting from the instant $t_0 = 0$, what will be the evolution of $r_1(t)$ for $t \geq t_0$ and its value at steady state?

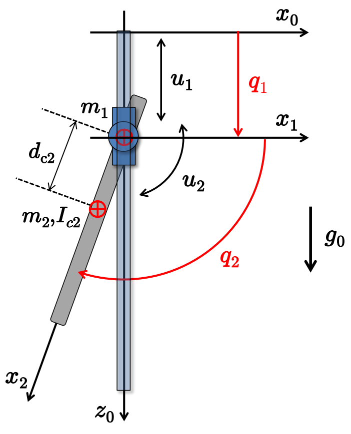


Figure 2: A PR planar robot under gravity.

[240 minutes (4 hours); open books]

Solution

June 12, 2024

Exercise 1

From the Denavit-Hartenberg table of parameters (Tab. 1), we prepare the vectors and matrices needed in each step of the recursive algorithm with moving frames for computing the kinetic energy.

Rotation matrix

$$ {}^0\boldsymbol{R}_1(q_1) = \begin{pmatrix} c_1 & 0 & s_1 \\ s_1 & 0 & -c_1 \\ 0 & 1 & 0 \end{pmatrix} \quad {}^1\boldsymbol{R}_2(q_2) = \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad {}^2\boldsymbol{R}_3(q_3) = \begin{pmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

Position vector between origins in previous frame

$$ {}^0\boldsymbol{r}_{01} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad {}^1\boldsymbol{r}_{12}(q_2) = \begin{pmatrix} a_2 c_2 \\ a_2 s_2 \\ 0 \end{pmatrix} \qquad {}^2\boldsymbol{r}_{23}(q_3) = \begin{pmatrix} a_3 c_3 \\ a_3 s_3 \\ 0 \end{pmatrix}. $$

Position vector between origins in moving frame

$$ {}^1\boldsymbol{r}_{01} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad {}^2\boldsymbol{r}_{12} = \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix} \qquad {}^3\boldsymbol{r}_{23} = \begin{pmatrix} a_3 \\ 0 \\ 0 \end{pmatrix}. $$

Based on the given assumptions on mass distribution of the links, we have also:

Position of link CoM in moving frame

$$ {}^1\boldsymbol{r}_{c1} = \begin{pmatrix} 0 \\ d_{c1} \\ 0 \end{pmatrix} \qquad {}^2\boldsymbol{r}_{c2} = \begin{pmatrix} d_{c2} - a_2 \\ 0 \\ 0 \end{pmatrix} \qquad {}^3\boldsymbol{r}_{c3} = \begin{pmatrix} d_{c3} - a_3 \\ 0 \\ 0 \end{pmatrix}, $$

being d_{ci} the distance from the origin O_{i-1} to the center of mass (CoM) of link i along the $\boldsymbol{x}_i$ axis.

Link inertia matrix in moving frame

$$ {}^1\boldsymbol{I}_{c1} = \begin{pmatrix} I_{c1,x} & 0 & 0 \\ 0 & I_{c1,y} & 0 \\ 0 & 0 & I_{c1,x} \end{pmatrix} \quad {}^2\boldsymbol{I}_{c2} = \begin{pmatrix} I_{c2,x} & 0 & 0 \\ 0 & I_{c2,z} & 0 \\ 0 & 0 & I_{c2,z} \end{pmatrix} \quad {}^3\boldsymbol{I}_{c3} = \begin{pmatrix} I_{c3,x} & 0 & 0 \\ 0 & I_{c3,z} & 0 \\ 0 & 0 & I_{c3,z} \end{pmatrix}, $$

where we used the symmetry of the cylindrical links with respect to the two minor axes that are transversal to their major axis (which are, respectively, y_1, x_2 and x_3).

With the above structures, the algorithm is initialized with ${}^0\boldsymbol{\omega}_0 = \boldsymbol{0}$, ${}^0\boldsymbol{v}_0 = \boldsymbol{0}$ and yields:

Step 1

$$ {}^1\boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ \dot{q}_1 \\ 0 \end{pmatrix} \qquad {}^1\boldsymbol{v}_1 = \boldsymbol{0} \qquad {}^1\boldsymbol{v}_{c1} = \boldsymbol{0} $$

Kinetic energy of link 1

$$ T_1 = \frac{1}{2} I_{c1,y} \dot{q}_1^2. $$

Step 2

$$ {}^2\boldsymbol{\omega}_2 = \begin{pmatrix} s_2\dot{q}_1 \\ c_2\dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \qquad {}^2\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ a_2\dot{q}_2 \\ -a_2c_2\dot{q}_1 \end{pmatrix} \qquad {}^2\boldsymbol{v}_{c2} = \begin{pmatrix} 0 \\ d_{c2}\dot{q}_2 \\ -d_{c2}c_2\dot{q}_1 \end{pmatrix} $$

Kinetic energy of link 2

$$ T_2 = \frac{1}{2}\left(I_{c2,x}\, s_2^2 + (I_{c2,z} + m_2 d_{c2}^2)\, c_2^2\right)\dot{q}_1^2 + \frac{1}{2}\left(I_{c2,z} + m_2 d_{c2}^2\right)\dot{q}_2^2. $$

Step 3

$$ {}^3\boldsymbol{\omega}_3 = \begin{pmatrix} s_{23}\dot{q}_1 \\ c_{23}\dot{q}_1 \\ \dot{q}_2 + \dot{q}_3 \end{pmatrix} \quad {}^3\boldsymbol{v}_3 = \begin{pmatrix} a_2s_3\dot{q}_2 \\ a_2c_3\dot{q}_2 + a_3\left(\dot{q}_2 + \dot{q}_3\right) \\ -\left(a_2c_2 + a_3c_{23}\right)\dot{q}_1 \end{pmatrix} \quad {}^3\boldsymbol{v}_{c3} = \begin{pmatrix} a_2s_3\dot{q}_2 \\ a_2c_3\dot{q}_2 + d_{c3}\left(\dot{q}_2 + \dot{q}_3\right) \\ -\left(a_2c_2 + d_{c3}c_{23}\right)\dot{q}_1 \end{pmatrix} $$

Kinetic energy of link 3

$$ \begin{aligned} T_3 = {} & \frac{1}{2}\left(I_{c3,x}\, s_{23}^2 + I_{c3,z}\, c_{23}^2 + m_3\left(a_2c_2 + d_{c3}c_{23}\right)^2\right)\dot{q}_1^2 + \frac{1}{2} m_3 a_2^2 \dot{q}_2^2 \\ & + \frac{1}{2}\left(I_{c3,z} + m_3 d_{c3}^2\right)\left(\dot{q}_2 + \dot{q}_3\right)^2 + \frac{1}{2}\left(2\, m_3 a_2 d_{c3} c_3\, \dot{q}_2\left(\dot{q}_2 + \dot{q}_3\right)\right). \end{aligned} $$

Thus, the kinetic energy of the 3R robot is

$$ T = T_1 + T_2 + T_3 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}. $$

In the expressions of the elements of the inertia matrix $\boldsymbol{M}(\boldsymbol{q})$, one can eliminate all appearances of square roots of the *sine* functions by setting

$$ s_3^2 = 1 - c_3^2 \qquad s_{23}^2 = 1 - c_{23}^2. $$

With these substitutions, one recognizes the presence of six independent dynamic coefficients ρ_i in the inertia matrix:

$$ \boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} \rho_1 + \rho_2\, c_2^2 + \rho_3\, c_{23}^2 + 2\rho_5\, c_2c_{23} & 0 & 0 \\ 0 & \rho_4 + 2\rho_5\, c_3 & \rho_6 + \rho_5\, c_3 \\ 0 & \rho_6 + \rho_5\, c_3 & \rho_6 \end{pmatrix}, $$

where

$$ \begin{aligned} \rho_1 &= I_{c1,y} + I_{c2,x} + I_{c3,x} \\ \rho_2 &= I_{c2,z} + m_2 d_{c2}^2 - I_{c2,x} + m_3 a_2^2 \\ \rho_3 &= I_{c3,z} + m_3 d_{c3}^2 - I_{c3,x} \\ \rho_4 &= I_{c2,z} + m_2 d_{c2}^2 + I_{c3,z} + m_3\left(a_2^2 + d_{c3}^2\right) \\ \rho_5 &= m_3 a_2 d_{c3} \\ \rho_6 &= I_{c3,z} + m_3 d_{c3}^2. \end{aligned} $$

Note that other parametrizations with the same number of coefficients can be defined, but the minimum number of parameters (six in the present case) is unique.

As a result, the inertial terms in the dynamic model can be linearly parametrized by

$$\boldsymbol{\rho} = \begin{pmatrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 & \rho_6 \end{pmatrix}^T$$

as

$$\boldsymbol{M}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} = \boldsymbol{Y}(\boldsymbol{q},\ddot{\boldsymbol{q}})\,\boldsymbol{\rho},$$

where the 3×6 regressor matrix $\boldsymbol{Y}$ is

$$\boldsymbol{Y}(\boldsymbol{q},\ddot{\boldsymbol{q}}) = \begin{pmatrix} \ddot{q}_1 & c_2^2\,\ddot{q}_1 & c_{23}^2\,\ddot{q}_1 & 0 & 2c_2c_{23}\,\ddot{q}_1 & 0 \\ 0 & 0 & 0 & \ddot{q}_2 & c_3\,(2\ddot{q}_2+\ddot{q}_3) & \ddot{q}_3 \\ 0 & 0 & 0 & 0 & c_2\,\ddot{q}_2 & \ddot{q}_2+\ddot{q}_3 \end{pmatrix}.$$

Exercise 2

The variables $\boldsymbol{r} \in \mathbb{R}^m$ that are used to define a robot task are always functions of the configuration $\boldsymbol{q} \in \mathbb{R}^n$, i.e., $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q})$. On the other hand, the desired behavior of the task variables (which needs to be imposed by the control law) is usually expressed by an exogenous, time-varying signal $\boldsymbol{r}_d(t)$ (a constant $\boldsymbol{r}_d$ for regulation tasks). However, the desired behavior of the primary task assigned to the robot in Fig. 1, i.e., pointing with the end effector to a (moving) target, has the peculiarity of depending also on the current robot configuration $\boldsymbol{q}(t)$, or $\boldsymbol{r}_d(t,\boldsymbol{q}(t))$. This requires some caution in the definition of the task control problem at a differential level.

