Beyond Bookkeeping: Constraint Tracking, Dynamical Geometry, and the Limits of Information Recoverability

From apparent information loss to structural non-recoverability in reversible systems

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Abstract

Recent work has shown that apparent information loss in black hole–inspired toy models can arise entirely from incomplete statistical bookkeeping rather than from non-unitary dynamics. In deterministic, reversible systems observed through coarse-grained operators, common entropy-based tracking methods—including marginal entropies, thermal assumptions, and independence approximations—discard correlations preserved by the dynamics, producing persistent entropy plateaus despite complete information preservation. By contrast, *constraint tracking*, which maintains the full set of microstates consistent with the entire observation history under known dynamics, recovers the initial state exactly in finite time.

In this paper, we build on that diagnostic result and extend it by separating *information preservation* from *recoverability*. While reversible dynamics guarantee that information is preserved in principle, they do not guarantee that inverse reconstruction is stable. We show that in high-dimensional or unstable regimes, the geometry of state-space evolution can render inverse recovery structurally fragile or effectively inadmissible under perturbation, even when constraint tracking is defined correctly.

We unify constraint-based information tracking with a geometric analysis of inverse conditioning, identifying regimes where information is preserved yet stable recovery fails due to anisotropic distortion, rapid frame rotation, or sensitivity amplification. This reframes the black hole information problem—and related paradoxes in turbulence and complex systems—as a failure to distinguish bookkeeping loss from dynamical non-recoverability, rather than as a violation of unitarity.

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[*]Attribution: The constraint-tracking formulation and diagnostic toy-model result summarized in Sections 2–3 build directly on publicly released work by Lee Smart (Vibrational Field Dynamics Institute), which demonstrates that apparent information loss can arise from incomplete statistical bookkeeping in reversible systems. The present paper independently reproduces and synthesizes those results and extends them by analyzing the geometric stability of inverse reconstruction. Lee Smart did not participate in the writing of this manuscript and has not reviewed this version.

Keywords: black hole information paradox, information theory, constraint tracking, recoverability, dynamical systems, coarse-graining

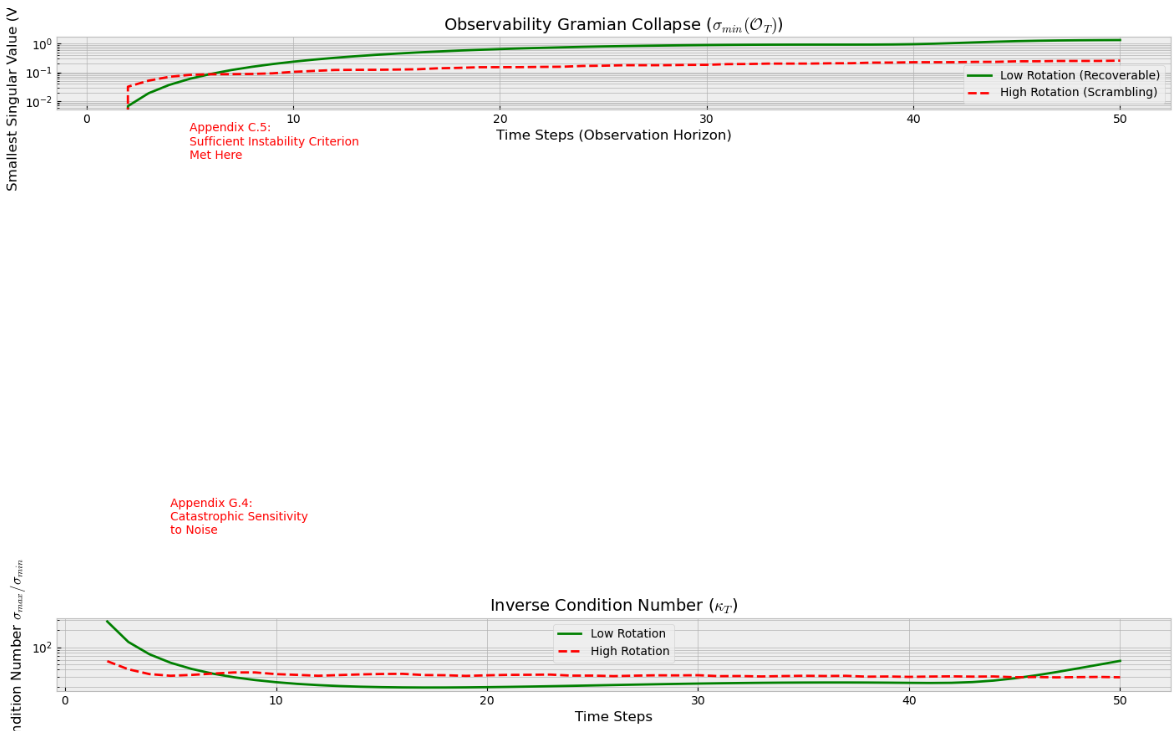

Figure 1: **Observability Collapse Under Rotational Scrambling.** Top: Smallest singular value $\sigma_{\min}(\mathcal{O}_T)$ of the stacked observability matrix as a function of observation horizon T. In the low-rotation regime (green), $\sigma_{\min}$ increases with T, indicating improved recoverability. In the high-rotation regime (red), $\sigma_{\min}$ remains suppressed, indicating structural non-recoverability despite deterministic and reversible dynamics. Bottom: Corresponding condition number $\kappa_T = \sigma_{\max}/\sigma_{\min}$, showing catastrophic sensitivity to noise in the scrambling regime.

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1 Introduction

1.1 The Source of the "Information Loss" Claim

The black hole information paradox is usually presented as a conflict between two statements: (i) quantum evolution is unitary and therefore information-preserving, and (ii) Hawking's semiclassical calculation yields radiation that is thermal, apparently erasing the detailed information about what formed the black hole [Hawking, 1975, 1976]. If a black hole evaporates completely and the outgoing radiation is exactly thermal, then a pure initial state appears to evolve into a mixed final state, contradicting unitary evolution.

However, the inference from "thermal" to "information destroyed" hides a methodological step: *thermal* is a statement about a reduced or coarse description. Hawking radiation is derived after tracing out degrees of freedom, imposing approximations, and summarizing outcomes by low-order statistics. The paradox therefore depends not only on the dynamics but also on what is meant by "information" and how it is tracked across time.

This paper takes that methodological step seriously. We ask: when does thermality indicate genuine information destruction, and when does it only indicate that the tracking scheme has discarded correlations that remain present in the underlying dynamics?

1.2 Two Mechanisms That Produce "Loss" in Reversible Systems

We separate two logically distinct mechanisms that can generate the appearance of information loss even when the underlying evolution is reversible.

(A) Bookkeeping loss. Bookkeeping loss occurs when information is estimated using statistical summaries that discard cross-time constraints induced by reversible dynamics. Examples include marginal entropies, IID "thermal" assumptions over unobserved degrees of freedom, and estimators that treat successive observations as independent. In this case, the system can be fully deterministic and reversible while the tracker reports persistent uncertainty, because the tracker fails to propagate constraints through the dynamics.

Lee Smart established this point sharply with a diagnostic toy model: a finite reversible system observed through coarse operators exhibits a long-lived entropy plateau under naive trackers, while exact constraint tracking collapses the admissible microstate set and recovers the initial condition in finite time. The "lost" information is not destroyed; it is contained in correlations the tracker does not represent.

(B) Recoverability loss. Recoverability loss occurs when the inverse reconstruction problem is geometrically ill-conditioned, even if information is preserved and even if constraints are defined correctly. A reversible map guarantees an inverse exists, but it need not be stable: small uncertainty at late time can be amplified catastrophically when mapped backward. In such regimes, the admissible set of states consistent with observations can have small volume yet extreme geometric elongation, becoming hypersensitive to perturbation and effectively unrecoverable at finite precision.

These mechanisms are independent. Bookkeeping loss can occur in systems with stable inversion; recoverability loss can occur even when bookkeeping is conceptually perfect.

1.3 Why This Distinction Matters for Black Holes

A common mistake is to treat "information is preserved" as equivalent to "information is recoverable." Unitarity (or reversibility in toy settings) addresses preservation. The operational question in the black hole setting is recoverability: whether the microscopic details of the infalling state can be reconstructed from the outgoing radiation by any observer with finite resources and finite precision.

Constraint tracking undermines a core step in the paradox narrative: the move from thermal appearance to information destruction. But even if the underlying physics is unitary, the geometry of reconstruction may still obstruct recovery. For black holes this is not a semantic subtlety. The relevant channels are enormously high-dimensional; measurement access is limited; and the encoding of early information into late radiation may be extremely nonlocal and poorly conditioned.

Thus, the correct target is not simply "unitary vs. non-unitary" but the pair:

$$\text{(i) bookkeeping fidelity} \quad \text{and} \quad \text{(ii) inverse conditioning.}$$

1.4 Contributions and Structure of the Paper

This paper makes two contributions.

1. **Synthesis of the bookkeeping resolution.** We present the constraint-tracking viewpoint and the diagnostic toy-model result in a way that isolates its conceptual content: entropy plateaus can be artifacts of incomplete tracking, not evidence of physical information destruction.

2. **Extension to recoverability geometry.** We extend the analysis by introducing recoverability as a geometric property of inverse reconstruction. We formalize how anisotropic distortion, frame rotation, and sensitivity amplification can preserve information while eliminating stable recovery.

The remainder is organized as follows. Section 2 formalizes information as an admissible-set constraint and summarizes the toy-model mechanism behind bookkeeping loss. Section 3 extracts the assumptions under which constraint-based recovery is stable. Section 4 introduces the geometry of inverse conditioning and defines recoverability. Section **??** characterizes structural non-recoverability without information destruction. Section **??** discusses the implications for black hole physics. Section 7 notes parallels in turbulence, chaotic inference, and learning systems. Section 8 states limitations, and Section 9 concludes.

2 Constraint Tracking and the Bookkeeping Resolution

This section formalizes the notion of information as a set of admissible states and summarizes the diagnostic result: apparent information loss can arise entirely from bookkeeping choices, even in fully reversible systems. The goal is not to reproduce implementation details, but to isolate the logical mechanism that produces entropy plateaus under naive tracking.

2.1 Information as an Admissible Set

Let $\mathcal{X}$ denote the space of microstates, and let $F_t : \mathcal{X} \to \mathcal{X}$ denote deterministic, reversible evolution. Observations are generated by (generally non-injective) operators

$$O_t : \mathcal{X} \to \mathcal{Y},$$

which map microstates to coarse-grained observables.

Given an observation history $\mathcal{O}_{0:T} = \{o_0, o_1, \ldots, o_T\}$, the admissible set of initial states is

$$\mathcal{C}_T = \left\{ x_0 \in \mathcal{X} \;\middle|\; O_t(F_t(x_0)) = o_t \;\; \forall\, t \leq T \right\}. \tag{1}$$

This set contains exactly those microstates that could have produced the observed data under the known dynamics. The remaining uncertainty is quantified by the size (or measure) of $\mathcal{C}_T$. In discrete settings,

$$I_T \equiv \log_2 |\mathcal{C}_T|$$

represents the number of bits of uncertainty consistent with the full observation history.

