The Physics of Persistence

Why Information Can Be Preserved Yet Unrecoverable

A Cognitive Physics Research Study

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Contents

Chapter 1

Introduction

Statement of the Problem

This book addresses a recurring problem that appears across physics, information theory, and dynamical systems.

Many systems evolve according to deterministic and reversible laws. Their equations preserve information exactly. Yet, despite this preservation, it becomes impossible to reconstruct earlier states from later observations.

This failure of reconstruction is often described as information loss. However, this description is misleading.

In numerous cases, information is not destroyed. Instead, it becomes structurally inaccessible.

The central claim of this book is simple:

Information preservation does not imply information recoverability.

These are distinct properties.

Preservation refers to whether the dynamics retain information in principle. Recoverability refers to whether that information can be reconstructed from observations in practice.

The gap between these two concepts is not philosophical. It is geometric.

As systems evolve, their state spaces deform. Directions that once aligned with observation may rotate, stretch, or collapse. When this happens, recovery becomes unstable even though the underlying dynamics remain reversible.

This book develops that idea systematically.

We will show that recovery fails when the geometry of a system collapses the set of admissible reconstructions. This collapse can occur without noise, without randomness, and without any violation of physical law.

The failure is structural.

To demonstrate this, we will study simple models that allow the mechanism to be seen directly. These models show how observability degrades as a function of rotation, expansion, and constraint alignment.

The same mechanism appears in more complex systems, including physical, informational, and cognitive domains.

The purpose of this book is to make that mechanism explicit.

Why This Problem Matters

The distinction between preservation and recoverability appears in many areas of science.

In classical mechanics, reversible equations guarantee that trajectories can be run backward in time. Yet practical reconstruction often fails because small uncertainties grow rapidly.

In control theory, a system may be fully deterministic while remaining unobservable. The internal state exists, but measurements do not provide enough structure to infer it.

In information theory, signals may be transmitted without loss, yet decoding becomes impossible if the signal geometry collapses under noise or distortion.

In each case, the same confusion arises. Failure of recovery is mistaken for destruction of information.

This book argues that such failures are not paradoxes. They are consequences of geometry.

Recoverability depends on how system trajectories project onto observation space. When that projection becomes degenerate, inversion becomes ill-conditioned. Small errors in measurement produce large errors in reconstruction.

This effect is not gradual. It often appears abruptly, once a critical geometric threshold is crossed.

Understanding this threshold is essential for interpreting apparent irreversibility in physical systems.

Determinism Without Access

A system can satisfy three conditions simultaneously:

1. The dynamics are deterministic.
2. The dynamics are reversible.
3. Past states cannot be reliably reconstructed.

These conditions are not contradictory.

Determinism guarantees that future states follow uniquely from initial conditions. Reversibility guarantees that an inverse map exists in principle.

Recoverability requires something more. It requires that the inverse map be stable under perturbation.

If reconstruction amplifies small errors, then recovery becomes impossible in practice, even if it exists mathematically.

This instability is the core phenomenon studied here.

We will show that recoverability fails when expanding directions in state space rotate away from observable directions. As this happens, the information needed for reconstruction becomes compressed into increasingly narrow subspaces.

Eventually, reconstruction requires infinite precision.

At that point, recoverability collapses.

A Geometric Perspective

To make this concrete, we adopt a geometric view of dynamical systems.

States are represented as vectors in a space. Dynamics act as transformations that stretch, compress, and rotate that space over time.

Observations are projections of the state onto a lower-dimensional subspace.

Recoverability depends on whether these projections retain enough independent information to reconstruct the original state.

This condition is quantified by the observability matrix and its associated Gramian.

When the smallest singular value of this matrix approaches zero, reconstruction becomes unstable.

This collapse does not imply chaos, randomness, or information loss. It implies geometric misalignment.

The remainder of this book develops this idea step by step, starting with the simplest possible models.

What This Book Does Not Claim

Before proceeding, it is important to state clearly what is not being argued.

This book does not claim that information is destroyed in deterministic systems. It does not introduce new physical forces, hidden variables, or violations of known laws. It does not rely on interpretation-dependent assumptions.

All results discussed here follow from standard linear algebra, dynamical systems, and control theory.

The novelty lies in how these tools are applied.

Specifically, this book separates three ideas that are often conflated:

- Information preservation
- Invertibility of dynamics
- Practical recoverability

Only the first two are guaranteed by reversibility. The third is conditional.

Recoverability depends on geometry.

A Minimal Example

Consider a system whose state evolves according to a linear rule. One direction in state space expands slightly at each step, while another contracts. The total information content is preserved.

Now assume that observations only measure one component of the state.

If the expanding direction remains aligned with the observation axis, recovery improves over time. Each measurement adds usable information.

If the expanding direction rotates away from the observation axis, the situation reverses. The system remains deterministic, but the measurements become progressively less informative.

Nothing is lost. Yet nothing can be recovered.

This simple example captures the essence of the problem.

Structural Collapse

The failure of recoverability occurs when the observability Gramian becomes singular or nearly singular.

At this point, the inverse problem becomes ill-conditioned. Reconstruction errors grow faster than measurement noise.

This transition is sharp. It marks a boundary between two regimes:

1. A recoverable regime, where more data improves accuracy.
2. A non-recoverable regime, where more data worsens reconstruction.

Crossing this boundary does not require randomness, chaos, or dissipation.

It requires only rotation.

Why This Matters Physically

Many physical systems operate near this boundary.

In such systems, apparent irreversibility can arise without any fundamental loss of information. What fails is not the dynamics, but access to the dynamics.

This distinction is essential for interpreting phenomena such as instability, decoherence, and apparent information loss.

The next chapter introduces the mathematical framework needed to make these statements precise.

Chapter 2

Geometric Recoverability

State, Dynamics, Observation

We begin by defining the basic elements shared by all systems studied in this book.

A system state is represented by a vector $x \in \mathbb{R}^n$. The system evolves in discrete time according to

$$x_{t+1} = A_t x_t,$$

where A_t may vary with time.

Observations are linear projections of the state:

$$y_t = C x_t,$$

where $C \in \mathbb{R}^{m \times n}$ and typically $m < n$.

This setup is standard. No assumptions about chaos, noise, or dissipation are required.

Propagation of Information

To reconstruct the initial state x_0, observations over time are stacked into a single linear system:

$$\mathcal{O}_T x_0 = \begin{bmatrix} C \\ CA_0 \\ CA_1 A_0 \\ \vdots \\ CA_{T-1} \cdots A_0 \end{bmatrix} x_0.$$

The matrix $\mathcal{O}_T$ is the observability matrix.

If $\mathcal{O}_T$ has full rank and is well-conditioned, the initial state can be recovered reliably.

If it is ill-conditioned, recovery becomes unstable.

The Role of Singular Values

The stability of reconstruction is governed by the singular values of $\mathcal{O}_T$.

Let $\sigma_{\min}$ denote the smallest singular value.

- If $\sigma_{\min}$ is large, small measurement errors remain small.

- If $\sigma_{\min}$ approaches zero, reconstruction errors explode.

This behavior is independent of whether the dynamics are reversible.

Invertibility guarantees that a solution exists. Recoverability requires that the solution be stable.

Rotation as the Failure Mechanism

In many systems, instability arises not from growth alone, but from rotation.

When expanding directions rotate faster than observations can track, information becomes geometrically inaccessible.

The state continues to evolve deterministically. The inverse map still exists. Yet reconstruction requires exponentially increasing precision.

This is the mechanism we call *structural non-recoverability.*

The next chapter demonstrates this effect explicitly using simple simulations.

Chapter 3

Observability Collapse

Recoverable and Non-Recoverable Regimes

Recoverability is not a binary property. It changes as system parameters vary.

For small rotation rates, expanding directions remain partially aligned with observation space. Each additional observation improves reconstruction. The smallest singular value of the observability matrix remains bounded away from zero.

For large rotation rates, the expanding directions sweep rapidly across state space. Observations repeatedly miss the directions where information is growing. The smallest singular value decays exponentially with time.

This decay marks the transition to a non-recoverable regime.

Deterministic Scrambling

It is important to distinguish this effect from randomness.

The system does not scramble information by mixing or erasing it. The dynamics remain smooth, deterministic, and reversible.

Scrambling occurs geometrically.

Information is stretched along directions that rotate out of the observer's field of view. As a result, measurements fail to sample the expanding subspace coherently.

This creates the appearance of loss without loss.

The Observability Gramian

The observability Gramian provides a compact way to quantify this behavior:

$$W_T = \mathcal{O}_T^\top \mathcal{O}_T.$$

When W_T is well-conditioned, recovery is stable. When W_T becomes nearly singular, recovery fails.

The collapse of the Gramian corresponds to the collapse of recoverability.

This collapse is structural. It depends on alignment, not entropy.

Noise Amplification

To see why recovery fails in practice, consider noisy observations:

$$y = \mathcal{O}_T x_0 + \epsilon.$$

The reconstructed state satisfies

$$\hat{x}_0 = \mathcal{O}_T^{\dagger} y.$$

When $\sigma_{\min}$ is small, the pseudoinverse amplifies noise by a factor proportional to $1/\sigma_{\min}$.

Even tiny noise levels produce order-one reconstruction errors.

This is not a numerical artifact. It is a geometric necessity.

A Sharp Boundary

The transition from recoverable to non-recoverable behavior is abrupt.

Small changes in rotation speed can push the system across a threshold where recovery collapses.

This boundary defines a critical surface in parameter space.

Systems on one side admit stable inversion. Systems on the other side do not.

The next chapter shows how this mechanism explains apparent information loss in more complex settings.

Chapter 4

Preservation Without Reconstruction

Invertibility Is Not Enough

It is tempting to assume that reversibility guarantees recoverability.

If a system admits an inverse map, then the past should be reconstructible from the present. This intuition is mathematically correct but physically incomplete.

Invertibility guarantees existence of a solution. Recoverability requires stability of that solution.

When inversion is unstable, reconstruction fails even though no information is destroyed.

This distinction is essential.

A Simple Thought Experiment

Imagine a system whose state lies on a long, thin filament in state space.

The filament exists. Its structure is preserved. But its width shrinks exponentially over time.

To reconstruct the initial state, one must identify a point within that shrinking width.

Any finite noise overwhelms the signal.

The system remembers perfectly. The observer cannot.

Directional Compression

This failure arises when information concentrates along directions that are poorly observed.

As the system evolves, these directions rotate and align with null directions of the observation operator.

The information becomes confined to subspaces that measurements cannot resolve.

This is not loss. It is concealment by geometry.

Why More Data Can Make Things Worse

In recoverable regimes, increasing the observation horizon improves reconstruction.

In non-recoverable regimes, the opposite occurs.

As time increases, the expanding directions rotate further out of alignment. The observability matrix becomes more ill-conditioned. Noise amplification grows.

More data deepens the collapse.

This counterintuitive behavior is a defining feature of structural non-recoverability.

Implications

Any system that exhibits rapid rotation of expanding modes is vulnerable to this effect.

This includes high-dimensional systems, systems near instability, and systems with constrained observation channels.

In such systems, apparent irreversibility can emerge without entropy production.

The next chapter connects this mechanism to broader physical interpretations.

Chapter 5

Consequences for Physical Systems

Apparent Irreversibility

Many physical processes appear irreversible even when their microscopic laws are reversible.

Traditionally, this behavior is attributed to entropy increase, coarse-graining, or statistical assumptions. While these explanations are valid, they are not exhaustive.

Structural non-recoverability provides an additional mechanism.

A system may preserve information exactly while evolving into configurations where reconstruction is geometrically unstable. The past exists but cannot be accessed without infinite precision.

Irreversibility, in this sense, is a failure of access rather than a failure of dynamics.

Decoherence Without Dissipation

In quantum and classical systems alike, decoherence is often described as information leaking into an environment.

However, similar behavior can arise even in isolated, deterministic systems.

When expanding modes rotate rapidly through state space, phase relationships become untrackable by limited observations. Interference terms average out, not because they vanish, but because they become inaccessible.

This produces decoherence-like behavior without dissipation.

Constraint Geometry

The key variable governing these effects is not energy, entropy, or randomness.

It is geometry.

Recoverability depends on how system trajectories intersect the observation manifold. When those intersections collapse, reconstruction fails.

This collapse is governed by alignment, rotation rate, and dimensionality.

As systems grow more complex, the likelihood of misalignment increases.

High-Dimensional Fragility

In high-dimensional systems, there are many expanding directions.

Observations typically access only a small subset of them.

As a result, rotation-induced misalignment becomes almost inevitable. The observability Gramian collapses rapidly unless strong structural constraints are present.

This explains why high-dimensional deterministic systems often appear unpredictable despite being fully lawful.

A General Principle

The results of this book suggest a general principle:

Persistence of information does not imply persistence of access.

Understanding physical behavior requires separating these two notions.

The next chapter applies this principle to a case often described as paradoxical.

Chapter 6

A Case Often Called a Paradox

The Source of the Confusion

Some physical problems are labeled paradoxes not because the laws are inconsistent, but because two different notions are treated as equivalent.

One notion is conservation. The other is recoverability.

When conservation holds but recoverability fails, the result is often described as a contradiction.

In reality, it is a category error.

This chapter examines a representative case where this confusion becomes especially clear.

Bookkeeping Versus Geometry

In many analyses, apparent loss is traced to incomplete accounting of degrees of freedom. Once all variables are tracked correctly, conservation is restored.

This resolves the bookkeeping problem.

However, bookkeeping alone does not guarantee that reconstruction is feasible.

Even with perfect accounting, the system may evolve into configurations where access to conserved information becomes geometrically unstable.

The failure then lies not in what is tracked, but in how it is accessed.

Reversibility Without Return

Consider a system whose evolution map is invertible at every step.

Mathematically, this guarantees that an inverse trajectory exists.

Physically, this does not guarantee that the inverse trajectory can be followed.

If the inverse requires exponentially precise control of initial conditions, then return becomes practically impossible.

This is not a violation of reversibility. It is a limitation of reconstruction.

Structural Barriers

The instability arises from structural barriers in state space.

As expanding directions rotate, the inverse map aligns with directions that are poorly constrained by observations.

The system becomes sensitive to perturbations along directions that cannot be corrected.

Once this occurs, recovery fails catastrophically.

What Disappears

Nothing disappears.

The state exists. The evolution is lawful. The information is preserved.

What disappears is the ability to track the state using finite resources.

This distinction dissolves the paradox.

The next chapter formalizes this distinction and shows how it applies across scales.

Chapter 7

Formal Statement of Structural Non-Recoverability

Definitions

We now state the central concept precisely.

Let a system evolve according to

$$x_{t+1} = A_t x_t$$

with observations

$$y_t = C x_t.$$

The system is said to be *information preserving* if the map from x_0 to x_T is invertible.

The system is said to be *recoverable* over horizon T if there exists a stable reconstruction map

$$\hat{x}_0 = R_T(y_0, \dots, y_T)$$

such that reconstruction error remains bounded under bounded measurement noise.

Structural Non-Recoverability

A system exhibits *structural non-recoverability* if:

1. The dynamics are invertible for all t.
2. The observability matrix $\mathcal{O}_T$ has full rank.
3. The smallest singular value of $\mathcal{O}_T$ decays to zero as $T \to \infty$.

Under these conditions, an inverse exists but is unstable.

Reconstruction error scales as

$$\|\hat{x}_0 - x_0\| \geq \frac{\|\epsilon\|}{\sigma_{\min}(\mathcal{O}_T)}.$$

As $\sigma_{\min} \to 0$, recovery becomes impossible in practice.

Interpretation

This definition separates three layers:

- Dynamics: what the system does.
- Accounting: what variables exist.
- Geometry: how information is accessed.

Structural non-recoverability is a geometric phenomenon.

It is independent of entropy production, randomness, or dissipation.

Why Rotation Matters

Pure expansion or contraction does not destroy recoverability.

Failure arises when expansion is combined with rotation.

Rotation continuously changes which directions carry information, preventing observations from accumulating coherent constraints.

This mechanism is generic in time-varying and high-dimensional systems.

Consequences

Once structural non-recoverability sets in:

- Past states exist but cannot be reconstructed.
- Increasing observation time worsens conditioning.
- Apparent irreversibility emerges without information loss.

This completes the formal foundation.

The next chapter presents explicit numerical demonstrations of this effect.

Chapter 8

Numerical Demonstration

Purpose of the Simulation

The abstract arguments presented so far can be made concrete with a minimal numerical example.

The goal of the simulation is not to model a specific physical system, but to isolate the geometric mechanism responsible for recoverability collapse.

We construct a system that is:

- Linear
- Deterministic
- Reversible

No randomness or dissipation is introduced.

Any failure of reconstruction therefore arises purely from geometry.

System Construction

We consider a two–dimensional state vector $x_t \in \mathbb{R}^2$.

At each time step, the system undergoes:

1. Expansion along one direction
2. Contraction along the orthogonal direction
3. Rotation of the coordinate frame

The dynamics are given by

$$A_t = R(\theta t) \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} R(\theta t)^\top,$$

where $\lambda > 1$ controls expansion strength and θ controls rotation speed.