The task assigned to the 4R planar robot ($n = 4$) involves the orientation angle α of the last link

$$\alpha = f_\alpha(\boldsymbol{q}) = q_1 + q_2 + q_3 + q_4,$$

as well as the position of the end effector

$$\boldsymbol{p} = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} c_1 + c_{12} + c_{123} + c_{1234} \\ s_1 + s_{12} + s_{123} + s_{1234} \end{pmatrix} = \boldsymbol{f}_p(\boldsymbol{q}).$$

Accordingly, we have at the differential level

$$\dot{\alpha} = \frac{\partial f_\alpha}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}\dot{\boldsymbol{q}} = \boldsymbol{J}_\alpha\,\dot{\boldsymbol{q}} \tag{5}$$

and

$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{f}_p}{\partial \boldsymbol{q}}\,\dot{\boldsymbol{q}} = \boldsymbol{J}_p(\boldsymbol{q})\,\dot{\boldsymbol{q}}, \tag{6}$$

with

$$\boldsymbol{J}_p(\boldsymbol{q}) = \begin{pmatrix} -(s_1+s_{12}+s_{123}+s_{1234}) & -(s_{12}+s_{123}+s_{1234}) & -(s_{123}+s_{1234}) & -s_{1234} \\ c_1+c_{12}+c_{123}+c_{1234} & c_{12}+c_{123}+c_{1234} & c_{123}+c_{1234} & c_{1234} \end{pmatrix}.$$

The two Jacobians $\boldsymbol{J}_\alpha$ and $\boldsymbol{J}_p$ will be needed in the following.

With reference to Fig. 3, the pointing task has dimension $m = 1$ and the desired behavior for the task variable $r = \alpha$ is expressed as

$$\alpha_d(t,\boldsymbol{q}(t)) = \text{atan2}\left\{p_{ty}(t) - p_y(\boldsymbol{q}(t)), p_{tx}(t) - p_x(\boldsymbol{q}(t))\right\},$$

where $\boldsymbol{p}_d(t) = (p_{tx}(t), p_{ty}(t))$ is the current position of the target. It is apparent that this scalar function depends both on the target motion (which is the exogenous part) and on the robot configuration $\boldsymbol{q}$. We need then to define the differential mapping between $\dot{\boldsymbol{q}}$ and $\dot{\alpha}_d$.

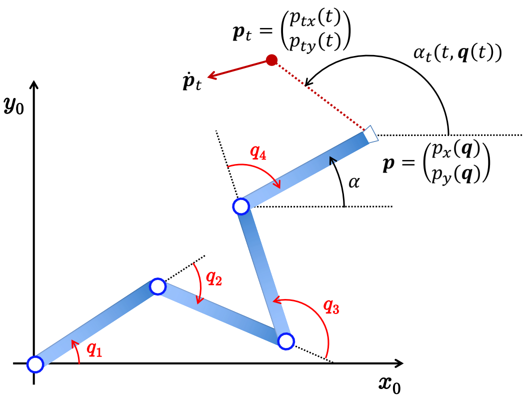


Figure 3: The task variable α and its desired value α_d at a given instant.

For this, remember that for a generic scalar function $g(z)$

$$\frac{d}{dz} \arctan g(z) = \frac{1}{1 + g^2(z)} \frac{dg(z)}{dz}$$

and note that the two functions atan2$\{y, x\}$ and $\arctan(y/x)$ behave in the same way at the differential level. After some lengthy but straightforward computations, we obtain

$$\dot{\alpha}_d = \frac{(\dot{p}_{ty} - \dot{p}_y)(p_{tx} - p_x) - (\dot{p}_{tx} - \dot{p}_x)(p_{ty} - p_y)}{(p_{tx} - p_x)^2 + (p_{ty} - p_y)^2}.$$

This can be reorganized as

$$\begin{aligned} \dot{\alpha}_d &= \begin{pmatrix} \dfrac{p_{ty} - p_y}{(p_{tx} - p_x)^2 + (p_{ty} - p_y)^2} & -\dfrac{p_{tx} - p_x}{(p_{tx} - p_x)^2 + (p_{ty} - p_y)^2} \end{pmatrix} \begin{pmatrix} \dot{p}_x - \dot{p}_{tx} \\ \dot{p}_y - \dot{p}_{ty} \end{pmatrix} \\ &= \boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\,(\dot{\boldsymbol{p}} - \dot{\boldsymbol{p}}_t) = \boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\boldsymbol{J}_p(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\dot{\boldsymbol{p}}_t, \end{aligned} \tag{7}$$

where we used the Jacobian in (6). Note that in order to evaluate the 1×2 Jacobian $\boldsymbol{J}_{\alpha_d}$, we need to know in general also the current position $\boldsymbol{p}_t$ of the target.

The task equation corresponding to the desired condition (i.e., with the robot end effector always pointing at the target) is then $\alpha(t) = \alpha_d(t)$, for all $t \geq 0$. In the nominal case, this equality holds true at the initial time $t = 0$ and thus it can be replaced by the identity $\dot{\alpha}(t) = \dot{\alpha}_d(t)$ at the differential level or, using also the Jacobian in (5),

$$\boldsymbol{J}_\alpha\,\dot{\boldsymbol{q}} = \boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\boldsymbol{J}_p(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\,\dot{\boldsymbol{p}}_t,$$

which can be reorganized as

$$\boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\,\dot{\boldsymbol{p}}_t = (\boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\boldsymbol{J}_p(\boldsymbol{q}) - \boldsymbol{J}_\alpha)\,\dot{\boldsymbol{q}} = \boldsymbol{J}_r(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{8}$$

with the 1×4 task matrix $\boldsymbol{J}_r$. This relation also shows that if the target is not moving ($\dot{\boldsymbol{p}}_t = \boldsymbol{0}$), any robot velocity $\dot{\boldsymbol{q}}$ that keeps task satisfaction would have to be in the null space of $\boldsymbol{J}_r$ (and not necessarily also in the null space of $\boldsymbol{J}_p$, i.e., without a change of the end effector position!).

The minimum norm solution to (8) is given by

$$\dot{\boldsymbol{q}}_0 = \boldsymbol{J}_r^{\#}(\boldsymbol{q})\boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\,\dot{\boldsymbol{p}}_t,$$

whereas the simultaneous minimization of an objective function $H(\boldsymbol{q})$ leads to the PG method

$$\begin{aligned}\dot{\boldsymbol{q}}_H &= \boldsymbol{J}_r^{\#}(\boldsymbol{q})\boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\,\dot{\boldsymbol{p}}_t - \big(\boldsymbol{I} - \boldsymbol{J}_r^{\#}(\boldsymbol{q})\boldsymbol{J}_r(\boldsymbol{q})\big)\beta\,\nabla H(\boldsymbol{q}) \\ &= -\beta\,\nabla H(\boldsymbol{q}) + \boldsymbol{J}_r^{\#}(\boldsymbol{q})\,\big(\boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\,\dot{\boldsymbol{p}}_t + \boldsymbol{J}_r(\boldsymbol{q})\,\beta\,\nabla H(\boldsymbol{q})\big),\end{aligned} \tag{9}$$

for some stepsize $\beta > 0$ in the direction of the negative gradient of H.

When there is a task error $e_r = \alpha_d - \alpha \neq 0$, a feedback action should be incorporated in the joint velocity command (9) as

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_r^{\#}(\boldsymbol{q})\,(\boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\,\dot{\boldsymbol{p}}_t - k e_r) - \big(\boldsymbol{I} - \boldsymbol{J}_r^{\#}(\boldsymbol{q})\boldsymbol{J}_r(\boldsymbol{q})\big)\,\beta\,\nabla H(\boldsymbol{q}), \tag{10}$$

for some scalar control gain $k > 0$. When the matrix $\boldsymbol{J}_r$ has full rank, i.e., when its single row does not vanish, then $\boldsymbol{J}_r\,\boldsymbol{J}_r^{\#} = 1$ and using (10) it follows that

$$\begin{aligned}\dot{e}_r = \dot{\alpha}_d - \dot{\alpha} &= \boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\boldsymbol{J}_p(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\dot{\boldsymbol{p}}_t - \boldsymbol{J}_{\alpha}\dot{\boldsymbol{q}} = \boldsymbol{J}_r(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\dot{\boldsymbol{p}}_t \\ &= \boldsymbol{J}_r(\boldsymbol{q})\big(\boldsymbol{J}_r^{\#}(\boldsymbol{q})\,(\boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\dot{\boldsymbol{p}}_t - k e_r) - \big(\boldsymbol{I} - \boldsymbol{J}_r^{\#}(\boldsymbol{q})\,\boldsymbol{J}_r(\boldsymbol{q})\big)\,\beta\,\nabla H(\boldsymbol{q})\big) - \boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\dot{\boldsymbol{p}}_t \\ &= \boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\dot{\boldsymbol{p}}_t - k e_r - \boldsymbol{J}_{\alpha_d}(\boldsymbol{q})\dot{\boldsymbol{p}}_t \\ &= -k e_r,\end{aligned}$$

showing exponential recovery of the task error.

In order to keep the joints close to the middle values $\bar{q}_i = (q_{m,i} + q_{M,i})/2$ of their ranges $[q_{m,i}, q_{M,i}]$, for $i = 1, \ldots, 4$, the following objective function (in our case, with $N = 4$)

$$H(\boldsymbol{q}) = \frac{1}{2N} \sum_{i=1}^{N} \left(\frac{q_i - \bar{q}_i}{q_{M,i} - q_{m,i}} \right)^2$$

will be minimized in the null space of the primary task. The gradient of H is

$$\nabla H(\boldsymbol{q}) = \left(\frac{\partial H}{\partial \boldsymbol{q}} \right)^T = \frac{1}{N} \begin{pmatrix} \dfrac{q_1 - \bar{q}_1}{(q_{M,1} - q_{m,1})^2} \\ \vdots \\ \dfrac{q_N - \bar{q}_N}{(q_{M,N} - q_{m,N})^2} \end{pmatrix}.$$

With the robot in the configuration $\boldsymbol{q} = (0, \pi/2, 0, -\pi/4)$, we can evaluate a number of terms useful for the control expressions (9) or (10):

$$\boldsymbol{p} = \begin{pmatrix} 1.7071 \\ 2.7071 \end{pmatrix} \qquad \boldsymbol{J}_p = \begin{pmatrix} -2.7071 & -2.7071 & -1.7071 & -0.7071 \\ 1.7071 & 0.7071 & 0.7071 & 0.7071 \end{pmatrix} \qquad \alpha = \pi/4.$$

Moreover, using the given joint limits we have

$$H = 0.0625 \qquad \nabla H = \begin{pmatrix} 0 \\ 0.0796 \\ 0 \\ -0.0796 \end{pmatrix}.$$

