Gauge Theories on Transitive Lie Algebroids

Sebastian Camilo Puerto Galindo

Advised by: Prof. Alexander Cardona Guio Ph.D.

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A mi Mamá. . .
Gracias por toda una vida de amor y apoyo incesante.

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Chapter 0

Introduction

The dynamics of the elementary particles of physics and their interactions are modeled by quantized gauge theories, a type of field theories. These include Quantum Electrodynamics, the Electroweak Interaction, Quantum Chromodynamics and the Standard Model of particle physics, which describe three of the four known fundamental forces and which classify all observed fundamental particles. The mathematical framework usually used for the formulation of gauge theories, prior to quantization, is that of the differential geometry of principal fiber bundles and vector bundles. Based on [3, 4, 5], in this document we study the use of the structure of transitive Lie algebroids as the mathematical framework for a possible generalization of the *formulation of a gauge theory through an action functional*: the integral of a differential form on the algebroid. From this, the standard formulation is derived as a particular case in which the underlying algebroid is the Atiyah Lie algebroid associated to the corresponding principal bundle. The formulation here presented naturally introduces new fields τ that induce the appearance of new coupling terms between these τ fields with both the gauge fields, as well as with the matter fields, implying, at least on the Atiyah Lie algebroid of a principal bundle, the presence of mass terms for these fields without having to appeal to mechanisms external to the theory like spontaneous symmetry breaking.

In Chapter 1, following Mackenzie [6], we review the needed fundamentals about Lie algebroids, vector bundles A over a manifold M with a Lie bracket field $[\cdot,\cdot]$ and a vector bundle morphism $a : A \to TM$ called the anchor.

Special emphasis is made on transitive Lie algebroids, those with fiberwise surjective anchor, meaning that they are the direct sum of the tangent bundle of the base space with a Lie algebra bundle called an adjoint Lie algebroid of A. The example that motivates the study of these algebroids is the Atiyah Lie algebroid TP/G associated to a principal bundle P with structure group G, which has $P \times \mathfrak{g}/G$ as adjoint Lie algebroid, where $\mathfrak{g} = \mathrm{Lie}(G)$; the principal bundle connections of P are in a bijective correspondence with the vector bundle morphisms from TP/G to $P \times \mathfrak{g}/G$. The transitive Lie algebroid $\mathfrak{D}(E)$ of a vector bundle enables the formulation of the concept of a connection on E, as well as a generalization of the concept of a vector bundle associated to a principal bundle through the concept of a Lie algebroid representation $\phi : A \to \mathfrak{D}(E)$ on E. A property of transitive Lie algebroids that simplifies their manipulation, specially for those familiar with the point of view used on gauge theories in the physics literature, is the fact that locally they are isomorphic to trivial Lie algebroids, of the form $TM \oplus (M \times \mathfrak{g})$ with M the base manifold. This allows the description of transitive Lie algebroids from a family of local descriptions identical to that of the traditional gauge theories over open sets $U_i \subseteq M$, plus two pasting functions on each non-empty intersection $U_i \cup U_j$; for TP/G, with G a matrix Lie group, this two pasting functions are the familiar $\alpha^i_j = g_{ij} \cdot g_{ij}^{-1}$ and $\chi^i_j = g_{ij} d g_{ij}^{-1}$, with g_{ij} transition functions of P. In this chapter an important effort is made to understand the local trivializations of the different concepts introduced throughout the chapter, since this point of view provides a way to build piece by piece all the necessary ingredients for the formulation of gauge theories on transitive Lie algebroids in concrete examples.

In Chapter 2 we define spaces of differential forms on a Lie algebroid based on [5], enabling the definition on the following chapters of connection forms and of integration on transitive Lie algebroids. Given a representation vector bundle E of an algebroid A, a differential form is a multilinear, alternating vector bundle morphism taking values in A and returning values on E; the space of such E-valued forms is denoted by $\Omega^\bullet(A, E)$, and on it a differential $\hat{d}_\phi$ and a wedge product $\wedge$ will be defined. Section 2.1 of this chapter is devoted to understand the structure of this spaces, and hence the properties that its elements and operations posses, providing a way to manipulate forms to obtain both theoretical and practical results. Since a transitive Lie algebroid is locally trivial, we first study in Section 2.2 important spaces of differential forms on trivial Lie algebroids, and then in Section

2.3 we conclude that it is indeed correct to study global forms on a transitive Lie algebroid A from the family of forms on trivial Lie algebroids that a trivialization of A provide. The majority of the chapter in focused on laying the theoretical background that justifies the manipulation of forms that is observed in the guiding articles.

Both the connections of a principal bundle P, and the connections of a vector bundle may be seen as sections of the anchor of the associated transitive Lie algebroids TP/G and $\mathfrak{D}(E)$, and this is called an ordinary connection on the Lie algebroids [6]. In Chapter 3 we define two notions of connection given a Lie algebroid A, each one generalizing one of the previously mentioned kinds of connection [5]. Generalizing principal bundle connections, the space of *generalized connections* on a transitive Lie algebroids is defined as the space of adjoint Lie algebroid-forms on A. Generalizing vector bundle connections on E, also called covariant derivatives, the A-connections are introduced, which, roughly, allow the derivation of sections of E with respect to directions not only tangent to the base manifold, but with respect to the "generalized directions" of A. Just as principal connections induce covariant derivatives on the associated vector bundles, a generalized connection on the transitive Lie algebroid A will induced an A-connection given a representation $\phi : A \to \mathfrak{D}(E)$ on E; under a trivialization of A and E and the base manifold M the produced A-connection decomposes in two types of derivations of sections of E: tangent $\hat{\nabla}^{E,loc}_{\partial_\mu} = \partial_\mu + B_\mu + A^b_\mu \phi^{loc}(E_b)$, and vertical $\hat{\nabla}^{E,loc}_{E_a} = -\tau^b_a \phi^{loc}(E_b)$, where the τ fields indicate the deviation of the connection on A from being an ordinary connection, E_a, E_b being the elements of a basis of $\mathfrak{g}$ and B is a Maurer-Cartan form coming from the trivialization of the representation ϕ. In particular, the extra degrees of freedom in which covariant derivates can be taken become a coupling between the sections of E, or "matter fields", with the tau fields.

Fournel et al., in [3], develop the framework to build the action functional of a gauge theory as an inner product of differential forms on A defined in Chapter 4 as a composition of metrics and integrals applied to forms. A metric on A will induce a notion of horizontallity on A, i.e. an ordinary connection, which then defines a differential form that we will call a volume form, that can be decomposed as a volume form for inner integration, or inner volume form, and a volume form on the base manifold. Having metrics on representation vector bundles E enables a multiplication of E-valued forms to get a scalar-valued form that may be integrated over the transitive Lie algebroid, resulting in an inner product on the space of E-valued forms.

Finally, on Chapter 5 a gauge theory will be defined on a transitive Lie algebroid A with a metric whose vertical part is Killing, together with a representation vector bundle E on which there is a metric compatible with the representation. With the current language only a Lie algebra of infinitesimal gauge transformation is defined in a way very much analogous to the traditional theory: as the space of sections of the adjoint Lie algebroid. The action functional of the theory is an operator which takes as input a connection on A and a matter field and it is defined as the sum of the norm of certain differential forms; the Lagrangian of the theory can be found by taking only the inner integrals defining the norm. These action functionals will be invariant under the action of the infinitesimal gauge transformations. The resulting theory when applied to the Atiyah Lie algebroid associated to a principal bundle is identical to a standard Yang-Mills theory when the connections are restricted to be ordinary, associated to τ fields identically 0, but additional terms appear on the Lagrangian in the general case when $\tau \neq 0$, and these terms describe the dynamics of the tau fields and their quadratic interaction with the connection and the matter field. In the general case, since all transitive Lie algebroids are locally trivial, and hence locally isomorphic to the Atiyah Lie algebroids of a principal bundle, the resulting Lagrangian has the same kind of decomposition, although no interpretation can yet be made since no further study on the extremization of the gauge functional or the quantization of the theory has been made so far.

Throughout the document two particular families of transitive Lie algebroids are studied in order to be able to apply in concrete cases the theory that has been put forward. These are the families of Atiyah Lie algebroids associated to the principal S^1-bundles over S^2 and S^3-bundles over S^4. The local version of some concepts was studied by us in order to have “ready to use” formulas for the action of a gauge theory, in particular for the Lagrangian density of the matter action, once a simple set of maps have been established.

Chapter 1

Basic Lie Algebroid Theory

The basic elements of a gauge theory under the traditional formalism are [8]:

- A principal bundle $G \to P \to M$, a fibration over spacetime M whose fibers encode inner degrees of freedom associated to a smooth symmetry G.
- Their associated vector bundles, whose sections are called matter fields, which can be interpreted as G-equivariant vector valued functions on the bigger space P.
- Connections on the principal bundle P, also called gauge potentials, that allow a formalization of the statement that T_mM is the horizontal part of T_pP for each $p \in \pi^{-1}(m)$, hence making possible to talk about covariant derivatives of matter fields in directions tangent to the base manifold M as directional derivatives of the equivariant functions in the corresponding horizontal direction on P.

It is precisely the principal connection-induced notion of covariant derivative for matter fields which encodes the principle of gauge invariance of the equations of motion under change of these inner degrees of freedom that characterizes gauge theories [2]. In an informal manner, we might say that the matter fields are functions defined G-equivariantly over the total space P. However, the equations of motion involve directional derivatives along the base manifold M; the connection on P solves this problem, since it is equivalent to a G-invariant notion of horizontallity on P, which enables the definition of directional derivatives along M as directional derivatives along the horizontal directions in P. We may thus say, informally, that the core of a gauge theory consists on:

1. a space of generalized directions, bigger than the directions tangent to the base manifold, along which
2. a matter field may vary, and where
3. a connection on the space of generalized directions, which determines a way to define directional derivatives of matter fields.

As we will see throughout this document, *the language of Lie algebroids provide an alternative, more general, formalization of the previous notions* through the following concepts:

1. A Lie algebroid A over a base manifold M.
2. A Lie algebroid representation of A on a vector bundle E over M.
3. A Lie algebroid connection on A (when A is transitive 1.4.4).

Having given some motivation for the use of Lie algebroids to formulate gauge theories, we embrace this possibility and start this document by introducing in this chapter the basic theory of Lie algebroid that is required to build gauge theories on top of them. The contents of this chapter were taken mainly from [6]. In sections 1.1 and 1.4 we study some fundamental concepts on Lie algebroids that provide the minimal basis to carry out a formulation of gauge theories on Lie algebroids. In section 1.2 some examples are studied to make more concrete the concept of a Lie algebroid. In section 1.3 we carefully build the Atiyah sequence associated to any principal bundle, since these represent the base case on which the formulation of gauge theories on transitive Lie algebroids is based. In order to have concrete ways to manipulate transitive Lie algebroids, based on [4] we study the trivialization of transitive Lie algebroids and of some of its concepts in section 1.5; particularly, we take some time developing some basic understanding of the local versions of Lie algebroid representations, since this will enable in Chapter 5 the concrete study of examples once a simple set of maps are established.

Let us fist introduce some notation. Throughout this document every construction will occur over the category of smooth paracompact manifolds, hence all vector bundles, sections of vector bundles, and any other function will be smooth. In addition, vector bundles will be considered of finite rank, and to be over the field $\mathbb{R}$. . A vector bundle E over a manifold M will usually be denoted by its projection map $p^E : E \to M$. A vector-bundle chart, or simply a chart, will make reference to a pair $(U \subseteq M, \psi : U \times V \to E|_U)$, where U is an open subset of M and ψ is a local trivialization which is C^∞-differentiable (and a vector bundle isomorphism, by definition of local trivialization). Additionally, throughout this document boldface letters will be used to denote sections of the vector bundle when additional clarity is desired, specially throughout this chapter. In addition, for a section $\boldsymbol{\mu} \in \Gamma(E)$ its value in the fiber $m \in M$ will be denoted both by $\boldsymbol{\mu}(m)$ and by $\boldsymbol{\mu}_m$.

1.1 Basic Definitions

Throughout this section, M will be a manifold.

Definition 1.1.1 (Lie Algebroid over M)**.** A Lie algebroid over M is a vector bundle $q : A \to M$ together with the following structures:

- $a : A \to TM$ is a vector bundle morphism called the anchor of A, and
- the bracket of A $[\cdot, \cdot]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ is a Lie algebra structure on the $\mathbb{R}$-vector space $\Gamma(A)$ (i.e. a $\mathbb{R}$-bilinear alternating map satisfying the Jacobi identity)

which satisfy the following compatibility conditions:

1. The $C^\infty(M)$ map induced by a on the sections $\boldsymbol{\mathfrak{X}} \mapsto (m \mapsto a(\boldsymbol{\mathfrak{X}}(m)))$, also referred to as the anchor, $a : \Gamma(A) \to \Gamma(TM)$, is a Lie algebra morphism; that is, for any $\boldsymbol{\mathfrak{X}}, \boldsymbol{\mathfrak{Y}} \in \Gamma(A)$,
$$a([\boldsymbol{\mathfrak{X}}, \boldsymbol{\mathfrak{Y}}]_A) = [a(\boldsymbol{\mathfrak{X}}), a(\boldsymbol{\mathfrak{Y}})]_A \in \Gamma(TM).$$
2. (Leibniz identity) For any $\boldsymbol{\mathfrak{X}}, \boldsymbol{\mathfrak{Y}} \in \Gamma(A)$, $f \in C^\infty(M)$:
$$[\boldsymbol{\mathfrak{X}}, f\boldsymbol{\mathfrak{Y}}]_A = f[\boldsymbol{\mathfrak{X}}, \boldsymbol{\mathfrak{Y}}]_A + a(\boldsymbol{\mathfrak{X}})(f)\,\boldsymbol{\mathfrak{Y}},$$

where $a(\mathfrak{X})(f)$ means the usual (Lie) derivative in the direction of $a(\mathfrak{X}) \in \Gamma(TM)$ of the vector valued function f on M.

Such a Lie algebroid will be represented by the following diagram

$$\begin{tikzcd} (A, [\cdot,\cdot]_A) \arrow[r, "a"] \arrow[dr, "q"'] & (TM, [\cdot,\cdot]) \arrow[d, "\pi"] \\ & M. \end{tikzcd} \tag{1.1}$$

However, to simplify notation we will often say instead that <u>A is a Lie algebroid over the manifold M with anchor a</u>, and the brackets will be usually be denoted simply by $[\cdot,\cdot]$ for all Lie algebroids, unless otherwise stated.

Proposition 1.1.2. Given U open subset of M and $\mathfrak{X}, \mathfrak{Y}$ sections of A, the value of $[\mathfrak{X}, \mathfrak{Y}] \in \Gamma(A)$ at each $m \in U$ depends only on the restrictions of $\mathfrak{X}$ and $\mathfrak{Y}$ to U.

Proof. By the alternating property of the bracket, it is enough to prove that if $\mathfrak{Y}_1, \mathfrak{Y}_2 \in \Gamma(A)$ coincide in U, then $[\mathfrak{X}, \mathfrak{Y}_1]_m = [\mathfrak{X}, \mathfrak{Y}_2]_m$. By the $\mathbb{R}$-linearity of the bracket, we may prove instead that if $\mathfrak{Y} \in \Gamma(A)$ vanishes in U, then $[\mathfrak{X}, \mathfrak{Y}]_m = 0$.

Let $\mathfrak{Y} \in \Gamma(A)$ as indicated above and $\mathfrak{X} \in \Gamma(A)$ be any section. Let $m \in M$ and $U' \subseteq M$ be the intersection of U with a chart neighborhood of m, so that U' is also a chart neighborhood of m where $\mathfrak{Y}$ vanishes. Let $f \in C^\infty(M)$ be a bump function such that $f(p) = 1$ and $supp(f) \subseteq U'$, then $f\mathfrak{Y} = 0 \in \Gamma(A)$. By the Leibniz property of the anchor

$$0 = [\mathfrak{X}, f\mathfrak{Y}]_m = f(m)[\mathfrak{X}, \mathfrak{Y}]_m + a(\mathfrak{X})(f)(m)\mathfrak{Y}_m = [\mathfrak{X}, \mathfrak{Y}]_m.$$

□

The previous result allows us to restrict the Lie bracket of A to the restriction vector bundle of A to U, $A|_U$, which will be important to justify the study of (transitive) Lie algebroids from local information.

Definition 1.1.3. Let U be an open subset of M. <u>The restriction of the Lie algebroid A to U</u> is the restriction vector bundle A to U $A|_U$ with projection map $q_U : A|_U \to U$, together with the anchor $a_U : A|_U \to TU$ and bracket $[\cdot,\cdot] : \Gamma(A|_U) \times \Gamma(A|_U) \to \Gamma(A|_U)$ that result from the restrictions to U of the anchor and bracket of A.

Perhaps the two most basic examples of Lie algebroids are the following:

Example 1.1.4. Let M be a manifold. Its tangent bundle TM with the bracket given by the commutator of vector fields on M is a Lie algebroid:

$$\begin{tikzcd}(TM,[\cdot,\cdot]) \arrow[r, "id_{TM}"] \arrow[dr, "q"] & (TM,[\cdot,\cdot]) \arrow[d, "\pi"] \\ & M.\end{tikzcd}$$

Example 1.1.5. Let $\mathfrak{g}$ be any (real) Lie algebra with Lie bracket $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$. When considered as a vector bundle over a point, $\mathfrak{g}$ is a Lie algebroid, where the anchor is necessarily null:

$$\begin{tikzcd}(\mathfrak{g},[\cdot,\cdot]) \arrow[r] \arrow[dr] & (\{0\},[\cdot,\cdot]) \arrow[d, "\pi"] \\ & \{\bullet\}.\end{tikzcd}$$

The last example illustrates a family of examples fundamentally different from the tangent bundle, or spaces of tangent directions on M, because the anchor completely nullifies. This example has the following important generalization:

Definition 1.1.6. Let $q : L \to M$ be a vector bundle over M.

- A <u>field of Lie algebra brackets on L</u> $[\cdot,\cdot] : \Gamma(L)\times\Gamma(L)\to\Gamma(L)$ is a section of the vector bundle $Alt^2(L;L)\subseteq Hom(L\otimes L;L)$ of alternating 2-forms with values on L such that, at any $m\in M$, the restriction $[\cdot,\cdot]_m : L_m\times L_m\to L_m$ makes L_m a Lie algebra.
- L together with a field of Lie brackets $[\cdot,\cdot] : \Gamma(L)\times\Gamma(L)\to\Gamma(L)$ is a <u>Lie algebra bundle</u> or <u>LAB</u> if L admits an atlas, called <u>an LAB atlas</u>, $\{(U_i,\psi_i : U_i\times\mathfrak{g}_i\to L|_{U_i})\}_{i\in I}$, where $\mathfrak{g}_i$ is a Lie algebra and each $\psi_{i,m} : \mathfrak{g}_i\to L_m$ is a Lie algebra isomorphism. An LAB is said to <u>have fiber type $\mathfrak{g}$</u> if there is an LAB atlas such that $\mathfrak{g}_i=\mathfrak{g}$ for all $i\in I$.

Notice that a field of Lie algebra brackets is C^∞-bilinear, thus allowing its pointwise restriction $[\cdot,\cdot]_m$ at each $m\in U$ to be well defined. Any *LAB*

L over a manifold M has a unique structure as a Lie algebroid on M given by the anchor $a = 0$, since a field of Lie algebra brackets is $C^\infty(M)$-linear in each component and so the Leibniz identity of Lie algebroids leaves us with no other choice for a:

$$\begin{tikzcd} (L, [\cdot,\cdot]) \arrow[r, "0"] \arrow[dr, "q"'] & (TM, [\cdot,\cdot]) \arrow[d, "\pi"] \\ & M. \end{tikzcd} \tag{1.2}$$

The emergence of Lie algebras on the fibers when $a = 0$ is in fact an important characteristic property of Lie algebroids.

Theorem 1.1.7. Let A be a Lie algebroid over the manifold M with anchor a, and suppose $ker(a)$ is a vector subbundle of A, i.e. the inclusion map $: ker(A) \to A$ is an embedding and $ker(A)$ is a vector bundle. Then $ker(a)$ is a Lie algebroid, called the vertical Lie subalgebroid of A with trivial anchor, where the bracket of A restricts to a field of Lie brackets on L, giving each fiber a Lie algebra structure.

Proof. $ker(a) \subseteq A$ implies that the sections of $ker(a)$ can be naturally considered sections of A. The first compatibility condition of Lie algebroids, i.e. that the anchor applied to sections is a Lie algebra morphism, imply that the bracket of A restricts to a bracket on $ker(a)$.

The second compatibility condition, Leibniz identity, becomes the statement of the $C^\infty(M)$ bilinearity of the bracket, which imply that the bracket is a point operator, and so the result follows. $\square$

Remark 1.1.8. Even though the bracket in $ker(A)$, when it is a vector bundle to begin with, gives the structure of a Lie algebra to each of its fibers, $ker(A)$ is not necessarily a LAB since there might not exist an atlas of local trivializations $\psi : U \times \mathfrak{g} \to ker(A)|_U$ compatible with the brackets.

So far we have worked from a topological perspective of Lie algebroids. However, an alternative view is possible:

Proposition 1.1.9. $\Gamma(A)$ is a finitely generated projective $C^\infty(M)$-module over M, i.e. A is a direct summand of a trivial vector bundle over M. Furthermore, the anchor a naturally induces a morphism of $C^\infty(M)$-modules also denoted by $a : \Gamma(A) \to \Gamma(TM)$, defined by $a(\mathfrak{X})_m := a(\mathfrak{X}_m)$ for all $\mathfrak{X} \in \Gamma(A)$ and $m \in M$.

Proof. That Γ is an equivalence the category of vector bundles over a smooth connected manifold and the category of finitely generated projective $C^\infty(M)$-modules is known as Serre-Swan's theorem. A morphism of $C^\infty(M)$-modules is simply a $C^\infty(M)$-linear map, so this statement is guaranteed by the equivalence between morphisms of vector bundles over M and $C^\infty(M)$-linear maps between their sections [10]. □

The previous result remarks the algebraic perspective which may also be used to study this topic. What enables this change of perspective, and therefore what is important to have in mind when shifting from one perspective to the other, is the *equivalence between morphisms of vector bundles over M and $C^\infty(M)$-linear maps between their sections* [10]. When doing explicit calculations the algebraic perspective is generally more useful, since it is here where fields, like those relevant in physics, make their appearance as sections of vector bundles. With this in mind, a rule of thumb that we will use throughout the document to use one or the other framework will be that local statements, related to the trivialization of a Lie algebroid (Section 1.5), will generally be studied in the algebraic framework (since vector sections are then represented by simple vector valued functions useful in calculations), and global statements will be studied from the topological perspective.