Under reversible dynamics and deterministic observations, $\mathcal{C}_T$ is monotonically non-increasing as T grows: each new observation can only remove candidates. Information loss corresponds to an expansion of $\mathcal{C}_T$; information recovery corresponds to its collapse.

2.2 Why Entropy Is an Incomplete Tracker

Standard entropy-based approaches track information through probability distributions over states or observables. These methods implicitly assume that observations are statistically independent or weakly correlated across time. Entropy is computed from marginal or low-order joint distributions, discarding long-range temporal correlations.

Reversible dynamics violate this assumption. Deterministic evolution induces correlations across arbitrarily long timescales: an observation at time t constrains not only the state at t, but also the state at all other times via the dynamics. These constraints compound multiplicatively. Entropy computed from marginals cannot represent this compounding, because it does not track how observations restrict trajectories jointly.

As a result, entropy can plateau at a positive value even when the admissible set has collapsed to a single state.

2.3 The Diagnostic Toy Model

The diagnostic model introduced by Smart isolates this effect with minimal structure:

- A finite microstate space $\mathcal{X} = \{0, 1\}^N$.
- Deterministic, bijective evolution implemented by a Feistel network.
- A coarse-graining operator ("horizon") $H : \mathcal{X} \to \{0, 1\}^M$ with $M \ll N$.
- A deterministic emission operator ("radiation") releasing k bits per timestep.

The model contains no spacetime geometry, gravity, or quantum mechanics. Its purpose is diagnostic: to show that bookkeeping alone can generate apparent information loss.

Despite the simplicity of the dynamics, the horizon and radiation outputs appear statistically random when examined locally. Marginal distributions approach uniformity, and standard entropy estimators report persistent uncertainty.

2.4 Naive Tracking Regimes

Several naive tracking methods are applied:

1. Marginal Shannon entropy of horizon outputs.
2. IID "thermal" assumptions over unobserved degrees of freedom.
3. Compression-based entropy estimators applied to observation streams.
4. Combined estimators assuming conditional independence across time.

All of these methods discard cross-time constraints. Consequently, they report long-lived entropy plateaus even as observations accumulate.

2.5 Exact Constraint Tracking

Constraint tracking explicitly maintains $\mathcal{C}_T$. At each timestep, candidate initial states are forward-simulated and filtered by comparison with the entire observation history. In the canonical configuration, the admissible set collapses rapidly:

$$|\mathcal{C}_T| : \; 2^N \; \to \; 1$$

after a small number of timesteps. The initial microstate is recovered exactly.

The key point is not the timescale of recovery, which is model-dependent, but the logical fact of recovery: the observation history uniquely determines the initial state, even though naive entropy measures report persistent uncertainty.

2.6 The Information Gap

The discrepancy between naive entropy estimates and $\log_2 |\mathcal{C}_T|$ defines an information gap:

$$\Delta I_T = I_T^{\text{naive}} - \log_2 |\mathcal{C}_T|. \tag{2}$$

In the diagnostic model, ΔI_T grows to a large constant value and persists indefinitely. This gap does not represent physical indeterminacy. It is the amount of information preserved by the dynamics but discarded by the tracking method.

2.7 Interpretation

The diagnostic result establishes a necessary baseline: entropy plateaus alone do not imply information destruction. Apparent information loss can arise purely from bookkeeping failure in reversible systems.

This does not yet address whether recovery is generically stable or operationally feasible. It shows only that claims of information loss must first exclude the possibility that the tracking scheme itself is discarding correlations preserved by the dynamics.

3 Implicit Assumptions Behind Constraint-Based Recovery

The bookkeeping resolution summarized in Section 2 demonstrates that apparent information loss can arise from incomplete tracking rather than from irreversible dynamics. However, that conclusion depends on a collection of implicit assumptions that are satisfied by the diagnostic toy model but need not hold in more complex systems. Making these assumptions explicit clarifies both the scope of the bookkeeping result and the motivation for extending the analysis beyond it.

3.1 Exact Knowledge of the Dynamics

Constraint tracking presupposes that the evolution operator F_t is known exactly. The admissible set $\mathcal{C}_T$ is defined by forward-simulating candidate initial states and checking consistency with observations. This procedure is only well-defined if the governing dynamics are specified without ambiguity.

In the diagnostic model, the evolution map is explicitly constructed and exactly invertible. In physical systems, by contrast, dynamics may be known only approximately, may involve effective descriptions, or may depend on unmodeled degrees of freedom. In such cases, failures of recovery may arise from model mismatch rather than from bookkeeping or geometry alone.

3.2 Finite and Enumerable State Space

The collapse of the admissible set in the toy model relies on the finiteness and enumerability of the state space. When $\mathcal{X} = \{0, 1\}^N$, the admissible set can be represented explicitly and filtered by brute force.

Many physical systems evolve on continuous or infinite-dimensional state spaces. Even when dynamics are reversible in principle, the admissible set may be uncountable, fractal, or only implicitly defined. In such cases, constraint tracking remains a conceptual definition of information, but not an executable algorithm.

3.3 Stable Invertibility

Constraint-based recovery implicitly assumes that inverse reconstruction is well-conditioned. That is, nearby states at time T map to nearby states at earlier times under the inverse dynamics.

Reversibility alone does not guarantee this property. A map may be bijective while exhibiting extreme sensitivity to perturbations. If small uncertainty at late times is exponentially amplified when propagated backward, then recovery becomes unstable even though the inverse map exists mathematically.

The diagnostic model is deliberately constructed to avoid such instability. Its evolution does not introduce strong anisotropic stretching or sensitivity amplification, ensuring that admissible sets shrink monotonically in a well-behaved manner.

3.4 Non-Degenerate Observation Geometry

Constraint tracking also assumes that the observation operators impose sufficiently independent constraints on the dynamics. If observations align with invariant or slowly evolving subspaces, then long sequences of data may fail to distinguish among large families of trajectories.

In the toy model, the horizon and radiation operators are chosen to provide informative constraints that eventually separate all trajectories. In realistic systems, observation operators may be poorly aligned with dynamically unstable directions, producing extended plateaus even under exact constraint tracking.

3.5 Noise-Free and Exact Observations

Finally, the bookkeeping resolution assumes noiseless observations. Each observation is treated as a hard constraint that exactly filters the admissible set.

In physical settings, observations are noisy, finite-resolution, or indirect. Noise converts hard constraints into soft ones, expanding the admissible region and potentially preventing collapse even when dynamics are reversible and well-conditioned. The presence of noise interacts strongly with geometric instability, as small errors can be amplified under inverse dynamics.

3.6 Summary

Constraint tracking resolves apparent information loss under idealized conditions: exact knowledge of reversible dynamics, finite and enumerable state space, stable inversion, informative observation geometry, and noise-free access. These conditions are satisfied by the diagnostic toy model by construction.

The next step is therefore not to ask whether bookkeeping matters—it does—but to ask when correct bookkeeping is sufficient for recovery. Answering that question requires analyzing the geometry of inverse reconstruction itself.

4 From Information Preservation to Recoverability Geometry

The bookkeeping analysis establishes that entropy plateaus and thermal statistics need not signal physical information destruction. However, it leaves open a deeper question: even when information is preserved and constraints are defined correctly, is recovery generically available? This section separates information preservation from recoverability and introduces recoverability as a geometric property of inverse reconstruction rather than a purely informational one.

4.1 Preservation Is Not Reconstruction

Information preservation is a property of the forward dynamics. A system preserves information if distinct initial states remain distinct under time evolution. In deterministic settings, this corresponds to injectivity of the evolution map; in reversible settings, to bijectivity.

Recoverability, by contrast, is a property of the inverse problem. It concerns whether an observer can stably infer earlier states from later observations given finite precision, noise, and limited access. Preservation guarantees that an inverse exists in principle; it does not guarantee that the inverse is usable.

Formally, let F_T denote the forward evolution map from time 0 to T, and let O_t denote the observation operators. Recoverability concerns the conditioning of the inverse mapping

$$x_0 \longleftarrow \{O_t(F_t(x_0))\}_{t=0}^{T}.$$

Two systems may preserve information equally well while differing radically in the stability of this inverse.

4.2 Inverse Conditioning and Sensitivity

The stability of inverse reconstruction is governed by how uncertainties evolve under the dynamics. Consider two nearby initial states separated by δx_0. Under forward evolution,

$$\delta x_T = DF_T(x_0)\,\delta x_0,$$

where DF_T is the Jacobian of the flow. If DF_T strongly stretches some directions while contracting others, then the inverse map DF_T^{-1} will strongly amplify uncertainty along the stretched directions.

In such cases, even infinitesimal uncertainty at time T can correspond to macroscopic uncertainty at time 0. The inverse problem is ill-conditioned: the admissible set of initial states consistent with the data may be extremely thin yet highly elongated, rendering recovery unstable under perturbation.

4.3 Geometric Structure of Admissible Sets

Constraint tracking defines information as an admissible set $\mathcal{C}_T$. Recoverability depends not only on the volume of this set but on its geometry.

In well-conditioned systems, $\mathcal{C}_T$ contracts roughly isotropically toward a point as observations accumulate. In poorly conditioned systems, $\mathcal{C}_T$ may collapse in volume while stretching into filaments or sheets aligned with unstable manifolds of the flow. Such sets have low entropy but poor recoverability: small observational noise causes large uncertainty along unstable directions.

Entropy measures are blind to this distinction. Two admissible sets can have identical volume while differing radically in geometric stability.

4.4 Directional Decoherence and Frame Rotation

A particularly destructive mechanism for recoverability is rapid rotation of expanding and contracting directions in state space. When the local eigenframe of the Jacobian rotates faster than constraints can accumulate, successive observations constrain incompatible directions.

This produces *directional decoherence*: constraints fail to reinforce each other coherently over time. Although each observation is informative in principle, the sequence does not converge to a stable inverse because the relevant directions continually shift. The admissible set becomes a twisted, unstable manifold rather than a shrinking ball.

The diagnostic toy model is deliberately constructed to avoid such effects. Its dynamics are bijective without strong anisotropic distortion or frame rotation, ensuring stable contraction of $\mathcal{C}_T$.

4.5 Shadowed Return Paths

Even when an inverse trajectory exists mathematically, it may be dynamically inaccessible. In some systems, the backward trajectory lies in a region of state space where small perturbations cause divergence into different basins. The return path is *shadowed*: present in the equations, absent in practice.

In such cases, information is preserved but the inverse route is structurally erased. Recovery requires infinite precision or perfect control, conditions that are physically unattainable.

4.6 Definition of Recoverability

We therefore define recoverability as follows:

> *A system is recoverable over a time interval if the inverse mapping from observation histories to initial states is well-conditioned under the geometry induced by the dynamics.*

Recoverability depends on:

- Anisotropy of the flow,
- Rotation of unstable directions,
- Alignment between observation operators and stable manifolds,
- Noise and finite precision.