Consider first the nominal initial condition in which the end effector is correctly pointing at the target (see Fig. 4). In this case, the actual position $\boldsymbol{p}_t$ of the target is not specified in the problem

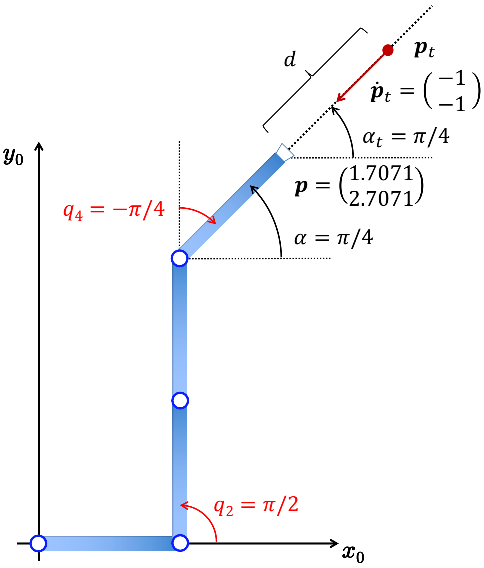


Figure 4: Nominal initial condition for the task, with the robot end effector pointing at the target. In addition, the target moves along the joining direction with $\dot{\boldsymbol{p}}_t = (-1, -1)$ [m/s].

and, as already mentioned, we cannot evaluate the Jacobian $\boldsymbol{J}_{\alpha_d}$ nor the task matrix $\boldsymbol{J}_r$. However, we are in a very special situation since the velocity $\dot{\boldsymbol{p}}_t$ will keep the target along the direction joining it to the robot end effector. Analytically, the target position is on the half-line

$$\boldsymbol{p}_t = \boldsymbol{p} + d \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} 1.7071 \\ 2.7071 \end{pmatrix} + d \begin{pmatrix} 0.7071 \\ 0.7071 \end{pmatrix} \qquad d > 0,$$

at a (unknown) distance d from the robot end effector. Since $\sin\alpha = \cos\alpha = 0.7071$ and the velocity of the target is $\dot{\boldsymbol{p}}_t = (-1, -1)$ [m/s], we have from (7)

$$\boldsymbol{J}_{\alpha_d}\, \dot{\boldsymbol{p}}_t = \begin{pmatrix} \dfrac{\sin\alpha}{d} & -\dfrac{\cos\alpha}{d} \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 0.$$

Moreover

$$\begin{aligned} \boldsymbol{J}_r = \boldsymbol{J}_{\alpha_d}\boldsymbol{J}_p - \boldsymbol{J}_\alpha &= \frac{0.7071}{d} \begin{pmatrix} -4.4142 & -3.4142 & -2.4142 & -1.4142 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \\ &= \frac{1}{d} \begin{pmatrix} -(3.2113 + d) & -(2.4142 + d) & -(1.7071 + d) & -(1 + d) \end{pmatrix}. \end{aligned}$$

Without pursuing further the numerical computation, note that since $\boldsymbol{J}_r$ is just a row, its pseudo-inverse is $\boldsymbol{J}_r^\# = \boldsymbol{J}_r^T / \| \boldsymbol{J}_r^T \|^2$. The solution (9) takes then the form

$$\dot{\boldsymbol{q}}_H = -\beta \left(\boldsymbol{I} - \frac{1}{\| \boldsymbol{J}_r^T \|^2} \boldsymbol{J}_r^T \boldsymbol{J}_r \right) \nabla H.$$

The distance $d > 0$, which is embedded in the product $\boldsymbol{J}_r^\# \boldsymbol{J}_r$, plays a limited role in the solution. On the other hand, the larger is $\beta > 0$, the more the joints will try to get close to their midrange.

In the second situation, i.e., when there is an initial pointing error $e_r \neq 0$, the target position is given and thus all numerical data are available for evaluating the control law (10). In fact, we have

$$\alpha_d = -3.0209 \qquad e_r = -3.8063 \qquad \text{[rad]}$$

and

$$\boldsymbol{J}_r = \begin{pmatrix} 0.1751 & -0.4022 & -0.4722 & -0.5423 \end{pmatrix}.$$

Choosing for instance $k = 0.2$ and $\beta = 20$ in eq. (10) yields finally

$$\dot{\boldsymbol{q}}_H = \begin{pmatrix} 0.3681 & -2.4370 & -0.9927 & 0.4516 \end{pmatrix}^T \qquad \text{[rad/s]}.$$

In the chosen configuration, the second joint is at its upper limit, i,e., $q_2 = q_{M,2} = \pi/2$, while the fourth joint is at its lower limit, i.e., $q_4 = q_{m,4} = -\pi/4$; therefore, in order to remain within the feasible range, the velocity of the second and fourth joints should be respectively $\dot{q}_2 \leq 0$ (remain at rest, or rotate clockwise) and $\dot{q}_4 \geq 0$ (remain at rest, or rotate counterclockwise), which is what happens with the solution $\dot{\boldsymbol{q}}_H$. Note however that this is obtained by fine tuning the gain k and (especially) the stepsize β in the control law (10), since the joint range limits are not considered as hard constraints in the problem but only within the (soft) objective H.

Exercise 3

Consider the closed-loop equations (2), (3) with the control law (4)

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{K}(\boldsymbol{q}-\boldsymbol{\theta}) = \boldsymbol{0} \tag{11}$$

$$\boldsymbol{B}_m\ddot{\boldsymbol{\theta}} + \boldsymbol{K}(\boldsymbol{\theta}-\boldsymbol{q}) = \boldsymbol{K}_P(\boldsymbol{\theta}_d-\boldsymbol{\theta}) - \boldsymbol{K}_D\dot{\boldsymbol{\theta}}, \tag{12}$$

where we replaced $\boldsymbol{c}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ using any factorization of the Coriolis and centrifugal terms. It is easy to see that $\boldsymbol{x} = (\boldsymbol{q},\boldsymbol{\theta},\dot{\boldsymbol{q}},\dot{\boldsymbol{\theta}}) = (\boldsymbol{\theta}_d,\boldsymbol{\theta}_d,\boldsymbol{0},\boldsymbol{0}) = \boldsymbol{x}_e$ is the unique equilibrium state of such a controlled robot with elastic joints (in the absence of gravity).

To show that this is a globally asymptotically stable equilibrium, define the energy-based Lyapunov candidate

$$V = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{\theta}}^T\boldsymbol{B}_m\dot{\boldsymbol{\theta}} + \frac{1}{2}(\boldsymbol{q}-\boldsymbol{\theta})^T\boldsymbol{K}(\boldsymbol{q}-\boldsymbol{\theta}) + \frac{1}{2}(\boldsymbol{\theta}_d-\boldsymbol{\theta})^T\boldsymbol{K}_P(\boldsymbol{\theta}_d-\boldsymbol{\theta}),$$

which contains the kinetic energy of the links and the motors, the potential energy due to the elasticity of the joints (quadratic in the joint deformation $\boldsymbol{\delta} = \boldsymbol{q}-\boldsymbol{\theta}$) , and a virtual potential energy introduced by the control (in terms of the motor position error, with $\boldsymbol{K}_P > 0$). For this function, it is $V \geq 0$ for all $\boldsymbol{x}$ and $V = 0$ if and only if $\boldsymbol{x} = \boldsymbol{x}_e$.

The time derivative of V, evaluated along the trajectories of the closed-loop system, is

$$\begin{aligned}
\dot{V} &= \dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\boldsymbol{\theta}}^T\boldsymbol{B}_m\ddot{\boldsymbol{\theta}} + (\dot{\boldsymbol{q}}-\dot{\boldsymbol{\theta}})^T\boldsymbol{K}(\boldsymbol{q}-\boldsymbol{\theta}) - \dot{\boldsymbol{\theta}}^T\boldsymbol{K}_P(\boldsymbol{\theta}_d-\boldsymbol{\theta}) \\
&= -\dot{\boldsymbol{q}}^T\left(\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{K}(\boldsymbol{q}-\boldsymbol{\theta})\right) + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\boldsymbol{\theta}}^T\left(\boldsymbol{K}(\boldsymbol{q}-\boldsymbol{\theta}) + \boldsymbol{K}_P(\boldsymbol{\theta}_d-\boldsymbol{\theta}) - \boldsymbol{K}_D\dot{\boldsymbol{\theta}}\right) \\
&\quad + \dot{\boldsymbol{q}}^T\boldsymbol{K}(\boldsymbol{q}-\boldsymbol{\theta}) - \dot{\boldsymbol{\theta}}^T\boldsymbol{K}(\boldsymbol{q}-\boldsymbol{\theta}) - \dot{\boldsymbol{\theta}}^T\boldsymbol{K}_P(\boldsymbol{\theta}_d-\boldsymbol{\theta}) - \dot{\boldsymbol{\theta}}^T\boldsymbol{K}_D\dot{\boldsymbol{\theta}} \\
&= -\dot{\boldsymbol{\theta}}^T\boldsymbol{K}_D\dot{\boldsymbol{\theta}} \leq 0,
\end{aligned}$$

where we used the identity

$$\dot{\boldsymbol{q}}^T\left(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{S}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)\dot{\boldsymbol{q}} = 0, \qquad \text{for all } \boldsymbol{q},\dot{\boldsymbol{q}}.$$

Thus, the closed-loop system is certainly stable.

To conclude about asymptotic stability, we use LaSalle theorem. Since

$$\dot{V} = 0 \qquad \Longleftrightarrow \qquad \dot{\boldsymbol{\theta}} = \boldsymbol{0},$$

we analyze the closed-loop equations for $\dot{\boldsymbol{\theta}} \equiv \mathbf{0}$. From eq. (12), since $\ddot{\boldsymbol{\theta}}$ must also vanish in order for any set contained in $\mathcal{S} = \{\boldsymbol{x} : \dot{V} = 0\}$ to be invariant, we have

$$\boldsymbol{K}\left(\boldsymbol{\theta} - \boldsymbol{q}\right) = \boldsymbol{K}_P\left(\boldsymbol{\theta}_d - \boldsymbol{\theta}\right) = \text{constant}. \tag{13}$$

Being $\boldsymbol{\theta}$ constant itself, this equation implies that also $\boldsymbol{q}$ must remain constant. Therefore, since $\dot{\boldsymbol{q}}$ and $\ddot{\boldsymbol{q}}$ must also vanish, from eq. (11) it follows that $\boldsymbol{K}(\boldsymbol{q}-\boldsymbol{\theta}) = \mathbf{0}$, and then $\boldsymbol{q} = \boldsymbol{\theta}$. Substituting this in (13) leads to $\boldsymbol{\theta} = \boldsymbol{\theta}_d$ (thus, the constant therein is necessarily zero). Summarizing, $\boldsymbol{q} = \boldsymbol{\theta} = \boldsymbol{\theta}_d$ and the maximal set of invariant states contained in $\mathcal{S}$ reduces to the singleton $\boldsymbol{x}_e = (\boldsymbol{\theta}_d, \boldsymbol{\theta}_d, \mathbf{0}, \mathbf{0})$, which is then a global, asymptotically stable equilibrium. This concludes the proof.

