

An Exploration of Order Theoretic Analysis

Cohen Dickey

November 2025

Definition. Path Ordering generated by r .

Let $r : T \subset \mathbb{R} \rightarrow \mathbb{R}^n$ be a path on \mathbb{R}^n where r is injective. Let $y_1, y_2 \in r(T)$ such that $y_1 \neq y_2$, then $y_1 \leq_r y_2 \iff r^{-1}(y_1) \leq r^{-1}(y_2)$. If $(y_1 \notin r(T)) \vee (y_2 \notin r(T))$ then y_1 is r incomparable to y_2 . Let $y_1 = y_2$ then $y_1 \leq_r y_2$. We will refer to this ordering as the path ordering of r on \mathbb{R}^n , and thus will be denoted \leq_r .

Theorem. \leq_r is a total ordering on the subposet $(r(T), \leq_r)$.

Proof. For any $a, b \in r(T)$, either $r^{-1}(a) \leq r^{-1}(b)$ or $r^{-1}(b) \leq r^{-1}(a)$ in \mathbb{R} , so either $a \leq_r b$ or $b \leq_r a$.

Remark: Ensure there is no confusion between total and well ordering. As usually the uncountable nature makes a dream of well orderliness infeasible.

Theorem. r embeds (T, \leq) into $(r(T), \leq_r)$. That is, $r : (T, \leq) \hookrightarrow (r(T), \leq_r)$

Proof. Let $\alpha, \beta \in T$. By definition of \leq_r , $r(\alpha) \leq_r r(\beta) \iff r^{-1}(r(\alpha)) \leq r^{-1}(r(\beta)) \iff \alpha \leq \beta$, where the last equivalence uses $r^{-1}(r(\alpha)) = \alpha$, since r is injective. Thus, r preserves the order and is injective, so it is an order embedding.

Note: This is equivalent to saying $(T, \leq) \cong (r(T), \leq_r)$

Where From Here? My advisor suggested that I explore how transformations of the path affect the lattice posets generated by a open set topology which encodes characteristics of the path. This is what I am currently working on.

1 Path Generated Lattices

Goal: Given the advice previously said, I need to find a lattice characteristic of the path we have in question. We will start by creating a characteristic topology on \mathbb{R}^n

Failure 1:

$$\tau = \{A \mid (A \cap r(T) \neq \emptyset) \vee (A = \emptyset)\}$$

Need:

1. $\emptyset, \mathbb{R}^n \in \tau$
2. $(\chi, \xi \subseteq \tau) \implies ((\chi \cup \xi) \cap r(T) \neq \emptyset) \vee (\chi \cup \xi = \emptyset)$
3. $\chi, \xi \subseteq \tau$, then if $\chi \cap \xi = \emptyset$ then the intersection is in τ , or if $\chi \cap \xi \neq \emptyset$ we need $(\chi \cap \xi) \in \tau$.

Condition 3 fails because consider two sets which contain pairwise disjoint sections of $r(T)$ yet have a nonempty intersection which contains no $r(T)$, then the intersection is not satisfactory.

How does the work in progress look? Check the files in the folder, all with AEOTA in it will be relevant.