Example D.1

记公理系统$\mathbf{LPGL}$的公式构成为

$$
\begin{aligned}
&u ::= c \mid x \mid (u \cdot u) \mid (u + u) \mid !u \\
&P ::= p \mid \neg P \mid P \to P \mid \Box P \mid u : P
\end{aligned}
$$

we try to proof $\mathbf{LPGL} \vdash u : \Box P \to P$

Proof.

$$
\begin{aligned}
&(1)\ u : \Box P \to \Box P \quad Factivity \\
&(2)\ \neg\Box P \to \neg u : \Box P \quad 1 \\
&(3)\ \neg u : \Box P \to \Box(\neg u : \Box P) \quad \text{weak negative introspection} \\
&(4)\ \Box(\neg u : \Box P) \to \Box(u : \Box P \to P) \quad TAUT, NEC, K \\
&(5)\ \neg\Box P \to \Box(u : \Box P \to P) \quad 2, 3, 4 \\
&(6)\ \Box P \to \Box(u : \Box P \to P) \quad TAUT, NEC, K \\
&(7)\ \Box(u : \Box P \to P) \quad 5, 6 \\
&(8)\ u : \Box P \to P \quad 7, \text{reflexivity rule}
\end{aligned}
$$

Example D.2

记公理系统$\text{hybrid-}\mathbf{S4_f}$的公式构成为

$$
F ::= i \mid P \mid \hat{\neg} F \mid F \hat{\to} F \mid \hat{\Box} F \mid @_i F
$$

外化公理：$\neg P \to \hat{\neg} P, (P \to Q) \hat{\to} (P \hat{\to} Q)$

Let $@_i$ be $@_{\mathbf{LPGL}}$, we try to proof $@_i(u : \Box P \to P)$ is a theorem.

Proof.

$$(1)\ @_i(u:\Box P \rightarrow \Box P) \quad \mathit{Factivity\ on}\ \mathbf{LPGL}$$

$$(2)\ @_i((u:\Box P \rightarrow \Box P) \rightarrow (\neg\Box P \rightarrow \neg u:\Box P)) \quad \mathit{TAUT\ on}\ \mathbf{LPGL}$$

$$(4)\ @_i(u:\Box P \rightarrow \Box P)\hat{\rightarrow}@_i(\neg\Box P \rightarrow \neg u:\Box P) \quad 2,(*),K_@$$

$$(5)\ @_i(\neg\Box P \rightarrow \neg u:\Box P) \quad 1,4,MP$$

$$(6)\ @_i(\neg u:\Box P \rightarrow \Box(\neg u:\Box P)) \quad \text{weak negative introspection } \mathit{on}\ \mathbf{LPGL}$$

$$(7)\ @_i(\neg u:\Box P \rightarrow (u:\Box P \rightarrow P)) \quad \mathit{TAUT\ on}\ \mathbf{LPGL}$$

$$(8)\ @_i(\Box(\neg u:\Box P) \rightarrow \Box(u:\Box P \rightarrow P)) \quad 7,NEC,K\ \mathit{on}\ \mathbf{LPGL}$$

$$(9)\ @_i\Big((\neg\Box P \rightarrow \neg u:\Box P) \rightarrow \Big((\neg u:\Box P \rightarrow \Box(\neg u:\Box P)) \rightarrow$$

$$\big((\Box(\neg u:\Box P) \rightarrow \Box(u:\Box P \rightarrow P)) \rightarrow (\neg\Box P \rightarrow \Box(u:\Box P \rightarrow P))\big)\Big)\Big)$$

$$\mathit{TAUT\ on}\ \mathbf{LPGL}$$

$$(11)\ @_i(\neg\Box P \rightarrow \neg u:\Box P)\hat{\rightarrow}\Big(@_i(\neg u:\Box P \rightarrow \Box(\neg u:\Box P))\hat{\rightarrow}$$

$$\big(@_i(\Box(\neg u:\Box P) \rightarrow \Box(u:\Box P \rightarrow P))\hat{\rightarrow}@_i(\neg\Box P \rightarrow \Box(u:\Box P \rightarrow P))\big)\Big)$$

$$10,(*),@NEC,K_@$$

$$(12)\ @_i(\neg\Box P \rightarrow \Box(u:\Box P \rightarrow P)) \quad 5,6,8,11,MP$$

$$(13)\ @_i(P \rightarrow (u:\Box P \rightarrow P)) \quad \mathit{TAUT\ on}\ \mathbf{LPGL}$$

