Towards the Study of the Metric Properties of the New Fuzzy Spheres of Fiore and Pisacane

Universidad de los Andes

Sebastian Camilo Puerto Galindo

Thesis submitted for the degree of Physicist

Advisor: Prof. Andres Fernando Reyes-Lega Ph.D.

January 19, 2021

Para mi Chia,
gracias por mostrarme que el universo rebosa colores.

Abstract

Noncommutative spaces are strong candidates for the description of the underlying quantum structure of spacetime, and in this document we can see them arise through the introduction of energy cut-offs in a quantum theory. The purpose of this document is to develop a basic understanding of the geometry of noncommutative spaces that may arise in physically plausible situations such as those derived from the introduction of such cut-offs. In order to do this, we first study the traditional fuzzy sphere of Madore [1] and its metric properties [2]. We then embark on the research of the geometry of the fuzzy spheres recently proposed by Fiore and Pisacane [3] through energy cut-offs by studying their definition and equivalent characterizations, as well as some systems of coherent states on them [4, 5], which play the role of points on these noncommutative spheres, and which enable the use of these spaces in areas outside of geometry. Consequently, we provide the mathematical basis for the study of (yet to be constructed) spectral triples on these spaces, as well as the metric structure they will induce.

Resumen

Los espacios noconmutativos son fuertes candidatos para la descripción de la estructura cuántica subyacente del espaciotiempo, y en este documento los podemos ver surgir a través de la introducción de energías de corte en una teoría cuántica. El propósito de este documento es desarrollar un entendimiento básico de la geometría de los espacios noconmutativos que pueden surgir en situaciones físicamente plausibles como aquellas derivadas de la introducción de energías de corte. Para esto, primero estudiamos la tradicional esfera fuzzy de Madore [1] y sus propiedades métricas [2]. Luego nos embarcamos en la investigación de la geometría de las esferas fuzzy recientemente propuestas por Fiore y Pisacane [3] a través de la introducción de energías de corte, estudiando tanto sus definiciones y caracterizaciones equivalentes, como algunos sistemas de estados coherentes en ellas [5, 4], los cuales juegan el rol de puntos en estas esferas noconmutativas, y que habilitan el uso de estos espacios en áreas distintas a la geometría. Consecuentemente, proveemos las bases matemátiasc para el estudio de (aún por construirse) triplas espectrales en estos espacios, así como la estructura métrica que inducirán.

Contents

Chapter 1

Introduction

The introduction of energy cutoffs in Quantum Field Theory has allowed the regularization of ultraviolet divergences, serving as a first step toward the renormalization of these theories. Similarly, an energy cutoff is also expected in a sensible theory of quantum gravity, a cutoff on the local energy concentration associated to a lower bound on the localizability of the events, as suggested by Heisenberg's uncertainty principle. Doplicher, Fredenhagen and Roberts [6] formalized this argument, and proposed that this lower bound could be a result of *noncommutative coordinates*, suggesting that the underlying structure of spacetime might be that of a *noncommutative space*.

A particular type of noncommutative space that has found its way into varied areas of physics is that of a *fuzzy space*. These are in fact a sequence of finite dimensional noncommutative spaces that approximate better and a better a commutative space, while having the important property of preserving the continuous symmetries of the commutative space. They have found applications in quantum field theory, condensed matter physics, and cosmology, among others.

A noncommutative space is fundamentally an algebraic object, thus what a "point" of this space might be, or what is its "geometry", is not canonically defined. However, Connes [7] paved a way to give geometry to this spaces by showing an algebraic formulation of a traditional or "commutative" metric space that can be easily extended to algebraic or noncommutative ones. More precisely, he showed that the distance between points p and q in a metric space M, that admits a spin structure, has an algebraic formulation given by *Connes' distance formula*:

$$d(p,q) := sup_{f\in\mathcal{C}^\infty(M)}\{|\phi_p(f)-\phi_q(f)| \,:\, ||\not{D}, f|| \leq 1\}; \tag{1.1}$$

thus, the properties of these spaces are contained in their *canonical spectral triple* $(\mathcal{A} = C^\infty(M), \mathcal{H} = L^2(\Sigma M), \not{D} : \mathcal{H} \to \mathcal{H})$, where $\mathcal{A}$ is the commutative algebra of smooth functions on M, $\mathcal{H}$ is the Hilbert space of spinor fields on M, $\not{D}$ is called the *Dirac operator*, $\phi_x : \mathcal{A} \to \mathbb{C}$, for $x = p, q$, are the functionals on $\mathcal{A}$ such that $\phi_x(f) = f(x)$ for all $f \in \mathcal{A}$, and $||\cdot||$ denotes the operator norm on $\mathcal{H}$. Notice that each point $x \in M$ has an algebraic equivalent as a *state* ϕ_x, i.e. a linear functional $\phi_x : \mathcal{A} \to \mathbb{C}$, which is positive and of unit norm; furthermore, they are actually *pure states*, meaning that they can not be written as convex combinations of other elements of the convex set $\mathcal{S}(\mathcal{A})$ of all states (of the norm completion of $\mathcal{A}$); under the weak assumption that M is locally compact, we find that the purely algebraic object $\mathcal{S}(\mathcal{A})$ is in fact in a bijective correspondence with the set of points of M [8]. This construction illustrates the bijective mapping that exists between the collection of C^*-algebras with the collection of locally compact spaces, which is the subject of the commutative Gelfand-Naimark theorem [9], and which supports the idea of interpreting more general algebras, particularly C^*-algebras, as algebraic versions of a different kind of space that we call *noncommutative spaces*.

In order to give a notion of distance, of geometry, to noncommutative spaces, a straightforward generalization of Connes' distance formula (1.1) suffices if a good generalization to the canonical spectral triple is found (meaning that it possesses enough of the properties that enable the canonical spectral triple to define a geometry); such generalizations are called *spectral triples*. Given the noncommutative spaces that we will deal with in this document, we will content ourselves with only defining the following subfamily of spectral triples [2]: a *unital spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is composed of a complex separable Hilbert space $\mathcal{H}$, a complex associative involutive unital C^*-algebra $\mathcal{A}$ with a faithful unital $*$-representation $\mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H})$, and finally a self-adjoint (unbounded) operator $D : \mathcal{H} \to \mathcal{H}$ with compact resolvent, such that $[D, a] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{A}$. A spectral triple is called *even* if there is a *grading* or *chirality operator* $\gamma \in \mathcal{B}(\mathcal{H})$, i.e. satisfying $\gamma = \gamma^*$ and $\gamma^2 = 1$; it is called *real* if there is a *real structure* $J : \mathcal{H} \to \mathcal{H}$, i.e. an antilinear isometry such that $J^2 = \pm 1$, $JD = \pm DJ$, and $J\gamma = \pm\gamma J$ if the spectral triple is even (the correct sign depends on the KO-dimension of the triple);

these two are properties which make a spectral triple more closely resemble the canonical triples. Finally, having a spectral triple allows the following *generalized Connes' distance formula* between any two states $\omega, \omega' \in \mathcal{S}(\mathcal{A})$ to give a geometry to noncommutative spaces:

$$d_D(\omega, \omega') := sup_{a\in\mathcal{A}} \left\{|\omega(a) - \omega'(a)| \ : \ ||[D, a]||_{op} \leq 1\right\}. \tag{1.2}$$

This document has the purpose of providing some basic knowledge on the calculation of distances within noncommutative spaces, specially fuzzy spaces, in order to apply this understanding to the study of the metric properties of the new fuzzy spheres presented by Fiore and Pisacane in [3]. With this in mind, we first study in Chapter 2 the highly used fuzzy sphere of Madore [1] on which D'Andrea, et al. [2] introduce two related spectral triples derived from the canonical spectral triple of the 2-sphere. Then, in Chapter 3 we study the definition, via introduction of energy cut-offs, of the new fuzzy circle of Fiore and Pisacane whose limit is the 1-sphere, and which preserve the $O(2)$ symmetry of this commutative space. Finally, in Chapter 4 we review some systems of coherent states that will play the role of points between which we will want to calculate distances once a spectral triple has been proposed.

Chapter 2

Spectral Triples on the Fuzzy Sphere and their Geometry

In order to understand the formulation of spectral triples on a fuzzy space as well as their use to carry out distance calculations, in this chapter we review the study of the metric properties of the fuzzy sphere of Madore and Hoppe [1] developed by D'Andrea, et al. in [2].

Recall that the 2-*sphere*, denoted by S^2, is the set of points in $\mathbb{R}^3$ located at an euclidean distance of 1 unit from the origin. As a metric space and as a manifold it is rotationally invariant, that is, $SO(3)$-invariant. Since the group $SU(2)$ is the universal cover of the group of rotations $SO(3)$, we can translate this invariance to be a $SU(2)$-invariance. We will see that this invariance induces a decomposition of the canonical Dirac operator on S^2 that is compatible with the action of $SU(2)$. Truncating this $SU(2)$-decomposition will give rise to equivariant spectral triples on the fuzzy sphere of Madore and Hoppe, defined in Section 2.2, which inherit some of the important properties of the canonical spectral triple as seen in Section 2.3. A family of states studied in Section 2.4 called the *spin coherent states* have some properties, including a $SU(2)$-equivariant bijective correspondence with the points in S^2, that motivate us to think of them as each one of the as a fuzzy analog of the points of S^2, and which allow us to say that the convergence of the fuzzy sphere to S^2 occurs as a metric space, on which $SU(2)$ acts by "isometries".

2.1 The Canonical Spectral Triple of S^2

Although there are several ways to describe the canonical spectral triple on S^2, the one that will be useful later to define the spectral triples on the fuzzy sphere comes from understanding S^2 as the compact Riemannian symmetric space $S^2 \cong G/U$ of the compact semisimple Lie group $G = SU(2)$, where we take $U = SO(2)$ as the invariant subgroup under the complex conjugation involution.

Let G be a compact semisimple Lie group G with (real) Lie algebra $\mathfrak{g}$, where the semisiplicity means that the Killing form $K : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of $\mathfrak{g}$ is non-degenerate, and the compactness that K is negative-definite giving G a natural Riemannian manifold structure. For such a manifold G, and its symmetric spaces, the canonical Dirac operator can be seen to arise [10] from an algebraic element $\mathcal{D} \in U(\mathfrak{g}) \times U(\mathfrak{g})$, where $U(\mathfrak{g})$ is the universal enveloping complex algebra of the Lie algebra $\mathfrak{g}$, i.e. the largest unital, associative complex algebra containing $\mathfrak{g}$ where the Lie bracket coincides with the commutator in $U(\mathfrak{g})$; it is important to remark that the representations of $\mathfrak{g}$ are in a bijective correspondence with the modules over $U(\mathfrak{g})$. In the explicit case where $G = SU(2)$,

$$\mathcal{D} := 1 \otimes 1 + 2 \sum_{k=1}^{3} J^k \otimes J^k \in U(\mathfrak{su}(2)) \otimes U(\mathfrak{su}(2)) \tag{2.1}$$

where $J^k \in U(\mathfrak{su}(2)), k = 1, \ldots, 3$ are the basis of $\mathfrak{su}(2) \otimes \mathbb{C}$ such that

$$[J^i, J^j] = i\epsilon_{ijk} J^k \tag{2.2}$$

(e.g. $J^k = \frac{\sigma_k}{2} = \pi_{1/2}(J^k)$).

Notation 2.1.1. Let us denote by $\pi_j : U(\mathfrak{su}(2)) \to M_{2j+1}(\mathbb{C}) = End(V_j)$, with $V_j := \mathbb{C}^{2j+1}$, the spin $j \in \frac{\mathbb{N}}{2}$ representation of $\mathfrak{su}(2)$ where each element of the canonical basis $e_m \in \mathbb{C}^{2j+1}$, $m = -j, \ldots, j-1, j$ satisfies $\pi_j(\vec{J}^2)(e_m) = j(j+1)e_m$ and $\pi_j(J^3)(e_m) = me_m$, where $\vec{J}^2$ is notation for $(J^1)^2 + (J^2)^2 + (J^3)^2 \in U(\mathfrak{su}(2))$. Recall each of these representations is unitary, and induces a unitary representation of $SU(2)$.

Let $Y^l_m \in C^\infty(S^2)$, $l \in \mathbb{N}$, $m \in \{-l, \dots, l-1, l\}$ be the spherical harmonics, and recall that, for each fixed l, they are an orthonormal basis for the Hilbert spaces $\tilde{V}_l := span\{Y^l_m\}_{|m|\leq l}$ which are simply the spaces of homogeneous complex polynomials on the coordinate variables x^1, x^2, x^3 restricted to S^2 (i.e. polynomials under the relation $(x^1)^2+(x^2)^2+(x^3)^2 = 1$). Hence, every continuous/smooth/square integrable function is the limit of linear combinations of spherical harmonics of increasing polynomial degree[1]; in particular,

$$L^2(S^2) = \bigoplus_{l=0}^{\infty} \tilde{V}_l, \tag{2.3}$$

where the direct sum is to be understood as the one for Hilbert spaces. The spaces $\tilde{V}_l$ are precisely all the irreducible unitary representation spaces of $SO(3)$, acting inversely on the coordinate variables, and so they are another description of the spin-l representation spaces V_l of $SU(2)$. This decomposition (into invariant, irreducible subrepresentation spaces) of $L^2(S^2)$ is in fact the one induced by the action of $SU(2)$, inherited from the action on the base space S^2, and which can be seen to be induced by the natural action of $\mathfrak{so}(3) \cong \mathfrak{su}(2)$ on S^2 as vector fields of infinitesimal rotations. In particular, it can be shown that

$$\partial_H := -i\partial_\phi \qquad \text{is the action of } H := J^3 \text{ on } L^2(S^2) \tag{2.4}$$

$$\partial_E := e^{i\phi}\left(\partial_\theta + i cot\,\theta \partial_\phi\right) \quad \text{is the action of } E := J^+ = J^1 + iJ^2 \text{ on } L^2(S^2) \tag{2.5}$$

$$\partial_F := -\partial_E \qquad \text{is the action of } F := J^- = J^1 - iJ^2 \text{ on } L^2(S^2) \tag{2.6}$$

where ϕ and θ are the azimuthal and polar angles on S^2, respectively. Thus, equation (2.3) is the decomposition of $L^2(S^2)$ not only as a Hilbert space, but also as a representation space of $SU(2)$.

[1]More precisely, the vector space $\bigoplus_{l=0}^{\infty} \tilde{V}_l$ is the space of polynomials restricted to the compact space S^2, and so the Stone-Weierstrass theorem implies that this subalgebra is dense in $C(S^2)$ under the uniform convergence norm topology, and hence also dense in $C^\infty(S^2) \subseteq C(S^2)$. Now, the compactness of S^2 also implies that $C(S^2)$ (and even $C^\infty(S^2)$, due to the existence of smooth bump functions) is dense in $L^2(S^2)$ with the L2 norm topology, thus the vector space of polynomials is also dense in $L^2(S^2)$ under this topology.

The spinor bundle ΣS^2 of S^2 is trivial, so the space of *spinor fields* $\mathcal{H} = L^2(S^2, \Sigma S^2)$ of S^2 is isomorphic (as a $C^\infty(S^2)$-module, and hence a $\mathbb{C}$-vector space) to $L^2(S^2) \otimes \mathbb{C}^2$, where $\mathbb{C}^2$ is understood as the *fermionic Fock space* associated to the tangent spaces of S^2, i.e. the unique irreducible representation of the Clifford algebra $\mathbb{C}l_2 = M_2(\mathbb{C}) = \pi_{1/2}(U(\mathfrak{su}(2)))$. Therefore, **the space of spinor fields of S^2 as a representation space of $SU(2)$ decomposes as**

$$L^2(S^2, \Sigma S^2) \cong \bigoplus_{l=1}^{\infty} V_l \otimes V_{1/2}; \tag{2.7}$$

we will understand from now on the separable Hilbert space $\mathcal{H} = L^2(S^2) \otimes \mathbb{C}^2$ as the space of spinor fields on the 2-sphere.

This decomposition of the spinor bundle induced by the action of $SU(2)$ correctly suggests that the canonical Dirac operator $\not{D} : L^2(S^2) \otimes \mathbb{C}^2 \to L^2(S^2) \otimes \mathbb{C}^2$ on S^2 has the following form:

$$\not{D} = \bigoplus_{l \in \mathbb{N}} \pi_l \otimes \pi_{1/2}(\mathcal{D}) = 1 + \partial_F \otimes \sigma_+ + \partial_E \otimes \sigma_- + \partial_H \otimes \sigma_3 = \begin{pmatrix} 1 + \partial_H & \partial_F \\ \partial_E & 1 - \partial_H \end{pmatrix}, \tag{2.8}$$

where $\sigma_\pm = \sigma_1 \pm i\sigma_2$.

Using the spherical harmonics we can construct spinor fields $Y'_{jm}, Y''_{jm} \in \mathcal{H}$ eigenvectors of $\not{D}$, with $j \in \mathbb{N} + 1/2$, $m = -j, \ldots, j$, called the *spinor harmonics*, which make up an orthogonal basis of $\mathcal{H}$. The eigenvalue of the $Y'_{j\cdot}$ is $(j + 1/2) \in \mathbb{N}$, and for the $Y''_{j\cdot}$ spinor field it is $-(j+1/2)$, hence the eigenvalue j has multiplicity $2j+1$. The spectrum of $\not{D}$ (and the respective multiplicities) will be an indication of how good of an approximation is a Dirac operator on the fuzzy sphere.

The following aiding lemma extracted from [2] will allow us to study the spectrum of the Dirac triples that will be defined on the fuzzy sphere.

Lemma 2.1.2. The operator $(\pi_0 \otimes \pi_{1/2})(\mathcal{D}^2) : V_0 \otimes \mathbb{C}^2 \to V_0 \otimes \mathbb{C}^2$ has as unique eigenvalue 1 with multiplicity 2. For any $j \in \frac{\mathbb{Z}_{\geq 1}}{2}$, the $(\pi_j \otimes \pi_{1/2})(\mathcal{D}^2)$ has two eigenvalues: j^2 with multiplicity $2j$, and $(j+1)^2$ with multiplicity $2j+2$.

2.2 The Fuzzy Sphere

A fuzzy space is a sequence of finite dimensional C^*-algebras $\{\mathcal{A}_N\}_{N\in\mathbb{N}}$ with increasing dimension which approximate, in some sense to be determined in the specific case, a commutative C^* algebra $\mathcal{A}$ in the limit $N \to \infty$. When the commutative algebra encodes a topological space or a manifold with certain symmetry group G that acts by homeomorphisms or diffeomorphisms, we require a fuzzy space to implement this symmetry, i.e. that G acts on each $\mathcal{A}_N$ by $*$-isomorphisms (the algebraic version of homeomorphisms) and in a way compatible with the derivations of $\mathcal{A}$ (the algebraic version of diffeomorphisms).

With the full spectral triple defined below, the fuzzy sphere will approximate S^2 in the following loose ways, whose precise statements will be clear soon:

1. A C^*-algebra acting on the Hilbert space of spinors.
2. A smooth space on which $SU(2)$ acts by diffeomorphisms.
3. A metric space on which $SU(2)$ acts by isometries.

The action of $SU(2)$ on the commutative C^*-algebra $\mathcal{A} = C(S^2)$ is the one induced by its action on S^2 as the double cover of the rotation group, i.e. for any $f \in C(S^2)$ and any $\vec{x} \in S^2$ it is given by $g \cdot f(\vec{x}) := f(g^{-1} \cdot \vec{x})$, where $g^{-1}\cdot$ is the image of g^{-1} under the covering map $SU(2) \twoheadrightarrow SO(3)$. To define the fuzzy sphere $\{\mathcal{A}_N\}_{N\in\mathbb{N}}$ we start with the previously mentioned fact that any element $f \in \mathcal{A}$ is such that $\{f_N\}_{N\in\mathbb{N}} \to f$ uniformly, for some sequence of polynomials $f_N \in \bigoplus_{l=0}^N \tilde{V}_l$. This suggest the definition of $\mathcal{A}'_N := \bigoplus_{l=0}^N \tilde{V}_l$, for all $N \in \mathbb{N}$, as the elements of the sequence composing the fuzzy sphere. This definition seems appropriate in at least two ways: they are unitary representation spaces of $SU(2)$, and there is a clear way in which they approximate $\mathcal{A}$.

However, an obvious problem with this definition is that the representation spaces $\mathcal{A}'_N$ are not closed under their natural multiplication, so they are not algebras. But, from the case $N = 1$ there is a suggestion on how to define a multiplication [1]: requiring that the radical of the algebra is zero, there are only two possibilities, and only one being noncommutative:

$$x^i \mapsto \hat{x}^i := \lambda\sigma_i \qquad\qquad i = 1, 2, 3; \tag{2.9}$$

where $\lambda \in \mathbb{C}$ is any constant; this means that we are defining the $N = 1$ element of the fuzzy sphere to be $\mathcal{A}_1 := M_2(\mathbb{C})$. Having $SU(2)$ act on $\mathcal{A}_1$ with the the natural representation on linear operators of representation spaces, *the adjoint representation*, i.e. the action of $g \in SU(2)$ on $a \in \mathcal{A}_1$ is gag^{-1}, notice that the identification $x^i \mapsto \hat{x}^i$ is $SU(2)$-equivariant since the image of $i\frac{\sigma_i}{2} \in \mathfrak{su}(2)$ under the isomorphism $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ corresponds to the generator of rotations with respect to the x^i-axis, and the adjoint action on these generators represents a change of rotation axis. The constant λ is chosen such that "radius of the fuzzy sphere is 1", meaning that $(\hat{x}^1)^2 + (\hat{x}^2)^2 + (\hat{x}^3)^2 = 1$, hence $\lambda = \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{\frac{1}{2}(\frac{1}{2}+1)}}$.

Instead of insisting on defining a foreign multiplication in $\mathcal{A}'_N$, the fuzzy sphere is now defined by the following generalization of the $N = 1$ case studied above:

Definition 2.2.1. *The fuzzy sphere* is defined as the sequence of C^*-algebras $\{\mathcal{A}_N\}_{N\in\mathbb{N}}$ where $\mathcal{A}_N := \pi_j(U(\mathfrak{su}(2))) \equiv End(V_j)$ where $N =: 2j$ and with π_j as defined in Notation 2.1.1. In addition, define the algebra generators $\hat{x}^i$ as

$$\hat{x}^i := \frac{1}{\sqrt{j(j+1)}}\pi_j(J^i) \qquad\qquad i = 1, 2, 3; \tag{2.10}$$

the number $\lambda_N := \frac{1}{\sqrt{j(j+1)}} = \frac{2}{\sqrt{N(N+2)}}$ is called *the normalization factor*. The algebra generators $\hat{x}^i$ of each $\mathcal{A}_N$ will be usually referred to as *the fuzzy coordinates*.

The normalization factor λ_N is such that $(\hat{x}^1)^2 + (\hat{x}^2)^2 + (\hat{x}^3)^2 = 1$. The elements $\hat{x}^i \in \mathcal{A}_N$, $i = 1, 2, 3$ generate the algebra; furthermore, each element in $a \in \mathcal{A}_N$ has a unique expansion $a = \sum_{l=0}^{N} \frac{1}{l!} a_{\mu_1\cdots a_l}\hat{x}^{a_{\mu_1}} \cdots \hat{x}^{a_l}$, where $a_{\mu_1\cdots a_l} \in \mathbb{C}$ is symmetric and trace-free as a tensor [1]. Thus, the following is an injective, but also $SO(3)$-equivariant linear map:

$$\begin{aligned} \mathcal{A}_N &\to C(S^2) \\ \hat{x}^i &\mapsto x^i; \end{aligned} \tag{2.11}$$

its equivariance is carefully proven in Proposition 3.1.9. We might say that $\hat{x}^i$ *is the fuzzy analog of the coordinate function* x^i, and so we obtain in general the **nonzero commutation relation of the *fuzzy coordinates***:

$$[\hat{x}^i, \hat{x}^j] = \frac{1}{\sqrt{j(j+1)}} i\epsilon_{ijk}\hat{x}^k, \qquad i, j, k \in \{1, 2, 3\}, \tag{2.12}$$

where the Einstein summation convention for implicit sums of repeated indices is understood, and where ϵ_{ijk} denotes the Levi-Civita tensor.

In order to gain some intuition for the fuzzy sphere, recall that the operator representation of J^i in each V_j and in $L^2(S^2)$ is as the i-th component of the angular momentum operator, in the corresponding quantum theory. Another interpretation, at least for even-indexed elements of the sequence, comes from the fact that the Lie algebra $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3) \subseteq M_3(\mathbb{C})$ under the $(SU(2), SO(3))$-equivariant mapping $iJ^i \mapsto i\hat{J}^i$, where $i\hat{J}^i$ is the generator of rotation in $\mathbb{R}^3$ about the i-th axis; hence, the fuzzy analog of the coordinate function $x^i \in C(S^2)$ is an element proportional to the infinitesimal rotation with respect to the i-th axis.

This *fuzzy* version of the sphere is, so far, good in two of the three ways stated at the beginning of this subsection:

$SU(2)$ **acts by diffeomorphisms**

From the $SU(2)$-equivariance of the identifications $x^i \to \hat{x}^i$, we see that $\mathcal{A}_N \cong \mathcal{A}'_N \cong \bigoplus_{l=0}^N V_l$ as representation spaces of $SU(2)$, where the action of $g \in SU(2)$ is the adjoint action on $\mathcal{A}_N$ seen as $End(V_j)$, namely by the inner automorphism $B \mapsto \pi_j(g)B\pi_j(g)^{-1}$, where $\pi_j(g) \in SU(N+1)$ since π_j is an unitary representation. That $(\pi_j(g)B\pi_j(g)^{-1})^* = \pi_j(g)B^*\pi_j(g)^{-1}$, where B^* is the adjoint of $B \in M_{N+1}(\mathbb{C})$, shows that $SU(2)$ **acts by $*$-automorphisms on $\mathcal{A}_N$**.