1.2 Examples

1.2.1 Fundamental Examples: Tangent Bundles and LABs

The tangent bundle of a manifold M is the most basic example of a Lie algebroid, as it should be if we want to understand Lie algebroids on M as generalized directions on M, since TM is the space of the directions tangent to M. The diagram representing this algebroid was seen in example 1.1.4. Notice how, evidently, the anchor is fiberwise surjective, meaning that every tangent direction on M has a "representative" in this Lie algebroid. Additionally, there are no "vertical directions", i.e. the vertical Lie subalgebroid of TM is the 0 bundle on M.

A contrasting family of examples is given by Lie Algebra Bundles, or LABS, over M as defined in 1.1.6, and whose diagram is given in (1.2). A LAB L over M necessarily has a trivial anchor $a = 0$, so L is its own vertical Lie subalgebroid and its elements are solely "vertical directions" on M, isomorphic in each fiber to a Lie algebra.

1.2.2 Involutive Distributions

The following family of examples exposes Lie algebroids with trivial vertical Lie subalgebroids, but which don't necessarily span all of the tangent directions of its base manifold M either.

Definition 1.2.1. Let $\mathcal{F} = \{L_\alpha\}_{\alpha \in A}$ for some indexing set A be a collection of disjoint connected regular submanifolds of M whose union is all of M. $\mathcal{F}$ is called a (smooth) foliation of M if every $m \in M$ is an element of a chart $(U, \phi : U \to \mathbb{R}^n)$ for which the coordinates of any $L_\alpha \in \mathcal{F}$ that intersects with U, $U \cap L_\alpha$ are described in the chart by the equations $x^{p+1} = c^{p+1}, \cdots, x^n = c^n$ for constants $c^{p+1}, \ldots, c^n \in \mathbb{R}$ for some natural number p. The elements $L_\alpha \in \mathcal{F}$ are called the leaves of the foliation $\mathcal{F}$.

Definition 1.2.2.

- A distribution Δ on M is a vector subbundle of TM; it is called q-dimensional if Δ is a vector bundle of rank q.
- A distribution Δ on M is called involutive or integrable if it is closed under the vector field commutator, i.e. every $\boldsymbol{\mu}, \boldsymbol{\nu} \in \Gamma(\Delta)$, when viewed as sections of TM, satisfy $[\boldsymbol{\mu}, \boldsymbol{\nu}] \in \Gamma(\Delta)$.

Let $\mathcal{F} = \{L_\alpha\}_{\alpha \in A}$ be a foliation of M. Each leaf L_α is a regular submanifold of M, therefore TL_α is embedded in TM and we may see TL_α as a submanifold of TM. The (disjoint) union $\Delta = \bigcup_{\alpha \in A} TL_\alpha \subseteq TM$ can be given a (smooth) vector bundle structure over M: around $m \in M$ there is a chart $(U, \psi : U \to \mathbb{R}^n)$ compatible with the foliation; from it we can construct a vector bundle chart $(U, \psi : U \times \mathbb{R}^n \to TM|_U)$ for TM which can be restricted to a chart $(U, \widetilde{\psi} : U \times \mathbb{R}^p \to \Delta|_U)$ for Δ, where $\Delta|_U = \bigcup_{\alpha : U \cap L_\alpha \neq \varnothing} TL_\alpha|_{U \cap L_\alpha}$. These charts constructed around all $m \in M$ also show that the inclusion $: \Delta \to TM$ is an embedding (since Δ can be described locally from vanishing

$m-p$ coordinates of a manifold chart of TM), making Δ a vector subbundle of TM.
Furthermore, any $\boldsymbol{\mu}, \boldsymbol{\nu} \in \Gamma(\Delta)$ has each of its integral lines completely contained in a single leaf of the foliation, so they can be restricted to vector fields on any leaf L_α, denoted by $\boldsymbol{\mu}|_{L_\alpha}$ and $\boldsymbol{\nu}|_{L_\alpha}$ respectively. They can also be considered vector fields on M and so their vector field commutator at $m \in M$ has the formula $[\boldsymbol{\mu}, \boldsymbol{\nu}]_m = (\mathcal{L}_{\boldsymbol{\mu}}\boldsymbol{\nu})_m = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\psi_{-t})_* \boldsymbol{\nu}_{\psi_t(m)} = [\boldsymbol{\mu}|_{L_\alpha}, \boldsymbol{\nu}|_{L_\alpha}]_m$, where $\psi_t(m)$ is the local flow of $\boldsymbol{\mu}$ at m and L_α is the unique leaf in which $m \in M$ is contained. Therefore, $\Gamma(\Delta)$ is closed under the vector field commutator and so we may consider its restriction to Δ a Lie algebroid bracket on Δ. In summary,

Proposition 1.2.3. A foliation of the manifold M gives rise to an integrable distribution Δ on M which, together with the anchor $a = id_{TM}|_\Delta$, becomes a Lie algebroid over M represented by:

$$\begin{tikzcd} (\Delta, [\cdot,\cdot]|_\Delta) \arrow[r, "id_{TM}|_\Delta"] \arrow[dr, "\pi|_\Delta"'] & (TM, [\cdot,\cdot]) \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

Notice that, if M is an n-dimensional manifold and $\mathcal{F}$ is a p-dimensional foliation with $p < n$, then Δ is a p-dimensional distribution and a is not fiberwise surjective, but it is of constant rank p.

The converse of Proposition 1.2.3 is also true, and it is the well known Frobenius theorem:

Proposition 1.2.4. Let Δ be a distribution in M. If Δ is integrable, then Δ arises from a foliation of M.

1.2.3 Trivial Lie Algebroid

The trivial Lie algebroid makes explicit the understanding of Lie algebroids as spaces of generalized directions on its base manifolds, as it is simply the combination of both the usual tangent directions of the manifold and a vertical Lie subalgebroid of Lie algebra valued directions. We will see in theorem 1.5.2 that any transitive Lie algebroid has this form locally. The transitive part, as we will see, simply means that every direction tangent to M is represented in the algebroid, i.e. the anchor is fiberwise surjective, which, as seen from the example of involutive distributions, isn't always the case.

Let M be a manifold, and $\mathfrak{g}$ a (real, finite dimensional) Lie algebra with Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Consider $p_1 : M \times \mathfrak{g} \to M$, $(m, \eta) \mapsto m$ as the trivial vector bundle with fiber $\mathfrak{g}$ and base M.

The space of sections of a trivial vector bundle $B \times V$ over B is a $C^\infty(B)$-module isomorphic to the space of smooth $\mathfrak{g}$-valued functions on B, denoted by $C^\infty(B, \mathfrak{g})$ and so there is a bijective correspondence between sections $\boldsymbol{\mu} \in \Gamma(B \times V)$ and functions $\widetilde{\mu} \in C^\infty(B, V)$, which is characterized by

$$\boldsymbol{\mu}(b) = (b, \widetilde{\boldsymbol{\mu}}(b)), \tag{1.3}$$

and, to any function $\widetilde{\boldsymbol{\nu}} \in C^\infty(B, V)$ corresponds a section $\boldsymbol{\nu} \in \Gamma(B \times V)$. Therefore, in general, we will indistinctively regard $C^\infty(B, V)$ of $\Gamma(B \times V)$ as the space of sections, so we might say that $\widetilde{\boldsymbol{\mu}} \in C^\infty(B, V)$ "is a section of $B \times V$". In our current situation, this means that may we regard the $\mathfrak{g}$-valued functions $C^\infty(M, \mathfrak{g})$ as the space of sections of $p_1 : M \times \mathfrak{g} \to M$.

Now, consider the internal direct sum over M of the vector bundles TM and $M \times \mathfrak{g}$ with projection map $\pi \oplus p_1 : TM \oplus (M \times \mathfrak{g}) \to M$. By abuse of notation we will also denote $\pi \oplus p_1$ as π. It has a vector bundle atlas $\left\{(U_i, \psi_i^{\oplus} : U_i \times (\mathbb{R}^{n_i} \oplus \mathfrak{g}) \to \pi^{-1}(U_i))\right\}_{i \in I}$, where n_i is some positive integer and U_i is a chart neighborhood of M that simultaneously trivializes the vector bundles TM and $M \times \mathfrak{g}$ through the coordinate maps $\psi_i^{TM} : U_i \times \mathbb{R}^{n_i} \to TU_i$ and $\psi^{\mathfrak{g}} = id_{U_i \times \mathfrak{g}}$.

The space of sections of $\Gamma(TM \oplus (M \times \mathfrak{g}))$ is a $C^\infty(M)$-module isomorphic to $\Gamma(TM) \oplus_{C^\infty(M)} \Gamma(M \times \mathfrak{g})$ which, in turn, is isomorphic to the module $\Gamma(TM) \oplus_{C^\infty(M)} C^\infty(M, \mathfrak{g})$. Therefore, the $C^\infty(M)$-module $\Gamma(TM) \oplus C^\infty(M, \mathfrak{g})$ may be regarded as the space of sections of $TM \oplus (M \times \mathfrak{g})$, and an element $\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}} \in \Gamma(TM) \oplus C^\infty(M, \mathfrak{g})$ will be called a section of this bundle.

We now define the main ingredients of the trivial Lie algebroid:

- The anchor $a = pr_1 : TM \oplus (M \times \mathfrak{g}) \to TM$, $X \oplus \eta \mapsto X$. This map is a vector bundle map as it clearly respects the fibers and, locally on a neighborhood chart, is $\widetilde{a} = (\psi_i^{TM})^{-1} \circ a \circ \psi_i^{\oplus} : (m, (v, \eta)) \mapsto v$ which is smooth since it is simply a projection, hence a is smooth.

- The bracket is given by

$$[\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}}, \boldsymbol{Y} \oplus \widetilde{\boldsymbol{\theta}}] := [\boldsymbol{X}, \boldsymbol{Y}] \oplus \{\boldsymbol{X}(\widetilde{\boldsymbol{\theta}}) - \boldsymbol{Y}(\widetilde{\boldsymbol{\eta}}) + [\widetilde{\boldsymbol{\eta}}, \widetilde{\boldsymbol{\theta}}]\}$$

  for all $\boldsymbol{X}, \boldsymbol{Y} \in \Gamma(TM)$, $\widetilde{\boldsymbol{\eta}}, \widetilde{\boldsymbol{\theta}} \in C^\infty(M, \mathfrak{g})$. Notice that this definition, by $\mathbb{R}$-linearity and alternation which are clear, means that $[\boldsymbol{X}, \widetilde{\boldsymbol{\theta}}] \equiv \boldsymbol{X}(\widetilde{\boldsymbol{\theta}})$ and $[\widetilde{\boldsymbol{\eta}}, \boldsymbol{Y}] = -[\boldsymbol{Y}, \widetilde{\boldsymbol{\eta}}] \equiv -\boldsymbol{Y}(\widetilde{\boldsymbol{\eta}})$, i.e. the bracket on this vector bundle is the Lie derivative applied to both vector fields and vector valued functions. To check the Jacobi identity,

$$[\boldsymbol{\mathfrak{X}}, [\boldsymbol{\mathfrak{Y}}, \boldsymbol{\mathfrak{Z}}]] = [[\boldsymbol{\mathfrak{X}}, \boldsymbol{\mathfrak{Y}}], \boldsymbol{\mathfrak{Z}}] + [\boldsymbol{\mathfrak{Y}}, [\boldsymbol{\mathfrak{X}}, \boldsymbol{\mathfrak{Z}}]],$$

  it is enough to check two cases: when two of the sections $\boldsymbol{\mathfrak{X}}, \boldsymbol{\mathfrak{Y}}, \boldsymbol{\mathfrak{Z}} \in \Gamma(TM \oplus (M \times \mathfrak{g}))$ are sections of TM and the remaining one is a section of $M \times \mathfrak{g}$, and, where the opposite is true, all of which can easily be done.

Let's check that the compatibility conditions between the anchor and the bracket are satisfied; let $\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}}, \boldsymbol{Y} \oplus \widetilde{\boldsymbol{\theta}} \in \Gamma(TM \oplus (M \times \mathfrak{g}))$, $f \in C^\infty(M)$:

1. $a : \Gamma(A) \to \Gamma(TM)$ is a Lie algebra morphism:

$$\begin{aligned} a([\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}}, \boldsymbol{Y} \oplus \widetilde{\boldsymbol{\theta}}]) &= a(\,[\boldsymbol{X}, \boldsymbol{Y}] \oplus \{\boldsymbol{X}(\widetilde{\boldsymbol{\theta}}) - \boldsymbol{Y}(\widetilde{\boldsymbol{\eta}}) + [\widetilde{\boldsymbol{\eta}}, \widetilde{\boldsymbol{\theta}}]\}) \\ &= [\boldsymbol{X}, \boldsymbol{Y}] = [a(\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}}), a(\boldsymbol{Y} \oplus \widetilde{\boldsymbol{\theta}})]. \end{aligned}$$

2. Lebniz identity:

$$\begin{aligned} &[\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}}, f\,(\boldsymbol{Y} \oplus \widetilde{\boldsymbol{\theta}})] = \\ &\qquad = [\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}}, f\boldsymbol{Y} \oplus f\widetilde{\boldsymbol{\theta}}] \\ &\qquad = [\boldsymbol{X}, f\boldsymbol{Y}] \oplus \{\boldsymbol{X}(f\widetilde{\boldsymbol{\theta}}) - f\boldsymbol{Y}(\widetilde{\boldsymbol{\eta}}) + [\widetilde{\boldsymbol{\eta}}, f\widetilde{\boldsymbol{\theta}}]\} \\ &\qquad = (f[\boldsymbol{X}, \boldsymbol{Y}] + \boldsymbol{X}(f)\,\boldsymbol{Y}) \oplus \{\boldsymbol{X}(f)\,\widetilde{\boldsymbol{\theta}} + f\,\boldsymbol{X}(\widetilde{\boldsymbol{\theta}}) - f\boldsymbol{Y}(\widetilde{\boldsymbol{\eta}}) + f[\widetilde{\boldsymbol{\eta}}, \widetilde{\boldsymbol{\theta}}]\} \\ &\qquad = f([\boldsymbol{X}, \boldsymbol{Y}] \oplus \{\boldsymbol{X}(\widetilde{\boldsymbol{\theta}}) - \boldsymbol{Y}(\widetilde{\boldsymbol{\eta}}) + [\widetilde{\boldsymbol{\eta}}, \widetilde{\boldsymbol{\theta}}]\}) + (\boldsymbol{X}(f)\boldsymbol{Y} \oplus \boldsymbol{X}(f)\widetilde{\boldsymbol{\theta}}) \\ &\qquad = f[\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}}, \boldsymbol{Y} \oplus \widetilde{\boldsymbol{\theta}}] + a(\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}})(f)\,(\boldsymbol{Y} \oplus \widetilde{\boldsymbol{\theta}}). \end{aligned}$$

The vector bundle $TM \oplus (M \times \mathfrak{g})$ together with the previously defined anchor and bracket is called <u>the trivial Lie algebroid on M with structure</u>

algebra $\mathfrak{g}$, and it is represented by:

$$\begin{tikzcd}
(TM \oplus (M \times \mathfrak{g}), [\cdot,\cdot]) \arrow[r, "pr_1"] \arrow[dr, "\pi \oplus p_1"] & (TM, [\cdot,\cdot]) \arrow[d, "\pi"] \\
& M.
\end{tikzcd} \tag{1.4}$$

Since the underlying set of this Lie algebroid is $TM \times \mathfrak{g}$, we will constantly refer to the latter as the trivial Lie algebroid.

1.2.4 The Algebroid of Derivations on a Vector Bundle

Let $p : E \to M$ be a vector bundle. Its algebroid of derivations $\mathfrak{D}(E)$ is analogous to the Lie algebra of infinitesimal linear transformations $\mathfrak{gl}(V)$ of a vector space V. In fact, if $E = M \times V$ is a trivial vector bundle, $\mathfrak{D}(E)$ is isomorphic to the trivial Lie algebroid $TM \oplus (M \times \mathfrak{gl}(V))$. This Lie algebroid will be important to define representations of a Lie algebroid on a vector bundle, a generalized notion of vector bundles associated to a principal bundle.

Let $End(E)$ denote the vector bundle whose sections are vector bundle endomorphisms or E, which has $End(E_m)$ as fiber at $m \in M$. We may regard the sections of $End(E)$ as $C^\infty(M)$-linear maps $\Gamma(E) \to \Gamma(E)$, also called 0th-order differential operators on E.

A 1st-order differential operator on E is an $\mathbb{R}$-linear map

$$D : \Gamma(E) \to \Gamma(E)$$

such that, for every $f \in C^\infty(M)$, the map

$$F : \Gamma(E) \to \Gamma(E), \qquad \boldsymbol{\mu} \mapsto D(f\boldsymbol{\mu}) - fD(\boldsymbol{\mu})$$

is a 0-th order differential operator. It is easy to check that this condition implies that the 1st order differential operators are local operators on E. They are sections of a vector bundle $Diff^1(E)$ on M [6]. Common examples include the usual partial derivatives of functions : $\mathbb{R}^n \to \mathbb{R}$ understood as sections of $E = \mathbb{R}^n \times \mathbb{R}$, the Lie derivative $\mathcal{L}_{\boldsymbol{X}}$ on $E = TM$, and the covariant derivatives $\nabla_{\boldsymbol{X}}$ on any vector bundle E, where $\boldsymbol{X}$ is a vector field on M.

Let $Hom(T^*M, End(E))$ be the vector bundle whose sections are vector bundle morphisms between T^*M and $End(E)$ over M or, equivalently, $C^\infty(M)$-linear maps : $\Omega^1(M) \to \Gamma(End(E))$. Define the symbol map to be the vector bundle morphism

$$\sigma : Diff^1(E) \to Hom(T^*M, End(E))$$

defined by the induced $C^\infty(M)$-linear action on 1-forms on M:

$$\sigma(D)(fdg)(\boldsymbol{\mu}) := f(D(g\boldsymbol{\mu}) - gD(\boldsymbol{\mu}))$$

for all $D \in \Gamma(\mathfrak{D}(E))$, $f, g \in C^\infty(M)$, $\boldsymbol{\mu} \in \Gamma(E)$. Note that the previous definitions suffices since $\sigma(D) \in \Gamma(Hom(T^*M, End(E)))$ is a local operator because it is a $C^\infty(M)$-linear map between sections of vector bundles over M. σ is a surjective submersion [6], and its kernel is precisely $End(E)$ since $F : \Gamma(E) \to \Gamma(E)$ vanishes precisely here.

The bundle TM is embedded into $Hom(T^*M, End(E))$ via

$$\begin{aligned} T_mM &\to Hom(T_m^*M, End(E_m)) \\ X_m &\mapsto (\omega_m \mapsto \omega_m(X_m)\cdot\ ) \end{aligned}$$

where $\omega_m(X_m)\cdot$ denotes the multiplication operator by $\omega_m(X_m) \in \mathbb{R}$ operator in $End(E_m)$. The bundle $\sigma^{-1}(TM)$ may be realized as the pullback on the category of (smooth) vector bundles over M

$$\begin{tikzcd} \mathfrak{D}(E) \arrow[r, dashed, "a"] \arrow[d, dashed] & TM \arrow[d, hook] \\ Diff^1(E) \arrow[r, two heads, "\sigma"] & Hom(T^*M, End(E)). \end{tikzcd}$$

This diagram implies that $\mathfrak{D}(E)$ can be regarded as a vector subbundle of $Diff^1(E)$ since the left vertical arrow must be an injective immersion, and $a : \mathfrak{D}(E) \to TM$ is a surjective vector bundle map since the bottom arrow is so. This means that we have an exact sequence of vector bundles over M

$$0 \to End(E) \to \mathfrak{D}(E) \xrightarrow{a} TM \to 0 \tag{1.5}$$

In conclusion, the sections of $\mathfrak{D}(E)$, called derivations on E, are characterized inside the 1st-order differential operators : $\Gamma(E) \to \Gamma(E)$ by the following property: for any $D \in \Gamma\mathfrak{D}(E)$ there exists a vector field $a(D)$ on M such that

$$D(f\boldsymbol{\mu}) = fD(\boldsymbol{\mu}) + a(D)(f)\boldsymbol{\mu} \tag{1.6}$$

for all $f \in C^\infty(M)$, $\boldsymbol{\mu} \in \Gamma(E)$.

To complete the structure of $\mathfrak{D}(E)$ as a Lie algebroid, the vector bundle endomoprhism commutator

$$[D, D'] = D \circ D' - D' \circ D$$

is also a derivation on E for any $D, D' \in \Gamma(\mathfrak{D}(E))$, and it is straightforward to verify the compatibility conditions

$$a([D, D']) = [a(D), a(D')] \tag{1.7}$$

$$[D, fD'] = f[D, D'] + a(D)(f)D' \tag{1.8}$$

for all $D, D' \in \Gamma(\mathfrak{D}(E)), f \in C^\infty(M)$ which allow us to conclude that, with a defined in equation (1.6) and with the endomorphism commutator,

$$\begin{tikzcd} (\mathfrak{D}(E), [\cdot,\cdot]) \arrow[r, "a"] \arrow[dr] & (TM, [\cdot,\cdot]) \arrow[d, "\pi"] \\ & M \end{tikzcd} \tag{1.9}$$

is a transitive Lie algebroid on M, called <u>the Lie algebroid of derivations on E</u>.

Example 1.2.5 (The derivations Lie algebroid of a trivial vector bundle)**.** Let E be a trivial vector bundle $M \times V$. Then $End(E) \cong M, \mathfrak{gl}(V)$. On such an E, the standard derivative with respect to $\boldsymbol{X}$ on vector V-valued functions is a derivation for any a vector field $\boldsymbol{X} \in \Gamma(TM)$; since this mapping is $C^\infty(M)$-linear it defines a vector bundle section of the anchor in (1.5) that respects the Lie bracket. This section acts as an injection of TM in $\mathfrak{D}(E)$ that defines an isomorphism

$$\mathfrak{D}(M \times V) \cong TM \oplus (M \times \mathfrak{gl}(V)), \tag{1.10}$$

which is a trivial Lie algebroid 1.2.3. Thus, the corresponding Lie algebroid is represented by:

$$\begin{tikzcd} (TM \oplus (M \times \mathfrak{gl}(V)), [\cdot,\cdot]) \arrow[r, "pr_1"] \arrow[dr, "\pi\oplus p_1"] & (TM, [\cdot,\cdot]) \arrow[d, "\pi"] \\ & M \end{tikzcd} \tag{1.11}$$

1.3 The Atiyah Lie Algebroid Sequence of a Principal Bundle

Throughout this section $G \to P \to M$ will denote a principal bundle over M with the (real) Lie group G (in particular, a finite dimensional, smooth manifold) as fiber which acts to the right by the smooth action $R : G \times P \to G$.