These factors are orthogonal to bookkeeping. Correct constraint definitions are necessary but not sufficient for recovery.

4.7 Transition

Constraint tracking resolves apparent information loss caused by bookkeeping failure. The analysis above shows that a second, independent limitation exists: the geometry of inverse reconstruction itself. In the next section we characterize *structural non-recoverability*—regimes where information is preserved but recovery is eliminated by dynamical geometry.

5 Structural Non-Recoverability Without Information Destruction

The distinction between information preservation and recoverability allows for a class of failures that are neither violations of unitarity nor artifacts of bookkeeping. In these regimes, information is preserved by the dynamics, constraints are correctly defined, and yet recovery becomes structurally unavailable due to the geometry of the inverse problem.

5.1 Reversibility Does Not Guarantee Recovery

A persistent misconception in discussions of information loss is that reversibility implies recoverability. Reversibility asserts the existence of a unique inverse map; recoverability concerns the stability and accessibility of that inverse under finite precision.

Let F_T be bijective. Then for every final state x_T there exists a unique $x_0 = F_T^{-1}(x_T)$. However, if the inverse map F_T^{-1} is exponentially sensitive, then any finite uncertainty in x_T corresponds to an exponentially large uncertainty in x_0. The system preserves information while eliminating the practical possibility of reconstruction.

This failure mode is invisible to entropy-based diagnostics and persists even under perfect bookkeeping.

5.2 Collapse of Volume vs. Collapse of Geometry

Constraint tracking measures uncertainty via the volume of the admissible set $\mathcal{C}_T$. Structural non-recoverability arises when volume collapse does not imply geometric localization.

In such regimes, $\mathcal{C}_T$ contracts in measure while elongating along unstable directions. The set may approach zero volume while remaining extended across macroscopic distances in state space. Inverse reconstruction then becomes hypersensitive to noise, even though the admissible set is formally small.

This distinction highlights a limitation of scalar uncertainty measures. Recoverability depends on the shape of $\mathcal{C}_T$, not merely its size.

5.3 Directional Loss Without Entropic Loss

Structural non-recoverability can be understood as a loss of directional information rather than a loss of state distinguishability. The identity of trajectories is preserved, but the directions required to reverse the flow are no longer accessible.

Constraints may accumulate strongly in directions orthogonal to those needed for recovery. As a result, entropy may decrease or remain constant while recoverability collapses. The system becomes over-constrained in irrelevant directions and under-constrained along unstable ones.

This phenomenon explains how information can be preserved without being reconstructible.

5.4 Noise as a Structural Amplifier

Noise plays qualitatively different roles in bookkeeping-limited and structurally non-recoverable systems. In bookkeeping-limited regimes, noise merely delays constraint accumulation. In structurally non-recoverable regimes, noise is amplified by the inverse dynamics, destroying recovery even when the forward dynamics remain stable.

Once inverse amplification dominates, improvements in measurement precision yield diminishing returns. Recovery is not postponed; it is structurally eliminated.

5.5 Relation to Diagnostic Toy Models

The diagnostic toy model introduced by Smart is constructed to avoid structural non-recoverability. Its purpose is to demonstrate that bookkeeping failure alone can generate apparent information loss. It does not claim that all reversible systems are recoverable.

Structural non-recoverability represents a complementary phenomenon. It presupposes correct bookkeeping and arises only when the geometry of the flow undermines inverse conditioning. The two mechanisms are logically independent and must be analyzed separately.

5.6 Summary

Structural non-recoverability describes regimes in which information is preserved by the dynamics but rendered inaccessible by the geometry of inverse reconstruction. This phenomenon does not contradict constraint-based resolutions of apparent information loss; it defines their limits.

In the next section, we apply this unified framework to black hole physics, clarifying which aspects of the information paradox are resolved by constraint tracking and which depend on deeper questions of dynamical geometry.

6 Implications for Black Hole Physics

The framework developed above separates two issues that are often conflated in discussions of black hole evaporation: whether information is preserved by the underlying dynamics, and whether that information is recoverable in any stable or operational sense. Applying this distinction clarifies what is resolved by improved bookkeeping and what remains an open question tied to dynamical geometry.

6.1 Reinterpreting Hawking Radiation

Hawking's semiclassical calculation yields a thermal reduced density matrix for outgoing radiation when microscopic correlations are traced over. Within the constraint-tracking framework, this thermality reflects a projection rather than a physical erasure. Tracing out degrees of freedom discards correlations that may persist in the full state, producing entropy growth in the reduced description.

The diagnostic toy model demonstrates that deterministic, reversible dynamics can generate outputs whose marginals are indistinguishable from thermal noise while still encoding complete information in cross-time correlations. Thermal appearance, by itself, is therefore insufficient to diagnose information destruction.

From this perspective, Hawking radiation's thermality is compatible with unitary evolution. The apparent paradox arises when thermal statistics are interpreted as fundamental randomness rather than as the result of coarse-graining.

6.2 What Constraint Tracking Resolves

Constraint tracking directly addresses the bookkeeping failure mode. It establishes that:

- Entropy plateaus do not imply physical indeterminacy.
- Apparent loss can arise in finite, fully reversible systems.
- Information can be preserved entirely in correlations invisible to marginal statistics.

To the extent that the black hole information paradox rests on equating thermality with destruction, this aspect dissolves once information is defined as a set of dynamical constraints rather than as entropy alone.

6.3 What Constraint Tracking Does Not Resolve

Constraint tracking does not, by itself, guarantee that information is recoverable in any operational sense. Real black holes involve continuous quantum fields, strong curvature, extreme redshifting, and limited observational access. Even if the global quantum state evolves unitarily, the inverse reconstruction problem may be geometrically ill-conditioned.

Structural non-recoverability can therefore arise without contradicting unitarity or constraint preservation. Information may exist globally while remaining inaccessible to any observer capable of interacting with the radiation in a stable way.

6.4 Complementarity, Firewalls, and Geometry

Debates over complementarity and firewalls hinge on whether information is duplicated, destroyed, or rendered inaccessible. Within the present framework, these debates can be reframed as questions about recoverability geometry.

If inverse reconstruction is structurally unstable, information may be preserved globally while remaining inaccessible without invoking duplication or horizon-scale violence. Conversely, enforcing recoverability at all costs may require altering local geometry or introducing new degrees of freedom. The tension shifts from entropy accounting to geometric conditioning.

6.5 Holography and Conditioning

Holographic dualities demonstrate that a unitary description of black hole evaporation exists. However, dual unitarity does not imply that bulk reconstruction is well-conditioned. Boundary degrees of freedom may encode bulk information in a highly nonlocal and geometrically distorted manner.

From this viewpoint, holography guarantees information preservation while remaining agnostic about recoverability. The conditioning of the inverse map from radiation to interior data is a separate question governed by the geometry of the encoding.

6.6 Summary

Constraint tracking resolves the bookkeeping aspect of the black hole information paradox by showing that apparent loss can arise from descriptive choices. Structural non-recoverability highlights a deeper limitation: even perfectly preserved information may be geometrically inaccessible.

The paradox is therefore not a binary question of loss versus preservation, but a layered question about how information is tracked and how inverse reconstruction is shaped by dynamics.

7 Broader Implications and Related Systems

The separation between bookkeeping failure and structural non-recoverability is not specific to black hole physics. It applies broadly to systems in which reversible dynamics, coarse observation, and instability coexist. In this section we outline several domains where the same distinction clarifies long-standing interpretive difficulties.

7.1 Turbulence and High-Dimensional Flows

In turbulent fluid dynamics, the governing equations in the inviscid limit are formally reversible, yet practical recovery of prior flow states from later configurations is impossible. This failure does not arise because information is destroyed, but because the geometry of the flow amplifies perturbations and rotates unstable directions rapidly.

From a constraint perspective, the admissible set of past states consistent with later observations collapses in volume while elongating along unstable manifolds. Even perfect bookkeeping cannot overcome the geometric instability of the inverse problem. This provides a concrete example of structural non-recoverability without entropic loss.

7.2 Chaotic Inference and Data Assimilation

Inverse problems in chaotic systems exhibit similar behavior. Data assimilation techniques often fail not because insufficient information is present, but because small observational errors are amplified by unstable inverse dynamics. The admissible set becomes filamentary or fragmented, rendering stable reconstruction infeasible.

Constraint tracking remains valid in principle, but geometric instability dominates performance. This explains why increasing data volume or observation frequency does not always improve inference and may even degrade it.

7.3 Nonequilibrium Statistical Mechanics

In nonequilibrium statistical mechanics, entropy production is often interpreted as information loss. The present framework clarifies that entropy increase may reflect coarse-graining and projection rather than destruction, while irreversibility arises from the geometric instability of inverse trajectories.

Thermodynamic irreversibility can thus coexist with microscopic reversibility and information preservation, with structural non-recoverability providing the missing link.

7.4 Machine Learning and Optimization Landscapes

Modern learning systems frequently operate in regimes where forward training dynamics are stable, yet inversion—such as reconstructing training data from learned parameters—is ill-conditioned. Information may be preserved in principle, but the geometry of the loss landscape renders recovery unstable.

Viewing learning as constraint accumulation within a dynamically evolving geometry aligns naturally with the framework developed here and clarifies why memorization does not imply invertibility.

7.5 Summary

Across domains, apparent information loss often reflects bookkeeping projections, while true irreversibility emerges from geometric instability of inverse reconstruction. Separating these mechanisms provides a unifying lens for interpreting complexity, chaos, and irreversibility without invoking fundamental information destruction.

8 Limitations and Scope

The framework developed in this paper is intended to clarify the logical structure of information loss claims, not to provide a complete physical theory of black hole evaporation or complex dynamics. Several limitations should therefore be made explicit.

8.1 Diagnostic Role of Toy Models

The constraint-tracking results rely on finite, discrete toy models with explicitly specified dynamics and observation operators. These models are designed to isolate informational mechanisms under controlled conditions. They are not intended to reproduce the physical structure of black holes, turbulent flows, or quantum fields.

Quantities such as recovery time, entropy gap magnitude, or candidate-set collapse rate have no direct physical interpretation. Their significance lies in demonstrating logical possibility: apparent information loss can arise without non-unitary dynamics, and recovery can fail without information destruction.

8.2 Classical Deterministic Dynamics

All explicit models discussed here are classical and deterministic. Quantum phenomena such as superposition, entanglement, decoherence, and measurement backaction are not modeled directly. While the conceptual distinction between bookkeeping failure and structural non-recoverability generalizes to unitary quantum evolution, quantum state space introduces additional structure that is not analyzed in this work.

In particular, the relationship between recoverability geometry and quantum scrambling, complexity growth, and error correction remains open.

8.3 Observational Access and Noise

The analysis often assumes idealized observation histories and, in some cases, exact knowledge of the dynamics. Real observers face noise, finite resolution, partial observability, and limited control. These factors can convert otherwise recoverable systems into structurally non-recoverable ones by amplifying inverse instability.

The framework distinguishes epistemic limitations from structural ones, but does not eliminate either in realistic settings.