Exercise 4

The method of residuals for fault detection is based on terms appearing in the robot dynamics. We need then to derive first the dynamic model of the PR robot. Using the variables $\boldsymbol{q} = (q_1, q_2)$ defined in Fig. 2, we have in particular

$$\boldsymbol{p}_{c2} = \begin{pmatrix} d_{c2}c_2 \\ 0 \\ q_1 - d_{c2}s_2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{v}_{c2} = \begin{pmatrix} -d_{c2}s_2\,\dot{q}_2 \\ 0 \\ \dot{q}_1 - d_{c2}c_2\,\dot{q}_2 \end{pmatrix} \quad \Rightarrow \quad \left\|\boldsymbol{v}_{c2}\right\|^2 = \dot{q}_1^2 + d_{c2}^2\,\dot{q}_2^2 - 2\,d_{c2}c_2\,\dot{q}_1\dot{q}_2.$$

Therefore, the kinetic and potential energies of the two links are

$$T_1 = \frac{1}{2}\,m_1\dot{q}_1^2 \qquad T_2 = \frac{1}{2}\,I_{c2}\,\dot{q}_2^2 + \frac{1}{2}\,m_2\left(\dot{q}_1^2 + d_{c2}^2\,\dot{q}_2^2 - 2\,d_{c2}c_2\,\dot{q}_1\dot{q}_2\right) \qquad \Rightarrow \qquad T = T_1 + T_2$$

and

$$U_1 = -m_1 g_0\,q_1 \qquad U_2 = -m_2 g_0\left(q_1 - d_{c2}s_2\right) \qquad \Rightarrow \qquad U = U_1 + U_2.$$

with $g_0 = 9.81$ m/s^2.

The dynamic model of the robot, assuming the possible presence of a fault $u_{f1}(t)$ on the first actuator, is

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u} - \boldsymbol{u}_f. \tag{14}$$

with inertia matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1 + m_2 & -m_2 d_{c2}\,c_2 \\ -m_2 d_{c2}\,c_2 & I_{c2} + m_2 d_{c2}^2 \end{pmatrix},$$

velocity terms (as computed from Christoffel symbols)

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} m_2 d_{c2} s_2\,\dot{q}_2^2 \\ 0 \end{pmatrix},$$

gravity vector

$$\boldsymbol{g}(\boldsymbol{q}) = \frac{\partial U}{\partial \boldsymbol{q}} = g_0 \begin{pmatrix} -(m_1 + m_2) \\ m_2 d_{c2} c_2 \end{pmatrix},$$

and fault vector

$$\boldsymbol{u}_f = \begin{pmatrix} u_{f1} \\ 0 \end{pmatrix}.$$

The expression of the scalar residual for a fault on the first actuator[1] is

$$r_1(t) = k\left(\int_0^t \left(u_1 - \alpha_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) - r_1\right) d\tau - p_1(\boldsymbol{q}, \dot{\boldsymbol{q}})\right) \qquad r_1(0) = 0, \tag{15}$$

[1]It is assumed that the robot starts at rest at $t = 0$, i.e., $\dot{\boldsymbol{q}}(0) = \mathbf{0}$.

where $k > 0$, u_1 is the commanded force for the actuator at the first prismatic joint, the function α_1 is defined as

$$\alpha_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\frac{1}{2}\,\dot{\boldsymbol{q}}^T\,\frac{\partial \boldsymbol{M}}{\partial q_1}\,\dot{\boldsymbol{q}} + g_1(\boldsymbol{q}),$$

and $p_1 = \boldsymbol{m}_1^T(\boldsymbol{q})\dot{\boldsymbol{q}}$ is the first component of the generalized momentum $\boldsymbol{p} = \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$, being $\boldsymbol{m}_1$ the first column of the inertia matrix. Since q_1 never appears in the inertia matrix, one has simply

$$\alpha_1 = -(m_1 + m_2)\, g_0,$$

while

$$p_1 = (m_1 + m_2)\,\dot{q}_1 - m_2 d_{c2}\, c_2\, \dot{q}_2.$$

Substituting these expressions in (15) gives finally

$$r_1(t) = k_1 \left(\int_0^t (u_1 + (m_1 + m_2)\, g_0 - r_1)\, d\tau + m_2 d_{c2}\, c_2\, \dot{q}_2 - (m_1 + m_2)\, \dot{q}_1 \right) \qquad r_1(0) = 0. \tag{16}$$

The theory says that the evolution of $r_1(t)$ is governed by $\dot{r}_1 = k_1\,(u_{f1} - r_1)$, allowing detection of the fault, whenever present, through the response of a first-order filter with time constant $1/k_1$ excited by the unknown input signal u_{f1}. One can also verify this property by differentiating (16) and using the model terms in (14). In fact

$$\begin{aligned}
\dot{r}_1 &= k_1 \left((u_1 + (m_1 + m_2)\, g_0 - r_1) + m_2 d_{c2}\, c_2\, \ddot{q}_2 - m_2 d_{c2}\, s_2\, \dot{q}_2^2 - (m_1 + m_2)\, \ddot{q}_1 \right) \\
&= k_1 \left((u_1 + (m_1 + m_2)\, g_0 - r_1) - (u_1 - u_{f1} + (m_1 + m_2)\, g_0) \right) \\
&= k_1 \left(u_{f1} - r_1 \right).
\end{aligned}$$

If the fault is $u_{f1}(t) = 2$, for $t \geq 0$, the solution trajectory $r_1(t)$ and its steady-state value are

$$r_1(t) = 2(1 - \exp(-k_1 t)) \qquad \Rightarrow \qquad r_{1,ss} = \lim_{t \to \infty} r_1(t) = 2.$$

* * * * *

Robotics 2

July 8, 2024

Exercise 1

Consider the robot in Fig. 1 with $n = 3$ joints, the first one prismatic and the other two revolute. Each link has uniformly distributed mass, center of mass on its major physical axis, and a diagonal barycentric inertia matrix. Assume that friction at the joints can be neglected. The robot is commanded at the joint level by a generalized vector of forces/torques $\boldsymbol{\tau} \in \mathbb{R}^3$.

a) Derive the dynamic model of the robot in the Lagrangian form $\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}.\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$.

b) Find a linear parametrization $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\,\boldsymbol{a} = \boldsymbol{\tau}$ of the robot dynamics in terms of a vector $\boldsymbol{a} \in \mathbb{R}^p$ of dynamic coefficients and of a $3 \times p$ regressor matrix $\boldsymbol{Y}$. Discuss the minimality of p.

c) Determine which of the $10n = 30$ standard dynamic parameters of the links are irrelevant for the describing the motion of the robot.

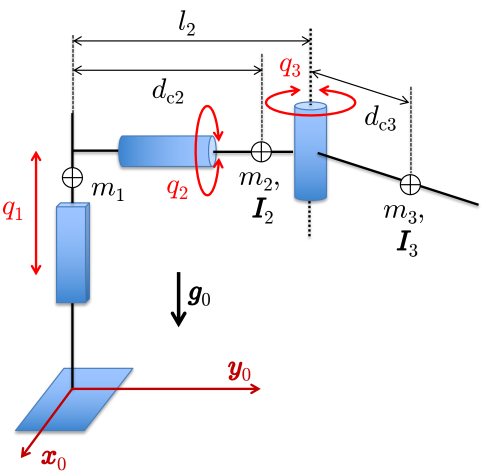


Figure 1: A PRR robot with its coordinates $\boldsymbol{q} = (q_1\ q_2\ q_3)^T$ and kinematic/dynamic parameters.

Exercise 2

The 2R planar robot in Fig. 2 has equal links of length l and is commanded by the joint acceleration $\ddot{\boldsymbol{q}} \in \mathbb{R}^2$. The robot end effector has to perform a one-dimensional trajectory task $r_d(t) \in \mathbb{R}$ specified *only* along the x-direction. In a given robot state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, a desired task acceleration $\ddot{r}_d$ is assigned.

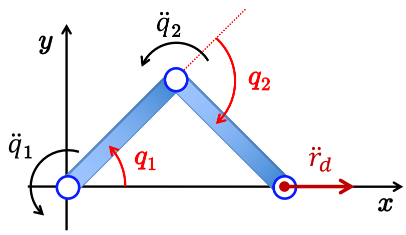


Figure 2: A 2R planar robot in a one-dimensional task.

Provide in symbolic form the command $\ddot{\boldsymbol{q}}$ that executes the task while minimizing the cost

$$H = \frac{1}{2}\left(\ddot{\boldsymbol{q}} - \ddot{\boldsymbol{q}}_0\right)^T\left(\ddot{\boldsymbol{q}} - \ddot{\boldsymbol{q}}_0\right), \qquad \ddot{\boldsymbol{q}}_0 = -\boldsymbol{K}_v\,\dot{\boldsymbol{q}},$$

for a diagonal matrix $\boldsymbol{K}_v > 0$. Evaluate then numerically the solution when the robot is in the configuration $\boldsymbol{q} = (\pi/4, -\pi/2)$ [rad] with joint velocity $\dot{\boldsymbol{q}} = (1, -1)$ [rad/s], the link length is $l = 1$ m, the task acceleration is $\ddot{r}_d = 1$ m/s^2, and $\boldsymbol{K}_v = \text{diag}\{2, 2\}$ [s^{-1}]. How would you modify the acceleration command if there was a task error in position and/or velocity?

Exercise 3

Figure 3 shows a mechanical system made of two masses B and M connected by a pulley and a damped elastic spring, viscous friction on the motion of the individual masses, an input force τ acting on the first mass, and gravity acting on the second mass only. The zero of the two position variables θ and q is associated to an undeformed spring. The spring has stiffness $K > 0$ and its elastic potential energy is quadratic in the deformation $q - \theta$.