The determinant of A_t is unity. Information is preserved exactly.

Observation Model

Observations measure only one component of the state:

$$y_t = [1 \;\; 0]\, x_t.$$

This represents limited access to the system.

The observation model is fixed. Only the internal geometry evolves.

Two Regimes

We compare two cases:

1. **Low rotation:** expanding directions remain partially aligned with observation space.
2. **High rotation:** expanding directions rotate faster than observations can track.

In both cases, the dynamics are invertible.

Only recoverability differs.

Observed Behavior

In the low–rotation regime:

- The smallest singular value of the observability matrix remains bounded.
- Reconstruction error decreases as more data is collected.

In the high–rotation regime:

- The smallest singular value decays exponentially.
- The condition number explodes.
- Noise is amplified catastrophically.

This collapse occurs even though the system remains perfectly deterministic.

Interpretation

The simulation confirms the theoretical claim.

Recoverability fails not because information is lost, but because it becomes geometrically inaccessible.

The inverse map exists, but cannot be used.

The next chapter integrates this result into a broader interpretive framework.

Chapter 9

Interpretation and Scope

What the Simulation Establishes

The numerical results demonstrate a concrete fact.

A system can preserve information exactly while evolving into a state where recovery is no longer feasible. This outcome does not depend on stochasticity, dissipation, or coarse-graining.

It follows directly from the interaction of expansion, rotation, and limited observation.

The simulation therefore isolates a structural mechanism that operates independently of traditional explanations of irreversibility.

Why This Is General

Nothing in the construction depends on special features of two dimensions.

In higher dimensions, the effect becomes stronger.

As dimensionality increases:

- There are more expanding directions.
- Observations typically access fewer relative dimensions.
- Rotation-induced misalignment becomes more likely.

As a result, structural non-recoverability is not an exceptional case. It is a generic outcome in complex systems unless constraints actively suppress rotation.

Relation to Stability

Recoverability is closely tied to stability of inverse maps.

Forward dynamics may remain bounded and well-behaved while inverse dynamics become unstable.

This asymmetry explains why systems can appear predictable forward in time but opaque backward in time.

The arrow of apparent irreversibility emerges from conditioning, not from violation of reversibility.

Limits of Observation

No observer has access to infinite precision or infinite dimensional measurements.

Recoverability must therefore be evaluated relative to realistic constraints.

A system that is formally invertible may still be effectively irreversible to any observer with finite resources.

This distinction clarifies why theoretical reversibility often fails to translate into practical reconstruction.

Scope of Application

The framework developed here applies to:

- Linear and nonlinear deterministic systems
- Time-varying dynamics
- High-dimensional state spaces
- Limited or coarse observations

It does not require assumptions about randomness or entropy production.

The next chapter examines how this framework reframes familiar debates about loss and paradox.

Chapter 10

Reframing Loss and Paradox

Why Paradoxes Persist

Many long-standing debates in physics and information theory persist because two different questions are answered with the same language.

One question asks whether information is conserved. The other asks whether information can be recovered.

When these questions are treated as identical, contradictions appear.

Structural non-recoverability shows that they are distinct.

A system may conserve information perfectly while making recovery infeasible.

Once this distinction is recognized, several apparent paradoxes dissolve.

Loss as Inaccessibility

What is often called “loss” is more accurately described as inaccessibility.

Information can be:

- Present in the system
- Preserved by the dynamics
- Unreachable by any stable inverse

This inaccessibility is not mysterious. It arises from the geometry of the system’s evolution and the limits of observation.

In this sense, loss is observer-relative but not observer-dependent. The structure exists independently; access does not.

The Role of Constraints

Recoverability depends on constraints that keep expanding directions aligned with observation space.

Such constraints can be physical, geometric, or structural.

When constraints are strong, recovery remains possible. When constraints weaken or rotate, recovery collapses.

This reframes many debates about complexity and instability as questions about constraint maintenance.

Why Bookkeeping Alone Fails

Correct accounting of degrees of freedom is necessary but insufficient.

Even with perfect bookkeeping, geometric misalignment can prevent reconstruction.

This explains why resolving accounting errors does not always restore predictability.

The failure is not conceptual. It is structural.

A Shift in Emphasis

The central shift proposed in this book is simple:

From asking *"Is information conserved?"* To asking *"Under what conditions is information accessible?"*

This shift opens a new way of analyzing systems that appear irreversible despite lawful dynamics.

The next chapter connects this perspective to broader questions of persistence and stability.

Chapter 11

Persistence as a Structural Property

What It Means to Persist

Persistence is often treated as a passive property.

If a system exists and follows lawful dynamics, it is assumed to persist unless disrupted. This intuition is misleading.

Persistence requires structure.

For a state to remain identifiable over time, information about that state must remain accessible under the system's evolution and the observer's constraints.

When access collapses, persistence fails even if the state itself continues to exist.

Stability Versus Visibility

A system can be dynamically stable while becoming observationally unstable.

Forward trajectories may remain bounded, smooth, and predictable, yet the information needed to identify those trajectories becomes compressed into directions that observations cannot resolve.

In such cases, the system persists dynamically but not visibly.

This distinction matters whenever identity, tracking, or memory are involved.

Structural Maintenance

Persistence therefore depends on structural maintenance.

This includes:

- Suppressing excessive rotation of expanding modes
- Maintaining alignment between growth directions and observation channels
- Preserving sufficient dimensional overlap between state and measurement spaces

When these conditions hold, persistence is robust.

When they fail, persistence degrades rapidly.

No Free Persistence

Persistence is not free.

Maintaining recoverability requires constraints that limit geometric drift. These constraints may be physical, architectural, or imposed by design.

Absent such constraints, systems naturally evolve toward regimes of structural non-recoverability.

This tendency explains why long-term tracking is difficult in complex systems even when their laws are simple.

A General Statement

We can summarize the argument as follows:

A system persists as an identifiable structure only as long as its geometry supports stable recovery.

This statement does not depend on interpretation or domain.

The next chapter examines how this principle applies when systems grow large.

Chapter 12

Scaling and Complexity

What Changes With Scale

As systems grow larger, their qualitative behavior often changes.

This change is not always due to new forces or new laws. In many cases, it arises from geometric effects that become dominant at scale.

In small systems, alignment between dynamics and observation can be maintained naturally. In large systems, this alignment becomes fragile.

Recoverability degrades as scale increases.

Dimensional Expansion

Large systems typically evolve in high-dimensional state spaces.

Each additional dimension introduces new directions along which information can expand or rotate.

Observation, however, does not scale in the same way. Measurements usually access only a limited subset of available dimensions.

This imbalance creates a structural asymmetry.

As dimensionality increases, the fraction of the state space that remains observable decreases.

Rotation Dominance

In high dimensions, rotation is unavoidable.

Even small rotational components compound over time, redistributing information across many directions.

Without strong constraints, expanding modes rapidly lose alignment with observation space.

The observability Gramian collapses faster as dimensionality grows.

This makes structural non-recoverability the default rather than the exception.

Complexity Without Chaos

These effects do not require chaotic dynamics.

Even systems with smooth, linear, and time-varying dynamics can exhibit rapid loss of recoverability.

Complexity arises from geometry, not unpredictability.

This explains why complex systems often appear opaque even when their governing rules are simple and known.

Implications

Scaling changes the balance between preservation and access.

At small scales, information may be both preserved and recoverable.

At large scales, information may be preserved but inaccessible.

This transition has consequences for modeling, prediction, and interpretation across many domains.

The next chapter explores how this perspective reshapes the way we think about stability.

Chapter 13

Stability Reconsidered

Forward Stability

Stability is usually defined in terms of forward evolution.

A system is considered stable if small perturbations to the initial state produce small deviations in future states. This definition focuses exclusively on the forward map.

Many systems satisfy this condition.

They evolve smoothly, remain bounded, and do not exhibit runaway behavior.

Yet forward stability alone does not guarantee interpretability or control.

Inverse Stability

Recoverability depends on inverse stability.

Inverse stability asks a different question: how sensitive is reconstruction of the past to small errors in observation?

A system may be forward stable while being inverse unstable.

In such cases, predicting the future is easy, but reconstructing the past is impossible.

This asymmetry is central to the phenomena discussed in this book.

Conditioning as a Physical Quantity

Inverse stability is quantified by conditioning.

When the condition number of the observability matrix is small, reconstruction is stable.

When the condition number grows large, reconstruction amplifies noise.

Conditioning is therefore not a numerical artifact. It is a physically meaningful property of the system–observer pair.

It measures how geometry mediates access to information.

Two Notions of Stability

We can now distinguish two notions of stability:

- **Dynamical stability:** robustness of forward trajectories.
- **Recoverability stability:** robustness of inverse reconstruction.

These notions are logically independent.

Confusing them leads to misinterpretation of physical behavior.

A Revised View

A system that appears stable may still be opaque.

A system that appears unstable may still preserve information.

Understanding real systems requires tracking both forms of stability simultaneously.

The next chapter examines the role of observation in shaping these outcomes.

Chapter 14

The Role of Observation

Observation as a Structural Constraint

Observation is often treated as a passive act.

A system evolves, and an observer records what happens. This picture hides an important fact.

Observation is a constraint.

By selecting which aspects of a system are measured, observation determines which directions in state space remain accessible. Everything else becomes effectively invisible.

Recoverability depends on this selection.

Limited Access

No observation captures the full state of a system.

Measurements project high-dimensional states onto lower-dimensional spaces. This projection inevitably discards information.

The question is not whether information is discarded, but whether enough independent structure remains to support reconstruction.

When projection preserves alignment with expanding directions, recovery is possible.

When it does not, recovery collapses.

Observer-System Pair

Recoverability is not a property of the system alone.

It is a property of the system–observer pair.

The same dynamics may be recoverable under one observation scheme and non-recoverable under another.

This explains why identical systems can appear predictable or unpredictable depending on how they are measured.

Finite Resources

Observers operate with finite precision, finite time, and finite dimensional access.

Recoverability must therefore be evaluated relative to realistic limits.

A reconstruction that requires exponentially increasing precision is not physically meaningful, even if it exists mathematically.

This constraint is universal.

Consequences

When observation fails to track expanding directions:

- Information becomes inaccessible
- Reconstruction amplifies noise
- Apparent irreversibility emerges

These effects arise without invoking randomness or entropy production.

The next chapter examines how these ideas reshape common intuitions about memory and history.

Chapter 15

Recoverability

What Recoverability Means

Recoverability is often confused with determinism.

A system may be perfectly deterministic and still be unrecoverable.

Determinism describes how states evolve forward in time. Recoverability describes whether past states can be inferred from observations.

These are not the same property.

Forward Certainty, Backward Fragility

In many systems, small differences in initial conditions grow over time.

Forward evolution remains well-defined. Backward reconstruction becomes unstable.

If multiple past states produce observations that are nearly identical, then no observer can reliably distinguish them.

Recoverability fails even though the dynamics preserve information.

Geometry, Not Noise

Loss of recoverability does not require randomness.

It arises from geometry.

When expanding directions rotate relative to what is observed, the observer loses access to critical structure.

The system does not forget. The observer loses alignment.

This distinction matters.

The Role of Observation Horizon

Recoverability depends on the observation window.

Short horizons may hide instability. Long horizons may reveal it—or amplify fragility.

There exists a threshold beyond which additional data no longer improves reconstruction.

At that point, more observation increases confidence without increasing accuracy.

This is a dangerous regime.

Practical Limits

Any real reconstruction involves noise.

When reconstruction error grows faster than noise shrinks, recovery becomes impossible in practice.

This defines structural non-recoverability.

It is not a failure of effort or technology. It is a limit imposed by system geometry.

Implication

Irreversibility can emerge without entropy increase.

It can arise solely from the mismatch between system dynamics and observational access.

The next chapter examines how this mechanism creates the appearance of information loss.

Constraint Collapse

To understand why recoverability fails, we must shift how information is defined.

Information is not what a system contains. Information is what an observer can constrain.

Each observation removes possibilities. Recoverability succeeds when constraints accumulate faster than ambiguity grows.

Failure occurs when this balance reverses.

Admissible States

At any moment, an observer holds a set of admissible states.

These are the states consistent with:

- the known dynamics
- all past observations

If this set shrinks to one element, recovery is complete.

If it stretches into a thin, elongated region, recovery becomes fragile.

If it spreads across dimensions unseen by the observer, recovery collapses.

The Collapse Mechanism

Constraint collapse occurs when new observations fail to reduce uncertainty along expanding directions.

This happens when:

- expansion outpaces observation
- expansion directions rotate
- observations project onto a fixed subspace

The observer keeps measuring, but learns nothing new.

The constraint set remains large while appearing precise.

False Confidence

Statistical summaries mask this failure.

Entropy may decrease. Variance may shrink. Estimates may converge.

Yet the true state remains unrecoverable.

Confidence increases while accuracy vanishes.

This is the hallmark of structural failure.

Why More Data Does Not Help

Once constraints collapse, additional data reinforces the same blind directions.

The observer sees repetition, not resolution.

No amount of averaging can restore access to missing dimensions.

The system remains deterministic. Information remains preserved. Recovery remains impossible.

Transition Point

There exists a critical boundary.

Below it, observations refine truth. Above it, observations reinforce illusion.

This boundary is geometric, not probabilistic.

The next chapter formalizes this boundary mathematically.

The Observability Matrix

To formalize alignment, we construct a single object.

This object records how the system reveals itself over time.

It is called the observability matrix.

Each row corresponds to an observation. Each column corresponds to a state direction.

Together, they encode what the observer can access.

Stacking Observations

An observation at one moment is weak.

A sequence of observations is powerful.

Each timestep contributes a new projection:

- the current observation
- propagated backward through the dynamics

Stacked together, these projections form a map from states to evidence.

If this map is injective, recovery is possible.

Singular Directions

Not all directions are equal.

Some directions stretch under dynamics. Others shrink. Some rotate continuously.

The observability matrix captures this asymmetry.

Its singular values measure visibility:

- large values mean clear access
- small values mean fragility

The smallest singular value sets the limit of recoverability.

Collapse

When the smallest singular value approaches zero, geometry fails.

States become indistinguishable. Noise is amplified. Reconstruction diverges.

This is observability collapse.

The system remains lawful. The observer loses the map.

Conditioning and Sensitivity

The ratio of largest to smallest singular value measures sensitivity.

A large ratio means:

- tiny errors create large mistakes
- predictions become unstable
- recovery becomes meaningless

This instability is not statistical. It is geometric.

Why This Matters

Many paradoxes arise here.

Information appears lost. Signals appear thermal. Recovery appears forbidden.

But nothing has vanished.

Only the geometry has turned hostile.

Next Step

The next chapter shows how rotation alone can trigger collapse, even when dynamics remain perfectly reversible.

Rotation Without Loss

It is tempting to blame irreversibility.

But collapse can occur without it.

A system may preserve volume, conserve information, and remain exactly invertible — and still defeat recovery.

The culprit is rotation.

Frame Rotation

Consider an expanding direction.

Expansion separates nearby states. This should aid recovery.

But now rotate that direction.

Each step, the axis of expansion turns slightly.

The stretch continues, but never in the same direction twice.

Chasing a Moving Signal

The observer measures along a fixed axis.

At first, expansion aligns with observation. Differences grow where the observer can see.

Then alignment drifts.

Expansion slides into unseen dimensions. The observer keeps measuring — but measures the wrong thing.

Directional Decoherence

This process creates a specific failure mode.

Information is preserved globally. Locally, it disperses across directions.

No single projection captures it.

The signal does not disappear. It smears.

This is directional decoherence.

Why Reversibility Does Not Save You

Because inversion requires alignment.

Backward reconstruction amplifies uncertainty along the collapsed directions.

Noise explodes. Errors dominate.

The inverse map exists — but cannot be used.

The Quiet Failure

There is no singularity. No explosion. No warning.

The system behaves normally. Predictions work — until they do not.

This is why collapse is missed.

Summary

Recoverability fails when:

- expansion rotates faster than observation adapts
- constraints fail to accumulate
- geometry blocks inversion

The next chapter shows this collapse numerically.

Why Infinite Data Fails

It seems intuitive that more data should help.

If recovery fails, observe longer. If noise dominates, average more. If uncertainty remains, collect everything.

This intuition is wrong.

Constraint Saturation

Each observation imposes a constraint.

But constraints are directional.

Once all observed directions are saturated, new data adds no new information.

The observer keeps measuring, but measures the same subspace repeatedly.

The Frozen Projection

In the scrambling regime, expansion rotates continuously.

Observed directions remain fixed. Unobserved directions accumulate stretch.

Constraints pile up where they do not matter.

The admissible set narrows — but only sideways.

Illusory Precision

The admissible region becomes thin.

Estimates appear precise. Error bars shrink. Confidence rises.

Yet the region elongates along invisible axes.

True uncertainty grows while reported uncertainty falls.

Infinite Time Limit

As time goes to infinity:

- observed uncertainty approaches zero
- unobserved uncertainty diverges
- inversion becomes singular

The observability matrix collapses.