The Atiyah Lie algebroid sequence of principal bundles is an exact sequence of Lie algebroids that arises naturally in traditional gauge theories, explicitly in the context of principal connections forms. This exact sequence will be generalized by the Lie algebroid sequences of transitive Lie algebroids, allowing the formulation of the concept of transitive Lie algebroid connections, among other things. The main component of this sequence is the Atiyah Lie algebroid TP/G. Our first goal is to give a smooth vector bundle structure to TP/G inherited from the corresponding structure on TP, and to define the smooth fiberwise-linear anchor and the Lie algebra structure on $\Gamma(TP/G)$, all of which will allow us to say that TP/G is indeed a Lie algebroid.

Then, we will follow a similar procedure to define the adjoint Lie algebroid $P \times \mathfrak{g}/G$ of TP/G and its embedding into TP/G as the kernel of the anchor, concluding the Atiyah Lie algebroid short exact sequence

$$0 \to P \times \mathfrak{g}/G \xrightarrow{j} TP/G \xrightarrow{a} TM \to 0,$$

formally introduced in Definition 1.3.21.

1.3.1 The Atiyah Lie algebroid

The Vector Bundle TP/G

Proposition 1.3.1. Let $p^E : E \to P$ be a (smooth) vector bundle over the principal bundle P. Suppose that G acts (smoothly) to the right on E in such a way that:

1. Each $g \in G$ acts by vector bundle isomorphism over the right translation map $R_g : P \to P$.

2. E is covered by G-equivariant π-saturated charts. That is, for each $p \in P$ there is a chart $(\mathcal{U}, \psi : \mathcal{U} \times V \to E|_{\mathcal{U}})$ of E, where $p \in \mathcal{U} = \pi^{-1}(U)$ for some open $U \subseteq M$, that satisfies

$$\psi(p, v)g = \psi(pg, v)$$

for all $p \in \mathcal{U}$, $v \in V$, $g \in G$.

Then *the quotient space E/G has a (smooth) vector bundle structure $p^{E/G} : E/G \to M$, called the quotient vector bundle of $p^E : E \to P$ by the action of G, which is unique* in the sense that:

- E/G has the unique manifold structure that makes the quotient map $\natural : E \to E/G$ a surjective submersion.
- E/G has the unique topological vector bundle structure that makes the quotient map $\natural$ linear over $\pi : P \to M$.

Furthermore, the diagram

$$\begin{tikzcd} E \arrow[r, two heads, "\natural"] \arrow[d, "p^E"] & E/G \arrow[d, "p^{E/G}"] \\ P \arrow[r, two heads, "\pi"] & M \end{tikzcd} \tag{1.12}$$

commutes and E is isomorphic to the pullback of E/G by π.

We will denote by $\langle \mu \rangle \in E/G$ the orbit of $\mu \in E$, and by $p^{E/G} : E/G \to M$ the projection map; the projection map of the vector bundle E/G over M satisfies

$$p^{E/G}(\langle \mu \rangle) = \pi(p^E(\mu)) \in M \tag{1.13}$$

for any $\mu \in E$, as indicated by the commutative diagram. In general, the action of $g \in G$ on some manifold will be denoted by concatenation if no confusion may arise; for example, pg means $R_g(p)$ for any $p \in P$.

Proof. Notice that conditions (1) and (2) tell us that, locally, the exact point in the fiber $\pi^{-1}(m) \in \mathcal{U} \subseteq P$ of $m \in M$ that we choose is not important to determine the vector space V nor the representing vector $v \in V$, therefore it should make sense to eliminate that redundancy, obtained by the group action, to make charts maps : $U \times V \to E/G|_U$ from $\psi : \mathcal{U} \times V \to E|_{\mathcal{U}}$. This is what we will now formalize.

Let $p^{E/G} : E/G \to M$, $\langle \mu \rangle \mapsto \pi(p^E(\mu))$. This function is well defined because, by (1), $p^E(\mu g)$ and $p^E(\mu)$ are in the same fiber of P, for all $g \in G$ (notice that condition (1) implies that $p^E(\mu g) = p^E(\mu)g$ for μ and g arbitrary).

Let $m \in M$ and let $\psi : \mathcal{U} \times V \to E|_{\mathcal{U}}$ be coordinate map of E such that $m \in U$ that satisfies condition (2); we may further assume that $U \subseteq M$ is a trivializing neighborhood of P (by intersecting one such neighborhood of m with U if necessary), so that there exists a local section $\sigma : U \to \mathcal{U} \subseteq P$ of π such that the local trivialization of P is described by $\psi^G : U \times G \to \mathcal{U}$, $(m', g) \mapsto \sigma(m')g$.

We will now see that $E/G|_m := p^{E/G}(m)$ can be given a vector space structure isomorphic to V:

- To define $\langle \mu \rangle + \langle \nu \rangle$ for $\langle \mu \rangle, \langle \nu \rangle \in E/G|_m$ arbitrary, let $g \in G$ be such that $p^E(\nu g) = p^E(\mu)$, which exists because of condition (1). This means that both μ and νg belong to the same vector space $E_{p^E(\mu)}$, so we may define $\langle \mu \rangle + \langle \nu \rangle := \langle \mu + \nu g \rangle$. To see that this is well defined it suffices to show that the result doesn't change if another representative of, say, the class of $\langle \nu \rangle$ is chosen: let $\nu' \in \langle \nu \rangle$, thus there is element $g_\nu \in G$ such that $\nu = \nu' g_\nu$; now let $g' \in G$ be such that $p^E(\nu' g') = p^E(\mu)$, therefore, by (1), $p^E(\nu) g_\nu g' = p^E(\nu g_\nu g') = p^E(\mu) \in P$, which allows us to conclude that $g_\nu g' = g$ as it is also true that $p^E(\mu) = p^E(\nu)g$ and G acts freely on P.

- We define $t\langle \mu \rangle := \langle t\mu \rangle$, for any $t \in \mathbb{R}$, $\langle \mu \rangle \in E/G$. This is well defined as, by condition (1), G acts by vector bundle isomorphism, in particular it restricts to linear maps between fibers. It is now clear that these operations make $E/G|_m$ a vector space.

- To see that $E/G|_m$ is isomorphic to V, we prove instead that it is isomorphic to E_p for any $p \in \pi^{-1}(m) \subseteq \mathcal{U}$, each of which is isomorphic to V by the restriction of the chart to the fiber of $p \in \mathcal{U}$, $\psi_p : V \to E_p$, which is a vector space isomorphism by definition. For any $p \in \pi^{-1}(m)$ the restriction of $\natural$ to its fiber, $\natural_p : E_p \to E/G|_m$, can easily be seen to be linear; it is clearly surjective, and it is also injective as, if $\mu, \nu \in E_p$ are such that $\natural_p(\mu) = \langle\mu\rangle = \langle\nu\rangle = \natural_p(\nu)$, then there exists a $g \in G$ such that $\mu = \nu g$, but that means that $p = p^E(\mu) = p^E(\nu g) = p^E(\nu)g = pg$ which can only happen if $g = e$, i.e. if $\mu = \nu \in E_p$.

To define the vector bundle of E/G suggested above, we first notice that the vector bundle coordinate map $\psi : \mathcal{U} \times V \to E_{\mathcal{U}}$ and the principal bundle trivialization $\rho : U \times G \to \mathcal{U} = P|_U$ associated to the local section $\sigma : U \times \mathcal{U}$ allow us to make the diffeomorphism $U \times G \times V \to E|_{\mathcal{U}}$, $(m', g, v) \mapsto \psi(\sigma(y)g, v)$; if we make G act, smoothly, to the right on $U \times G \times V$ by $(y, g, v)g' := (m', gg', v)$ for all $g' \in G$, it is clear that the quotient topological space $(U \times G \times V)/G$ is homeomorphic to $U \times V$, and so, taking the quotients by G on both sides we obtain the homeomorphism:

$$\psi^G : U \times V \to E/G|_U, \qquad\qquad (y, v) \mapsto \langle\psi(\sigma(y), v)\rangle$$

Notice that this definition is independent of which local section is chosen, in the sense that if σ' is another local section around $m \in M$, in the intersection of the domains of σ and σ', condition (2) implies that $\langle\psi(\sigma(y), v)\rangle = \langle\psi(\sigma'(y), v)\rangle$, because $\sigma(y) = \sigma(y')g$ for some $g \in G$. This map is, fiberwise, a vector space isomorphism which can be seen by rewriting it as the composition of the isomorphisms $\natural_{\sigma(m)} \circ \psi_{\sigma(m)} : V \to E_{\sigma(m)} \to E/G|_m$, so we can give E/G a topological vector bundle structure over M by stating that the $\psi^G : U \times V \to E/G|_U$ maps are local trivializations.

To give a smooth manifold structure to E/G we notice that the transition function between this trivialization and any other local trivialization around m is smooth in their common domain. Let $\psi'^G : U' \times V \to E/G|_{U'}$ with $m \in U'$ come from an equivariant coordinate map $\psi' : \mathcal{U}' \times V \to E_{\mathcal{U}'}$ of E with respect to a local section σ' of P; the transition function on E between this two charts $\mathrm{tr} : \mathcal{U} \cap \mathcal{U}' \to GL(V)$, $p \mapsto \psi_p^{-1} \circ \psi'_p$ is smooth, so its precomposition with $\sigma : U \cap U' \to \mathcal{U} \cap \mathcal{U}'$ is also smooth, but by condition (2) we obtain the same map if precompose with σ' instead, and so we see that this smooth map is precisely the transition function $\mathrm{tr}^G : U \cap U' \to GL(V)$, $m' \mapsto (\psi^G_{m'})^{-1} \circ \psi'^G_{m'}$ on E/G, which, therefore, might be written by the formula $\mathrm{tr}^G_{m'} = \psi^{-1}_{\sigma(m')} \circ \psi'_{\sigma(m')}$. Finally, the family of charts $\{(U, \psi^G : U \times V \to E/G|_U)\}$ constructed from the equivariant charts of E, with smooth transition functions, give a smooth structure to the topological vector bundle E/G.

We now prove that the (surjective) quotient map $\natural : E \to E/G$ is also smooth and a submersion. Let $\psi : \mathcal{U} \times V \to E|_{\mathcal{U}}$ be an equivariant chart of E with $U = \pi(\mathcal{U})$ trivializing for P, as above. Notice that the restriction of $\natural$ to U may be written as $\natural_U = \psi^G \circ (\pi \times id_V) \circ \psi^{-1} : E|_{\mathcal{U}} \to E/G|_U$, where ψ and $\psi^{/G}$ are diffeomorphisms: this equates to the formula

$$\natural_U \circ \psi = \psi^{/G} \circ (\pi \times id_V) : \mathcal{U} \times V \to E/G|_U,$$

which for $(p, v) \in \mathcal{U} \times V$, in the LHS gives us $\natural_U \circ \psi(p, v) = \langle \psi(p, v) \rangle$, and for the RHS $\psi^G \circ (\pi \times id_V)(p, v) = \psi^G(\pi(m), v) := \langle \psi(\sigma(\pi(m)), v) \rangle$ which coincide because $p = \sigma(\pi(p))g$ for some $g \in G$. This formula for $\natural_U$ implies that it inherits the properties of $\pi \times id_V$ as they factor through the diffeomorphisms ψ and ψ^G. Therefore, $\natural_U$ is smooth and also a submersion as both π and id_V are submersions, since the pushforward of $\pi \times id_V$, which is diffeomorphic to the product of the pushforwards, is surjective.

The uniqueness assertions follow from the fact that there is at most one manifold structure in the image of a surjection which makes it a submersion, and at most one topological vector bundle structure in the image of a surjection which makes the map a topological vector bundle homomorphism.

E is clearly in a bijective correspondence with the pullback bundle $\pi^* E/G$ equal to $\{(p, \langle \mu \rangle) \mid \pi(p) = p^{E/G}(\langle \mu \rangle)\}$. Furthermore, the pullback bundle can be characterized by the fact that it is locally isomorphic to the product bundle $p_1 : \pi^{-1}(U) \times V \to \pi^{-1}(U)$ for some coordinate chart U of M, precisely as the atlas of equivariant charts on E show. Therefore, we see that E is isomorphic to the pullback bundle. $\square$

Theorem 1.3.2. Let $g \in G$ act to the right on the vector bundle $\pi : TP \to P$ by the push-forward of the right translation map $R_g : P \to P$ on P. Then, the quotient space TP/G has a (smooth) vector bundle structure over M, with the projection map called $\pi^G : TP/G \to M$, such that the quotient map $\natural : TP \to TP/G$ is a smooth, linear, surjective submersion, and for any $\mathcal{X} \in TP$

$$\pi^G(\langle \mathcal{X} \rangle) = \pi(\mathcal{X}). \tag{1.14}$$

The bundle $\pi^G : TP/G \to M$ is called the vector bundle of the Atiyah Lie algebroid associated to the principal bundle P.

Proof. This theorem reduces to the verification that the conditions in 1.3.1 are satisfied by $\pi : TP \to P$. Condition (1) is satisfied since, for any $g \in G$, the map $R_{g*} : TP \to TP$ is a fiberwise linear map over R_g with inverse $R_{g^{-1}*}$.

Given $p_0 \in P$, to construct an equivariant π-saturated chart near p we will start with a chart $\psi : U \times G \to \mathcal{U} = \pi^{-1}(U)$ for P where U is the range of a chart $\theta : \mathbb{R}^n \to U$ for M. The desired chart $\Psi : \mathcal{U} \times (\mathbb{R}^n \oplus \mathfrak{g}) \to TP|_{\mathcal{U}}$ for TP will essentially be the pushforward of ψ, after we identify (via diffeomorphisms)

- $U \times \mathbb{R}^n$ with TU by the map $\theta_* : \mathbb{R}^n \times \mathbb{R}^n \to T_m U$,
- $G \times \mathfrak{g}$ with TG via $R_{\cdot *,1} : G \times \mathfrak{g} \to TG$,
- and, therefore, the vector bundle $T(U \times G)$, understood as the external Whitney sum $T(U) \boxplus T(G)$, with $(U \times \mathbb{R}^n) \boxplus (G \times \mathfrak{g})$ and so with $\mathcal{U} \times (\mathbb{R}^n \oplus \mathfrak{g})$ via $\psi_* : TU \boxplus TG \to TP|_{\mathcal{U}}$.

Concretely, let

$$\begin{aligned} \widetilde{\Psi} :&(U \times G) \times (\mathbb{R}^n \oplus \mathfrak{g}) && \xrightarrow{\cong} && TP|_{\mathcal{U}} \\ &(m, g, v, \eta) && \mapsto && \psi_{*,(m,g)}(\theta_{*,\theta^{-1}(m)} v, R_{g*,1}\eta). \end{aligned}$$

Now, define the coordinate map for TP/G $\Psi : \mathcal{U} \times (\mathbb{R}^n \oplus \mathfrak{g}) \to T_{\mathcal{U}}P$ by $\Psi = \widetilde{\Psi} \circ (\psi^{-1} \times id_{\mathbb{R}^n \oplus \mathfrak{g}})$. The equivariance of this chart means that for any $p \in \mathcal{U}$, which can be written as $p = \psi(m, g)$ for some $m \in M$ and $g \in G$,

$$\begin{aligned} \Psi(p, v, \eta)g' &= R_{g'*,p} \circ \psi_{*,p}(X_m, R_{g*,1}\eta) \\ &= \psi_{*,(m,gg')}(X_m, R_{gg'*,1}\eta) = \Psi(pg', v, \eta). \end{aligned}$$

where $X_m = \theta_{*,\theta^{-1}(m)} v \in T_m M$ and $pg' = \psi(m, gg')$.

This follows from the of the identity $\psi(m, g)g' = \psi(m, gg')$, when it is expressed as the equality of the functions $R_{g'} \circ \psi = \psi \circ (Id_U \times R_{g'}) : U \times G \to TP|_{\mathcal{U}}$: deriving each formula at (m, g) using the chain rule and the equality $R_g \circ R_{g'} = R_{gg'}$ we get the desired equation.

$\square$

The Anchor

The following proposition and its corollary allow us to define smooth vector bundle maps between quotient vector bundles. In particular, it will allow us to define the anchor $\pi_* : TP/G \to TM$ of what will be the Atiyah Lie algebroid, and later on to define the embedding of the adjoint Lie algebroid into TP/G.

Proposition 1.3.3. Let $p^E : E \to P$ and $p^{E'} : E' \to P'$ be vector bundles over principal bundles $G \to P \to M$ and $G' \to P' \to M'$, respectively. Let G and G' act on E and E' respectively satisfying the conditions of Proposition 1.3.1. Let $\Psi : E \to E'$ be a vector bundle morphism, over the principal bundle morphism $F(f, \psi)$ where $f : M \to M'$ and $\psi : G \to G'$, that satisfies

$$\Psi(\mu g) = \Psi(\mu)\psi(g)$$

for all $\mu \in E$, $g \in G$, i.e. $\underline{\Psi \text{ is } G\text{-equivariant}}$. Then, there *exists a unique (smooth) vector bundle morphism* $\Psi^G : E/G \to E'/G'$ over $f : M \to M'$ such that $\Psi^G \circ \natural = \natural' \circ \Psi$, where $\natural$ and $\natural'$ denote the quotient maps in E and E' respectively. In particular, the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "\Psi"] \arrow[d, "\natural"] & E' \arrow[d, "\natural'"] \\
E/G \arrow[r, dashed, "\Psi^G"] \arrow[d, "p^{E/G}"] & E'/G' \arrow[d, "p^{E'/G'}"] \\
M \arrow[r, "f"] & M'
\end{tikzcd} \tag{1.15}$$

Proof. The top square of the above diagram implies that Ψ^G is uniquely defined by $\Psi^G(\langle\mu\rangle) := \langle\Psi(\mu)\rangle$. This map is well defined thanks to the G-equivariance of Ψ.

To see that Ψ^G is fiberwise linear, for $\langle\mu\rangle, \langle\nu\rangle \in E/G$ let $g \in G$ be such that μ and νg are in the same vector fiber of E; notice that this also means that $\Psi(\mu)$ and $\Psi(\nu g) = \Psi(\nu)\psi(g)$ are in the same vector fiber of E' because $\Psi : E \to E'$ is fiber preserving. Then $\Psi^G(\langle\mu\rangle + t\langle\nu\rangle) = \Psi^G(\langle\mu + t\nu g\rangle) = \langle\Psi(\mu+t\nu g)\rangle = \langle\Psi(\mu)+t\Psi(\nu)\psi(g)\rangle = \langle\Psi(\mu)\rangle+t\langle\Psi(\nu)\rangle = \Psi^G(\langle\mu\rangle)+t\Psi^G(\langle\nu\rangle)$.

Now let us prove that Ψ^G is smooth. Given that $F := \Psi^G \circ \natural = \natural' \circ \Psi : E \to E'/G'$ is a smooth mapping and that $\natural$ is a surjective submersion, the submersion theorem allows us to see the local smoothness of Ψ^G, and therefore conclude that Ψ^G is smooth. Explicitly, we may apply the submersion theorem to $\natural$ to conclude that for any $\mu \in E$ there exist manifold charts

$(\mathcal{U}, \Phi : \mathcal{U} \to \mathbb{R}^N)$ of E near μ, $(U, \phi : U \to \mathbb{R}^n)$ of E/G near $\langle \mu \rangle$, and $(U', \phi' : U' \to \mathbb{R}^m)$ of E'/G' near $\Psi^G(\langle e \rangle)$ where $n \leq N$, such that:

$$
\begin{tikzcd}
(x_1, \ldots, x_N) \arrow[d, mapsto, "\phi\circ\natural\circ\Phi^{-1}"] \arrow[dr, mapsto, "\phi'\circ(\natural'\circ\Psi)\circ\Phi^{-1}"] & \\
(x_1, \ldots, x_n) \arrow[dr, mapsto, "\phi'\circ\Psi^G\circ\phi^{-1}"'] & (\widetilde{F}^1(x_1, \ldots, x_N), \ldots, \widetilde{F}^m(x_1, \ldots, x_N)) \arrow[d, equal] \\
& (\widetilde{\Psi^G}^1(x_1, \ldots, x_n), \ldots, \widetilde{\Psi^G}^m(x_1, \ldots, x_n))
\end{tikzcd}
$$

the smoothness of F implies that the functions $F^i : \Phi(\mathcal{U}) \to \phi'(U')$ are smooth, but $\widetilde{F}^i(x_1, \ldots, x_N) = \widetilde{\Psi^G}^i(x_1, \ldots, x_n)$, so the functions $\widetilde{\Psi^G}^i : \phi(U) \to \phi'(U')$ that locally trivialize Φ^G are smooth, therefore so is Ψ^G.

Finally, Ψ^G is a smooth vector bundle morphism over $f : M \to M'$ since $p^{E'/G'} \circ \Psi^G(\langle \mu \rangle) = p^{E'/G'}(\langle \Psi(\mu) \rangle) = p^{E'}(\Psi(\mu)) = f \circ p^E(\mu) = f \circ p^{E/G}(\langle \mu \rangle)$.

□

Corollary 1.3.4. Let $p^E : E \to P$ be a vector bundle over the principal bundle $G \to P \to M$, and let G act on E satisfying the conditions of Proposition 1.3.1. Let $p^{E'} : E' \to M'$ be another vector bundle. Let $\Psi : E \to E'$ be a vector bundle morphism over a map $f : P \to M'$ that satisfies

$$\Psi(\mu g) = \Psi(\mu)$$

and $f(pg) = f(p)$ for all $\mu \in E$, $p \in P$ and $g \in G$. Then there exists a unique (smooth) vector bundle morphism $\Psi^G : E/G \to E'$ over a map $f^G : M \to M'$ such that $f = f^G \circ \pi$, that satisfies

$$\Psi^G \circ \natural = \Psi,$$

where $\natural : E \to E/G$ is the quotient map. In particular, the following diagram commutes

$$\begin{tikzcd} E \arrow[d, "\natural"] \arrow[dr, "\Psi"] & \\ E/G \arrow[r, dashed, "\Psi^G"] \arrow[d, "p^{E/G}"] & E' \arrow[d, "p^{E'}"] \\ M \arrow[r, "f^G"] & M' \end{tikzcd} \tag{1.16}$$

Proof. Consider M' as the principal bundle $P(M', \{1\})$, let $\psi : G \to \{1\}$ be trivial and apply theorem 1.3.3. $\square$

Applying the previous corollary to the derivative of $\pi : P \to M$, gives us immediately the following theorem.

