Physics C–M–1

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1 Introduction

Remember when we told you F=ma? Well, we lied. Newton's laws actually point to a change of the momentum vector.

$$\sum_i F = \frac{\Delta p_i}{\Delta t} = \frac{\Delta(mv)}{\Delta t} = m\frac{\Delta v}{\Delta t} = ma : i = 1, 2, 3$$

Is a more honest version if and ONLY IF mass isn't dynamical [1]. This is how we get a change in momentum from a force over time. For notation sake I am doing it the way Dirac notates his "canonical coordinates", "i" is simply one of our three basis axis such as x, y, z, or r, ρ, θ, etc.
The general form would be:

$$\sum_i F_i = \frac{\Delta m}{\Delta t} v + m\frac{\Delta v}{\Delta t}$$

Now let's try to figure out how we can get the aforementioned change in P. Our first equation carries the following implication:

$$\Delta p_i = \sum_i F_i \Delta t$$

2 But these are approximate!

Let's think about this analytically. Consider any graph that is continuous and has a slope at every point. What if we take that slope and follow it for a small step in x and add the difference to our point's y after that step in distance. Now we arrive on an approximation fairly close, we notice also that as our step gets bigger, so does our error of our approximation.
Consider when we make the step size very small, we get a near exact curve

[1] Dynamical means no change with time, also mass technically changes relativistically with speed but we aren't going anywhere close to the speed of light — yet.

identical to our initial one. [2]
Traditionally a small slope is denoted as:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

This is our *exact* slope. Now how do we get our "exact sum". If dy is infinitely small, then we must add an infinite amount of them.

$$\lim_{\Delta x \to 0} \sum_{i}^{\infty} \Delta x = x$$

We however don't like writing the limit. So instead we write it with a cursive S for "summa":

$$\lim_{\Delta x \to 0} \sum_{i}^{\infty} \Delta x = \int dx = x$$

This means if we apply it to Newton's laws:

$$\int F dt = \Delta p \text{ where delta is the total change in p}$$

3 Investigate the math:

- How would you justify canceling out the Δx in the sum?
- How do you expect the summa to act for a changing $f(x)$?
- Try taking the limit slope formula, expanding, and try to find formula for certain functions like $x^2, sin(x), x^n, e^x$....
  Hint[3]: $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{x + \Delta x - x}$
- Play with the sum, and more precisely try the following formula for any function.

  $$\int_0^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x + i\Delta x)\Delta x$$

  Where $\Delta x = \frac{b-a}{n}$. This is where a and b represent endpoints. [4]
- How would you move the lower bound of our infinite sum?

[2] This analogy is very similar to that of Euler's Method
[3] Skill: Differentiation
[4] This is a Riemann sum, the formal representation of an integral. This is assumed to be learned alongside in a calculus class. See 3Blue1Brown or other resources to learn.

4 Investigate the physics

The Humble Point Mass Box Suppose we have a box with known mass of 5 Kilograms and a velocity of 20 meters per second forwards. We decide to kick it! Applying a force equal to $5x^2$ over 2 seconds in the same direction as motion.
1. What do you expect to happen?

2. Which quantity will change the most, explain without numbers but rather through your knowledge of kinetic energy and momentum as functions of velocity.

3. Calculate the total change in momentum using the formula earlier which we derived, the bounds will be from 0 to the finish time of the force. If you have trouble, ask a peer for help.

Challenge Question [5]: change the function of force to e^{-x^2} and calculate the total contribution to the change of momentum over all time (0 to infinity).

Hint: this is the gaussian integral in the quarter positive plane in statistics!

5 Further Inquiry and Tips:

- Ask questions, and most importantly try to answer them yourself!
- Ask peers things, there is no shame.
- The d in dt is just a very tiny Δ so we made it a d.
- try a situation in multiple dimensions. If you are comfortable with vectors, try making vector equations. If not then break up into components, whether they be xy, $r\theta$, or even tangential and normal (this can be a very smart thing to do!).
- If you are struggling yet progressing, then you are learning!
- I highly suggest you start trying to see vectors for your forces and velocity, because in a couple months they will be the only thing you and I see.

[5] You are not expected to know how to do this, it is just an opportunity to learn a method for integration. If this is your first time with calculus do not worry about this question!