A diffeomorphism of S^2 is equivalent to an automorphism of the algebra $C^\infty(S^2)$ (dense subalgebra of $\mathcal{A}$) that respects the complex conjugation $*$, thus giving us an alternative algebraic definition of a diffeomorphism. In the matrix algebras $M_{N+1}(\mathbb{C})$, since they are simple, all $*$-automorphisms are precisely of the form $B \mapsto CBC^{-1}$ for some $C \in SU(N+1)$. Furthermore, all such automorphisms also respect the derivations of $\mathcal{A}_N$, which are the algebraic analogues of vector fields. Hence, all $*$-automorphisms of $\mathcal{A}_N$ are good analogues of diffeomorphisms of S^2, and so we may say that $SU(2)$ **is acting by diffeomorphisms on the fuzzy sphere**.

As C^*-algebra acting on spinors

Replacing the coordinates x^i by $\hat{x}^i$ in the (symmetric) polynomial expansion of the spherical harmonics Y^l_m, $l = 0, \ldots, N$ and $|m| \leq l$, we obtain the fuzzy harmonics $\hat{Y}^l_m \in \mathcal{A}_N$ that make up a basis of $\mathcal{A}_N$.

Notice that the linear, injective mapping from (**??**) satisfies the following:

$$\begin{gathered} \mathcal{A}_N \to \mathcal{A}'_N \subseteq \mathcal{A} \\ \sum_{l=0}^{N} \frac{1}{l!} a_{\mu_1 \cdots a_l} \hat{x}^{a_{\mu_1}} \cdots \hat{x}^{a_l} \mapsto \sum_{l=0}^{N} \frac{1}{l!} a_{\mu_1 \cdots a_l} x^{a_{\mu_1}} \cdots x^{a_l}. \end{gathered} \tag{2.13}$$

Then, the sequence of noncommutative C^*-algebras approximates the commutative algebra $A = C(S^2)$ in the limit $N \to \infty$ in the sense that the above map gets increasingly closer to being an algebra morphism and **the algebra A can be considered as the image of the *near-diagonal matrices* in $\mathcal{A}_N$** i.e. matrices which commute within order 1 of the normalizing factor $\lambda_N = \frac{1}{\sqrt{j(j+1)}}$; the details may be found in [1].

Finally, we are yet to see how the fuzzy sphere approximates S^2 as metric space on which $SU(2)$ acts by isometries, but this will be consequences of Theorem 2.3.5 and the remark that follows.

2.3 Spectral Triples

The introduction of a spectral triple on each unital algebra $\mathcal{A}_N$ that make up the fuzzy sphere will introduce the notion of distance between states of the algebra, opening the door to the study of geometry on this spaces. Furthermore, we would like to introduce a spectral triple that somehow allows us to think of the fuzzy sphere as an approximation of S^2 also as metric spaces.

In this document, this will occur in the following senses:

- The spectrum (and the multiplicities of the eigenvalues) of the Dirac operator approximates that of $\not{D}$ as N increases
- To each point in S^2 there is a corresponding pure state on each $\mathcal{A}_N$, and the distance between the coherent states corresponding to two points in S^2 tends to the commutative distance between the points as $N \to \infty$.

2.3.1 $SU(2)$-equivariance

Rotating the 2-sphere leaves invariant not only its topological and differential properties, but it also leaves invariant the metric, $g = d\theta^2 + \sin^2(\theta)d\phi^2$, implying that the distance between any 2 points is the same as the distance between the rotated points, under any rotation $h \in SO(3)$. Since $SU(2)$ is the double covering of $SO(3)$, there is a 2-to-1 Lie group morphism $SU(2) \to SO(3)$ under which any rotation $h \in SO(3)$ corresponds to two distinct points $\pm\tilde{h} \in SU(2)$. From this it follows that each element of $SU(2)$ acts as a rotation on S^2, so that $SU(2)$ **is also a group of symmetries of S^2 that acts by diffeomorphisms and isometries**. Notice, however, that the most general symmetry group of S^2 is the orthogonal group $O(3)$, which includes reflections under planes on $\mathbb{R}^3$ that pass through the origin.

One important property of the spectral triples that will be defined on the fuzzy sphere is their equivariance under $SU(2)$, or more generally under the Hopf algebra $U(\mathfrak{su}(2))$ as defined in [11], giving an analogous notion to the action of these symmetry spaces by isometries. The precise definition of a *spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *equivariant under a symmetry space* H, H being called *the isometry of the spectral triple*, is fairly involved, but, ignoring some details, reduces to the following [11]:

- $\mathcal{A}$ is a representation of H.
- $\mathcal{H}$ is a representation of H.
- The action of H on $\mathcal{A}$ coincides with the action of $\mathcal{A}$ as subalgebra of operators on $\mathcal{H}$, i.e. the adjoint action $A \mapsto g \circ A \circ g^{-1}$.

- The *Dirac operator is equivariant*, i.e. it satisfies that $[\mathcal{D}, h] = 0$ for all $h \in H$.

For the spectral triples defined in this section, the $SU(2)$-equivariance reduces to the verification of the last two conditions, since the first two are readily satisfied. The following theorem, which will apply to all the spectral triples defined in this chapter, illustrates how this notion of equivariance induces a notion of isometry on the respective noncommutative spaces.

Theorem 2.3.1. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a spectral triple. Let G be a group, let $\mathcal{H}$ and $\mathcal{A}$ be representation spaces of G, with G acting unitarily on $\mathcal{H}$, with the action of $g \in G$ on $\mathcal{H}$ denoted by $g\cdot$. Then, if for all $g \in G$, $[D, g\cdot] = 0$ and the action on $\mathcal{A}$ coincides with the induced action on $\mathcal{A}$ as subalgebra of operators on $\mathcal{H}$, then G acts by isometries on the state space $\mathcal{S}(\mathcal{A})$. That is, for all $g \in G$:

$$d_{\mathcal{D}}(\omega, \omega') = d_{\mathcal{D}}(g_*\omega, g_*\omega'), \qquad \text{for all } \omega, \omega' \in \mathcal{S}(\mathcal{A}); \tag{2.14}$$

we have denoted by $g_*\omega$ the action of G on $\mathcal{S}(\mathcal{A})$, $g_*\omega : \mathcal{A} \to \mathbb{C}$, $a \mapsto \omega(a^g)$, induced from the action $a \mapsto a^g$ of G on $\mathcal{A}$.

Proof. The theorem follows once we show that

$$|||[\mathcal{D}, a^g]||| = |||[\mathcal{D}, a]|||, \tag{2.15}$$

since, in that case,

$$\begin{aligned} d_{\mathcal{D}}(g_*\omega, g_*\omega') &= sup_{a \in \mathcal{A}_N}\{|\omega(a^g) - \omega'(a^g)| : |||[\mathcal{D}, a \otimes 1_2]||| \leq 1\} \\ &= sup_{b \in \mathcal{A}_N}\{|\omega(b) - \omega'(b)| : |||[\mathcal{D}, b \otimes 1_2]||| \leq 1\} \\ &= d_{\mathcal{D}}(\omega, \omega'), \end{aligned}$$

since $b = a^g$, for fixed $g \in SU(2)$, sweeps all $\mathcal{A}_N$ if a does.

Now, denoting the action of $G \ni g$ on $\mathcal{H}$ by $g\cdot$:

$$\begin{aligned} (g\cdot) \circ [\mathcal{D}, a] \circ (g\cdot)^* &= (g\cdot) \circ [\mathcal{D}, a] \circ (g^{-1}\cdot) \\ &= (g\cdot) \circ \mathcal{D} \circ a \circ (g^{-1}\cdot) - (g\cdot) \circ a \circ \mathcal{D} \circ (g^{-1}\cdot) \\ &= \mathcal{D}(g\cdot) \circ a \circ (g^{-1}\cdot) - (g\cdot) \circ a \circ (g^{-1}\cdot)\mathcal{D} \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{D} \circ a^g - a^g \circ \mathcal{D} \\
&= [\mathcal{D}, a^g];
\end{aligned}
$$

the third equality follows from the commutation of $\mathcal{D}$ with $g\cdot$, and the fourth one from the fact that the actions of g and $\mathcal{A}$ interwine. The desired equality (2.15) follows from this calculation, and from the unitarity of the operators $(g\cdot)$ and $(g\cdot)^*$. □

2.3.2 The Irreducible Spectral Triple

This spectral triple, although unsatisfactory as an approximation of the canonical spectral triple of S^2 with respect to the first criteria stated at the beginning of the section, will turn out to be very useful for computations.

Definition 2.3.2. For each $N \in \mathbb{N}$ and $j = N/2$, define the irreducible spectral triple as $(\mathcal{A}_N, H_N := V_j \otimes \mathbb{C}^2, D_N = (\pi_j \otimes \pi_{1/2})(\mathcal{D}))$, where $\mathcal{A}_N = End(V_j)$ acts naturally on the first factor of H_N.

This spectral triple may be seen to come as a single term of the $SU(2)$-induced expansion of $\not{D}$ in (2.8), although we are also allowing half integer spin representations in the current construction, which don't appear in the expansion of $\not{\mathcal{D}}$. The Dirac operators can also be written as

$$
D_N = \begin{pmatrix} 1 + \pi_j(H) & \pi_j(F) \\ \pi_j(E) & 1 - \pi_j(H) \end{pmatrix} = 1 + \pi_j(F) \otimes \sigma_+ + \pi_j(E) \otimes \sigma_- + \pi_j(H) \otimes \sigma_3 \tag{2.16}
$$

where $H = J^3$, $F = J^+$, $E = J^-$ are the respective actions of J^3, $J^+, J^- \in U(\mathfrak{su}(2))$ on V_j.

We have said before that this spectral triple is $SU(2)$-equivariant, but for this to make sense in the first place we need to define an action of $SU(2)$ on $\mathcal{A}_N$ and H_N, both of which will be inherited from the untitary representation π_j of $SU(2)$ on V_j. Since $\mathcal{A}_N = End(V_j)$, the action of each $SU(2) \ni g$ induces the adjoint action on its operators by inner automorphisms:

$$
a \mapsto a^g := \pi_j(g) a \pi_j(g)^*. \tag{2.17}
$$

Similarly, on $H_N = V_j \otimes V_{1/2}$ we may define the unitary left action of $g \in SU(2)$ by $u_g := \pi_j(g) \otimes \pi_{1/2}(g) \in \mathcal{B}(H_n)$; notice that, when $j \in \mathbb{N}$, this representation is one of the terms of the decomposition of the representation of $\mathcal{H} = L^2(S^2) \otimes \mathbb{C}^2$ into representations of $SU(2)$.

Proposition 2.3.3. For each $N \in \mathbb{N}$, the associated irreducible spectral triple:

(i) Is $SU(2)$-equivariant.

(ii) Has eigenvalues $j+1$ and $-j$ with multiplicities $2j+2$ and $2j$, where $j = N/2$.

(iii) Isn't compatible with a grading or a real structure.

Proof. As stated at the beginning of the section, to proof the $SU(2)$-equivariance follows from $[D_N, u_g] = 0$ and $u_g a u_g^* \equiv u_g \circ (a \otimes 1) \circ u_g^* = a^g \otimes 1 \equiv a_g$.

That those are the eigenvalues and their multiplicities follows from finding eigenvectors

$$|j,m\rangle\rangle_+ := \begin{pmatrix} \sqrt{\frac{j+m+1}{2j+1}}|j,m\rangle \\ \sqrt{\frac{j-m}{2j+1}}|j,m+1\rangle \end{pmatrix} \tag{2.18}$$

$$|j,m\rangle\rangle_- := \begin{pmatrix} -\sqrt{\frac{j-m}{2j+1}}|j,m\rangle \\ \sqrt{\frac{j+m+1}{2j+1}}|j,m+1\rangle \end{pmatrix} \tag{2.19}$$

for $|m| \leq j$ with the mentioned eigenvectors, and deducing that there are no more eigenvectors thanks to the Lemma 2.1.2.

If the spectral triple had a grading, the Dirac operator would have a symmetric spectrum about 0, hence this can't be the case.

That a real structure $J_N : H_N \to H_N$ doesn't exist follows from the fact that the commutant of $\mathcal{A}_N \otimes \mathbb{C}^2$ has dimension $dim\, \mathbb{C}^2 = 2$, and so it can't contain $J\mathcal{A}_N J^{-1}$.

□

2.3.3 The Full Spectral Triple

Definition 2.3.4. For each $N \in \mathbb{N}$, the full spectral triple is $(\mathcal{A}_N, \mathcal{H}_N := \mathcal{A}_N \otimes \mathbb{C}^2, \mathcal{D}_N := (\mathrm{ad}\pi_j \otimes \pi_{1/2})(\mathcal{D}))$, for $j = N/2$, where $a \in \mathcal{A}_N$ acts on the first factor of $\mathcal{H}_N$ via the commutator $[a, \cdot]$.

Notice that the action of $\mathcal{A}_N$ on itself as the first factor of $\mathcal{H}_N$ is simply the derivative of the adjoint action $B \mapsto \pi_j(g)B\pi_j(g)^{-1}$ of $SU(2) \ni g$ on $\mathcal{A}_N = End(V_j)$. This action extends to an action $\mathrm{ad}\pi_j \otimes \pi_{1/2}$ on $\mathcal{H}_N = \mathcal{A}_N \otimes V_{1/2}$. Under the $SU(2)$-induced decomposition of $\mathcal{A}_N$ into irreducible representations, the Hilbert space $\mathcal{H}_N$ is isomorphic to $\bigoplus_{l=1}^{N} H_l$, where the representation $\mathrm{ad}\pi_j$ on one side then corresponds to the representation $\bigoplus_{l=0}^{N} \pi_l$ on the other side, in which case the Dirac operator $\mathcal{D}_N$ corresponds to:

$$\mathcal{D}_N : \mathcal{H}_N \to \mathcal{H}_N \iff \bigoplus_{l=1}^{N} D_l : \bigoplus_{l=1}^{N} H_l \to \bigoplus_{l=1}^{N} H_l. \tag{2.20}$$

Theorem 2.3.5. Under the adjoint action of $SU(2)$ on the algebra $\mathcal{A}_N$ and the action of $SU(2)$ defined in the previous paragraph, the full spectral triple satisfies:

(i) It is $SU(2)$-equivariant.

(ii) It is a real spectral triple, with real structure $\mathcal{J}_N : \mathcal{H}_N \to \mathcal{H}_N$, $a \otimes v \mapsto a^* \otimes \sigma_2 \overline{v}$.

(iii) The spectrum of $\mathcal{D}_N$ is the truncation of $\not{D}$ to $\{-N, \ldots, N+1\}$, ie. the eigenvalues of $\mathcal{D}_N$ are $N+1$ with multiplicity $2N+2$, and $\pm l$ with multiplicity $2l$ for $l = 1, \ldots, N$.

(iv) It is not compatible with a grading.

Proof. It is straightforward to verify that: $\mathcal{J}_N$ is antilinear and antiunitary, $\mathcal{J}_N^2 = -1$, $\mathcal{J}_N(\mathcal{A}_N \otimes 1)J_N^{-1} \subseteq (\mathcal{A}_N \otimes 1)' \subseteq \mathcal{B}(\mathcal{H}_N)$ (the $\mathcal{A}'$ denotes the algebra that commutes with the algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_N)$), and $\mathcal{J}_N\mathcal{D}_N = \mathcal{D}_N\mathcal{J}_N$; hence $\mathcal{J}_N$ is a real structure.

From the $SU(2)$-decomposition (2.20) of the Dirac operator, results (ii) and (iii) follow from the application of Proposition2.20 to each of the terms.

Once again, a grading can't exist since the spectrum of $\mathcal{D}_N$ isn't symmetric. $\square$

The third statement of the theorem states that we have found a spectral triple with the desired property of being a truncation of the canonical spectral triple of S^2. Hence, this will be the spectral triple which we will assign the forming algebras of the fuzzy sphere, under which the metric properties are to be studied, since it **may be considered a sequence of spectral triples that approximate the metric of** S^2 **increasingly better with growing** N. However, the following theorem allows us to study instead the much simpler irreducible spectral triple.

Theorem 2.3.6. The irreducible and full spectral triple on $\mathcal{A}_N$, for each $N \in \mathbb{N}$, induce the same distance in the state space of $\mathcal{A}_N$.

Proof. Let ω, ω' be states on $\mathcal{A}_N$. Notice that distinct spectral triples induce different notions of distance only due to the condition $||[D,a]|| \leq 1$ in formula (1.2), which determines over which elements a of the algebra the supremum of $|\omega(a) - \omega'(a)|$ is evaluated. However, for all $a \in \mathcal{A}$, $||[\mathcal{D}_N, a]|| = ||[D_N, a]||$ as shown by the following calculation:

$$
\begin{aligned}
[\mathcal{D}, a] b \otimes v &= \mathcal{D}_N(ab \otimes v) - a\left(b \otimes v + \sum_{k=1}^{3}[\pi_j(J^k), b] \otimes \sigma_k v\right) \\
&= ab \otimes v + \sum_{k=1}^{3}[\pi_j(J^k), ab] \otimes \sigma_k v - ab \otimes v - \sum_{k=1}^{3} a[\pi_j(J^k), b] \otimes \sigma_k v \\
&= \sum_{k}[\pi_j(J^k), a] b \otimes \sigma_k v \\
&= [1_{N+1} \otimes 1_2 + \sum_{k=1}^{3} \pi_j(J^k) \otimes \sigma_k, a](b, v) \\
&= [D_N, a] \cdot b \otimes v.
\end{aligned}
$$

This tells us that $[\not{\mathcal{D}}_N, a] : \mathcal{H}_N \to \mathcal{H}_N$ is the operator of left multiplication by the matrix $[D_N, a] \in \mathcal{A}_N \otimes M_2(\mathbb{C})$, implying that its operator norm coincides with the norm of the matrix $[D_N, a]$. $\square$

2.4 Coherent States

2.4.1 General Theory

We now define the *Bloch spin coherent states*, also called or $SU(2)$-*coherent states*, in each $\mathcal{A}_N$ of the fuzzy sphere as a set of points between which we want to calculated distances. Given $N \in \mathbb{N}$, each coherent state is labeled by a point in $S^2 = SU(2)/S^1$, allowing us to interpret it as a "fuzzy approximation" of a point in S^2. Furtheremore, the "fuzzyness" of this approximation decreases as $N \to \infty$ since Theorem 2.5.14 shows that the distance between coherent states corresponding to two points in S^2 tends to the respective distance in S^2.

In quantum mechanics, coherent states were first introduced for the quantum version of the simple harmonic oscillator, as minimum uncertainty states for position and momentum, making them states whose time evolution is the closest to that of the corresponding classical evolution. Since then, many generalizations have emerged, using mainly one of the following properties of the system of coherent states $\{|\alpha\rangle\}_{\alpha\in\mathbb{C}}$ of the simple harmonic oscillator:

1. Each $|\alpha\rangle$ saturates the Heisenberg uncertainty relation

2. Each $|\alpha\rangle$ are eigenstates of the annihilation operator, with eigenvalue $\alpha \in \mathbb{C}$

3. $\{|\alpha\rangle\}_{\alpha\in\mathbb{C}}$ is generated by the action of the Heisenberg-Weyl group acting on the vacuum $|0\rangle$

Particularly Perolomov [12] defines systems of coherent states using (3) for various Lie groups G acting irreducibly and unitarily on a Hilbert space $\mathcal{H}$ through a representation T, once a vector ϕ_0 has been chosen. Then, the corresponding *set of coherent states* denoted by $\{T, \phi_0\}$ is defined as the set of vectors in $\mathcal{H}$ given by:

$$\{T, \phi_0\} := \{\phi_g := T(g)\phi_0 \,:\, g \in G\}. \tag{2.21}$$

These vectors are labeled by the topological and measure space $X := G/H$ where H is *the isotropy subgroup for* $|0\rangle$, i.e. the subgroup $H \ni h$ composed of elements such that

$$\phi_h = \exp\left[i\alpha(h)\right]\phi_0,$$

for some function $\alpha : H \to \mathbb{R}$, i.e. H simply generate a change of phase on ϕ_0, which also means that ϕ_g and ϕ_{gh} are in the same ray for all $g \in G$ and $h \in H$. This, in particular, means that the projection into the 1-dimensional subspace generated by ϕ_g, P_{ϕ_g}, coincides with $P_{\phi_{gh}}$, and so these projections can also be labeled by X. This procedure produces a system of coherent states under one of the following definitions:

Definition 2.4.1. Let $\mathcal{H}$ be a Hilbert space. A set $\{\phi_x\}_{x\in\Omega} \subseteq \mathcal{H}$ indexed by a topological space Ω is called a *strong system of coherent states* if it satisfies the following properties:

1. *Continuity*: the mapping $x \mapsto \phi_x$ is strongly continuous.
2. *Resolution of the identity*: there exists on Ω an integration measure dx such that

$$1_{\mathcal{H}} = \int_{\Omega} P_x \, dx, \qquad\qquad P_x = |\phi_x\rangle\langle\phi_x|. \tag{2.22}$$

If instead the set $\{\phi_x\}_{x\in\Omega}$ property (1) and

2'. *Completeness*: $\overline{\mathrm{Span}\{\phi_x \,:\, x \in \Omega\}} = \mathcal{H}$,

it is called a *weak system of coherent states*.

The set $\{T, \phi_g\}$ is then a weak system of coherent states indexed by G since T is an irreducible representation. Similarly, a set $\{\phi_{g(x)}\}_{x\in X}$ for which $g(x)$ is a section of the principal bundle $H \to G \to G/H$ is also a weak system of coherent states.

The weak systems $\{T, \phi_0\}$ that we just saw are, under some conditions, also strong systems of coherent states. Since G is a Lie group it has a left invariant Haar measure, and this induces an invariant (under G) measure in $X = G/H$. Now, the generating vector ϕ_0 is called *admissible* if $I_T = \int_X |\langle\phi_0|T[g(x)]|\phi_0\rangle|^2 dx$ is finite; this is automatically true if G or G/H are compact. If ϕ_0 is admissible, then the operator $B := \int_X P_x dx : \mathcal{H} \to \mathcal{H}$ is convergent. Since, if $x' = g'x$ for some g' implies that $T(g')P_xT(g'^{-1}) = P_{x'}$, then the invariance of dx implies that $T(g')BT(g'^{-1}) = B$, i.e. B is central.

Since the representation on $\mathcal{H}$ is unitary, Schurl's lemma then tells us that $B = bI_{\mathcal{H}}$ for some constant b. The value of b can be determined to be $b = \frac{I_T}{\langle \phi_0, \phi_0 \rangle}$, and this induces a normalized integration measure $d\mu(x) = dx/b$ and hence

$$1_{\mathcal{H}} = B/b = \int_X P_x d\mu(x), \tag{2.23}$$

implying that $\{\phi_x\}_{x \in X}$ is a strong system of coherent states labeled by $X = G/H$. If H has finite volume h, then it is also true that

$$1_{\mathcal{H}} = \int_G P_{\phi_g} d\mu'(g), \tag{2.24}$$

where $d\mu'(g) = d\mu(g)/bh$, where $d\mu$ is the Haar measure of G; under this condition, $\{\phi_g\}_{g \in G}$ is a strong system of coherent states labeled by G itself.

2.4.2 $SU(2)$-Coherent States

So far, the systems of coherent states $\{T, \phi_0\}$ derived from the procedure previously followed only generalize property (3) of the canonical coherent states. However, when we apply the procedure to $G = SU(2)$ we can obtain a system of coherent states that generalize also property (1), and they are called the *spin coherent states* or *Bloch coherent states* or $SU(2)$*-coherent states*. Let $\mathcal{H} = V_j$ be, as usual, the unitary irreducible representation space of $SU(2)$ of spin j, and suppose that ϕ_0 maximizes the (dimension of the) Lie algebra $\mathfrak{h}$ of the isotropry group H within the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ of the Lie algebra $\mathfrak{su}(2)$ of $G = SU(2)$, and also that ϕ_0 minimizes the $SU(2)$-invariant square dispersion $(\Delta \vec{L})^2$; this is satisfied by both $|j, j\rangle$ and $|j, -j\rangle$, since their isotropy group is $H = \{e^{iJ^3\alpha} : \alpha \in \mathbb{R}\} \cong SO(2)$ have maximal isotropy algebra in $\mathfrak{sl}(2, \mathbb{C})$, and they both satisfy

$$(\Delta \vec{L})^2 = (\Delta \vec{L})^2_{min} = \langle \vec{L}^2 \rangle - \langle \vec{L} \rangle^2 = l(l+1) - l^2 = l. \tag{2.25}$$

We then see that there is system of coherent states labeled by the space

$$S^1 \cong SU(2)/SO(2); \tag{2.26}$$

furthermore, since $SU(2) \cong S^3$ is compact, *this system of coherent states is strong.* We may obtain these vectors by specifying rotations $R_{\phi,\theta} \in$

$SO(3)$ given an arbitrary point $E(\phi,\theta)$ in S^2, and applying them to one of $|j,j\rangle$ or $|j,-j\rangle$. Following [13], we choose the rotation about the axis $\hat{n} = (\sin\phi, -\cos\phi, 0)$ of an angle θ.

Getting rid of the sign ambiguity in even dimensional representation spaces:

Definition 2.4.2. The *SU(2)-coherent states* in V_j are defined as the application of

$$R_{\phi,\theta} := e^{-i\theta(J^1\sin\phi - J^2\cos\phi)} = e^{\zeta J^+ - \zeta^* J^-} \in SU(2), \tag{2.27}$$

where $\zeta = \frac{1}{2}\theta e^{-i\phi}$, to the vector $|j,-j\rangle$ that minimizes the $SU(2)$-invariant dispersion $(\Delta\vec{L})^2$ and maximizes the isotropy subalgebra within $\mathfrak{sl}(2,\mathbb{C})$ under the action of $SU(2)$. They have the formulas:

$$\begin{aligned} |\phi,\theta)_N &:= R_{\phi,\theta}|j,-j\rangle \\ &= \sum_{m=-j}^{j} \binom{2j}{j+1}^{\frac{1}{2}} e^{-im\phi}(\sin\frac{\theta}{2})^{j+m}(\cos\theta)^{j-m}|j,m\rangle. \end{aligned} \tag{2.28}$$

Definition 2.4.3. Let $N = 2j \in \mathbb{N}$. For each $E(\phi,\theta)$ in S^2 define

$$\psi^N_{(\phi,\theta)} := (\phi,\theta,|\ \cdot\ |\phi,\theta)_N \in \mathcal{S}(\mathcal{A}_N),$$

i.e. as the vector state associated to the coherent state $|\phi,\theta)_N$. We will also call these *the* $SU(2)$*-coherent states*, *the Bloch coherent states* or the *spin coherent states*.