Theorem 1.3.5. There exists a unique (smooth) vector bundle morphism $\pi_*^G : TP/G \to TM$ over M such that the following diagram commutes:

$$\begin{tikzcd} TP \arrow[d, "\natural"] \arrow[dr, "\pi_*"] & \\ TP/G \arrow[r, dashed, "\pi_*^G"] \arrow[d, "\pi_*^G"] & TM \arrow[d, "\pi"] \\ M \arrow[r, "id_M"] & M \end{tikzcd} \tag{1.17}$$

We call the vector bundle morphism $\pi_*^G : TP/G \to TM$ the anchor of the Atiyah Lie algebroid of P.

The previous theorem implies that the anchor of the Atiyah Lie algebroid is well defined by the formula

$$\pi_*^G(\langle \mathcal{X} \rangle) = \pi_*(\mathcal{X}) \in TM \tag{1.18}$$

for any $\mathcal{X} \in TP$.

To finish the construction of the algebroid we need to define a Lie bracket in its space of sections. We devote the next subsections to this purpose.

The Space of Sections

Now, the following proposition gives a description of the sections of a quotient vector vector bundle by a group action.

Proposition 1.3.6. Let $p^E : E \to P$ on which G acts to the right satisfying the properties of Proposition 1.3.1. Let $U \subseteq M$ be open and $\mathcal{U} = \pi^{-1}(U)$ open in P. The extension of the quotient $\natural : E \to E/G$ map to local sections of these vector bundles is an isomorphism of $C^\infty(U)$-modules between the local sections $\Gamma_U(E/G)$ of E and the G-invariant local sections of E on $\mathcal{U}$

$$\Gamma^G_{\mathcal{U}}(E) = \{\boldsymbol{\mu} \in \Gamma_{\mathcal{U}}(E) \mid \forall p \in P, \forall g \in G : \ \boldsymbol{\mu}_{pg} = \boldsymbol{\mu}_p g\},$$

where the action of a function is given by $f\boldsymbol{\mu}(p) := (f \circ \pi)(p)\boldsymbol{\mu}(p)$ for any $\boldsymbol{\mu} \in \Gamma^G_{\mathcal{U}}(E)$, $p \in \mathcal{U}$ via . Explicitly, the isomorphism is given by

$$\begin{aligned} \Gamma^G_{\mathcal{U}}(E) &\to \Gamma_U(E/G) \\ \boldsymbol{\mu} &\mapsto \langle \boldsymbol{\mu} \rangle : m \mapsto \langle \boldsymbol{\mu}_p \rangle \in E/G|_m \end{aligned} \tag{1.19}$$

where $p \in \mathcal{U}$ is any point in the fiber of $m \in U$, which has as inverse the map

$$\begin{aligned} \Gamma_U(E/G) &\to \Gamma^G_{\mathcal{U}}(E) \\ \boldsymbol{\mu} &\mapsto \overline{\boldsymbol{\mu}} : p \mapsto \natural_p^{-1}(\boldsymbol{\mu}_{\pi(p)}) \in E|_p. \end{aligned} \tag{1.20}$$

It is important to realize that:

- the first map can not be extended to non-invariant sections $\boldsymbol{\mu} \in \Gamma_{\mathcal{U}}(E)$ because it is not true, in general, that $\langle \boldsymbol{\mu}(p) \rangle = \langle \boldsymbol{\mu}(p') \rangle$ even when $p, p' \in \pi^{-1}(m)$;
- the second map can not be extended to elements $\mu \in E/G_m$ to return elements of E because there is not a canonical choice of $p \in \pi^{-1}(m)$ where "$\overline{\mu}$" should belong to .

Proof. Before showing smoothness, let us check that that the above isomorphisms indeed produce sections that satisfy the necessary properties. The map $\boldsymbol{\mu} \mapsto \langle \boldsymbol{\mu} \rangle$ is well defined due to the G-invariance of $\boldsymbol{\mu} \in \Gamma^G_{\mathcal{U}}(E)$, hence $\langle X(pg) \rangle = \langle X(p)g \rangle = \langle X(p) \rangle$. Now, given $\boldsymbol{\mu} \in \Gamma_U(E/G)$, $\overline{\boldsymbol{\mu}}$ is G-invariant since, if $m = \pi(p) = \pi(pg)$ for some $p \in \mathcal{U}$, $\overline{\boldsymbol{\mu}}_{pg} = \natural^{-1}_{pg}(\boldsymbol{\mu}_m) =$

$\natural_p^{-1}(\boldsymbol{\mu}_m)g \in E_{pg}$. The $\mathbb{R}$-linearity of the pointwise vector space isomorphism $\natural_p : E_p \to (E/G)|_{\pi(p)}$ for any $p \in \mathcal{U}$ implies the $C^\infty(M)$-linearity of the isomorphisms. Finally, these two functions as defined by the formulas above are inverses of each other since, for $\boldsymbol{\mu} \in \Gamma_U(E/G)$ and $m \in u$,

$$\langle \overline{\boldsymbol{\mu}} \rangle (m) = \langle \overline{\boldsymbol{\mu}}(p) \rangle = \left\langle \natural_p^{-1}(\boldsymbol{\mu}(m)) \right\rangle = \natural \circ \natural_p^{-1}(\boldsymbol{\mu}(m)) = \boldsymbol{\mu}(m)$$

where p is any in $\pi^{-1}(m)$; on the other hand, for $\boldsymbol{\mu} \in \Gamma^G_{\mathcal{U}}(E)$ and $p \in P$ with $m = \pi(p)$,

$$\overline{\langle \boldsymbol{\mu} \rangle}(p) = \natural_p^{-1}(\langle \boldsymbol{\mu} \rangle (m)) = \natural_p^{-1}(\langle \boldsymbol{\mu}(p') \rangle) = \natural_p^{-1} \circ \natural(\boldsymbol{\mu}(pg)) = \natural_p^{-1} \circ \natural(\boldsymbol{\mu}(p)) = \boldsymbol{\mu}(p)$$

for any $p' \in \pi^{-1}(m)$, with formula $p' = pg$ for some $g \in G$.

We now proceed to prove the smoothness of the sections. Given $\boldsymbol{\mu} \in \Gamma_U(E/G)$, notice that the pullback (1.12) shows that $\overline{\boldsymbol{\mu}}$ is precisely the pullback section of $\boldsymbol{\mu}$, so it is a smooth section. On the other hand, take $\boldsymbol{\mu} \in \Gamma^G_{\mathcal{U}}(E)$, so $\langle \boldsymbol{\mu} \rangle$ is a smooth section since $\langle \boldsymbol{\mu} \rangle \circ \pi = \natural \circ \boldsymbol{\mu}$ is smooth and π is a surjective submersion.

□

Definition 1.3.7. Let $p^E : E \to B$ and $p^{E'} : E' \to B'$ be vector bundles and $F : E \to E'$ be a vector bundle morphism over $f : B \to B'$. Let $\boldsymbol{\mu} \in \Gamma(E)$ and $\boldsymbol{\nu}' \in \Gamma(E')$, then

- <u>$\boldsymbol{\mu}$ is F-related to $\boldsymbol{\nu}'$</u> if, for all $b \in B$, $F(\boldsymbol{\mu}_b) = \boldsymbol{\nu}'_{f(b)}$
- <u>$\boldsymbol{\mu}$ is F-projectable</u> if there exists $\boldsymbol{\nu}' \in \Gamma(E')$ such that $\boldsymbol{\mu}$ is F-related to $\boldsymbol{\nu}'$.
- If $f : B \to B'$ is surjective, and $\boldsymbol{\mu} \in \Gamma(E)$ is F-projectable, define <u>the section $F(\boldsymbol{\mu}) \in \Gamma(E')$</u> to be the smooth section such that, for any $b' \in B$

$$F(\boldsymbol{\mu})_{b'} := F(\boldsymbol{\mu}_b) \tag{1.21}$$

where b is any element in $f^{-1}(b')$, and this is equal to

$$F(\boldsymbol{\mu})_{b'} = \boldsymbol{\nu}'_{b'}$$

for any $\boldsymbol{\nu}' \in \Gamma(E')$ to which $\boldsymbol{\mu}$ is F-related.

Notice that whenever $f : B \to B'$ is bijective, e.g. an identity map, then every $\boldsymbol{\mu} \in \Gamma(E)$ is F-projectable. Also notice that in Proposition 1.3.6 any section $\boldsymbol{\mu} \in \Gamma(E)$ is $\natural$-related to $\langle \boldsymbol{\mu} \rangle$ and the base map $\pi : P \to M$ is surjective, hence the definition of $\langle \boldsymbol{\mu} \rangle$ given in that proposition coincides with the notation $\langle \boldsymbol{\mu} \rangle$.

Theorem 1.3.8. Let $\mathfrak{X} \in \Gamma_U(TP/G)$. $\mathfrak{X}$ is projectable over the surjective map $\pi_*^G : T\mathcal{U}/G \to TU$, $\overline{\mathfrak{X}} \in \Gamma_{\mathcal{U}}^G(TP)$ is projectable over the surjective map $\pi_* : T\mathcal{U} \to TU$ and

$$\pi_*^G(\mathfrak{X}) = \pi_*(\overline{\mathfrak{X}}) \in \Gamma(TU). \tag{1.22}$$

Notice that we are understanding TU as a subset of TM and $T\mathcal{U}$ as subset of TP.

Proof. Notice that the last statement makes sense thanks to the first part of the theorem, which allows the maps π_*^G and π_* to be extended to sections.

$\mathfrak{X}$ is π_*^G-projectable since the corresponding base space map is the bijective map $Id_U : U \to U$.

The following calculation proves the remaining statements. Let $m \in U$ and $p \in \pi^{-1}(m)$, so any other element $p' \in P$ such that $m = \pi(p)$ may be written as $p' = pg$ for some $g \in G$. Now $\pi_*(\overline{\mathfrak{X}}_{pg}) = \pi_*(\natural_{pg}^{-1}(\mathfrak{X}_{\pi(pg)})) = a(\natural \circ \natural_{pg}^{-1}(\mathfrak{X}_m)) = a(\mathfrak{X}_m)$, where the second to last equality is the defining property of a, as seen in the Diagram (1.17). $\square$

The following theorem gives a convenient way to understand the space of sections of the Atiyah Lie algebroid that, additionally, will be used in the next section to define the Lie algebra structure on $\Gamma(TP/G)$ that completes the construction of the Atiyah Lie algebroid.

Theorem 1.3.9. Let $U \subseteq M$ be open, e.g. $U = M$, and $\mathcal{U} = \pi^{-1}(U) \subseteq P$. The $C^\infty(U)$-module of local sections $\Gamma_U(TP/G)$ is isomorphic to the space of local G-invariant sections $\Gamma_{\mathcal{U}}^G(TP)$.

Based on this theorem, for any $U \subseteq M$ open we will usually understand $\Gamma_{\mathcal{U}}^G(TP)$ as the space of sections of $T\mathcal{U}/G$, and we might say that a G-invariant section $\overline{\mathfrak{X}} \in \Gamma_{\mathcal{U}}^G(TP)$ "is a local section of the Atiyah Lie algebroid TP/G".

The Bracket

As we saw in theorem 1.3.9, the space of (local) sections of TP/G can be identified as the subset of G-invariant (local) sections of TP, so we may use the bracket of $\Gamma(TP)$ to induce a bracket in $\Gamma(TP/G)$. Furthermore, Proposition 1.1.2 applied to the Lie algebroid TP enables the definition of the bracket on TP/G only from local information without any sort of ambiguity.

Lemma 1.3.10. If $\boldsymbol{\mu}, \boldsymbol{\nu} \in \Gamma^G_{\mathcal{U}}(TP) \subseteq \Gamma_{\mathcal{U}}(TP)$, then $[\boldsymbol{\mu}, \boldsymbol{\nu}] \in \Gamma^G_{\mathcal{U}}(TP)$.

Proof. The invariance of a section $\boldsymbol{\mu} \in \Gamma^G_{\mathcal{U}}(TP) \subseteq \Gamma_{\mathcal{U}}(TP)$ means that for $f \in C^\infty(P)$, $g \in G$ and $p \in \mathcal{U}$,

$$\boldsymbol{\mu}_{pg}(f) = \boldsymbol{\mu}_p(f \circ R_g) \tag{1.23}$$

because $\boldsymbol{\mu}_{pg} = R_{g*}\boldsymbol{\mu}_p$ by the definition of $\Gamma^G_{\mathcal{U}}(TP)$. It also means that

$$(\boldsymbol{\mu}(f)) \circ R_g = \boldsymbol{\mu}(f \circ R_g) \tag{1.24}$$

since $((\boldsymbol{\mu} f) \circ R_g)(p) = \boldsymbol{\mu} f(pg) = \boldsymbol{\mu}_{pg} f = \boldsymbol{\mu}_p(f \circ R_g) = (\boldsymbol{\mu}(f \circ R_g))(p)$, where the second to last equation follows from the invariance of $\boldsymbol{\mu}$.

So, for $\boldsymbol{\mu}, \boldsymbol{\nu} \in \Gamma^G_{\mathcal{U}}(TP)$, $p \in \mathcal{U}$, $g \in G$ arbitrary,

$$\begin{aligned}
[\boldsymbol{\mu}, \boldsymbol{\nu}]_{pg}(f) =& \boldsymbol{\mu}_{pg}(\boldsymbol{\nu}(f)) - \boldsymbol{\nu}_{pg}(\boldsymbol{\mu}(f)) \\
=& \boldsymbol{\mu}_p(\boldsymbol{\nu}(f) \circ R_g) - \boldsymbol{\nu}_p(\boldsymbol{\mu}(f) \circ R_g) && \text{by (1.23)} \\
[\boldsymbol{\mu}, \boldsymbol{\nu}]_p(f \circ R_g) =& \boldsymbol{\mu}_p(\boldsymbol{\nu}(f \circ R_g)) - \boldsymbol{\nu}_p(\boldsymbol{\mu}(f \circ R_g)) && \text{by (1.24)}
\end{aligned}$$

from which the G-invariance of $[\boldsymbol{\mu}, \boldsymbol{\nu}]$ follows by (1.23). □

The previous lemma shows that the following is a good definition.

Definition 1.3.11. For any $\boldsymbol{\mathfrak{X}}, \boldsymbol{\mathfrak{Y}} \in \Gamma_U(TP/G)$, define

$$[\boldsymbol{\mathfrak{X}}, \boldsymbol{\mathfrak{Y}}] := \left\langle [\overline{\boldsymbol{\mathfrak{X}}}, \overline{\boldsymbol{\mathfrak{Y}}}] \right\rangle \in \Gamma_U(TP/G). \tag{1.25}$$

This is a Lie algebra structure on $\Gamma_U(E/G)$ called the Lie bracket of the Atiyah Lie algebroid associated to the principal bundle P.

This bracket inherits the $\mathbb{R}$-linearity, the Jacobi identity and the alternating property from the bracket on $\Gamma_{\mathcal{U}}(TP)$, meaning that it is a Lie algebra structure on $\Gamma_U(TP/G)$.

Theorem 1.3.12. The vector bundle TP/G with anchor $\pi_*^G : TP/G \to TM$ of Definition 1.3.5 and bracket of Definition 1.3.11 is a Lie algebroid over M, called <u>the Atiyah Lie algebroid associated to the principal bundle P</u>:

$$\begin{tikzcd} (TP/G,[\cdot,\cdot]) \arrow[r, "\pi_*^G"] \arrow[dr, "\pi"'] & (TM,[\cdot,\cdot]) \arrow[d, "\pi"] \\ & M. \end{tikzcd} \tag{1.26}$$

Proof. All what is left to see is that this definitions of the anchor and the bracket of the Lie algebroid satisfy the two compatibility conditions of Definition 1.1.1. Let $\mathfrak{X}, \mathfrak{Y} \in \Gamma_U(TP/G)$ and let $f \in C^\infty(M)$ and recall that theorem 1.3.8 tells us that $\overline{\mathfrak{X}}, \overline{\mathfrak{Y}}$ are π_*-related to $\pi_*^G(\mathfrak{X}), \pi_*^G(\mathfrak{Y})$ respectively, so, when equation (1.22) is applied to $f \in C^\infty(U)$, at $m \in U$ we conclude that $\pi_*^G(\mathfrak{X})_m f = \pi_*(\overline{\mathfrak{X}})_m f = \overline{\mathfrak{X}}_p(f \circ \pi) \in \mathbb{R}$ for any $p \in \pi^{-1}(m)$, so

$$\overline{\mathfrak{X}}(f \circ \pi) = \pi_*^G(\mathfrak{X})(f) \circ \pi \in C^\infty(\mathcal{U}). \tag{1.27}$$

1. $\pi_*^G : \Gamma(TP/G) \to \Gamma(TM)$ is a Lie algebra morphism: the result follows from the following calculation

$$\begin{aligned} &\pi_*^G([\mathfrak{X},\mathfrak{Y}])(f) \circ \pi = \overline{[\mathfrak{X},\mathfrak{Y}]}(f \circ \pi) = [\overline{\mathfrak{X}},\overline{\mathfrak{Y}}](f \circ \pi) \\ &= \overline{\mathfrak{X}}(\overline{\mathfrak{Y}}(f\circ\pi)) - \overline{\mathfrak{Y}}(\overline{\mathfrak{X}}(f\circ\pi)) = \overline{\mathfrak{X}}(\pi_*^G(\mathfrak{Y})(f)\circ\pi) - \overline{\mathfrak{Y}}(\pi_*^G(\mathfrak{X})(f)\circ\pi) \\ &= \pi_*^G(\mathfrak{X})(\pi_*^G(\mathfrak{Y})(f)) \circ \pi - \pi_*^G(\mathfrak{Y})(\pi_*^G(\mathfrak{X})(f)) \circ \pi \\ &= [\pi_*^G(\mathfrak{X}), \pi_*^G(\mathfrak{Y})](f) \circ \pi. \end{aligned}$$

2. Leibniz identity:

$$\begin{aligned} &\overline{[\mathfrak{X}, f\mathfrak{Y}]} = [\overline{\mathfrak{X}}, (f \circ \pi)\overline{\mathfrak{Y}}] \\ &= (f\circ\pi)[\overline{\mathfrak{X}},\overline{\mathfrak{Y}}] + \overline{\mathfrak{X}}(f\circ\pi)\overline{\mathfrak{Y}} = (f\circ\pi)[\overline{\mathfrak{X}},\overline{\mathfrak{Y}}] + (\pi_*^G(\mathfrak{X})(f)\circ\pi)\overline{\mathfrak{Y}} \end{aligned}$$

so

$$[\mathfrak{X}, f\mathfrak{Y}] = \left\langle [\overline{\mathfrak{X}}, (f\circ\pi)\overline{\mathfrak{Y}}] \right\rangle = f[\mathfrak{X},\mathfrak{Y}] + \pi_*^G(\mathfrak{X})(f)\mathfrak{Y}$$

$\square$

1.3.2 The Adjoint Lie algebroid

A principal connection form on the principal bundle $G \to P \to M$ is usually defined as $\mathfrak{g}$-valued 1-form $w : TP \to P \times \mathfrak{g}$ that satisfies:

- $w(\eta_p^*) = \eta$ for any $\eta \in \mathfrak{g}$, $p \in P$ and where $\eta_p^* = \frac{d}{dt}\big|_{t=0} p \exp(\eta t) \in T_pP$ is the fundamental field on P generated by η at p.
- G-equivariant: $w(R_{g*}\mathcal{X}) = Ad_{g^{-1}}(w(\mathcal{X}))$ for any $\in TP$, $g \in G$.

We will see that the G-equivariance of a connection suggests the use of TP/G and $P \times \mathfrak{g}/G$ instead of TP and $P \times \mathfrak{g}$ respectively, and the first property suggests a connection as a section of the Atiyah sequence of a principal bundle.

The Lie algebroid $T^\pi P/G$

Lemma 1.3.13. Define $T^\pi P := ker(\pi_*) \subseteq TP$. $T^\pi P$ is a vector subbundle of TP and the action of G on TP restricts to an action on $T^\pi P$.

Proof. The projection $\pi : P \to M$ is a submersion, since it is locally a projection map $U \times G \to U$ for $U \subseteq M$ open, therefore π_* too is a submersion because it is locally the projection $TU \times TG \to TG$. This means that π_* is of locally constant rank, and so $ker(\pi_*)$ is a subbundle.

The action of $g \in G$ applied to $\mathcal{X} \in T^\pi P$ falls again on $T^\pi P$ because $\pi_*(\mathcal{X}g) = \pi_*(R_{g*}\mathcal{X}) = (\pi \circ R_g)_*\mathcal{X}$, but $\pi \circ R_g = \pi$, so $\pi_*(\mathcal{X}g) = 0$. □

Theorem 1.3.14. The quotient vector bundle $(T^\pi P/G, \pi_*^G, M)$ by the action of G of the bundle $(T^\pi P, \pi_*|_{T^\pi P}, P)$ is well defined. The inclusion map $i : T^\pi P \to TP$ over P induces the inclusion map $j : T^\pi P/G \to TP/G$ which is the unique vector bundle morphism that makes the following diagram commute:

$$
\begin{tikzcd}
T^\pi P \arrow[r, "i"] \arrow[d] & TP \arrow[d] \\
T^\pi P/G \arrow[r, dashed, "j"] \arrow[d, "\pi"] & TP/G \arrow[d, "\pi"] \\
M \arrow[r, "id_M"] & M
\end{tikzcd}
\tag{1.28}
$$

where the top vectical arrows are the quotient maps, and, abusing notation, π is also used to denote the restriction of $\pi : TP/G \to M$ to $T^\pi P/G$.

