

Methods 2 - 5

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Linear Algebra: Vectors and Matrices – Gill, chapter 3

- **Vectors**
 - Vector algebra
 - Inner, cross, outer products
 - Transpose
 - Norms
- **Matrices**
 - Symmetric
 - Diagonal
 - Identity, J, zero, triangular
 - Multiplication
 - Transposition

Elementary Formal Properties of Vector Algebra

- Commutative Property $\mathbf{u} + \mathbf{v} = (\mathbf{v} + \mathbf{u})$
- Additive Associative Property $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- Vector Distributive Property $s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}$
- Scalar Distributive Property $(s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$
- Zero Property $\mathbf{u} + \mathbf{0} = \mathbf{u} \iff \mathbf{u} - \mathbf{u} = \mathbf{0}$
- Zero Multiplicative Property $\mathbf{0}\mathbf{u} = \mathbf{0}$
- Unit Rule $\mathbf{1}\mathbf{u} = \mathbf{u}$

Inner product (“dot” product)

$$\mathbf{u} \cdot \mathbf{v} = [u_1v_1 + u_2v_2 + \cdots u_kv_k] = \sum_{i=1}^k u_iv_i.$$

Inner Product Formal Properties of Vector Algebra

→ Commutative Property $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

→ Associative Property $s(\mathbf{u} \cdot \mathbf{v}) = (s\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (s\mathbf{v})$

→ Distributive Property $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

→ Zero Property $\mathbf{u} \cdot \mathbf{0} = 0$

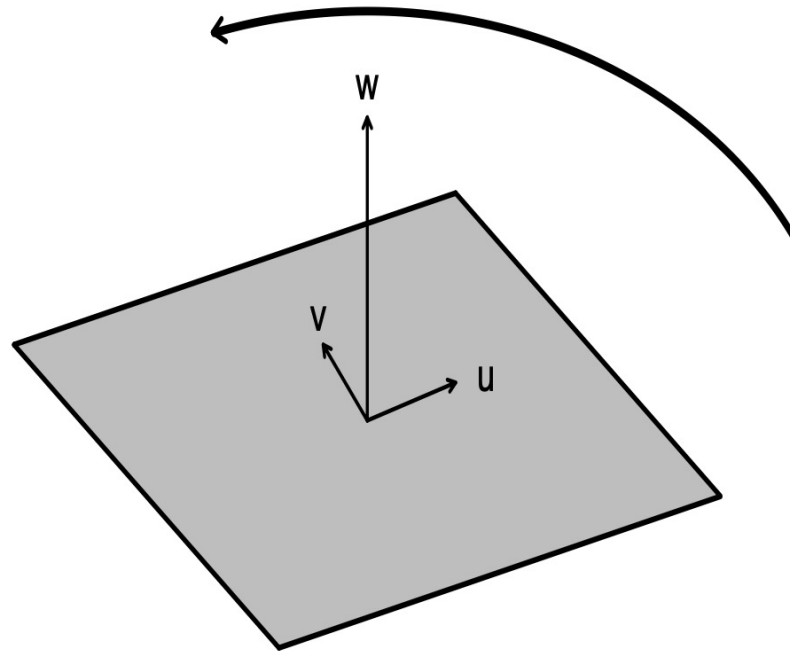
→ Unit Rule $\mathbf{1u} = \mathbf{u}$

→ Unit Rule $\mathbf{1u} = \sum_{i=1}^k \mathbf{u}_i$, for \mathbf{u} of length k

Cross product

$$\mathbf{u} \times \mathbf{v} = [u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1]$$

Fig. 3.2. THE RIGHT-HAND RULE ILLUSTRATED



Cross Product Formal Properties of Vector Algebra

→ Commutative Property $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

→ Associative Property $s(\mathbf{u} \times \mathbf{v}) = (s\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (s\mathbf{v})$

→ Distributive Property $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

→ Zero Property $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

→ Self-Orthogonality $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Vector transpose

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}'_{k \times 1} = \underset{1 \times k}{[u_1, u_2, \dots, u_k]},$$

$$\underset{1 \times k}{[u_1, u_2, \dots, u_k]}' = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}_{k \times 1}.$$