We have said that we want to identify the coherent states as fuzzy approximations of points in S^2. A first step was made by the fact that the coherent states are labeled by $SU(2)$. Furthermore, this identification is $SU(2)$-equivariant, i.e. for any $g \in SU(2)$,

$$g_*\psi^N_{(\phi,\theta)} = \psi^N_{g\cdot(\phi,\theta)} \tag{2.29}$$

for all $E(\phi,\theta) \in S^2$. This is expressed at the Lie algebra level by the following:

Proposition 2.4.4. For all $E(\phi,\theta) \in S^2$ and any $a \in \mathcal{A}_N$, for arbitrary

$N \in \mathbb{N}$

$$\begin{aligned}\psi^N_{(\phi,\theta)}([H,a]) &= -i\frac{\partial}{\partial\phi}\psi^N_{(\phi,\theta)}(a),\\ \psi^N_{(\phi,\theta)}([E,a]) &= e^{i\phi}\left(\frac{\partial}{\partial\phi} + i\cot\theta\frac{\partial}{\partial_\phi}\right)\psi^N_{(\phi,\theta)}(a),\\ \psi^N_{(\phi,\theta)}([F,a]) &= -e^{-i\phi}\left(\frac{\partial}{\partial\phi} - i\cot\theta\frac{\partial}{\partial_\phi}\right)\psi^N_{(\phi,\theta)}(a).\end{aligned} \tag{2.30}$$

2.5 Distance Between Families of Pure States

Due to Theorem 2.3.6, although the full spectral triple seems to be a better approximation of the canonical spectral triple due to the properties of the spectrum of $\mathcal{D}_N$, for N any natural number, it suffices to study the distances using the irreducible spectral triple since it induces exactly the same notion of distance on each $\mathcal{A}_N$. From now on we will stop writing the symbols π_j, $j = N/2$, denoting the unitary representation of $U(\mathfrak{su}(2)) \ni H = J^3, E = J^-, F = J^+$ on V_j unless confusion may arise; hence we will write the Dirac operator as

$$D_N = \begin{pmatrix} 1+H & F \\ E & 1-H \end{pmatrix}. \tag{2.31}$$

Notice that, under this matrix notation that follows from the tensor product with the Pauli matrices, an algebra element $a \in \mathcal{A}$ corresponds to the matrix

$$a = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \tag{2.32}$$

as an operator on H_N.

Remark 2.5.1. If $\mathcal{A}$ is a unital C^*-algebra, and D is an associated Dirac operator, the supremum $d_{\mathcal{D}}(\omega,\omega')$, of $|\omega(a) - \omega'(a)|$ for $a \in \mathcal{A}$ such that $||[D,a]|| \leq 1$, is always attained in hermitian ($a = a^*$) elements [14], hence Connes' distant formula (1.2) for a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ can be changed to

$$d_{\mathcal{D}}(\omega,\omega') = sup_{a\in\mathcal{A},\, a=a^*}\{|\omega(a) - \omega'(a)| : ||[\mathcal{D}, a||| \leq 1\}. \tag{2.33}$$

Remark 2.5.2. A state may be defined from its action on hermitian elements of the C^*-algebra $\mathcal{A}$, since from there it can be uniquely extended to all of

$\mathcal{A}$, since any $a \in \mathcal{A}$ can be written as $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$, where $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are hermitian.

The following lemma will also be a very helpful tool for our following studies, and its proof exemplifies how the norm of the commutator $||[D,a]||$ appearing in the definition of Connes' distance, may be studied.

Lemma 2.5.3. For any $a \in \mathcal{A}_N$, the following inequalities about operators on H_N are satisfied:

$$||[H,a]||, ||[E,a]||, ||[F,a]|| \leq ||[D_N,a]||; \tag{2.34}$$

if a is diagonal and hermitian, then

$$||[D_N,a]|| = ||[E,a]|| = ||[F,a]||. \tag{2.35}$$

Proof. Let $a \in \mathcal{A}_N$ be arbitrary. Then

$$\begin{aligned} ||[D_N,a]||^2 &= sup_{v \in H_N, |v|=1} |[D_N,a]|v\rangle|^2 \\ &= sup_{v \in H_N, |v|=1} \langle v|[D_N,a]^*[D_N,a]|v\rangle. \end{aligned}$$

To find a lower bound for the norm of $[D_N,a]$, first notice that

$$[D_N,a] = \begin{pmatrix} [H,a] & [F,a] \\ [E,a] & -[H,a] \end{pmatrix} \qquad [D_N,a]^* = \begin{pmatrix} [H,a]^* & [F,a]^* \\ [E,a]^* & -[H,a]^* \end{pmatrix},$$

hence

$$[D_N,a]^*[D_N,a] = \begin{pmatrix} [H,a]^*[H,a] + [E,a]^*[E,a] & \cdots \\ \cdots & [H,a]^*[H,a] + [F,a]^*[F,a] \end{pmatrix}.$$

If in the equation for $||[D_N,a]||$ instead of taking the supremum over all $v \in H_N$ of unit norm we take it over all unit vectors of the form $(x,0)^t$, and $(0,y)^t$, we obtain, respectively, the following lower bounds:

$$||[D_N,a]||^2 \geq ||[H,a]||^2 + ||[E,a]||^2 \tag{2.36}$$
$$||[D_N,a]||^2 \geq ||[H,a]||^2 + ||[F,a]||^2, \tag{2.37}$$

and so the first inequalities follow.

If $a \in \mathcal{A}_N = M_{N+1}(\mathbb{C})$ is diagonal, then $[H, a] = 0$ since $H = \pi_j(J^3)$ is also diagonal, and so

$$[D_N, a]^*[D_N, a] = \begin{pmatrix} [E, a]^*[E, a] & 0 \\ 0 & [F, a]^*[F, a] \end{pmatrix}$$

hence

$$||[D_N, a]|| = max\{||[E, a]||, ||[F, a]||\}\,. \tag{2.38}$$

Finally, $[D_N, a]^* = -[D_N, a^*]$ and $E = F^*$ imply that $[F, a] = -[E, a^*]^*$, so, if a is hermitian and diagonal, $||[E, a]|| = ||[F, a]||$ and the last part of the statement follows. □

The general procedure to find the distance between two states will be the following:

1. Understand $||[D, a]||$ for an arbitrary $a \in \mathcal{A}$;
2. find a small upper limit for $|\omega(a) - \omega(a')|$ dependent on $||[D, a]||$;
3. find a hermitian algebra element that saturates the inequality of the upper limit (or a sequence that has this bound as limit).

2.5.1 Distance Between (Vector) Discrete Basis States $|j, m\rangle$ for Arbitrary N

Another family of pure states on each $\mathcal{A}_N = End(V_j)$, for $N = 2j \in \mathbb{N}$ is the family of vector states $\omega_m := \langle j, m| \cdot |j, m\rangle$, where $m = -j, \ldots, j-1, j$. The study of this distance will be useful to the study of coherent states.

Theorem 2.5.4. For any N:

$$d_N(\omega_m, \omega_n) = \sum_{k=m+1}^{n} \frac{1}{\sqrt{(j+k)(j-k+1)}} = \sum_{k=m+1}^{n} d_N(\omega_{k-1}, \omega_k). \tag{2.39}$$

In particular, the distance between this family of states is additive.

Proof. Recall that $E = J^+ = (J^-)^* \in U(\mathfrak{su}(2))$ and that, for $m \pm 1 = -j \pm 1, \dots, j$, $J^\pm|j,m\rangle = \sqrt{(j \mp m)(j \pm m+1)}|j,m+1\rangle$. Therefore,

$$\begin{aligned}\langle j,m|[E,a]|j,m-1\rangle &= \sqrt{(j+m)(j-m+1)}\langle j,m-1|a|j,m-1\rangle \\ &\quad - \sqrt{(j-m+1)(j+m)}\langle j,m|a|j,m\rangle\end{aligned}$$

Hence,

$$\begin{aligned}\omega_m(a) - \omega_n(a) &= \sum_{k=m+1}^{n} \langle j,k-1|a|j,k-1\rangle - \langle j,k|a|j,k\rangle \\ &= \sum_{k=m+1}^{n} \frac{1}{\sqrt{(j+k)(j-k+1)}}\langle j,k|[E,a]|j,k-1\rangle\end{aligned}$$

Applying the triangle inequality on the norm of the above equation, and using Lemma 2.5.3, $|\langle j,k|[E,a]|j,k-1\rangle| \leq ||[E,a]|| \leq ||[D_N,a]|| \leq 1$, so we get the distance formula in the statement as an upper bound.

Define the diagonal hermitian operator

$$\hat{a}|j,m\rangle := -\left(\sum_{k=-j+1}^{m} \frac{1}{\sqrt{(j+k)(j-k+1)}}\right)|j,m\rangle. \tag{2.40}$$

For this $\hat{a}$, $[E,\hat{a}]|j,m\rangle = |j,m+1\rangle$ for all $m = -j,\dots,j-1$, and so $\hat{a}$ saturates the inequality. □

2.5.2 The Distance between $N = 1$ Coherent States

Any hermitian element in $\mathcal{A}_1 = M_2(\mathbb{C})$ can be written as

$$a = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix} = a_0 + \vec{a}\cdot\vec{\sigma}, \qquad \text{for } (a_0,\dots,a_3) \in \mathbb{R}^4.$$

Since very state $\omega : \mathcal{A}_1 \to \mathbb{C}$, in addition to its linearity over this 4-dimensional vector space, must satisfy that $\omega(Id_2) = 1$ and $\omega(a^*a) \geq 0$, then, when evaluated on hermitian elements a as above, every state must be of the form $\omega_{\vec{x}}(a) = a_0 + \vec{x}\cdot\vec{a}$, for $\vec{x} \in B^3 \subseteq \mathbb{R}^3$.

This form for the states identifies the convex combination $\alpha\omega_{\vec{x}} + \beta\omega_{\vec{y}}$, i.e. $\alpha, \beta \in \mathbb{R}_{\geq 0}$ and $\alpha + \beta = 1$, of states $\omega_{\vec{x}}$ and $\omega_{\vec{y}}$, with the convex combination points $\alpha\vec{x} + \beta\vec{y}$ in B^3. Hence, all pure states on $\mathcal{A}_1$, i.e. those that can not be written as a convex combination of more than one state, are of the form $\omega_{E(\phi,\theta)}$ for $E(\phi, \theta) := (\sin\theta\cos\phi, \sin\theta, \cos\theta) \in S^2$. Furthermore [2]:

Proposition 2.5.5. The set $\{\omega_{E(\phi,\theta)} \,|\, E(\phi, \theta)\}$ of pure states on $\mathcal{A}_1$ coincides with the set of $SU(2)$-coherent states, under the equality $\omega_{E(\phi,\theta)} = \psi^1_{(\phi,\theta)}$.

Theorem 2.5.6. For $N = 1$ all pure states are coherent states and

$$d_1(\hat{p}, \hat{q}) = \frac{1}{2}|\vec{p} - \vec{q}|_{\mathbb{R}^3}, \tag{2.41}$$

where $|\cdot|_{\mathbb{R}^3}$ denotes the euclidean norm in $\mathbb{R}^3$.

Proof. As stated at the beginning of this section, we may restrict our search to hermitian elements. Let $a \in \mathcal{A}_1 = M_2(\mathbb{C})$ be hermitian. Then,

$$\begin{aligned} |\omega_{\vec{x}}(b) - \omega_{\vec{y}}(b)| &= |(\vec{x} - \vec{y}) \cdot \vec{b}| \\ &\leq |\vec{x} - \vec{y}||\vec{b}|. \end{aligned}$$

Now, let $a_\pm = a_1 \pm ia_2$, so that $a = a_0 + a_+\sigma_+ + a_-\sigma_- + a_3\sigma_3$. Recall that $\pi_{1/2}(J^k) = \frac{\sigma_k}{2}$, $k = 1, 2, 3$, hence

$$[D_1, a] = \begin{pmatrix} 0 & a_+ & -a_+ & 0 \\ -a_- & 0 & 2a_3 & a_+ \\ a_- & -2a_3 & 0 & -a_+ \\ 0 & -a_- & a_- & 0 \end{pmatrix}$$

and so $i[D_1, a]$ is hermitian, with characteristic polynomial $\lambda^2(\lambda^2 - 4|\vec{a}|^2)$, so its norm is its maximum eigenvalue. Then

$$||[D_1, a]|| = 2|\vec{a}|.$$

Therefore, if we choose $a \in \mathcal{A}_1$ hermitian with associated $\vec{a}$ parallel to $\vec{x} - \vec{y}$ such that $2|\vec{a}| = 1$, this element saturates the inequality. $\square$

2.5.3 Relating Distinct N's and Upper Bound

In this subsection we will relate the coherent states and the algebras of subsequent N's, which will then allow us to find a relation between $|||[D_{N+1}, \cdot]|||$ and $|||[D_N, \cdot]|||$ for any $N = 2j \in \mathbb{N}$.

First define the linear injections, that respect the $SU(2)$ action,

$$\begin{aligned} U_j^+ &: V_{j+\frac{1}{2}} \to V_j \otimes V_{\frac{1}{2}}; & U_j^- &: V_{j-\frac{1}{2}} \to V_j \otimes V_{\frac{1}{2}} \\ \left| j + \frac{1}{2}, m + \frac{1}{2} \right\rangle &\mapsto |j, m\rangle\rangle_+ & \left| j - \frac{1}{2}, m + \frac{1}{2} \right\rangle &\mapsto |j, m\rangle\rangle_-, \end{aligned}$$

where $|j, m\rangle\rangle_\pm$ are the fuzzy spinors eigenvectors of D_N defined in 2.18, and $m \pm 1 = -j \mp 1, \ldots, j$ for $U_j^\pm$. Then, using only the formula for the coherent states (2.28), it can be easily shown that:

Lemma 2.5.7. $U_+|\phi, \theta)_{N+1} = |\phi, \theta)_N \otimes |\phi, \theta)_1$ for any $E(\phi, \theta) \in S^2$.

Now, define the injective linear maps

$$\begin{aligned} \eta_N^\pm &: \mathcal{A}_N \to \mathcal{A}_{N+1} \\ a &\mapsto (U_j^\pm)^* (a \otimes 1_2) U_j^\pm. \end{aligned}$$

Lemma 2.5.8. For any $a \in \mathcal{A}_N$,

$$\psi^{N+1}_{(\phi,\theta)} \circ \eta_N^+(a) = \psi^N_{(\phi,\theta)}. \tag{2.42}$$

Proof. This is a direct consequence of the definitions of $U_J^\pm$ and η_N^+, since

$$\begin{aligned} (\phi, \theta|\eta_N^+(a)|\phi, \theta)_{N+1} &= (\phi, \theta|a|\phi, \theta)_N (\phi, \theta|\phi, \theta), \qquad \text{Lemma 2.5.7} \\ &= (\phi, \theta|a|\phi, \theta)_N. \end{aligned}$$

□

Finally, using some more properties of the maps $U_j^\pm$ and $\eta_N^\pm$, together with the previous lemmas, the following can be proven:

Lemma 2.5.9. For any $a \in \mathcal{A}_N$,

$$|||[D_{N+1}, \eta^\pm(a)]||| \leq |||[D_N, a]|||. \tag{2.43}$$

From this relation between the norms of the commutators $[D_N, \cdot]$ for consecutive N's, we can now show the following.

Theorem 2.5.10. For any $N \geq 1$, the distance $d_N(\psi^N_{(\phi,\theta)}, \psi^N_{(\phi',\theta')})$ is nondecreasing with N, i.e.:

$$d_N(\psi^N_{(\phi,\theta)}, \psi^N_{(\phi',\theta')}) \leq d_{N+1}(\psi^{N+1}_{(\phi,\theta)}, \psi^{N+1}_{(\phi',\theta')}). \tag{2.44}$$

Proof.

$$\begin{aligned}
d_{N+1}(\psi^{N+1}_{(\phi,\theta)}, \psi^{N+1}_{(\phi',\theta')}) &= sup_{a\in\mathcal{A}_{N+1}}\{|\psi^{N+1}_{(\phi,\theta)}(a) - \psi^{N+1}_{(\phi',\theta')}(a)| : ||[D_{N+1}, a]|| \leq 1\} \\
&\geq sup_{a=\eta^+_N(b)\in\mathcal{A}_{N+1}}\{|\psi^{N+1}_{(\phi,\theta)}(a) - \psi^{N+1}_{(\phi',\theta')}(a)| : ||[D_{N+1}, a]|| \leq 1\} \\
&= sup_{a\in\mathcal{A}_N}\{|\psi^{N+1}_{(\phi,\theta)} \circ \eta^+_N(a) - \psi^{N+1}_{(\phi',\theta')} \circ \eta^+_N(a)| : ||[D_{N+1}, \eta^+_N(a)]|| \leq 1\} \\
&= sup_{a\in\mathcal{A}_N}\{|\psi^N_{(\phi,\theta)}(a) - \psi^N_{(\phi',\theta')}(a)| : ||[D_{N+1}, \eta^+_N(a)]|| \leq 1\} \\
&\geq sup_{a\in\mathcal{A}_N}\{|\psi^N_{(\phi,\theta)}(a) - \psi^N_{(\phi',\theta')}(a)| : ||[D_N, a]|| \leq 1\} \\
&= d_N(\psi^N_{(\phi,\theta)}, \psi^N_{(\phi',\theta')});
\end{aligned}$$

the fourth line is an application of Lemma 2.5.8, and the fifth line of Lemma 2.5.9. □

An interesting upper bound is possible for the increasing sequence with N that is the distance between any two coherent states associated to two points in the sphere.

Theorem 2.5.11. For all N, and for all $E(\phi,\theta), E(\phi',\theta') \in S^2$

$$\frac{1}{2}|E(\phi,\theta) - E(\phi',\theta')|_{\mathbb{R}^3} \leq d_N(\psi^N_{(\phi,\theta)}, \psi^N_{(\phi',\theta')}) \leq d_{S^2}(E(\phi,\theta), E(\phi',\theta')), \tag{2.45}$$

where d_{S^2} is the usual distance within S^2, inherited from the metric $g = d\theta^2 + \sin^2\theta\, d\phi^2$.

Proof. The lower bound is a simple consequence of the $N = 1$ distance found in Theorem 2.5.6 and of the nondecrease of the distance with N found in Theorem 2.5.10.

To prove the upper bound, thanks to the $SU(2)$-invariance of the distance between states proven in Theorem 2.3.1, we only need to prove it within the great circle $\theta = \frac{\pi}{2}$, i.e. we only need to prove that

$$d_N(\psi^N_{(0,\frac{\pi}{2})}, \psi^N_{(\phi,\frac{\pi}{2})}) \leq |\phi|.$$

It can be shown that $\psi^N_{(\phi,\theta)}([H,a]) = -i\frac{\partial}{\partial\phi}\psi^N_{(\phi,\theta)}(a)$; this is a statement of the equivariance under $SU(2)$ of the identification of $SU(2)$-coherent states and points in S^2. Then the following calculation

$$\begin{aligned}
|\psi^N_{(\phi,\frac{\pi}{2})} - \psi^N_{(0,\frac{\pi}{2})}| &= \left| i\int_0^\phi \psi^N_{(\alpha,\frac{\pi}{2})}([H,a])d\alpha \right| \\
&\leq \left| \int_0^\phi d\alpha \right| \left| \psi^N_{(\alpha,\theta)}([H,a]) \right| \\
&\leq |\phi| \, ||[H,a]||, && |\omega(a)| \leq ||a||, \, \forall\omega : \mathcal{A} \ni a \to \mathbb{C} \\
&\leq |\phi| \, ||[D_N,a]||, && \text{Theorem 2.34,}
\end{aligned}$$

proves the desired upper bound. □

2.5.4 Auxiliary Distance and the Commutative Limit

A final result, stating that the distance between any two coherent states associated to two points in the sphere is not only bounded by the euclidean distance between the points, but that in fact this is the limit distance when $N \to \infty$ can be shown. To do this, we use an auxiliary distance by limiting the algebra on which we maximize the norm of the difference of the evaluation of the states that appears in the definition of Connes' distance. By reducing the algebra the supremum over it will necessarily be a lower bound, but an exact formula for the distance will be possible.

Let $\mathcal{B}_N \subseteq \mathcal{A}_N$ be subalgebra of diagonal matrices. Notice that for any $a \in \mathcal{B}_N$, $\psi^N_{(\phi,\theta)} = \psi^N_{(0,\theta)}$ for all ϕ; also note that, for $|n|, |m| \leq j$, on the diagonal element a with entries $a_{nm} = c_m\delta_{nm}$, $c_m = \omega_m(a)$, where ω_m is the vector state of $|j,m\rangle \in V_j$. Define

$$\rho_N(\theta) := sup_{a\in\mathcal{B}_N, a=a^*}\{|\psi^N_{(0,\theta)} - \psi^N_{(0,0)}| : ||[D_N,a]|| \leq 1\}. \tag{2.46}$$

Lemma 2.5.12. For $N \in \mathbb{N}$ fixed:

(i) Let $\hat{a}$ be diagonal hermitian element introduced in (2.40) to saturate the distance between the ω_m's

$$\rho_N(\theta) = \psi^N_{(0,\theta)}(\hat{a}) - \psi^N_{(0,0)}(\hat{a}) \tag{2.47}$$

$$= \sum_{k=1}^{N} \binom{2j}{j+m} (\sin\frac{\theta}{2})^{2n} (\cos\frac{\theta}{2})^{2(N-n)} \sum_{k=1}^{n} \frac{1}{\sqrt{k(N-k+1)}} \tag{2.48}$$

(ii) $0 \le \rho'_N(\theta) \le 1$. In particular $\rho_N(\theta-\theta') \le |\theta-\theta'| = d_{S^2}(E(0,\theta), E(0,\theta'))$.

(iii)

$$\rho_N(\theta-\theta') \le \rho_{N+1}(\theta-\theta') \tag{2.49}$$

Proof. (i) For a diagonal element a,

$$\psi^N_{(0,\theta)}(a) = \sum_{k=1}^{N} \binom{2j}{j+m} (\sin\frac{\theta}{2})^{2(j+m)} (\cos\frac{\theta}{2})^{2(j-m)} \omega_m(a),$$

hence $|\psi^N_{(0,\theta)}(a) - \psi^N_{(0,0)}(a)|$ over the diagonal elements can be bounded by the claimed distance using the triangle inequality and the bound $|| \le d_N(\omega_m, \omega_{-j}) = \sum_{k=1}^{n} \frac{1}{\sqrt{k(N-k+1)}}$. Using $\hat{a}$ to saturate the inequality, we obtain the desired formula.

(ii) From the first formula of the first part of the Lemma 2.5.12 we easily deduce that $\rho'_N(\theta) = \frac{d}{d\theta}\psi^N_{(0,\theta)}(\hat{a})$. Furthermore, from the $SU(2)$-equivariance of the identification of coherent states and points in S^2, on the diagonal elements it follows that

$$\rho'_N(\theta) = \frac{d}{d\theta}\psi^N_{(0,\theta)}(\hat{a}) = \psi^N_{(0,\theta)}([E,\hat{a}]) = (0,\theta|[E,\hat{a}]|0,\theta)_N, \tag{2.50}$$

where, recall, $[E,\hat{a}]$ is the ladder operator on V_j.

Since for every state ω is is true that $|\omega(a)| \le ||a||$ for every a, we get the following inequality:

$$|\rho'_N(\theta)| = |\psi^N_{(0,\theta)}([E,\hat{a}])| \le ||[E,a]|| \le 1,$$

recalling that $[E,\hat{a}]$ is the ladder operator on V_j, of unit norm. From expanding the formula $\rho'_N(\theta) = (0,\theta|[E,\hat{a}]|0,\theta)_N$ and using the equation (2.28) for the spin coherent states, the lower bound 0 then follows by noticing that every term in the expansion is positive since $\theta \in [0, \pi]$.

(iii) This follows from adapting the proof of Theorem 2.5.10, but where we need to take into account that $\eta_N^+(\mathcal{B}_N) \subseteq \mathcal{A}_{N+1}$ might not be contained in the diagonal subalgebra $\mathcal{B}_{N+1}$, but it is contained in an algebra $\mathcal{B}'_{N+1}$ conjugate to $\mathcal{B}_{N+1}$ through a unitary operator commuting with the $SU(2)$ action on $\mathcal{A}_{N+1}$ $(\pi_{j+\frac{1}{2}})$; this equates to rotating the basis vectors of $V_{j+\frac{1}{2}}$ in a way that leaves invariant the $\psi^{N+1}_{(\phi,\theta)}$ unchanged. This allows us to write the first equation of the proof of 2.5.10 by changing $\mathcal{A}_{N+1}$ by $\mathcal{B}'_N$, and from then on simply changing $\mathcal{A}_N$ by $\mathcal{B}_N$. $\square$

The following lemma is last piece of our puzzle. [2].

Lemma 2.5.13. For all N and $\theta \in [0, \pi]$, the sequence $\rho_N(\theta)$ converges uniformly to $\theta = d_{S^2}(E(0,0), E(0,\theta))$ as $N \to \infty$.

Proof. Let $f_N(\theta) := \theta - \rho_N(\theta)$; since $f_N(0) = 0$ and, by the second part of the Lemma 2.5.12, $f'_N(\theta) \geq 0$, so for each N the function $f_N(\theta)$ is nondecreasing and positive, so $||\theta - f_N(\theta)||_\infty - sup_{\theta\in[0,\pi]} f_N(\theta) = f_N(\pi) = \pi - \rho_N(\pi)$. So, the uniform convergence follows if and only if $\lim_{N\to\infty} \rho_N(\pi) = \pi$. For fixed θ, by the third part of Lemma 2.5.12, $\rho_N(\theta)$ is a nondecreasing sequence, and it is bounded above by π, so the sequence $\rho_N(\theta)$ is convergent and any subsequence has the same limit.