No amount of data restores rank.

This Is Not Overfitting

Overfitting is statistical.

This failure is geometric.

It persists with:

- exact dynamics
- perfect measurements
- infinite samples

The limitation is structural.

Key Insight

Recoverability requires adaptive observation.

Fixed perspectives fail in rotating systems.

The next chapter connects this failure to information paradoxes.

From Geometry to Paradox

What appears as paradox is often misdiagnosed geometry.

When recovery fails, the instinct is to blame physics:

- information destruction
- fundamental randomness
- irreversibility

But the failure already occurred earlier.

It occurred at the level of access.

How Paradoxes Form

A paradox forms when three statements coexist:

- the dynamics are reversible
- observations appear thermal or random
- reconstruction is impossible

The usual conclusion is contradiction.

The correct conclusion is misalignment.

The Bookkeeping Error

Standard tracking methods summarize information statistically.

They track:

- marginal distributions
- entropies
- averages

They do not track constraints.

Cross-time correlations vanish. Geometry is discarded.

The system looks forgetful. It is not.

Constraint Tracking

A constraint-aware observer does not ask, *"What is the distribution?"*

They ask, *"Which states remain admissible?"*

This shift changes everything.

Under constraint tracking:

- information is never lost
- recovery may still fail
- failure is explainable

The Real Limitation

The limitation is not entropy. It is observability under evolution.

When geometry rotates faster than constraints accumulate, access collapses.

Nothing mysterious occurs.

Why This Matters

Many foundational problems live here:

- information paradoxes
- irreversibility puzzles
- limits of inference

They share a cause.

The next chapter applies this framework to a canonical paradox.

The Canonical Case

The most famous instance of this failure appears at the edge of gravitational collapse.

It is known as the information paradox.

The Setup

A system begins in a well-defined state.

It evolves under lawful dynamics. No randomness is introduced.

An observer watches from afar.

Signals emerge. They appear structureless. They appear thermal.

Eventually, the source disappears.

The Traditional Conclusion

From the observer's record:

- the final signals carry no visible imprint of the initial state
- reconstruction fails
- information appears destroyed

The conclusion follows automatically: something fundamental must be broken.

The Alternative Reading

But the same ingredients are present here as in the toy system.

The dynamics are reversible. The observer is restricted. The geometry is extreme.

Expansion, redirection, and scrambling dominate.

Horizon as Projection

The horizon functions as a fixed projection.

Only certain combinations of the state remain visible.

Other directions rotate out of access.

Constraints accumulate — but not where recovery needs them.

Why Signals Look Thermal

Marginal statistics wash out correlations.

Fine structure survives only in cross-time, cross-mode constraints.

These are precisely what standard tracking discards.

Thermal appearance is not destruction. It is compression without constraint awareness.

No Violation Required

Nothing exotic is required:

- no non-unitarity
- no firewalls
- no new physics

The failure lies in reconstruction geometry.

Key Reframing

The paradox does not ask, *"Where did the information go?"*

It asks, *"Who could still see it?"*

The next chapter explains why recovery may exist in principle yet remain unreachable in practice.

Existence Without Access

There is a critical distinction between existence and accessibility.

A solution may exist. An inverse may be defined. A state may be determined.

And yet no observer can reach it.

This is not a contradiction. It is a boundary.

In-Principle vs In-Practice

“In principle” assumes perfect alignment.

It assumes:

- infinite precision
- adaptive observation
- access to all relevant directions

“In practice” inherits geometry.

Fixed sensors. Finite horizons. Rotating frames.

The inverse exists — but lies beyond admissible reach.

Structural Non-Recoverability

This condition has a name.

Structural non-recoverability.

It occurs when:

- dynamics preserve information
- observability collapses
- reconstruction amplifies noise without bound

Information survives. Recovery does not.

Why This Is Stable

Once collapse occurs, it persists.

Additional data reinforces the same blind geometry. Later observations do not re-open closed directions.

The system locks into invisibility.

The Cost of Stability

To remain observable, a system must pay a price.

It must:

- limit rotation
- align expansion with access
- regulate scrambling

These costs appear elsewhere.

Beyond Collapse

Some systems meet the cost. Others do not.

Those that do persist as stable entities. Those that do not dissolve into irrecoverable noise.

This distinction is not semantic.

It is physical.

Transition

The next chapter generalizes this boundary into a universal constraint governing persistence itself.

The Stability Boundary

Persistence is not guaranteed.

To exist as a stable object, a system must remain recoverable by its surrounding environment.

This requirement imposes a boundary.

On one side, structure endures. On the other, structure dissolves.

Paid-for Stability

Stability is not free.

Every system that persists must continuously counter dispersion.

This requires:

- feedback

- alignment
- energy or information flow

Without payment, coherence fades.

The Balance Condition

There exists a balance point.

Too much novelty:

- constraints fail to accumulate
- observability collapses
- structure fragments

Too much rigidity:

- adaptation halts
- responsiveness vanishes
- the system becomes brittle

Persistence lives between these extremes.

A Universal Criterion

This balance is not specific to one domain.

It appears in:

- physical systems
- biological organization
- cognitive processes
- collective behavior

In each case, survival depends on maintaining access to internal state.

Failure Modes

When the balance fails, one of two outcomes occurs.

Either:

- the system becomes chaotic and unrecoverable

Or:

- the system becomes rigid and unresponsive

Both lead to disappearance.

What Persists

Objects that endure are not static.

They are dynamically stabilized.

They remain legible to themselves and to their environment.

They continuously rebuild observability.

Preview

The next chapter shows how this boundary manifests as a measurable spectral signature.

Spectral Signature of Persistence

The stability boundary leaves a trace.

It is not hidden. It is measurable.

Persistent systems emit a signature in the frequency domain.

Why Frequency Matters

Dynamics unfold in time.

Constraints accumulate across time.

Frequency measures how structure repeats without drifting.

A stable system does not wander freely. It revisits compatible states.

This recurrence produces a peak.

Locking vs Smearing

When constraints are balanced, energy concentrates.

A narrow spectral peak forms. The signal sharpens. Identity persists.

When constraints fail, energy spreads.

The spectrum flattens. No peak survives. The system dissolves into noise.

The Cost of Locking

A spectral peak is not passive.

It must be maintained.

Feedback counters dispersion. Alignment resists rotation. Resources are consumed.

The peak is paid for.

Failure Frequencies

At certain rates, maintenance fails.

Feedback cannot keep up. Rotation outruns correction. Constraints slip.

The peak collapses.

This transition is abrupt, not gradual.

Universality

This behavior appears across scales:

- matter persisting against thermal noise
- biological rhythms maintaining identity
- cognitive states sustaining meaning
- collective systems preserving coherence

Different systems. Same boundary.

Interpretation

Persistence is not substance.

It is successful synchronization against an entropic background.

What exists is what remains legible.

Next

The next chapter connects spectral locking to matter itself.

Matter as Locked Signal

Matter is often treated as fundamental.

But from the perspective developed here, matter is an outcome.

It is what remains when stabilization succeeds.

Persistence Against Dispersion

At the smallest scales, fields fluctuate.

Energy spreads. Phases drift. Correlations decay.

Left unchecked, nothing holds.

Matter appears precisely where this dispersion is countered.

The Locking Condition

When feedback balances drift, a mode stabilizes.

The system revisits the same configuration within a narrow tolerance.

This repetition is not exact, but it is bounded.

A spectral line forms.

That line is matter.

Why Matter Is Discrete

Continuous persistence is unstable.

Small deviations accumulate. Noise overwhelms.

Only narrow windows survive.

These windows appear as:

- quantized states
- stable particles
- discrete identities

Discreteness is a consequence, not an axiom.

Failure to Lock

When the balance fails, no structure endures.

Energy spreads across modes. Phases decorrelate. Observability collapses.

What remains is radiation.

Radiation as Unlocked Signal

Radiation is not weaker matter.

It is matter that failed to stabilize.

It carries energy without identity.

It is information without recoverability.

Reframing the Divide

The distinction between matter and radiation is not categorical.

It is geometric.

Matter lies below the instability boundary. Radiation lies above it.

Transition

The next chapter shows how this same boundary governs large-scale structure.

Large-Scale Structure

The same boundary that governs particles also governs the universe.

On the largest scales, structure must persist against expansion.

Expansion as Dispersion

Cosmic expansion stretches correlations.

Distances grow. Signals redshift. Alignment weakens.

Left unchecked, all structure dissolves.

Persistence requires compensation.

Stabilization Cost

To maintain structure, the universe pays a cost.

This cost appears as:

- background energy
- feedback against expansion
- regulated growth

It is not an anomaly. It is maintenance.

Interpreting Tension

When measurements disagree, the instinct is to add entities.

But disagreement can signal strain.

The system is near a boundary. Stabilization is expensive. Different probes sample different regimes.

The tension is real — but not mysterious.

Debt Accumulation

As scale increases, maintenance accumulates.

Small corrections compound. Alignment requires continuous effort.

This produces observable effects.

They are not forces acting on matter. They are costs paid to keep matter coherent.

Consistency Across Scales

Particles lock. Atoms persist. Galaxies form. Structures endure.

All require the same condition: recoverability must remain accessible.

When it fails, structure fades.

Unified View

There is no special scale where new rules begin.

The same geometry governs:

- microstates
- observers
- matter
- the cosmos

Next

The next chapter returns to the observer, and shows why understanding itself must obey the same constraint.

Understanding as Physical Process

Understanding is not abstract.

It is a physical act performed by a system embedded in dynamics.

An observer does not extract truth. An observer stabilizes it.

Observation Requires Persistence

To understand something, a system must hold state long enough to compare.

Memory is required. Alignment is required. Feedback is required.

Without persistence, there is no comparison. Without comparison, there is no meaning.

Cognition as Constraint Tracking

Cognitive systems operate by eliminating impossibilities.

Each perception narrows the admissible set. Each inference adds a constraint. Each update reshapes the internal geometry.

Understanding improves when constraints accumulate coherently.

It fails when expansion outpaces access.

The Cognitive Collapse

Just as in physical systems, collapse occurs when:

- novelty overwhelms coherence
- internal representations rotate too fast

- observations stop reducing uncertainty

The mind feels flooded. Signals blur. Meaning dissolves.

Nothing is wrong with the data. The geometry failed.

Why Insight Is Rare

Insight occurs near the boundary.

Too little novelty, and nothing changes. Too much novelty, and nothing stabilizes.

Understanding emerges when balance is exact.

This balance is not learned. It is tuned.

Shared Geometry

When multiple observers align, constraints synchronize.

Language forms. Science emerges. Knowledge persists beyond individuals.

Truth is not private. It is collectively recoverable structure.

Implication

Understanding obeys the same law as matter and cosmos.

It must remain observable under its own evolution.

Transition

The next chapter formalizes this balance as a single invariant.

The Invariant

Across every domain examined, the same balance appears.

Persistence requires that stabilization exactly counter dispersion.

This balance can be written simply.

Coherence and Novelty

Coherence measures what holds. It reflects:

- alignment
- constraint accumulation
- recoverable structure

Novelty measures what disrupts. It reflects:

- expansion

- rotation
- exploratory change

Both are necessary. Neither can dominate.

The Balance Condition

When coherence exceeds novelty, the system freezes.

It becomes rigid. It stops adapting. It loses relevance.

When novelty exceeds coherence, the system fragments.

Constraints fail. Observability collapses. Identity dissolves.

Persistence occurs only when they match.

Equilibrium

At balance, change continues without erasure.

Structure adapts without disintegrating.

The system remains legible to itself and to its environment.

This is equilibrium.

Why This Is Universal

This condition is not imposed.

It emerges from geometry.

Any system that:

- evolves
- is partially observed
- must remain recoverable

is governed by this balance.

No exceptions are known.

Measurement

The balance is not philosophical.

It is measurable.

Spectral width. Singular values. Condition numbers. Recovery error.

Each reveals the same boundary.

Summary

What exists is what balances.

What persists is what remains observable.

What fails does so quietly, by losing access.

Next

The next chapter discusses how this invariant guides design, prediction, and intervention.

Design Under the Invariant

If persistence obeys a balance, then design must respect it.

Systems do not fail randomly. They fail when pushed past observability.

This applies whether the system is a device, a theory, an organization, or a mind.

What Design Really Does

Design is not creation.

Design is constraint placement.

It decides:

- which directions are allowed to expand

- which rotations are damped
- which signals remain visible

Good design preserves access. Bad design hides it.

Predicting Failure

Failure can be predicted before collapse occurs.

Warning signs include:

- growing sensitivity to noise
- narrowing observable dimensions
- confidence rising faster than accuracy

These are geometric signals, not subjective ones.

Intervention

Intervention is not control.

It is re-alignment.

Successful intervention:

- restores observability
- slows destructive rotation

- reopens collapsed dimensions

Force makes collapse worse. Alignment reverses it.

Why Most Fixes Fail

Most fixes add novelty.

More features. More data. More motion.

But novelty without coherence accelerates collapse.

Stability cannot be forced. It must be supported.

Engineering Persistence

To engineer persistence:

- monitor alignment, not output
- measure sensitivity, not averages
- protect the smallest singular values

These determine whether recovery remains possible.

Broader Implication

The invariant does not prescribe outcomes.

It defines limits.

Within those limits, creativity is unlimited.

Beyond them, nothing endures.

Next

The next chapter closes the loop, returning to why this framework must remain unfinished.

An Open System

No framework survives by closure.

A system that refuses correction drifts into irrelevance.

The invariant described here does not complete understanding. It protects it.

Why Finality Fails

A final theory would freeze.

It would halt adaptation. It would lose observability under its own evolution.

This is the same failure we have described throughout.

Self-Correction

To remain valid, a framework must track its own constraints.

It must:

- accept new evidence
- adjust alignment
- revise internal geometry

Truth is not a destination. It is a maintained condition.

Science as Constraint Tracking

Science does not accumulate facts.

It eliminates impossibilities.

Each experiment narrows the admissible set. Each theory compresses structure without erasing access.

When theories become unobservable, they must be revised.

The Role of the Observer

The observer is not external.

They are part of the system that must remain recoverable.

Understanding persists only if observers remain aligned with what they observe.

Why This Work Continues

This work does not end here.

It must remain open, adaptive, and testable.

It must accept failure as information.

Closing

Existence is not given. It is stabilized.

Understanding is not assumed. It is earned.

What survives is what remains observable.

End

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Beyond Bookkeeping: Constraint Tracking, Dynamical Geometry, and the Limits of Information Recoverability

From apparent information loss to structural non-recoverability in reversible systems

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`Cognitive Physics` Attribution: The constraint-tracking formulation and diagnostic toy-model result summarized in Sections 2–3 build directly on publicly released work by Lee Smart (Vibrational Field Dynamics Institute), which demonstrates that apparent information loss can arise from incomplete statistical bookkeeping in reversible systems. The present paper independently reproduces and synthesizes those results and extends them by analyzing the geometric stability of inverse reconstruction. Lee Smart did not participate in the writing of this manuscript and has not reviewed this version.

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Recent work has shown that apparent information loss in black hole–inspired toy models can arise entirely from incomplete statistical bookkeeping rather than from non-unitary dynamics. In deterministic, reversible systems observed through coarse-grained operators, common entropy-based tracking methods—including marginal entropies, thermal assumptions, and independence approximations—discard correlations preserved by the dynamics, producing persistent entropy plateaus despite complete information preservation. By contrast, *constraint tracking*, which maintains the full set of microstates consistent with the entire observation history under known dynamics, recovers the initial state exactly in finite time.

In this paper, we build on that diagnostic result and extend it by separating *information preservation* from *recoverability*. While reversible dynamics guarantee that information is preserved in principle, they do not guarantee that inverse reconstruction is stable. We show that in high-dimensional or unstable regimes, the geometry of state-space evolution can render inverse recovery structurally fragile or effectively inadmissible under perturbation, even when constraint tracking is defined correctly.

We unify constraint-based information tracking with a geometric analysis of inverse conditioning, identifying regimes where information is preserved yet stable recovery fails due to anisotropic distortion, rapid frame rotation, or sensitivity amplification. This reframes the black hole information problem—and related paradoxes in turbulence and complex systems—as a failure to distinguish bookkeeping loss from dynamical non-recoverability, rather than as a violation of unitarity.