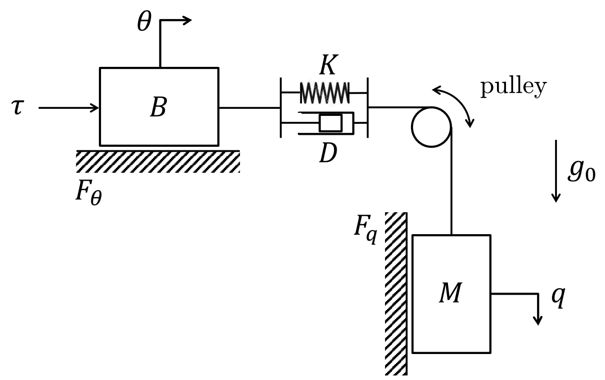


Figure 3: Two masses connected by a pulley and a damped elastic spring under gravity.

a) Derive the dynamic model of this system, including all non-conservative terms due to viscous friction (with coefficients $F_\theta > 0$ and $F_q > 0$, respectively) affecting the motion of the two masses and damping (with coefficient $D \geq 0$) on the time-varying deformation of the spring.

b) Provide the simplest feedback law that is able to asymptotically stabilize the position of the mass M to a constant desired height q_d. Prove the result using any preferred analysis method (linearization by Taylor expansion, Lyapunov/LaSalle, etc.).

c) Set now $D = 0$. Solve the inverse dynamics problem for a desired, sufficiently smooth trajectory $q_d(t)$. Provide the explicit expression of $\tau_d(t)$ as a function of $q_d(t)$ and its (higher order) time derivatives only.

Exercise 4

Suppose that a 2R planar robot that is moving in a vertical plane has only one actuator at the first joint providing a torque τ. Find the expression of all forced equilibria $(\bar{\boldsymbol{q}}, \boldsymbol{0})$ associated to a generic constant input torque $\bar{\tau}$. Are you able to find a state feedback control law $\tau = f(\bar{\boldsymbol{q}}, \boldsymbol{q}, \dot{\boldsymbol{q}})$ that asymptotically stabilizes one of these equilibria, at least locally?

[240 minutes (4 hours); open books]

Solution

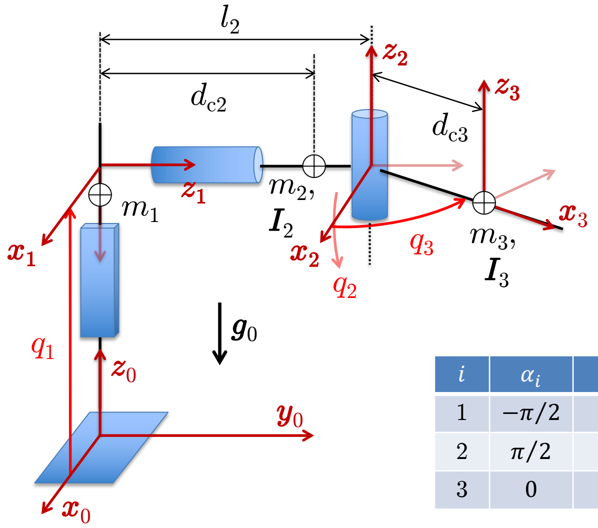
July 8, 2024

Exercise 1

The kinetic energies of the first two links are easy to compute:

$$T_1 = \frac{1}{2}\, m_1 \dot{q}_1^2 \qquad\qquad T_2 = \frac{1}{2}\, m_2 \dot{q}_1^2 + \frac{1}{2}\, I_{c2,yy}\, \dot{q}_1^2.$$

For the third link, one can use the moving frame recursive algorithm for obtaining ${}^3\boldsymbol{v}_{c3}$ and ${}^3\boldsymbol{\omega}_3$, or use the direct kinematics of the robot for computing the position $\boldsymbol{p}_{c3}(\boldsymbol{q})$ of the center of mass, and then differentiating it, as well as the orientation $\boldsymbol{R}_3(\boldsymbol{q})$ of the last frame, extracting then the angular velocity from its derivative. In any event, one should attach frames to the robot arm according to the Denavit-Hartenberg (DH) convention, as done in Fig. 4. Note that the last frame has been placed conveniently at the center of mass of the third link: this is reflected in the D-H parameter $a_3 = d_{c3}$.



i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	q_1	0
2	$\pi/2$	0	l_2	q_2
3	0	d_{c3}	0	q_3

Figure 4: DH frames and associated table of parameters for the PRR robot of Fig. 1.

Following the second approach, we compute via the DH direct kinematics

$$\boldsymbol{p}_{c3} = \begin{pmatrix} d_{c3} c_2 c_3 \\ l_2 + d_{c3} s_3 \\ q_1 - d_{c3} s_2 c_3 \end{pmatrix},$$

so that its velocity in the absolute (zero) frame and in the local (third) DH frame are

$$\boldsymbol{v}_{c3} = \dot{\boldsymbol{p}}_{c3} = \begin{pmatrix} -d_{c3}\,(s_2 c_3\, \dot{q}_2 + c_3 s_3\, \dot{q}_3) \\ d_{c3} c_3\, \dot{q}_3 \\ \dot{q}_1 + d_{c3}\,(s_2 s_3\, \dot{q}_3 - c_2 c_3\, \dot{q}_2) \end{pmatrix} \quad \Rightarrow \quad {}^3\boldsymbol{v}_{c3} = {}^0\boldsymbol{R}_3^T(\boldsymbol{q})\, \boldsymbol{v}_{c3} = \begin{pmatrix} -s_2 c_3\, \dot{q}_1 \\ s_2 s_3\, \dot{q}_1 + d_{c3}\, \dot{q}_3 \\ c_2\, \dot{q}_1 - d_{c3} c_3\, \dot{q}_2 \end{pmatrix}.$$

Similarly, using the relationship

$$\boldsymbol{S}(\boldsymbol{\omega}_3) = {}^0\dot{\boldsymbol{R}}_3(\boldsymbol{q})\, {}^0\boldsymbol{R}_3^T(\boldsymbol{q}),$$

the angular velocity of the third link is

$$\boldsymbol{\omega}_3 = \begin{pmatrix} -S_{23}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ S_{13}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ -S_{12}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{pmatrix} = \begin{pmatrix} s_2\,\dot{q}_3 \\ \dot{q}_2 \\ c_2\,\dot{q}_3 \end{pmatrix} \qquad \Rightarrow \qquad {}^3\boldsymbol{\omega}_3 = {}^0\boldsymbol{R}_3^T(\boldsymbol{q})\,\boldsymbol{\omega}_3 = \begin{pmatrix} s_3\,\dot{q}_2 \\ c_3\,\dot{q}_2 \\ \dot{q}_3 \end{pmatrix}.$$

Thus

$$\begin{aligned} T_3 &= \frac{1}{2}\,m_3\left\|{}^3\boldsymbol{v}_{c3}\right\|^2 + \frac{1}{2}\,{}^3\boldsymbol{\omega}_3^T\,{}^3\boldsymbol{I}_3\,{}^3\boldsymbol{\omega}_3 \\ &= \frac{1}{2}\,m_3\left((c_2\,\dot{q}_1 - d_{c3}c_3\,\dot{q}_2)^2 + (s_2 s_3\,\dot{q}_1 - d_{c3}\,\dot{q}_3)^2 + s_2^2 c_3^2\,\dot{q}_1^2\right) + \frac{1}{2}\left((I_{c3,xx}s_3^2 + I_{c3,yy}c_3^2)\,\dot{q}_2^2 + I_{c3,zz}\,\dot{q}_3^2\right). \end{aligned}$$

After substituting $s_3^2 = 1 - c_3^2$, the inertia matrix of the robot is found from the total kinetic energy

$$T = T_1 + T_2 + T_3 = \frac{1}{2}\,\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}},$$

as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} m_1 + m_2 + m_3 & -m_3 d_{c3}\,c_2 c_3 & m_3 d_{c3}\,s_2 s_3 \\ -m_3 d_{c3}\,c_2 c_3 & I_{c2,yy} + I_{c3,xx} + \left(I_{c3,yy} - I_{c3,xx} + m_3 d_{c3}^2\right)c_3^2 & 0 \\ m_3 d_{c3}\,s_2 s_3 & 0 & I_{c3,zz} + m_3 d_{c3}^2 \end{pmatrix}.$$

Introducing the following $p = 5$ dynamic coefficients

$$\begin{aligned} \boldsymbol{a} &= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix}^T \\ a_1 &= m_1 + m_2 + m_3 \\ a_2 &= I_{c2,yy} + I_{c3,xx} \\ a_3 &= I_{c3,yy} - I_{c3,xx} + m_3 d_{c3}^2 \\ a_4 &= m_3 d_{c3} \\ a_5 &= I_{c3,zz} + m_3 d_{c3}^2, \end{aligned} \tag{1}$$

the inertia matrix is rewritten more compactly as

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{pmatrix} a_1 & -a_4\,c_2 c_3 & a_4\,s_2 s_3 \\ -a_4\,c_2 c_3 & a_2 + a_3\,c_3^2 & 0 \\ a_4\,s_2 s_3 & 0 & a_5 \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}_1(\boldsymbol{q}) & \boldsymbol{M}_2(\boldsymbol{q}) & \boldsymbol{M}_3(\boldsymbol{q}) \end{pmatrix}.$$

Using Christoffel's symbols, the Coriolis and centrifugal terms are computed as

$$\boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{pmatrix} a_4\left(s_2 c_3\,(\dot{q}_2^2 + \dot{q}_3^2) + 2\,c_2 s_3\,\dot{q}_2\dot{q}_3\right) \\ -2a_3\,s_3 c_3\,\dot{q}_2\dot{q}_3 \\ a_3\,s_3 c_3\,\dot{q}_2^2 \end{pmatrix}.$$

Finally, the potential energy of the three links due to gravity is $U = U_1 + U_2 + U_3$, with

$$U_1 = m_1 g_0\,(q_1 - d_{c1}) \qquad U_2 = m_2 g_0\,q_1 \qquad U_3 = m_3 g_0\,(q_1 - d_{c3}s_2 c_3)\,.$$

Thus

$$\boldsymbol{g}(\boldsymbol{q}) = \left(\frac{\partial U}{\partial \boldsymbol{q}}\right)^T = \begin{pmatrix} g_0\,(m_1 + m_2 + m_3) \\ -m_3 d_{c3} g_0\,c_2 c_3 \\ m_3 d_{c3} g_0\,s_2 s_3 \end{pmatrix} = \begin{pmatrix} a_1 g_0 \\ -a_4 g_0\,c_2 c_3 \\ a_4 g_0\,s_2 s_3 \end{pmatrix}, \tag{2}$$

where $g_0 = 9.81$ m/s^2 is assumed to be known (this allows to use in (2) the previously introduced coefficients a_1 and a_4).

