

# An Exploration of Order Theoretic Analysis

Cohen Dickey

November 2025

**Definition.** Path Ordering generated by  $r$ .

Let  $r : T \subset \mathbb{R} \rightarrow \mathbb{R}^n$  be a path on  $\mathbb{R}^n$  where  $r$  is injective. Let  $y_1, y_2 \in r(T)$  such that  $y_1 \neq y_2$ , then  $y_1 \leq_r y_2 \iff r^{-1}(y_1) \leq r^{-1}(y_2)$ . If  $(y_1 \notin r(T)) \vee (y_2 \notin r(T))$  then  $y_1$  is  $r$  incomparable to  $y_2$ . Let  $y_1 = y_2$  then  $y_1 \leq_r y_2$ . We will refer to this ordering as the path ordering of  $r$  on  $\mathbb{R}^n$ , and thus will be denoted  $\leq_r$ .

**Theorem.**  $\leq_r$  is a total ordering on the subposet  $(r(T), \leq_r)$ .

**Proof.** For any  $a, b \in r(T)$ , either  $r^{-1}(a) \leq r^{-1}(b)$  or  $r^{-1}(b) \leq r^{-1}(a)$  in  $\mathbb{R}$ , so either  $a \leq_r b$  or  $b \leq_r a$ .

**Remark:** Ensure there is no confusion between total and well ordering. As usually the uncountable nature makes a dream of well orderliness infeasible.

**Theorem.**  $r$  embeds  $(T, \leq)$  into  $(r(T), \leq_r)$ . That is,  $r : (T, \leq) \hookrightarrow (r(T), \leq_r)$

**Proof.** Let  $\alpha, \beta \in T$ . By definition of  $\leq_r$ ,  
 $r(\alpha) \leq_r r(\beta) \iff r^{-1}(r(\alpha)) \leq r^{-1}(r(\beta)) \iff \alpha \leq \beta$ , where the last equivalence uses  $r^{-1}(r(\alpha)) = \alpha$ , since  $r$  is injective. Thus,  $r$  preserves the order and is injective, so it is an order embedding.

**Note:** This is equivalent to saying  $(T, \leq) \cong (r(T), \leq_r)$

**Where From Here?** My advisor suggested that I explore how transformations of the path affect the lattice posets generated by a open set topology which encodes characteristics of the path. This is what I am currently working on.

## 1 Path Generated Lattices

**Goal:** Given the advice previously said, I need to find a lattice characteristic of the path we have in question. We will start by creating a characteristic topology on  $\mathbb{R}^n$

**Failure 1:**

$$\tau = \{A \mid (A \cap r(T) \neq \emptyset) \vee (A = \emptyset)\}$$

Need:

1.  $\emptyset, \mathbb{R}^n \in \tau$
2.  $(\chi, \xi \subseteq \tau) \implies ((\chi \cup \xi) \cap r(T) \neq \emptyset) \vee (\chi \cup \xi = \emptyset)$
3.  $\chi, \xi \subseteq \tau$ , then if  $\chi \cap \xi = \emptyset$  then the intersection is in  $\tau$ , or if  $\chi \cap \xi \neq \emptyset$  we need  $(\chi \cap \xi) \in \tau$ .

Condition 3 fails because consider two sets which contain pairwise disjoint sections of  $r(T)$  yet have a nonempty intersection which contains no  $r(T)$ , then the intersection is not satisfactory.

**How does the work in progress look?** Check the files in the folder, all with AEOTA in it will be relevant.