Proof. π_* is G-equivariant since $\pi \circ R_g = \pi$, therefore the existence of the vector bundle TP/G will follow if we find π-saturated G-equivariant charts, by theorem 1.3.1. Using the notation in the proof of theorem 1.3.2, where equivariant charts for TP were built, we now construct the desired charts. Notice that $\pi_* \circ \psi_*(X, R_{g*,1}\eta) = 0$, where $\psi_* : TU \boxplus TG \to T_\mathcal{U}$ is the isomorphism induced by the principal bundle chart $\psi : U \times G \to P_\mathcal{U}$, is equivalent to the statement $X = 0$ since $\pi \circ \psi : U \times G \to U$ is the projection to the first coordinate. Therefore, the fiber of $T^\pi P$ is only $\mathfrak{g}$ and the following map is a vector bundle isomorphism:

$$
\begin{aligned}
\widetilde{\Psi} :(U \times G) \times \mathfrak{g} \quad &\overset{\cong}{\to} \quad T^\pi P|_\mathcal{U} \\
(m, g, \eta) \quad &\mapsto \quad \psi_{*,(m,g)}(0, R_{g*,1}\eta).
\end{aligned}
$$

Now, $\Psi : \mathcal{U} \times \mathfrak{g} \to T_\mathcal{U} P$ by $\Psi = \widetilde{\Psi} \circ (\psi^{-1} \times id_\mathfrak{g})$ is the desired G-equivariant coordinate map by the same arguments used in the proof of theorem 1.3.2.

The existence of the vector bundle morphism $j : T^\pi P/G \to TP/G$ follows from the clear G-equivariance of the inclusion i.

Momentarily, let $\natural^\pi : T^\pi P \to T^\pi/G$ and $\natural : TP \to TP/G$ be the quotient maps. Then, for any $\mathcal{X} \in T^\pi P$, $\natural^\pi(\mathcal{X}) = \{\mathcal{X}\, g \mid g \in G\} = \natural(\mathcal{X})$, so

$$
j(\natural^\pi(\mathcal{X})) = \natural(\mathcal{X}) = \natural^\pi(\mathcal{X}),
$$

so j is indeed an inclusion of sets. $\square$

Theorem 1.3.15. The bracket of the Lie algebroid TP/G induces a bracket on $T^\pi P/G$, and $T^\pi P/G = ker(\pi_*^G)$. Therefore

$$\begin{tikzcd}
(T^\pi P/G, [\cdot,\cdot]) \arrow[r, "0"] \arrow[dr, "\pi|_{T^\pi P/g}"'] & (TM, [\cdot,\cdot]) \arrow[d, "\pi"] \\
& M
\end{tikzcd}$$

is an Atiyah Lie algebroid with trivial anchor.

Proof. Notice that $T^\pi P/G = ker(\pi_*^G)$ due to the top square of the commutative diagram (1.17) that defines the anchor of TP/G. By the remark following theorem 1.1.7 all we need to prove is that the bracket of sections of $T^\pi P/G \subseteq TP/G$ is again a section of this kind. Suppose that $\mathfrak{X}, \mathfrak{Y} \in \Gamma(T^\pi P/G) \subseteq \Gamma(TP/G)$, then $\pi_*^G([\mathfrak{X}, \mathfrak{Y}]) = [\pi_*^G(\mathfrak{X}), \pi_*^G(\mathfrak{Y})]$ $= 0$, and so $[\mathfrak{X}, \mathfrak{Y}] \in \Gamma(T^\pi P/G)$. $\square$

The Adjoint Bundle

Although we could consider the exact sequence of Atiyah Lie algebroids

$$0 \to T^\pi P/G \to TP/G \to TM \to 0$$

as the Atiyah sequence of the principal bundle $G \to P \to M$, it will be useful to replace $T^\pi P/G$ by $P \times \mathfrak{g}/G$ once we see they are isomorphic Lie algebroids.

The Vector Bundle $P \times \mathfrak{g}/G$

Theorem 1.3.16. Let V be a vector space on which G acts to the left. Let $g \in G$ act to the right on the trivial vector bundle $p_1 : P \times V \to P$ by $(p, v)g = (pg, g^{-1}v)$. Let $\langle p, v\rangle$ or denote the orbit of $(p, v) \in P \times V$. The space $P \times V/G$ has a unique vector bundle structure over M, with projection map $p : P \times V/G \to M$, such that

$$p(\langle p, v\rangle) = \pi(p) \tag{1.29}$$

for any $(p, v) \in P \times V$.

Furthermore, let $U \subseteq M$ be open and $\mathcal{U} = \pi^{-1}(U) \subseteq P$. Then, the following is a $C^\infty(U)$-module isomorphism:

$$\begin{aligned} \Gamma_U(E) &\to C^\infty_G(\mathcal{U}, V) \\ \boldsymbol{\mu} &\mapsto \widetilde{\boldsymbol{\mu}} \\ (p, \widetilde{\boldsymbol{\mu}}(p)) &:= \boldsymbol{\mu}_{\pi(p)}, \end{aligned} \tag{1.30}$$

for any $p \in \mathcal{U}$, where $C^\infty_G(\mathcal{U}, V)$ is the $C^\infty(u)$-module of G-equivariant functions on $\mathcal{U} \subseteq P$ with values on V, on which $f \in C^\infty(U)$ acts by pointwise multiplication with $f \circ \pi \in C^\infty_G(U)$.

Proof. Let $\sigma : U' \to \mathcal{U}'$ be a local section of P and $\psi : U' \times G \to \mathcal{U}' = \pi^{-1}(U')$ be the associated chart that satisfies $\psi(m, g) = \sigma(m)g$ for $m \in U', g \in G$. Let the π-saturated chart $\Psi : \mathcal{U} \times V \to E|_{\mathcal{U}}$ be defined by

$$\begin{tikzcd} \mathcal{U} \times V \arrow[r, dashed, "\Psi"] & E|_{\mathcal{U}} = \mathcal{U} \times V \\ U \times G \times V \arrow[u, "\psi \times id_V"] \arrow[r, "(m,g,v)\mapsto(x,g,g^{-1}v)"] & U \times G \times V \arrow[u, "\psi \times id_V"'] \end{tikzcd}$$

for any $m \in U$, $v \in V$, $g \in G$ which, therefore, has formula

$$\Psi(\sigma(m)g, v) = (\sigma(m)g, g^{-1}v).$$

The G-equivariance of Ψ follows from $\Psi(\sigma(m)g, v)g' = (\sigma(m)g, g^{-1}v)g' = (\sigma(m)gg', (gg')^{-1}v) = \Psi(\sigma(m)gg', v)$.

The theorem then follows from theorems 1.3.1 and 1.3.6, noticing that $\Gamma^G_{\mathcal{U}}(P \times V) \cong C^\infty_G(U)$ and that $\widetilde{f\boldsymbol{\mu}} = (f \circ \pi)\widetilde{\boldsymbol{\mu}}$. □

In $V = \mathfrak{g}$, $g \in G$ acts smoothly to the right by $Ad_g = AD_{g*,1} : \mathfrak{g} \to \mathfrak{g}$, $\eta \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} g \exp(t\eta) g^{-1}$, where $AD_g : G \to G$, $h \mapsto ghg^{-1}$. Applying the previous theorem to this vector space we obtain $p : P \times \mathfrak{g}/G \to M$, the adjoint bundle of the principal bundle P. The second part of the theorem enables us to regard $C^\infty_G(P, \mathfrak{g})$ as the space of sections of $P \times \mathfrak{g}/G$, and so to call a function $\widetilde{\boldsymbol{\eta}} \in C^\infty_G(P, \mathfrak{g})$ "a section of the adjoint bundle".

The Inclusion

Theorem 1.3.17. For any $p \in P$ and $g \in G$, denote by $m_p : G \to P$, $g \mapsto gp = R_g(p)$. The map

$$-m_{p*,1} : \mathfrak{g} \to T_pG, \qquad \eta \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p\exp(-t\eta) \equiv -\eta_p^* \tag{1.31}$$

induces an injective vector bundle morphism $j : P \times \mathfrak{g}/G \to TP/G$ such that

$$j(\langle p,\eta\rangle) = \left\langle \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p\exp(-t\eta) \right\rangle \in T^\pi P/G \subseteq TP/G \tag{1.32}$$

for any $(p,\eta) \in P \times \mathfrak{g}$. Furthermore, the vector bundle $P \times \mathfrak{g}/G$ is isomorphic to $T^\pi P/G$ through j.

The vector bundle morphism j induces a $C^\infty(M)$-linear map between sections, but since $\Gamma(P \times g/G) \cong C_G^\infty(P,\mathfrak{g})$ and $\Gamma(TP/G) \cong \Gamma^G(TP)$, for $\widetilde{\boldsymbol{\eta}} \in C_G^\infty(P)$ denote by $\bar{j}$ the induced map:

$$\begin{aligned} \bar{j} &: C_G^\infty(P,\mathfrak{g}) \to \Gamma^G(TP) \\ \bar{j}(\widetilde{\boldsymbol{\eta}})(p) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p\exp(-t\widetilde{\boldsymbol{\eta}}(p)) \end{aligned} \tag{1.33}$$

Proof. Let $m : P \times G \to P$ be the Lie group action of G on P, and identify $T(P \times G)$ with $TP \boxplus TG$. Then the map $F : P \times \mathfrak{g} \to TP$, $(p,\eta) \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} p\exp(-t\eta)$ induced by $-m_{p*,1}$ may be rewritten as $F : (p,\eta) \mapsto -m_{*,(p,1)}(0_p,\eta)$, showing that F is a smooth vector bundle morphism over P.

Now, since for any $g \in G$, $p \in P$ and $\eta \in \mathfrak{g}$ it is true that $R_g \circ m_p = m_{pg} \circ AD_{g^{-1}} : G \to P$, then

$$\begin{aligned} F((p,\eta)g) = F(pg, Ad_{g^{-1}}\eta) &= -m_{pg*,1}(Ad_{g^{-1}}\eta) = -(m_{pg} \circ AD_{g^{-1}})_{*,1}(\eta) \\ &= -(R_g \circ m_p)_{*,1}(\eta) = -(m_{p*,1}(\eta))g = F(p,\eta)g, \end{aligned}$$

which means that $F : P \times \mathfrak{g} \to TP$ is G-equivariant, so we may apply theorem 1.3.3 to get the vector bundle morphism $j : P \times \mathfrak{g}/G \to TP/G$ where the formula (1.32) is nothing but the commutativity of the top square of diagram (1.15); in particular, the formula (1.32) is well defined.

The injectivity of j in inherited from the injectivity of $F : P \times \mathfrak{g} \to TP$, which, in turn, is consequence of the injectivity of each $m_{p*,1}$. The image of j is $T^\pi P/G$ because $\pi_*^G(j\langle p, \eta\rangle) = \pi_*^G(\langle \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} p \exp(-t\eta)\rangle) = \pi_*(\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} p \exp(-t\eta)) = \frac{d}{dt}_{t=0} p = 0$. Finally, since the vector bundles $P \times \mathfrak{g}/G$ and $im(j) = T^\pi P/G$ have the same fiber $\mathfrak{g}$, j being an injective vector bundle map implies that j is an isomorphism in each fiber, and so j is a vector bundle isomorphism between $P \times \mathfrak{g}/G$ and $T^\pi P/G$. $\square$

The Bracket

Since $P \times \mathfrak{g}/G$ and $T^\pi P/G$ are isomorphic as vector bundles, we can define a bracket on $P \times \mathfrak{g}/G$ by the bracket on $T^\pi P/G$, however, in $P \times \mathfrak{g}/G$ we also have a natural Lie algebra structure inherited from the Lie algebra bracket of $\mathfrak{g}$. Our current purpose is to conclude that they coincide.

Definition 1.3.18. Given $\eta \in \mathfrak{g}$, let <u>the left-invariant vector field on G generated by η</u>, denoted by $\boldsymbol{\eta}^L \in \Gamma(TG)$, be defined by

$$\boldsymbol{\eta}_g^L := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g \exp(t\eta) \in T_gG \qquad\qquad g \in G. \tag{1.34}$$

<u>The (left-hand) bracket on $\mathfrak{g}$</u> is the Lie bracket induced by the left-invariant vector fields, namely:

$$[\eta, \theta]^L := [\boldsymbol{\eta}^L, \boldsymbol{\theta}^L]_1 \qquad\qquad \eta, \theta \in \mathfrak{g} \tag{1.35}$$

It can be proven that the left-hand bracket $[\cdot,\cdot]^L$ on $\mathfrak{g}$ is the negative of the right-hand Lie algebra bracket $[\cdot,\cdot]^R$, defined instead using the left-invariant vector fields on G, i.e.

$$[\eta, \theta]^R = -[\eta, \theta]^L \qquad\qquad \eta, \theta \in \mathfrak{g} \tag{1.36}$$

Theorem 1.3.19. The Lie algebra structure induced on $P \times \mathfrak{g}/G$ by its isomorphism through $j : P \times \mathfrak{g}/G \to T^\pi P/G$

$$\begin{aligned} j[\langle\boldsymbol{\eta}\rangle, \langle\boldsymbol{\theta}\rangle] &:= [j\langle\boldsymbol{\eta}\rangle, j\langle\boldsymbol{\theta}\rangle] \in \Gamma(T^\pi P/G) \\ &\equiv \left\langle [\overline{j\langle\boldsymbol{\eta}\rangle}, \overline{j\langle\boldsymbol{\theta}\rangle}] \right\rangle \end{aligned} \tag{1.37}$$

for any $\langle \boldsymbol{\eta} \rangle, \langle \boldsymbol{\theta} \rangle \in \Gamma(P \times \mathfrak{g}/G)$ coincides with the Lie algebra structure induced by the (left-hand) bracket of $\mathfrak{g}$, namely

$$\begin{aligned} [\widetilde{\langle \boldsymbol{\eta} \rangle, \langle \boldsymbol{\theta} \rangle}]^L := [\widetilde{\boldsymbol{\eta}}, \widetilde{\boldsymbol{\theta}}]^L \in C^\infty_G(P, \mathfrak{g}) \\ p \mapsto [\widetilde{\boldsymbol{\eta}}(p), \widetilde{\boldsymbol{\theta}}(p)]^L. \end{aligned} \tag{1.38}$$

In particular,

$$\begin{tikzcd} (P \times \mathfrak{g}/G, [\cdot,\cdot]^L) \arrow[r, "0"] \arrow[dr, "p"'] & (TM, [\cdot,\cdot]) \arrow[d, "\pi"] \\ & M \end{tikzcd} \tag{1.39}$$

is a Lie algebroid isomorphic to the Lie algebroid $T^\pi P/G$ subalgebroid of TP/G.

Proof. First notice that a section on $P \times \mathfrak{g}/G$ is j-projectable since the base map of the vector bundle morphism j is the identity in M, and so it may be extended to $j : \Gamma(P \times \mathfrak{g}/G) \to \Gamma(TP/G)$. Now, given $\langle \boldsymbol{\eta} \rangle \in \Gamma(P \times \mathfrak{g}/G)$, for $m \in M$ and $p \in \pi^{-1}(m)$, $j(\langle \boldsymbol{\eta} \rangle)_m := j(\langle \boldsymbol{\eta} \rangle_m) = j(\langle p, \widetilde{\boldsymbol{\eta}}(p) \rangle)$ by definition of $\widetilde{\boldsymbol{\eta}}$, so $j(\langle \boldsymbol{\eta} \rangle)_m = \langle m_{p*,1}(-\widetilde{\boldsymbol{\eta}}(p)) \rangle$ by equation (1.32), and so

$$\overline{j(\langle \boldsymbol{\eta} \rangle)}_p = \natural_p^{-1}(j(\langle \boldsymbol{\eta} \rangle_{\pi(p)})) = m_{p*,1}(-\widetilde{\boldsymbol{\eta}}(p)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p \exp(-t \widetilde{\boldsymbol{\eta}}(p)). \tag{1.40}$$

Furthermore, for any section $\langle \boldsymbol{\eta} \rangle \in \Gamma(P \times \mathfrak{g}/G)$, $\overline{j(\langle \boldsymbol{\eta} \rangle)}$ has global flow $\phi_t(p) = p \exp(-t \widetilde{\boldsymbol{\eta}}(p))$ since, using the fact that $\widetilde{\boldsymbol{\eta}}(pg) = Ad_{g^{-1}} \widetilde{\boldsymbol{\eta}}(p)$ for any $g \in G$, it can be shown that $\phi_t \circ \phi_s = \phi_{t+s}$ for any $t, s \in \mathbb{R}$.

Therefore, for any $\langle \boldsymbol{\eta} \rangle, \langle \boldsymbol{\theta} \rangle \in \Gamma(P \times \mathfrak{g}/G)$

$$\begin{aligned} [\overline{j(\langle \boldsymbol{\eta} \rangle)}, \overline{j(\langle \boldsymbol{\theta} \rangle)}]_p &= \lim_{t \to 0} \frac{\phi_{-t*}(\overline{j(\langle \boldsymbol{\theta} \rangle)}_{\phi_t(p)}) - \overline{j(\langle \boldsymbol{\theta} \rangle)}_p}{t} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi_{-t*}(\overline{j(\langle \boldsymbol{\theta} \rangle)}_{\phi_t(p)}) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\phi_{-t} \circ m_{\phi_t(p)})_{*,1}(-\widetilde{\boldsymbol{\theta}}(\phi_t(p))) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (m_p)_{*,1}(-\widetilde{\boldsymbol{\theta}}(\phi_t(p))), \quad \phi_{-t} \circ m_{\phi_t(p)} = m_p \end{aligned}$$

$$
\begin{aligned}
&= (m_p)_{*,1}\left(-\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \widetilde{\boldsymbol{\theta}}(p\exp(-t\widetilde{\boldsymbol{\eta}}(p)))\right) \\
&= (m_p)_{*,1}\left(-\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} Ad_{\exp(t\widetilde{\boldsymbol{\eta}}(p))}\widetilde{\boldsymbol{\theta}}(p)\right) \\
&= (m_p)_{*,1}\left(-\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} ad_{\widetilde{\boldsymbol{\eta}}(p)}\widetilde{\boldsymbol{\theta}}(p)\right) \\
&= m_{p*,1}(-[\widetilde{\boldsymbol{\eta}}(p),\widetilde{\boldsymbol{\theta}}(p)]^L)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
j([\langle\boldsymbol{\eta}\rangle, \langle\boldsymbol{\theta}\rangle]^L)_m &= \left\langle m_{p*,1}(-[\widetilde{\boldsymbol{\eta}}(p),\widetilde{\boldsymbol{\theta}}(p)]^L)\right\rangle && \text{by definition of } j \\
&= \left\langle [\overline{j(\langle\boldsymbol{\eta}\rangle)},\overline{j(\langle\boldsymbol{\theta}\rangle)}]_p\right\rangle && \text{by the above calculation} \\
&= \left\langle \overline{[j(\langle\boldsymbol{\eta}\rangle),j(\langle\boldsymbol{\theta}\rangle)]}_p\right\rangle && \text{by definition of bracket in } TP/G \\
&= \left\langle \overline{j[\langle\boldsymbol{\eta}\rangle, \langle\boldsymbol{\theta}\rangle]}_p\right\rangle && \text{by equation (1.37)} \\
&= j([\langle\boldsymbol{\eta}\rangle, \langle\boldsymbol{\theta}\rangle])_m && \text{inverse definitions (1.3.6)}
\end{aligned}
$$

And so the two Lie algebra structures $[\cdot,\cdot]^L$ and $[\cdot,\cdot]$ defined on $P\times\mathfrak{g}/G$ coincide. $\square$

Remark 1.3.20. If the map $j : P\times\mathfrak{g}/G \to TP/G$ is defined instead by $j(\langle p,\eta\rangle) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p\exp(+t\eta)$, then the bracket induced by $\Gamma(T^\pi P/G)$ through j on $\Gamma(P\times\mathfrak{g}/G)$ corresponds to the right-hand bracket induced by $C^\infty_G(P,\mathfrak{g})$.

The Atiyah Sequence

Definition 1.3.21.

- With the Lie algebra structure on $P\times\mathfrak{g}/G$ from theorem 1.3.19, denoted from now on by $[\cdot,\cdot] : \Gamma(P\times\mathfrak{g}/G, P\times\mathfrak{g}/G) \to \Gamma(P\times\mathfrak{g}/G)$ and called the bracket of $P\times\mathfrak{g}/G$, and trivial anchor $a=0$, $P\times\mathfrak{g}/G$ is a Lie algebroid over M called the adjoint Lie algebroid to the Atiyah Lie algebroid associated to the principal bundle P.

- The exact sequence of Lie algebroids over M

$$0 \to P \times \mathfrak{g}/G \xrightarrow{j} TP/G \xrightarrow{\pi_*^G} TM \to 0 \tag{1.41}$$

is called the Atiyah sequence of the principal bundle $G \to P \to M$.

From the algebraic perspective, we could also say that the Atiyah sequence of P is the induced short exact sequence of $C^\infty(M)$-modules and Lie algebras

$$0 \to \Gamma(P \times \mathfrak{g}/G) \xrightarrow{j} \Gamma(TP/G) \xrightarrow{\pi_*^G} \Gamma(TM) \to 0. \tag{1.42}$$

but also the sequence

$$0 \to C_G^\infty(P, \mathfrak{g}) \xrightarrow{\bar{j}} \Gamma^G(TP) \xrightarrow{\pi_*} \Gamma(TM) \to 0. \tag{1.43}$$

1.3.3 Examples

Throughout the document we will develop some explicit examples of transitive Lie algebroids and of the different structures built on them. We will do so parting from two big families of principal bundles given a normed division algebra $\mathbb{F}$ over $\mathbb{R}$ which is one of $\mathbb{R}, \mathbb{C}, \mathbb{H}$: let d be the dimension of $\mathbb{F}$ as an $\mathbb{R}$-vector space, then for any $n \in \mathbb{Z}_{\geq 1}$ the sphere $S^{dn-1} \subseteq \mathbb{F}^n$ on which the Lie group [1] S^{d-1} acts is a principal bundle over $\mathbb{F}P^{n-1}$. We will focus in the two particular fibrations obtained by restricting to $n = 2$ and $\mathbb{F}$ either $\mathbb{C}$ or $\mathbb{H}$, called the complex Hopf fibration and quaternionic Hopf fibration respectively. In these cases the base manifolds $\mathbb{C}P^1$ and $\mathbb{H}P^1$ can be regarded instead as S^2 and S^4 respectively, and so these principal fibrations may be seen as follows:

$$S^1 \longrightarrow S^3 \longrightarrow S^2,$$

$$S^3 \longrightarrow S^7 \longrightarrow S^4.$$

[1] It is well known that the only spheres which admit a topological group structure are S^0, S^1 and S^3, explaining why a similar construction, like the octonionic Hopf fibration, for the remaining normed division algebra, the octonion algebra, can't be a principal bundle.