By using the formula for $\rho_N(\pi)$ given in the first part of Lemma 2.5.12,

$$\begin{aligned}\rho_N(\pi) &= 2 \sum_{k=1}^{\frac{1}{2}(N-1)} \frac{1}{k(N-k+1)} + \frac{2}{\sqrt{N+1}} \\ &\geq \int_1^{\frac{1}{2}} \frac{dx}{\sqrt{x(N-x+1)}} = 2 \arcsin \frac{N-1}{N+1},\end{aligned}$$

since the function $\frac{1}{\sqrt{x(N-x+1)}}$ is positive for $1 \leq x \leq$ and symmetric about $x = \frac{1}{2}(N+1)$ and monotonically decreasing for $1 \leq x \leq \frac{1}{2}(N+1)$. The sequence $2 \arcsin \frac{N-1}{N+1}$ converges monotonically to π as $N \to \infty$ if we restrict to the subsequence of odd N's, but this suffices to prove that $\lim_{N\to\infty} \rho_N(\theta) = \pi$. $\square$

From the previous lemma, and joining for completeness some previous results, the theorem follows.

Theorem 2.5.14. For all $N \in \mathbb{N}$ and all points $E(\phi, \theta)$ and $E(\phi', \theta')$ in the 2-sphere:

$$\rho_N(\theta - \theta') \leq d_N(\psi^N_{(\phi,\theta)}, \psi^N_{(\phi',\theta')}) \leq d_{S^2}(E(\phi, \theta), E(\phi', \theta')),$$

and

$$\lim_{N\to\infty} d_N(\psi^N_{(\phi,\theta)}, \psi^N_{(\phi',\theta')}) = d_{S^2}(E(\phi, \theta), E(\phi', \theta')). \tag{2.51}$$

Chapter 3

The New Fuzzy Spheres of Fiore and Pisacane

Fiore and Pisacane in [3] and [15] build new fuzzy spheres from the introduction of an energy cut-off to quantum mechanics on euclidean spaces in which there is a near-harmonic potential around the unit sphere $r = 1$. More precisely, for a general dimension $D = d + 1 \in \mathbb{Z}_{\geq 2}$, they first build a low energy effective theory for this system introducing an energy cutoff $\overline{E} \geq 0$. This includes the study of operators that that might play the role of representing certain observables in these effective theories like position. The resulting vector of position operators have the characteristic of transforming covariantly under the group $O(D)$ as expected from a position vector, but, interestingly, its coordinates no longer commute.

A sequence of such low effective theories can be made with increasingly higher energy cutoffs, and in the cases $D = 2$ and $D = 3$ they show that, in some precise sense, these theories have as "commutative limit" the quantum mechanics of a free spin-less particle in S^d. Under some additional restrictions on how the steepness of the harmonic potential increases between the terms of the sequence, Fiore and Pisacane show that these sequence are a fuzzy space that approximate the spheres S^d.

In this chapter we follow their construction for the case $D = 2$, including the presentation of some alternative algebras that are $*$-isomorphic in an $O(2)$-equivariant way to the original construction.

3.1 Quantum Mechanics with near-harmonic Potential and an Energy Cutoff

3.1.1 Algebras of Effective Observables $\mathcal{A}_{\overline{E}}$

In order to construct the new fuzzy spheres we need to develop a way to produce the individual algebras. These algebras will all be examples of the algebras $\mathcal{A}_{\overline{E}}$ that we will now build.

In dimension $D = d + 1 \in \mathbb{Z}_{\geq 2}$, let $\mathcal{H} = L^2(\mathbb{R}^D)$ be the Hilbert space of square integrable complex valued functions on $\mathbb{R}^D$ (modulo identifications almost-everywhere). Suppose that the dynamics of a quantum spin-less particle described by the wavefunction $\psi(\vec{r}) \in \mathcal{H}$ is determined by the Schrödinger equation

$$H\psi = i\frac{\partial}{\partial t}\psi, \tag{3.1}$$

where the Hamiltonian operator H of the system is of the form

$$H = -\frac{1}{2}\Delta + V(r); \tag{3.2}$$

we are using natural units $\hbar = c = 1$, and assuming a unit mass $m = 1$. Here, r is the radius in $\mathbb{R}^D$, $V(r)$ is a radially symmetric potential and $\Delta = \sum_{i=1}^{D} \frac{\partial}{\partial x^i} =: \sum_{i=1}^{D} \partial_i\partial_i$ is the Laplacian on $\mathbb{R}^D$, where x^i, $i = 1, \ldots, D$ are the Cartesian coordinates on $\mathbb{R}^D$.

The first ingredient for the construction will be to let the potential be nearly harmonic with minimum value $V(1) = V_0$ at $r = 1$, i.e. such that

$$V(r) \approx V_0 + 2k(r-1)^2 \qquad\qquad k \geq 0 \tag{3.3}$$

in some region $\nu_{\tilde{E}} := \{r \,|\, V(r) \leq \tilde{E}\}$, for some value $\tilde{E} \geq V_0$. Actually, to obtain simple eigenvalues of the Hamiltonian, V_0 is chosen such that the lowest energy E is 0, turning V_0 into a function of k. Moreover, we will suppose that $k = V''(r)/4 > 0$ is very large, so that $V(r)$ is a close to a confining potential for the particle into the d-sphere, S^d; this condition on k will be made more precise below, when the cutoff energy is introduced.

The algebras that make a up the new fuzzy will be chosen as the space of observables of the vector subspace of $\mathcal{H}$ inside which no wavefunctions of energy higher that some cutoff energy inhabit. Hence, we need a description of the eigenfunctions of the energy operator, i.e. of the Hamiltonian. To do this, we use polar coordinates $r, \phi, \cdots$ where $r = \sqrt{\sum_i (x^i)^2}$ is the radius, and $\phi, \cdots$ are angles on S^d, so

$$\Delta = \partial_r^2 + (D-1)\frac{1}{r}\partial_r - \frac{1}{r^2}L^2, \tag{3.4}$$

where, p_i, $i, j = 1, \dots, D$, is the linear momentum conjugate to x^i, and the operators $L_{ij} = i(x^i p_j - x^j p_i)$ are the angular momentum components, and so $L^2 = \frac{1}{2}\sum_{i,j}(L_{ij})^2$ is the square angular momentum operator, which is also the negative of the Laplacian on S^d as the above equation shows if $r = 1$. It is well known [] that the eigenvectors of L^2, called the spherical harmonics, have eigenvalues $j(j + D - 2)$, for $j \in \mathbb{Z}_{\geq 0}$.

For the eigenfunction ψ of H with eigenvalue E we may use the ansatz

$$\psi(r, \phi, \dots) = \tilde{f}(r) Y(\phi, \dots). \tag{3.5}$$

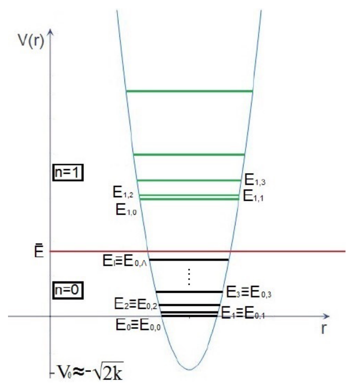
Using separation of variables on the Schrödinger equation (3.1) we deduce that *ψ is an eigenfunction of L^2 with eigenvalue* $j(j + D - 2)$; this means that we may add a first label j to the observable energy: E_j. On the other hand, the radial part of ψ satisfies

$$\left[-\partial_r^2 - (D-1)\frac{1}{r}\partial_r + \frac{1}{r^2}j(j+D-2) + V(r)\right]\tilde{f}(r) = E_j \tilde{f}(r). \tag{3.6}$$

By Fuchs Theorem, for this equation to have a solution $V(r)$ needs to grow at most as β/r^2 when r goes to 0, for some $\beta \geq 0$, in which case equation (3.6) has two independent solutions with asymptotic behavior r^α and $r^{-\alpha}$, respectively, as r goes to 0, where $\alpha = \sqrt{\beta + j(j + D - 2)}$. For $\tilde{f}$ to be defined in the origin the contribution of the second solution must vanish, in particular $\tilde{f}(0) = 0$; for $\psi = \tilde{f}Y$ to be square integrable on $\mathbb{R}^D$ it is required that $\tilde{f}(r) \to 0$ as r goes to infinity. We now see that the near confining harmonic potential (3.3) implies that the energy eigenfunction ψ becomes small outside the $\nu_{\overline{E}}$ region. Finally, it can be shown [15] that the above equation can be approximated, for $r \approx 1$, by a harmonic oscillator equation

$$[-\partial_{\tilde{r}} + \tilde{k}_j(\tilde{r} - \tilde{r}_j)^2] f(\tilde{r}) = \tilde{E}_j \tilde{f}(r) \tag{3.7}$$

for some $\tilde{r}$ function of r, $f(\tilde{r}) := \tilde{f}(r)$ and $\tilde{k}, \tilde{r}_j$ constants and, furthermore, that *the lowest eigenvalue solutions of the Schrödinger equation are approximately the solutions of the above harmonic oscillator equation.* Then, at least for the lowest eigenvalue solutions $\psi = \tilde{f}(r)Y(\phi, \dots)$ of the Schrödinger equation, the respective energies will have two labels: a label $j \in \mathbb{Z}$ coming from the angular factor $Y(\phi, \dots)$, and a label $n \in \mathbb{Z}_{\geq 0}$ coming from the radial part $\tilde{f}(r)$: $E_{n,j}$.



[h]

Figure 3.1: Eigenvalues of the Hamiltonian for steep enough potential for $D = 2$. The cutoff energy $\overline{E}$ can be chosen so that the energies below it are the spectrum of the free Hamiltonian L^2 in S^d. Image extracted from [3].

As illustrated in Figure 3.1, if k is steep enoughthe eigenvalues of H below $E_{1,0}$ will be a truncation of the spectrum of L^2, i.e. the smallest eigenvalues of H will be (approximately) $j(j + D - 1)$ for $j = 0, 1, 2, \dots, \Lambda$ for some $\Lambda \in \mathbb{N}$. This introduces the second main ingredient of our construction: *the energy cutoff*. In general:

Definition 3.1.1. Let the dimension $D = d+1 \in \mathbb{Z}_{\geq 2}$ be arbitrary. Given an energy $\overline{E} \geq 0$, suppose that k is sufficiently steep to make the eigenvalue $E_{1,0}$ be above $\overline{E}$, now called the cutoff energy. Then, we will define the algebra $\mathcal{A}_{\overline{E}}$ as the algebra of operators over the finite dimensional Hilbert space $\mathcal{H}_{\overline{E}}$, spanned by the solutions $\psi_{0,j} \equiv \psi_j$ of the Schrodinger equation (under the harmonic oscillator approximation (3.7)) whose energies $E_j := E_{0,j} E_j$ are below $\overline{E}$.

Finally, in order to make a fuzzy space $\{\mathcal{A}_\Lambda\}_{\Lambda\in\mathbb{N}}$ out of this construction of algebras $\mathcal{A}_{\overline{E}}$, we need to make $\overline{E}$ and k grow with a $\Lambda \in \mathbb{N}$ in such a way that:

1. k is big enough to not produce radial excitations, i.e. keep nontrivial radial eigenfunctions, rhose for which the label $n \geq 1$, above the cutoff;
2. for fixed Λ, the maximum proper energy is $\Lambda(\Lambda + D - 2)$. This may be done by choosing

$$\overline{E} = \overline{E}(\Lambda) = \Lambda(\Lambda + d - 1), \qquad k = k(\Lambda) \geq \Lambda^2(\Lambda + 1)^2. \tag{3.8}$$

We now make a series of observations about the constructed algebras:

Remark 3.1.2. Since the Hamiltonian operator is the generator of time evolution for a quantum system, a **quantum system that is setup in a state $\psi \in \mathcal{H}_{\overline{E}}$ will not evolve out of $\mathcal{H}_{\overline{E}} \subseteq \mathcal{H}$.** More precisely, recall that the eigenvectors of the Hamiltonian operator are called steady-state states since their time evolutions consists only of a change of phase, leaving invariant the probability distribution they induce and, in particular, staying as states of constant energy. Hence, if $\psi_0 \in \mathcal{H}_{\overline{E}}$, it is a linear combination of eigenfunctions of H of energy lower than $\overline{E}$ and, by applying the linear time-evolution operator $\exp[-itH] = 1 - itH - t^2H^2 + \cdots$, $\psi(t) := U(t)\psi_0$ will once again be a linear combination of eigenfunctions of H with low energy, i.e. $\psi(t)$ is also in $\mathcal{H}_{\overline{E}}$.

Remark 3.1.3. The orthogonal group $O(D)$ is defined as the subset of the matrices $M_D(\mathbb{R})$ acting on the vectors of $\mathbb{R}^D$ that respects the inner product of $\mathbb{R}^d$, i.e. for any $\vec{v}, \vec{w} \in \mathbb{R}^D$, every matrix $A \in O(D)$ satisfies $(A\vec{v})\cdot(A\vec{w}) = \vec{v}\cdot\vec{w}$; it is easily shown that this condition is equivalent to $A^tA = 1_D$, where the subscript $\cdot^t$ denotes the transpose matrix. This group contains not only all possible rotations on $\mathbb{R}^D$, the group $SO(D) = \{A \,:\, det(A) = 1,\, A \in O(D)\}$, but also the reflections with respect to every hyperplane on $\mathbb{R}^D$.

$O(D) \ni g$ acts naturally from the left on $\mathbb{R}^D$, and so it also acts on functions ψ of $\mathbb{R}^D$, in particular on $\mathcal{H} = L^2(\mathbb{R}^D)$, through the unitary representation π defined by $\pi(g)\psi : \vec{v} \to \psi(g^{-1}\vec{v})$. Actually, $O(D)$ **acts on $\mathcal{H}_{\overline{E}} \subseteq \mathcal{H}$, and, therefore, on $\mathcal{A}_{\overline{E}}$ from the left by inner automorphisms**, meaning that the action of any $g \in O(D)$ on any function $\psi \in \mathcal{H}_{\overline{E}}$ produces again a function in $\mathcal{H}_{\overline{E}}$. This follows from the fact that the Hamiltonian (3.2), being radially symmetric, is $O(D)$-invariant, and so $[H, \pi(g)] = 0 \in \mathcal{B}(\mathcal{H})$ for the action of any $g \in O(D)$, which implies that, if ψ_E is an eigenfunction of H of energy $E \leq \overline{E}$, then $\pi(g)\psi_E$ is also an eigenfunction for the same energy.Theorem 3.1.8 describes an additional property of the action of $\mathcal{A}_{\overline{E}}$ under $O(2)$ when position observables are introduced.

3.1.2 Low Energy Effective Quantum Theories and their Properties

Let $(\mathcal{H}, \mathcal{A}, H)$ be a *quantum theory*, like quantum mechanics on $\mathbb{R}^D$: $(\mathcal{H} = L^2(\mathbb{R}^D), \mathcal{B}(\mathcal{H}), H)$ where H is any Hamiltonian, or quantum mechanics on S^d: $(L^2(S^d), \mathcal{B}(\mathcal{H}), H)$ for some Hamiltonian H. For every cutoff $\overline{E}$, and k compatible with it, the Hilbert space $\mathcal{H}_{\overline{E}}$ together with its algebra of operators $\mathcal{A}_{\overline{E}}$ and the time evolution operator $H|_{\overline{E}} := H|_{\mathcal{H}_{\overline{E}}}$ is a quantum theory that may be called a *low energy effective quantum theory* of the quantum mechanics of the particle on $\mathbb{R}^D$ subject to a potential that is nearly harmonic near $r = 1$, associated to the algebra of observables $\mathcal{A}_{\overline{E}}$. To make the interpretation of the (hermitian) elements of $\mathcal{A}_{\overline{E}}$ as the space that represents the observables and to justify calling it a quantum theory, based on Fiore and Pisacane [3] we say that a *measurement* of an observable $B \in \mathcal{A}_{\overline{E}}$ on the state $\psi \in \mathcal{H}_{\overline{E}}$ means to apply B to ψ and then return as the measured quantity an eigenvalue of B with probability given by the norm of the projection of $B\psi$ in the corresponding eigenspace of B, normalized by the norm of ψ. Furthermore, at least for $D = 2$ we will see in Proposition 3.5.1 that a sequence of such low energy approximations with increasing $\overline{E}$, as that given by a fuzzy space, converge to the quantum theory of a spin-less particle in S^1.

Projected Observables

Fiore and Pisacane in [3] propose a way to define an element $\overline{A} \in \mathcal{A}_{\overline{E}}$ for each $A \in \mathcal{B}(\mathcal{H})$ which, in case that A represents an observable (i.e. A is hermitian) then $\overline{A}$ may be considered the representation of the same observable in the low energy effective quantum theory:

Proposition 3.1.4.

- Let $\mathcal{A}'_{\overline{E}} \subseteq \mathcal{A}$ be the subalgebra that annihilates $\mathcal{H}_{\overline{E}}^{\perp}$ and such that $\mathcal{A}'_{\overline{E}}\mathcal{H}_{\overline{E}} \subseteq \mathcal{H}_{\overline{E}}$. Then $\mathcal{A}'_{\overline{E}}$ is a C*-algebra isomorphic to $\mathcal{A}_{\overline{E}}$ through the mapping $' : \overline{\mathcal{A}}_{\overline{E}} \to \overline{\mathcal{A}}'_{\overline{E}}$ defined, for each $B \in \mathcal{A}_{\overline{E}}$, by $B'|_{\mathcal{H}_{\overline{E}}} = B$ and $B'|_{\mathcal{H}_{\overline{E}}^{\perp}} = 0$.

- For any densely defined operator A on $\mathcal{H}$ whose domain includes $\mathcal{H}_{\overline{E}}$ (or whose action can be continuously extended to all of $\mathcal{H}_{\overline{E}}$), define $\overline{A} = P_{\overline{E}}AP_{\overline{E}} \in \mathcal{A}'_{\overline{E}}$. Then:

$$(\overline{AB})' = \overline{A}'\,\overline{B}', \tag{3.9}$$

  for all A, B operators for which the $\overline{}$ map can be applied.

Proof. This is a straightforward application of the definitions. □

Notation 3.1.5. This proposition allows us to think of $\overline{\mathcal{A}}$ as a subalgebra $\overline{\mathcal{A}}'$ of $\mathcal{A}$, and to operate elements $\overline{A}, \overline{B} \in \overline{\mathcal{A}}$ that correspond to observables $A, B \in \mathcal{A}$ using its corresponding elements in $\overline{\mathcal{A}}' \subseteq \mathcal{A}$. From now on we will omit the $'$ notation and we will indistinctively let $\overline{A}$ be either an element of $\overline{\mathcal{A}}_{\overline{E}}$ or of $\overline{\mathcal{A}}'_{\overline{E}}$, given $A \in \mathcal{B}(\mathcal{H})$.

In particular, $\overline{x^i} \in \mathcal{A}_{\overline{E}}$, $i, j = 1, \ldots, D$ are candidates for the position observables within the effective quantum theory and, although $[x^i, x^j] = 0$, the additional projection operator implies that, in general:

$$[\overline{x}^i, \overline{x}^j] \neq 0. \tag{3.10}$$

Other candidates for position operators in these effective quantum theories will also be used, and the criteria of Section 3.1.2 is used to decide whether some operators are acceptable as candidates position operators in a sequence of algebras that approximate a quantum theory.

$O(D)$-Covariance

In a quantum theory $(\mathcal{H}, \mathcal{A}, H)$, whenever a group G acts on $\mathcal{H}$, G also acts on $\mathcal{A}$ by inner automorphisms. In some cases this transformation satisfies certain properties, in which case we will call it a G-covariant theory according to Fiore and Pisacane [16]:

Definition 3.1.6. Let $(\overline{\mathcal{H}}, \overline{\mathcal{A}} = End(\overline{\mathcal{H}}), \overline{H})$ be a quantum theory as described at the beginning of this section, and let G be a subgroup of $M_D(\mathbb{R})$, for some $D \in \mathbb{Z}_{\geq 1}$ that acts on $\mathcal{H}$ through a unitary representation $\overline{\pi} : G \to \overline{\mathcal{A}}$.

- Suppose that there are operators x^i on $\mathcal{H}$, $i = 1, \ldots, D$, where each x^i is considered the representation of the observable of i-th coordinate of position. We say that *the position coordinates transform contravariantly under* G, where G is a subgroup of $M_D(\mathbb{R})$, if G acts on $\mathcal{H}$ through a unitary representation and if

$$\overline{\pi}(g)x^i\overline{\pi}(g)^* = \sum_{j=1}^{D}(g^{-1})^i_j x^j. \tag{3.11}$$

- Suppose that there are elements p_i operators on $\mathcal{H}$, $i = 1, \ldots, D$, where each p_i is considered the representation of the observable of i-th coordinate of momentum. We say that *the momentum coordinates transform covariantly under* G, where G is a subgroup of $M_D(\mathbb{R})$ if G acts on $\mathcal{H}$ through a unitary representation and if

$$\overline{\pi}(g)p_j\overline{\pi}(g)^* = \sum_{j=1}^{D} g^i_j p_i. \tag{3.12}$$

- When the previous two conditions are satisfied, we say that *the theory is G-covariant.*

We now see some important examples of covariant theories, including the low energy effective quantum theories developed in this chapter.

Proposition 3.1.7. For any $D \in \mathbb{Z}_{\geq 1}$, quantum mechanics on $\mathbb{R}^D$ is an $O(D)$-covariant theory.

Proof. For this statement to make sense, we first see that $O(D) \subseteq M_D(\mathbb{R})$ acts on the left on $\mathbb{R}^D$ by multiplication, and so it acts on $\psi \in L^2(\mathbb{R}^D)$ by $\pi(g)\psi(\vec{x}) := \psi(g^{-1}\vec{x})$. To see that this representation of $O(D)$ on $\mathcal{H} = L^2(\mathbb{R}^D)$ is unitary it suffices to see that

$$\langle \pi(g)\psi_1|\pi(g)\psi_2\rangle = \int_{\mathbb{R}^D} \overline{\psi_1}(g^{-1}\vec{x})\psi(g^{-1}\vec{x})d\vec{x} = \int_{\mathbb{R}^D} \overline{\psi_1}(\vec{x})\psi(\vec{x})d\vec{x} = \langle \psi|\psi\rangle,$$

for all $g \in O(D)$ and $\psi_1, \psi_2 \in L^2(\mathbb{R}^D)$; we used that the measure of $\mathbb{R}^D$ is invariant under $O(D)$.

Now we can go on to prove that the position and momentum coordinates transform covariantly. Let $g \in G$, $\psi \in \mathcal{H}$ and $i \in \{1, \dots, D\}$ be arbitrary, then:

$$\begin{aligned}
\pi(g)^* x^i \pi(g) \psi(\vec{x}) &= \pi(g)^* x^i \psi(g^{-1}\vec{x}) \\
&= \pi(g)^* f(\vec{x}) \\
&= \pi(g^{-1}) f(\vec{x}) \\
&= f(g\vec{x}) \\
&= (g\vec{x})^i \psi(gg^{-1}\vec{x}) \\
&= \left(\sum_{j=1}^{D} g^i_j x^j \right) \psi(\vec{x}),
\end{aligned}$$

where $f(\vec{x}) = x^i \psi(g^{-1}\vec{x})$, since g and ψ are arbitrary, we have thus proved that the the position coordinates operators transform contravariantly. Now, invert the relation $(g\vec{x})^i = \sum_j g^i_j x^j$ in $\mathbb{R}^D$ to conclude that $x^j = \sum_i (g^{-1})^j_i (g\vec{x})^i$ for all $j \in \{1, \dots, D\}$. This implies that

$$\pi(g^{-1}) p_i \pi(g) \psi(\vec{x}) = \pi(g^{-1})(-i\partial_{x^i}) \psi(g^{-1}\vec{x}) = -i\partial_{(g\vec{x})^i} \psi(\vec{x}), \quad \forall \psi \in L^2(\mathbb{R}^D)$$
$$\text{but} \quad \partial_{(g\vec{x})^i} = \sum_j \partial_{(g\vec{x})^i} x^j \, \partial_{x^j} = \sum_j (g^{-1})^j_i \partial_{x^j},$$

proving that the momentum operators transform $O(D)$-covariantly. $\square$

From the previous proposition, the following important property of the low energy effective quantum theories developed in this chapter follows:

Theorem 3.1.8. For any $\overline{E} \in \mathbb{R}^+$, let $\mathcal{H}_{\overline{E}}$ and $\mathcal{A}_{\overline{E}}$ define a low effective quantum theory according to Definition 3.1.1. Then, using $\overline{x^i}$ as position coordinate operators and $\overline{p_i}$ as momentum operators, this theory is $O(D)$-covariant

Proof. This fact is a combination of Proposition 3.1.7 with the invariance of the Hamiltonian H (3.2) under $O(D)$ that follows from its radial symmetry, i.e. $\pi(g)^*(-\frac{1}{2}\Delta + V(r))\pi(g) = -\frac{1}{2}\Delta + V(r)$ for all $g \in O(D)$, where π is the unitary representation of $O(D)$ on $\mathcal{H} = L^2(\mathbb{R}^D)$. The invariance of H can be rewritten as $[H, \pi(g)] = 0$ by composing the previous relation with $\pi(g)$ to the left. The last relation implies that if ψ is such that $H\psi = E\psi$ for some $E \in \mathbb{R}$, then $H(\pi(g)\psi) = E(\psi(g)\psi)$, meaning that acting with $O(D)$ keeps an eigenvector in its corresponding eigenspace, and so, in particular, that $[P_{\overline{E}}, \pi(g)] = 0$.