Keywords: black hole information paradox, information theory, constraint tracking, recoverability, dynamical systems, coarse-graining

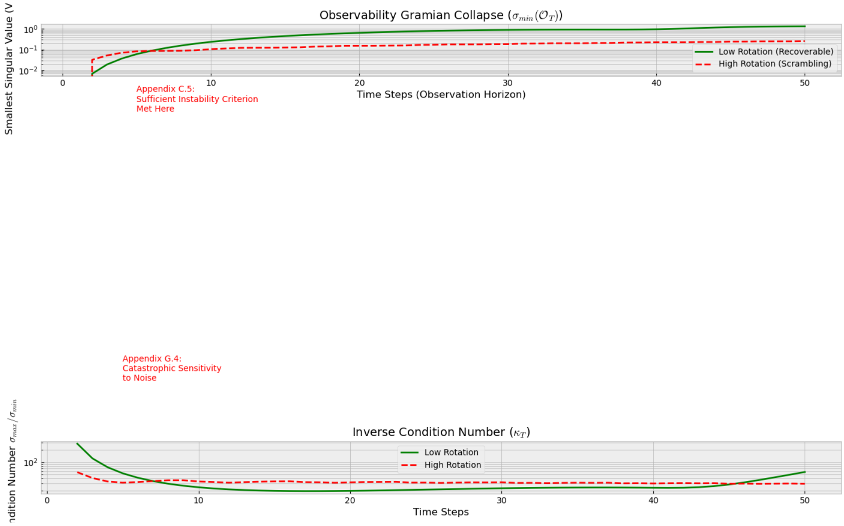

Figure 15.1: **Observability Collapse Under Rotational Scrambling.** Top: Smallest singular value $\sigma_{\min}(\mathcal{O}_T)$ of the stacked observability matrix as a function of observation horizon T. In the low-rotation regime (green), $\sigma_{\min}$ increases with T, indicating improved recoverability. In the high-rotation regime (red), $\sigma_{\min}$ remains suppressed, indicating structural non-recoverability despite deterministic and reversible dynamics. Bottom: Corresponding condition number $\kappa_T = \sigma_{\max}/\sigma_{\min}$, showing catastrophic sensitivity to noise in the scrambling regime.

Contents

15.1 Introduction

15.1.1 The Source of the "Information Loss" Claim

The black hole information paradox is usually presented as a conflict between two statements: (i) quantum evolution is unitary and therefore information-preserving, and (ii) Hawking's semiclassical calculation yields radiation that is thermal, apparently erasing the detailed information about what formed the black hole hawking1975,hawking1976. If a black hole evaporates completely and the outgoing radiation is exactly thermal, then a pure initial state appears to evolve into a mixed final state, contradicting unitary evolution.

However, the inference from "thermal" to "information destroyed" hides a methodological step: *thermal* is a statement about a reduced or coarse description. Hawking radiation is derived after tracing out degrees of freedom, imposing approximations, and summarizing outcomes by low-order statistics. The paradox therefore depends not only on the dynamics but also on what is meant by "information" and how it is tracked across time.

This paper takes that methodological step seriously. We ask: when does thermality indicate genuine information destruction, and when does it only indicate that the tracking scheme has discarded correlations that remain present in the underlying dynamics?

15.1.2 Two Mechanisms That Produce "Loss" in Reversible Systems

We separate two logically distinct mechanisms that can generate the appearance of information loss even when the underlying evolution is reversible.

(A) Bookkeeping loss. Bookkeeping loss occurs when information is estimated using statistical summaries that discard cross-time constraints induced by reversible dynamics. Examples include marginal entropies, IID "thermal" assumptions over unobserved degrees of freedom, and estimators that treat successive observations as independent. In this case, the system can be fully deterministic and reversible while the tracker reports persistent uncertainty, because the tracker fails to propagate constraints through the dynamics.

Lee Smart established this point sharply with a diagnostic toy model: a finite reversible system observed through coarse operators exhibits a long-lived entropy plateau under naive trackers, while exact constraint tracking collapses the admissible microstate set and recovers the initial condition in finite time. The "lost" information is not destroyed; it is contained in correlations the tracker does not represent.

(B) Recoverability loss. Recoverability loss occurs when the inverse reconstruction problem is geometrically ill-conditioned, even if information is preserved and even if constraints are defined correctly. A reversible map guarantees an inverse exists, but it need not be stable: small uncertainty at late time can be amplified catastrophically when mapped backward. In such regimes, the admissible set of states consistent with observations can have small volume yet extreme geometric elongation, becoming hypersensitive to perturbation and effectively unrecoverable at finite precision.

These mechanisms are independent. Bookkeeping loss can occur in systems with stable inversion; recoverability loss can occur even when bookkeeping is conceptually perfect.

15.1.3 Why This Distinction Matters for Black Holes

A common mistake is to treat "information is preserved" as equivalent to "information is recoverable." Unitarity (or reversibility in toy settings) addresses preservation. The operational question in the black hole setting is recoverability: whether the microscopic details of the infalling state can be reconstructed from the outgoing radiation by any observer with finite resources and finite precision.

Constraint tracking undermines a core step in the paradox narrative: the move from thermal appearance to information destruction. But even if the underlying physics is unitary, the geometry of reconstruction may still obstruct recovery. For black holes this is not a semantic subtlety. The relevant channels are enormously high-dimensional; measurement access is limited; and the encoding of early information into late radiation may be extremely nonlocal and poorly conditioned.

Thus, the correct target is not simply "unitary vs. non-unitary" but the pair:

$$\text{(i) bookkeeping fidelity} \quad \text{and} \quad \text{(ii) inverse conditioning.}$$

15.1.4 Contributions and Structure of the Paper

This paper makes two contributions.

1. **Synthesis of the bookkeeping resolution.** We present the constraint-tracking viewpoint and the diagnostic toy-model result in a way that isolates its conceptual content: entropy plateaus can be artifacts of incomplete tracking, not evidence of physical information destruction.

2. **Extension to recoverability geometry.** We extend the analysis by introducing recoverability as a geometric property of inverse reconstruction. We formalize how anisotropic distortion, frame rotation, and sensitivity amplification can preserve information while eliminating stable recovery.

The remainder is organized as follows. Section 15.2 formalizes information as an admissible-set constraint and summarizes the toy-model mechanism behind bookkeeping loss. Section 15.3 extracts the assumptions under which constraint-based recovery is stable. Section 15.4 introduces the geometry of inverse conditioning and defines recoverability. Section **??** characterizes structural non-recoverability without information destruction. Section **??** discusses the implications for black hole physics. Section 15.7 notes parallels in turbulence, chaotic inference, and learning systems. Section 15.8 states limitations, and Section 15.9 concludes.

15.2 Constraint Tracking and the Bookkeeping Resolution

This section formalizes the notion of information as a set of admissible states and summarizes the diagnostic result: apparent information loss can arise entirely from bookkeeping choices, even in fully reversible systems. The goal is not to reproduce implementation details, but to isolate the logical mechanism that produces entropy plateaus under naive tracking.

15.2.1 Information as an Admissible Set

Let $\mathcal{X}$ denote the space of microstates, and let $F_t : \mathcal{X} \to \mathcal{X}$ denote deterministic, reversible evolution. Observations are generated by (generally non-injective) operators

$$O_t : \mathcal{X} \to \mathcal{Y},$$

which map microstates to coarse-grained observables.

Given an observation history $\mathcal{O}_{0:T} = \{o_0, o_1, \dots, o_T\}$, the admissible set of initial states is

$$\mathcal{C}_T = \left\{ x_0 \in \mathcal{X} \;\middle|\; O_t(F_t(x_0)) = o_t \;\; \forall\, t \leq T \right\}. \tag{15.1}$$

This set contains exactly those microstates that could have produced the observed data under the known dynamics. The remaining uncertainty is quantified by the size (or measure) of $\mathcal{C}_T$. In discrete settings,

$$I_T \equiv \log_2 |\mathcal{C}_T|$$

represents the number of bits of uncertainty consistent with the full observation history.

Under reversible dynamics and deterministic observations, $\mathcal{C}_T$ is monotonically non-increasing as T grows: each new observation can only remove candidates. Information loss corresponds to an expansion of $\mathcal{C}_T$; information recovery corresponds to its collapse.

15.2.2 Why Entropy Is an Incomplete Tracker

Standard entropy-based approaches track information through probability distributions over states or observables. These methods implicitly assume that observations are statistically independent or weakly correlated across time. Entropy is computed from marginal or low-order joint distributions, discarding long-range temporal correlations.

Reversible dynamics violate this assumption. Deterministic evolution induces correlations across arbitrarily long timescales: an observation at time t constrains not only the state at t, but also the state at all other times via the dynamics. These constraints compound multiplicatively. Entropy computed from marginals cannot represent this compounding, because it does not track how observations restrict trajectories jointly.

As a result, entropy can plateau at a positive value even when the admissible set has collapsed to a single state.

15.2.3 The Diagnostic Toy Model

The diagnostic model introduced by Smart isolates this effect with minimal structure:

- A finite microstate space $\mathcal{X} = \{0, 1\}^N$.
- Deterministic, bijective evolution implemented by a Feistel network.
- A coarse-graining operator ("horizon") $H : \mathcal{X} \to \{0, 1\}^M$ with $M \ll N$.
- A deterministic emission operator ("radiation") releasing k bits per timestep.

The model contains no spacetime geometry, gravity, or quantum mechanics. Its purpose is diagnostic: to show that bookkeeping alone can generate apparent information loss.

Despite the simplicity of the dynamics, the horizon and radiation outputs appear statistically random when examined locally. Marginal distributions approach uniformity, and standard entropy estimators report persistent uncertainty.

15.2.4 Naive Tracking Regimes

Several naive tracking methods are applied:

1. Marginal Shannon entropy of horizon outputs.
2. IID "thermal" assumptions over unobserved degrees of freedom.
3. Compression-based entropy estimators applied to observation streams.
4. Combined estimators assuming conditional independence across time.

All of these methods discard cross-time constraints. Consequently, they report long-lived entropy plateaus even as observations accumulate.

15.2.5 Exact Constraint Tracking

Constraint tracking explicitly maintains $\mathcal{C}_T$. At each timestep, candidate initial states are forward-simulated and filtered by comparison with the entire observation history. In the canonical configuration, the admissible set collapses rapidly:

$$|\mathcal{C}_T| : \; 2^N \; \rightarrow \; 1$$

after a small number of timesteps. The initial microstate is recovered exactly.

The key point is not the timescale of recovery, which is model-dependent, but the logical fact of recovery: the observation history uniquely determines the initial state, even though naive entropy measures report persistent uncertainty.

15.2.6 The Information Gap

The discrepancy between naive entropy estimates and $\log_2 |\mathcal{C}_T|$ defines an information gap:

$$\Delta I_T = I_T^{\text{naive}} - \log_2 |\mathcal{C}_T|. \tag{15.2}$$

In the diagnostic model, ΔI_T grows to a large constant value and persists indefinitely. This gap does not represent physical indeterminacy. It is the amount of information preserved by the dynamics but discarded by the tracking method.

15.2.7 Interpretation

The diagnostic result establishes a necessary baseline: entropy plateaus alone do not imply information destruction. Apparent information loss can arise purely from bookkeeping failure in reversible systems.

This does not yet address whether recovery is generically stable or operationally feasible. It shows only that claims of information loss must first exclude the possibility that the tracking scheme itself is discarding correlations preserved by the dynamics.

15.3 Implicit Assumptions Behind Constraint-Based Recovery

The bookkeeping resolution summarized in Section 15.2 demonstrates that apparent information loss can arise from incomplete tracking rather than from irreversible dynamics. However, that conclusion depends on a collection of implicit assumptions that are satisfied by the diagnostic toy model but need not hold in more complex systems. Making these assumptions explicit clarifies both the scope of the bookkeeping result and the motivation for extending the analysis beyond it.

15.3.1 Exact Knowledge of the Dynamics

Constraint tracking presupposes that the evolution operator F_t is known exactly. The admissible set $\mathcal{C}_T$ is defined by forward-simulating candidate initial states and checking consistency with observations. This procedure is only well-defined if the governing dynamics are specified without ambiguity.

In the diagnostic model, the evolution map is explicitly constructed and exactly invertible. In physical systems, by contrast, dynamics may be known only approximately, may involve effective descriptions, or may depend on unmodeled degrees of freedom. In such cases, failures of recovery may arise from model mismatch rather than from bookkeeping or geometry alone.

15.3.2 Finite and Enumerable State Space

The collapse of the admissible set in the toy model relies on the finiteness and enumerability of the state space. When $\mathcal{X} = \{0, 1\}^N$, the admissible set can be represented explicitly and filtered by brute force.

Many physical systems evolve on continuous or infinite-dimensional state spaces. Even when dynamics are reversible in principle, the admissible set may be uncountable, fractal, or only implicitly defined. In such cases, constraint tracking remains a conceptual definition of information, but not an executable algorithm.

15.3.3 Stable Invertibility

Constraint-based recovery implicitly assumes that inverse reconstruction is well-conditioned. That is, nearby states at time T map to nearby states at earlier times under the inverse dynamics.

Reversibility alone does not guarantee this property. A map may be bijective while exhibiting extreme sensitivity to perturbations. If small uncertainty at late times is exponentially amplified when propagated backward, then recovery becomes unstable even though the inverse map exists mathematically.

The diagnostic model is deliberately constructed to avoid such instability. Its evolution does not introduce strong anisotropic stretching or sensitivity amplification, ensuring that admissible sets shrink monotonically in a well-behaved manner.

15.3.4 Non-Degenerate Observation Geometry

Constraint tracking also assumes that the observation operators impose sufficiently independent constraints on the dynamics. If observations align with invariant or slowly evolving subspaces, then long sequences of data may fail to distinguish among large families of trajectories.

In the toy model, the horizon and radiation operators are chosen to provide informative constraints that eventually separate all trajectories. In realistic systems, observation operators may be poorly aligned with dynamically unstable directions, producing extended plateaus even under exact constraint tracking.

15.3.5 Noise-Free and Exact Observations

Finally, the bookkeeping resolution assumes noiseless observations. Each observation is treated as a hard constraint that exactly filters the admissible set.

In physical settings, observations are noisy, finite-resolution, or indirect. Noise converts hard constraints into soft ones, expanding the admissible region and potentially preventing collapse even when dynamics are reversible and well-conditioned. The presence of noise interacts strongly with geometric instability, as small errors can be amplified under inverse dynamics.

15.3.6 Summary

Constraint tracking resolves apparent information loss under idealized conditions: exact knowledge of reversible dynamics, finite and enumerable state space, stable inversion, informative observation geometry, and noise-free access. These conditions are satisfied by the diagnostic toy model by construction.

The next step is therefore not to ask whether bookkeeping matters—it does—but to ask when correct bookkeeping is sufficient for recovery. Answering that question requires analyzing the geometry of inverse reconstruction itself.

15.4 From Information Preservation to Recoverability Geometry

The bookkeeping analysis establishes that entropy plateaus and thermal statistics need not signal physical information destruction. However, it leaves open a deeper question: even when information is preserved and constraints are defined correctly, is recovery generically available? This section separates information preservation from recoverability and introduces recoverability as a geometric property of inverse reconstruction rather than a purely informational one.

15.4.1 Preservation Is Not Reconstruction

Information preservation is a property of the forward dynamics. A system preserves information if distinct initial states remain distinct under time evolution. In deterministic settings, this corresponds to injectivity of the evolution map; in reversible settings, to bijectivity.

Recoverability, by contrast, is a property of the inverse problem. It concerns whether an observer can stably infer earlier states from later observations given finite precision, noise, and limited access. Preservation guarantees that an inverse exists in principle; it does not guarantee that the inverse is usable.

Formally, let F_T denote the forward evolution map from time 0 to T, and let O_t denote the observation operators. Recoverability concerns the conditioning of the inverse mapping

$$x_0 \longleftarrow \{O_t(F_t(x_0))\}_{t=0}^{T}.$$

Two systems may preserve information equally well while differing radically in the stability of this inverse.

15.4.2 Inverse Conditioning and Sensitivity

The stability of inverse reconstruction is governed by how uncertainties evolve under the dynamics. Consider two nearby initial states separated by δx_0. Under forward evolution,

$$\delta x_T = DF_T(x_0)\,\delta x_0,$$

where DF_T is the Jacobian of the flow. If DF_T strongly stretches some directions while contracting others, then the inverse map DF_T^{-1} will strongly amplify uncertainty along the stretched directions.

In such cases, even infinitesimal uncertainty at time T can correspond to macroscopic uncertainty at time 0. The inverse problem is ill-conditioned: the admissible set of initial states consistent with the data may be extremely thin yet highly elongated, rendering recovery unstable under perturbation.

15.4.3 Geometric Structure of Admissible Sets

Constraint tracking defines information as an admissible set $\mathcal{C}_T$. Recoverability depends not only on the volume of this set but on its geometry.

In well-conditioned systems, $\mathcal{C}_T$ contracts roughly isotropically toward a point as observations accumulate. In poorly conditioned systems, $\mathcal{C}_T$ may collapse in volume while stretching into filaments or sheets aligned with unstable manifolds of the flow. Such sets have low entropy but poor recoverability: small observational noise causes large uncertainty along unstable directions.

Entropy measures are blind to this distinction. Two admissible sets can have identical volume while differing radically in geometric stability.

15.4.4 Directional Decoherence and Frame Rotation

A particularly destructive mechanism for recoverability is rapid rotation of expanding and contracting directions in state space. When the local eigenframe of the Jacobian rotates faster than constraints can accumulate, successive observations constrain incompatible directions.

This produces *directional decoherence*: constraints fail to reinforce each other coherently over time. Although each observation is informative in principle, the sequence does not converge to a stable inverse because the relevant directions continually shift. The admissible set becomes a twisted, unstable manifold rather than a shrinking ball.