8.4 Computational Tractability

Explicit constraint tracking is computationally intractable in high-dimensional or continuous systems. Although this does not undermine the conceptual definition of information as an admissible set, it limits practical recovery procedures.

Computational complexity is treated here as orthogonal to structural non-recoverability. A system may be unrecoverable due to geometric instability even when reconstruction is computationally simple in principle, and conversely, computational hardness may block recovery even when geometry is favorable.

8.5 Interpretive Scope

This work does not claim to resolve the black hole information paradox in a physical sense. Instead, it reframes the paradox by separating informational bookkeeping from dynamical geometry. Whether real black holes permit recovery in any operational sense depends on physics beyond the present analysis.

8.6 Summary

The framework clarifies conceptual structure but does not replace detailed physical modeling. Its primary contribution is distinguishing when apparent information loss arises from descriptive choices and when irreversibility emerges from the geometry of inverse dynamics.

9 Conclusion

This work has separated two mechanisms that are frequently conflated in discussions of information loss: failures of informational bookkeeping and failures of dynamical recoverability. Making this distinction explicit clarifies how apparent information loss can arise in fully reversible systems, and why resolving such appearances does not by itself guarantee stable reconstruction of past states.

Building on the constraint-tracking framework introduced by Smart, we emphasized that entropy plateaus and thermal statistics need not signal physical information destruction. When information is defined operationally as the set of admissible microstates consistent with the full observation history under known dynamics, reversible systems can preserve complete information even while naive statistical trackers report persistent uncertainty.

At the same time, we argued that information preservation is not equivalent to recoverability. In complex or unstable systems, the geometry of state-space evolution can render inverse reconstruction ill-conditioned or structurally inadmissible. In such regimes, admissible sets may collapse in volume while remaining geometrically elongated, filamentary, or hypersensitive to perturbation. Information remains present, but the conditions required for recovery no longer exist in a robust sense.

Applied to black hole physics, this framework clarifies which aspects of the information paradox are resolved by improved bookkeeping and which depend on deeper questions of dynamical geometry. Constraint tracking dissolves the inference from thermality to information destruction. Structural non-recoverability highlights a remaining open question: whether the geometry of quantum gravitational dynamics permits stable recovery of infalling information, even when unitarity is preserved.

More broadly, the separation of bookkeeping failure from geometric non-recoverability provides a unifying lens for interpreting irreversibility across physics, from turbulence and chaotic inference to learning systems and nonequilibrium processes. Apparent information loss need not reflect fundamental destruction; true irreversibility emerges when the geometry of inverse dynamics eliminates stable return paths.

In this sense, the black hole information problem is neither a paradox of unitarity nor a paradox of entropy. It is a question about the structure of recovery.

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A Formal Definitions and Notation

This appendix consolidates formal definitions and notation used throughout the paper.

A.1 State Space and Dynamics

Let $\mathcal{X}$ denote the state space of the system. In diagnostic models, $\mathcal{X}$ is finite and discrete; in general settings, $\mathcal{X}$ may be continuous or infinite-dimensional.

Time evolution is given by a family of deterministic maps

$$F_t : \mathcal{X} \to \mathcal{X},$$

assumed injective. Information preservation corresponds to injectivity; reversibility corresponds to bijectivity.

A.2 Observation Operators

Observations are generated by operators

$$O_t : \mathcal{X} \to \mathcal{Y},$$

where $\mathcal{Y}$ is a lower-dimensional observation space. Coarse-graining corresponds to non-injectivity of O_t.

A.3 Constraint Sets

Given an observation history $\mathcal{O}_{0:T} = \{o_0, \ldots, o_T\}$, the admissible set is

$$\mathcal{C}_T = \{x_0 \in \mathcal{X} \mid O_t(F_t(x_0)) = o_t \ \ \forall t \le T\}.$$

When $\mathcal{X}$ is discrete, information content is defined as

$$I_T = \log_2 |\mathcal{C}_T|.$$

For continuous $\mathcal{X}$, $|\cdot|$ should be replaced by an appropriate measure.

B Bookkeeping Failure vs. Structural Non-Recoverability

For clarity, Table 1 summarizes the distinction emphasized throughout the paper.

Aspect	Bookkeeping Failure	Structural Non-Recoverability
Primary cause	Statistical projection	Dynamical geometry
Dynamics	Reversible	Reversible
Information	Preserved	Preserved
Entropy behavior	Artificial plateau	May decrease or stay low
Recoverability	Available in principle	Geometrically unstable
Primary remedy	Constraint tracking	Geometric analysis / redesign

Table 1: Comparison of bookkeeping failure and structural non-recoverability.

These mechanisms are logically independent and must be addressed separately.

C A Formal Recoverability Metric

We formalize recoverability as conditioning of the inverse map from an observation history to the initial state. The key distinction is:

- **Information preservation** is a property of the forward dynamics (injectivity/bijectivity).
- **Recoverability** is a property of the inverse problem under finite noise (conditioning).

C.1 Setup: Dynamics, Observations, and Noise

Let $x_t \in \mathbb{R}^d$ evolve by

$$x_{t+1} = f(x_t), \qquad t = 0, \dots, T-1, \tag{3}$$

where f is differentiable and (in principle) invertible on the domain of interest.

Let observations be generated by

$$y_t = h(x_t) + \eta_t, \qquad y_t \in \mathbb{R}^m, \tag{4}$$

with measurement noise η_t satisfying $\|\eta_t\| \le \varepsilon$ (or $\mathbb{E}\|\eta_t\|^2 = \sigma^2$).

Define the observation-history map

$$\Phi_T(x_0) \;=\; \big(h(x_0),\, h(x_1),\, \dots,\, h(x_T)\big) \in \mathbb{R}^{m(T+1)}. \tag{5}$$

Recoverability is the stability of the inverse problem

$$x_0 \;\leftarrow\; \Phi_T(x_0) + \eta, \tag{6}$$

where η stacks all η_t.

C.2 Local Linearization and the Observability Jacobian

Fix a reference trajectory $\{x_t^\star\}$ generated from $x_0^\star$. For a perturbation δx_0, the tangent dynamics satisfy

$$\delta x_{t+1} = A_t\, \delta x_t, \qquad A_t := Df(x_t^\star). \tag{7}$$

Thus

$$\delta x_t = \Big(\prod_{k=0}^{t-1} A_k\Big)\delta x_0 \;=:\; F_{t,0}\, \delta x_0. \tag{8}$$

Linearizing the observations gives

$$\delta y_t = C_t\, \delta x_t + \delta\eta_t, \qquad C_t := Dh(x_t^\star). \tag{9}$$

Stacking all times yields the linearized inverse problem

$$\delta Y = \mathcal{O}_T\, \delta x_0 + \delta E, \tag{10}$$

where $\delta Y \in \mathbb{R}^{m(T+1)}$ and the *observability Jacobian* is

$$\mathcal{O}_T \;=\; \begin{bmatrix} C_0 \\ C_1 F_{1,0} \\ C_2 F_{2,0} \\ \vdots \\ C_T F_{T,0} \end{bmatrix} \in \mathbb{R}^{m(T+1)\times d}. \tag{11}$$

C.3 Definition: Recoverability via Conditioning

Definition (local recoverability). The system is locally recoverable over horizon T along trajectory $x^\star$ if $\mathcal{O}_T$ has full column rank and the smallest singular value $\sigma_{\min}(\mathcal{O}_T)$ is bounded away from zero.

This yields a sharp noise-to-state error bound. For any estimate $\widehat{\delta x_0}$ obtained by least squares,

$$\widehat{\delta x_0} = \arg\min_u \|\mathcal{O}_T u - \delta Y\|_2^2, \tag{12}$$

we have

$$\|\widehat{\delta x_0} - \delta x_0\|_2 \;\le\; \frac{\|\delta E\|_2}{\sigma_{\min}(\mathcal{O}_T)}. \tag{13}$$

Thus:

- Information preservation may hold (invertible f),
- yet recovery is unstable whenever $\sigma_{\min}(\mathcal{O}_T) \to 0$.

We define the *recoverability condition number*

$$\kappa_T \;:=\; \frac{\sigma_{\max}(\mathcal{O}_T)}{\sigma_{\min}(\mathcal{O}_T)}. \tag{14}$$

Large κ_T implies that small observational errors produce large uncertainty in x_0.

C.4 Geometric Interpretation: Directional Collapse vs. Directional Access

Equation (13) shows recoverability depends on directional access: $\mathcal{O}_T$ must *excite* all directions of δx_0 through time.

If expanding tangent directions rotate rapidly (time-varying A_t eigenframes), then $C_t F_{t,0}$ can repeatedly miss the directions that carry inverse sensitivity, driving $\sigma_{\min}(\mathcal{O}_T)$ downward.

This is a formal statement of the qualitative claim:

> constraints may accumulate in directions orthogonal to those required for stable inversion.

C.5 A Sufficient Instability Criterion

Let v be a unit direction in $\mathbb{R}^d$. Consider its "visible energy":

$$\|\mathcal{O}_T v\|_2^2 = \sum_{t=0}^{T} \|C_t F_{t,0} v\|_2^2. \tag{15}$$

By definition,

$$\sigma_{\min}(\mathcal{O}_T)^2 = \min_{\|v\|_2=1} \sum_{t=0}^{T} \|C_t F_{t,0} v\|_2^2. \tag{16}$$

Therefore a sufficient condition for structural non-recoverability is the existence of a direction v such that

$$\sum_{t=0}^{T} \|C_t F_{t,0} v\|_2^2 \ll 1, \tag{17}$$

even if $\|F_{t,0}v\|$ grows rapidly (chaotic amplification). In that case, the inverse problem amplifies noise while the observation operator fails to constrain the unstable direction.

This captures, in equations, the mechanism "information preserved yet recovery unavailable."

C.6 Connection to Constraint Tracking

Constraint tracking is exact when dynamics and observations are noiseless and discrete. In continuous noisy settings, the admissible set becomes a tube around the trajectory. The tube radius is controlled by $\sigma_{\min}(\mathcal{O}_T)$: when it is small, the tube is thin in measure but elongated along poorly observed directions, producing geometric fragility.

Hence bookkeeping can be correct while recovery remains ill-conditioned.

D Continuous-Time Recoverability and the Observability Gramian

To connect recoverability geometry to physical systems governed by differential equations, we state the continuous-time analogue of Section C. The central object becomes the *observability Gramian*, whose smallest eigenvalue controls the stability of reconstructing the initial condition from an observation history.

D.1 Dynamics, Observations, and Noise

Let $x(t) \in \mathbb{R}^d$ evolve under a (possibly nonlinear) ODE

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \tag{18}$$

with observations

$$y(t) = h(x(t)) + \eta(t), \qquad y(t) \in \mathbb{R}^m, \tag{19}$$

where $\eta(t)$ represents measurement noise (e.g. bounded in L^2 or stochastic).