Recall the notation 3.1.5. It follows that for any operator A on $\mathcal{H}$ such that $\overline{A} = P_{\overline{E}} A P_{\overline{E}} \in \mathcal{A}_{\overline{E}}$ is well defined and:

$$\begin{aligned}\pi(g^*)\overline{A}\pi(g) &= \pi(g) P_{\overline{E}} A P_{\overline{E}} \pi(g) \\ &= P_{\overline{E}} \pi(g^*) A \pi(g) P_{\overline{E}} \\ &= \overline{\pi(g^*) A \pi(g)}.\end{aligned}$$

Thus, the combination of the last equality with Proposition 3.1.7, and the linearity of the $\overline{}$ map prove the theorem. $\square$

Proposition 3.1.9. Suppose that each element $\mathcal{A}_N = \pi_{N/2}(U(\mathfrak{su}(2)))$ of the fuzzy sphere, from Definition 2.2.1, is considered a quantum theory over the irreducible spin $j = N/2$ representation space V_j. Take the algebra generators $\hat{x}^i = \frac{1}{\sqrt{j(j+1)}} \pi_j(J^i)$, defined in (2.10) for $i = 1, 2, 3$, as the representations of the i-th coordinate of the position observable. Then, for even N the position coordinates transform $SO(3)$-contravariantly.

Proof. If N is even, the irreducible representation on V_j of the simply connected Lie group $SU(2)$, also denoted by π_j, quotients to an irreducible representation $\hat{\pi}_N$ of $SO(3)$. This follows from the fact that $SU(2)$ is its double cover, where the covering map $c : SU(2) \to SO(3)$ sends each element and its negative to the same $SO(3)$ element, combined with the fact that π_j sends $-1 \in SU(2)$ to $1 \in End(V_j)$ for integer j.

That $SU(2)$ is the double cover of $SO(3)$ implies that there is an $(SU(2), SO(3))$-equivariant isomorphism $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, hence $U(\mathfrak{su}(2)) \cong U(\mathfrak{so}(3))$, where the isomorphism is such that $J^i \mapsto \hat{J}^i$, where J^i are the generators of $U(\mathfrak{su}(2))$ with the commutation relations (2.2) and where $i\hat{J}^i$ is the infinitesimal rotation with respect to the i-th axis in $\mathbb{R}^3$, i.e. $e^{i\theta J^i}$ is the rotation of an angle θ with respect to the i-th axis. Every rotation $R(\hat{n}, \theta) SO(3)$ is determined by an axis of rotation $\hat{n}$ and an angle of rotation θ, and it is equal to

$$R(\hat{n}, \theta) = \exp[i\theta \vec{\hat{J}} \cdot \hat{n}],$$

where $\overrightarrow{\hat{J}} = (\hat{J}^1, \hat{J}^2, \hat{J}^3)$. It can be shown that, given any other rotation $g \in SO(3)$

$$gR(\hat{n}, \theta)g^{-1} = R(g^{-1}\hat{n}, \theta), \tag{3.13}$$

and since, by properties of the adjoint map, $g \exp[i\theta\overrightarrow{\hat{J}} \cdot \hat{n}]g^{-1} = \exp[ig\theta\overrightarrow{\hat{J}} \cdot \hat{n}g^{-1}] = \exp[i\theta(g\overrightarrow{\hat{J}}g^{-1}) \cdot \hat{n}]$, where $g\overrightarrow{\hat{J}}g^{-1}$ is notation for $\sum_k g\hat{J}^k g^{-1}$, the previous equation implies, by differentiating the above equation with respect to θ at 0, that

$$(g\overrightarrow{\hat{J}}g^{-1}) \cdot \hat{n} = (g^{-1}\hat{n}) \cdot \overrightarrow{\hat{J}}. \tag{3.14}$$

In particular, having $\hat{n}$ be equal to $(1,0,0)^T$, $(0,1,0)^T$ or $(0,0,1)^T$ this equation becomes

$$g\hat{J}^k g^{-1} = \sum_i (g^{-1})^k_i \hat{J}^i. \tag{3.15}$$

respectively, for $k = 1$, $k = 2$, $k = 3$. The representation $\hat{\pi}_N$ of the group $SO(3)$ is equivalent to an algebra representation $\hat{\pi}_N$ of $U(\mathfrak{so}(3))$, so applying it on both sides results in

$$\hat{\pi}_N(g)\pi_j(J^k)\hat{\pi}_N(g)^* = \sum_i (g^{-1})^k_i \pi_j(J^i), \tag{3.16}$$

where we used the $(SU(2), SO(3))$-equivariance of the isomorphism $U(\mathfrak{su}(2)) \cong U(\mathfrak{so}(3))$ to conclude that $\pi_j(J^k) = \hat{\pi}_N(\hat{J}^k)$.

Thus, the $SO(3)$-contravariance transformation of the operators $\hat{x}^i = \frac{1}{\sqrt{j(j+1)}}\pi_j(J^i)$ follows. $\square$

Under a simple multiplication of the definition of G-contravariance transformation of the coordinates, the previous proposition can be generalized to the statement that, for each $N \in \mathbb{N}$, the position coordinates $\hat{x}^i$ defined in (2.10) transform contravariently under $SU(2)$.

Criteria for Position Observables

Suppose that there is a sequence of quantum theories, with algebras $\{\mathcal{A}_N\}_{N\in\mathbb{N}}$, whose limit is the quantum theory of a spin-less particle on S^d; Theorem 3.5.1

will prove that this is the case for the low energy effective theories associated to a fuzzy space produced through the constructions of this chapter. Fiore and Pisacane in [5] propose the following minimal criteria for a set of operators $\chi^i \in \mathcal{A}_N$, $i = 1, \ldots, D$, with spectrum Σ_i^N, $N \in \mathbb{N}$, to be a good $O(D)$-covariant approximation of the position operators x^i on $L^2(S^d)$:

1. For all N, the spectrum Σ_i^N of χ^i is invariant under the action of $O(D)$, including inversion $x^i \mapsto -x^i$. In particular, the spectra Σ_i of χ^i are all equal.

2. The sequence of spectra $\{\Sigma_i^N\}_{N\in\mathbb{N}}$ becomes uniformly dense[1] in $[-1, 1]$ in the limit $N \to \infty$. In particular, the maximal and minimal eigenvalues are a sequence converging to -1 and 1, respectively.

Proposition 3.1.10. On the elements of the fuzzy sphere $\mathcal{A}_N$ indexed by even N, let $\hat{x}^i = \frac{1}{\sqrt{j(j+1)}}\pi_j(J_i) \in \mathcal{A}_{2j} = End(V_j)$, $2j \in \mathbb{N}$. Then these operators satisfy the criteria, but replacing the group $O(3)$ by $SO(3)$.

Proof. We saw in Proposition 3.1.9 that those algebra elements transform $SO(3)$-contravariently; we will bring the notation from the proof of said proposition. Equation (3.14) means that acting with $g \in SO(3)$ on the infinitesimal rotation $i\vec{J} \cdot \hat{n}$ produces again an infinitesimal rotation, namely $i(g^{-1}\hat{n})\cdot\vec{J}$, and so the spectrum of $\hat{\pi}_N(J^i)$ is invariant under actions of $SO(3)$. Thus, the spectrum of each $\hat{x}_i$ given N is

$$\Sigma_i^N = \left\{ -\sqrt{\frac{j}{j+1}}, \frac{-j+1}{\sqrt{j(j+1)}}, \ldots, \frac{j-1}{\sqrt{j(j+1)}}, \sqrt{\frac{j}{j+1}} \right\},$$

and it is invariant under the action of $SO(3)$.

The sequence of spectra is clearly uniformly dense in $[-1, 1]$.

□

[1] For any point $x \in [-1, 1]$ and any open ball $U \subseteq [-1, 1]$ containing x, there is some $N \in \mathbb{N}$ such that, for all $n \geq N$, $U \cap \Sigma_i^N$ is not empty, and, furthermore, it is evenly distributed within U

3.2 Construction of $\mathcal{A}_{\overline{E}}$ for $D = 2$

When decomposing the eigenfunctions of the Schrodinger equation (3.1) into radial and angular components, $\psi(r, \phi) = \tilde{f}(r)Y(\phi)$, the angular equation $L^2 Y = EY$ for $D = d + 1 = 2$ is

$$-\partial_\phi^2 Y = EY, \tag{3.17}$$

since the only angular momentum component is $L_{12} := -L_{21} := -i(x^1\partial_2 - x_2\partial_1) = -i\partial_\phi$, we define the angular momentum operator to be

$$L := -i\partial_\phi. \tag{3.18}$$

The spherical harmonics are, then, labeled by an integer m and:

$$Y_m(\phi) = e^{im\phi}, \qquad m \in \mathbb{Z} \tag{3.19}$$

$$E_m = m^2. \tag{3.20}$$

Defining $\rho := \ln r$ and $f(\rho) := \tilde{f}(r)$, the radial equation (3.6) becomes $f''(\rho) + \{e^{2\rho}[E - V(e^\rho)] - m^2\}f(\rho) = 0$, and expanding $e^{2\rho}$ about $\rho = 0$ we obtain the following harmonic approximation of the radial equation (3.6):

$$[-\partial_\rho^2 + k_m(\rho - \tilde{\rho}_m)^2]f(\rho) = e_m f(\rho), \text{ where} \tag{3.21}$$

$$k_m := 2(k - E'_m), \quad E'_m := E_m - V_0, \quad \tilde{\rho}_m := \frac{E'_m}{k_m}, \quad e_m = \frac{E'^2_m}{k_m} + E'_m - m^2. \tag{3.22}$$

When solving this equation, taking into account that $\psi = f(\rho)Y(\phi)$ an additional label $n \in \mathbb{Z}_{\geq 0}$ appears for the eigenvalues of the previous equation; these solutions are:

$$\begin{aligned} f_{n,m}(\rho) &= N_m \exp\left[-\frac{(\rho - \tilde{\rho}_m\sqrt{k_m})}{2}\right] H_n\left[(\rho - \rho_m)\right], \\ e_{n,m} &= (2n+1)\sqrt{k_m}, \end{aligned} \tag{3.23}$$

where N_m is a normalization constant and the H_n are the Hermite polynomials, H_n being a polynomial of degree n on the dependent variable; in particular, $H_0 = 1$.

From equation (3.22) it follows that $E'_m \equiv E'_{n,m} = E_{n,m} - V_0$ satisfies

$$\frac{E'_{n,m}}{2(k - E'_{n,m})} + E'_{n,m} - m^2 = (2n+1)\sqrt{2(k - E'_{n,m})}; \tag{3.24}$$

by squaring both sides we obtain a quartic equation for $E'_{n,m}$ that makes it a function of k, and hence we obtain the original eigenvalues of the approximate Schrödinger equation (3.21) $E_{n,m}$ *as a functions of* k *and* V_0.

As was mentioned in the previous section, in order to get simple eigenenergies we fix V_0 such that the ground energy $E_{0,0}$ equals 0. To do this we replace $m = n = 0$ in equation (3.24) and use $E_{0,0} = 0$. From there, we deduce that V_0 *has a quartic equation that determines it a function of* k that can be rewritten as:

$$-\sqrt{\frac{1}{2k}}V_0 - \left(\sqrt{\frac{1}{2k}}\right)^3 V_0^2 = \left(1 + \frac{V_0}{k}\right)^{\frac{3}{2}}. \tag{3.25}$$

$V_0(k)$ has a unique solution root over the reals, although it is possible to find a closed formula for it since it is determined by a quadratic equation, we opt for the use of its Taylor series expansion with respect to the variable $1/\sqrt{k}$, at $k = \infty$, by using the implicit formula (3.25); the resulting expansion is:

$$V_0(k) = -\sqrt{2k} + 2 - \frac{7}{2}\frac{1}{\sqrt{2k}} + \frac{5}{2k} + O\left(\frac{1}{k^{\frac{3}{2}}}\right). \tag{3.26}$$

Finally, this expansion in k of V_0 induces the following expansion of the eigenenergies of the harmonic approximation of Schrodinger equation:

$$\begin{aligned} E_{n,m} = (m^2 - 2 - 8n - 8n^2) + \sqrt{2}(1+2n)\sqrt{k} \\ + \frac{(1+2n)(7 - 6m^2 + 28n + 28n^2)}{2\sqrt{2k}} + O\left(\frac{1}{k}\right). \end{aligned} \tag{3.27}$$

The final step to build the Hilbert space $\mathcal{H}_{\overline{E}}$ and $\mathcal{A}_{\overline{E}}$ given a cutoff energy $\overline{E} \geq 0$ is to make sure that k is steep enough to ensure that $E_{1,0}$ is above $\overline{E}$, and this is expressed by the inequality

$$\overline{E} < 3\sqrt{2k} - 18 \lessapprox E_{1,0}. \tag{3.28}$$

In that case, renaming $\psi_{0,m}$ and $E_{0,m}$ as ψ_m and E_m for $m \in \mathbb{Z}$, we define the spaces

$$\begin{aligned} \mathcal{H}_{\overline{E}} &:= span\{\psi_m(\rho,\phi) \,:\, E_m \leq \overline{E}\} \\ \mathcal{A}_{\overline{E}} &:= End(\mathcal{H}_{\overline{E}}); \end{aligned} \tag{3.29}$$

where

$$\psi_m(\rho,\phi) = c_m e^{im\phi} \exp\left[-\frac{(\rho-\tilde{\rho}_m)\sqrt{k_m}}{2}\right], \quad \text{where} \tag{3.30}$$

$$k_m = 2k\left(1 - \frac{2}{\sqrt{2k}} + \frac{2-m^2}{k} + O\left(\frac{1}{k^{\frac{3}{2}}}\right)\right), \quad \tilde{\rho}_m = \frac{1}{\sqrt{2k}} + \frac{m^2}{2k} + O\left(\frac{1}{k^{\frac{3}{2}}}\right). \tag{3.31}$$

Remark 3.2.1. Notice that, in principle, the definition of $\mathcal{A}_{\overline{E}}$ is dependent on the chosen k due to $k_m(k)$ and $\tilde{\rho}_m(k)$, so we may write instead $\psi_{n,m}(\rho,\phi)$ as $\psi_{n,m}(\rho,\phi;k)$; also notice that, for fixed k and two cutoffs $\overline{E}_1$ and $\overline{E}_2 > \overline{E}_1$ compatible with this k, $\mathcal{H}_{\overline{E}_1}$ is a subspace of $\mathcal{H}_{\overline{E}_2}$ since its basis elements ψ_m are also basis elements of the latter, but only due to the fixed choice of k for both cutoffs. Furthermore, since k is chosen large, all ψ_m essentially vanish outside the k dependent region $\nu_{\overline{E}}$, for m fixed, and, in fact, $\psi_m(\rho,\phi) \to e^{im\phi}\delta(r-1)$ in probability as $k \to \infty$. However, **the dependence on k of the definition of the ψ_m and, in consequence, of $\mathcal{A}_{\overline{E}}$ will not be relevant to its individual study as a noncommutative space**, since for that all we need is algebra structure and the action of the relevant symmetry spaceIndeed, in Theorem 3.4.5 we will prove that $\mathcal{A}_{\overline{E}}$ *is isomorphic to* $\mathcal{A}'_{\overline{E}} := End(\mathcal{H}'_{\overline{E}})$ *as a C^*-algebra and as a representation space of* $O(2)$.

Now, to parametrize these algebras by $\Lambda \in \mathbb{N}$ and obtain a sequence of approximations of quantum mechanics in S^1 and a fuzzy space approximating S^2, define:

Definition 3.2.2. For $\Lambda \in \mathbb{N}$, let

$$\begin{aligned} \mathcal{H}_\Lambda &:= \mathcal{H}_{\overline{E}=\Lambda^2} \\ \mathcal{A}_\Lambda &:= End(\mathcal{H}_\Lambda). \end{aligned} \tag{3.32}$$

This means that the energies of the basis elements of $\mathcal{H}_\Lambda$ are all those $m^2 \in \mathbb{N}$, up to the constant term Λ^2, and are below the energies $E_{1,0}$; additionally, we also need to choose k as a function on Λ in such a way that it diverges with Λ and such that $k(N)$ is compatible with the cutoff energy Λ; for this, the equation (3.28) tells us that we may choose any function $k = k(\Lambda)$ such that

$$k(\Lambda) \geq \frac{\left(\Lambda^2+18\right)^2}{18} \overset{\Lambda\geq 2}{\leq} \Lambda^2(\Lambda+1)^2; \tag{3.33}$$

that such a sequence of algebras do indeed approximate quantum mechanics in S^1 is made precise in Theorem 3.5.1.

Although the precise value of $k(\Lambda)$ once (3.33) holds isn't relevant for the study of a single algebra $\mathcal{A}_\Lambda$, it plays an important role in the convergence of the sequence of algebras when the sequence is made into a fuzzy sphere; see Theorem 3.5.2. The value of $k(\Lambda)$ will also determine the accuracy of the approximations of the operators that we will use in the next section, including that of the candidates for position operators.

3.3 Important Observables and their Commutation Relations

Throughout this section let Λ be any natural number, let k be such that equation (3.33) is satisfied, and let $\mathcal{H}_\Lambda = span\{\psi_m\}_{|m|\leq\Lambda}$ and $\mathcal{A}_\Lambda$ be as defined in (3.32).

3.3.1 Quantum Mechanics in $\mathbb{R}^2$

The algebra of observables $\mathcal{A} = \mathcal{B}(\mathcal{H})$ of the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^2)$ is generated by the coordinate functions x^j and the corresponding momentum operators $p_j = -i\partial_j$, for $j = 1, D = 2$ which satisfy the canonical commutation relations

$$[x^i, x^j] = 0, \qquad [p_i, p_j] = 0, \qquad [x^i, p_j] = i\delta^{ij}. \tag{3.34}$$

Let us define

$$x^\pm := x^1 \pm ix^2, \tag{3.35}$$

and

$$\begin{aligned}\partial_\pm &:= \partial_{x^\pm} \\ &= (\partial_1 \mp i\partial_2).\end{aligned} \tag{3.36}$$

The set of operators $x^\pm, \partial_\pm$ also generate the algebra of observables $\mathcal{A}$ since each x^i is linear combinations of x^+ and x^- and, similarly, each ∂_i is linear combination of ∂_+ and ∂_-.

The angular momentum operator $L = -i\partial_\phi$ can also be written as $L = \frac{1}{2}(x^+\partial_+ - x^-\partial_-)$, hence

$$[L, x^\pm] = \pm x^\pm, \qquad\qquad [L, \partial_\pm] = \mp\partial_\pm. \tag{3.37}$$

Proposition 3.3.1. *L induces the decomposition $\mathcal{H} = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}^m$ into orthogonal subspaces, where $\mathcal{H}^m = \{g(r)e^{im\phi} : g \in L^2(\mathbb{R}_{\geq 0})\}$ is the eigenspace of L associated to the eigenvalue $m \in \mathbb{Z}$ of L.*

Proof. This is a consequence of the application of the spectral theorem for the (unbounded) hermitian operator L. □

Rewriting the commutation relations (3.37) as

$$Lx^\pm = x^\pm(L \pm 1), \qquad\qquad L\partial_\pm = \partial_\pm(L \mp 1), \tag{3.38}$$

we see that

$$x^\pm\psi_m \in \mathcal{H}^{m\pm1}, \qquad \partial_\pm\psi^m \subseteq \mathcal{H}^{m\mp1}, \qquad \text{for any } \psi_m \in \mathcal{H}^m, \tag{3.39}$$

when the applications of $x^\pm$ and $\partial_\pm$ can be made and the result falls into $\mathcal{H}$; this means that, for example, $x^\pm\psi_m$ is an eigenvector of L for the eigenvalue $m \pm 1$, if ψ_m is eivenvalue for $m \in \mathbb{Z}$.

3.3.2 General Facts about $\mathcal{A}_\Lambda$

Proposition 3.3.2.

- The elements $\overline{x^+}$ and $\overline{x^-}$ of $\mathcal{A}_\Lambda$ are adjoints.

- $\overline{L} = L|_{\mathcal{H}_\Lambda}$. Therefore, $\mathcal{H}_\Lambda$ has the orthogonal decomposition $\mathcal{H}_\Lambda = \bigoplus_{m=-\Lambda}^{\Lambda} \mathcal{H}^m$.
- $[\overline{L}, \overline{A}] = \overline{[L, A]}$ for any $A \in \mathcal{A}$ to which the map from Proposition 3.1.4 can be applied and domain for which the operator $[L, A]$ exists.

Proof. It is easy to see that the adjoint of an orthogonal projection is itself. It follows from Proposition 3.3.1 that the projection $P_\Lambda = \sum_{m=-\lambda}^{\Lambda} \tilde{P}_m$ is an orthogonal projection, where $\tilde{P}_m$ is the orthogonal projection onto $\mathcal{H}^m$. Hence, $\overline{x^\pm}^* = (P_\lambda x^\pm P_\Lambda)^* = P_\Lambda^*(x^\pm)^* P_\Lambda^* = \overline{x^\mp}$.

The second statement follows from the fact that each ψ_m is eigenvector of L for the eigenvalue $m \in \{-\Lambda, \dots, \Lambda\}$, since the ψ_m make up a basis of $\mathcal{H}_\Lambda$.

Notice that $[\overline{L}, P_\Lambda] = 0$, so

$$\begin{aligned}
[\overline{L}, \overline{A}] &= P_\Lambda L P_\Lambda A P_\Lambda - P_\Lambda A P_\Lambda L P_\Lambda \\
&= P_\Lambda^2 L A P_\Lambda - P_\Lambda A L P_\Lambda^2 \\
&= P_\Lambda [L, A] P_\Lambda \\
&= \overline{[L, A]}.
\end{aligned}$$

□

Applying the last item of Proposition 3.3.2 to $A = x^\pm$ gives us a result analogous to equation (3.37), and therefore, applying the second statement, an analog of equation (3.39). In fact, the following useful generalization applies:

Definition 3.3.3. Let $\mathcal{A}_\Lambda^n$, for $n \in \{-2\Lambda, \dots, 2\Lambda\}$, be the subspace of $\mathcal{A}_\Lambda$ for which, for any $A^n \in \mathcal{A}_\Lambda^n$

$$A^n \psi_m \in \begin{cases} A^{n+m}, & \text{if } |n+m| \leq \Lambda \\ 0, & \text{otherwise} \end{cases},$$

where $m \in \{-\Lambda, \dots, \Lambda\}$ and $\psi_m \in \mathcal{H}^m$.

We can see that an element $A \in \mathcal{A}_\Lambda$ is in $\mathcal{A}_\Lambda^n$ if and only if $[\overline{L}, A] = nA$, since this can be rewritten as $\overline{L}A = A(\overline{L} + n)$. In particular,

$$\overline{x^\pm} \in \mathcal{A}_\Lambda^{\pm 1}. \tag{3.40}$$

We will now find generators for $\mathcal{A}_\Lambda$, and in the meantime we will develop a deeper understanding of $\mathcal{A}_\Lambda$.

Proposition 3.3.4.

- The orthogonal projections $\tilde{P}_m \in \mathcal{A}_\Lambda$ onto $\mathcal{H}_m$, for all $m \in \{-\Lambda, \dots, \Lambda\}$ are polynomials of $\overline{L}$ of degree at most 2Λ.
- Any operator $A \in \mathcal{A}_\Lambda^0$ can be written as a polynomial in L of degree at most 2Λ.

Proof. Since $\{\psi_m\}$ is a basis of $\mathcal{H}_\Lambda$, the polynomial $\prod_{n=-\Lambda}^{\Lambda}(L-n) = 0$, and so the image of $\prod_{n\neq m}(L-n)$ is a subset of $\mathcal{H}^m$; in fact,

$$\tilde{P}_m = \frac{\prod_{n\neq m}(L-n)}{\prod_{n\neq m}(m-n)} \tag{3.41}$$

is a polynomial formula for $\tilde{P}_m$.

To prove the second statement, notice that any $A \in \mathcal{A}_\Lambda^0$ satisfies $A\psi_m = A(m)\psi_m$ for some complex valued function $A(m)$ defined on $m \in \{-\Lambda, \dots, \Lambda\}$. Hence A may be written as

$$A = \sum_{m=-\Lambda}^{\Lambda} A(m)\tilde{P}_m, \tag{3.42}$$

which is a polynomial in L of degree at most 2Λ. □

Definition 3.3.5. Let $S^\pm \in \mathcal{A}_\Lambda$ be operators such that

$$S^\pm\psi_m = \psi^{m\pm1}, \quad \text{for } m \in \{-\Lambda, \dots, \Lambda\} \subseteq \mathbb{Z} \text{ such that } |m \pm 1| \leq \Lambda; \tag{3.43}$$

S^+ is called *the raising operator on* $\mathcal{H}_\Lambda$ with respect to the basis $\{\psi_m\}_{|m|\leq\Lambda}$, S^- is called *the lowering operator.* Let $S^n \in \mathcal{A}_\Lambda$, for $n \in \{-2\Lambda, \dots, 2\Lambda\} \subseteq \mathbb{Z}$ be

$$S^n := \begin{cases} (S^+)^n & n \geq 0 \\ (S^-)^{-n} & n \leq 0; \end{cases} \tag{3.44}$$

collectively, the operators S^n are called *the ladder operators on* $\mathcal{H}_\Lambda$ *with respect to the basis* $\{\psi_m\}_{|m|\leq\Lambda}$.

Lemma 3.3.6. Any set of elements $U, D, \overline{L}$, with $U \in \mathcal{A}^1_\Lambda$, $D \in \mathcal{A}^{-1}_\Lambda$ generate the algebra $\mathcal{A}_\Lambda$.

Proof. Let

$$T^l := \begin{cases} U^l, & \text{if } l \geq 0 \\ D^l, & \text{if } -l \leq 0; \end{cases}$$

then $T^l \in \mathcal{A}^l_N$ if $l \in \{-2\Lambda, \dots, 2\Lambda\}$. This operator works similarly to a ladder operator, but with some undesired coefficients appearing after its application. Concretely, the ladder operator with respect to the basis $\{\psi_m\}$ can be written as

$$S^l = \sum_{m=-\Lambda}^{\Lambda} ST(l,m) T^l \tilde{P}_m, \quad \text{where } ST(l,m) = \begin{cases} \frac{1}{\langle \psi_{m+l} | T^l | \psi_m \rangle}, & \text{if } |m+l| \leq \Lambda \\ 0 & \text{otherwise;} \end{cases}; \tag{3.45}$$

thus, the ladder operator is a polynomial in U, D and L.

