An Exploration of Order Theoretic Analysis

Cohen Dickey

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Definition. Path Ordering generated by r.
Let $r : T \subset \mathbb{R} \to \mathbb{R}^n$ be a path on $\mathbb{R}^n$ where r is injective. Let $y_1, y_2 \in r(T)$ such that $y_1 \neq y_2$, then $y_1 \leq_r y_2 \iff r^{-1}(y_1) \leq r^{-1}(y_2)$. If $(y_1 \notin r(T)) \vee (y_2 \notin r(T))$ then y_1 is r incomparable to y_2. Let $y_1 = y_2$ then $y_1 \leq_r y_2$. We will refer to this ordering as the path ordering of r on $\mathbb{R}^n$, and thus will be denoted $\leq_r$.

Theorem. $\leq_r$ is a total ordering on the subposet $(r(T), \leq_r)$.

Proof. For any $a, b \in r(T)$, either $r^{-1}(a) \leq r^{-1}(b)$ or $r^{-1}(b) \leq r^{-1}(a)$ in $\mathbb{R}$, so either $a \leq_r b$ or $b \leq_r a$.

Remark: Ensure there is no confusion between total and well ordering. As usually the uncountable nature makes a dream of well orderliness infeasible.

Theorem. r embeds $(T, \leq)$ into $(r(T), \leq_r)$. That is, $r : (T, \leq) \hookrightarrow (r(T), \leq_r)$

Proof. Let $\alpha, \beta \in T$. By definition of $\leq_r$,
$r(\alpha) \leq_r r(\beta) \iff r^{-1}(r(\alpha)) \leq r^{-1}(r(\beta)) \iff \alpha \leq \beta$, where the last equivalence uses $r^{-1}(r(\alpha)) = \alpha$, since r is injective. Thus, r preserves the order and is injective, so it is an order embedding.

Note: This is equivalent to saying $(T, \leq) \cong (r(T), \leq_r)$

Where From Here? My advisor suggested that I explore how transformations of the path affect the lattice posets generated by a open set topology which encodes characteristics of the path. This is what I am currently working on.

1 Path Generated Lattices

Goal: Given the advice previously said, I need to find a lattice characteristic of the path we have in question. We will start by creating a characteristic topology on $\mathbb{R}^n$

Failure 1:

$$\tau = \{A \mid (A \cap r(T) \neq \emptyset) \vee (A = \emptyset)\}$$

Need:

1. $\emptyset, \mathbb{R}^n \in \tau$
2. $(\chi, \xi \subseteq \tau) \implies ((\chi \cup \xi) \cap r(T) \neq \emptyset) \vee (\chi \cup \xi = \emptyset)$
3. $\chi, \xi \subseteq \tau$, then if $\chi \cap \xi = \emptyset$ then the intersection is in τ, or if $\chi \cap \xi \neq \emptyset$ we need $(\chi \cap \xi) \in \tau$.

Condition 3 fails because consider two sets which contain pairwise disjoint sections of$r(T)$ yet have a nonempty intersection which contains no $r(T)$, then the intersection is not satisfactory.

Basis It is easier to use a basis to generate our topology.
Let $\mathcal{B} = \{U \subseteq r((a, b)) \mid (a, b) \subseteq T\}$, then we will define $\mathcal{P} = \{U \subseteq r(T) \mid \forall x \in U\ (\exists B \in \mathcal{B}(x \in B \wedge B \subset U))\}$. By Basis Generation Theorem, we know our basis $\mathcal{B}$ generates topology $\mathcal{P}$.

Lattices. The easiest lattice to make from a topology is well the topology under subset inclusion. Let $\mathfrak{P} = (\mathcal{P}, \subseteq)$. If it is not obvious this is a lattice, lets check off two requirements.

Theorem. $\mathfrak{P} = (\mathcal{P}, \subseteq)$ is a lattice.
First, consider $\{x, y\} \subseteq \mathcal{P}$, we want for there to exist a meet and a join for every set of this form. Recall, $\emptyset$ and $r(T)$ are both in $\mathcal{P}$. Moreover, the empty set is the trivial subset, and by definition everything in our basis is in our path. Thus $x \wedge y$ is either the empty set or an arbitrary element closer. Meanwhile $x \vee y$ is either $r(T)$ or something arbitrary and closer. We have proven that $\mathfrak{P}$ is a meet semi-lattice and a join semi-lattice, thus we know $\mathfrak{P}$ is a lattice.

How does the work in progress look? Check the files in the folder, all with AEOTA in it will be relevant.