One way to describe the associated Atiyah Lie algebroid sequences 1.3.21 that will be most useful to use will be through their local trivializations as transitive Lie algebroids studied in the next section, and this requires a local description of both the base manifolds and the principal bundles, which we will now develop.

The Underlying Manifolds

First, let's define an atlas for $S^n \subseteq \mathbb{R}^{n+1}$ for $n \geq 1$. Let $U_N = S^n \setminus \{(0, \dots, 0, -1)\}$ and $U_S = S^n \setminus \{(0, \dots, 0, 1)\}$. Notice that these two sets are open subsets missing only one point of S^n, and so they make an open cover of S^n. A C^∞ atlas is given by $\{(U_S, \phi_S), (U_N, \phi_N)\}$ where

$$\vec{x} : U_S \to \mathbb{R}^n, \quad (x^1, \dots, x^{n+1}) \mapsto \frac{(x^1, \dots, x^n)}{1 - x^{n+1}} \tag{1.44}$$

$$\phi_S := x^{-1} : \mathbb{R}^n \to U_S, \qquad \vec{\xi} \mapsto \frac{(2\xi^1, \dots, 2\xi^n, ||\vec{\xi}||^2 - 1)}{1 + ||\vec{\xi}||^2} \tag{1.45}$$

and

$$\vec{y} : U_N \to \mathbb{R}^n \quad (x^1, \dots, x^{n+1}) \mapsto \frac{(x^1, \dots, x^n)}{1 + x^{n+1}} \tag{1.46}$$

$$\phi_N := \vec{y}^{-1} : \mathbb{R}^n \to U_N \qquad \vec{\xi} \mapsto \frac{(2\xi^1, \dots, 2\xi^n, 1 - ||\vec{\xi}||^2)}{1 + ||\vec{\xi}||^2}. \tag{1.47}$$

In the intersection $\mathbb{R}^n - 0 = \vec{x}(U_S \cap U_N) = \vec{y}(U_S \cap U_N) \ni \vec{\xi}$ they satisfy:

$$\vec{x} \circ \vec{y}^{-1}(\vec{\xi}) = \frac{\vec{\xi}}{||\vec{\xi}||^2} = \vec{y} \circ \vec{x}^{-1}(\vec{\xi}). \tag{1.48}$$

When $n = 2$ we write $\mathbb{C}$ instead of $\mathbb{R}^2$ and when $n = 4$ we write $\mathbb{H}$ instead of $\mathbb{R}^4$, which also allows the previous formula to be written in terms of the conjugation of this normed division algebras, i.e. for $y \in \mathbb{F}$

$$\vec{x} \circ \vec{y}^{-1}(\xi) = \overline{\xi}^{-1} = \vec{y} \circ \vec{x}^{-1}(\xi) \tag{1.49}$$

where $\mathbb{F}$ can, in fact, be either $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

In S^2 a very useful and intuitive set of coordinates are the polar coordinates. In $U_{SN} := U_S \cap U_N$ the polar angle $\phi \in [0, \pi]$ and the azimuthal angle $\theta \in [0, 2\pi]$ induce the local frame $\{\partial_\phi, \partial_\theta\} \subseteq \Gamma_{U_{SN}}(TS^2)$ of the tangent bundle which are properly defined as coordinate vector fields once at two appropriate charts that cover U_{SN}; one possibility is the chart in which $\phi \in (0, \pi)$ and $\theta \in (0, 2\pi)$ and another chart where $\phi \in (0, \pi)$ and $\theta \in (-\pi, \pi)$. The inverses of these coordinate maps will be denoted by E, or simply E, with formula

$$E : (\phi, \theta) \mapsto (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \in S^2. \tag{1.50}$$

Similarly, in S^3 there is a set of coordinates which allow us to develop an intuition of the complex Hopf bundle when combined with the previously defined coordinates for S^2. Notice that any point (z^1, z^2) in S^3 can be written as $(z_1, z_2) = (\cos\left(\frac{\phi}{2}\right) e^{i\xi_1}, \sin\left(\frac{\phi}{2}\right) e^{i\xi_2})$ with $\phi \in [0, \pi]$ and $\xi_1, \xi_2 \in [0, 2\pi]$, in particular notice that indeed $|z^1|^2 + |z^2|^2 = \cos^2\left(\frac{\phi}{2}\right) + \sin^2\left(\frac{\phi}{2}\right) = 1$. Again, once appropriate coordinate maps are chosen, these coordinates induce the local frame $\{\partial_\phi, \partial_{\xi_1}, \partial_{\xi_2}\} \subseteq \Gamma_{U_{SN}}(TS^3)$ where $U_{SN} = U_S \cap U_N$ is the set where $\phi \in (0, \pi)$. We will denote by T, or simply T, the map

$$T : (\phi, \xi_1, \xi_2) \mapsto (\cos\left(\frac{\phi}{2}\right) e^{i\xi_1}, \sin\left(\frac{\phi}{2}\right) e^{i\xi_2}) \in S^3. \tag{1.51}$$

We now find a smooth atlas for the projective space $\mathbb{F}P^1$, where $\mathbb{F}$ may be either $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The sets $U_i := \{[\xi^1, \xi^2] \mid \xi^i \neq 0\}$, for $i = 1, 2$, are open subsets of $\mathbb{F}P^1$ missing only one point of the total space, which make up an open cover. Writing the not necessarily commutative multiplication $\xi^1(\xi^2)^{-1}$ as ξ^1/ξ^2 an atlas for $\mathbb{F}P^1$ is given by $\{(U_2, \vec{x}), (U_1, \vec{y})\}$ where

$$\vec{x} : U_2 \to \mathbb{F} \qquad\qquad \phi_2 := \vec{x}^{-1} : \mathbb{F} \to U_2 \tag{1.52}$$
$$(\xi^1, \xi^2) \mapsto \xi^1/\xi^2 \qquad\qquad \xi \mapsto [\xi, 1] \tag{1.53}$$

and

$$\vec{y} : U_1 \to \mathbb{F} \qquad\qquad \phi_1 := \vec{y}^{-1} : \mathbb{F} \to U_1 \tag{1.54}$$
$$(\xi^1, \xi^2) \mapsto \overline{\xi^2/\xi^1} \qquad\qquad \xi \mapsto [1, \overline{\xi}]. \tag{1.55}$$

In the intersection $\mathbb{F} - 0 = \phi_i(U_2 \cap U_1) \ni \xi$ it is then true that

$$\vec{x} \circ \vec{y}^{-1}(\xi) = \overline{\xi}^{-1} = \vec{y} \circ \vec{x}^{-1}(\xi). \tag{1.56}$$

This construction can be easily modified to generate a smooth atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$ for $\mathbb{F}\,P^n$ for any $n \geq 1$ where $U_k = \left\{[\xi^1, \ldots, \xi^{n+1}] \mid \xi^k \neq 0\right\}$ and $\phi_k : U_k \to \mathbb{F}^n : [\xi^1, \ldots, \hat{\xi}^k, \ldots, \xi^{n+1}] \mapsto (\xi^1, \ldots, \hat{\xi}^k, \ldots, \xi^{n+1}) \cdot (\xi^k)^{-1}$.

The atlases given for $\mathbb{F}\,P^1$ and S^d where $d = dim_{\mathbb{R}}\,\mathbb{F}$ suggests a bijective map $\mathbb{F}\,P^1 \to S^d$ given by $\phi_S \circ \vec{x}$ in U_2 and $\phi_N \circ \vec{y}$ in U_1 if they were compatible in $U_2 \cap U_1$; it can be easily shown that these two maps indeed coincide there and that the resulting map has the formula

$$\begin{aligned} F : \mathbb{F}\,P^1 &\to S^d \\ [\xi^1, \xi^2] &\mapsto \frac{(2\xi^1\overline{\xi^2}, |\xi^1|^2 - |\xi^2|^2)}{||(\xi^1, \xi^2)||^2} \end{aligned} \tag{1.57}$$

and it is a diffeomorphism with inverse

$$\begin{aligned} F^{-1} : S^d &\to \mathbb{F}\,P^1 \\ (x^1, \ldots, x^{d+1}) &\mapsto [(x^1, \ldots, x^d), 1 - x^{d+1}] \\ &= [1 + x^{d+1}, (x^1, \ldots, x^d)]. \end{aligned} \tag{1.58}$$

The Principal Bundles

The complex Hopf bundle is obtained as the case $n = 2$ of the family of S^1-principal bundles $S^1 \to S^{2n-1} \to \mathbb{C}P^{n-1}$ that we will now define. $S^1 \subseteq \mathbb{C}$ acts naturally on the right on $S^{2n-1} \subseteq \mathbb{C}^n$ by coordinate-wise multiplication. For $n \in \mathbb{Z}_{\geq 1}$ let

$$\widetilde{\pi} : S^{2n-1} \to \mathbb{C}P^{n-1}$$

be the restriction of the smooth quotient map $p : \mathbb{C}^n \to \mathbb{C}P^{n-1}$ to S^{2n-1}; by the definition of the projective spaces, for any $g \in S^1$ and $p \in S^{2n-1} \subseteq \mathbb{C}^n$, the projection map π satisfies $\widetilde{\pi}(pg) = \widetilde{\pi}(p)$. To see that this is a principal bundle all we need to do is define smooth local sections of the projection map $\widetilde{\pi}$ for all the neighborhoods of the previously defined atlas of $\mathbb{C}P^{n-1}$. For $k \in \{1, \ldots, n\}$, let

$$\begin{aligned} \sigma_k : U_k &\to S^{2n-1} \\ [z^1, \ldots, z^n] &\mapsto \frac{(z^1, \ldots, z^n)|z^k|/z^k}{\sqrt{|z^1|^2 + \cdots + |z^n|^2}}; \end{aligned} \tag{1.59}$$

notice that this section takes us to points in S^{2n-1} whose k-th entry is strictly real, and so any point of the fiber of $m \in U_k \subseteq \mathbb{C}P^{n-1}$ can be obtained by letting a phase $g \in S^1$ act on $\sigma_k(m)$, meaning that S^{2d-1} is locally homeomorphic to $U_k \times S^1$ on U_k. The transition functions $g_{ij} : U_i \cap U_j \to S^1$ for $i, j \in \{1, 2\}$ are defined by the relations

$$\sigma_j(m) g_{ji}(m) = \sigma_j(m)$$

for all $m \in U_i \cap U_j$, and have the explicit formula

$$g_{ji}([z^1, \ldots, z^n]) = \frac{z^j/|z^j|}{z^i/|z^i|}. \tag{1.60}$$

In the particular case when $n = 2$, define $\pi = F \circ \widetilde{\pi} : S^3 \to S^2$. Since $\mathbb{C}P^1$ is diffeomorphic to S^2 through F, π is the projection map of a principal bundle S^3 over S^2 with fiber S^1, called the <u>complex Hopf bundle</u>. By plugging any point in S^3 using the special coordinates defined for S^2 and S^3, and using equation (1.57), we obtain the following nice formula for this projection map of the complex Hopf bundle:

$$\begin{aligned} \pi : S^3 &\to S^2 \\ T(\phi, \xi_1, \xi_2) &\mapsto E(\phi, \xi_1 - \xi_2). \end{aligned} \tag{1.61}$$

The action of $e^{i\alpha} \in S^1$ on $T(\phi, \xi_1, \xi_2) = (\cos\left(\frac{\phi}{2}\right) e^{i\xi_1}, \sin\left(\frac{\phi}{2}\right) e^{i\xi_2})$ returns $T(\phi, \xi_1 + \alpha, \xi_2 + \alpha)$, which indeed leaves invariant the quantity $\xi_1 - \xi_2$, and this implies that

$$\pi^{-1}(E(\phi, \theta)) = \{T(\phi, \xi_1, \xi_2) \mid \xi_1 - \xi_2 = \theta\}.$$

The induced local trivializations $\sigma_S := F^{-1} \circ \sigma_2$ and $\sigma_N := F^{-1} \circ \sigma_1$ have the formulas:

$$\sigma_S : U_S \to S^3, \qquad E(\phi, \theta) \mapsto T(\phi, \theta, 0) \tag{1.62}$$

$$\sigma_N : U_N \to S^3, \qquad E(\phi, \theta) \mapsto T(\phi, 0, -\theta). \tag{1.63}$$

This principal bundle, due to the Reconstruction Theorem [8], is completely determined by the transition function

$$g_{NS}(E(\phi, \theta)) = e^{i\theta}. \tag{1.64}$$

An identical procedure as the one used for the principal bundles $S^1 \to S^{2n-1} \to \mathbb{C}P^{n-1}$ can be used to define a family of S^3-principal bundles $S^3 \to S^{4n-1} \to \mathbb{H}\,P^{n-1}$ for all $n \in \mathbb{Z}_{\geq 1}$, where S^3 is understood as the unit sphere in the quaternions, also denoted by $S(\mathbb{H})$ when the distinction is necessary, and it acts on the right by coordinate-wise multiplication on S^{4n-1} seen a subset of $\mathbb{H}^n$, in which case the right action is once again given by coordinate-wise multiplication. The relevant ingredients of each of these principal bundles are its smooth projection map that respects the $S(\mathbb{H})$ action

$$\widetilde{\pi} : S^{4n-1} \to \mathbb{H}\,P^{n-1}$$

equal to the restriction of the quotient map $p : \mathbb{H}^n \to \mathbb{H}\,P^{n-1}$ to S^{4n-1}, the local sections

$$\begin{aligned} \sigma_k : U_k \subseteq \mathbb{H}\,P^{n-1} &\to S^{4n-1} \\ [q^1, \ldots, q^n] &\mapsto \frac{(q^1, \ldots, q^n)|q^k|/q^k}{\sqrt{|q^1|^2 + \cdots + |q^n|^2}}. \end{aligned} \tag{1.65}$$

and its transition functions $g_{ij} : U_i \cap U_j \to S^3$ for $i, j \in \{1, 2\}$

$$g_{ji}([q^1, \ldots, q^n]) = \frac{q^j/|q^j|}{q^i/|q^i|}. \tag{1.66}$$

When $n = 2$ we recall from equation (1.57) that $\mathbb{H}\,P^1 \cong S^4$, then the quaternionic Hopf bundle is the induced S^3-principal bundle $S^3 \to S^7 \to S^4$ with projection map $\pi := F \circ \widetilde{\pi} : S^7 \to S^4$.

Yet another family of principal bundles which we will use throughout the document is the set of principal bundles over S^2 with fiber S^1; the complex Hopf bundle is also an element of this family. Up to isomorphism of principal bundles, the S^1-principal bundles over S^2 are in bijective correspondence with $\mathbb{Z} = \pi_1(S^1)$ [8], where π_1 denoted the fundamental group of the topological space S^1. It can be shown [7] that the transitive Lie algebroids over S^2 with vertical fiber $\mathfrak{g}$ are, up to isomorphism, in corresponds with the center of the simply connected group G whose Lie algebra is $\mathfrak{g}$; we will not, however, study these other Lie algebroids on this document. Since we know an atlas of S^2 with only two neighborhooods, U_S and U_N, each of these bundles is completely determined [8] by its transition function $g_{NS} : U_{SN} \to S^1$. More precisely, the non-homotopic equivalent functions indexed by $k \in \mathbb{Z}$

$$\begin{aligned} g^k_{NS} : U_{NS} \subseteq S^2 &\to S^1 \\ E(\phi, \theta) &\mapsto e^{ik\theta} \end{aligned} \tag{1.67}$$

determine each a different S^1-principal bundle over S^2.

Let P^k be the S^1-principal bundle over S^2 determined by the transition function g^k_{NS}; as implied by equation (1.64), the complex Hopf bundle corresponds to the index $k = 1$. The Reconstruction Theorem of principal bundles [8] shows that P^k can be defined by

$$P^k := \left(\bigsqcup_{l \in N,S} U_i \times S^1 \right) \Big/ \sim \quad \ni [E(\phi, \theta), e^{i\theta}, l]$$

where the equivalence relation $\sim$ is defined, for any $E(\phi, \theta) \in S^2$ and $e^{i\alpha_N}, e^{i\alpha_N}$ in S^1, by

$$\begin{aligned} (E(\phi, \theta), e^{i\alpha_N}, N) \sim (E(\phi, \theta), e^{i\alpha_S}, S) &\iff e^{i\alpha_N} = g^k_{NS}(E(\phi, \theta))e^{i\alpha_S} \\ &\iff \alpha_N \equiv \alpha_S + k\theta \pmod{2\pi}, \end{aligned}$$

and the right group action by

$$[m, g, l]g' := [m, gg', l]$$

for all $m \in S^2$, $g, g' \in S^1$ and $l \in S, N$.

The regular atlas of S^2 can be used to define local sections for these principal bundles:

$$\begin{aligned} \sigma^k_S &: U_S \to P^k \\ E(\phi, \theta) &\mapsto [E(\phi, \theta), 1, S] = [E(\phi, \theta), e^{ik\theta}, N] \end{aligned} \tag{1.68}$$

and

$$\begin{aligned} \sigma^k_N &: U_N \to P^k \\ E(\phi, \theta) &\mapsto [E(\phi, \theta), 1, N] = [E(\phi, \theta), e^{-ik\theta}, S]; \end{aligned} \tag{1.69}$$

notice that it is indeed satisfied the relation

$$\sigma_N^k(E(\phi,\theta))g_{NS}^k(E(\phi,\theta)) = \sigma_S^k(E(\phi,\theta)).$$

An analogous family of principal bundles $S^3 \to P^k \to S^4$ also exists in the quaternionic case, since the S^3-principal bundles on S^4 are in a bijective correspondence with $\pi_3(S^3) \cong \mathbb{Z}$ [8]. The distinct vector bundles are characterized by the following transition functions over $U_{12} \subseteq \mathbb{H}\,P^1 \cong S^4$:

$$\begin{aligned} g_{NS}^k : U_{12} \subseteq \mathbb{H}\,P^1 &\to S^3 \\ [q^1, \dots, q^n] &\mapsto \left(\frac{q^j/|q^j|}{q^i/|q^i|}\right)^k ; \end{aligned} \tag{1.70}$$

just as in the complex case, the (quaternionic) Hopf bundle is obtained as the special case $k = 1$.

We leave here our task of constructing the Atiyah Lie algebroids associated to the principal bundles studied in this section. We will continue in Section 1.5.4, since the way that will be practical to us to understand a transitive Lie algebroid will be through its local trivializations studied in 1.5.11.

1.4 Other Basic Definitions about Lie Algebroids

Definition 1.4.1 ((Base preserving) morphism of Lie algebroids)**.** Given two algebroids over the manifold M, A with anchor a and A' with anchor a', a base preserving Lie algebroid morphism or Lie algebroid morphism over M

$$\psi : A \to A'$$

is a vector bundle morphism over M such that

$$a' \circ \psi = a : A \to TM, \tag{1.71}$$

i.e. it is compatible with the anchors, and such that

$$\psi[\mathfrak{X}, \mathfrak{Y}] = [\psi(\mathfrak{X}), \psi(\mathfrak{Y})]$$

for all $\mathfrak{X}, \mathfrak{Y} \in \Gamma(A)$, i.e. it is a Lie algebra morphism when applied to sections.

Remark 1.4.2. Notice that a morphism of Lie algebroids over M may also be defined algebraically as a map

$$\psi : \Gamma(A) \to \Gamma(A')$$

which is $C^\infty(M)$-linear, a Lie algebra morphism, and such that

$$a' \circ \psi = a, : \Gamma(A) \to \Gamma(TM)$$

This can be done due to the equivalence between vector bundle maps over M and $C^\infty(M)$-linear maps between sections.

Example 1.4.3 (Lie algebroid morphisms between Trivial Lie algebroid over M)**.** Let $TM \times \mathfrak{g}$ and $TM \times \mathfrak{g}'$ be trivial Lie algebroids, and suppose that ψ is a Lie algebroid morphisms between them.

Since ψ is $C^\infty(M)$-linear and it respects the anchor, it must have the form

$$\psi(X \oplus \eta) = X \oplus (\omega(X) + \psi_+(\eta)), \tag{1.72}$$

where $\omega : TM \to TM \times \mathfrak{g}'$ and $\phi_L : M \times \mathfrak{g} \to M \times \mathfrak{g}'$ are vector bundle morphisms. Applying the fact that ϕ respects the Lie brackets give us further properties of ω and ψ_+:

- When applied to $\boldsymbol{X}, \boldsymbol{Y} \in \Gamma(TM)$ arbitrary, we conclude that ω is a *Maurer-Cartan form*, i.e.

$$[\boldsymbol{X}, \omega(\boldsymbol{Y})] - [\boldsymbol{Y}, \omega(\boldsymbol{X})] - \omega[\boldsymbol{X}, \boldsymbol{Y}] = 0. \tag{1.73}$$

- When applied to $\eta, \theta \in M \times \mathfrak{g}$ arbitrary we conclude that ψ_+ is a LAB bundle.

- When applied to arbitrary $X \in TM$ and $\eta \in M \times \mathfrak{g}$ we conclude that ω and ψ_+ satisfy the following compatibility condition:

$$X(\psi_+(\eta)) - \psi_+(X(\eta)) + [\omega(X), \psi_+(\eta)] = 0. \tag{1.74}$$

Definition 1.4.4. Let A be a Lie algebroid with anchor a.

- A is a transitive Lie algebroid if a is fiberwise surjective anchor.
- A is a regular Lie algebroid if a is of locally constant rank.
- A is totally intransitive if $a = 0$.

Remark 1.4.5. If A is a regular Lie algebroid with anchor a, both $im(a)$ and $ker(a)$ are vector subbundles of A and TM respectively[2]. By theorem 1.1.7 $ker(a)$ is in fact a totally intransitive Lie algebroid. The image of the anchor is, then, an involutive distribution on M and, in consequence, induces a foliation of the base manifold called the characteristic foliation of A. Then, the restriction of A to each leaf is a transitive Lie algebroid.

Additionally, we can apply theorem 1.1.7 to see that, around any $m \in M$ there is a neighborhood $U \subseteq M$ such that a regular Lie algebroid is isomorphic to the vector bundle $U \times (V \oplus \mathfrak{g})$, where $V = a(A_m)$, although this local form of the Lie algebroid might not be consistent with the brackets.

Remark 1.4.6. Any LAB is a totally intransitive Lie algebroid, but the converse is not true since there might not be an atlas adapted to the Lie brackets.

Transitive Lie algebroids will be a main concern for us throughout this document. In this case, since the anchor is fiberwise surjective, it is a surjective submersion, and, in particular, it is locally constant rank implying that $ker(a)$ is a vector bundle, so theorem 1.1.7 may be applied to conclude that its kernel is a Lie algebroid. However, more is true, as shown below.