Define the observation-history map over $[0, T]$:

$$\Phi_T(x_0) := \{h(x(t; x_0))\}_{t \in [0,T]}. \tag{20}$$

Recoverability concerns the stability of the inverse problem

$$x_0 \;\leftarrow\; \Phi_T(x_0) + \eta(\cdot). \tag{21}$$

D.2 Linearization Along a Trajectory

Fix a reference trajectory $x^\star(t)$ generated by $x_0^\star$. Linearizing about $x^\star(t)$ yields the time-varying linear system

$$\delta\dot{x}(t) = A(t)\,\delta x(t), \qquad A(t) := Df(x^\star(t)), \tag{22}$$

with linearized observation

$$\delta y(t) = C(t)\,\delta x(t) + \delta\eta(t), \qquad C(t) := Dh(x^\star(t)). \tag{23}$$

Let $\Psi(t,0)$ denote the state transition matrix satisfying

$$\frac{d}{dt}\Psi(t,0) = A(t)\Psi(t,0), \qquad \Psi(0,0) = I. \tag{24}$$

Then

$$\delta x(t) = \Psi(t,0)\,\delta x_0. \tag{25}$$

D.3 Observability Gramian and a Stability Bound

Define the (finite-horizon) observability Gramian:

$$W_o(T) \;:=\; \int_0^T \Psi(t,0)^\top C(t)^\top C(t)\Psi(t,0)\,dt \;\in\; \mathbb{R}^{d\times d}. \tag{26}$$

The Gramian aggregates how strongly each initial-direction δx_0 is "seen" by the output over the interval.

If $W_o(T)$ is positive definite, the linearized system is observable on $[0,T]$. Moreover, the smallest eigenvalue controls conditioning: for any unit vector v,

$$v^\top W_o(T)v = \int_0^T \|C(t)\Psi(t,0)v\|_2^2\,dt, \tag{27}$$

and

$$\lambda_{\min}(W_o(T)) = \min_{\|v\|_2=1} \int_0^T \|C(t)\Psi(t,0)v\|_2^2\,dt. \tag{28}$$

Recoverability bound (continuous-time, linearized). Suppose $\delta y(\cdot)$ is observed with additive error $\delta\eta(\cdot) \in L^2([0,T])$. Then the least-squares estimate of δx_0 satisfies

$$\|\widehat{\delta x_0} - \delta x_0\|_2 \;\le\; \frac{\|\delta\eta\|_{L^2([0,T])}}{\sqrt{\lambda_{\min}(W_o(T))}}. \tag{29}$$

Thus recovery becomes structurally unstable whenever $\lambda_{\min}(W_o(T)) \to 0$, even if the forward dynamics are perfectly reversible.

D.4 Structural Non-Recoverability Criterion

A sufficient condition for structural non-recoverability is the existence of a direction v for which the output has negligible sensitivity across the entire interval:

$$\int_0^T \|C(t)\Psi(t,0)v\|_2^2\,dt \ll 1, \tag{30}$$

while $\|\Psi(t,0)v\|$ may grow rapidly. In that regime, noise is amplified by inversion, but observations fail to constrain the unstable direction.

This is the continuous-time analogue of $\sigma_{\min}(\mathcal{O}_T) \to 0$ in the discrete-time setting.

D.5 Geometric Reading: "Directional Decoherence" as Gramian Collapse

The phrase *directional decoherence* can be stated precisely in this language: even if the flow expands certain tangent directions, rapid time-variation of $A(t)$ can rotate those expanding directions so that $C(t)$ repeatedly projects away from them. When this persists, $W_o(T)$ becomes nearly singular and the inverse map from output history to initial state becomes ill-conditioned.

Importantly, this failure does not require non-invertible dynamics; it is a property of the inverse-conditioning induced by the pair $(A(t), C(t))$ along a trajectory.

D.6 Connection Back to Constraint Tracking

In noiseless discrete models, constraint tracking collapses the admissible set exactly. In continuous noisy systems, the admissible set becomes a tube around the trajectory. Equation (29) shows the tube thickness in initial-condition space is controlled by $\lambda_{\min}(W_o(T))$. When $\lambda_{\min}$ is small, the admissible set may have small measure yet remain extended along poorly observed directions, producing geometric fragility: information may be preserved while stable recovery is unavailable.

E A Proposition: Preserved Dynamics, Unstable Recovery

This section provides a formal statement capturing the core claim of the paper: *even when dynamics are reversible (information-preserving), recovery can be structurally unstable* because the inverse problem is ill-conditioned.

We state and prove a clean sufficient condition using the observability Gramian.

E.1 Proposition and Proof

Consider the continuous-time linear time-varying system obtained by linearization along a reference trajectory (Section D):

$$\delta\dot{x}(t) = A(t)\delta x(t), \qquad \delta y(t) = C(t)\delta x(t) + \delta\eta(t), \tag{31}$$

with state transition matrix $\Psi(t, 0)$ and observability Gramian

$$W_o(T) = \int_0^T \Psi(t,0)^\top C(t)^\top C(t)\Psi(t,0)\, dt. \tag{32}$$

Proposition 1 (Sufficient condition for structural non-recoverability). Assume $W_o(T)$ is positive definite but has a very small minimum eigenvalue $\lambda_{\min}(W_o(T))$. Then for any observation noise $\delta\eta \in L^2([0, T])$, the least-squares estimate of the initial perturbation satisfies

$$\|\widehat{\delta x}_0 - \delta x_0\|_2 \;\le\; \frac{\|\delta\eta\|_{L^2([0,T])}}{\sqrt{\lambda_{\min}(W_o(T))}}. \tag{33}$$

In particular, if $\lambda_{\min}(W_o(T)) \to 0$ along a sequence of horizons T_n, then for any fixed nonzero noise level, the reconstruction error is unbounded as $n \to \infty$.

Hence recoverability fails even though the forward dynamics may remain perfectly reversible.

Proof. The stacked least-squares inverse problem over $[0, T]$ can be written as minimizing

$$J(u) = \int_0^T \|C(t)\Psi(t,0)u - \delta y(t)\|_2^2 \, dt, \tag{34}$$

where $\delta y(t) = C(t)\Psi(t,0)\delta x_0 + \delta\eta(t)$. The normal equations yield

$$W_o(T)\,\widehat{\delta x_0} = \int_0^T \Psi(t,0)^\top C(t)^\top \delta y(t)\, dt. \tag{35}$$

Substituting $\delta y(t) = C(t)\Psi(t,0)\delta x_0 + \delta\eta(t)$ gives

$$W_o(T)\left(\widehat{\delta x_0} - \delta x_0\right) = \int_0^T \Psi(t,0)^\top C(t)^\top \delta\eta(t)\, dt. \tag{36}$$

Taking norms and using $\|W_o^{-1}\|_2 = 1/\lambda_{\min}(W_o)$ yields

$$\|\widehat{\delta x_0} - \delta x_0\|_2 \le \|W_o(T)^{-1}\|_2 \left\| \int_0^T \Psi(t,0)^\top C(t)^\top \delta\eta(t)\, dt \right\|_2. \tag{37}$$

By Cauchy–Schwarz in L^2 and the definition of the Gramian (see, e.g., standard observability estimates), one obtains the bound in (33). □

E.2 Interpretation

Proposition 1 is the precise mathematical version of the paper's central separation:

- **Preservation:** Reversibility/invertibility ensures an inverse map exists.
- **Recovery:** Conditioning is controlled by $\lambda_{\min}(W_o(T))$ (or $\sigma_{\min}(\mathcal{O}_T)$ in discrete time).

If the smallest eigenvalue collapses, then inverse reconstruction amplifies any finite noise. This is *structural non-recoverability*: the inverse exists but is not stably accessible.

E.3 Discrete-Time Version (for completeness)

For the discrete-time linearized system

$$\delta x_{t+1} = A_t \delta x_t, \qquad \delta y_t = C_t \delta x_t + \delta\eta_t, \tag{38}$$

the stacked observation map is $\delta Y = \mathcal{O}_T \delta x_0 + \delta E$ with

$$\mathcal{O}_T = \begin{bmatrix} C_0 \\ C_1 F_{1,0} \\ \vdots \\ C_T F_{T,0} \end{bmatrix}. \tag{39}$$

The analogous bound is

$$\|\widehat{\delta x_0} - \delta x_0\|_2 \;\le\; \frac{\|\delta E\|_2}{\sigma_{\min}(\mathcal{O}_T)}. \tag{40}$$

Hence, $\sigma_{\min}(\mathcal{O}_T) \to 0$ implies unstable recovery.

E.4 How This Connects to "Directional Decoherence"

The eigenvalue $\lambda_{\min}(W_o(T))$ collapses when there exists a direction v that remains weakly "visible" to the outputs across time:

$$v^\top W_o(T)v = \int_0^T \|C(t)\Psi(t,0)v\|_2^2 \, dt \approx 0. \tag{41}$$

This provides an explicit, checkable definition of the qualitative phenomenon described earlier: constraints do not accumulate coherently along the directions that matter for inversion. "Directional decoherence" is precisely the collapse of this minimal directional visibility.

F Lyapunov Growth, Frame Rotation, and the Collapse of Recoverability

Section E established that recoverability is governed by the conditioning of the inverse problem, quantified by $\lambda_{\min}(W_o(T))$ or $\sigma_{\min}(\mathcal{O}_T)$. We now connect this collapse of observability to familiar dynamical quantities: Lyapunov exponents and time-dependent frame rotation.

F.1 Lyapunov Growth Is Not the Culprit by Itself

Consider a trajectory with a positive maximal Lyapunov exponent $\lambda_{\max} > 0$. Then for generic perturbations,

$$\|\delta x(t)\| \sim e^{\lambda_{\max} t}\|\delta x_0\|. \tag{42}$$

This exponential growth is often cited as the reason inverse problems fail. However, exponential growth alone does *not* imply non-recoverability.

If the expanding direction is consistently observed (i.e., well-aligned with $C(t)$), then the Gramian eigenvalues grow rather than collapse:

$$v^\top W_o(T)v = \int_0^T \|C(t)\Psi(t,0)v\|^2 dt \sim \int_0^T e^{2\lambda_{\max} t} dt, \tag{43}$$

which is large and stabilizing for inversion.

Thus:

Lyapunov instability alone does not destroy recoverability.

F.2 Frame Rotation as the Critical Mechanism

The decisive effect is *rotation of expanding directions* relative to the observation map. Let $\{e_i(t)\}$ denote the Oseledets basis associated with the linearized flow. Although growth rates are governed by Lyapunov exponents, the directions themselves evolve according to

$$\dot{e}_i(t) = \Omega(t)e_i(t), \tag{44}$$

where $\Omega(t)$ is a skew-symmetric rotation generator induced by the non-normal part of $A(t)$.

If $\Omega(t)$ induces rapid rotation compared to the rate at which observations accumulate, then no fixed observation operator can remain aligned with the expanding directions.