The diagnostic toy model is deliberately constructed to avoid such effects. Its dynamics are bijective without strong anisotropic distortion or frame rotation, ensuring stable contraction of $\mathcal{C}_T$.

15.4.5 Shadowed Return Paths

Even when an inverse trajectory exists mathematically, it may be dynamically inaccessible. In some systems, the backward trajectory lies in a region of state space where small perturbations cause divergence into different basins. The return path is *shadowed*: present in the equations, absent in practice.

In such cases, information is preserved but the inverse route is structurally erased. Recovery requires infinite precision or perfect control, conditions that are physically unattainable.

15.4.6 Definition of Recoverability

We therefore define recoverability as follows:

> *A system is recoverable over a time interval if the inverse mapping from observation histories to initial states is well-conditioned under the geometry induced by the dynamics.*

Recoverability depends on:

- Anisotropy of the flow,
- Rotation of unstable directions,
- Alignment between observation operators and stable manifolds,
- Noise and finite precision.

These factors are orthogonal to bookkeeping. Correct constraint definitions are necessary but not sufficient for recovery.

15.4.7 Transition

Constraint tracking resolves apparent information loss caused by bookkeeping failure. The analysis above shows that a second, independent limitation exists: the geometry of inverse reconstruction itself. In the next section we characterize *structural non-recoverability*—regimes where information is preserved but recovery is eliminated by dynamical geometry.

15.5 Structural Non-Recoverability Without Information Destruction

The distinction between information preservation and recoverability allows for a class of failures that are neither violations of unitarity nor artifacts of bookkeeping. In these regimes, information is preserved by the dynamics, constraints are correctly defined, and yet recovery becomes structurally unavailable due to the geometry of the inverse problem.

15.5.1 Reversibility Does Not Guarantee Recovery

A persistent misconception in discussions of information loss is that reversibility implies recoverability. Reversibility asserts the existence of a unique inverse map; recoverability concerns the stability and accessibility of that inverse under finite precision.

Let F_T be bijective. Then for every final state x_T there exists a unique $x_0 = F_T^{-1}(x_T)$. However, if the inverse map F_T^{-1} is exponentially sensitive, then any finite uncertainty in x_T corresponds to an exponentially large uncertainty in x_0. The system preserves information while eliminating the practical possibility of reconstruction.

This failure mode is invisible to entropy-based diagnostics and persists even under perfect bookkeeping.

15.5.2 Collapse of Volume vs. Collapse of Geometry

Constraint tracking measures uncertainty via the volume of the admissible set $\mathcal{C}_T$. Structural non-recoverability arises when volume collapse does not imply geometric localization.

In such regimes, $\mathcal{C}_T$ contracts in measure while elongating along unstable directions. The set may approach zero volume while remaining extended across macroscopic distances in state space. Inverse reconstruction then becomes hypersensitive to noise, even though the admissible set is formally small.

This distinction highlights a limitation of scalar uncertainty measures. Recoverability depends on the shape of $\mathcal{C}_T$, not merely its size.

15.5.3 Directional Loss Without Entropic Loss

Structural non-recoverability can be understood as a loss of directional information rather than a loss of state distinguishability. The identity of trajectories is preserved, but the directions required to reverse the flow are no longer accessible.

Constraints may accumulate strongly in directions orthogonal to those needed for recovery. As a result, entropy may decrease or remain constant while recoverability collapses. The system becomes over-constrained in irrelevant directions and under-constrained along unstable ones.

This phenomenon explains how information can be preserved without being reconstructible.

15.5.4 Noise as a Structural Amplifier

Noise plays qualitatively different roles in bookkeeping-limited and structurally non-recoverable systems. In bookkeeping-limited regimes, noise merely delays constraint accumulation. In structurally non-recoverable regimes, noise is amplified by the inverse dynamics, destroying recovery even when the forward dynamics remain stable.

Once inverse amplification dominates, improvements in measurement precision yield diminishing returns. Recovery is not postponed; it is structurally eliminated.

15.5.5 Relation to Diagnostic Toy Models

The diagnostic toy model introduced by Smart is constructed to avoid structural non-recoverability. Its purpose is to demonstrate that bookkeeping failure alone can generate apparent information loss. It does not claim that all reversible systems are recoverable.

Structural non-recoverability represents a complementary phenomenon. It presupposes correct bookkeeping and arises only when the geometry of the flow undermines inverse conditioning. The two mechanisms are logically independent and must be analyzed separately.

15.5.6 Summary

Structural non-recoverability describes regimes in which information is preserved by the dynamics but rendered inaccessible by the geometry of inverse reconstruction. This phenomenon does not contradict constraint-based resolutions of apparent information loss; it defines their limits.

In the next section, we apply this unified framework to black hole physics, clarifying which aspects of the information paradox are resolved by constraint tracking and which depend on deeper questions of dynamical geometry.

15.6 Implications for Black Hole Physics

The framework developed above separates two issues that are often conflated in discussions of black hole evaporation: whether information is preserved by the underlying dynamics, and whether that information is recoverable in any stable or operational sense. Applying this distinction clarifies what is resolved by improved bookkeeping and what remains an open question tied to dynamical geometry.

15.6.1 Reinterpreting Hawking Radiation

Hawking's semiclassical calculation yields a thermal reduced density matrix for outgoing radiation when microscopic correlations are traced over. Within the constraint-tracking framework, this thermality reflects a projection rather than a physical erasure. Tracing out degrees of freedom discards correlations that may persist in the full state, producing entropy growth in the reduced description.

The diagnostic toy model demonstrates that deterministic, reversible dynamics can generate outputs whose marginals are indistinguishable from thermal noise while still encoding complete information in cross-time correlations. Thermal appearance, by itself, is therefore insufficient to diagnose information destruction.

From this perspective, Hawking radiation's thermality is compatible with unitary evolution. The apparent paradox arises when thermal statistics are interpreted as fundamental randomness rather than as the result of coarse-graining.

15.6.2 What Constraint Tracking Resolves

Constraint tracking directly addresses the bookkeeping failure mode. It establishes that:

- Entropy plateaus do not imply physical indeterminacy.
- Apparent loss can arise in finite, fully reversible systems.
- Information can be preserved entirely in correlations invisible to marginal statistics.

To the extent that the black hole information paradox rests on equating thermality with destruction, this aspect dissolves once information is defined as a set of dynamical constraints rather than as entropy alone.

15.6.3 What Constraint Tracking Does Not Resolve

Constraint tracking does not, by itself, guarantee that information is recoverable in any operational sense. Real black holes involve continuous quantum fields, strong curvature, extreme redshifting, and limited observational access. Even if the global quantum state evolves unitarily, the inverse reconstruction problem may be geometrically ill-conditioned.

Structural non-recoverability can therefore arise without contradicting unitarity or constraint preservation. Information may exist globally while remaining inaccessible to any observer capable of interacting with the radiation in a stable way.

15.6.4 Complementarity, Firewalls, and Geometry

Debates over complementarity and firewalls hinge on whether information is duplicated, destroyed, or rendered inaccessible. Within the present framework, these debates can be reframed as questions about recoverability geometry.

If inverse reconstruction is structurally unstable, information may be preserved globally while remaining inaccessible without invoking duplication or horizon-scale violence. Conversely, enforcing recoverability at all costs may require altering local geometry or introducing new degrees of freedom. The tension shifts from entropy accounting to geometric conditioning.

15.6.5 Holography and Conditioning

Holographic dualities demonstrate that a unitary description of black hole evaporation exists. However, dual unitarity does not imply that bulk reconstruction is well-conditioned. Boundary degrees of freedom may encode bulk information in a highly nonlocal and geometrically distorted manner.

From this viewpoint, holography guarantees information preservation while remaining agnostic about recoverability. The conditioning of the inverse map from radiation to interior data is a separate question governed by the geometry of the encoding.

15.6.6 Summary

Constraint tracking resolves the bookkeeping aspect of the black hole information paradox by showing that apparent loss can arise from descriptive choices. Structural non-recoverability highlights a deeper limitation: even perfectly preserved information may be geometrically inaccessible.

The paradox is therefore not a binary question of loss versus preservation, but a layered question about how information is tracked and how inverse reconstruction is shaped by dynamics.

15.7 Broader Implications and Related Systems

The separation between bookkeeping failure and structural non-recoverability is not specific to black hole physics. It applies broadly to systems in which reversible dynamics, coarse observation, and instability coexist. In this section we outline several domains where the same distinction clarifies long-standing interpretive difficulties.

15.7.1 Turbulence and High-Dimensional Flows

In turbulent fluid dynamics, the governing equations in the inviscid limit are formally reversible, yet practical recovery of prior flow states from later configurations is impossible. This failure does not arise because information is destroyed, but because the geometry of the flow amplifies perturbations and rotates unstable directions rapidly.

From a constraint perspective, the admissible set of past states consistent with later observations collapses in volume while elongating along unstable manifolds. Even perfect bookkeeping cannot overcome the geometric instability of the inverse problem. This provides a concrete example of structural non-recoverability without entropic loss.

15.7.2 Chaotic Inference and Data Assimilation

Inverse problems in chaotic systems exhibit similar behavior. Data assimilation techniques often fail not because insufficient information is present, but because small observational errors are amplified by unstable inverse dynamics. The admissible set becomes filamentary or fragmented, rendering stable reconstruction infeasible.

Constraint tracking remains valid in principle, but geometric instability dominates performance. This explains why increasing data volume or observation frequency does not always improve inference and may even degrade it.

15.7.3 Nonequilibrium Statistical Mechanics

In nonequilibrium statistical mechanics, entropy production is often interpreted as information loss. The present framework clarifies that entropy increase may reflect coarse-graining and projection rather than destruction, while irreversibility arises from the geometric instability of inverse trajectories.

Thermodynamic irreversibility can thus coexist with microscopic reversibility and information preservation, with structural non-recoverability providing the missing link.

15.7.4 Machine Learning and Optimization Landscapes

Modern learning systems frequently operate in regimes where forward training dynamics are stable, yet inversion—such as reconstructing training data from learned parameters—is ill-conditioned. Information may be preserved in principle, but the geometry of the loss landscape renders recovery unstable.

Viewing learning as constraint accumulation within a dynamically evolving geometry aligns naturally with the framework developed here and clarifies why memorization does not imply invertibility.

15.7.5 Summary

Across domains, apparent information loss often reflects bookkeeping projections, while true irreversibility emerges from geometric instability of inverse reconstruction. Separating these mechanisms provides a unifying lens for interpreting complexity, chaos, and irreversibility without invoking fundamental information destruction.

15.8 Limitations and Scope

The framework developed in this paper is intended to clarify the logical structure of information loss claims, not to provide a complete physical theory of black hole evaporation or complex dynamics. Several limitations should therefore be made explicit.

15.8.1 Diagnostic Role of Toy Models

The constraint-tracking results rely on finite, discrete toy models with explicitly specified dynamics and observation operators. These models are designed to isolate informational mechanisms under controlled conditions. They are not intended to reproduce the physical structure of black holes, turbulent flows, or quantum fields.

Quantities such as recovery time, entropy gap magnitude, or candidate-set collapse rate have no direct physical interpretation. Their significance lies in demonstrating logical possibility:

apparent information loss can arise without non-unitary dynamics, and recovery can fail without information destruction.

15.8.2 Classical Deterministic Dynamics

All explicit models discussed here are classical and deterministic. Quantum phenomena such as superposition, entanglement, decoherence, and measurement backaction are not modeled directly. While the conceptual distinction between bookkeeping failure and structural non-recoverability generalizes to unitary quantum evolution, quantum state space introduces additional structure that is not analyzed in this work.

In particular, the relationship between recoverability geometry and quantum scrambling, complexity growth, and error correction remains open.

15.8.3 Observational Access and Noise

The analysis often assumes idealized observation histories and, in some cases, exact knowledge of the dynamics. Real observers face noise, finite resolution, partial observability, and limited control. These factors can convert otherwise recoverable systems into structurally non-recoverable ones by amplifying inverse instability.

The framework distinguishes epistemic limitations from structural ones, but does not eliminate either in realistic settings.

15.8.4 Computational Tractability

Explicit constraint tracking is computationally intractable in high-dimensional or continuous systems. Although this does not undermine the conceptual definition of information as an admissible set, it limits practical recovery procedures.

Computational complexity is treated here as orthogonal to structural non-recoverability. A system may be unrecoverable due to geometric instability even when reconstruction is computationally simple in principle, and conversely, computational hardness may block recovery even when geometry is favorable.

15.8.5 Interpretive Scope

This work does not claim to resolve the black hole information paradox in a physical sense. Instead, it reframes the paradox by separating informational bookkeeping from dynamical geometry. Whether real black holes permit recovery in any operational sense depends on physics beyond the present analysis.

15.8.6 Summary

The framework clarifies conceptual structure but does not replace detailed physical modeling. Its primary contribution is distinguishing when apparent information loss arises from descriptive choices and when irreversibility emerges from the geometry of inverse dynamics.

15.9 Conclusion

This work has separated two mechanisms that are frequently conflated in discussions of information loss: failures of informational bookkeeping and failures of dynamical recoverability. Making this distinction explicit clarifies how apparent information loss can arise in fully reversible systems, and why resolving such appearances does not by itself guarantee stable reconstruction of past states.

Building on the constraint-tracking framework introduced by Smart, we emphasized that entropy plateaus and thermal statistics need not signal physical information destruction. When information is defined operationally as the set of admissible microstates consistent with the full observation history under known dynamics, reversible systems can preserve complete information even while naive statistical trackers report persistent uncertainty.

At the same time, we argued that information preservation is not equivalent to recoverability. In complex or unstable systems, the geometry of state-space evolution can render inverse reconstruction ill-conditioned or structurally inadmissible. In such regimes, admissible sets may collapse in volume while remaining geometrically elongated, filamentary, or hypersensitive to perturbation. Information remains present, but the conditions required for recovery no longer exist in a robust sense.

Applied to black hole physics, this framework clarifies which aspects of the information paradox are resolved by improved bookkeeping and which depend on deeper questions of dynamical geometry. Constraint tracking dissolves the inference from thermality to information destruction. Structural non-recoverability highlights a remaining open question: whether the geometry of quantum gravitational dynamics permits stable recovery of infalling information, even when unitarity is preserved.

More broadly, the separation of bookkeeping failure from geometric non-recoverability provides a unifying lens for interpreting irreversibility across physics, from turbulence and chaotic inference to learning systems and nonequilibrium processes. Apparent information loss need not reflect fundamental destruction; true irreversibility emerges when the geometry of inverse dynamics eliminates stable return paths.

In this sense, the black hole information problem is neither a paradox of unitarity nor a paradox of entropy. It is a question about the structure of recovery.

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.1 Formal Definitions and Notation

This appendix consolidates formal definitions and notation used throughout the paper.

.1.1 State Space and Dynamics

Let $\mathcal{X}$ denote the state space of the system. In diagnostic models, $\mathcal{X}$ is finite and discrete; in general settings, $\mathcal{X}$ may be continuous or infinite-dimensional.

Time evolution is given by a family of deterministic maps

$$F_t : \mathcal{X} \to \mathcal{X},$$

assumed injective. Information preservation corresponds to injectivity; reversibility corresponds to bijectivity.

.1.2 Observation Operators

Observations are generated by operators

$$O_t : \mathcal{X} \to \mathcal{Y},$$

where $\mathcal{Y}$ is a lower-dimensional observation space. Coarse-graining corresponds to non-injectivity of O_t.

.1.3 Constraint Sets

Given an observation history $\mathcal{O}_{0:T} = \{o_0, \ldots, o_T\}$, the admissible set is

$$\mathcal{C}_T = \{x_0 \in \mathcal{X} \mid O_t(F_t(x_0)) = o_t \;\; \forall t \leq T\}.$$

When $\mathcal{X}$ is discrete, information content is defined as

$$I_T = \log_2 |\mathcal{C}_T|.$$

For continuous $\mathcal{X}$, $|\cdot|$ should be replaced by an appropriate measure.

.2 Bookkeeping Failure vs. Structural Non-Recoverability

For clarity, Table 1 summarizes the distinction emphasized throughout the paper.

Aspect	Bookkeeping Failure	Structural Non-Recoverability
Primary cause	Statistical projection	Dynamical geometry
Dynamics	Reversible	Reversible
Information	Preserved	Preserved
Entropy behavior	Artificial plateau	May decrease or stay low
Recoverability	Available in principle	Geometrically unstable
Primary remedy	Constraint tracking	Geometric analysis / redesign

Table 1: Comparison of bookkeeping failure and structural non-recoverability.

These mechanisms are logically independent and must be addressed separately.

.3 A Formal Recoverability Metric

We formalize recoverability as conditioning of the inverse map from an observation history to the initial state. The key distinction is:

- **Information preservation** is a property of the forward dynamics (injectivity/bijectivity).
- **Recoverability** is a property of the inverse problem under finite noise (conditioning).