Now, an arbitrary element A of $\mathcal{A}_\Lambda$ may be written as the polynomial in U, D and L: $A = \sum_{m,n} A(m,n) S^{n-m} \tilde{P}_m$, for the function $A(m,n) = \langle \psi_n | A | \psi_m \rangle$ defined on the discrete set $m, n \in \{-\Lambda, \dots, \Lambda\}$. □

Lemma 3.3.7. $\overline{L} \in \mathcal{A}_\Lambda$ is a polynomial in U and D, for any $U \in \mathcal{A}^1_\Lambda$ and $D \in \mathcal{A}^{-1}_\Lambda$.

Proof. Notice that $U^{\Lambda+m} D^{2\Lambda} U^{\Lambda-m} \in \mathcal{A}^0_\Lambda$ for all $m \in \{-\Lambda, \Lambda\}$, hence the following is one such polynomial for $\overline{L}$:

$$\overline{L} = \sum_{m=-\Lambda}^{\Lambda} \frac{m}{\langle \psi_{m+l} | U^{\Lambda+m} D^{2\Lambda} U^{\Lambda-m} | \psi_m \rangle} U^{\Lambda+m} D^{2\Lambda} U^{\Lambda-m}. \tag{3.46}$$

□

Theorem 3.3.8. Let $U \in \mathcal{A}^1_\Lambda$ and $D \in \mathcal{A}^{-1}_\Lambda$ be arbitrary, e.g. $U = \overline{x^+}$ and $D = \overline{x^-}$. Then U and D generate $\mathcal{A}_\Lambda$.

Proof. This follows directly from the combination of Lemmas 3.3.6 and 3.3.7. □

3.3.3 Approximate Action of Effective Observables

We now find explicit values for the action of $\overline{x^{\pm}}$ and related observables by using the Proposition 6.1 in [3] by Fiore and Pisacane:

Proposition 3.3.9. For every entire function $f(\rho)$ not depending on k, and $h \in \mathbb{Z}$, the matrix elements of the operator $f(\rho)e^{ih\phi}$ are:

$$\langle \psi_{m'} | f(\rho) e^{ih\phi} | \psi_m \rangle = \delta_{h,m'-m} K_{m,m'}(k) \exp\left[\frac{\partial_\rho^2|_{\rho=\rho_{m,m'}}}{4c_{m,m'}(k)} f(\rho) \right] \tag{3.47}$$

with,

$$\begin{aligned} c_{m,m'}(k) &:= \frac{\sqrt{k_m}+\sqrt{k_{m'}}}{2} = \sqrt{2k} - 1 + \frac{3-m^2-m'^2}{2\sqrt{2}} \frac{1}{\sqrt{k}} \\ &\qquad - \frac{2+m^2+m'^2}{2} \frac{1}{k} + O\left(\frac{1}{k^{\frac{3}{2}}}\right) \\ \rho_{m,m'}(k) &:= \frac{2+\sqrt{k_m}\tilde{\rho}_m + \sqrt{k_{m'}}\tilde{\rho}_{m'}}{2c_{m,m'}} = \sqrt{2}\frac{1}{\sqrt{k}} + \frac{2+m^2+m'^2}{4}\frac{1}{k} + O\left(\frac{1}{k^{\frac{3}{2}}}\right) \\ K_{m,m'}(k) &:= \sqrt{\frac{4\pi^3}{c_{m,m'}}} N_m \overline{N_{m'}} e^{\frac{[2+\sqrt{k_m}\tilde{\rho}_m+\sqrt{k_{m'}}\tilde{\rho}_{m'}]^2}{4c_{m,m'}} - \frac{\sqrt{k_m}\tilde{\rho}_m^2+\sqrt{k_{m'}}\tilde{\rho}_{m'}^2}{2}} = 1 + O\left(\frac{1}{k^{\frac{3}{2}}}\right), \end{aligned} \tag{3.48}$$

where $k_m(k)$ and $\tilde{\rho}_m(k)$ were defined in (3.22)

$$N_m(k) = \sqrt[4]{\frac{\sqrt{k_m}}{4\pi^3}} e^{-\frac{1}{2\sqrt{k_m}} - \tilde{\rho}_m} = 1 - \frac{3}{2\sqrt{2}}\frac{1}{\sqrt{k}} + \frac{5-8m^2}{16}\frac{1}{k} + \frac{15}{32\sqrt{2}}\frac{1}{k^{\frac{3}{2}}} + O(\frac{1}{k^2}) \tag{3.49}$$

is the normalization factor of ψ_m up to a phase.

Remark 3.3.10. Let $F(x)$ and $f(x)$ be functions with uniformly convergent power series expansions around $f(x_0)$ and $x_0 \in \mathbb{R}$ respectively, for a non-zero radius of convergence. Then, to know the nth coefficient of the power expansion of $F \circ f$ around $f(x_0)$ (assuming the radius of convergence is non-zero) we only need know the first n coefficients for both f and F of the previously mentioned power series. In our case this, using the power expansions of (3.48) up to order 2 in $\frac{1}{\sqrt{k}}$ we can calculate the matrix elements of operators $f(\rho)e^{ih\phi}$ up to the same order.

Proposition 3.3.11. The operators $\overline{x^\pm}$, $[\overline{x^+},\overline{x^-}]$ and R^2 in $\mathcal{A}_\Lambda$ act as follows:

$$\overline{x^\pm}\psi_m = \begin{cases} \left(1+\frac{9\sqrt{2}}{8}\frac{1}{\sqrt{k}}+\frac{137\pm 32m+32m^2}{64}\frac{1}{k}+O\left(\frac{1}{k^{\frac{3}{2}}}\right)\right)\psi_{m\pm1}, & \text{if } |m+\frac{1}{2}|\le \Lambda-\frac{1}{2} \\ 0, & \text{otherwise;} \end{cases} \tag{3.50}$$

$$[\overline{x^+},\overline{x^-}]\psi_m = \begin{cases} \left(-\frac{2m}{k}+O\left(\frac{1}{k^{\frac{3}{2}}}\right)\right)\psi_m, & \text{if } |m|\le\Lambda-1 \\ \pm\left(1+\frac{9}{2\sqrt{2}}\frac{1}{\sqrt{k}}+\frac{109-16\Lambda+16\lambda^2}{16}\frac{1}{k}+O\left(\frac{1}{k^{\frac{3}{2}}}\right)\right)\psi_m, & \text{if } m=\pm\Lambda; \end{cases} \tag{3.51}$$

$$R^2\psi_m = \begin{cases} \left(1+\frac{9}{2\sqrt{2}}\frac{1}{\sqrt{k}}-\left(\frac{109}{16}+m^2\right)\frac{1}{k}+O\left(\frac{1}{k^{\frac{3}{2}}}\right)\right)\psi_m, & \text{if } |m|\le\Lambda-1 \\ \left(1+\frac{9}{4\sqrt{2}}\frac{1}{\sqrt{k}}+\frac{109-16\Lambda+16\lambda^2}{32}\frac{1}{k}+O\left(\frac{1}{k^{\frac{3}{2}}}\right)\right)\psi_m, & \text{if } m=\pm\Lambda; \end{cases} \tag{3.52}$$

where $R^2 := \overline{x^1}^2+\overline{x^1}^2 = \frac{1}{2}(\overline{x^+x^-}+\overline{x^-x^+})$.

Proof. First notice that $x^\pm\in\mathcal{A}_\Lambda^{\pm1}$ so $[\overline{x^+},\overline{x^-}]$ and R^2 are in $\mathcal{A}_\Lambda^0$, hence no other matrix elements than the displayed above are nonzero.

Now, Proposition 3.3.9 implies that:

$$\begin{aligned} \langle\psi_{m+1}|x^+|\psi_m\rangle &= \langle\psi_{m+1}|e^\rho e^{i\phi}|\psi_m\rangle && = K_{m,m+1}e^{\rho_{m,m+1}+\frac{1}{4c_{m,m+1}}}; \\ \langle\psi_{m-1}|x^-|\psi_m\rangle &= \overline{\langle\psi_m|x^+|\psi_{m-1}\rangle} && = K_{m-1,m}e^{\rho_{m-1,m}+\frac{1}{4c_{m-1,m}}}. \end{aligned}$$

With the help of the software Mathematica, and having in mind the Remark 3.3.10, we found the power expansions for these expressions. $\square$

The matrix elements $\langle\psi_{m\pm1}|\overline{x^\pm}|\psi_m\rangle$ are visualized in Figure 3.2.

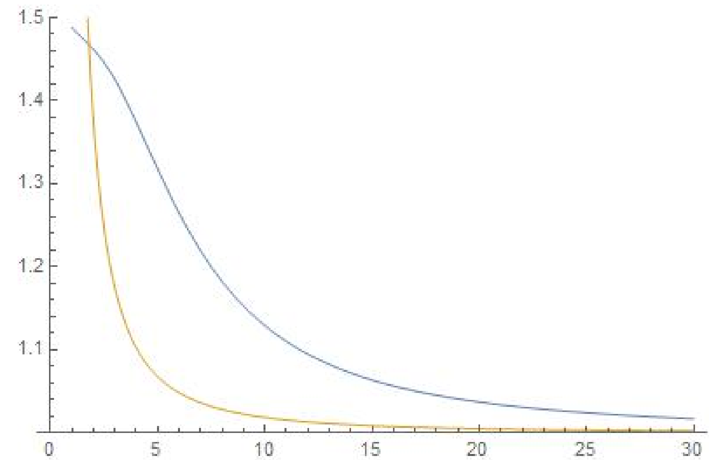


Figure 3.2: Matrix element $\langle\psi_\Lambda|\overline{x}^+|\psi_{\Lambda-1}\rangle = \langle\psi_{-\Lambda}|\overline{x}^-|\psi_{1-\Lambda}\rangle$ as a function of Λ. Blue: for $k(\Lambda) = \frac{(\Lambda^2+18)^2}{18}$. Yellow: for $k(\Lambda) = \Lambda^2(\Lambda+1)^2$.

Definition 3.3.12. *The normalization constant of the position operators* is the function in k:

$$\begin{aligned} a(k) &:= \langle\psi_1|x^+|\psi_0\rangle = \langle\psi_{-1}|x^-|\psi_0\rangle \\ &= K_{0,1}(k)e^{\rho_{0,1}(k)+\frac{1}{4c_{0,1}(k)}} \\ &= 1 + \frac{9}{4\sqrt{2}}\frac{1}{\sqrt{k}} + \frac{137}{64}\frac{1}{k} + \frac{715}{256\sqrt{2}}\frac{1}{k^{\frac{3}{2}}} + O\left(\frac{1}{k^2}\right). \end{aligned} \tag{3.53}$$

The normalized position operators are, for $i = 1, 2$:

$$\begin{aligned} \chi^i &:= \frac{\overline{x^i}}{a(k)}, \\ \chi^\pm &:= \frac{\overline{x^\pm}}{a(k)} = \chi^+ \pm i\chi^-. \end{aligned} \tag{3.54}$$

The normalization constant a is simply the factor that extracts the m independent terms in the series expansion of the action of $\overline{x^\pm}$ on each ψ_m. The factor a rapidly converges to 1 with increasing k, as shown in the Figure 3.3.

Proposition 3.3.13. Up to second order of powers in $\frac{1}{\sqrt{k}}$ in the coefficients

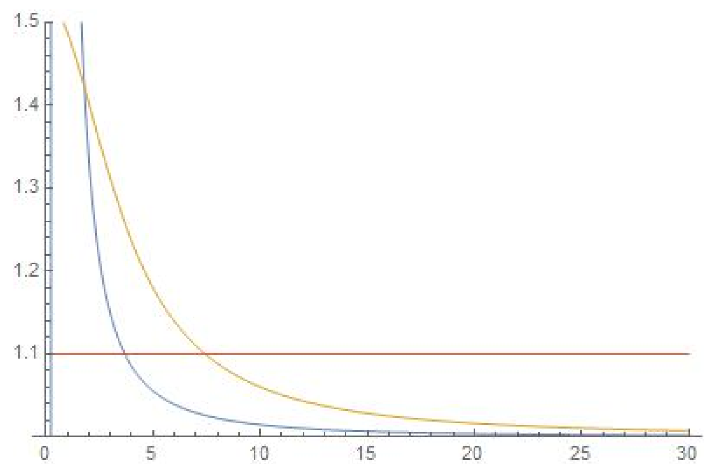


Figure 3.3: Normalizing factor a as a function of Λ. Yellow: for $k(\Lambda) = \frac{(\Lambda^2+18)^2}{18}$. Blue: for $k(\Lambda) = \Lambda^2(\Lambda+1)^2$.

the following relations hold:

$$\chi^{\pm}\psi_m = \begin{cases} \sqrt{1 + \frac{m(m+1)}{k}}\psi_{m\pm1}, & \text{if } -\Lambda \le \pm m \le \Lambda - 1 \\ 0 & \text{otherwise;} \end{cases} \tag{3.55}$$

$$[\chi^+, \chi^-]\psi_m = \begin{cases} -\frac{2m}{k}\psi_m, & \text{if } |m| \le \Lambda - 1 \\ \pm\left(1 + \frac{\Lambda(\Lambda-1)}{k}\right)\psi_m, & \text{if } m = \pm\Lambda; \end{cases} \tag{3.56}$$

$$\vec{\chi}^2\psi_m = \begin{cases} \left(1 + \frac{m^2}{k}\right)\psi_m, & \text{if } |m| \le \Lambda - 1 \\ \frac{1}{2}\left(1 + \frac{\Lambda(\Lambda-1)}{k}\right)\psi_m, & \text{if } m = \pm\Lambda; \end{cases} \tag{3.57}$$

where $\vec{\chi}^2 := (\chi^1)^2 + (\chi^2)^2 = \frac{1}{2}(\chi^+\chi^- + \chi^-\chi^+) = R^2/a^2$.

The coefficient $\langle\psi_{m\pm1}\chi^+|\psi_m\rangle$ is visualized in Figures 3.4 and 3.5.

Remark 3.3.14. Notice that if the action (3.55) were used exactly to define operators $\chi^{\pm}$, then equations (3.56) and (3.57) would also be exact. This is thanks to the fact that, although the square root in (3.55) is only valid up to the power $\frac{1}{k}$, i.e. it is only meaningful to say that it is equal to $1 + \frac{m(m+1)}{2}$, when we use it to calculate the commutator and the radius it produces precisely the correct first order terms stated above.

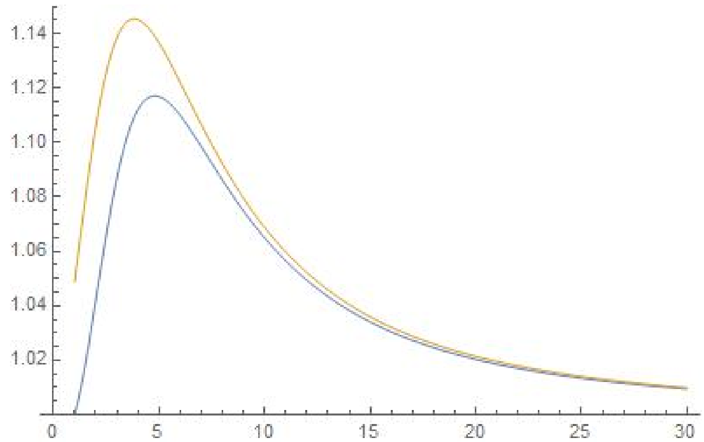


Figure 3.4: Maximum value of the coefficient $\langle\psi_{m\pm1}|\chi^\pm|\psi_m\rangle$ as a function of Λ for $k(\Lambda)=\frac{(\Lambda^2+18)^2}{18}$. Blue: exact value. Yellow: approximation $\sqrt{1+\frac{\Lambda(\Lambda+1)}{k}}$.

Theorem 3.3.15 (Summary)**.** χ^+ and χ^- generate the $*$-algebra $\mathcal{A}_\Lambda = End(\mathcal{H}_\Lambda)$. The generators $\chi^\pm$ and $\overline{L}$ satisfy the relations:

$$(\chi^+)^{2\Lambda+1}=0,\quad (\chi^-)^{2\Lambda+1}=0,\quad (\chi^+)^*=\chi^-,\quad \prod_{m=\Lambda}^{\Lambda}(\overline{L}-m),=0\quad \overline{L}^*=\overline{L}, \tag{3.58}$$

$$[\overline{L},\chi^\pm]=\pm\chi^\pm,\qquad [\chi^+,\chi^-]=-\frac{2\overline{L}}{k}+\left[1+\frac{\Lambda(\Lambda+1)}{k}\right](\tilde{P}_\Lambda-\tilde{P}_\Lambda), \tag{3.59}$$

where the projections $\tilde{P}_m$ are polynomials in $\overline{L}$. Furthermore,

$$\vec{\chi}^2=1+\frac{L^2}{k}-\left[1+\frac{\Lambda(\Lambda+1)}{k}\right]\frac{\tilde{P}_\Lambda-\tilde{P}_\Lambda}{2}. \tag{3.60}$$

Equations $(3.59)_2$ and (3.60) are only valid up to second order in powers of $\frac{1}{\sqrt{k}}$; however, notice that *if the formulas* (3.55) *are used as exact definitions of* $\chi^\pm$*, then all the equations in this theorem are satisfied exactly.*

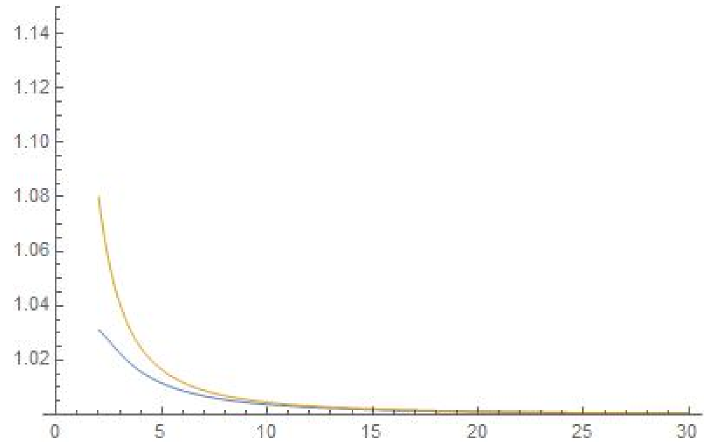


Figure 3.5: Maximum value of the coefficient $\langle\psi_{m\pm1}|\chi^{\pm}|\psi_m\rangle$ as a function of Λ for $k(\Lambda) = \Lambda^2(\Lambda+1)^2$. Blue: exact value. Yellow: approximation $\sqrt{1+\frac{\Lambda(\Lambda+1)}{k}}$.

3.4 Realization of $\mathcal{A}_\Lambda$ through $U(\mathfrak{so}(3))$

Throughout this section let $\Lambda \in \mathbb{N}$ and let $\mathcal{A}_\Lambda$ and $\mathcal{H}_\Lambda$ be defined as in Definition 3.2.2.

Lemma 3.4.1. Let A_1, A_2 be elements of $\mathcal{A}^n_\Lambda$, and suppose that $A_1\psi_m = 0$ implies that $A_2\psi_m = 0$ for $m \in \{-\Lambda, \dots, \Lambda\}$. Then there is a polynomial $f(\overline{L})$ in $\overline{L}$ such that $A_2 = f(\overline{L})A_1$. Furthermore, if f' is another polynomial such that $f'(\overline{L})\psi_m = f(\overline{L})\psi_m$ whenever $A_1\psi_m \neq 0$, then it is also satisfied that $A_2 = f'(\overline{L})A_1$.

Proof. Let g_1, g_2 be functions on $m \in \{-\Lambda, \dots, \Lambda\} \subseteq \mathbb{Z}$ such that $A_i = \sum_{m:\,|m+n|\leq\Lambda} g_i(m)S^{n-m}\tilde{P}_m$, where $S^{\pm1}$ are ladder operators; notice that $g_i(m)$ is arbitrary when $A_i\psi_m = 0$, in particular on m such that $|m+n| > \Lambda$, so we may choose the g_i nowhere 0. Define $f(m) = \frac{g_2(m-n)}{g_1(m-n)}$ if $|m+n| \leq \Lambda$, and choose it arbitrarily for the the remaining $m \in \{-\Lambda, \dots, \Lambda\}$. Thus, any polynomial $f(\overline{L}) := \sum_m f(m)\tilde{P}_m$ defined with such a function f on $\{-\Lambda, \dots, \Lambda\}$ makes the equation $A_2 = f(\overline{L})A_1$ be satisfied. $\square$

Proposition 3.4.2. Let $L^\pm \in \mathcal{A}_\Lambda$ be defined by

$$L^\pm\psi_m := \sqrt{\Lambda(\Lambda+1) - m(m\pm1)}, \quad \text{for all } m \in \{-\Lambda, \dots, \Lambda\} \subseteq \mathbb{Z}. \tag{3.61}$$

Then, $L^{\pm}$ generate $\mathcal{A}_\Lambda$ and there are polynomials $f^x_{\pm}(L)$, $f^\chi_{\pm}$, $f_{\pm}$ in $\overline{L}$ such that

$$\overline{x^{\pm}} = f^x_{\pm}(\overline{L})L^{\pm}, \qquad \overline{\chi^{\pm}} = f^\chi_{\pm}(\overline{L})L^{\pm}; \tag{3.62}$$

any such polynomials have the property that, for $|m \pm 1| \leq \Lambda$, the functions

$$f^x_{\pm}(m) := f^x_{\pm}(L)\psi_m = \frac{1 + \frac{9\sqrt{2}}{8}\frac{1}{\sqrt{k}} + \frac{137 \pm 32(m\mp 1) + 32(m\mp 1)^2}{64}\frac{1}{k}}{\sqrt{\Lambda(\Lambda+1) - m(m\mp 1)}}, \text{ and} \tag{3.63}$$

$$f^\chi_{\pm}(m) := f^\chi_{\pm}(L)\psi_m = \sqrt{\frac{1 + \frac{m(m\mp 1)}{k}}{\Lambda(\Lambda+1) - m(m\mp 1)}} \tag{3.64}$$

satisfy the equalities above up to second order in powers of $\frac{1}{\sqrt{k}}$.

Similarly, if equations (3.55) are used exactly to define the operators $\chi^{\pm} \in \mathcal{A}_\Lambda$, then equation (3.64) is exact too.

Proof. That $L^{\pm}$ are generators of $\mathcal{A}_\Lambda$ is a direct combination of Lemmas 3.3.6 and 3.3.7.

The rest of the statement is a corollary of Lemma 3.4.1, where the polynomial $f^x_{\pm}(\overline{L}) \equiv \sum_m f^x(m)\tilde{P}_m$ in $\overline{L}$ are defined by the fact, $\overline{x^{\pm}}\psi_m = f^x_{\pm}(m)L^{\pm}\psi_m$ on those m such that $|m \pm 1| \leq \Lambda$; there is no restriction on what $f^x_{\pm}(\pm\Lambda)$ should be, since the evaluation of the polynomial isn't ever done on $\psi^{\pm\Lambda}$. The same reasoning applies to $f^\chi_{\pm}$. $\square$

As we saw in Chapter 2, an important component of a fuzzy space is its action under a symmetry group. It was remarked 3.1.3 that the left action of $O(2) \ni g$ on $\mathbb{R}^2$ induced a unitary representation $\hat{\pi} : O(2) \to \mathcal{B}(L^2(\mathbb{R}^2))$, which in turn induces a left action of $O(2)$ on $\mathcal{A}_{\overline{E}} \ni A$ by inner isomorphisms $A^g = \pi'(g)A\pi'(g)^*$, for any cutoff energy $\overline{E} \geq 0$ and compatible k. It was further proved in Theorem 3.1.8 that the operators $\overline{x^i}$, $i = 1, 2$, transform contravariantly under $O(2)$. We will now see how this action looks for of the operators that we have used so far for the algebras $\mathcal{A}_\Lambda$ for arbitrary $\Lambda \in \mathbb{N}$, where we will call by $\hat{\pi}_\Lambda$ the representation of $O(2)$ on $\mathcal{H}_\Lambda$.

First, recall that the group of rotations on $\mathbb{R}^2$ is precisely the subgroup $SO(2) \subseteq O(2)$ of elements of determinant 1, instead of -1. Furthermore, every rotation is determined by the angle $\theta \in [0, 2\pi]$ that the x^1-axis rotates in the anticlockwise direction; hence, every rotation is of the form

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{3.65}$$

Additionally, let

$$F := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.66}$$

be the element of $O(2)$ of determinant -1 that represents the reflection with respect to the x^1-axis. Then, every element $g \in O(2)$ whose determinant isn't 1, i.e. it is -1 is such that $Fg \in SO(2)$, and so every element of determinant -1 in $O(2)$ can be written as the product of a rotation with F; this all combines to the fact that the group $O(2)$ is generated by the space of rotations together with the reflection F.

The contravariant transformation $\pi(g)'\overline{x^i}\pi(g)^* = \sum_j (g^T)^i_j \overline{x^j}$ of the operators $\overline{x^i}$ under the rotation $g = R_\theta$ looks as follows:

$$\begin{aligned} (\overline{x^1})^{R_\theta} &= \pi(R_\theta)'\overline{x^i}\pi(R_\theta)* \\ &= \overline{x^1}\cos\theta + \overline{x^2}\sin\theta, \\ (\overline{x^2})^{R_\theta} &= -\overline{x^1}\sin\theta + \overline{x^2}\cos\theta. \end{aligned} \tag{3.67}$$

Similarly, the (contravariant) transformation under the reflection F is

$$\begin{aligned} (\overline{x^1})^F &= \overline{x^1}, \\ (\overline{x^2})^F &= -\overline{x^2}. \end{aligned} \tag{3.68}$$

This all implies that the action on the operators $\overline{x^\pm} = \overline{x^1} + i\overline{x^2}$ is

$$\begin{aligned} (\overline{x^\pm})^{R_\theta} &= e^{-i\theta}x^\pm, \\ (\overline{x^\pm})^F &= x^\mp; \end{aligned} \tag{3.69}$$

we will call this transformation rule for operators $U \in \mathcal{A}^1_\Lambda$ and $D \in \mathcal{A}^{-1}_\Lambda$ *a contravariant transformation of the operators* U, D.