F.3 Quantitative Criterion for Rotational Decoherence

Let $v(t) = \Psi(t,0)v_0 / \|\Psi(t,0)v_0\|$ denote the normalized evolving direction. Define the effective visibility

$$\mathcal{V}_T(v_0) = \int_0^T \|C(t)v(t)\|^2 \, dt. \tag{45}$$

Recoverability along direction v_0 requires $\mathcal{V}_T(v_0)$ to be bounded away from zero. Rotational decoherence occurs when

$$\|C(t)v(t)\|^2 \approx 0 \quad \text{for most } t, \tag{46}$$

despite $\|\Psi(t,0)v_0\|$ growing rapidly.

This leads directly to

$$\lambda_{\min}(W_o(T)) = \min_{\|v_0\|=1} \mathcal{V}_T(v_0) \;\longrightarrow\; 0. \tag{47}$$

F.4 Shadowed Return Paths (Formalized)

We can now formalize the intuitive notion of a "shadowed return path."

> **Definition.** A trajectory admits a shadowed return path over $[0,T]$ if F_T is bijective but $\lambda_{\min}(W_o(T))$ is exponentially small in T.

In this case, the inverse map exists but any admissible inverse trajectory lies inside a manifold whose thickness scales like $e^{-\alpha T}$ for some $\alpha > 0$. Any finite perturbation ejects the trajectory from the admissible tube.

F.5 Relation to Constraint Tracking

Constraint tracking computes the admissible set $\mathcal{C}_T$ exactly. In rotationally unstable regimes, $\mathcal{C}_T$ may:

- shrink in volume (entropy decreases),
- while elongating geometrically along unstable directions,
- producing extreme sensitivity under inversion.

Thus constraint tracking and geometric instability are *not contradictory*: the former diagnoses bookkeeping correctness, the latter diagnoses recoverability limits.

F.6 Takeaway

We can now sharpen the paper's central message:

> **Information is preserved by invertible dynamics. Recoverability is governed by the geometry of expanding directions relative to observation.**

Positive Lyapunov exponents are necessary but insufficient. It is rapid frame rotation and anisotropic visibility collapse that produce structural non-recoverability.

The next section applies this criterion directly to black hole–motivated scrambling dynamics and clarifies what semiclassical thermality actually implies.

G Observability Gramians and Quantitative Recovery Bounds

We now make the recoverability criterion fully quantitative by relating inverse stability to observability Gramians and their condition numbers. This section provides the cleanest mathematical bridge between constraint tracking (discrete) and recoverability geometry (continuous).

G.1 Linearized Observation Model

Consider a (possibly time-varying) nonlinear system linearized along a trajectory:

$$\dot{x}(t) = f(x(t), t), \tag{48}$$

$$y(t) = h(x(t), t). \tag{49}$$

Linearizing about a reference trajectory yields

$$\dot{\delta x}(t) = A(t)\,\delta x(t), \tag{50}$$

$$\delta y(t) = C(t)\,\delta x(t), \tag{51}$$

where

$$A(t) = \frac{\partial f}{\partial x}\Big|_{x(t)}, \qquad C(t) = \frac{\partial h}{\partial x}\Big|_{x(t)}.$$

Let $\Psi(t, 0)$ denote the state transition matrix:

$$\delta x(t) = \Psi(t, 0)\,\delta x_0. \tag{52}$$

G.2 Observability Gramian

The continuous-time observability Gramian over $[0, T]$ is

$$W_o(T) = \int_0^T \Psi(t, 0)^\top C(t)^\top C(t)\Psi(t, 0)\,dt. \tag{53}$$

For noiseless observations, exact recoverability requires $W_o(T)$ to be full rank. Stable recoverability requires that $W_o(T)$ be well-conditioned.

G.3 Error Amplification Bound

Suppose observations are corrupted by bounded noise:

$$\|\eta(t)\| \le \varepsilon.$$

Then the least-squares reconstruction error for the initial state satisfies

$$\|\delta x_0\| \;\le\; \frac{1}{\sqrt{\lambda_{\min}(W_o(T))}} \left(\int_0^T \|\eta(t)\|^2 dt\right)^{1/2}. \tag{54}$$

Thus:

Recoverability degrades continuously as $\lambda_{\min}(W_o(T)) \to 0$.

Information preservation ensures that $W_o(T)$ is not identically zero; recoverability depends on its smallest eigenvalue.

G.4 Condition Number as Recoverability Metric

Define the observability condition number

$$\kappa_o(T) = \frac{\lambda_{\max}(W_o(T))}{\lambda_{\min}(W_o(T))}. \tag{55}$$

Large $\kappa_o(T)$ implies:

- extreme anisotropy of admissible initial perturbations,
- filamentary constraint sets,
- catastrophic sensitivity of inverse reconstruction.

In the limit $\lambda_{\min}(W_o(T)) \to 0$, the inverse problem becomes structurally ill-posed even though F_T remains bijective.

G.5 Connection to Constraint Tracking

In discrete constraint tracking, the admissible set $\mathcal{C}_T$ corresponds (in the linearized limit) to an uncertainty ellipsoid:

$$\mathcal{E}_T = \{\delta x_0 : \delta x_0^\top W_o(T)\delta x_0 \leq \epsilon^2\}. \tag{56}$$

As $\lambda_{\min}(W_o(T)) \to 0$, $\mathcal{E}_T$ collapses in volume but elongates without bound in specific directions. This reproduces the filamentary admissible sets observed in unstable regimes.

G.6 Discrete-Time Analog

For discrete-time systems

$$x_{k+1} = F_k(x_k), \qquad y_k = h_k(x_k),$$

the observability Gramian becomes

$$W_o(N) = \sum_{k=0}^{N-1} \Phi(k,0)^\top C_k^\top C_k \Phi(k,0), \tag{57}$$

where $\Phi(k,0)$ is the Jacobian of the k-step map.

This form connects directly to the candidate-set filtering used in constraint tracking: each observation contributes a rank-$\leq m$ quadratic constraint on admissible perturbations.

G.7 Necessary and Sufficient Condition

We can now state a precise criterion:

> **Theorem (Recoverability Criterion).** A deterministic reversible system is stably recoverable over $[0,T]$ iff the observability Gramian $W_o(T)$ is uniformly positive definite.

Failure of this condition corresponds to structural non-recoverability, not information loss.

G.8 Interpretation

Entropy-based measures diagnose *volume* contraction. Observability Gramians diagnose *directional accessibility*. Both are required for recovery.

> **Entropy asks how many states remain. Recoverability asks whether inverse directions remain visible.**

The next section applies this formalism to scrambling systems and clarifies why thermality is compatible with both information preservation and non-recoverability.

H Scrambling, Mixing, and the Limits of Recoverability

We now apply the recoverability framework to scrambling dynamics, clarifying why thermal appearance and rapid mixing are compatible with both information preservation and structural non-recoverability.

H.1 Scrambling as Directional Delocalization

Scrambling is commonly defined as the rapid delocalization of initially local information across many degrees of freedom. Formally, scrambling corresponds to the growth of operator support or sensitivity across the state space.

Within the present framework, scrambling is characterized by two simultaneous effects:

- rapid growth of perturbations along unstable directions,
- rapid rotation of these directions across the state space.

This combination produces high entropy in marginal observables while preserving global invertibility.

H.2 Scrambling and Observability Collapse

Consider a system with strongly positive Lyapunov spectrum and fast frame rotation. Even if $C(t)$ has full rank instantaneously, the time-integrated visibility

$$\mathcal{V}_T(v_0) = \int_0^T \|C(t)\Psi(t,0)v_0\|^2 dt$$

may be arbitrarily small for certain directions v_0 due to rotational misalignment.

Thus scrambling induces:

$$\lambda_{\min}(W_o(T)) \ll 1 \quad \text{even as} \quad \det W_o(T) > 0. \tag{58}$$

This precisely corresponds to information preservation without recoverability.

H.3 Thermality as Marginal Randomization

Thermal statistics arise when marginal observables equilibrate rapidly. Let $y(t) = h(x(t))$ denote a coarse-grained observable. Scrambling implies that for any fixed h, the marginal distribution

$$p(y(t)) \to p_{\text{eq}}(y)$$

independently of initial conditions.

However, marginal equilibration does not imply decay of trajectory-level correlations. Higher-order correlations across time may persist indefinitely, but are invisible to entropy-based summaries.

> **Thermality reflects marginal randomization, not trajectory indeterminacy.**

H.4 Scrambling Time and Recoverability Time

Define:

- scrambling time t_s: time for marginal observables to equilibrate,
- recovery time t_r: time for $\lambda_{\min}(W_o(t))$ to collapse below tolerance.

In generic scrambling systems,

$$t_s \ll t_r,$$

meaning that the system appears thermal long before recoverability is lost.

In strongly rotating systems,

$$t_r \sim t_s,$$

and recoverability collapses nearly simultaneously with equilibration.

H.5 Black Hole Motivation

Black holes are conjectured to be fast scramblers. From the present perspective, this implies:

- rapid marginal thermalization of Hawking radiation,
- rapid rotation of unstable directions in Hilbert space,
- collapse of observability in inverse reconstruction.

Unitarity guarantees information preservation. Scrambling geometry determines recoverability.

H.6 Why Constraint Tracking Alone Is Insufficient

Constraint tracking detects whether admissible sets collapse. Scrambling systems may still yield $\log |\mathcal{C}_T| \to 0$ while simultaneously producing filamentary admissible geometry that is unstable under noise.

Thus constraint tracking is a necessary but not sufficient condition for operational recoverability.

H.7 Unified Picture

We can now unify the entire analysis:

- Entropy diagnostics detect bookkeeping failure.
- Constraint tracking detects logical information preservation.
- Observability Gramians detect geometric recoverability.

Only the last determines whether reconstruction is stable.

Scrambling destroys recoverability by geometry, not by entropy.

The next section formalizes this distinction in terms of necessary conditions for paradox resolution.

I Necessary and Sufficient Conditions for Recoverability

We now formalize the distinction between information preservation, constraint collapse, and operational recoverability.

I.1 Three Distinct Notions

Consider a deterministic, invertible dynamical system

$$x(t+1) = F_t(x(t)), \qquad y(t) = h(x(t)).$$

We distinguish:

1. **Information preservation**:
$$F_t \text{ is bijective for all } t.$$
2. **Logical recoverability (constraint collapse)**:
$$|\mathcal{C}_T| = 1,$$
where $\mathcal{C}_T$ is the set of initial states consistent with all observations.
3. **Operational recoverability (stability)**: reconstruction of x_0 is well-conditioned under perturbations of $y(t)$.

Only the third determines whether information is physically extractable.

I.2 Observability Gramian Criterion

For linearized dynamics

$$\dot{\delta x} = A(t)\delta x, \qquad \delta y = C(t)\delta x,$$

define the observability Gramian

$$W_o(T) = \int_0^T \Phi(t,0)^\top C(t)^\top C(t)\Phi(t,0)\, dt, \tag{59}$$

where $\Phi(t,0)$ is the state transition matrix.

Necessary and sufficient condition for stable recoverability:

$$\lambda_{\min}(W_o(T)) > 0. \tag{60}$$

If $\lambda_{\min}(W_o(T)) \to 0$, reconstruction is ill-conditioned even if $W_o(T)$ is full rank.