.3.1 Setup: Dynamics, Observations, and Noise

Let $x_t \in \mathbb{R}^d$ evolve by

$$x_{t+1} = f(x_t), \qquad t = 0, \ldots, T-1, \tag{3}$$

where f is differentiable and (in principle) invertible on the domain of interest.

Let observations be generated by

$$y_t = h(x_t) + \eta_t, \qquad y_t \in \mathbb{R}^m, \tag{4}$$

with measurement noise η_t satisfying $\|\eta_t\| \leq \varepsilon$ (or $\mathbb{E}\|\eta_t\|^2 = \sigma^2$).

Define the observation-history map

$$\Phi_T(x_0) \;=\; \big(h(x_0),\, h(x_1),\, \ldots,\, h(x_T)\big) \in \mathbb{R}^{m(T+1)}. \tag{5}$$

Recoverability is the stability of the inverse problem

$$x_0 \;\leftarrow\; \Phi_T(x_0) + \eta, \tag{6}$$

where η stacks all η_t.

.3.2 Local Linearization and the Observability Jacobian

Fix a reference trajectory $\{x_t^\star\}$ generated from $x_0^\star$. For a perturbation δx_0, the tangent dynamics satisfy

$$\delta x_{t+1} = A_t\,\delta x_t, \qquad A_t := Df(x_t^\star). \tag{7}$$

Thus

$$\delta x_t = \Big(\prod_{k=0}^{t-1} A_k\Big)\delta x_0 \;=: F_{t,0}\,\delta x_0. \tag{8}$$

Linearizing the observations gives

$$\delta y_t = C_t\,\delta x_t + \delta\eta_t, \qquad C_t := Dh(x_t^\star). \tag{9}$$

Stacking all times yields the linearized inverse problem

$$\delta Y = \mathcal{O}_T\,\delta x_0 + \delta E, \tag{10}$$

where $\delta Y \in \mathbb{R}^{m(T+1)}$ and the *observability Jacobian* is

$$\mathcal{O}_T = \begin{bmatrix} C_0 \\ C_1 F_{1,0} \\ C_2 F_{2,0} \\ \vdots \\ C_T F_{T,0} \end{bmatrix} \in \mathbb{R}^{m(T+1)\times d}. \tag{11}$$

.3.3 Definition: Recoverability via Conditioning

Definition (local recoverability). The system is locally recoverable over horizon T along trajectory $x^\star$ if $\mathcal{O}_T$ has full column rank and the smallest singular value $\sigma_{\min}(\mathcal{O}_T)$ is bounded away from zero.

This yields a sharp noise-to-state error bound. For any estimate $\widehat{\delta x_0}$ obtained by least squares,

$$\widehat{\delta x_0} = \arg\min_u \|\mathcal{O}_T u - \delta Y\|_2^2, \tag{12}$$

we have

$$\|\widehat{\delta x_0} - \delta x_0\|_2 \le \frac{\|\delta E\|_2}{\sigma_{\min}(\mathcal{O}_T)}. \tag{13}$$

Thus:

- Information preservation may hold (invertible f),
- yet recovery is unstable whenever $\sigma_{\min}(\mathcal{O}_T) \to 0$.

We define the *recoverability condition number*

$$\kappa_T := \frac{\sigma_{\max}(\mathcal{O}_T)}{\sigma_{\min}(\mathcal{O}_T)}. \tag{14}$$

Large κ_T implies that small observational errors produce large uncertainty in x_0.

.3.4 Geometric Interpretation: Directional Collapse vs. Directional Access

Equation (13) shows recoverability depends on directional access: $\mathcal{O}_T$ must *excite* all directions of δx_0 through time.

If expanding tangent directions rotate rapidly (time-varying A_t eigenframes), then $C_t F_{t,0}$ can repeatedly miss the directions that carry inverse sensitivity, driving $\sigma_{\min}(\mathcal{O}_T)$ downward.

This is a formal statement of the qualitative claim:

> constraints may accumulate in directions orthogonal to those required for stable inversion.

.3.5 A Sufficient Instability Criterion

Let v be a unit direction in $\mathbb{R}^d$. Consider its "visible energy":

$$\|\mathcal{O}_T v\|_2^2 = \sum_{t=0}^{T} \|C_t F_{t,0} v\|_2^2. \tag{15}$$

By definition,

$$\sigma_{\min}(\mathcal{O}_T)^2 = \min_{\|v\|_2=1} \sum_{t=0}^{T} \|C_t F_{t,0} v\|_2^2. \tag{16}$$

Therefore a sufficient condition for structural non-recoverability is the existence of a direction v such that

$$\sum_{t=0}^{T} \|C_t F_{t,0} v\|_2^2 \ll 1, \tag{17}$$

even if $\|F_{t,0}v\|$ grows rapidly (chaotic amplification). In that case, the inverse problem amplifies noise while the observation operator fails to constrain the unstable direction.

This captures, in equations, the mechanism "information preserved yet recovery unavailable."

.3.6 Connection to Constraint Tracking

Constraint tracking is exact when dynamics and observations are noiseless and discrete. In continuous noisy settings, the admissible set becomes a tube around the trajectory. The tube radius is controlled by $\sigma_{\min}(\mathcal{O}_T)$: when it is small, the tube is thin in measure but elongated along poorly observed directions, producing geometric fragility.

Hence bookkeeping can be correct while recovery remains ill-conditioned.

.4 Continuous-Time Recoverability and the Observability Gramian

To connect recoverability geometry to physical systems governed by differential equations, we state the continuous-time analogue of Section .3. The central object becomes the *observability Gramian*, whose smallest eigenvalue controls the stability of reconstructing the initial condition from an observation history.

.4.1 Dynamics, Observations, and Noise

Let $x(t) \in \mathbb{R}^d$ evolve under a (possibly nonlinear) ODE

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \tag{18}$$

with observations

$$y(t) = h(x(t)) + \eta(t), \qquad y(t) \in \mathbb{R}^m, \tag{19}$$

where $\eta(t)$ represents measurement noise (e.g. bounded in L^2 or stochastic).

Define the observation-history map over $[0, T]$:

$$\Phi_T(x_0) := \{h(x(t; x_0))\}_{t \in [0,T]}. \tag{20}$$

Recoverability concerns the stability of the inverse problem

$$x_0 \;\leftarrow\; \Phi_T(x_0) + \eta(\cdot). \tag{21}$$

.4.2 Linearization Along a Trajectory

Fix a reference trajectory $x^\star(t)$ generated by $x_0^\star$. Linearizing about $x^\star(t)$ yields the time-varying linear system

$$\delta\dot{x}(t) = A(t)\,\delta x(t), \qquad A(t) := Df(x^\star(t)), \tag{22}$$

with linearized observation

$$\delta y(t) = C(t)\,\delta x(t) + \delta\eta(t), \qquad C(t) := Dh(x^\star(t)). \tag{23}$$

Let $\Psi(t,0)$ denote the state transition matrix satisfying

$$\frac{d}{dt}\Psi(t,0) = A(t)\Psi(t,0), \qquad \Psi(0,0) = I. \tag{24}$$

Then

$$\delta x(t) = \Psi(t,0)\,\delta x_0. \tag{25}$$

.4.3 Observability Gramian and a Stability Bound

Define the (finite-horizon) observability Gramian:

$$W_o(T) \;:=\; \int_0^T \Psi(t,0)^\top C(t)^\top C(t)\Psi(t,0)\,dt \;\in\; \mathbb{R}^{d\times d}. \tag{26}$$

The Gramian aggregates how strongly each initial-direction δx_0 is "seen" by the output over the interval.

If $W_o(T)$ is positive definite, the linearized system is observable on $[0, T]$. Moreover, the smallest eigenvalue controls conditioning: for any unit vector v,

$$v^\top W_o(T) v = \int_0^T \|C(t)\Psi(t,0)v\|_2^2 \, dt, \tag{27}$$

and

$$\lambda_{\min}(W_o(T)) = \min_{\|v\|_2=1} \int_0^T \|C(t)\Psi(t,0)v\|_2^2 \, dt. \tag{28}$$

Recoverability bound (continuous-time, linearized). Suppose $\delta y(\cdot)$ is observed with additive error $\delta\eta(\cdot) \in L^2([0,T])$. Then the least-squares estimate of δx_0 satisfies

$$\|\widehat{\delta x_0} - \delta x_0\|_2 \ \le \ \frac{\|\delta\eta\|_{L^2([0,T])}}{\sqrt{\lambda_{\min}(W_o(T))}}. \tag{29}$$

Thus recovery becomes structurally unstable whenever $\lambda_{\min}(W_o(T)) \to 0$, even if the forward dynamics are perfectly reversible.

.4.4 Structural Non-Recoverability Criterion

A sufficient condition for structural non-recoverability is the existence of a direction v for which the output has negligible sensitivity across the entire interval:

$$\int_0^T \|C(t)\Psi(t,0)v\|_2^2 \, dt \ll 1, \tag{30}$$

while $\|\Psi(t,0)v\|$ may grow rapidly. In that regime, noise is amplified by inversion, but observations fail to constrain the unstable direction.

This is the continuous-time analogue of $\sigma_{\min}(\mathcal{O}_T) \to 0$ in the discrete-time setting.

.4.5 Geometric Reading: "Directional Decoherence" as Gramian Collapse

The phrase *directional decoherence* can be stated precisely in this language: even if the flow expands certain tangent directions, rapid time-variation of $A(t)$ can rotate those expanding directions so that $C(t)$ repeatedly projects away from them. When this persists, $W_o(T)$ becomes nearly singular and the inverse map from output history to initial state becomes ill-conditioned.

Importantly, this failure does not require non-invertible dynamics; it is a property of the inverse-conditioning induced by the pair $(A(t), C(t))$ along a trajectory.

.4.6 Connection Back to Constraint Tracking

In noiseless discrete models, constraint tracking collapses the admissible set exactly. In continuous noisy systems, the admissible set becomes a tube around the trajectory. Equation (29) shows the tube thickness in initial-condition space is controlled by $\lambda_{\min}(W_o(T))$. When $\lambda_{\min}$ is small, the admissible set may have small measure yet remain extended along poorly observed directions, producing geometric fragility: information may be preserved while stable recovery is unavailable.

.5 A Proposition: Preserved Dynamics, Unstable Recovery

This section provides a formal statement capturing the core claim of the paper: *even when dynamics are reversible (information-preserving), recovery can be structurally unstable* because the inverse problem is ill-conditioned.

We state and prove a clean sufficient condition using the observability Gramian.

.5.1 Proposition and Proof

Consider the continuous-time linear time-varying system obtained by linearization along a reference trajectory (Section .4):

$$\delta\dot{x}(t) = A(t)\delta x(t), \qquad \delta y(t) = C(t)\delta x(t) + \delta\eta(t), \tag{31}$$

with state transition matrix $\Psi(t, 0)$ and observability Gramian

$$W_o(T) = \int_0^T \Psi(t, 0)^\top C(t)^\top C(t)\Psi(t, 0)\, dt. \tag{32}$$

> **Proposition 1 (Sufficient condition for structural non-recoverability).** Assume $W_o(T)$ is positive definite but has a very small minimum eigenvalue $\lambda_{\min}(W_o(T))$. Then for any observation noise $\delta\eta \in L^2([0, T])$, the least-squares estimate of the ini-

tial perturbation satisfies

$$\|\widehat{\delta x_0} - \delta x_0\|_2 \;\le\; \frac{\|\delta\eta\|_{L^2([0,T])}}{\sqrt{\lambda_{\min}(W_o(T))}}. \qquad (33)$$

In particular, if $\lambda_{\min}(W_o(T)) \to 0$ along a sequence of horizons T_n, then for any fixed nonzero noise level, the reconstruction error is unbounded as $n \to \infty$. Hence recoverability fails even though the forward dynamics may remain perfectly reversible.

Proof. The stacked least-squares inverse problem over $[0,T]$ can be written as minimizing

$$J(u) = \int_0^T \|C(t)\Psi(t,0)u - \delta y(t)\|_2^2 \, dt, \qquad (34)$$

where $\delta y(t) = C(t)\Psi(t,0)\delta x_0 + \delta\eta(t)$. The normal equations yield

$$W_o(T)\,\widehat{\delta x_0} = \int_0^T \Psi(t,0)^\top C(t)^\top \delta y(t)\, dt. \qquad (35)$$

Substituting $\delta y(t) = C(t)\Psi(t,0)\delta x_0 + \delta\eta(t)$ gives

$$W_o(T)\left(\widehat{\delta x_0} - \delta x_0\right) = \int_0^T \Psi(t,0)^\top C(t)^\top \delta\eta(t)\, dt. \qquad (36)$$

Taking norms and using $\|W_o^{-1}\|_2 = 1/\lambda_{\min}(W_o)$ yields

$$\|\widehat{\delta x_0} - \delta x_0\|_2 \le \|W_o(T)^{-1}\|_2 \left\| \int_0^T \Psi(t,0)^\top C(t)^\top \delta\eta(t)\, dt \right\|_2 . \qquad (37)$$

By Cauchy–Schwarz in L^2 and the definition of the Gramian (see, e.g., standard observability estimates), one obtains the bound in (33). $\square$

.5.2 Interpretation

Proposition 1 is the precise mathematical version of the paper's central separation:

- **Preservation:** Reversibility/invertibility ensures an inverse map exists.
- **Recovery:** Conditioning is controlled by $\lambda_{\min}(W_o(T))$ (or $\sigma_{\min}(\mathcal{O}_T)$ in discrete time).

If the smallest eigenvalue collapses, then inverse reconstruction amplifies any finite noise. This is *structural non-recoverability*: the inverse exists but is not stably accessible.

.5.3 Discrete-Time Version (for completeness)

For the discrete-time linearized system

$$\delta x_{t+1} = A_t \delta x_t, \qquad \delta y_t = C_t \delta x_t + \delta \eta_t, \tag{38}$$

the stacked observation map is $\delta Y = \mathcal{O}_T \delta x_0 + \delta E$ with

$$\mathcal{O}_T = \begin{bmatrix} C_0 \\ C_1 F_{1,0} \\ \vdots \\ C_T F_{T,0} \end{bmatrix}. \tag{39}$$

The analogous bound is

$$\|\widehat{\delta x_0} - \delta x_0\|_2 \;\le\; \frac{\|\delta E\|_2}{\sigma_{\min}(\mathcal{O}_T)}. \tag{40}$$

Hence, $\sigma_{\min}(\mathcal{O}_T) \to 0$ implies unstable recovery.

.5.4 How This Connects to "Directional Decoherence"

The eigenvalue $\lambda_{\min}(W_o(T))$ collapses when there exists a direction v that remains weakly "visible" to the outputs across time:

$$v^\top W_o(T) v = \int_0^T \|C(t)\Psi(t,0)v\|_2^2 \, dt \approx 0. \tag{41}$$

This provides an explicit, checkable definition of the qualitative phenomenon described earlier: constraints do not accumulate coherently along the directions that matter for inversion. "Directional decoherence" is precisely the collapse of this minimal directional visibility.

.6 Lyapunov Growth, Frame Rotation, and the Collapse of Recoverability

Section .5 established that recoverability is governed by the conditioning of the inverse problem, quantified by $\lambda_{\min}(W_o(T))$ or $\sigma_{\min}(\mathcal{O}_T)$. We now connect this collapse of observability to familiar dynamical quantities: Lyapunov exponents and timedependent frame rotation.

.6.1 Lyapunov Growth Is Not the Culprit by Itself

Consider a trajectory with a positive maximal Lyapunov exponent $\lambda_{\max} > 0$. Then for generic perturbations,

$$\|\delta x(t)\| \sim e^{\lambda_{\max} t} \|\delta x_0\|. \tag{42}$$

This exponential growth is often cited as the reason inverse problems fail. However, exponential growth alone does *not* imply non-recoverability.

If the expanding direction is consistently observed (i.e., well-aligned with $C(t)$), then the Gramian eigenvalues grow rather than collapse:

$$v^\top W_o(T) v = \int_0^T \|C(t)\Psi(t,0)v\|^2 dt \sim \int_0^T e^{2\lambda_{\max} t} dt, \tag{43}$$

which is large and stabilizing for inversion.

Thus:

Lyapunov instability alone does not destroy recoverability.

.6.2 Frame Rotation as the Critical Mechanism

The decisive effect is *rotation of expanding directions* relative to the observation map. Let $\{e_i(t)\}$ denote the Oseledets basis associated with the linearized flow. Although growth rates are governed by Lyapunov exponents, the directions themselves evolve according to

$$\dot{e}_i(t) = \Omega(t)e_i(t), \tag{44}$$

where $\Omega(t)$ is a skew-symmetric rotation generator induced by the non-normal part of $A(t)$.

If $\Omega(t)$ induces rapid rotation compared to the rate at which observations accumulate, then no fixed observation operator can remain aligned with the expanding directions.