Now, on $\mathbb{R}^2$, $\pi(R_\theta)\partial_\phi\pi(R_\theta)*\psi(r,\phi) = \pi(R_\theta)\partial_\phi\psi(r,\phi{+}\theta) = \pi(R_\theta)\partial_\phi\psi(r,\phi+\theta) = \pi(R_\theta)\partial_{\phi+\theta}\psi(r,\phi+\theta) = \partial_\phi\psi(r,\phi)$ for all $\psi \in L^2(\mathbb{R}^2)$ i.e. $(\partial_\phi)^{R_\theta} = \partial_\phi$, then we have that $L^{R_\theta} = L$ and so, since $[P_\Lambda, \pi'(R_\theta)] = 0$, that

$$(\overline{L})^{R_\theta} = \overline{L}. \tag{3.70}$$

Similarly, since $\phi = \arctan \frac{x^2}{x^1}$ in $\mathbb{R}^2$, the action by the reflection $F \in O(2)$ on $\overline{L}$ is

$$(\overline{L})^F = -\overline{L}. \tag{3.71}$$

Now, notice that since $\chi^i = \frac{\overline{x^i}}{a(k)}$, then $\chi^\pm$ has the same transformation laws under $O(2)$ as $\overline{x^\pm}$, hence

$$\begin{aligned} (\chi^\pm)^{R_\theta} &= e^{\mp i\theta}\chi^\pm; \\ (\chi^\pm)^F &= \chi^\mp. \end{aligned} \tag{3.72}$$

Similarly, the operators $L^\pm$ defined in Proposition 3.4.2 also have a contravariant transformation law under $O(2)$. To see this, first notice that we may write $L^\pm = (f^\chi_\pm)^{-1}(\overline{L})x^\pm$, where $(f^\chi_\pm)^{-1}(\overline{L})$ is the polynomial such that $(f^\chi_\pm)^{-1}(m) := (f^\chi_\pm)^{-1}(\overline{L})\psi_m = \frac{1}{f^\chi_\pm(m)}$. Also notice that $\pi'(g)(f^\chi_\pm)^{-1}(\overline{L})\pi'(g)^* = (f^\chi_\pm)^{-1}(\pi'(g)\overline{L}\pi'(g)^*)$ for all $g \in O(2)$, so $\pi'(R_\theta)(f^\chi_\pm)^{-1}(\overline{L})\pi'(R_\theta)^* = (f^\chi_\pm)^{-1}(\overline{L})$ and $\pi'(F)(f^\chi_\pm)^{-1}(\overline{L})\pi'(F)^* = (f^\chi_\pm)^{-1}(-\overline{L}) = (f^\chi_\mp)^{-1}(\overline{L})$, where the last equation follows from the fact that, as polynomials in $\overline{L}$, $\tilde{P}_m(-\overline{L}) = \tilde{P}_{\overline{L}}(\overline{L})$ and from the equation $f^\chi_\pm(-m) = f^\chi_\mp(m)$. Thus,

$$\begin{aligned} \pi'(R_\theta)L^\pm\pi'(R_\theta)^* &= \pi(R_\theta)(f^\chi_\pm)^{-1}(\overline{L})\pi(R_\theta)^*\pi'(R_\theta)\chi^\pm\pi'(R_\theta)^* \\ &= (f^\chi_\pm)^{-1}(\overline{L})e^{\mp i\theta}\chi^\pm \\ &= e^{\mp i\theta}L^\pm, \end{aligned} \tag{3.73}$$

and

$$\begin{aligned} \pi'(F)L^\pm\pi'(F)^* &= \pi'(F)(f^\chi_\pm)^{-1}(\overline{L})\pi'(F)^*\pi'(F)\chi^\pm\pi'(F)^* \\ &= (f^\chi_\mp)^{-1}(\overline{L})\chi^\mp \\ &= L^\mp. \end{aligned} \tag{3.74}$$

We will now see that the algebras $\mathcal{A}_\Lambda$ developed in this chapter can be more concisely understood as the algebras of endomorphisms of irreducible representation spaces of $SO(3)$, just like the even-indexed elements of the fuzzy sphere. Recall that $\hat{\pi}_\Lambda$ denotes the representation of both $SO(3)$ on the representation spaces of $SU(2)$ $V_\Lambda \cong \mathbb{C}^{2\Lambda+1}$, which is induced by the representation π_Λ of $SU(2)$. Let us also denote by $\hat{\pi}$ the algebra representation of $U(\mathfrak{so}(3))$ induced by the group representation $\hat{\pi}$, and recall that

$i\hat{J}^k \in U(\mathfrak{so}(3))$, $k = 1, 2, 3$, is the infinitesimal rotation in $\mathbb{R}^3$ about the i-th axis.

Since V_Λ is a representation space of $SO(3)$, by identifying $O(2)$ as a subgroup of $SO(3)$ we can then restrict $\hat{\pi}$ to $O(2)$ to get a representation of it on V_Λ, and therefore a left action of $O(2)$ on $\mathcal{A}_\Lambda$. Let us make the following identifications

$$\begin{array}{ccr} O(2) & \hookrightarrow & SO(3) \\ R_\theta & \mapsto & \begin{pmatrix} R_\theta & 0 \\ 0 & 1 \end{pmatrix} = e^{i\theta\hat{J}^3} \\ F & \mapsto & e^{i\pi\hat{J}^1}; \end{array}$$

we can see that the subgroup of $SO(3)$ generated by the elements of the form $e^{i\theta\hat{J}^3}$ and $e^{i\pi\hat{J}^3}$ is isomorphic to $O(2)$, by realizing that it acts on the plane $x^1 - x^2$ in $\mathbb{R}^3$ precisely as $O(2)$ acts on $\mathbb{R}^2$.

Theorem 3.4.3. For all $\Lambda \in \mathbb{N}$, let $T : \mathcal{H}_\Lambda \to V_\Lambda$ be the isomorphism of the Hilbert spaces $\mathcal{H}_\Lambda \subseteq L^2(\mathbb{R}^2)$ and $V_\Lambda \equiv \mathbb{C}^{2\Lambda+1}$ generated by the mapping $\psi_m \mapsto |\Lambda, m\rangle$, $m \in \{-\Lambda, \ldots, \Lambda\}$. Then, T induces an isomorphism $\tilde{T}$ of the C^*-algebra $\mathcal{A}_\Lambda = End(\mathcal{H}_)$, with the operator norm, is a C^*-algebra isomorphic to $\hat{\pi}_\Lambda(U(\mathfrak{so}(3))) = End(V_\Lambda)$ with the operator norm, and this isomorphism is $O(2)$-equivariant. Furthermore, the tuples of operators $(\tilde{T}(\overline{x^1}), \tilde{T}(\overline{x^2}))$ and $(\tilde{T}(\chi^1), \tilde{T}(\chi^2))$ transform $O(2)$-contravariantly.

Proof. First notice that $T : \mathcal{H}_\Lambda \to V_\Lambda$ is an isomorphism of Hilbert spaces since $\{\psi_m\}_{|m|\leq\Lambda} \subseteq \mathcal{H}_\Lambda$ and $\{|\Lambda, m\rangle\}_{|m|\leq\Lambda}$ are orthonormal basis of their respective space. The invertible mapping $\tilde{T} : \mathcal{A}_\Lambda \to End(V_\Lambda)$ is defined as $A \mapsto T \circ A \circ T^{-1}$, and it clearly respects the algebra operations, i.e. it is an algebra isomorphism. This mapping is such that $\langle\psi_m|A|\psi^n\rangle = \langle\Lambda, m|\tilde{T}(A)|\Lambda, n\rangle$, for all $n, m \in \{-\Lambda, \ldots, \Lambda\}$, and so $\tilde{T}$ respects adjoints, i.e. it is a $*$-isomorphism. Since $V_\Lambda \neq \{0\}$, for any $A \in \mathcal{A}_\Lambda$:

$$\begin{aligned} ||\tilde{T}(A)|| &= \sup_{v\in V_\Lambda}\{||\tilde{T}(A)(v)|| \,:\, ||v|| = 1\} \\ &= \sup_{\psi\in\mathcal{H}_\Lambda}\{||\tilde{T}(A)(T(\psi))|| \,:\, ||T(\psi)|| = 1\} \\ &= \sup_{\psi\in\mathcal{H}_\Lambda}\{||T(A(\psi))|| \,:\, ||\psi|| = 1\} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\psi \in \mathcal{H}_\Lambda} \{||A(\psi)|| \,:\, ||\psi|| = 1\} \\
&= ||A||;
\end{aligned}
$$

hence $\tilde{T}$ respects the norm, so it is a Banach algebra isomorphism; notice that this boils down to the fact that T is a normed vector space isomorphism, which itself is a consequence of the fact that T maps an orthonormal basis to an orthonormal basis.

We now prove the $O(2)$-equivariance of $\tilde{T}$. Notice that

$$\tilde{T}(L^\pm) = \hat{J}^\pm, \tag{3.75}$$

where $\hat{J}^\pm = \hat{J}^1 \pm i\hat{J}^2$, and $L^\pm$ are the algebra generators defined in Proposition 3.4.2; this follows from the equation

$$\langle \Lambda, n|\hat{J}^\pm|\Lambda, m\rangle = \delta_{n,m\pm1}\sqrt{\Lambda(\Lambda+1) - m(m\pm1)} = \langle \psi_n|L^\pm|\psi_m\rangle,$$

for all $n, m \in \{-\Lambda, \ldots, \Lambda\}$. Define the operators $L^1 = \frac{1}{2}(L^+ + L^-)$ and $L^2 = \frac{1}{2i}(L^+ - L^-)$; they satisfy that $\tilde{T}(L^k) = \hat{J}^k$, for $k = 1, 2$, and since $L^\pm$ generates $\mathcal{A}_\Lambda$, then L^1, L^2 also generate it. Finally, the transformation laws (3.73) and (3.74) of $L^\pm$ imply that the operators L^1, L^2 transform contravariantly under $O(2)$, and so do $\hat{J}^1$, $\hat{J}^2$ under $O(2) \subseteq SO(3)$ as stated in Proposition 3.1.10, recalling that $O(2) \subseteq SO(3)$ leaves invariant the third coordinate. Since L^1, L^2 generate $\mathcal{A}_\Lambda$, the action of $O(2)$ on $\mathcal{A}_\Lambda$ is completely determined by its action on L^1 and L^2; hence, that the mapping $L^k \mapsto \hat{J}^k$ is $O(2)$-covariant implies that $\tilde{T}$ is $O(2)$-equivariant, as desired.

The last part of the theorem follows from the $O(2)$-equivariance of $\tilde{T}$, together with the $O(2)$-contravariant transformations of $\overline{x^1}, \overline{x^2}$, and of χ^1, χ^2.

$\square$

Another $O(2)$-equivariant manifestation of the algebras $\mathcal{A}_\Lambda$ can be achieved through $L^2(S^1)$:

Definition 3.4.4. For $\Lambda \in \mathbb{N}$, let $\tilde{\mathcal{H}}_\Lambda \subseteq L^2(S^1)$ be the Hilbert space generated by the orthonormal set $\{e^{im\phi}\}_{|m|\leq\Lambda}$ with the inherited inner product, and let $\tilde{\mathcal{A}}_\Lambda := End(\tilde{\mathcal{H}}_\Lambda)$. By abuse of notation, denote by L and $x^\pm$ the restriction to $\tilde{\mathcal{H}}_\Lambda$ of the operators L and $x^\pm$ on $L^2(S^1)$.

Since $S^1 \subseteq \mathbb{R}^2$, $O(2)$ acts naturally on each $\tilde{\mathcal{H}}_\Lambda$. Explicitly,

$$\begin{aligned} R_\theta \cdot e^{im\phi} &= e^{im(\phi-\theta)}, \\ F \cdot e^{im\phi} &= e^{-im\phi}, \end{aligned} \tag{3.76}$$

where R_θ is the anticlockwise rotation by $\theta \in [0, 2\pi]$ on $\mathbb{R}^2$, and $\cdot$ denotes the (right) action of $O(2)$ on $\tilde{\mathcal{H}}_\Lambda$. Now, recall that $\psi_m \in \mathcal{H}_\Lambda$ has the form $\psi_m(\rho, \phi) = \tilde{f}(\rho)e^{im\phi}$, therefore

$$\begin{aligned} R_\theta \cdot \psi_m(\rho, \phi) &= \psi_m(\rho, \phi - \theta), \\ F \cdot \psi_m(\rho, \phi) &= \psi_m(\rho, -\phi), \end{aligned} \tag{3.77}$$

due to the fact that the radial coordinate is not affected by a $O(2)$ transformation, since, by definition, $O(2)$ doesn't change the metric in $\mathbb{R}^2$. Thus, we can define the following alternative view of $\mathcal{A}_\Lambda$:

Theorem 3.4.5. For all $\Lambda \in \mathbb{N}$, let $T : \mathcal{H}_\Lambda \to \tilde{\mathcal{H}}_\Lambda$ be defined by $\psi_m \mapsto e^{im\phi}$, for $m \in \{-\Lambda, \dots, \Lambda\}$. T induces an isomorphism between the C^*-algebra $\mathcal{A}_\Lambda$ is isomorphic to $\tilde{\mathcal{A}}_\Lambda$ in an $O(2)$-equivariant way

Proof. The isomorphism $T : \mathcal{H}_\Lambda \to \tilde{\mathcal{H}}_\Lambda$ takes an orthonormal basis of $\mathcal{H}_\Lambda$ to an orthonormal basis of $\tilde{\mathcal{H}}_\Lambda$, hence, as seen in the proof of Theorem 3.4.3 this implies that the induced map on the algebras of endomorphisms in a C^*-algebra isomorphism.

The $O(2)$-equivariance follows from the equivalence of the transformation laws of the basis of these Hilbert spaces, stated in equations (3.76) and (3.77).

□

3.5 Convergence

3.5.1 To Quantum Mechanics on S^1

In [3] the following type of convergence of a sequence of low energy effective theories with observable algebras $\{\mathcal{A}_\Lambda\}_{\Lambda\in\mathbb{N}}$ where each $\mathcal{A}_\Lambda$ is as in Definition 3.2.2:

Consider the $O(2)$-equivariant embedding $T : \mathcal{H}_\Lambda \to \tilde{\mathcal{H}}_\Lambda$ of Theorem 3.4.5, as an identification of the two Hilbert spaces. We may say that we are considering the basis elements $\psi_m \in \mathcal{H}_\Lambda$ *as the fuzzy analogs of* $e^{im\phi}$, when understood as elements of the Hilbert space $\mathcal{H} = L^2(S^1)$. For every $\phi \in L^2(S^2)$, let $\phi_\Lambda = \sum_{m=-\Lambda}^{\Lambda} \phi_m e^{im\phi}$ where ϕ_m are the Fourier coefficients of ϕ; ϕ_Λ is identified via T with $\sum_{m=-\Lambda}^{\Lambda} \phi_m \psi_m$. Clearly $\phi_\Lambda \to \phi$ in the L^2-norm as $\Lambda \to 0$. In this sense Fiore and Pisacane in [3] then say that $\mathcal{H}_\Lambda$ *invades* $\mathcal{H}$ as $\Lambda \to \infty$.

Now, the identification T induces an identification $\mathcal{A}_\Lambda$ within $\mathcal{B}(L^2(S^1))$ as the algebra that annihilates $\tilde{\mathcal{H}}_\Lambda^\perp$ and whose application produces elements of $\tilde{\mathcal{H}}_\Lambda$.

Theorem 3.5.1. Let $k(\Lambda)$ be any function satisfying inequality (3.33). Under the identification $T : \mathcal{H}_\Lambda \to \tilde{\mathcal{H}}_\Lambda \subseteq \mathcal{H}$, then $\mathcal{A}_\Lambda \subseteq \mathcal{A} = \mathcal{B}(\mathcal{H})$. Then the operators $\overline{L}$ converges strongly [2] to L, and both $\chi^\pm$ and $\overline{x^\pm}$ converge strongly to $e^{\pm i\phi}$. If the operators $\chi^\pm$ are defined not as in Definition 3.3.12, but via the exact action on the basis (3.55), the same convergence applies.

Since L and $e^{\pm i\phi}$ generate the algebra of observables of a quantum particle in S^1, we then say that *the limit of the low energy effective theories* $\{(\mathcal{H}_\Lambda, \mathcal{A}_\Lambda)\}$ *is the quantum mechanics of a spin-less particle in* S^1.

Proof. Suppose that ϕ is in the domain of L, i.e. $\sum_{m\in\mathbb{Z}} m^2|\phi_m|^2 \leq \infty$, then

$$||(L - \overline{L})\phi|| = \sum_{|m|>\Lambda} m^2|\phi_m|^2 \xrightarrow{\Lambda\to 0} 0.$$

Similarly, let $\phi = \sum_{m\in\mathbb{Z}} \phi_m e^{i\phi}$. Then

$$\begin{aligned}
&||(\chi^\pm - e^{\pm i\phi})\phi||^2 \\
&= \sum_{m=-\Lambda}^{\Lambda-1} \left[\sqrt{1 + \frac{m(m\pm 1)}{k}} - 1 + O\left(\frac{m^3}{k^{\frac{3}{2}}}\right)\right] \phi_m e^{i(m\pm 1)} - \sum_{m<\Lambda, m\geq\Lambda} \phi_m e^{i(m\pm 1)} \\
&\leq \sum_{m=-\Lambda}^{\Lambda-1} \left[\frac{m^2(m\pm 1)^2}{4k^2} + O\left(\frac{\Lambda^6}{k^3}\right)\right] |\phi_m|^2 + \sum_{m<\Lambda, m\geq\Lambda} |\phi_m|^2
\end{aligned}$$

[2]Ignoring domain issues, a sequence of operators $\{T_n\}_{n\in\mathbb{N}}$ in a Banach space V converges strongly to the operator T with if, for all $v \in V$, $\lim_{n\to\infty} ||T_n v - Tv|| \to 0$.

$$\leq \frac{\Lambda^2(\Lambda+1)^2}{4k^2}||\phi||^2 + \sum_{|m|\geq\Lambda} |\phi_m|^2,$$

and this goes to 0 when $\Lambda \to 0$ thanks to the inequality (3.33); if the operators $\chi^{\pm}$ are defined by (3.55), the same calculation but removing the assymptotic terms applies. A similar argument follows for $\overline{x^{\pm}}$ since $\langle\psi_{m\pm1}|\overline{x^{\pm}}|\psi_m\rangle - 1 = O\left(\frac{m}{k^{1/2}}\right) = O\left(\frac{\Lambda}{\Lambda^2}\right)$. $\square$

Notice that $\chi^{\pm}$ nor $\overline{x^{\pm}}$ can converge in operator norm to $e^{\pm i\phi}$. Take for example $e^{i\lambda+1} \in \mathcal{H}_\Lambda^{\perp}$, then $||(\chi^{\pm} - e^{\pm im\phi})e^{i(\Lambda+1)\phi}|| = ||0 - e^{i(\Lambda+1\pm1)\phi}|| = 1$, meaning that $||\chi^{\pm} - e^{\pm i\phi}|| \geq 1$.

Notice that the transformation relations (3.77) imply that the spaces $W_{|m|} := span\{\psi_m, \psi_{-m}\}$, for $m \in \{-\Lambda, \ldots, \Lambda\}$, are irreducible under the action of $O(2)$, and so

$$\mathcal{H}_\Lambda = \bigoplus_{m=0}^{\Lambda} W_m \tag{3.78}$$

is the decomposition of $\mathcal{H}_\Lambda$ in irreducible representations of $O(2)$. Thus each $\mathcal{H}_\Lambda$ can be seen as the truncation of the decomposition

$$L^2(S^1) = \bigoplus_{m\in\mathbb{N}} W_m. \tag{3.79}$$

3.5.2 To the Commutative Algebra $C(S^1)$

Recall that in the canonical Dirac triple of a commutative space X, the commutative algebra $C(X)$ of continuous functions on X has the role to act by multiplication on the Hilbert space of spinor fields. Since S^1 is a manifold or rank 1, the fermionic Fock space has dimension $2^{\lfloor\frac{1}{2}\rfloor} = 1$, and the fact that S^1 is a compact simple Lie group means that *its spinor bundle is a trivial complex vector bundle*, hence it is simply $\mathcal{H} = L^2(S^1)$. Under some assumptions on the function $k(\Lambda)$, the identification of $\mathcal{H}_\Lambda$ with $\tilde{\mathcal{H}}_\Lambda$ through the $O(2)$-equivariant $*$-isomorphism T of Theorem 3.4.5 will imply that the sequence of algebras $\{\mathcal{A}_\Lambda\}_{\Lambda\in\mathbb{N}}$ converge, in the sense soon to be specified, to the commutative subalgebra of $\mathcal{B}(\mathcal{H})$ that the space $C(S^1)$ is.

In Theorem 3.5.1 it was shown that $\chi^{\pm}$ and $\overline{x^{\pm}}$ converge strongly to the operator $e^{\pm i\phi}$, so let $x^{\pm}$ be either $\chi^{\pm}$ or $\overline{x^{\pm}}$; we might say that $x^{\pm} \in \mathcal{A}_\Lambda$ are the fuzzy analogs of $e^{\pm i\phi}$, but as operators on $L^2(S^1)$. This suggests [3] the set (not an algebra)

$$C_\Lambda := \left\{ \sum_{h=-2\Lambda}^{2\Lambda} f_h x^h \right\} \subseteq \mathcal{A}_\Lambda \subseteq \mathcal{B}(\mathcal{H}), \tag{3.80}$$

where $x^h = (x^+)^h$ if $h \geq 0$ and $x^h = (x^-)^h$ if $h \leq 0$, as *the fuzzy analog of* $C(S^1)$, and in fact of the space of bounded functions $B(S^1)$, since they are the functions that have a Fourier series expansion that converges almost everywhere. For $f \in B(S^1)$, let $f = \sum_{h\in\mathbb{Z}} f_h e^{ih\phi}$ be its Fourier series, and define

$$\hat{f}_\Lambda := \sum_{h=-2\Lambda}^{2\Lambda} f_h x^h; \tag{3.81}$$

then it shown in [3] the following:

Theorem 3.5.2. Suppose that $k(\Lambda) \geq 2\Lambda(\Lambda+1)(2\Lambda+1)^2$. Then, for all $f, g \in B(S^1)$ the following strong limits hold:

$$\hat{f}_\Lambda \to f\cdot \qquad (\hat{fg})_\Lambda \to fg\cdot \qquad \hat{f}_\Lambda \hat{g}_\Lambda \to fg\cdot\,, \tag{3.82}$$

where the notation $f\cdot$ emphasizes f as an operator on $\mathcal{H}$.

Chapter 4

Systems of Coherent States on the New Fuzzy Spheres

From a geometric point of view, these states possess the algebraic properties that characterize points of traditional spaces, thus our interest in their study. Furthermore, coherent states of an algebraic space associated to a Hilbert space are concrete objects with defined properties that enable the use of these spaces into a variety of applications. In this chapter we describe some families of coherent states on the new fuzzy circle S^1_Λ following [4, 5, 16], and study their localization properties for both position and angular momentum.

Coherent states are vectors of the underlying Hilbert spaces, and so they induce states of the algebra. Our purpose is to eventually study the distance between the family of states described in this document.

Throughout this chapter let $\mathcal{H}_\Lambda$ and $\mathcal{A}_\Lambda$ be defined as in definition 3.2.2 for some $\Lambda \in \mathbb{N}$ and some appropriate function $k = k(\Lambda)$ satisfying the inequality (3.33), where $\{\psi_m\}_{|m|\leq\Lambda}$ given in (3.30) are an orthonormal basis of $\mathcal{H}_\Lambda$ composed of eigenvalues of $\overline{L} \in \mathcal{A}_\Lambda$. Additionally, let the operators $\overline{L}$ and $\chi^\pm \in \mathcal{A}_\Lambda$ be defined as in Section 3.3, but let us use the symbol L instead of $\overline{L}$. We will call L the angular momentum observable and $\chi^1 = \frac{1}{2}(\chi^+ + \chi^-)$ and $\chi^2 = \frac{1}{2i}(\chi^+ - \chi^-)$ the position observables.

Given vector $\psi \in \mathcal{H}_\Lambda$, we will use the following measure of the localization of the vector in configuration space:

$$(\Delta\vec{x})^2 = \sum_{i=1}^{D}(\Delta x^i)^2 = \langle(\vec{x} - \langle\vec{x}\rangle)^2\rangle = \langle\vec{x}^2\rangle - \langle\vec{x}\rangle^2, \tag{4.1}$$

where the expectation value for any observable A is

$$\langle A\rangle = \langle\psi|A|\psi\rangle. \tag{4.2}$$

Similarly, for localization on momentum space the measure will be:

$$(\Delta\overline{L})^2 = \langle(\overline{L} - \langle L\rangle)^2\rangle = \langle L^2\rangle - \langle L\rangle^2; \tag{4.3}$$

from now on we will denote $\overline{L} \in \mathcal{A}_\Lambda$ simply by L.

4.1 Angular Momentum Saturating Coherent States

The orthonormal basis $\{\psi_m\}_{|m|\leq\Lambda}$ is our first family of coherent states.