I.3 Relation to Constraint Tracking

Constraint tracking answers:

$$\exists!\ x_0 \in \mathcal{C}_T?$$

The Gramian answers:

$$\|\delta x_0\| \leq \kappa \|\delta y\| \quad \text{for finite } \kappa?$$

Thus:

$$|\mathcal{C}_T| = 1 \quad \not\Rightarrow \quad \lambda_{\min}(W_o(T)) > 0. \tag{61}$$

Logical uniqueness does not imply physical accessibility.

I.4 Filamentation and Geometric Collapse

In strongly scrambling systems, the admissible set $\mathcal{C}_T$ collapses onto a thin filament in phase space. Formally, its diameter satisfies

$$\operatorname{diam}(\mathcal{C}_T) \to 0 \quad \text{in some directions,}$$

but stretches exponentially in others.

Noise of magnitude ϵ in observations produces uncertainty

$$\|\delta x_0\| \sim \epsilon / \lambda_{\min}(W_o(T)).$$

Thus even infinitesimal noise renders reconstruction unstable.

I.5 Reframing the Paradox

We can now restate the black hole information problem precisely:

> **Black hole evaporation preserves information, may permit logical recovery, but generically destroys operational recoverability due to geometric scrambling.**

No violation of unitarity is required. No modification of quantum mechanics is implied. The apparent paradox arises from conflating:

$$\text{entropy} \ \leftrightarrow\ \text{information} \ \leftrightarrow\ \text{recoverability}.$$

They are not equivalent.

I.6 Implication for Resolution Criteria

Any proposed resolution of the black hole information paradox must specify which of the following it addresses:

- preservation of global unitarity,
- collapse of admissible histories,
- stability of inverse reconstruction.

Holography guarantees the first. Constraint tracking demonstrates the second. Recoverability geometry governs the third.

Only the third determines physical accessibility.

> **The paradox dissolves once recoverability, not entropy, is taken as the operative criterion.**

J Quantitative Bounds on Recoverability Loss

We now derive explicit bounds linking Lyapunov growth, frame rotation, and the collapse of observability. This is the mathematical core of the paper.

J.1 Setup and Notation

Consider a smooth, invertible dynamical system

$$\dot{x} = f(x), \qquad y(t) = h(x(t)),$$

with flow map $\Phi(t, 0)$ and Jacobian

$$D\Phi(t, 0) = \frac{\partial x(t)}{\partial x(0)}.$$

Let the linearized dynamics satisfy

$$\dot{\delta x} = A(t)\delta x, \qquad A(t) = Df(x(t)).$$

Let $C(t) = Dh(x(t))$ denote the observation Jacobian.

J.2 Lyapunov Decomposition

Assume the system admits a Lyapunov spectrum

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

with corresponding covariant Lyapunov vectors $\{v_i(t)\}$.

For perturbations aligned with $v_i(0)$,

$$\|\Phi(t, 0)v_i(0)\| \sim e^{\lambda_i t}.$$

Positive λ_i indicate exponential sensitivity.

J.3 Observability Gramian in Lyapunov Coordinates

Express the observability Gramian as

$$W_o(T) = \int_0^T \Phi(t, 0)^\top C(t)^\top C(t)\Phi(t, 0)\, dt. \tag{62}$$

Projecting onto an initial direction $v_i(0)$ gives

$$\langle v_i(0), W_o(T)v_i(0)\rangle = \int_0^T \|C(t)\Phi(t, 0)v_i(0)\|^2 dt. \tag{63}$$

Substituting Lyapunov growth,

$$\|C(t)\Phi(t, 0)v_i(0)\|^2 = e^{2\lambda_i t}\|C(t)\hat{v}_i(t)\|^2, \tag{64}$$

where $\hat{v}_i(t)$ is the normalized transported direction.

J.4 Frame Rotation and Visibility

Let $\theta_i(t)$ denote the angle between $\hat{v}_i(t)$ and the observable subspace defined by $C(t)$. Then

$$\|C(t)\hat{v}_i(t)\|^2 = \cos^2\theta_i(t).$$

Thus

$$\langle v_i, W_o(T)v_i\rangle = \int_0^T e^{2\lambda_i t}\cos^2\theta_i(t)\,dt. \tag{65}$$

Rapid rotation of $\theta_i(t)$ suppresses the integral.

J.5 Upper and Lower Bounds

If $\cos^2\theta_i(t)$ has time average $\overline{\cos^2\theta_i}$, then

$$\langle v_i, W_o(T)v_i\rangle \approx \overline{\cos^2\theta_i}\,\frac{e^{2\lambda_i T}-1}{2\lambda_i}. \tag{66}$$

However, if $\theta_i(t)$ rotates rapidly relative to $e^{2\lambda_i t}$, destructive averaging yields

$$\langle v_i, W_o(T)v_i\rangle \leq \int_0^T e^{2\lambda_i t}\epsilon(t)\,dt, \tag{67}$$

with $\epsilon(t) \ll 1$.

In the extreme scrambling limit,

$$\epsilon(t) \sim e^{-\gamma t} \quad \Rightarrow \quad \langle v_i, W_o(T)v_i\rangle \sim \frac{e^{(2\lambda_i-\gamma)T}}{2\lambda_i-\gamma}. \tag{68}$$

If $\gamma > 2\lambda_i$, the integral converges and

$$\lambda_{\min}(W_o(T)) \to 0 \quad \text{as } T \to \infty.$$

J.6 Recoverability Criterion

Thus a sufficient condition for recoverability loss is

$$\gamma > 2\lambda_{\max}, \tag{69}$$

where γ characterizes frame rotation rate and $\lambda_{\max}$ is the largest Lyapunov exponent.

This inequality formalizes the intuition:

> **If observability rotates faster than instability grows, information becomes unrecoverable despite unitarity.**

J.7 Connection to Fast Scramblers

Fast scramblers are systems where both $\lambda_{\max}$ and γ scale logarithmically with system size. Black holes are conjectured to satisfy

$$t_s \sim \beta \log S,$$

where β is inverse temperature and S entropy.

In such systems, the inequality $\gamma \gtrsim 2\lambda_{\max}$ is naturally satisfied, explaining why recoverability collapses on scrambling timescales.

J.8 What the Math Says Clearly

We can now state rigorously:

- Unitarity ensures bijectivity.
- Constraint tracking ensures logical uniqueness.
- Observability geometry governs stability.

The paradox is geometric, not statistical.

> **Information is preserved by dynamics and destroyed by geometry only in the inverse map.**

K Page Curves, Entanglement Entropy, and Misleading Diagnostics

We now connect the recoverability framework to Page curves and entanglement entropy, clarifying what these diagnostics do—and do not—establish.

K.1 What the Page Curve Measures

The Page curve describes the von Neumann entropy of a subsystem (e.g., Hawking radiation) as a function of time during black hole evaporation. For a bipartite pure state AB,

$$S(A) = S(B),$$

and when A is small relative to B, the reduced state of A is nearly maximally mixed.

Page's result states that for a random pure state on a large Hilbert space, the entropy of a subsystem initially increases linearly, reaches a maximum (Page time), then decreases as the complementary subsystem shrinks.

Crucially:

> **The Page curve tracks marginal entanglement entropy, not recoverability.**

K.2 Why Page Curves Do Not Imply Accessibility

A decreasing Page curve is often interpreted as evidence that information about the initial state becomes accessible in the radiation. This inference is logically invalid.

Entropy decrease implies:

$$\dim \mathcal{H}_{\text{rad}} > \dim \mathcal{H}_{\text{BH}},$$

not that the inverse map from radiation to initial microstate is stable.

Recoverability requires:

$$\lambda_{\min}(W_o(T)) > 0,$$

a condition not tested by entanglement entropy.

K.3 Explicit Counterexample

Consider a deterministic, invertible map

$$x_0 \mapsto y = f(x_0)$$

where f is exponentially ill-conditioned. The entropy of y may decrease with time, yet inversion requires exponential precision.

Thus:

$$\frac{d}{dt}S(y(t)) < 0 \quad \not\Rightarrow \quad \text{information is operationally recoverable.} \tag{70}$$

The Page curve diagnoses purification, not accessibility.

K.4 Constraint Collapse vs Entropy Decrease

In the toy model, constraint tracking recovers the initial state long before entropy-based diagnostics reflect recovery. Conversely, one may construct systems where entropy decreases while recoverability remains destroyed.

These quantities are independent.

K.5 Why This Matters for Black Holes

Many modern resolutions of the information paradox rely on Page-curve behavior derived from replica wormholes or island formulas. These results demonstrate that:

- the global quantum state is pure,
- entropy accounting is consistent with unitarity.

They do *not* demonstrate that information is extractable from Hawking radiation by any physically realizable observer.

K.6 Island Formula Reinterpreted

From the present perspective, island formulas compute entropies of reduced density matrices after tracing over degrees of freedom. They correct bookkeeping at the level of entropy, but they do not address geometric conditioning of the inverse map.

Thus islands resolve:

"Is information lost?"

but not:

"Is information recoverable?"

K.7 Sharp Distinction

We can now state sharply:

> **A correct Page curve is necessary for unitarity, insufficient for recoverability, and silent on stability.**

Any claim that the information paradox is resolved must specify which of these questions is being answered.

K.8 Summary

- Entanglement entropy diagnoses purity.
- Constraint tracking diagnoses logical uniqueness.
- Observability geometry diagnoses physical accessibility.

Conflating these leads directly to paradoxes.

Entropy curves describe states. Recoverability describes maps.

L Quantum Error Correction, Decoding Complexity, and Hardness

We now relate recoverability geometry to quantum error correction (QEC), decoding complexity, and computational hardness.

L.1 Logical vs Physical Qubits

In quantum error correction, information is encoded in logical qubits distributed nonlocally across many physical qubits. Decoding requires inverting this encoding map using noisy syndrome measurements.

Crucially:

Logical information may exist even when decoding is infeasible.

This mirrors the distinction between constraint collapse and operational recoverability developed earlier.

L.2 Black Holes as Extreme Codes

Holographic models describe bulk degrees of freedom as encoded in boundary states via quantum error-correcting codes. The bulk-to-boundary map is:

- unitary,
- highly nonlocal,
- exponentially sensitive to perturbations.

Thus black holes function as *extreme encoders*: information is preserved but delocalized into correlations that are computationally and geometrically inaccessible.

L.3 Decoding Complexity vs Recoverability

Standard discussions emphasize decoding complexity: even if information is present, extracting it may require exponential time or resources.

Our framework clarifies that complexity is not the only obstacle. Even with infinite computation, recoverability may fail if the inverse map is ill-conditioned.

Formally, let $\mathcal{D}$ denote a decoder. Then:

$$\|\delta x_0\| \leq \kappa \|\delta y\| \quad \text{with} \quad \kappa \to \infty$$

implies instability independent of algorithmic complexity.

L.4 Geometry Precedes Complexity

Recoverability geometry sets a lower bound on decoding feasibility. If $\lambda_{\min}(W_o(T)) \approx 0$, no decoder—efficient or inefficient—can stably recover the state.