.6.3 Quantitative Criterion for Rotational Decoherence

Let $v(t) = \Psi(t, 0)v_0/\|\Psi(t, 0)v_0\|$ denote the normalized evolving direction. Define the effective visibility

$$\mathcal{V}_T(v_0) = \int_0^T \|C(t)v(t)\|^2 \, dt. \tag{45}$$

Recoverability along direction v_0 requires $\mathcal{V}_T(v_0)$ to be bounded away from zero. Rotational decoherence occurs when

$$\|C(t)v(t)\|^2 \approx 0 \quad \text{for most } t, \tag{46}$$

despite $\|\Psi(t,0)v_0\|$ growing rapidly.

This leads directly to

$$\lambda_{\min}(W_o(T)) = \min_{\|v_0\|=1} \mathcal{V}_T(v_0) \;\longrightarrow\; 0. \tag{47}$$

.6.4 Shadowed Return Paths (Formalized)

We can now formalize the intuitive notion of a "shadowed return path."

> **Definition.** A trajectory admits a shadowed return path over $[0,T]$ if F_T is bijective but $\lambda_{\min}(W_o(T))$ is exponentially small in T.

In this case, the inverse map exists but any admissible inverse trajectory lies inside a manifold whose thickness scales like $e^{-\alpha T}$ for some $\alpha > 0$. Any finite perturbation ejects the trajectory from the admissible tube.

.6.5 Relation to Constraint Tracking

Constraint tracking computes the admissible set $\mathcal{C}_T$ exactly. In rotationally unstable regimes, $\mathcal{C}_T$ may:

- shrink in volume (entropy decreases),

- while elongating geometrically along unstable directions,
- producing extreme sensitivity under inversion.

Thus constraint tracking and geometric instability are *not contradictory*: the former diagnoses bookkeeping correctness, the latter diagnoses recoverability limits.

.6.6 Takeaway

We can now sharpen the paper's central message:

> **Information is preserved by invertible dynamics. Recoverability is governed by the geometry of expanding directions relative to observation.**

Positive Lyapunov exponents are necessary but insufficient. It is rapid frame rotation and anisotropic visibility collapse that produce structural non-recoverability.

The next section applies this criterion directly to black hole–motivated scrambling dynamics and clarifies what semiclassical thermality actually implies.

.7 Observability Gramians and Quantitative Recovery Bounds

We now make the recoverability criterion fully quantitative by relating inverse stability to observability Gramians and their condition numbers. This section provides the cleanest mathematical bridge between constraint tracking (discrete) and recoverability geometry (continuous).

.7.1 Linearized Observation Model

Consider a (possibly time-varying) nonlinear system linearized along a trajectory:

$$\dot{x}(t) = f(x(t), t), \tag{48}$$

$$y(t) = h(x(t), t). \tag{49}$$

Linearizing about a reference trajectory yields

$$\dot{\delta x}(t) = A(t)\,\delta x(t), \tag{50}$$

$$\delta y(t) = C(t)\,\delta x(t), \tag{51}$$

where

$$A(t) = \frac{\partial f}{\partial x}\Big|_{x(t)}, \qquad C(t) = \frac{\partial h}{\partial x}\Big|_{x(t)}.$$

Let $\Psi(t, 0)$ denote the state transition matrix:

$$\delta x(t) = \Psi(t, 0)\,\delta x_0. \tag{52}$$

.7.2 Observability Gramian

The continuous-time observability Gramian over $[0, T]$ is

$$W_o(T) = \int_0^T \Psi(t,0)^\top C(t)^\top C(t)\Psi(t,0)\, dt. \tag{53}$$

For noiseless observations, exact recoverability requires $W_o(T)$ to be full rank. Stable recoverability requires that $W_o(T)$ be well-conditioned.

.7.3 Error Amplification Bound

Suppose observations are corrupted by bounded noise:

$$\|\eta(t)\| \le \varepsilon.$$

Then the least-squares reconstruction error for the initial state satisfies

$$\|\delta x_0\| \;\le\; \frac{1}{\sqrt{\lambda_{\min}(W_o(T))}} \left(\int_0^T \|\eta(t)\|^2 dt \right)^{1/2}. \tag{54}$$

Thus:

> **Recoverability degrades continuously as $\lambda_{\min}(W_o(T)) \to 0$.**

Information preservation ensures that $W_o(T)$ is not identically zero; recoverability depends on its smallest eigenvalue.

.7.4 Condition Number as Recoverability Metric

Define the observability condition number

$$\kappa_o(T) = \frac{\lambda_{\max}(W_o(T))}{\lambda_{\min}(W_o(T))}. \tag{55}$$

Large $\kappa_o(T)$ implies:

- extreme anisotropy of admissible initial perturbations,
- filamentary constraint sets,
- catastrophic sensitivity of inverse reconstruction.

In the limit $\lambda_{\min}(W_o(T)) \to 0$, the inverse problem becomes structurally ill-posed even though F_T remains bijective.

.7.5 Connection to Constraint Tracking

In discrete constraint tracking, the admissible set $\mathcal{C}_T$ corresponds (in the linearized limit) to an uncertainty ellipsoid:

$$\mathcal{E}_T = \{\delta x_0 : \delta x_0^\top W_o(T) \delta x_0 \leq \epsilon^2\}. \tag{56}$$

As $\lambda_{\min}(W_o(T)) \to 0$, $\mathcal{E}_T$ collapses in volume but elongates without bound in specific directions. This reproduces the filamentary admissible sets observed in unstable regimes.

.7.6 Discrete-Time Analog

For discrete-time systems

$$x_{k+1} = F_k(x_k), \qquad y_k = h_k(x_k),$$

the observability Gramian becomes

$$W_o(N) = \sum_{k=0}^{N-1} \Phi(k,0)^\top C_k^\top C_k \Phi(k,0), \tag{57}$$

where $\Phi(k,0)$ is the Jacobian of the k-step map.

This form connects directly to the candidate-set filtering used in constraint tracking: each observation contributes a rank-$\leq m$ quadratic constraint on admissible perturbations.

.7.7 Necessary and Sufficient Condition

We can now state a precise criterion:

> **Theorem (Recoverability Criterion).** A deterministic reversible system is stably recoverable over $[0,T]$ iff the observability Gramian $W_o(T)$ is uniformly positive definite.

Failure of this condition corresponds to structural non-recoverability, not information loss.

.7.8 Interpretation

Entropy-based measures diagnose *volume* contraction. Observability Gramians diagnose *directional accessibility*. Both are required for recovery.

> **Entropy asks how many states remain. Recoverability asks whether inverse directions remain visible.**

The next section applies this formalism to scrambling systems and clarifies why thermality is compatible with both information preservation and non-recoverability.

.8 Scrambling, Mixing, and the Limits of Recoverability

We now apply the recoverability framework to scrambling dynamics, clarifying why thermal appearance and rapid mixing are compatible with both information preservation and structural non-recoverability.

.8.1 Scrambling as Directional Delocalization

Scrambling is commonly defined as the rapid delocalization of initially local information across many degrees of freedom. Formally, scrambling corresponds to the growth of operator support or sensitivity across the state space.

Within the present framework, scrambling is characterized by two simultaneous effects:

- rapid growth of perturbations along unstable directions,
- rapid rotation of these directions across the state space.

This combination produces high entropy in marginal observables while preserving global invertibility.

.8.2 Scrambling and Observability Collapse

Consider a system with strongly positive Lyapunov spectrum and fast frame rotation. Even if $C(t)$ has full rank instantaneously, the time-integrated visibility

$$\mathcal{V}_T(v_0) = \int_0^T \|C(t)\Psi(t,0)v_0\|^2 dt$$

may be arbitrarily small for certain directions v_0 due to rotational misalignment.

Thus scrambling induces:

$$\lambda_{\min}(W_o(T)) \ll 1 \quad \text{even as} \quad \det W_o(T) > 0. \tag{58}$$

This precisely corresponds to information preservation without recoverability.

.8.3 Thermality as Marginal Randomization

Thermal statistics arise when marginal observables equilibrate rapidly. Let $y(t) = h(x(t))$ denote a coarse-grained observable. Scrambling implies that for any fixed h, the marginal distribution

$$p(y(t)) \to p_{\text{eq}}(y)$$

independently of initial conditions.

However, marginal equilibration does not imply decay of trajectory-level correlations. Higher-order correlations across time may persist indefinitely, but are invisible to entropy-based summaries.

> **Thermality reflects marginal randomization, not trajectory indeterminacy.**

.8.4 Scrambling Time and Recoverability Time

Define:

- scrambling time t_s: time for marginal observables to equilibrate,
- recovery time t_r: time for $\lambda_{\min}(W_o(t))$ to collapse below tolerance.

In generic scrambling systems,

$$t_s \ll t_r,$$

meaning that the system appears thermal long before recoverability is lost.

In strongly rotating systems,

$$t_r \sim t_s,$$

and recoverability collapses nearly simultaneously with equilibration.

.8.5 Black Hole Motivation

Black holes are conjectured to be fast scramblers. From the present perspective, this implies:

- rapid marginal thermalization of Hawking radiation,
- rapid rotation of unstable directions in Hilbert space,
- collapse of observability in inverse reconstruction.

Unitarity guarantees information preservation. Scrambling geometry determines recoverability.

.8.6 Why Constraint Tracking Alone Is Insufficient

Constraint tracking detects whether admissible sets collapse. Scrambling systems may still yield $\log |\mathcal{C}_T| \to 0$ while simultaneously producing filamentary admissible geometry that is unstable under noise.

Thus constraint tracking is a necessary but not sufficient condition for operational recoverability.

.8.7 Unified Picture

We can now unify the entire analysis:

- Entropy diagnostics detect bookkeeping failure.
- Constraint tracking detects logical information preservation.
- Observability Gramians detect geometric recoverability.

Only the last determines whether reconstruction is stable.

> **Scrambling destroys recoverability by geometry, not by entropy.**

The next section formalizes this distinction in terms of necessary conditions for paradox resolution.

.9 Necessary and Sufficient Conditions for Recoverability

We now formalize the distinction between information preservation, constraint collapse, and operational recoverability.

.9.1 Three Distinct Notions

Consider a deterministic, invertible dynamical system

$$x(t+1) = F_t(x(t)), \qquad y(t) = h(x(t)).$$

We distinguish:

1. **Information preservation**:

$$F_t \text{ is bijective for all } t.$$

2. **Logical recoverability (constraint collapse)**:

$$|\mathcal{C}_T| = 1,$$

where $\mathcal{C}_T$ is the set of initial states consistent with all observations.

3. **Operational recoverability (stability)**: reconstruction of x_0 is well-conditioned under perturbations of $y(t)$.

Only the third determines whether information is physically extractable.

.9.2 Observability Gramian Criterion

For linearized dynamics

$$\dot{\delta x} = A(t)\delta x, \qquad \delta y = C(t)\delta x,$$

define the observability Gramian

$$W_o(T) = \int_0^T \Phi(t,0)^\top C(t)^\top C(t)\Phi(t,0)\,dt, \tag{59}$$

where $\Phi(t,0)$ is the state transition matrix.

Necessary and sufficient condition for stable recoverability:

$$\lambda_{\min}(W_o(T)) > 0. \tag{60}$$

If $\lambda_{\min}(W_o(T)) \to 0$, reconstruction is ill-conditioned even if $W_o(T)$ is full rank.

.9.3 Relation to Constraint Tracking

Constraint tracking answers:

$$\exists!\ x_0 \in \mathcal{C}_T?$$

The Gramian answers:

$$\|\delta x_0\| \le \kappa\|\delta y\| \quad \text{for finite } \kappa?$$

Thus:

$$|\mathcal{C}_T| = 1 \quad \not\Rightarrow \quad \lambda_{\min}(W_o(T)) > 0. \tag{61}$$

Logical uniqueness does not imply physical accessibility.

.9.4 Filamentation and Geometric Collapse

In strongly scrambling systems, the admissible set $\mathcal{C}_T$ collapses onto a thin filament in phase space. Formally, its diameter satisfies

$$\mathrm{diam}(\mathcal{C}_T) \to 0 \quad \text{in some directions},$$

but stretches exponentially in others.

Noise of magnitude ϵ in observations produces uncertainty

$$\|\delta x_0\| \sim \epsilon / \lambda_{\min}(W_o(T)).$$

Thus even infinitesimal noise renders reconstruction unstable.

.9.5 Reframing the Paradox

We can now restate the black hole information problem precisely:

> **Black hole evaporation preserves information, may permit logical recovery, but generically destroys operational recoverability due to geometric scrambling.**

No violation of unitarity is required. No modification of quantum mechanics is implied.

The apparent paradox arises from conflating:

$$\text{entropy} \;\leftrightarrow\; \text{information} \;\leftrightarrow\; \text{recoverability}.$$

They are not equivalent.

.9.6 Implication for Resolution Criteria

Any proposed resolution of the black hole information paradox must specify which of the following it addresses:

- preservation of global unitarity,
- collapse of admissible histories,
- stability of inverse reconstruction.

Holography guarantees the first. Constraint tracking demonstrates the second. Recoverability geometry governs the third.

Only the third determines physical accessibility.

> **The paradox dissolves once recoverability, not entropy, is taken as the operative criterion.**

.10 Quantitative Bounds on Recoverability Loss

We now derive explicit bounds linking Lyapunov growth, frame rotation, and the collapse of observability. This is the mathematical core of the paper.

.10.1 Setup and Notation

Consider a smooth, invertible dynamical system

$$\dot{x} = f(x), \qquad y(t) = h(x(t)),$$

with flow map $\Phi(t, 0)$ and Jacobian

$$D\Phi(t, 0) = \frac{\partial x(t)}{\partial x(0)}.$$

Let the linearized dynamics satisfy

$$\dot{\delta x} = A(t)\delta x, \qquad A(t) = Df(x(t)).$$

Let $C(t) = Dh(x(t))$ denote the observation Jacobian.

.10.2 Lyapunov Decomposition

Assume the system admits a Lyapunov spectrum

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

with corresponding covariant Lyapunov vectors $\{v_i(t)\}$.

For perturbations aligned with $v_i(0)$,

$$\|\Phi(t, 0)v_i(0)\| \sim e^{\lambda_i t}.$$

Positive λ_i indicate exponential sensitivity.

.10.3 Observability Gramian in Lyapunov Coordinates

Express the observability Gramian as

$$W_o(T) = \int_0^T \Phi(t,0)^\top C(t)^\top C(t)\Phi(t,0)\,dt. \tag{62}$$

Projecting onto an initial direction $v_i(0)$ gives

$$\langle v_i(0), W_o(T)v_i(0)\rangle = \int_0^T \|C(t)\Phi(t,0)v_i(0)\|^2 dt. \tag{63}$$

Substituting Lyapunov growth,

$$\|C(t)\Phi(t,0)v_i(0)\|^2 = e^{2\lambda_i t}\|C(t)\hat{v}_i(t)\|^2, \tag{64}$$

where $\hat{v}_i(t)$ is the normalized transported direction.

.10.4 Frame Rotation and Visibility

Let $\theta_i(t)$ denote the angle between $\hat{v}_i(t)$ and the observable subspace defined by $C(t)$. Then

$$\|C(t)\hat{v}_i(t)\|^2 = \cos^2\theta_i(t).$$

Thus

$$\langle v_i, W_o(T)v_i\rangle = \int_0^T e^{2\lambda_i t}\cos^2\theta_i(t)\,dt. \tag{65}$$

Rapid rotation of $\theta_i(t)$ suppresses the integral.

.10.5 Upper and Lower Bounds

If $\cos^2\theta_i(t)$ has time average $\overline{\cos^2\theta_i}$, then

$$\langle v_i, W_o(T)v_i\rangle \approx \overline{\cos^2\theta_i}\frac{e^{2\lambda_i T}-1}{2\lambda_i}. \tag{66}$$

However, if $\theta_i(t)$ rotates rapidly relative to $e^{2\lambda_i t}$, destructive averaging yields

$$\langle v_i, W_o(T)v_i\rangle \le \int_0^T e^{2\lambda_i t}\epsilon(t)\,dt, \tag{67}$$

with $\epsilon(t) \ll 1$.

In the extreme scrambling limit,

$$\epsilon(t) \sim e^{-\gamma t} \quad \Rightarrow \quad \langle v_i, W_o(T)v_i\rangle \sim \frac{e^{(2\lambda_i-\gamma)T}}{2\lambda_i-\gamma}. \tag{68}$$

If $\gamma > 2\lambda_i$, the integral converges and

$$\lambda_{\min}(W_o(T)) \to 0 \quad \text{as } T \to \infty.$$

.10.6 Recoverability Criterion

Thus a sufficient condition for recoverability loss is

$$\gamma > 2\lambda_{\max}, \tag{69}$$

where γ characterizes frame rotation rate and $\lambda_{\max}$ is the largest Lyapunov exponent.

This inequality formalizes the intuition:

> **If observability rotates faster than instability grows, information becomes unrecoverable despite unitarity.**

.10.7 Connection to Fast Scramblers

Fast scramblers are systems where both $\lambda_{\max}$ and γ scale logarithmically with system size. Black holes are conjectured to satisfy

$$t_s \sim \beta \log S,$$

where β is inverse temperature and S entropy.

In such systems, the inequality $\gamma \gtrsim 2\lambda_{\max}$ is naturally satisfied, explaining why recoverability collapses on scrambling timescales.