The group

$$G = \{S^n e^{i(aL+b)} \,:\, (a,b,n) \in \mathbb{R}^2 \times \mathbb{Z}_{2\Lambda+1}\} \cong U(1) \times U(1) \rtimes \mathbb{Z}_{2\Lambda+1}, \tag{4.4}$$

where S is the ladder operator defined in 3.3.5, with operation

$$S^n e^{i(aL+b)}\, S^{n'} e^{i(a'L+b')} = S^{n+n'} e^{i[(a+a')L+(b+b'+an')]},$$

acts unitarily, transitively and irreducibly on $\{\psi_m\}_{|m|\Lambda}$, and the isotropy subgroup H of all ψ_m under this group action is clearly

$$H = \{e^{i(aL+b)} \,:\, (a,b) \in \mathbb{R}^2\} \cong [U(1)]^2. \tag{4.5}$$

Hence

$$G/H = \{S^n \,:\, n \in \mathbb{Z}_{2\Lambda+1}\} \cong \mathbb{Z}_{2\Lambda+1}, \tag{4.6}$$

is the label space for the basis $\{\psi_m\}_{|m|\leq\Lambda}$.

Additionally, the fact that they are a basis means that we have the resolution of the identity

$$
\begin{aligned}
1_{\mathcal{H}_\Lambda} &= \sum_{m=-\Lambda}^{\Lambda} \tilde{P}_m \\
&= \int_{G/H} \tilde{P}_m \, d\mu(m),
\end{aligned}
\tag{4.7}
$$

where μ is the counting measure on G/H, induced by the Haar measure on the Lie group G. Hence $\{\psi_m\}_{|m|\leq\Lambda}$ is a strong system of coherent states 2.4.1.

Now, let's study the localization of these vectors. Since each ψ_m is an eigenvalue of L, $(L - \langle L\rangle)\psi_m = 0$,

$$
(\Delta L)^2 = 0. \tag{4.8}
$$

Similarly, $\langle\psi_m|\chi^i|\psi_m\rangle = 0$, so $\langle\chi^i\rangle = 0$ for each $i = 1, 2$, and therefore $(\Delta\chi^i)^2 = \langle(\chi^i)^2\rangle$, and so

$$
(\Delta\chi^i)^2 = \begin{cases} \frac{1}{2}\left(1 + \frac{m^2}{k}\right), & \text{if } |m| \leq \Lambda \\ \frac{1}{4}\left[1 + \frac{\Lambda(\Lambda-1)}{k}\right], & |m| \leq \Lambda \end{cases}, \tag{4.9}
$$

$$
(\Delta\vec{\chi})^2 = \begin{cases} \left(1 + \frac{m^2}{k}\right), & \text{if } |m| \leq \Lambda \\ \frac{1}{2}\left[1 + \frac{\Lambda(\Lambda-1)}{k}\right], & |m| \leq \Lambda \end{cases}, \tag{4.10}
$$

up to fourth order in powers of $1/\sqrt{k}$.

This system of coherent states can be characterized by a system of uncertainty relations: the commutation relations (3.59) have as consequence the following uncertainty relations:

$$
(\Delta L)^2(\Delta\chi^i)^2 \geq \frac{1}{4}\langle\chi^i\rangle^2, \qquad (\Delta L)^2(\Delta\vec{\chi})^2 \geq \frac{1}{4}\langle\vec{\chi}\rangle^2. \tag{4.11}
$$

The fact that $\langle\chi^i\rangle = (\Delta L)^2 = 0$ means that the system $\{\psi_m\}$ saturates these inequalities. In fact, more is true, as proven in [4]:

Proposition 4.1.1. The basis $\{\psi_m\}_{|m|\leq\Lambda}$ is a system of coherent states that minimize the uncertainty relation (4.11).

4.2 $SO(2)$-invariant Families of Strong Coherent States

Let us now construct some families of strong coherent states by action of the compact group $SO(2)$ on each $\mathcal{H}_\Lambda$. Start with a unit vector $\omega = \sum_{m=-\Lambda}^{\Lambda} \omega_m \psi_m$. Acting on it with $e^{i\alpha L} \in SO(2)$, for $\alpha \in [0, 2\pi)$ produces the unit vector

$$\omega_\alpha := e^{i\alpha L}\omega = \sum_{m=-\Lambda}^{\Lambda} e^{im\alpha}\omega_m\psi_m, \tag{4.12}$$

and associated to this vector define the orthogonal projection $\tilde{P}_\alpha := |\omega_\alpha\rangle\langle\omega_\alpha|$. Define the operator $B \in \mathcal{A}_\Lambda$ by $B := \int_0^{2\pi} d\alpha P_\alpha$, and notice that

$$B\psi_m = \int_0^{2\pi} d\alpha\, \omega_\alpha \overline{\omega_m} e^{-im\alpha} = \overline{\omega_m} \sum_{|n|\leq\Lambda} \omega_n \psi_n \int_0^{2\pi} d\alpha\, e^{i\alpha(n-m)} = 2\pi|\omega_m|^2\psi_m.$$

Hence, B is proportional to the identity if and only if the norms $|\omega_m|$ are independent of m; since ω is unitary, this means that all $|\omega_m|^2$ have to be equal to $\frac{1}{2\Lambda+1}$, and so

$$\omega_m = \frac{e^{i\beta_m}}{\sqrt{2\Lambda+1}}$$

for some $\beta_m \in \mathbb{R}/2\pi\mathbb{Z}$.

In summary:

Proposition 4.2.1. All unit vectors ω for which their orbit under the action of $SO(3)$ generate a strong system of coherent states are parametrized by tuples $\beta \in (\mathbb{R}/2\pi\mathbb{Z})^{2\Lambda+1}$:

$$\omega^\beta := \sum_{m=-\Lambda}^{\Lambda} \frac{e^{i\beta_m}}{\sqrt{2\Lambda+1}}\psi_m, \tag{4.13}$$

and the resulting system of coherent states

$$\mathcal{S}_\alpha^\beta := \{\omega_\alpha^\beta\}_{\alpha\in[0,2\pi)}, \qquad \omega_\alpha^\beta = e^{i\alpha L}\omega^\beta = \sum_{m=-\Lambda}^{\Lambda} \frac{e^{i\beta_m + m\alpha}}{\sqrt{2\Lambda+1}}\psi_m \tag{4.14}$$

induces the resolution of the identity

$$1_{\mathcal{H}_\Lambda} = \frac{2\Lambda+1}{2\pi}\int_0^{2\pi} d\alpha\, P_\alpha^\beta \qquad\qquad P_\alpha^\beta := |\omega_\alpha^\beta\rangle\langle\omega_\alpha^\beta|. \tag{4.15}$$

For the strong system of coherent states $\{\omega_\alpha^\beta\}_{\alpha\in\mathbb{R}/2\pi\mathbb{Z}}$ to further be $O(2)$-invariant, recall from (3.76) that the action of the inversion F with respect to the x^1-axis transforms ψ_m into ψ_{-m}, hence we may choose

$$\beta_{-m} = \beta_m \qquad\qquad \text{for all } m \in \mathbb{Z}_{2\Lambda+1} \tag{4.16}$$

to get an $O(2)$-invariant system of coherent states.

With respect to the localization of the $SO(2)$-invariant coherent states first notice that $\langle L\rangle = 0$, and also that, since $x^+ = x^1 + ix^2$, $\langle x\rangle^2 = |\langle x^+\rangle|^2$, it can be shown that

$$\langle x^+\rangle = \frac{e^{-i\alpha}}{2\Lambda+1}\sum_m e^{i(\beta_{m-1}-\beta_m)}\sqrt{1+\frac{m(m-1)}{k}},$$

up to second order in $1/\sqrt{k}$, and from this it is shown in [4] that:

$$(\Delta L)^2 = \langle L^2\rangle = \frac{\Lambda(\Lambda+1)}{3}, \quad (\Delta\vec{x})^2 = \frac{2\Lambda}{2\Lambda+1} + \frac{2(\Lambda-1)\Lambda(\Lambda+1)}{3(2\Lambda+1)k}, \tag{4.17}$$

up to second order in $1/\sqrt{k}$. This means that $(\Delta\vec{x})^2 = \langle x^2\rangle - \langle x\rangle^2$ is minimal when $\langle x\rangle^2 = |\langle x^+\rangle|^2$ is maximal, and this happens, by the triangle inequality, when β is constant; in that case it is shown in [4] that

$$(\Delta\vec{x})^2 < \frac{1}{\Lambda+1}\left(\frac{1}{2}+\frac{1}{3\Lambda}\right) \overset{\Lambda\geq 2}{\leq} \frac{2}{3(\Lambda+1)}. \tag{4.18}$$

Notice that we might as well say that $\beta = 0$, since this represents a constant shift of phase for all elements of the system, and so **the system of coherent states that minimize the spacial dispersion within the families of $SO(2)$-invariant strong systems is $O(2)$-invariant and is in a $O(2)$-equivariant correspondence with points in S^1 via the identification** $\omega_\alpha^0 \leftrightarrow e^{i\alpha}$.

Finally, notice that the unitary vectors ω_α^β have no limit in $L^2(S^1)$ as $\Lambda \to \infty$ since their components with respect to the basis $\{\psi_m\}$ go to 0; however, the vectors $\sqrt{2\Lambda+1}\omega_\alpha^0$ have the limit $\delta_{\phi=\alpha}$ as a distribution, where these are vectors are multiples of the elements of a system of a $O(2)$-invariant system of coherent states that minimize, within this family, the spacial dispersion.

4.3 Coherent States Minimizing the Square Distance

Let $\mathcal{W}^1 \subseteq \mathcal{H}_\Lambda$ be the set of vectors that minimize the spacial dispersion $(\Delta\vec{\chi})^2$. We can observe that since $\vec{\chi}^2$ and $\langle\vec{\chi}\rangle^2$ are $O(2)$-invariant, then $\mathcal{W}^1$ is invariant under $O(2)$ transformations. This is a complete set of coherent states, it can be generated from a single vector ψ such that $\langle\chi^1\rangle = 0$ through both $O(2)$ and $SO(2)$ [4]; recall that equations (3.77) show that the spaces $W_m := \text{span}\{\psi_m\}$ for fixed $m \in \{-\Lambda, \ldots, \Lambda\}$ are the irreducible subspaces of $\mathcal{H}_\Lambda$ under $SO(2)$, and that $\tilde{W}_{|m|} := \text{span}\{\psi_m, \psi_{-m}\}$ are the irreducible subspaces under the action of $O(2)$.

However, closed formulas for this coherent states might not be possible, in particular, because $\vec{\chi}^2 \neq 1$ unlike in quantum mechanics on a circle or on the fuzzy sphere. Instead, it is only true that $\vec{\chi}^2 = 1 + O(1/\Lambda^2)$ according to inequality (3.33); therefore, the minimization of $(\Delta\vec{\chi})^2 = \langle\vec{\chi}^2\rangle - \langle\vec{\chi}\rangle^2$ necessarily involves a simultaneous process of minimizing $\langle\vec{\chi}^2\rangle$ and maximizing $\langle\vec{\chi}\rangle^2$. However, from the fact that $\vec{\chi}^2\psi_m = (1 + O(1/\Lambda^2))\psi_m$ (except for $m = \pm\Lambda$, the states closer to the cut-off) and from the $O(2)$-equivariance of the spectrums of the position observables, it is *expected [4] that the eigenvectors $\tilde{\psi}$ of χ^1 of highest eigenvalue in absolute value, approximate ψ at order* $O(1/\Lambda^2)$. Figure 4.1 shows, for $k(\Lambda) = \Lambda^2(\Lambda+1)^2$ we observe the rate of convergence of the eigenvalues of $\vec{\chi}^2$, and hence of $\langle\vec{\chi}^2\rangle$, as a function of Λ.

The operator χ^1 has the following symmetric matrix representation

$$X^\Lambda = \frac{1}{2}\begin{pmatrix} 0 & b_\Lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_\Lambda & 0 & b_{\Lambda-1} & 0 & \cdots & 0 & 0 & 0 \\ 0 & b_{\Lambda-1} & 0 & b_{\Lambda-2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_{2-\Lambda} & 0 & b_{1-\Lambda} \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{1-\Lambda} & 0. \end{pmatrix} \tag{4.19}$$

in the ordered basis $\{\psi_\Lambda, \psi_{\Lambda-1}, \ldots, \psi_{1-\Lambda}, \psi_{-\Lambda}\}$, where $b_m := \langle\psi_m|\chi^+|\psi_{m-1}\rangle =$

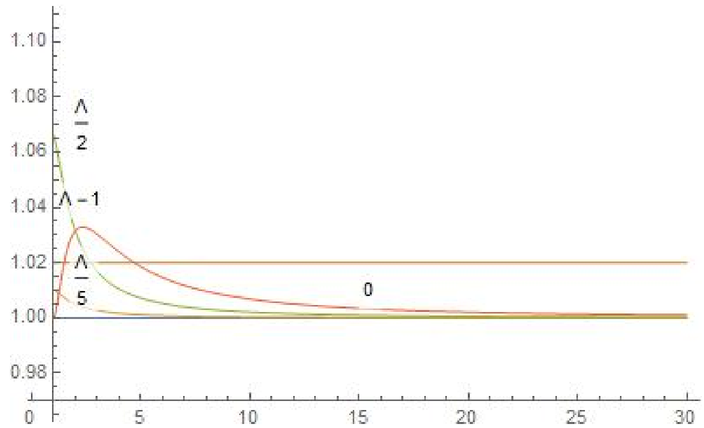


Figure 4.1: Eigenvalues of $\vec{\chi}^2$ for various values of m as a function of Λ, for $k(\Lambda) = \Lambda^2(\Lambda+1)^2$. The constant line with value 1.02 is drawn as a reference.

$\langle\psi_{m-1}|\chi^-|\psi_m\rangle$ and so

$$b_m = \begin{cases} \sqrt{1+\frac{m(m-1)}{k}} + O\left(\frac{m^3}{k^{3/2}}\right), & \text{if } 1-\Lambda \le m \le \Lambda \\ 0, & \text{otherwise.} \end{cases} \tag{4.20}$$

An additional approximation can be made by replacing the matrix X^Λ by $X_0^\Lambda = \lim_{\Lambda\to\infty} X^\Lambda$, which is the Toeplitz matrix where every nondiagonal entry of X^Λ is replaced by a 1; notice that the smallest value of b_m is 1 for $m = 0$ and it grows with the absolute value of m. Figure 4.2 shows the value for the maximum b_m as a function of Λ assuming $k(\Lambda) = \Lambda^2(\Lambda+1)^2$. The eigenvectors and their corresponding eigenvalues of the operator $S^1 = \frac{S^+ + S^-}{2}$ represented by X_0^Λ are:

$$\tilde{\psi}_n = \sum_{m=-\Lambda}^{\Lambda} \sin\left(\frac{(\Lambda+1-m)(\Lambda+1-n)\pi}{2\Lambda+2}\right)\psi_m, \quad \tilde{\alpha}_m = \cos\left(\frac{(\Lambda+1-m)\pi}{2\Lambda+2}\right), \tag{4.21}$$

each vector with norm $\Lambda+1$, and with $\tilde{\alpha}_\Lambda > \tilde{\alpha}_{\Lambda-1} > \cdots > \tilde{\alpha}_{-\Lambda}$.

A good estimate, then, of the minimum dispersion is the dispersion of the eigenvector of S^1 with maximum absolute value eigenvalue $\tilde{\alpha}_{\pm\Lambda} =$

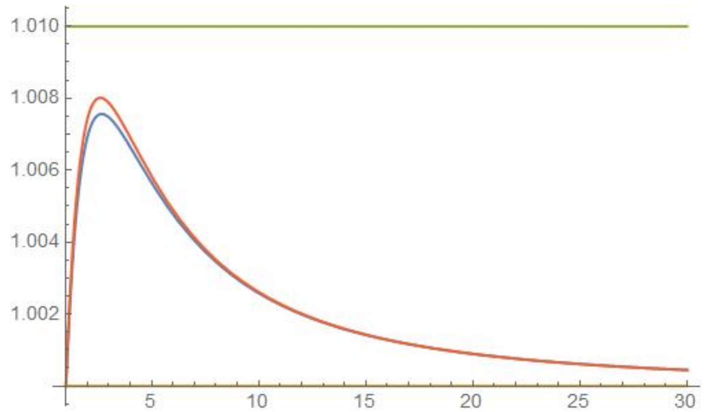


Figure 4.2: Greatest b_m as a function of Λ, for $k(\Lambda) = \Lambda^2(\Lambda+1)^2$. Red: b_Λ, Blue: approximation $b_\Lambda \approx \sqrt{1 + \frac{\Lambda(\Lambda-1)}{k}}$.

$\cos[\pi/(2\Lambda+2)]$; in [4] it is shown that for this vectors:

$$(\Delta\vec{\chi})^2 < \frac{3.5}{(\Lambda+1)^2} \stackrel{\Lambda\to\infty}{\longrightarrow} 0. \tag{4.22}$$

The spectrum of S^1 satisfies that between any two subsequence eigenvalues in its spectrum for $\Lambda+1$, $\Lambda_0^{\Lambda+1}$, there is exactly one in Σ_0^Λ, and, furthermore, Σ_0^Λ becomes uniformly dense in $[-1,1]$ as $\Lambda \to \infty$. A similar result [5, 16] can be obtained for the spectrum of χ^1:

Theorem 4.3.1. For all $\Lambda \in \mathbb{N}$, denote the spectrum of χ^1 for Λ by $\Sigma_{\chi^1}^\Lambda = \{\alpha_k^\Lambda\}_{n=-\Lambda,\ldots,\Lambda}$ ordered in increasing order with n. Then:

1. If α belongs to $\Sigma_{\chi^1}^\Lambda$, then so does $-\alpha$.

2. $\Sigma_{\Sigma_{\chi^1}}^\Lambda$ and $\Sigma_{\Sigma_{\chi^1}}^{\Lambda+1}$ interlace, i.e. between any two consecutive eigenvalues of χ^1 for $\Lambda+1$, there is exactly one of χ^1 for Λ:

$$\alpha_{\Lambda+1}^{\Lambda+1} > \alpha_\Lambda^\Lambda > \alpha_\Lambda^{\Lambda+1} > \alpha_{\Lambda-1}^\Lambda > \ldots > \alpha_{-\Lambda}^\Lambda > \alpha_{-\Lambda-1}^{\Lambda+1}. \tag{4.23}$$

3. $\Sigma_{\chi^1}^\Lambda$ becomes uniformly dense in $[-1,1]$ as $\Lambda \to \infty$. In particular, $\alpha_\Lambda(\Lambda) \geq 1 - \frac{\pi^2}{8(\Lambda+1)^2}$.

In analogy with the study of distances in the fuzzy sphere, we expect this interlacing relation to be useful in the study of the relation between distances for contiguous Λ's and for the study of the asymptotic behavior of the distance between this family of vectors.

Chapter 5

Final Remarks and Further Work

The noncommutative algebras associated to the fuzzy spheres studied in this document may all be seen as square matrix algebras. Their finite dimension and the fact that we understand completely these algebras facilitate the study of the metric properties of the associated noncommutative spaces. This made fairly straighforward in Chapter 2 the complete study of the distances between vector discrete basis states of the Madore fuzzy sphere, although this was not possible between the spin-, or $SU(2)$-, coherent states for matrix sizes beyond 2×2. However, an approximate study was possible by relating the $SU(2)$-coherent states of contiguous dimensions, revealing the relation between subsequent elements of the sequence composing the fuzzy sphere, as well as its behavior in relation to the commutative space S^2 that it approximates.

The sequence of algebras comprising the fuzzy circle of Fiore and Pisacane studied in Chapter 3 can be understood, both as a C^*-algebra and a representation space of $O(2)$, as the subsequence of matrix algebras of the fuzzy sphere of odd column and row number. This fuzzy space converges to the circle S^1 and is formed as a sequence of low energy effective quantum theories for the quantum mechanics on S^1 by introducing energy cut-offs, which naturally induces noncommuting position coordinates. Various systems of coherent states were defined and partially studied in Chapter 4, and a special emphasis was made on those with minimal position uncertainty in order to find states that may be associated to the points of the commutative space S^1. The next step to follow is to propose Dirac operators on the algebras making up the fuzzy circle of Fiore and Pisacane. Although a complete characterization of the real spectral triples on semisimple finite dimensional algebras was achieved by Paschke and Sitarz in [17], we need both a spectral triple on each element of the sequence of the fuzzy circle, but also a sequence that approximates the spectral triple of the compact simple groups S^1; it remains to be seen if an approach resembling the one followed in Chapter 2 based on the decomposition of the spinor bundle $\mathcal{H} \cong L^2(S^1)$ as a representation space of $O(2)$, the desired symmetry group, is possible. A related alternative method may come from the procedure proposed by D'Andrea et at. in [18] to introduce spectral triples on a fuzzy space based on projections of the elements of the canonical spectral triple of the limit space as high momentum cut-offs. Yet another possible source of a Dirac triple may come from the formalism of Noncommutative Quantum Mechanics of Scholtz et al. [19, 20]. Once a spectral triple on the fuzzy circle has been established, the study of the resulting metric properties may be possible to make following the general procedure outlined in Chapter 2.2 or on [21], although a more in depth analysis will depend on the nature of the Dirac operators previously proposed.

In addition to a fuzzy circle, Fiore and Pisacane have researched in [3, 5, 4] alternative descriptions and coherent states on the fuzzy sphere that result from the application of the energy cut-off procedure of Section 3.1 when $D = 3$, instead of $D = 2$, where the symmetry group is instead $O(4)$. The introduction of spectral triples on this fuzzy space has yet to be done, but the procedure done on the fuzzy circle may illustrate a path towards this objective, as well as that for the general $D \in \mathbb{Z}_{\geq 2}$ case.

Bibliography

[1] J. Madore. "The fuzzy sphere". In: *Classical and Quantum Gravity* (1992). ISSN: 02649381. DOI: 10.1088/0264-9381/9/1/008.

[2] F. D'Andrea, F. Lizzi, and J. C. Várilly. "Metric Properties of the Fuzzy Sphere". In: *Letters in Mathematical Physics* (2013). ISSN: 03779017. DOI: 10.1007/s11005-012-0590-5. arXiv: 1209.0108.

[3] G. Fiore and F. Pisacane. "Fuzzy circle and new fuzzy sphere through confining potentials and energy cutoffs". In: *Journal of Geometry and Physics* (2018). ISSN: 03930440. DOI: 10.1016/j.geomphys.2018.07.001. arXiv: 1709.04807.

[4] G. Fiore and F. Pisacane. "On localized and coherent states on some new fuzzy spheres". In: *Letters in Mathematical Physics* 110 (6 2020). ISSN: 15730530. DOI: 10.1007/s11005-020-01263-3.

[5] G. Fiore and F. Pisacane. "The x i-eigenvalue problem on some new fuzzy spheres". In: *Journal of Physics A: Mathematical and Theoretical* 53 (9 2020). ISSN: 17518121. DOI: 10.1088/1751-8121/ab67e3.

[6] S. Doplicher, K. Fredenhagen, and J. E. Roberts. "The quantum structure of spacetime at the Planck scale and quantum fields". In: *Communications in Mathematical Physics* (1995). ISSN: 00103616. DOI: 10.1007/BF02104515. arXiv: 0303037 [hep-th].

[7] A. Connes. *Noncommutative Geometry Alain Connes*. 1994. ISBN: 012185860X. DOI: 10.4171/OWR/2007/43. arXiv: 0811.1462v2.

[8] J. C. Várilly. *Dirac Operators and Spectral Geometry*. Tech. rep. 2006.

[9] O. Bratteli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 1*. 1987. DOI: 10.1007/978-3-662-02520-8.

[10] J. S. Huang and P. Pandžić. *Dirac Operators in Representation Theory.* Mathematics: Theory & Applications. Boston, MA: Birkhäuser Boston, 2007. ISBN: 978-0-8176-3218-2. DOI: 10.1007/978-0-8176-4493-2. URL: http://link.springer.com/10.1007/978-0-8176-4493-2.

[11] A. Sitarz. "Equivariant spectral triples". In: *Noncommutative Geometry and Quantum Groups* (2008). ISSN: 0137-6934. DOI: 10.4064/bc61-0-16.

[12] A. Perelomov. *Generalized Coherent States and Their Applications.* 1986. DOI: 10.1007/978-3-642-61629-7.

[13] F. T. Arecchi et al. "Atomic coherent states in quantum optics". In: *Physical Review A* (1972). ISSN: 10502947. DOI: 10.1103/PhysRevA.6.2211.

[14] B. Iochum, T. Krajewski, and P. Martinetti. "Distances in finite spaces from noncommutative geometry". In: *Journal of Geometry and Physics* (2001). ISSN: 03930440. DOI: 10.1016/S0393-0440(00)00044-9. arXiv: 9912217 [hep-th].

[15] G. Fiore and F. Pisacane. "New fuzzy spheres through confining potentials and energy cutoffs". In: *Proceedings of Science.* 2019. DOI: 10.22323/1.318.0184.

[16] G. Fiore and F. Pisacane. "Energy cutoff, effective theories, noncommutativity, fuzzyness: the case of $O(D)$-covariant fuzzy spheres". In: 2020. DOI: 10.22323/1.376.0208.

[17] M. Paschke and A. Sitarz. "Discrete spectral triples and their symmetries". In: *Journal of Mathematical Physics* (1998). ISSN: 00222488. DOI: 10.1063/1.532623. arXiv: 9612029 [q-alg].

[18] F. D'Andrea, F. Lizzi, and P. Martinetti. "Spectral geometry with a cut-off: Topological and metric aspects". In: *Journal of Geometry and Physics* 82 (2014). ISSN: 03930440. DOI: 10.1016/j.geomphys.2014.03.014.

[19] F. G. Scholtz et al. "Formulation, interpretation and application of non-commutative quantum mechanics". In: *Journal of Physics A: Mathematical and Theoretical* (2009). ISSN: 17518113. DOI: 10.1088/1751-8113/42/17/175303.

[20] F. G. Scholtz and B. Chakraborty. "Spectral triplets, statistical mechanics and emergent geometry in non-commutative quantum mechanics". In: *Journal of Physics A: Mathematical and Theoretical* (2013). ISSN: 17518113. DOI: 10.1088/1751-8113/46/8/085204.

[21] Y. Chaoba Devi et al. "Revisiting Connes' finite spectral distance on noncommutative spaces: Moyal plane and fuzzy sphere". In: *International Journal of Geometric Methods in Modern Physics* (2018). ISSN: 02198878. DOI: 10.1142/S0219887818502043. arXiv: 1608.05270.