An Exploration of Order Theoretic Analysis

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Definition. Path Ordering generated by r.
Let $r : T \subset \mathbb{R} \to \mathbb{R}^n$ be a path on $\mathbb{R}^n$ where r is injective. Let $y_1, y_2 \in r(T)$ such that $y_1 \neq y_2$, then $y_1 \leq_r y_2 \iff r^{-1}(y_1) \leq r^{-1}(y_2)$. If $(y_1 \notin r(T)) \vee (y_2 \notin r(T))$ then y_1 is r incomparable to y_2. Let $y_1 = y_2$ then $y_1 \leq_r y_2$. We will refer to this ordering as the path ordering of r on $\mathbb{R}^n$, and thus will be denoted $\leq_r$.

Theorem. $\leq_r$ is a total ordering on the subposet $(r(T), \leq_r)$.

Proof. For any $a, b \in r(T)$, either $r^{-1}(a) \leq r^{-1}(b)$ or $r^{-1}(b) \leq r^{-1}(a)$ in $\mathbb{R}$, so either $a \leq_r b$ or $b \leq_r a$.

Remark: Ensure there is no confusion between total and well ordering. As usually the uncountable nature makes a dream of well orderliness infeasible.

Theorem. r embeds $(T, \leq)$ into $(r(T), \leq_r)$. That is, $r : (T, \leq) \hookrightarrow (r(T), \leq_r)$

Proof. Let $\alpha, \beta \in T$. By definition of $\leq_r$,
$r(\alpha) \leq_r r(\beta) \iff r^{-1}(r(\alpha)) \leq r^{-1}(r(\beta)) \iff \alpha \leq \beta$, where the last equivalence uses $r^{-1}(r(\alpha)) = \alpha$, since r is injective. Thus, r preserves the order and is injective, so it is an order embedding.

Note: This is equivalent to saying $(T, \leq) \cong (r(T), \leq_r)$

Where From Here? My advisor suggested that I explore how transformations of the path affect the lattice posets generated by a open set topology which encodes characteristics of the path. This is what I am currently working on.

How does the work in progress look? Check the files in the folder, all with AEOTA in it will be relevant.