$$(14)\ @_i(\Box P \rightarrow \Box(u:\Box P \rightarrow P)) \quad 13,NEC,K\ \mathit{on}\ \mathbf{LPGL}$$

$$(15)\ @_i\Big((\neg\Box P \rightarrow \Box(u:\Box P \rightarrow P)) \rightarrow$$

$$\big((\Box P \rightarrow \Box(u:\Box P \rightarrow P)) \rightarrow \Box(u:\Box P \rightarrow P)\big)\Big)\mathit{TAUT\ on}\ \mathbf{LPGL}$$

$$(17)\ @_i(\neg\Box P \rightarrow \Box(u:\Box P \rightarrow P))\hat{\rightarrow}$$

$$\big(@_i(\Box P \rightarrow \Box(u:\Box P \rightarrow P))\hat{\rightarrow}@_i\Box(u:\Box P \rightarrow P)\big) \quad 16,(*),K_@$$

$$(18)\ @_i\Box(u:\Box P \rightarrow P) \quad 12,14,17,MP$$

$$(19)\ @_i(u:\Box P \rightarrow P) \quad \text{reflexivity rule } \mathit{on}\ \mathbf{LPGL}$$

我们没法使用下层逻辑的规则，但是公理可以用，规则被外化到主语言中了，用term上的函数表示

用term内化了模型中系统的可证的公式的具体的证明过程，这是有趣的。因为term提供了一种证明的核心直觉，非常的简洁，清晰的刻画了证明的关键步骤。
term不仅能够内化外部证明，也能在给出下层系统之后，内化内部证明
成为可以用的推理规则

Example D.3

下层逻辑$\mathbf{LPGL}$:
$\hat{u} ::= c \mid x \mid (u \cdot u) \mid (u + u) \mid !u$
$P ::= p \mid \neg P \mid P \to P \mid \Box P \mid \hat{u} : P$
函数:
$f_{nec}(t)$ 对应$\mathbf{LPGL}$中的NEC规则;
$f_r(t)$ 对应$\mathbf{LPGL}$中的reflexivity rule规则;
$f_k(t)$ 对应$\mathbf{LPGL}$中的K规则;

$$
\begin{aligned}
&(1)\ c : (\hat{u} : \Box P \to \Box P) \quad Factivity \\
&(2)\ d : ((\hat{u} : \Box P \to \Box P) \to (\neg\Box P \to \neg\hat{u} : \Box P)) \quad TAUT \\
&(3)\ dc : (\neg\Box P \to \neg\hat{u} : \Box P) \quad 2, 1, Application \\
&(4)\ h : (\neg\hat{u} : \Box P \to \Box(\neg\hat{u} : \Box P)) \quad \text{weak negative introspection} \\
&(5)\ j : (\neg\hat{u} : \Box P \to (\hat{u} : \Box P \to P)) \quad TAUT \\
&(6)\ f_k f_{nec}(j) : (\Box(\neg\hat{u} : \Box P) \to \Box(\hat{u} : \Box P \to P)) \\
&(7)\ m : \Big((\neg\Box P \to \neg\hat{u} : \Box P) \to \Big((\neg\hat{u} : \Box P \to \Box(\neg\hat{u} : \Box P)) \to \\
&\qquad \big((\Box(\neg\hat{u} : \Box P) \to \Box(\hat{u} : \Box P \to P)) \to (\neg\Box P \to \Box(\hat{u} : \Box P \to P))\big)\Big)\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad TAUT \\
&(8)\ m(dc)h(f_k f_{nec}(j)) : (\neg\Box P \to \Box(\hat{u} : \Box P \to P)) \quad 7, 3, 4, 6, Application \\
&(9)\ n : (P \to (\hat{u} : \Box P \to P)) \quad TAUT \\
&(10)\ o : \Big((\neg\Box P \to \Box(\hat{u} : \Box P \to P)) \to \\
&\qquad \big((\Box P \to \Box(\hat{u} : \Box P \to P)) \to \Box(\hat{u} : \Box P \to P)\big)\Big) \quad TAUT \\
&(11)\ o\big(m(dc)h(f_k f_{nec}(j))\big)(f_k f_{nec}(n)) : \Box(\hat{u} : \Box P \to P) \quad 10, 8, 9, application \\
&(12)\ f_r\Big(o\big(m(dc)h(f_k f_{nec}(j))\big)(f_k f_{nec}(n))\Big) : (\hat{u} : \Box P \to P) \quad 11, \text{reflexivity rule}
\end{aligned}
$$