Definition 1.4.7. Let A be a transitive Lie algebroid on M with anchor a. A Lie algebroid sequence of a transitive Lie algebroid A is an exact sequence of Atiyah Lie algebroids

$$0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0 \tag{1.75}$$

where L is a Lie algebroid isomorphic to $ker(a)$ by a Lie algebroid morphism $j : L \to ker(a) \subseteq A$. The lie algebroid L is called an adjoint Lie algebroid of (the transitive Lie algebroid) A.

[2] The kernel of a fiber bundle map is a subbundle if and only if the image is a subbundle if and only if it has locally constant rank.

Notice that an adjoint Lie algebroid of A must be a totally intransitive Lie algebroid with a bracket compatible, through some j, with the bracket inherited by $ker(a)$ from A.

Remark 1.4.8. A sequence of a transitive Lie algebroid A induces the exact sequence of $C^\infty(M)$-modules and Lie algebras

$$0 \xrightarrow{j} \Gamma(L) \to \Gamma(A) \xrightarrow{a} \Gamma(TM) \to 0. \tag{1.76}$$

When doing computations, it will be important to have in mind this sequences, perhaps replacing some of the elements by isomorphic ones, like we did in the Atiyah sequence of a principal bundle 1.3.21.

Example 1.4.9. A trivial Lie algebroid induces the sequence

$$0 \to M \times \mathfrak{g} \xrightarrow{j} TM \oplus M \times \mathfrak{g} \xrightarrow{p_1} TM \to 0 \tag{1.77}$$

where j is simply the inclusion. The algebraic version of this sequence that we will work with is

$$0 \to C^\infty(M, \mathfrak{g}) \xrightarrow{j} \Gamma(TM) \oplus C^\infty(M, \mathfrak{g}) \xrightarrow{p_1} \Gamma(TM) \to 0 \tag{1.78}$$

Example 1.4.10. A principal bundle induces the sequence

$$0 \to P \times \mathfrak{g}/G \xrightarrow{j} TP/G \xrightarrow{\pi_*^G} TM \to 0 \tag{1.79}$$

as shown in section 1.3.

Example 1.4.11. A vector bundle E over M induces the Atiyah sequence

$$0 \to 0 \to End(E) \xrightarrow{j} \mathfrak{D}(E) \xrightarrow{a} TM \to 0, \tag{1.80}$$

as was proven leading to equation (1.5).

The following theorem shows the structure of an adjoint Lie algebroid of a transitive Lie algebroid; its proof, however, requires knowledge beyond the scope of this document, so we refer to [6] for it.

Theorem 1.4.12. Let $0 \to L \to A \to TM \to 0$ be a transitive Lie algebroid sequence. Then L is a Lie algebra bundle with respect to the bracket structure on $\Gamma(L)$ induced by that on $\Gamma(A)$.

Example 1.4.13. A LAB trivialization of $End(E)$ when E has the vector space V as typical fiber is given by a family of bracket preserving maps $\psi_i : U_i \times End(V) \to End(E)|_{U_i}$, where the bracket on $End(V)$ is the commutator of operators, for i in some index set I.

Definition 1.4.14. Let $0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0$ and $0 \to L' \xrightarrow{j'} A' \xrightarrow{a'} TM \to 0$ be transitive Lie algebroids sequences over M. Let $\psi : A \to A'$ be a morphism of Lie algebroids. Since, $a' \circ \psi = a$, ψ induces a morphism between the adjoint Lie algebroids denoted by

$$\psi_L : L \to L',$$

called <u>the vertical part of ψ</u>.

Proposition 1.4.15. Let $0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0$ and $0 \to L' \xrightarrow{j'} A' \xrightarrow{a'} TM \to 0$ be transitive Lie algebroids sequences over M. Let $\psi : A \to A'$ be a morphism of Lie algebroids. Then ψ is a surjection, injection or bijection if and only if $\psi^+ : L \to L'$ has the respective property.

Proof. We may think of L and L' as the kernels of the anchors, so j and j' are simply the inclusion maps and $\psi^+ = \psi|_L$. Since $a' \circ \psi = a$, $ker(\psi) \subseteq \ker(a) = L$, and so, by linearity, ψ is injective if and only if ψ^+ is injective.

By the same property, only elements of L can be mapped to L', therefore ψ being surjective implies that ψ^+ is surjective. For the converse, we recall that the anchors are fiberwise surjective, so given any $\mathfrak{Y} \in A'$, $a'(\mathfrak{Y}) = Y \in TM$, we can find an element $\mathfrak{X}_1 \in A$ with $a(\mathfrak{X}_1) = Y$; let $\nu = \mathfrak{Y} - \psi(\mathfrak{X}_1)$, so $\nu \in L'$ and if ψ^+ is surjective there must exist $\mu \in L$ such that $\psi^+(\mu) = \nu$ and so $\mathfrak{X} = \mathfrak{X}_1 + j\mu$ satisfies $\psi(X) = \mathfrak{Y}$, which allows us to conclude that ψ must also be surjective. □

The following concept generalizes that of associated vector bundles of a principal bundle $G \to P \to M$. In the principal bundle theory, sections of an associated vector bundle with fiber V are equivalent to G-equivariant functions $P \to V$ or, equivalently, to sections in $\Gamma(P \times V/G)$ as Proposition 1.3.16 shows, and G-equivariant vector fields in $\Gamma^G(TP)$ or, equivalently sections $\Gamma(TP/G)$, act as derivations on the sections of E. Similarly, sections of a Lie algebroid A act as derivations on representation vector bundles E. In particular, a (Lie algebroid connection) on A will induce a vector bundle connection and covariant derivatives on a representation vector bundle E, just as principal bundle connections induce covariant derivatives in associated vector bundles.

Definition 1.4.16. Let A be a Lie algebroid on M and let E be a vector bundle also on M. A representation of A on E is a Lie algebroid morphism over M,

$$\phi : A \to \mathfrak{D}(E). \tag{1.81}$$

Given $\mathfrak{X} \in A$, $\phi(\mathfrak{X})$ is usually denoted by $\phi_{\mathfrak{X}}$.

Using an algebraic language, notice how a representation $\rho : A \to \mathfrak{D}(E)$ may be defined instead as a $C^\infty(M)$-linear mapping that, to every $\boldsymbol{\mathfrak{X}} \in \Gamma(A)$, assigns a derivation on E

$$\phi(\boldsymbol{\mathfrak{X}}) : \Gamma(E) \to \Gamma(E) \tag{1.82}$$

in a way that is compatible with the anchors and the brackets.

Definition 1.4.17. (Adjoint representation of a transitive Lie algebroid) Let A be a transitive Lie algebroid on M with adjoint bundle L. The adjoint representation of A is the representation on L

$$ad : A \to \mathfrak{D}(L),$$

such that

$$ad(\boldsymbol{\mathfrak{X}})(\boldsymbol{\mu}) = j^{-1}[\boldsymbol{\mathfrak{X}}, j(\boldsymbol{\mu})]$$

for all $\boldsymbol{\mathfrak{X}} \in \Gamma(A)$, $\boldsymbol{\mu} \in \Gamma(L)$.

The adjoint action is well defined since the anchor commutes with the bracket.

Definition 1.4.18. Let A be a Lie algebroid on M and let V be a vector space. The trivial representation of A on $M \times V$ is defined by identifying $\Gamma(M \times V)$ with $C^\infty(M, V)$:

$$\rho^0(\boldsymbol{\mathfrak{X}})(f) := a(\boldsymbol{\mathfrak{X}})(f), \qquad f : \Gamma(M \times V) \cong C^\infty(M, V),\ \boldsymbol{\mathfrak{X}} \in \Gamma(A).$$

Example 1.4.19 (All representations of a TLA on a Trivial Vector Bundle)**.** Let $TM \times \mathfrak{g}$ be a trivial vector bundle and let $E = M \times V$ be a trivial vector bundle. We saw in example 1.2.5 that $\mathfrak{D}(E)$ is the trivial vector bundle $TM \times (gl)(V)$, so a representation of $TM \times \mathfrak{g}$ on $M \times V$, as illustrated by 1.4.3, has the form

$$\phi(X \oplus \eta) = X \oplus (B(X) + \phi_L(\eta)), \tag{1.83}$$

where $B : TM \to M \times \mathfrak{gl}(V)$ is a Maurer-Cartan form, $\phi_L : M \times \mathfrak{g} \to M \times \mathfrak{gl}(V)$ is a LAB morphism, and B and ϕ_L satisfies the compatibility condition indicated in 1.4.3.

Example 1.4.20 (Group induced representation)**.** Let $TM \times \mathfrak{g}$ be a trivial Lie algebroid and $E = M \times V$ a trivial vector bundle. Furthermore, suppose that G is a Lie group with Lie algebra $\mathfrak{g}$, and that $\pi : G \to Aut(V)$ is a representation of G. This allows us to define a Lie algebroid representation ϕ with vertical part ϕ_L being constantly equal to the induced Lie representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. Explicitly, for all $X \oplus \eta \in TM \times \mathfrak{g}$ and $f \in C^\infty(M, V)$:

$$\phi(X \oplus \eta)(f) := X(f) + \pi(\eta)f; \tag{1.84}$$

when the representation is clear from the context, we will usually rewrite the previous equation as

$$\phi(X \oplus \eta)(f) := X(f) + \eta \cdot f. \tag{1.85}$$

This is called the G-induced representation of the trivial Lie algebroid on the trivial vector bundle $M \times V$.

Example 1.4.21. For a principal bundle $G \to P \to M$ and an associated vector bundle $E = P \times V/G$ over M with fiber V on which G acts on the left, there is a natural representation

$$\phi : TP/M \to \mathfrak{D}(E),$$

or, equivalently

$$\phi : \Gamma(TP/M) \to \Gamma(\mathfrak{D}(E)),$$

which is best understood recalling, from Propositions 1.3.9 and 1.3.16, that $\Gamma(TP/G) \cong \Gamma^G(TP)$ and $\Gamma(E) \cong \Gamma^G(P \times V) \cong C^\infty_G(P, V)$.

Recall the notation introduced in Proposition 1.3.6 and 1.3.16. Given $\boldsymbol{\mathfrak{X}} \in \Gamma(TP/G)$ and $\boldsymbol{\mu} \in \Gamma(E) = \Gamma(P \times V/G)$, the G-invariant section $\overline{\boldsymbol{\mathfrak{X}}} \in \Gamma^G(TP)$ acts naturally on the equivariant function $\widetilde{\boldsymbol{\mu}} \in C^\infty_G(M)$ by the Lie derivative. It is not hard to show that $\overline{\boldsymbol{\mathfrak{X}}}(\widetilde{\boldsymbol{\mu}}) \in C^\infty(M)$ is also G-equivariant, and therefore there is a corresponding element in $\Gamma(E)$, which we call $\phi(\boldsymbol{\mathfrak{X}})(\boldsymbol{\mu})$. In other words, the representation of TP/G on E is given by

$$\phi(\boldsymbol{\mathfrak{X}})(\boldsymbol{\mu}) := \overline{\boldsymbol{\mathfrak{X}}}(\widetilde{\boldsymbol{\mu}}) \in C^\infty_G(P, V). \tag{1.86}$$

It is clear that ϕ is $C^\infty(M)$-linear in $\boldsymbol{\mathfrak{X}}$ and $\mathbb{R}$-linear in $\boldsymbol{\mu}$. That $\phi(\boldsymbol{\mathfrak{X}}) : \Gamma(E) \to \Gamma(E)$ is a derivation, i.e. $\mathbb{R}$-linear and satisfies Leibniz, follows once we recall the definition of the anchor in $\Gamma(TP/G)$ in equation (1.3.5), which tells us that $a(\boldsymbol{\mathfrak{X}}) = \overline{\boldsymbol{\mathfrak{X}}}(f \circ \pi)$, therefore:

$$\begin{aligned}
&\rho\widetilde{(\boldsymbol{\mathfrak{X}})(f\boldsymbol{\mu}})=\overline{\boldsymbol{\mathfrak{X}}}(\widetilde{f\boldsymbol{\mu}}) = \\
&\qquad \overline{\boldsymbol{\mathfrak{X}}}((f\circ\pi)\widetilde{\boldsymbol{\mu}}) = (f\circ\pi)\overline{\boldsymbol{\mathfrak{X}}}(\widetilde{\boldsymbol{\mu}}) + \overline{\boldsymbol{\mathfrak{X}}}(f\circ\pi)\widetilde{\boldsymbol{\mu}} = f\widetilde{\rho(\boldsymbol{\mathfrak{X}})(\boldsymbol{\mu}}) + \widetilde{a(\boldsymbol{\mathfrak{X}})(f)}\boldsymbol{\mu}.
\end{aligned}$$

This equation also shows us that

$$a(\phi(\boldsymbol{\mathfrak{X}})) = a(\boldsymbol{\mathfrak{X}}),$$

so ρ is compatible with the anchors. The fact that, by definition of the bracket on $\Gamma(TP/G)$, the map $\boldsymbol{\mathfrak{X}} \mapsto \overline{\boldsymbol{\mathfrak{X}}}$ is a Lie algebra morphism, by definition, implies the respective property for ϕ.

All of these combined mean that $\rho : \Gamma(TP/G) \to \Gamma(\mathfrak{D}(E))$ is indeed a Lie algebra representation of TP/G in $E = P \times V/G$.

Example 1.4.22. Let $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ be a connection on a a vector bundle E. That statement is equivalent to: given any $\boldsymbol{X} \in \Gamma(TM)$,

$$\Gamma(TM) \to \Gamma(\mathfrak{D}(E)) \qquad\qquad \boldsymbol{X} \mapsto \nabla_{\boldsymbol{X}}$$

is a $C^\infty(M)$-linear mapping from the vector fields on M to the derivations on E. The Leibniz identity of the derivation $\nabla_{\boldsymbol{X}}$ implies that $a(\nabla_{\boldsymbol{X}}) = \boldsymbol{X}$. This means that a connection on E is only missing the compatibility with the brackets to be a representation, but this compatibility is equivalent to the property of the connection being flat. Therefore flat connections on the vector bundle E give us a representation of TM on E.

1.5 Local Description of Transitive Lie Algebroids

1.5.1 Lie Algebroid Atlas

Recall that Lemma 1.1.2 allowed us to restrict a Lie algebroid to an open subset U of the base manifold. We now study how a transitive Lie algebroid is a collection of locally defined trivial Lie algebroids correctly pasted together.

Definition 1.5.1. Let $0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0$ be a transitive Lie algebroid sequence, and let $\mathfrak{g}$ be a Lie algebra. Given an open cover $\{U_i\}_{i\in I}$ of M, a Lie algebroid atlas for A is a collection $\{U_i, \psi_i : U_i \times \mathfrak{g} \to L|_{U_i}, \nabla^{0,i} : TU_i \to A|_{U_i}\}_{i\in I}$ such that:

- each $\nabla^{0,i} : TU_i \to A_{u_i}$ is a Lie algebroid morphism,
- the family $\{\psi_i : U_i \times \mathfrak{g} \to L|_{U_i}\}$ is a LAB atlas for the adjoint bundle
- for all $\boldsymbol{X} \in \Gamma(TU_i)$, $\widetilde{\boldsymbol{\eta}} \in C^\infty(U_i, \mathfrak{g}) \cong \Gamma(U_i \times \mathfrak{g})$

$$[\nabla^{0,i}(\boldsymbol{X}), j\psi_i(\boldsymbol{\eta})] = j\psi_i([\boldsymbol{X} \oplus 0, 0 \oplus \boldsymbol{\eta}]) \equiv j\psi_i(\boldsymbol{X}(\boldsymbol{\eta})) \tag{1.87}$$

In Chapter 3 we will see that the $\nabla^{0,i}$ are flat connections in $A|_{U_i}$ and so their image gives a notion of horizontallity in $A|_{U_i}$ and $\mathfrak{X} \mapsto \nabla^{0,i}(a(\mathfrak{X})) \in A|_{U_i}$ is the horizontal projection of $\mathfrak{X}$ induced by $\nabla^{0,i}$.

The following theorem will be key to our document, and will also enable the treatment of gauge theories on transitive Lie algebroids in the way familiar in the physics literature, since it will imply that locally every transitive Lie algebroid looks like the Atiyah Lie algebroid of a principal bundle; a proof may be found in [6].

Theorem 1.5.2. Every transitive Lie algebroid has a Lie algebroid atlas.

Throughout this chapter $0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0$ will be a sequence of the transitive Lie algebroid A with Lie algebroid atlas $\{U_i, \psi_i : U_i \times \mathfrak{g} \to L|_{U_i}, \nabla^{0,i} : TU_i \to A|_{U_i}\}_{i\in I}$.

Let

$$S_i : \qquad TU_i \oplus (U_i \times \mathfrak{g}) \qquad \to \qquad A|_{U_i} \tag{1.88}$$

$$X \oplus \eta \qquad \mapsto \qquad \nabla^{0,i}(X) + j\psi_i(\eta); \tag{1.89}$$

this vector bundle morphisms inherits the compatibility with the anchors from the respective property of the Lie algebroid morphisms $\nabla^{0,i}$, and are compatible with the bracket due to the compatibility equation (1.87) which is equivalent to the equation

$$S_i[\boldsymbol{X} \oplus 0, 0 \oplus \boldsymbol{\eta}] = [S_i(X \oplus 0), S_i(0 \oplus \eta)],$$

thus we proved the following:

Theorem 1.5.3. The maps $S_i : TU_i \oplus (U_i \times \mathfrak{g}) \to A|_{U_i}$ defined in equation (1.88) are Lie algebroid isomorphisms between the trivial Lie algebroids $TU_i \oplus (U_i \times \mathfrak{g})$ and $A|_{U_i}$.

Furthermore for each $i \in I$

$$\psi_i = S_i^+ : U_i \times \mathfrak{g} \to L|_{U_i}$$

and

$$\nabla^{0,i}(X) = S_i(X \oplus 0)$$

for all $X \in TU_i$.

This means that for every $\mathfrak{X} \in A|_{U_i}$, $X = a(\mathfrak{X}) \in TM$ and so there is some $\eta^i \in U_i \times \mathfrak{g}$ such that

$$\mathfrak{X} = S_i(X \oplus \eta^i). \tag{1.90}$$

Definition 1.5.4. The set $\{X \oplus \eta^i \in TU_i \oplus (U_i \times \mathfrak{g})\}_{i \in I}$ will be called <u>a family of trivializations of $\mathfrak{X} \in A$</u>. Analogously, for $\boldsymbol{\mathfrak{X}} \in \Gamma(A)$, a <u>family of trivializations of $\boldsymbol{\mathfrak{X}}$</u> will be a set $\{\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}}^i \in \Gamma(TU_i) \oplus C^\infty(U_i, \mathfrak{g})\}_{i \in I}$.

We now study the transition between charts of a Lie algebroid atlas.

Let

$$\begin{array}{cccc} \alpha^i_j : & U_{ij} = U_i \cap U_j & \to & Aut(\mathfrak{g}) \\ & m & \mapsto & \alpha^i_{j,m} := \psi^{-1}_{i,m} \circ \psi_{j,m} \end{array}$$

be the transition functions of L with respect to the trivializing maps $\{\psi_i\}$. Since any element $\eta \in U_i \times \mathfrak{g}$ is of the form $\eta = (m, \eta_m)$ for some $m \in U_{ij}$ and $\eta_m \in \mathfrak{g}$, denote also by α^i_j the maps

$$\alpha^i_j : \quad U_{ij} \times \mathfrak{g} \quad \to \quad U_{ij} \times \mathfrak{g} \tag{1.91}$$

$$\eta = (m, \eta_m) \quad \mapsto \quad \alpha^i_j(\eta) := \psi_i^{-1} \circ \psi_j(\eta); \tag{1.92}$$

notice that $\alpha^i_j(\eta) = (m, \alpha^i_{j,m}(\eta_m))$.

Notice, however, that for $\mathfrak{X} = S_j(X \oplus \eta^j)$ in $A|_{U_{ij}}$, $\mathfrak{X} \neq S_i(X \oplus \alpha^i_j(\eta^j))$ since $S_i(X \oplus \alpha^i_j(\eta^j)) = \nabla^{0,i}(X) + j\psi_i(\alpha^i_j(\eta^j)) = \nabla^{0,i}(X) + j\psi^j(\eta^j) = S_j(X \oplus \eta^j) + (\nabla^{0,i}(X) - \nabla^{0,j}(X))$. But $a(S_j(X \oplus \eta^j)) = a(S_i(X \oplus \alpha^i_j(\eta^j)))$ so $\nabla^{0,i}(X) - \nabla^{0,j}(X) \in ker(a)$ and we may define the $L|_{Uij}$-valued 1-form

$$l^i_j : TU_{ij} \to L|_{U_{ij}} \tag{1.93}$$

$$X \mapsto j^{-1}(\nabla^{0,j}(X) - \nabla^{0,i}(X)), \tag{1.94}$$

from where the following the following $\mathfrak{g}$-valued 1-forms, called the transition forms for the Lie algebroid atlas are defined:

$$\begin{aligned} \chi^i_j : TU_{ij} &\to U_{ij} \times \mathfrak{g} \\ X &\mapsto \psi_i^{-1} \circ l^i_j(X). \end{aligned} \tag{1.95}$$

With this maps, in $A|_{U_{ij}}$ the way to change from a local expression from another is given by

$$S_j(X \oplus \eta^j) = S_i(X \oplus [\alpha^i_j(\eta^j) + \chi^i_j(X)]). \tag{1.96}$$

In summary, for a Lie algebroid element that in the coordinate frame j has formula $X \oplus \eta^j$ and formula $X \oplus \eta^i$ in the coordinate frame j, two corrections in the vertical components are needed when the coordinate change from i to j is applied:

- one for the distinct notions of horizontality induced by the $\nabla^{0,i}$'s: $\chi^i_j(X)$,
- another for the distinct ways the adjoint Lie algebroid $L|_{U_{ij}}$ is trivialized as the bundle $U_{ij} \times \mathfrak{g}$: $\alpha^i_j(\eta^j)$.