The complete dynamic model is

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{c}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{Y}(\boldsymbol{q}.\dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\, \boldsymbol{a} = \boldsymbol{\tau},$$

with the 3×5 regressor matrix $\boldsymbol{Y}$ of the linear parametrization given by

$$\boldsymbol{Y} = \begin{pmatrix} \ddot{q}_1 + g_0 & 0 & 0 & s_2c_3\,(\dot{q}_2^2 + \dot{q}_3^2) + 2c_2s_3\,\dot{q}_2\dot{q}_3 - c_2c_3\,\ddot{q}_2 + s_2s_3\,\ddot{q}_3 & 0 \\ 0 & \ddot{q}_2 & c_3\,\ddot{q}_2 - 2s_3c_3\,\dot{q}_2\dot{q}_3 & -(\ddot{q}_1 + g_0)\,c_2c_3 & 0 \\ 0 & 0 & s_3c_3\,\dot{q}_2^2 & (\ddot{q}_1 + g_0)\,s_2s_3 & \ddot{q}_3 \end{pmatrix} \tag{3}$$

and the dynamic coefficients $\boldsymbol{a} \in \mathbb{R}^5$ defined in (1).

Summarizing, out of the $3 \times 10 = 30$ standard dynamic parameters of the three robot links, half of them (15) has been removed from the beginning because of the simplifying assumptions made (center of mass on the kinematic/physical link axis, diagonal barycentric inertia matrix); other 7 parameters never appear, and play thus no role in the robot dynamics; the remaining 8 dynamic parameters appear in suitable combinations, generating the 5 dynamic coefficients in (1) — note that $a_4 = m_3 d_{c3}$ is both a standard dynamic parameter and a dynamic coefficient.

Exercise 2

The one-dimensional task kinematics of the 2R planar robot is

$$r = p_x(\boldsymbol{q}) = l_1 \cos q_1 + l_2 \cos(q_1 + q_2).$$

Therefore, we have

$$\dot{r} = \frac{\partial p_x}{\partial \boldsymbol{q}}\, \dot{\boldsymbol{q}} = \boldsymbol{J}_r(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix} -\,(l_1 \sin q_1 + l_2 \sin(q_1 + q_2)) & -l_2 \sin(q_1 + q_2) \end{pmatrix} \dot{\boldsymbol{q}},$$

with the 1×2 task Jacobian matrix $\boldsymbol{J}_r$, and

$$\ddot{r} = \boldsymbol{J}_r(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_r(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{J}_r(\boldsymbol{q})\ddot{\boldsymbol{q}} + h(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

where

$$\begin{aligned} h(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \begin{pmatrix} -\,(l_1 \cos q_1 \dot{q}_1 + l_2 \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2)) & -l_2 \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2) \end{pmatrix} \dot{\boldsymbol{q}} \\ &= -\left(l_1 \cos q_1\, \dot{q}_1^2 + l_2 \cos(q_1 + q_2)(\dot{q}_1 + \dot{q}_2)^2\right). \end{aligned}$$

Note that the y-component of the end-effector position is not assigned by the task and its acceleration can take any value (this is why the 2R planar robot is redundant with respect to the given one-dimensional task).

The joint acceleration command $\ddot{\boldsymbol{q}}$ that realizes the desired task $\ddot{r}_d$ (out of singularities, i.e., where $\boldsymbol{J}_r(\boldsymbol{q})$ does not vanish) while minimizing instantaneously the complete quadratic objective H, where $\ddot{\boldsymbol{q}}_0 = -\boldsymbol{K}_v\dot{\boldsymbol{q}}$ is the preferred joint acceleration, is given by

$$\begin{aligned} \ddot{\boldsymbol{q}} &= \boldsymbol{J}_r^{\#}(\boldsymbol{q})\,(\ddot{r}_d - h(\boldsymbol{q}, \dot{\boldsymbol{q}})) - \left(\boldsymbol{I} - \boldsymbol{J}_r^{\#}(\boldsymbol{q})\boldsymbol{J}_r(\boldsymbol{q})\right)\boldsymbol{K}_v\dot{\boldsymbol{q}} \\ &= -\boldsymbol{K}_v\dot{\boldsymbol{q}} + \boldsymbol{J}_r^{\#}(\boldsymbol{q})\,(\ddot{r}_d - h(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{J}_r(\boldsymbol{q})\boldsymbol{K}_v\dot{\boldsymbol{q}}), \end{aligned} \tag{4}$$

with $\boldsymbol{J}_r^{\#}(\boldsymbol{q}) = \boldsymbol{J}_r^T(\boldsymbol{q})(\boldsymbol{J}_r(\boldsymbol{q})\boldsymbol{J}_r^T(\boldsymbol{q}))^{-1}$ (if $\boldsymbol{J}_r(\boldsymbol{q}) \neq \boldsymbol{0}^T$). Substituting the numerical values of the problem in (4) gives

$$\ddot{\boldsymbol{q}} = \begin{pmatrix} -2 \\ 2.4142 \end{pmatrix} \text{[rad/s}^2\text{]}. \tag{5}$$

Note that the two components of the joint acceleration both oppose their respective velocity (i.e., $\ddot{q}_i \dot{q}_i < 0$, for $i = 1, 2$), which confirms that the damping action specified by $\ddot{\boldsymbol{q}}_0$ is being pursued. On the other hand, with the acceleration (5) produced in the given state $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, the resulting acceleration of the end-effector is

$$\ddot{\boldsymbol{p}} = \begin{pmatrix} \ddot{p}_x \\ \ddot{p}_y \end{pmatrix} = \begin{pmatrix} 1 \\ -1.8284 \end{pmatrix} \ [\text{m/s}^2]$$

showing that $\ddot{p}_x = 1 = \ddot{r}_d$ has been correctly realized (while $\ddot{p}_y \neq 0$ is just a result of the chosen redundancy resolution scheme).

In the presence of a task error, the command (4) is modified by adding a PD action to the desired task acceleration,

$$\ddot{\boldsymbol{q}} = -\boldsymbol{K}_v \dot{\boldsymbol{q}} + \boldsymbol{J}_r^{\#}(\boldsymbol{q}) \left(\ddot{r}_d + K_D(\dot{r}_d - \boldsymbol{J}_r(\boldsymbol{q})\dot{\boldsymbol{q}}) + K_P(r_d - p_x(\boldsymbol{q})) - h(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{J}_r(\boldsymbol{q})\boldsymbol{K}_v \dot{\boldsymbol{q}} \right), \qquad (6)$$

so as to bring the task error back to zero at an exponential rate governed by the choice of the two gains $K_P > 0$ and $K_D > 0$.

Exercise 3

The dynamic model of the system in Fig. 3 is given by the two second-order differential equations (one for each mass)

$$B\,\ddot{\theta} + K\,(\theta - q) + D\,(\dot{\theta} - \dot{q}) + F_\theta\,\dot{\theta} = \tau \qquad (7)$$

$$M\,\ddot{q} + K\,(q - \theta) - Mg_0 + D\,(\dot{q} - \dot{\theta}) + F_q\,\dot{q} = 0, \qquad (8)$$

which can be easily derived either from an energy-based Lagrangian approach with

I think this should be theta and q without the dot

$$\mathcal{T}_B = \frac{1}{2}\,B\dot{\theta}^2 \qquad \mathcal{T}_M = \frac{1}{2}\,M\dot{q}^2 \qquad \mathcal{U}_g = -Mg_0 q \qquad \mathcal{U}_e = \frac{1}{2}\,K\,(\dot{\theta} - \dot{q})^2,$$

including all non-conservative terms, or simply by the balance of forces acting on the two masses in a Newton approach. Note that the dynamics (7) is *linear*, whereas eq. (8) is *affine* since it contains the offset constant term $-Mg_0$ due to gravity.

Under the action of a constant force $\bar{\tau}$, any forced equilibrium configuration $(\bar{\theta}, \bar{q})$ for eqs. (7), (8) should satisfy

$$K\,(\bar{\theta} - \bar{q}) = \bar{\tau} \qquad K\,(\bar{\theta} - \bar{q}) - Mg_0 = 0,$$

and thus

$$\bar{\tau} = -Mg_0 \qquad \bar{\theta} = \bar{q} - K^{-1}Mg_0. \qquad (9)$$

With the above in mind, for the regulation problem consider the simplest linear feedback law with constant feedforward

$$\tau = \tau_d + K_P(\theta_d - \theta) \qquad K_P > 0, \qquad (10)$$

where

$$\tau_d = -Mg_0 \qquad \theta_d = q_d - K^{-1}Mg_0 \qquad (11)$$

satisfy the conditions (9) for achieving an equilibrium; θ_d is the required position of mass B when mass M is in equilibrium at q_d. Note that no derivative term is present in (10), as customary instead in a PD control law: the presence of various sources of dissipation in the system (viscous friction with coefficients F_θ and F_q, damping D on the elastic spring) makes this additional control action no longer needed for stabilization purposes.

The closed-loop equations (7), (8) with the control law (10) are

$$B\,\ddot{\theta} + K\,(\theta - q) + D\,(\dot{\theta} - \dot{q}) + F_\theta\,\dot{\theta} = -Mg_0 + K_P(\theta_d - \theta) \tag{12}$$

$$M\,\ddot{q} + K\,(q - \theta) - Mg_0 + D\,(\dot{q} - \dot{\theta}) + F_q\,\dot{q} = 0. \tag{13}$$

At steady state, $\dot{\theta} = \ddot{\theta} = \dot{q} = \ddot{q} = 0$, it is

$$K\,(\bar{\theta} - \bar{q}) = -Mg_0 + K_P(\theta_d - \bar{\theta}) \qquad K\,(\bar{q} - \bar{\theta}) - Mg_0 = 0 \tag{14}$$

which imply $K_P(\theta_d - \bar{\theta}) = 0$, so that $(\bar{\theta}, \bar{q}) = (\theta_d, q_d) = (q_d - K^{-1}Mg_0, q_d)$ is the unique equilibrium configuration.