Example D.4

上层逻辑hybrid-$\mathbf{LPS4_f}$:
$t ::= c \mid x \mid t \cdot t \mid t + t \mid !t \mid !_i t \mid ?_i t \mid f_{nec}(t) \mid f_r(t)$
$F ::= i \mid P \mid \hat{\neg} F \mid F \hat{\to} F \mid \hat{\Box} F \mid @_i F \mid t : F$
并使用函数$g_{nec}(t)$,$g_k(t)$简化证明过程:
$g_k(t)$ 对应hybrid-$\mathbf{LPS4_f}$中的$K_@$规则;
Proof. Let $@_i$ be $@_{\mathbf{LPGL}}$

$(1)\ c : @_i(\hat{u} : \Box P \to \Box P) \quad \mathit{Factivity\ on}\ \mathbf{LPGL}$

$(2)\ d : @_i((\hat{u} : \Box P \to \Box P) \to (\neg\Box P \to \neg\hat{u} : \Box P)) \quad \mathit{TAUT\ on}\ \mathbf{LPGL}$

$(3)\ g_k(d)c : @_i(\neg\Box P \to \neg\hat{u} : \Box P) \quad 2, (*), g_k, 1, \mathit{Application}$

$(4)\ h : @_i(\neg\hat{u} : \Box P \to \Box(\neg\hat{u} : \Box P)) \quad \text{weak negative introspection}\ \mathit{on}\ \mathbf{LPGL}$

$(5)\ j : @_i(\neg\hat{u} : \Box P \to (\hat{u} : \Box P \to P)) \quad \mathit{TAUT\ on}\ \mathbf{LPGL}$

$(6)\ f_k f_{nec}(j) : @_i(\Box(\neg\hat{u} : \Box P) \to \Box(\hat{u} : \Box P \to P))$

$$(7)\ m : @_i\Big((\neg\Box P \to \neg\hat{u} : \Box P) \to \big((\neg\hat{u} : \Box P \to \Box(\neg\hat{u} : \Box P)) \to$$
$$\big((\Box(\neg\hat{u} : \Box P) \to \Box(\hat{u} : \Box P \to P)) \to (\neg\Box P \to \Box(\hat{u} : \Box P \to P))\big)\big)\Big)$$
$$\mathit{TAUT\ on}\ \mathbf{LPGL}$$

$$(8)\ g_k\Big(g_k\big(g_k(m)(g_k(d)c)\big)h\Big)f_k f_{nec}(j) : @_i(\neg\Box P \to \Box(\hat{u} : \Box P \to P))$$
$$7, (*), g_k, 3, g_k, 4, g_k, 6, g_k, \mathit{Application}$$

$(9)\ n : @_i(P \to (\hat{u} : \Box P \to P)) \quad \mathit{TAUT\ on}\ \mathbf{LPGL}$

$$(10)\ o : @_i\Big((\neg\Box P \to \Box(\hat{u} : \Box P \to P)) \to$$
$$\big((\Box P \to \Box(\hat{u} : \Box P \to P)) \to \Box(\hat{u} : \Box P \to P)\big)\Big) \quad \mathit{TAUT\ on}\ \mathbf{LPGL}$$

$$(11)\ g_k\Big(g_k(o)\Big(g_k\Big(g_k\big(g_k(m)(g_k(d)c)\big)h\Big)f_k f_{nec}(j)\Big)\Big)f_k f_{nec}(n) :$$
$$@_i\Box(\hat{u} : \Box P \to P) \qquad 10, (*), g_k, 8, g_k, 9, \mathit{application}$$

$$(12)\ f_r\Big(g_k\Big(g_k(o)\Big(g_k\Big(g_k\big(g_k(m)(g_k(d)c)\big)h\Big)f_k f_{nec}(j)\Big)\Big)f_k f_{nec}(n)\Big) :$$
$$@_i(\hat{u} : \Box P \to P) \qquad 11, \text{reflexivity rule}\ \mathit{on}\ \mathbf{LPGL}$$

这个例子体现出混合算子的必要性。让系统中的定理转化为公理系统中的定理。从而能够公理化方法推理。

四个例子构成平行关系，总结内化定理。

给出的证明项，要求都是从公理出发的，这保证了证明的可靠性，不会出现跳步