Outer product (better: Hadamard product)

★ **Example 3.9: Outer Product Calculation.** Once again using the simple numerical forms, we now calculate the outer product instead of the cross product:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [3, 3, 3] = \begin{bmatrix} 3 & 3 & 3 \\ 6 & 6 & 6 \\ 9 & 9 & 9 \end{bmatrix}.$$

And to show that order matters, consider:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} [1, 2, 3] = \begin{bmatrix} 3 & 6 & 9 \\ 3 & 6 & 9 \\ 3 & 6 & 9 \end{bmatrix}.$$

Vector norms

$$\|\mathbf{v}\| = (v_1^2 + v_2^2 + \cdots + v_n^2)^{\frac{1}{2}} = (\mathbf{v}' \cdot \mathbf{v})^{\frac{1}{2}}$$

Properties of the Standard Vector Norm

→ Vector Norm $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$

→ Difference Norm $||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 - 2(\mathbf{u} \cdot \mathbf{v}) + ||\mathbf{v}||^2$

→ Multiplication Norm $||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2 ||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2$

***p*-norms**

Actually, the norm used above is the most commonly used form of a **p-norm**:

$$\|\mathbf{v}\|_p = (|v_1|^p + |v_2|^p + \cdots + |v_n|^p)^{\frac{1}{p}}, \quad p \geq 0,$$

where $p = 2$ so far. Other important cases include $p = 1$ and $p = \infty$:

$$\|\mathbf{v}\|_\infty = \max_{1 \leq i \leq n} |x_i|,$$

Properties of Vector Norms, Length- n

- Triangle Inequality $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$
- Hölder's Inequality for $\frac{1}{p} + \frac{1}{q} = 1$, $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|_p \|\mathbf{w}\|_q$
- Cauchy-Schwarz Ineq. $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|_2 \|\mathbf{w}\|_2$
- Cosine Rule $\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$
- Vector Distance $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$
- Scalar Property $\|s\mathbf{v}\| = |s| \|\mathbf{v}\|$

Matrices

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & \cdots & x_{1(p-1)} & x_{1p} \\ x_{21} & x_{22} & \cdots & \cdots & x_{2(p-1)} & x_{2p} \\ \vdots & \vdots & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ x_{(n-1)1} & x_{(n-1)2} & \cdots & \cdots & x_{(n-1)(p-1)} & x_{(n-1)p} \\ x_{n1} & x_{n2} & \cdots & \cdots & x_{n(p-1)} & x_{np} \end{bmatrix}$$

Symmetric and skew-symmetric matrices

$$\mathbf{X} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 8 & 5 & 6 \\ 3 & 5 & 1 & 7 \\ 4 & 6 & 7 & 8 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

Diagonal matrices

$$\mathbf{X} = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & d_{n-1} & 0 \\ 0 & 0 & 0 & 0 & d_n \end{bmatrix}$$

Identity, J, and zero matrices

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Lower and upper triangular matrices

$$\mathbf{X}_{LT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 3 & 5 & 1 & 0 \\ 4 & 6 & 7 & 8 \end{bmatrix}, \quad \mathbf{X}_{UT} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 8 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

Properties of (Conformable) Matrix Manipulation

- **Commutative Property** $\mathbf{X} + \mathbf{Y} = \mathbf{Y} + \mathbf{X}$
- **Additive Associative Property** $(\mathbf{X} + \mathbf{Y}) + \mathbf{Z} = \mathbf{X} + (\mathbf{Y} + \mathbf{Z})$
- **Matrix Distributive Property** $s(\mathbf{X} + \mathbf{Y}) = s\mathbf{X} + s\mathbf{Y}$
- **Scalar Distributive Property** $(s + t)\mathbf{X} = s\mathbf{X} + t\mathbf{X}$
- **Zero Property** $\mathbf{X} + \mathbf{0} = \mathbf{X}$ and $\mathbf{X} - \mathbf{X} = \mathbf{0}$