To verify the asymptotic stability of the closed-loop equilibrium, we present next two alternative methods: the first is a global approach based on Lyapunov analysis, completed by the use of LaSalle theorem as done during the course; the second analyzes the linearized version of the system dynamics, obtained by a first-order Taylor expansion around the desired closed-loop equilibrium point, and has in general only a local validity.[1]

1. Lyapunov analysis. Define the following energy-based function

$$V = \frac{1}{2}\,B\,\dot{\theta}^2 + \frac{1}{2}\,M\,\dot{q}^2 + \frac{1}{2}\,K\,(\theta - q)^2 - Mg_0\,q + \frac{1}{2}\,K_P\,(\theta_d - \theta)^2 - Mg_0\,(\theta_d - \theta)\,, \tag{15}$$

which contains the kinetic energy of the two masses, the potential energy of the elastic spring, the potential energy due to gravity (acting only on mass M), together with a virtual potential energy introduced by control (in terms of the position error of the mass B, with $K_P > 0$), and an additional term that is linear in the position error. The addition of this last term guarantees that $V \geq 0$ for all states $\boldsymbol{x} = (\theta, q, \dot{\theta}, \dot{q}) \in \mathbb{R}^4$ and $V = 0$ if and only if $\boldsymbol{x} = \boldsymbol{x}_e = (\theta_d, q_d, 0, 0)$, namely the desired equilibrium specified in (14), so that (15) is a Lyapunov candidate. On one hand, V is quadratic and positive definite with respect to the velocities $\dot{\theta}$ and $\dot{q}$. On the other hand, consider the remaining terms of V collected in the configuration-dependent function

$$P(\theta, q) = V|_{\dot{\theta}=\dot{q}=0} = \frac{1}{2}\,K\,(\theta - q)^2 + \frac{1}{2}\,K_P\,(\theta_d - \theta)^2 - Mg_0\,(\theta_d + q - \theta)\,.$$

It is easy to see that P is a convex function with global minimum at (θ_d, q_d): in fact, its gradient is

$$\nabla P = \begin{pmatrix} \nabla_\theta P \\ \nabla_q P \end{pmatrix} = \begin{pmatrix} K\,(\theta - q) - K_P\,(\theta_d - \theta) + Mg_0 \\ K\,(q - \theta) - Mg_0 \end{pmatrix}$$

and the stationarity condition $\nabla P = \mathbf{0}$ holds if and only if $(\theta, q) = (\theta_d, q_d)$ in agreement with (14); moreover, being the Hessian of P

$$\nabla^2 P = \begin{pmatrix} \nabla^2_{\theta\theta} P & \nabla^2_{\theta q} P \\ \nabla^2_{q\theta} P & \nabla^2_{qq} P \end{pmatrix} = \begin{pmatrix} K + K_P & -K \\ -K & K \end{pmatrix} > 0,$$

the desired configuration is a minimum for P. As a result, V is a valid Lyapunov candidate.

The time derivative of (15) evaluated along the trajectories of the closed-loop system is

$$\begin{aligned}
\dot{V} &= B\ddot{\theta}\,\dot{\theta} + M\ddot{q}\,\dot{q} + K\,(q - \theta)\,(\dot{q} - \dot{\theta}) - Mg_0\,(\dot{q} - \dot{\theta}) - K_P\,(\theta_d - \theta)\,\dot{\theta} \\
&= \Big(K\,(q - \theta) + D\,(\dot{q} - \dot{\theta}) - F_\theta\,\dot{\theta} - Mg_0 + K_P(\theta_d - \theta)\Big)\,\dot{\theta} \\
&\quad + \Big(K\,(\theta - q) + Mg_0 + D\,(\dot{\theta} - \dot{q}) - F_q\,\dot{q}\Big)\,\dot{q} \\
&\quad + K\,(q - \theta)\,(\dot{q} - \dot{\theta}) - Mg_0\,(\dot{q} - \dot{\theta}) - K_P\,(\theta_d - \theta)\,\dot{\theta} \\
&= -D\,(\dot{\theta} - \dot{q})^2 - F_\theta\,\dot{\theta}^2 - F_q\,\dot{q}^2 \leq 0.
\end{aligned}$$

[1] This method should be part of the background knowledge of any student exposed to linear dynamical systems.

Thus, the closed-loop system is certainly stable. To conclude about asymptotic stability, we use LaSalle theorem. Since

$$\dot{V} = 0 \quad \Longleftrightarrow \quad \dot{\theta} = \dot{q} = 0,$$

we analyze the closed-loop eqs. (12) and (13) under this condition:

$$\begin{aligned} B\,\ddot{\theta} + K\,(\theta - q) &= -Mg_0 + K_P(\theta_d - \theta) \\ M\,\ddot{q} + K\,(q - \theta) - Mg_0 &= 0. \end{aligned}$$

Since for any set of states contained in $\mathcal{S} = \{\boldsymbol{x} : \dot{V} = 0\}$ to be invariant both $\ddot{\theta}$ and $\ddot{q}$ must also vanish, we have

$$\begin{aligned} K\,(\theta - q) &= -Mg_0 + K_P(\theta_d - \theta) \\ K\,(q - \theta) - Mg_0 &= 0. \end{aligned}$$

As shown in (14), these equations have $(\theta, q) = (\theta_d, q_d)$ as the only solution. Thus, the maximal set of invariant states contained in $\mathcal{S}$ reduces to the singleton $\boldsymbol{x}_e = (\theta_d, q_d, 0, 0)$, which is then a global, asymptotically stable equilibrium. This concludes the proof.

It should be noted that a control law of the form

$$\tau = -Mg_0 + K_P(q_d - q) \qquad K_P > 0, \tag{16}$$

similar to eq. (10) and maybe more natural at first sight, produces the same desired equilibrium point. However, showing asymptotic stability with a Lyapunov argument is quite difficult — either when trying to define a correct candidate function $V \geq 0$, with $V = 0$ only at the desired equilibrium, or in proving that $\dot{V} \leq 0$ is obtained. Moreover, a restriction to the maximum gain in the control law (16) would apply, while a global result would be hard if not impossible to obtain. The reason for this behavior will become clearer when pursuing the alternative approach.

2. Analysis by local approximate linearization. A simpler and systematic approach consists in linearizing the dynamic equations (7), (8) around the desired equilibrium point $\boldsymbol{x}_d = (\theta_d, q_d, 0, 0)$. In this case, the approximate linearization procedure by Taylor expansion boils down to simply removing the constant offset term $-Mg_0$ due to gravity. Furthermore, one can apply Laplace transforms to the linearized equations and then conveniently use a SISO transfer function for describing the process to be controlled.

Let $\Delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_d = (\theta - \theta_d, q - q_d, \dot{\theta}, \dot{q}) = (\Delta\theta, \Delta q, \Delta\dot{\theta}, \Delta\dot{q})$ and $\Delta\tau = \tau - \tau_d = \tau + Mg_0$. Then, replacing in eqs. (7), (8)

$$\theta = \theta_d + \Delta\theta \qquad q = q_d + \Delta q \qquad \dot{\theta} = \Delta\dot{\theta} \qquad \dot{q} = \Delta\dot{q} \qquad \tau = \tau_d + \Delta\tau = -Mg_0 + \Delta\tau,$$

as well as $\ddot{\theta} = \Delta\ddot{\theta}$ and $\ddot{q} = \Delta\ddot{q}$, yields

$$\begin{aligned} B\,\Delta\ddot{\theta} + K\,(\Delta\theta - \Delta q) + D\,(\Delta\dot{\theta} - \Delta\dot{q}) + F_\theta\,\Delta\dot{\theta} &= \Delta\tau \\ M\,\Delta\ddot{q} + K\,(\Delta q - \Delta\theta) - Mg_0 + D\,(\Delta\dot{q} - \Delta\dot{\theta}) + F_q\,\Delta\dot{q} &= 0, \end{aligned}$$

where the identity $K(\theta_d - q_d) + Mg_0 = 0$ coming from (11) has been used. In the Laplace domain, we obtain

$$\begin{aligned} \big(Bs^2 + (D + F_\theta)s + K\big)\,\Delta\theta(s) - (Ds + K)\,\Delta q(s) &= \Delta\tau(s) \\ \big(Ms^2 + (D + F_q)s + K\big)\,\Delta q(s) - (Ds + K)\,\Delta\theta(s) &= 0. \end{aligned}$$

Thus, after some algebraic manipulation, the transfer function from $\Delta\tau$ to Δq is

$$P_q(s) = \frac{\Delta q(s)}{\Delta\tau(s)} = \frac{Ds + K}{s\,\mathrm{den}_3(s)}, \tag{17}$$

with the third-order polynomial in the denominator

$$\mathrm{den}_3(s) = BMs^3 + ((D{+}F_\theta)M + (D{+}F_q)B)\,s^2 + ((B{+}M)K + (F_\theta{+}F_q)D + F_\theta F_q)\,s + (F_\theta{+}F_q)K.$$

On the other hand, the transfer function from $\Delta\tau$ to $\Delta\theta$ is

$$P_\theta(s) = \frac{\Delta\theta(s)}{\Delta\tau(s)} = \frac{\Delta\theta(s)}{\Delta q(s)}\,P_q(s) = \frac{Ms^2 + (D + F_q)s + K}{s\,\mathrm{den}_3(s)}. \tag{18}$$

The transfer function $P_\theta(s)$ has a pole-zero excess (also called relative degree) equal to two. Being all physical coefficients positive, the two zeros of its numerator have negative real part and the three poles from $\mathrm{den}_3(s)$ (one certainly real) have all negative real parts —as can be shown from the Routh table this polynomial; finally, the fourth pole is at the origin. According to elementary feedback theory and using the properties of the root locus method, if one considers a proportional feedback of the form

$$\Delta\tau = -K_P\Delta\theta \qquad K_P > 0, \tag{19}$$

the four closed-loop poles will remain in the left-hand side of the complex plane *for all* positive values of the gain K_P. In particular, when increasing this gain, two poles converge to the open-loop zeros, while the other two approach the vertical asymptotes (whose center is located at a value $s_0 < 0$). A numerical example of such behavior is shown in Fig. 5.

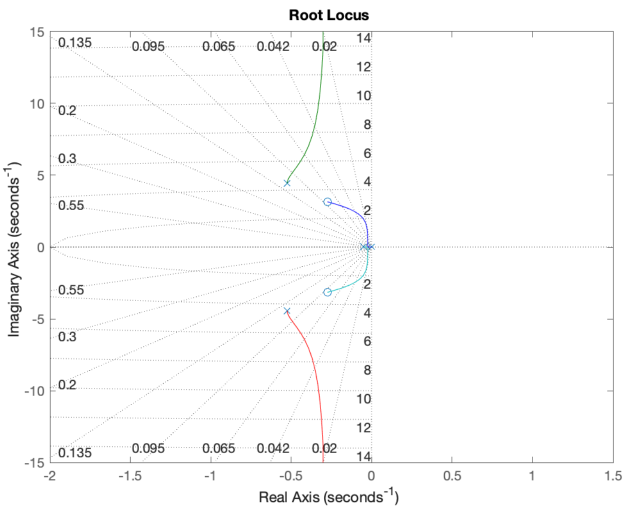


Figure 5: The root locus of the process (18) for varying $K_P > 0$.