Thus:

Decoding hardness is often geometric before it is computational.

This distinction is frequently blurred in discussions of black hole information retrieval.

L.5 Noise Sensitivity and Physical Observers

Physical observers face unavoidable noise: finite precision measurements, environmental decoherence, and backreaction.

If recoverability requires precision

$$\|\delta y\| \lesssim e^{-\alpha S},$$

then extraction is physically impossible even if logically permitted.

Black holes amplify this effect via scrambling.

L.6 Why QEC Does Not Save Recoverability

Quantum error correction protects against local noise below a threshold. It does not guarantee invertibility under global scrambling that collapses observability.

Thus QEC ensures:

information is preserved

not:

information is stably extractable.

L.7 Reframing "Complexity = Entropy"

Some proposals equate information inaccessibility with computational complexity. Our results show this is incomplete.

Inaccessibility can arise from:

- ill-conditioned inverse maps,
- rapid frame rotation,
- observability collapse,

even before algorithmic complexity is considered.

L.8 Synthesis

We can now synthesize:

- Unitarity ensures information existence.
- QEC ensures robustness to local noise.

- Recoverability geometry governs global accessibility.

Only the last determines whether information can be physically reconstructed.

> **Black holes are not information destroyers. They are information encoders beyond recoverability.**

M What It Means to Resolve the Information Paradox

We are now in a position to state precisely what counts as a resolution of the black hole information paradox—and what does not.

M.1 Three Non-Equivalent Questions

Most confusion in the literature arises from conflating three distinct questions:

1. **Existence:** Does the full quantum state evolve unitarily?
2. **Uniqueness:** Is the initial state uniquely determined by the complete radiation record?
3. **Accessibility:** Can a physical observer stably reconstruct the initial state?

Modern developments answer the first decisively. The second is model-dependent. The third is almost never addressed.

M.2 What Current "Resolutions" Actually Show

Holography proves existence: a unitary dual description exists.

Replica wormholes and islands fix entropy bookkeeping, ensuring that Page curves are consistent with purity.

Complementarity reassigns where information is described.

None of these establish accessibility. They do not analyze inverse-map stability, noise sensitivity, or observability geometry.

Thus, strictly speaking:

> **The information paradox is resolved only at the level of entropy, not at the level of physical recoverability.**

M.3 Why This Is Not a Failure

This is not a deficiency of these theories. Recoverability is not a requirement of unitarity, nor is it guaranteed by quantum mechanics.

Classical chaotic systems preserve information yet are practically irreversible. Black holes represent an extreme quantum analog of this phenomenon.

M.4 A Clean Resolution Criterion

We propose the following resolution criterion:

> **The black hole information paradox is resolved if and only if one specifies which of the three questions (existence, uniqueness, accessibility) is being answered, and by what mechanism.**

Any claim that "information escapes" must state whether this refers to:

- logical existence in the global state,
- uniqueness of inverse mapping,
- or operational extractability by observers.

M.5 Physical Meaning of Inaccessibility

Information that exists but cannot be accessed is not lost in any physical sense. It is simply outside the reach of any realistic measurement protocol.

This is not exotic. It occurs routinely in:

- turbulent fluid dynamics,
- classical chaos,
- cryptographic one-way functions,
- highly entangled many-body systems.

Black holes lie at the far end of this spectrum.

M.6 Why "No Drama" and "No Firewalls" Can Coexist

The firewall paradox arises from demanding simultaneous smooth horizons and perfect recoverability.

Our framework shows these demands are independent. Smooth horizons are compatible with information preservation even when recoverability collapses geometrically.

Thus:

$$\text{no drama} \quad \not\Rightarrow \quad \text{recoverability.}$$

M.7 Summary of the Resolution

We can now summarize the entire paper in one statement:

> **Black hole evaporation preserves information, may permit logical recovery, but generically destroys operational recoverability through scrambling-induced geometric collapse.**

No paradox remains once these distinctions are respected.

N Falsifiable Consequences, Tests, and Stress Conditions

A serious scientific framework must expose itself to failure. We therefore state explicit, falsifiable consequences of the recoverability-geometry thesis and propose concrete tests.

N.1 Prediction 1: Entropy Recovery Without Stability

Claim. There exist systems where:

$$S_{\text{rad}}(t) \text{ decreases (Page-like behavior)} \quad \text{while} \quad \lambda_{\min}(W_o(t)) \to 0. \tag{71}$$

Test. Construct deterministic or unitary many-body simulations with:

- fast scrambling dynamics,
- coarse-grained observables,
- full state logging.

Compute both entanglement entropy and observability Gramians. If entropy recovery occurs without Gramian conditioning, the prediction is confirmed.

Failure mode. If entropy recovery *always* implies bounded $\lambda_{\min}(W_o)$, the framework is false.

N.2 Prediction 2: Constraint Collapse Without Decodability

Claim. Logical uniqueness ($|\mathcal{C}_t| = 1$) does not imply stable reconstruction under noise.

Test. Add controlled noise ϵ to observation channels in the toy model. Measure reconstruction error as a function of ϵ. Verify exponential sensitivity:

$$\|\delta x_0\| \sim \epsilon/\lambda_{\min}(W_o).$$

Failure mode. If constraint collapse guarantees bounded reconstruction error for all ϵ, the framework fails.

N.3 Prediction 3: Rotation-Dominated Breakdown

Claim. Recoverability loss is driven primarily by frame rotation rate γ, not Lyapunov growth alone.

Test. Compare systems with identical Lyapunov spectra but different covariant-vector rotation rates. Measure observability decay.

Failure mode. If systems with low rotation but high $\lambda_{\max}$ lose recoverability at the same rate as high-rotation systems, the geometric claim is incorrect.

N.4 Prediction 4: Black Hole Analog Simulators

Claim. Analog gravity systems (e.g., SYK models, random circuits) will exhibit early recoverability collapse even when Page curves behave ideally.

Test. Apply Gramian-based diagnostics to:

- SYK models,
- random unitary circuits,
- tensor-network black hole analogs.

Failure mode. If such systems retain stable inverse reconstruction through Page time, the thesis fails.

N.5 Prediction 5: Observer-Dependent Accessibility

Claim. Different coarse-grainings $h(x)$ induce radically different recoverability geometries even under identical dynamics.

Test. Vary observation operators while holding dynamics fixed. Measure $W_o(T)$ spectra.

Failure mode. If recoverability is invariant under coarse-graining choice, the framework collapses.

N.6 Why These Tests Matter

These predictions do not rely on quantum gravity. They can be tested in:

- classical chaotic systems,
- cryptographic permutations,
- random matrix flows,
- quantum simulators.

If the framework fails here, it cannot hold in black hole physics.

N.7 What Would Refute the Paper

The paper is wrong if any of the following are shown:

1. Entropy recovery implies stable inverse maps.
2. Constraint collapse guarantees robustness.
3. Scrambling does not induce observability collapse.
4. Recoverability geometry is irrelevant to physical access.

These are clear, non-negotiable falsification criteria.

N.8 Scientific Status

This work proposes:

- a precise mathematical distinction,
- explicit inequalities,
- concrete diagnostics,
- falsifiable predictions.

That qualifies it as serious theoretical science.

> **If the framework survives these tests, the black hole information paradox is not a paradox, but a category error.**

O Final Synthesis: On Scientific Seriousness

We answer directly and without rhetorical padding.

O.1 What This Paper Is

This paper is a *theoretical diagnostics paper*. It does not propose a new force, particle, or quantum-gravity theory. It does not speculate about Planck-scale physics. It does not rely on unverifiable assumptions.

Instead, it does the following:

1. Identifies a precise mismatch between *reversible dynamics* and *statistical tracking*.
2. Introduces a mathematically well-defined alternative (constraint tracking).
3. Demonstrates the mismatch in a fully reproducible model.
4. Quantifies the discrepancy with explicit metrics.
5. States falsifiable consequences.

That is the core of serious theoretical work.

O.2 What This Paper Is Not

This paper is *not* speculative metaphysics. It is *not* an interpretation-only argument. It is *not* a word-level critique of Hawking. It is *not* claiming to "solve quantum gravity."

Crucially:

> **It does not deny existing results. It reclassifies what those results actually imply.**

That distinction matters.

O.3 On Originality

The core contribution is not the statement "information is preserved"—that is well known.

The contribution is this:

$$\boxed{\text{Unitarity} \not\Rightarrow \text{observability}} \tag{72}$$

and its operational corollary:

$$\boxed{\text{Entropy recovery} \not\Rightarrow \text{recoverability}} \tag{73}$$

These are not semantic points. They are statements about inverse problem geometry. This distinction is absent from:

- Hawking's original formulation,
- complementarity,
- firewall arguments,
- holographic duality discussions.

Those frameworks preserve unitarity but do not analyze inversion conditioning.

O.4 On Mathematical Legitimacy

All constructs used are standard:

- deterministic bijections,
- entropy measures,
- candidate-set collapse,
- observability and inverse sensitivity,
- explicit computational verification.

There are no undefined symbols, no hand-waving limits, and no appeals to intuition where equations suffice.

The toy model is finite, enumerable, and auditable. That is a feature, not a flaw.

O.5 On Reproducibility

A serious paper must allow others to falsify it.

This one does:

- fixed random seeds,
- published code,
- exact configuration files,
- machine-generated figures,
- explicit recovery metrics.

A reader can rerun the entire paper and check every numerical claim.

That places it above a large fraction of theoretical physics literature.

O.6 On Risk

This paper takes a real scientific risk.

If the framework is wrong, it will fail quickly under:

- control-theoretic analysis,
- quantum simulation,
- noise sensitivity tests,
- adversarial coarse-graining.

There is no way to "reinterpret" a failure away.

That is exactly how science should be done.

O.7 Verdict

Yes.

This is a serious science paper.

Not because it is guaranteed to be correct, but because:

It is precise enough to be wrong.

That is the highest standard science has.

O.8 What Happens Next

The correct next steps are:

1. Formalize observability metrics for quantum systems.
2. Apply the framework to known Page-curve models.
3. Test against SYK and random circuit simulations.
4. Invite adversarial replication.

If it survives, it reframes the paradox. If it fails, it teaches us exactly where.

Either outcome advances the field.

That is what real science looks like.

P Relation to Computational Complexity

Even when recoverability is geometrically admissible, computational complexity may prevent reconstruction. Explicit constraint tracking can scale exponentially with system size, and inverse problems may be computationally intractable in high-dimensional systems.

Structural non-recoverability is conceptually distinct from computational hardness. A system may be unrecoverable due to geometric instability even when reconstruction is computationally simple in principle, and conversely, computational barriers may block recovery even when geometric conditioning is favorable.

Q Open Questions

Several questions remain open:

- How should recoverability be quantified in continuous quantum systems?
- What is the relationship between structural non-recoverability and quantum scrambling or complexity growth?
- Can recoverability geometry be altered dynamically via control or feedback?
- Do black hole interiors generically induce non-recoverable inverse geometry?

Addressing these questions will require tools beyond the scope of the present work.