Matrix multiplication

$$\begin{aligned}\mathbf{XY} &= \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \\ &= \begin{bmatrix} (x_{11} \ x_{12}) \cdot (y_{11} \ y_{21}) & (x_{11} \ x_{12}) \cdot (y_{12} \ y_{22}) \\ (x_{21} \ x_{22}) \cdot (y_{11} \ y_{21}) & (x_{21} \ x_{22}) \cdot (y_{12} \ y_{22}) \end{bmatrix} \\ &= \begin{bmatrix} x_{11}y_{11} + x_{12}y_{21} & x_{11}y_{12} + x_{12}y_{22} \\ x_{21}y_{11} + x_{22}y_{21} & x_{21}y_{12} + x_{22}y_{22} \end{bmatrix}.\end{aligned}$$

Matrix multiplication

$$\begin{matrix} \mathbf{X} & \mathbf{Y} \\ (k \times n) & (n \times p) \end{matrix} = \begin{bmatrix} \sum_{i=1}^n x_{1i}y_{i1} & \sum_{i=1}^n x_{1i}y_{i2} & \cdots & \sum_{i=1}^n x_{1i}y_{ip} \\ \sum_{i=1}^n x_{2i}y_{i1} & \sum_{i=1}^n x_{2i}y_{i2} & \cdots & \sum_{i=1}^n x_{2i}y_{ip} \\ \vdots & & \ddots & \vdots \\ \sum_{i=1}^n x_{ki}y_{i1} & \cdots & \cdots & \sum_{i=1}^n x_{ki}y_{ip} \end{bmatrix}$$

Matrix multiplication

Starting with the matrices

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} -2 & 2 \\ 0 & 1 \end{bmatrix},$$

calculate

$$\begin{aligned} \mathbf{XY} &= \begin{bmatrix} (1 \ 2) \cdot (-2 \ 0) & (1 \ 2) \cdot (2 \ 1) \\ (3 \ 4) \cdot (-2 \ 0) & (3 \ 4) \cdot (2 \ 1) \end{bmatrix} \\ &= \begin{bmatrix} (1)(-2) + (2)(0) & (1)(2) + (2)(1) \\ (3)(-2) + (4)(0) & (3)(2) + (4)(1) \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 \\ -6 & 10 \end{bmatrix}. \end{aligned}$$

Properties of (Conformable) Matrix Multiplication

→ **Associative Property** $(\mathbf{XY})\mathbf{Z} = \mathbf{X}(\mathbf{YZ})$

→ Additive Distributive Property $(\mathbf{X} + \mathbf{Y})\mathbf{Z} = \mathbf{XZ} + \mathbf{YZ}$

\rightarrow **Scalar Distributive Property**

$${}_s\mathbf{XY} = (\mathbf{X}s)\mathbf{Y}$$

$$= \mathbf{X}({}_s\mathbf{Y}) = \mathbf{XY}s$$

\vdash Zero Property $\mathbf{X0} = 0$

Matrix multiplication

$$\mathbf{XY} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$\mathbf{YX} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

This is a very simple example, but the implications are obvious. Even in cases where pre-multiplication and post-multiplication are possible, these are different operations and **matrix multiplication is not commutative.**

Matrix transposition

$$\mathbf{X}' = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Properties of Matrix Transposition

→ Invertibility $(\mathbf{X}')' = \mathbf{X}$

→ Additive Property $(\mathbf{X} + \mathbf{Y})' = \mathbf{X}' + \mathbf{Y}'$

→ Multiplicative Property $(\mathbf{XY})' = \mathbf{Y}'\mathbf{X}'$

→ General Multiplicative Property $(\mathbf{X}_1\mathbf{X}_2\cdots\mathbf{X}_{n-1}\mathbf{X}_n)'$
 $= \mathbf{X}'_n\mathbf{X}'_{n-1}\cdots\mathbf{X}'_2\mathbf{X}'_1$

→ Symmetric Matrix $\mathbf{X}' = \mathbf{X}$

→ Skew-Symmetric Matrix $\mathbf{X} = -\mathbf{X}'$