As a result, the desired equilibrium is asymptotically (actually, exponentially) stabilized by control laws of the form

$$\tau = \tau_d + \Delta\tau = \tau_d - K_P\Delta\theta = -Mg_0 + K_P(\theta_d - \theta), \qquad \forall K_P > 0,$$

just like in eq. (10). Although an approach based on approximate linearization has usually only a local validity, in the present case the method was only used to remove a constant bias (and no other nonlinear terms). The analysis can then be considered of global validity, as already confirmed by the previous Lyapunov method.

The transfer function $P_q(s)$ in (17) also explains why it is preferable to close the proportional feedback loop (19) on the position variable θ of the first mass B (using the modified reference θ_d, as computed from q_d in (11), rather than closing a feedback loop $K_P(q_d - q)$ directly on the desired position of the second mass M. In fact, while the transfer function $P_q(s)$ shares most of the characteristics of $P_\theta(s)$, its pole-zero excess is instead equal to three (there is only one zero in the numerator). Again from elementary feedback theory and the properties of the root locus, such a feedback would lead to a stable closed-loop system only for a very limited positive range of K_P, going unstable for larger values. This is a consequence of the physical *non-colocation* between the control input τ and the output q to be controlled, because of the presence of elastic dynamics in the mechanical transmission between the two masses B and M.

Finally, assume that $D = 0$ in eqs. (7), (8) and that we would like to reproduce a trajectory $q_d(t)$ that is *four* times differentiable. Setting $q = q_d(t)$ in (8) and solving for θ gives

$$\theta_d(t) = q_d(t) + K^{-1}(M\,\ddot{q}_d(t) + F_q\,\dot{q}_d(t) - Mg_0).$$

Differentiating this once and twice provides

$$\dot{\theta}_d(t) = \dot{q}_d(t) + K^{-1}(M\,\dddot{q}_d(t) + F_q\,\ddot{q}_d(t)).$$

and

$$\ddot{\theta}_d(t) = \ddot{q}_d(t) + K^{-1}(M\,\ddddot{q}_d(t) + F_q\,\dddot{q}_d(t)).$$

By replacing these expressions for θ, $\dot{\theta}$ and $\ddot{\theta}$ in (7), we obtain the required inverse dynamics torque

$$\begin{aligned}
\tau_d(t) &= B\,\ddot{\theta}_d(t) + K(\theta_d(t) - q_d(t)) + F_\theta\,\dot{\theta}_d(t) \\
&= B\,\ddot{q}_d(t) + BK^{-1}(M\,\ddddot{q}_d(t) + F_q\,\dddot{q}_d(t)) + M\,\ddot{q}_d(t) + F_q\,\dot{q}_d(t) - Mg_0 \\
&\quad + F_\theta\,\dot{q}_d(t) + F_\theta K^{-1}(M\,\dddot{q}_d(t) + F_q\,\ddot{q}_d(t)) \\
&= BMK^{-1}\,\ddddot{q}_d(t) + (F_\theta M + F_q B)\,K^{-1}\,\dddot{q}_d(t) \\
&\quad + ((B+M) + F_\theta F_q)K^{-1})\,\ddot{q}_d(t) + (F_\theta + F_q)\,\dot{q}_d(t) - Mg_0,
\end{aligned}$$

which is eventually expressed only in terms of $q_d(t)$ and its first four time derivatives, as requested.

Exercise 4

The dynamic model of a 2R robot in the vertical plane is given by (see, e.g., the lecture slides)

$$\begin{aligned}
&\begin{pmatrix} a_1 + 2a_2\cos q_2 & a_3 + a_2\cos q_2 \\ a_3 + a_2\cos q_2 & a_3 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -a_2\sin q_2\left(\ddot{q}_2^2 + 2\dot{q}_1\dot{q}_2\right) \\ a_2\sin q_2\ddot{q}_1^2 \end{pmatrix} \\
&\qquad\qquad + \begin{pmatrix} a_4\cos q_1 + a_5\cos(q_1+q_2) \\ a_5\cos(q_1+q_2) \end{pmatrix} = \begin{pmatrix} \tau \\ 0 \end{pmatrix},
\end{aligned} \tag{20}$$

for suitable dynamic coefficients a_i, $i = 1, \ldots, 5$, and with a zero in the right-hand side of the second scalar equation due to the missing actuation (this underactuated robot is usually called Pendubot). The forced equilibrium conditions for this mechanical system are determined by setting $\tau = \bar{\tau}$ (constant) and $\dot{\boldsymbol{\theta}} = \ddot{\boldsymbol{\theta}} = \mathbf{0}$ in (20). This gives

$$\begin{aligned}
a_4\cos\bar{q}_1 + a_5\cos(\bar{q}_1 + \bar{q}_2) &= \bar{\tau} \\
a_5\cos(\bar{q}_1 + \bar{q}_2) &= 0,
\end{aligned} \tag{21}$$

implying

$$\bar{\tau} = a_4 \cos \bar{q}_1 = g_0 (m_1 d_{c1} + m_2 l_1) \cos \bar{q}_1. \tag{22}$$

Therefore, we have two continuum families of equilibria, each parametrized by the value $\bar{q}_1$, with

$$\bar{q}_2 = -\bar{q}_1 \pm \frac{\pi}{2},$$

i.e., the second link is vertical upward or downward, and with the corresponding equilibrium torque $\bar{\tau}$ given by (22).

The global stabilization of any such equilibrium by a PD control law with gravity compensation/cancellation is made very hard by the fact that the robot is underactuated: no torque command can be delivered at joint 2. Therefore, in order to define a state feedback control law that *locally* asymptotically stabilizes one of these equilibria, we use the approximate linearization of (20) around $(\bar{\boldsymbol{q}}, \boldsymbol{0})$, with $\bar{\boldsymbol{q}} = (\bar{q}_1, \bar{q}_2)$ satisfying (21) and with a (small) control action $\Delta\tau$ around the equilibrium torque $\bar{\tau}$ given by (22). Substituting in (20)

$$\boldsymbol{q} = \bar{\boldsymbol{q}} + \Delta \boldsymbol{q} \qquad \dot{\boldsymbol{q}} = \Delta \dot{\boldsymbol{q}} \qquad \ddot{\boldsymbol{q}} = \Delta \ddot{\boldsymbol{q}} \qquad \tau = \bar{\tau} + \Delta\tau,$$

and neglecting second- and higher-order terms in the variations $\Delta(\cdot)$, we obtain

$$\begin{aligned} &\begin{pmatrix} a_1 + 2a_2 \cos \bar{q}_2 & a_3 + a_2 \cos \bar{q}_2 \\ a_3 + a_2 \cos \bar{q}_2 & a_3 \end{pmatrix} \begin{pmatrix} \Delta \ddot{q}_1 \\ \Delta \ddot{q}_2 \end{pmatrix} \\ -&\begin{pmatrix} a_4 \sin \bar{q}_1 + a_5 \sin(\bar{q}_1 + \bar{q}_2) & a_5 \sin(\bar{q}_1 + \bar{q}_2) \\ a_5 \sin(\bar{q}_1 + \bar{q}_2) & a_5 \sin(\bar{q}_1 + \bar{q}_2) \end{pmatrix} \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix} = \begin{pmatrix} \Delta\tau \\ 0 \end{pmatrix}, \end{aligned} \tag{23}$$

or in compact form

$$\bar{\boldsymbol{M}}\, \Delta \ddot{\boldsymbol{q}} + \bar{\boldsymbol{G}}\, \Delta \boldsymbol{q} = \begin{pmatrix} \Delta\tau \\ 0 \end{pmatrix}, \qquad \text{with} \quad \bar{\boldsymbol{M}} = \boldsymbol{M}(\bar{\boldsymbol{q}}) \qquad \bar{\boldsymbol{G}} = \boldsymbol{G}(\bar{\boldsymbol{q}}) = \left. \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}} \right|_{\boldsymbol{q} = \bar{\boldsymbol{q}}}.$$

The system can be put in state-space format by choosing, e.g.,

$$\Delta \boldsymbol{x} = \begin{pmatrix} \Delta \boldsymbol{x}_1 \\ \Delta \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} \Delta \boldsymbol{q} \\ \bar{\boldsymbol{M}} \Delta \dot{\boldsymbol{q}} \end{pmatrix},$$

leading to

$$\Delta \dot{\boldsymbol{x}} = \boldsymbol{A}\, \Delta \boldsymbol{x} + \boldsymbol{b}\, \Delta\tau, \qquad \text{with} \quad \boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \bar{\boldsymbol{M}}^{-1} \\ -\bar{\boldsymbol{G}} & \boldsymbol{O} \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} \boldsymbol{0} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}.$$

At this stage, provided that the 4×4 controllability matrix

$$\mathcal{C} = \begin{pmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \boldsymbol{A}^2\boldsymbol{b} & \boldsymbol{A}^3\boldsymbol{b} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} & \bar{\boldsymbol{M}}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \boldsymbol{0} & -\boldsymbol{M}^1 \bar{\boldsymbol{G}} \bar{\boldsymbol{M}}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \boldsymbol{0} & -\bar{\boldsymbol{G}} \bar{\boldsymbol{M}}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \boldsymbol{0} \end{pmatrix}$$

is nonsingular,[2] we can obtain a local asymptotic stabilization of the system at the chosen equilibrium state $\boldsymbol{x}_e = (\bar{\boldsymbol{q}}, \boldsymbol{0})$ (and actually also assign all the closed-loop eigenvalues as desired), by means of the full state feedback law

$$\Delta\tau = -\boldsymbol{K} \Delta \boldsymbol{x} = -\boldsymbol{K}_1 \Delta \boldsymbol{q} - \boldsymbol{K}_2 \bar{\boldsymbol{M}} \Delta \dot{\boldsymbol{q}} = -K_1 \Delta q_1 - K_2 \Delta q_2 - K_3 \begin{pmatrix} 1 & 0 \end{pmatrix} \bar{\boldsymbol{M}} \Delta \dot{\boldsymbol{q}} - K_4 \begin{pmatrix} 0 & 1 \end{pmatrix} \bar{\boldsymbol{M}} \Delta \dot{\boldsymbol{q}},$$

[2] The controllability condition is generically satisfied.

with $\boldsymbol{K}$ such that $\sigma(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{K}) \in \mathbb{C}^-$. As a result, the required complete control torque will be

$$\tau = \bar{\tau} + \Delta\tau = a_4 \cos \bar{q}_1 + \begin{pmatrix} K_1 & K_2 \end{pmatrix} \begin{pmatrix} \bar{q}_1 - q_1 \\ \bar{q}_2 - q_2 \end{pmatrix} - \begin{pmatrix} K_3 & K_4 \end{pmatrix} \bar{\boldsymbol{M}} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix},$$

in the form of a PD-type feedback with constant feedforward.

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