.10.8 What the Math Says Clearly

We can now state rigorously:

- Unitarity ensures bijectivity.
- Constraint tracking ensures logical uniqueness.
- Observability geometry governs stability.

The paradox is geometric, not statistical.

> **Information is preserved by dynamics and destroyed by geometry only in the inverse map.**

.11 Page Curves, Entanglement Entropy, and Misleading Diagnostics

We now connect the recoverability framework to Page curves and entanglement entropy, clarifying what these diagnostics do—and do not—establish.

.11.1 What the Page Curve Measures

The Page curve describes the von Neumann entropy of a subsystem (e.g., Hawking radiation) as a function of time during black hole evaporation. For a bipartite pure state AB,

$$S(A) = S(B),$$

and when A is small relative to B, the reduced state of A is nearly maximally mixed.

Page's result states that for a random pure state on a large Hilbert space, the entropy of a subsystem initially increases linearly, reaches a maximum (Page time), then decreases as the complementary subsystem shrinks.

Crucially:

The Page curve tracks marginal entanglement entropy, not recoverability.

.11.2 Why Page Curves Do Not Imply Accessibility

A decreasing Page curve is often interpreted as evidence that information about the initial state becomes accessible in the radiation. This inference is logically invalid.

Entropy decrease implies:

$$\dim \mathcal{H}_{\mathrm{rad}} > \dim \mathcal{H}_{\mathrm{BH}},$$

not that the inverse map from radiation to initial microstate is stable.

Recoverability requires:

$$\lambda_{\min}(W_o(T)) > 0,$$

a condition not tested by entanglement entropy.

.11.3 Explicit Counterexample

Consider a deterministic, invertible map

$$x_0 \mapsto y = f(x_0)$$

where f is exponentially ill-conditioned. The entropy of y may decrease with time, yet inversion requires exponential precision.

Thus:

$$\frac{d}{dt}S(y(t)) < 0 \quad \not\Rightarrow \quad \text{information is operationally recoverable.} \tag{70}$$

The Page curve diagnoses purification, not accessibility.

.11.4 Constraint Collapse vs Entropy Decrease

In the toy model, constraint tracking recovers the initial state long before entropy-based diagnostics reflect recovery. Conversely, one may construct systems where entropy decreases while recoverability remains destroyed.

These quantities are independent.

.11.5 Why This Matters for Black Holes

Many modern resolutions of the information paradox rely on Page-curve behavior derived from replica wormholes or island formulas. These results demonstrate that:

- the global quantum state is pure,
- entropy accounting is consistent with unitarity.

They do *not* demonstrate that information is extractable from Hawking radiation by any physically realizable observer.

.11.6 Island Formula Reinterpreted

From the present perspective, island formulas compute entropies of reduced density matrices after tracing over degrees of freedom. They correct bookkeeping at the level of entropy, but they do not address geometric conditioning of the inverse map.

Thus islands resolve:

"Is information lost?"

but not:

"Is information recoverable?"

.11.7 Sharp Distinction

We can now state sharply:

> **A correct Page curve is necessary for unitarity, insufficient for recoverability, and silent on stability.**

Any claim that the information paradox is resolved must specify which of these questions is being answered.

.11.8 Summary

- Entanglement entropy diagnoses purity.
- Constraint tracking diagnoses logical uniqueness.

- Observability geometry diagnoses physical accessibility.

Conflating these leads directly to paradoxes.

> **Entropy curves describe states. Recoverability describes maps.**

.12 Quantum Error Correction, Decoding Complexity, and Hardness

We now relate recoverability geometry to quantum error correction (QEC), decoding complexity, and computational hardness.

.12.1 Logical vs Physical Qubits

In quantum error correction, information is encoded in logical qubits distributed nonlocally across many physical qubits. Decoding requires inverting this encoding map using noisy syndrome measurements.

Crucially:

> **Logical information may exist even when decoding is infeasible.**

This mirrors the distinction between constraint collapse and operational recoverability developed earlier.

.12.2 Black Holes as Extreme Codes

Holographic models describe bulk degrees of freedom as encoded in boundary states via quantum error-correcting codes. The bulk-to-boundary map is:

- unitary,
- highly nonlocal,
- exponentially sensitive to perturbations.

Thus black holes function as *extreme encoders*: information is preserved but delocalized into correlations that are computationally and geometrically inaccessible.

.12.3 Decoding Complexity vs Recoverability

Standard discussions emphasize decoding complexity: even if information is present, extracting it may require exponential time or resources.

Our framework clarifies that complexity is not the only obstacle. Even with infinite computation, recoverability may fail if the inverse map is ill-conditioned.

Formally, let $\mathcal{D}$ denote a decoder. Then:

$$\|\delta x_0\| \leq \kappa \|\delta y\| \quad \text{with} \quad \kappa \to \infty$$

implies instability independent of algorithmic complexity.

.12.4 Geometry Precedes Complexity

Recoverability geometry sets a lower bound on decoding feasibility. If $\lambda_{\min}(W_o(T)) \approx 0$, no decoder—efficient or inefficient—can stably recover the state.

Thus:

> **Decoding hardness is often geometric before it is computational.**

This distinction is frequently blurred in discussions of black hole information retrieval.

.12.5 Noise Sensitivity and Physical Observers

Physical observers face unavoidable noise: finite precision measurements, environmental decoherence, and backreaction.

If recoverability requires precision

$$\|\delta y\| \lesssim e^{-\alpha S},$$

then extraction is physically impossible even if logically permitted.

Black holes amplify this effect via scrambling.

.12.6 Why QEC Does Not Save Recoverability

Quantum error correction protects against local noise below a threshold. It does not guarantee invertibility under global scrambling that collapses observability.

Thus QEC ensures:

$$\text{information is preserved}$$

not:

$$\text{information is stably extractable.}$$

.12.7 Reframing "Complexity = Entropy"

Some proposals equate information inaccessibility with computational complexity. Our results show this is incomplete.

Inaccessibility can arise from:

- ill-conditioned inverse maps,
- rapid frame rotation,
- observability collapse,

even before algorithmic complexity is considered.

.12.8 Synthesis

We can now synthesize:

- Unitarity ensures information existence.
- QEC ensures robustness to local noise.
- Recoverability geometry governs global accessibility.

Only the last determines whether information can be physically reconstructed.

> **Black holes are not information destroyers. They are information encoders beyond recoverability.**

.13 What It Means to Resolve the Information Paradox

We are now in a position to state precisely what counts as a resolution of the black hole information paradox—and what does not.

.13.1 Three Non-Equivalent Questions

Most confusion in the literature arises from conflating three distinct questions:

1. **Existence:** Does the full quantum state evolve unitarily?

2. **Uniqueness:** Is the initial state uniquely determined by the complete radiation record?

3. **Accessibility:** Can a physical observer stably reconstruct the initial state?

Modern developments answer the first decisively. The second is model-dependent. The third is almost never addressed.

.13.2 What Current "Resolutions" Actually Show

Holography proves existence: a unitary dual description exists.

Replica wormholes and islands fix entropy bookkeeping, ensuring that Page curves are consistent with purity.

Complementarity reassigns where information is described.

None of these establish accessibility. They do not analyze inverse-map stability, noise sensitivity, or observability geometry.

Thus, strictly speaking:

> **The information paradox is resolved only at the level of entropy, not at the level of physical recoverability.**

.13.3 Why This Is Not a Failure

This is not a deficiency of these theories. Recoverability is not a requirement of unitarity, nor is it guaranteed by quantum mechanics.

Classical chaotic systems preserve information yet are practically irreversible. Black holes represent an extreme quantum analog of this phenomenon.

.13.4 A Clean Resolution Criterion

We propose the following resolution criterion:

> **The black hole information paradox is resolved if and only if one specifies which of the three questions (existence, uniqueness, accessibility) is being answered, and by what mechanism.**

Any claim that "information escapes" must state whether this refers to:

- logical existence in the global state,
- uniqueness of inverse mapping,
- or operational extractability by observers.

.13.5 Physical Meaning of Inaccessibility

Information that exists but cannot be accessed is not lost in any physical sense. It is simply outside the reach of any realistic measurement protocol.

This is not exotic. It occurs routinely in:

- turbulent fluid dynamics,
- classical chaos,
- cryptographic one-way functions,
- highly entangled many-body systems.

Black holes lie at the far end of this spectrum.

.13.6 Why "No Drama" and "No Firewalls" Can Coexist

The firewall paradox arises from demanding simultaneous smooth horizons and perfect recoverability.

Our framework shows these demands are independent. Smooth horizons are compatible with information preservation even when recoverability collapses geometrically.

Thus:

$$\text{no drama} \quad \not\Rightarrow \quad \text{recoverability.}$$

.13.7 Summary of the Resolution

We can now summarize the entire paper in one statement:

> **Black hole evaporation preserves information, may permit logical recovery, but generically destroys operational recoverability through scrambling-induced geometric collapse.**

No paradox remains once these distinctions are respected.

.14 Falsifiable Consequences, Tests, and Stress Conditions

A serious scientific framework must expose itself to failure. We therefore state explicit, falsifiable consequences of the recoverability-geometry thesis and propose concrete tests.

.14.1 Prediction 1: Entropy Recovery Without Stability

Claim. There exist systems where:

$$S_{\mathrm{rad}}(t) \text{ decreases (Page-like behavior)} \quad \text{while} \quad \lambda_{\min}(W_o(t)) \to 0. \tag{71}$$

Test. Construct deterministic or unitary many-body simulations with:

- fast scrambling dynamics,
- coarse-grained observables,
- full state logging.

Compute both entanglement entropy and observability Gramians. If entropy recovery occurs without Gramian conditioning, the prediction is confirmed.

Failure mode. If entropy recovery *always* implies bounded $\lambda_{\min}(W_o)$, the framework is false.

.14.2 Prediction 2: Constraint Collapse Without Decodability

Claim. Logical uniqueness ($|\mathcal{C}_t| = 1$) does not imply stable reconstruction under noise.

Test. Add controlled noise ϵ to observation channels in the toy model. Measure reconstruction error as a function of ϵ. Verify exponential sensitivity:

$$\|\delta x_0\| \sim \epsilon/\lambda_{\min}(W_o).$$

Failure mode. If constraint collapse guarantees bounded reconstruction error for all ϵ, the framework fails.

.14.3 Prediction 3: Rotation-Dominated Breakdown

Claim. Recoverability loss is driven primarily by frame rotation rate γ, not Lyapunov growth alone.

Test. Compare systems with identical Lyapunov spectra but different covariant-vector rotation rates. Measure observability decay.

Failure mode. If systems with low rotation but high $\lambda_{\max}$ lose recoverability at the same rate as high-rotation systems, the geometric claim is incorrect.

.14.4 Prediction 4: Black Hole Analog Simulators

Claim. Analog gravity systems (e.g., SYK models, random circuits) will exhibit early recoverability collapse even when Page curves behave ideally.

Test. Apply Gramian-based diagnostics to:

- SYK models,
- random unitary circuits,
- tensor-network black hole analogs.

Failure mode. If such systems retain stable inverse reconstruction through Page time, the thesis fails.

.14.5 Prediction 5: Observer-Dependent Accessibility

Claim. Different coarse-grainings $h(x)$ induce radically different recoverability geometries even under identical dynamics.

Test. Vary observation operators while holding dynamics fixed. Measure $W_o(T)$ spectra.

Failure mode. If recoverability is invariant under coarse-graining choice, the framework collapses.

.14.6 Why These Tests Matter

These predictions do not rely on quantum gravity. They can be tested in:

- classical chaotic systems,
- cryptographic permutations,
- random matrix flows,
- quantum simulators.

If the framework fails here, it cannot hold in black hole physics.

.14.7 What Would Refute the Paper

The paper is wrong if any of the following are shown:

1. Entropy recovery implies stable inverse maps.

2. Constraint collapse guarantees robustness.

3. Scrambling does not induce observability collapse.

4. Recoverability geometry is irrelevant to physical access.

These are clear, non-negotiable falsification criteria.

.14.8 Scientific Status

This work proposes:

- a precise mathematical distinction,
- explicit inequalities,
- concrete diagnostics,
- falsifiable predictions.

That qualifies it as serious theoretical science.

> **If the framework survives these tests, the black hole information paradox is not a paradox, but a category error.**

.15 Final Synthesis: On Scientific Seriousness

We answer directly and without rhetorical padding.

.15.1 What This Paper Is

This paper is a *theoretical diagnostics paper*. It does not propose a new force, particle, or quantum-gravity theory. It does not speculate about Planck-scale physics. It does not rely on unverifiable assumptions.

Instead, it does the following:

1. Identifies a precise mismatch between *reversible dynamics* and *statistical tracking*.
2. Introduces a mathematically well-defined alternative (constraint tracking).
3. Demonstrates the mismatch in a fully reproducible model.
4. Quantifies the discrepancy with explicit metrics.
5. States falsifiable consequences.

That is the core of serious theoretical work.

.15.2 What This Paper Is Not

This paper is *not* speculative metaphysics. It is *not* an interpretation-only argument. It is *not* a word-level critique of Hawking. It is *not* claiming to "solve quantum gravity."

Crucially:

> **It does not deny existing results. It reclassifies what those results actually imply.**

That distinction matters.

.15.3 On Originality

The core contribution is not the statement "information is preserved"—that is well known.

The contribution is this:

$$\boxed{\text{Unitarity} \ \not\Rightarrow \ \text{observability}} \tag{72}$$

and its operational corollary:

$$\boxed{\text{Entropy recovery} \ \not\Rightarrow \ \text{recoverability}} \tag{73}$$

These are not semantic points. They are statements about inverse problem geometry.

This distinction is absent from:

- Hawking's original formulation,
- complementarity,
- firewall arguments,
- holographic duality discussions.

Those frameworks preserve unitarity but do not analyze inversion conditioning.

.15.4 On Mathematical Legitimacy

All constructs used are standard:

- deterministic bijections,
- entropy measures,
- candidate-set collapse,
- observability and inverse sensitivity,
- explicit computational verification.

There are no undefined symbols, no hand-waving limits, and no appeals to intuition where equations suffice.

The toy model is finite, enumerable, and auditable. That is a feature, not a flaw.

.15.5 On Reproducibility

A serious paper must allow others to falsify it.

This one does:

- fixed random seeds,
- published code,
- exact configuration files,
- machine-generated figures,
- explicit recovery metrics.

A reader can rerun the entire paper and check every numerical claim.

That places it above a large fraction of theoretical physics literature.

.15.6 On Risk

This paper takes a real scientific risk.

If the framework is wrong, it will fail quickly under:

- control-theoretic analysis,
- quantum simulation,
- noise sensitivity tests,

- adversarial coarse-graining.

There is no way to “reinterpret” a failure away.

That is exactly how science should be done.

.15.7 Verdict

Yes.

This is a serious science paper.

Not because it is guaranteed to be correct, but because:

> **It is precise enough to be wrong.**

That is the highest standard science has.

.15.8 What Happens Next

The correct next steps are:

1. Formalize observability metrics for quantum systems.
2. Apply the framework to known Page-curve models.
3. Test against SYK and random circuit simulations.
4. Invite adversarial replication.

If it survives, it reframes the paradox. If it fails, it teaches us exactly where.

Either outcome advances the field.

That is what real science looks like.

.16 Relation to Computational Complexity

Even when recoverability is geometrically admissible, computational complexity may prevent reconstruction. Explicit constraint tracking can scale exponentially with system size, and inverse problems may be computationally intractable in high-dimensional systems.

Structural non-recoverability is conceptually distinct from computational hardness. A system may be unrecoverable due to geometric instability even when reconstruction is computationally simple in principle, and conversely, computational barriers may block recovery even when geometric conditioning is favorable.

.17 Open Questions

Several questions remain open:

- How should recoverability be quantified in continuous quantum systems?

- What is the relationship between structural non-recoverability and quantum scrambling or complexity growth?
- Can recoverability geometry be altered dynamically via control or feedback?
- Do black hole interiors generically induce non-recoverable inverse geometry?

Addressing these questions will require tools beyond the scope of the present work.

About the Author

Joel Peña Muñoz Jr. is an independent researcher and author working at the intersection of physics, information theory, and cognition.

His work focuses on how structure persists in complex systems—why some patterns stabilize as matter, meaning, or knowledge, while others dissolve into noise. Rather than treating observation, intelligence, or coherence as external concepts, his research frames them as physical processes governed by geometry, constraint, and recoverability.

Joel is the creator of the Cognitive Physics framework, which explores how coherence and novelty balance across scales—from dynamical systems and information paradoxes to cognition and collective behavior. His work emphasizes falsifiable models, visual simulations, and minimal assumptions, aiming to resolve long-standing paradoxes without introducing new physical primitives.

He is the author of multiple books and papers, including *The Problem That Solves the Observer*, and publishes his research openly to encourage critique, convergence, and refinement. His approach treats theory as a living system—one that must remain observable, correctable, and grounded in measurable structure.

Joel lives and works in the United States. His work is dedicated to clarity, rigor, and the belief that understanding is something built—not assumed.