The addition of both corrections gives $\eta^i = \chi^i_j(X) + \alpha^i_j(\eta^j)$, i.e.

$$\begin{aligned} S^i_j := S_i^{-1} \circ S_j : TU_{ij} \oplus (U_{ij} \times \mathfrak{g}) &\to TU_{ij} \oplus (U_{ij} \times \mathfrak{g}) \\ X \oplus \eta &\mapsto X \oplus (\chi^i_j(X) + \alpha^i_j(\eta)) \end{aligned} \tag{1.97}$$

The following properties must be satisfied by the pasting maps of a transitive Lie algeboid atlas:

Proposition 1.5.5. For the transitive Lie algebroid A with Lie algebroid atlas $\{(U_i, \psi_i : U_i \times \mathfrak{g} \to L|_{U_i}, \nabla^{0,i} : TU_i \to A|_{U_i})\}_{i\in I}$. On every $U_{ij} \neq \emptyset$:

- The transition forms $\chi^i_j : U_{ij} \to U_{ij} \times \mathfrak{g}$ defined in (1.95) are Maurer-Cartan forms.
- The transition functions of the adjoint Lie algebroid $\alpha^i_j : U_{ij} \times \mathfrak{g} \to U_{ij} \times \mathfrak{g}$ defined in (1.91) are LAB morphisms.
- Each α^i_j and χ^i_j are related by the following compatibility condition:

$$X(\alpha^i_j(\eta)) - \alpha^i_j(X(\eta)) + [\chi^i_j(X), \alpha^i_j(\eta)] = 0,$$

  for all $X \in TU_{ij}$ and $\eta \in U_{ij} \times \mathfrak{g}$.

Proof. This is nothing but the statement that the change of coordinate maps S^i_j defined in equation (1.97) are morphisms between trivial Lie algebroids (see example 1.4.3), which is the case because both S_i and S_j are invertible Lie algebroid morphisms. □

When making practical calculations with transitive Lie algebroids, the algebraic perspective of Lie algebroids is usually simpler to work with and more useful. Based on the fact that that the space of section $\Gamma(TU \times \mathfrak{g})$ of a transitive Lie algebroid over a manifold U is canonically isomorphic to $\Gamma(TU) \oplus C^\infty(U, \mathfrak{g})$, the following are the algebraic versions of the vector bundle maps defined in this section:

$$\begin{aligned} \psi_i : \quad & C^\infty(U_i, \mathfrak{g}) \to \Gamma_{U_i}(L) \\ \nabla^{0,i} : \quad & \Gamma(TU_i) \to \Gamma_{U_i}(A) \end{aligned}$$

$$\begin{aligned}
S_i : &\qquad \Gamma(TU_i) \oplus C^\infty(U_i, \mathfrak{g}) \to \Gamma_{U_i}(A)\\
\alpha^i_j : &\qquad C^\infty(U_{ij}, \mathfrak{g}) \to C^\infty(U_{ij}, \mathfrak{g})\\
\text{or } \alpha^i_j &\qquad \in C^\infty(U_{ij}, \mathfrak{gl}(\mathfrak{g}))\\
l^i_j : &\qquad \Gamma(TU_i) \to \Gamma_{U_i}(L)\\
\chi^i_j : &\qquad \Gamma(TU_i) \to C^\infty(U_{ij}, \mathfrak{g})\\
S^i_j : &\qquad \Gamma(TU_i) \oplus C^\infty(U_{ij}, \mathfrak{g}) \to \Gamma(TU_i) \oplus C^\infty(U_{ij}, \mathfrak{g}).
\end{aligned}$$

1.5.2 Local version of some concepts

Let us now see the local version of some important concepts.

Let $\psi : A \to A'$ be a morphism between transitive Lie algebroids. Once trivializing neighborhoods $\{U_i\}_{i\in I}$ for both A and A' have been chosen, ψ can be encoded as a family $\{(\omega_i, \psi_{+,i})\}_{i\in I}$ of representations $\psi_i = a \oplus (\omega_i + \psi_{+,i})$ between trivial Lie algebroids as suggested by example 1.4.19.

Similarly, let $\phi : A \to \mathfrak{D}(E)$ be a representation for the transitive Lie algebroid A, and suppose that $\{U_i\}_{i\in I}$ is an open cover trivializing both A and E. Based on example 1.4.19, the representation ϕ can be encoded as a family $\{(B_i, \phi_{L,i})\}_{i\in I}$ of representations $\phi_i = a \oplus (B_i + \phi_{L,i})$ of trivial Lie algebroids $TU_i \times \mathfrak{g}$ on trivial vector bundles $U_i \times V$, where $\mathfrak{g}$ is the typical fiber of the vertical part of A, and V is the typical fiber of E. The following examples show that the representations of the trivializing Lie algebroids of an algebroid that locally trivialize the trivial and adjoint representations, respectively, are already known representations.

Example 1.5.6. Let A be acting on the trivial vector bundle $E = M \times V$ by the trivial representation $\phi = a$. On a trivializing neighborhood U_i of A the trivialization of ϕ is once again a trivial representation:

$$\begin{aligned}
\phi_i(X \oplus \eta) &= \phi(S(X \oplus \eta))\\
&= a(S(X \oplus \eta))\\
&= a(X \oplus \eta);
\end{aligned}$$

to pass to the second line we used the fact that S respects the anchor. This, in particular, means that every B_i and $\phi_{L,i}$ are 0.

Example 1.5.7. Let A act on L through the adjoint representation $\phi = ad$. Then the local trivialization of ϕ over a neighborhood that trivializes A is again an adjoint representation. Assume without lose of generality that the injection j is an inclusion:

$$\begin{aligned}
\phi_i(X \oplus \eta)\theta &= \psi_i^{-1} \circ \phi(S_i(X \oplus \eta)) \circ \psi_i(\theta) \\
&= \psi_i^{-1}([S_i(X \oplus \eta), \psi_i(\theta)]) \\
&= \psi_i^{-1}([\nabla_X^{0,i} \oplus \psi_i(\eta), \psi_i(\theta)]) \\
&= \psi_i^{-1}[\nabla_X^{0,i}, \psi(\theta)] + \psi^{-1}[\psi_i(\eta), \psi_i(\theta)] \\
&= X(\theta) + [\eta, \theta] \\
&= [X \oplus \eta, \theta] \\
&= ad(X \oplus \eta, \theta).
\end{aligned}$$

An important family of representations are associated to the representation of a Lie group as seen in example 1.85. Let G be a group with Lie algebra $\mathfrak{g}$, and the vector space V be a representation space of G. Suppose that A is a transitive Lie algebroid with $\mathfrak{g}$ as the typical fiber of its vertical part, and let V be a vector space on which G acts. Let U_i and U_j, $U_{ij} \neq \emptyset$ be open subsets of M that trivialize A. If we allow $TU_i \times \mathfrak{g}$ and $TU_j \times \mathfrak{g}$ to be represented in $U_i \times V$ and $U_j \times V$, respectively, via the group representation (1.85), we can ask ourselves *¿what conditions need to be satisfied for those representations to be the trivializations of a global representation of A on a vector bundle E that trivializes over U_i and U_j?* Letting $\beta_j^i = \beta_i^{-1} \circ \beta_j : U_{ij} \times V \to U_{ij} \times V$ be the transition function of E from U_j to U_i, a quick analysis shows some conditions are imposed on them: let μ be a section on E such that $f \in C^\infty(U_j, V)$ is its trivialization over U_i, i.e. $\beta_i(f)$ be the trivialization of a section $\mu \in \Gamma(E)$, hence $\beta_j(f) = \mu|_{U_j}$, then the compatibility condition

$$\beta_j^i[(X \oplus \eta)f] = X \oplus (\chi_j^i(X) + \alpha_j^i(\eta))\beta_j^i(f) \tag{1.98}$$

must be satisfied for any $X \oplus \eta \in \Gamma(TU_i) \oplus C^\infty(U_{ij}, \mathfrak{g})$, where χ_j^i and α_j^i are the pasting maps (1.95) and (1.91) for A over U_{ij}. The previous equation decomposes in the following equations restricting the values of β_j^i:

$$\chi_j^i(X) \cdot \beta_j^i(f) = X(\beta_j^i)f \tag{1.99}$$

$$\alpha_j^i(\eta)(\beta_j^i(f)) = \beta_j^i(\eta \cdot f). \tag{1.100}$$

The fact that χ^i_j is a Maurer-Cartan form, that α^i_j is a LAB morphism, and the compatibility conditions between χ^i_j and α^i_j should impose further restrictions on what the transition function β^i_j can be, perhaps even characterizing them; however, on the literature that was reviewed no further analysis was found.

Definition 1.5.8. Let A be a transitive Lie algebroid with $\mathfrak{g}$ the typical fiber of the vertical subalgebroid, and let $\phi : A \to \mathfrak{D}(E)$ be a representation. We say that $\underline{\phi \text{ is the group induced representation on } E}$ if there is a representation $\pi : G \to Aut(V)$ on V of some Lie group whose Lie algebra is $\mathfrak{g}$, and if there is some Lie algebroid atlas over an open cover $\{U_i\}_{i\in I}$ that also trivializes E such that the local trivialization of ϕ over each U_i is $\phi_i = a \oplus \pi$, i.e. $B_i = 0$ and $\phi_{L,i} = \pi$ in the notation of example 1.4.19.

We will see in example 1.5.16 that in the case of the standard representation of a principal bundle on an associated vector bundle the representation is the group induced one.

1.5.3 Local Description of the Atiyah Lie algebroid

Let $G \to P \to M$ be a (smooth) principal bundle, and let $\mathfrak{g}$ be the Lie algebra of the Lie group G. Since this is the family of examples that appear in traditional gauge theories, some time will be devoted to its understanding.

Throughout this section $\{(U_i, \rho_i : U_i \times G \to P|_{U_i})\}_{i\in I}$ will denote a principal bundle atlas, $\mathcal{U}_i = P|_{U_i} = \pi^{-1}(U_i)$ and $\sigma_i : U_i \to \mathcal{U}_i$ will denote the corresponding trivializing local sections, which satisfy

$$\begin{aligned} \rho_i : U_i \times G &\to \mathcal{U}_i \\ (m, g) &\mapsto \sigma_i(m) g. \end{aligned}$$

Finally, let $g_{ij} : U_{ij} = U_i \cap U_j \to G$ be the transition functions of the atlas satisfying, for any $m \in U_{ij}$,

$$\begin{aligned} \sigma_i(m) &= \sigma_j(m) g_{ji}(m) \\ \rho_i(m, g) &= \rho_j(m, g_{ji}(m) g). \end{aligned} \tag{1.101}$$

From this principal bundle atlas, we will now build a Lie algebroid atlas for the Atiyah Lie algebroid TP/G. Since we are identifying the spaces of sections of the Atiyah sequence with some other $C^\infty(M)$-modules, we will also understand the relevant maps in terms of this spaces.

We will now build a Lie algebroid atlas on TP/G based on a trivialization of the principal bundle P. For the rest of this section suppose that the open cover $\{U_i\}_{i\in I}$ of M is associated to a manifold trivialization of M and to a principal bundle trivialization of P with local sections $\sigma_i : U_i \to P|_{U_i}$.

Recall the map j of the Atiyah sequence 1.3.21. For any $p \in P$ and $\eta \in \mathfrak{g}$, the immersion and injective Lie algebroid morphism j is defined by equation (1.32):

$$\begin{aligned} j : P \times \mathfrak{g}/G &\to T^\pi P/G \subseteq TP/G \\ \langle p, \eta\rangle &\mapsto \left\langle \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p \exp\left(-t\eta\right) \right\rangle . \end{aligned}$$

It induces the $C^\infty(M)$-module and Lie algebra morphism of equation (1.33):

$$\begin{aligned} \overline{j} : C^\infty_G(P, \mathfrak{g}) &\to \Gamma^G(TP) \\ \overline{j}(\widetilde{\boldsymbol{\eta}})(p) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p \exp(-t\widetilde{\boldsymbol{\eta}}(p)) \end{aligned}$$

Definition 1.5.9. For each $i \in I$, define the LAB morphism, i.e. vector bundle morphism that respects the fiberwise Lie brackets,

$$\begin{aligned} \psi_i : U_i \times \mathfrak{g} \to& \mathcal{U}_i \times \mathfrak{g}/G \\ (m, \eta) \mapsto& \langle \sigma_i(m), \eta\rangle \\ &= \langle \sigma_i(m)g, Ad_{g^{-1}}\eta\rangle \quad \text{for any } g \in G. \end{aligned} \tag{1.102}$$

It induces $C^\infty(M)$-module and Lie algebra morphism

$$\begin{aligned} \widetilde{\psi}_i : C^\infty(U_i, \mathfrak{g}) &\to C^\infty_G(\mathcal{U}_i, \mathfrak{g}) \\ \widetilde{\boldsymbol{\eta}} &\mapsto (\sigma_i(m)g \mapsto Ad_{g^{-1}}(\widetilde{\boldsymbol{\eta}}(m))) \quad \text{for any } m \in M,\, g \in G. \end{aligned} \tag{1.103}$$

Definition 1.5.10. For each $i \in I$, define the Lie algebroid morphism

$$\begin{aligned} \nabla^{0,i} : TU_i \to& T\mathcal{U}_i/G \\ X \mapsto& \langle \sigma_{i*}(X)\rangle . \end{aligned} \tag{1.104}$$

It induces $C^\infty(M)$-module and Lie algebra morphism

$$\begin{aligned}\overline{\nabla}^{0,i} : \Gamma(TU_i) &\to \Gamma^G(T\mathcal{U}_i) \\ \boldsymbol{X} &\mapsto (\sigma_i(m)g \mapsto R_{g*}\sigma_{i*}(X)),\end{aligned} \tag{1.105}$$

for all $m \in M$, $g \in G$.

Theorem 1.5.11. An open cover of M that corresponds to trivializes both the principal bundle $G \to P \to M$ and the manifold atlas of M produce a Lie algebroid atlas on $0 \to P \times \mathfrak{g}/G \xrightarrow{j} TP/G \xrightarrow{a} TM \to 0$ through the maps $\psi_i : U_i \times \mathfrak{g} \to P \times \mathfrak{g}/G|_{U_i}$ defined in 1.5.9, and the maps $\nabla^{0,i} : TU_i \to TP/G|_{U_i}$ defined in 1.5.10.

Proof. For any $m \in M$ and $\eta, \theta \in \mathfrak{g}$

$$\psi_i(m, [\eta, \theta]) = \langle \sigma_i(m), [\eta, \theta] \rangle = [\langle \sigma_i(m), \eta \rangle, \langle \sigma_i(m), \theta \rangle],$$

hence ψ is indeed a LAB morphism.

Since σ_i is a local section of the projection $\pi : P \to M$, $\pi_*(\sigma_{i*}(X)) = X$, hence $\nabla^{0,i}$ respects the anchor. is Lie algebroid morphism. That's what it induces

Given the definitions we have studied throught this chapter of the Atiyah Lie algebroid and its bracket, the compatibility condition of the suggested atlas reduces to verifying that $[\overline{\nabla^{0,i}}(\boldsymbol{X}), \overline{j\psi(\boldsymbol{\eta})}] = \overline{j}(\widetilde{\psi}_i(\boldsymbol{X}(\widetilde{\boldsymbol{\eta}}))) \in C^\infty_G(P)$ for all $\boldsymbol{X} \oplus \boldsymbol{\eta} \in \Gamma(TU_i) \oplus C^\infty(U_i, \mathfrak{g})$. This is done using the flow of $\overline{\nabla^{0,i}}(\boldsymbol{X})$ and the definition of the Lie derivative of vector fields on P. $\square$

The components of a Lie algebroid atlas combine to give the trivialization morphism:

Definition 1.5.12. For any $i \in I$, define the Lie algebroid morphism

$$\begin{aligned}S_i : TU_i \oplus (U_i \times \mathfrak{g}) \to& T\mathcal{U}_i/G \\ X \oplus \eta \mapsto& \nabla^{0,i}(X) + j\psi_i(\eta) \\ =& \left\langle \sigma_{i*}(X) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma_i(m) \exp(-t\eta) \right\rangle.\end{aligned} \tag{1.106}$$

This map induces the $C^\infty(M)$-linear and Lie algebra morphism:

$$\overline{S_i} : \Gamma(TU_i) \oplus C^\infty(U_i, \mathfrak{g}) \to \Gamma^G(T\mathcal{U}_i)$$

given by

$$\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}} \mapsto \left(\sigma_i(m)g \mapsto R_{g*}\sigma_{i*}(X) + \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \sigma_i(m)\exp\left(-t\widetilde{\boldsymbol{\eta}}(m)\right)g\right) \tag{1.107}$$

for any $m \in U_i$ and $g \in G$. This is called the trivializing Lie algebroid morphism of TP/G over U_i.

We now study the pasting maps between trivializations of the Atiyah Lie algebroid associated to the Lie algebroid atlas previously defined.

Proposition 1.5.13. For any $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, the transition maps of the adjoint Lie algebroid associated to the LAB atlas $\{\psi_i : U_i \times \mathfrak{g} \to \mathcal{U}_i \times \mathfrak{g}/G\}$ are

$$\begin{aligned} \alpha^i_j : U_{ij} \times \mathfrak{g} &\to U_{ij} \times \mathfrak{g} \\ (m, \eta) &\mapsto (m, Ad_{g_{ij}(m)}\eta). \end{aligned} \tag{1.108}$$

On sections it can induces the map,

$$\begin{aligned} \alpha^i_j : C^\infty(U_{ij}, \mathfrak{g}) &\to C^\infty(U_{ij}, \mathfrak{g}) \\ \widetilde{\boldsymbol{\eta}} &\mapsto Ad_{g_{ij}}(\widetilde{\boldsymbol{\eta}}) \end{aligned} \tag{1.109}$$

for all $\widetilde{\boldsymbol{\eta}} \in C^\infty(U_{ij}, \mathfrak{g})$. If G is matrix Lie group, this can also be written as:

$$\alpha^i_j(\widetilde{\boldsymbol{\eta}}) = g_{ij}\widetilde{\boldsymbol{\eta}}{g_{ij}}^{-1}. \tag{1.110}$$

Proof. For any $m \in U_{ij}$ and $\eta \in \mathfrak{g}$,

$$\begin{aligned} \alpha^i_j(m, \eta) &= \psi_i^{-1}(\langle \sigma_j(m), \eta \rangle) \\ &= \psi_i^{-1}(\langle \sigma_i(m)g_{ij}(m), \eta \rangle) \\ &= \psi_i^{-1}(\big\langle \sigma_i(m), Ad_{g_{ij}(m)}\eta \big\rangle) \\ &= (m, Ad_{g_{ij}(m)}\eta), \end{aligned}$$

as desired. □

Proposition 1.5.14. For any $i, j \in I$ such that $U_{ij} \neq \emptyset$, define the $\mathcal{U}_{ij} \times \mathfrak{g}/G$-valued forms on U_{ij}

$$
\begin{aligned}
l^i_j : TU_{ij} \to & \mathcal{U}_{ij} \times \mathfrak{g}/G \\
X \mapsto & j^{-1}(\nabla^{0,j}(X) - \nabla^{0,i}(X)) \\
= & \left\langle \sigma_i(m), L_{g_{ij}(m)*}(g^{-1}_{ij*}(X)) \right\rangle
\end{aligned}
\tag{1.111}
$$

for $m = \pi(X)$. If G is a matrix Lie group, it is then true that:

$$
l^i_j(X) = \left\langle \sigma_i(m), g_{ij}(m) dg^{-1}_{ij}(X) \right\rangle , \tag{1.112}
$$

where dg^{-1}_{ij} denotes the 1-form-valued matrix that results from applying on each entry the differential d of differential forms on U_{ij}.

Proof. Let $X \in T_m U_{ij}$ be tangent to the path γ on U_{ij} such that $\gamma'(0) = m \in TU_{ij}$. Then

$$
\begin{aligned}
\sigma_{i*}(X) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma_i(\gamma(t)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma_j(\gamma(t)) g^{-1}_{ij}(\gamma(t)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma_j(m) \gamma^{-1}_{ij}(\gamma(t)) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma_j(\gamma(t)) g^{-1}_{ij}(m) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma_i(m) \gamma_{ij}(m) \gamma^{-1}_{ij}(\gamma(t)) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma_j(\gamma(t)) g^{-1}_{ij}(m) \\
&= \left[L_{g_{ij}(m)*}(g^{-1}_{ij*}(X))\right]^*_{\sigma_i(m)} + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma_j(\gamma(t)) g^{-1}_{ij}(m);
\end{aligned}
$$

recall that the notation $[\eta]^*_p$ denotes the left invariant vector field on TP generated by $\eta \in \mathfrak{g}$ at $p \in P$. Now,

$$
\begin{aligned}
\nabla^{0,j}_X - \nabla^{0,i}_X &= \langle \sigma_{j*}(X) \rangle - \langle \sigma_{i*}(X) \rangle \\
&= \left\langle \sigma_{j*}(X) g^{-1}_{ij}(X) - \sigma_{i*}(X) \right\rangle \\
&= \left\langle -[L_{g_{ij}(m)*}(g^{-1}_{ij*}(X))]^*_{\sigma_i(m)} \right\rangle .
\end{aligned}
$$

Hence

$$
l^i_j(X) = j^{-1}(\nabla^{0,j}_X - \nabla^{0,i}_X)
$$

$$= \left\langle \sigma_i(m), L_{g_{ij}(m)*}(g_{ij*}^{-1}(X)) \right\rangle .$$

Recall that $L_{g_{ij}(m)*}(g_{ij*}^{-1}(X)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g_{ij}(m) g_{ij}^{-1}(\gamma(t))$. If G is a matrix Lie group, we can take $g_{ij}(m)$ outside the derivative and apply the derivative entry-wise, hence the last result follows. $\square$

Having a formula for the maps l_j^i the pasting 1-forms of the local trivialization are now clear:

Proposition 1.5.15. For any $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, the transition $\mathfrak{g}$-valued Maurer-Cartan forms over U_{ij} associated to the given Lie algebroid atlas of the Atiyah lie algebroid sequence $0 \to P \times \mathfrak{g}/G \to TP/G \to TM \to 0$ are

$$\begin{aligned} \chi_j^i : TU_{ij} \to & U_{ij} \times \mathfrak{g} \\ X \mapsto & \psi_i^{-1} \circ l_j^i(X) \\ = & \left(\sigma_i(m), L_{g_{ij}(m)*}(g_{ij*}^{-1}(X))\right) \end{aligned} \tag{1.113}$$

where $m = \pi(X)$ and L is the left multiplication map in G. If G is a matrix Lie group, on a vector field $\boldsymbol{X} \in \Gamma(TU_{ij})$ this becomes:

$$\chi_j^i(X) = \left(m, g_{ij}(m) d g_{ij}^{-1}(X)\right) . \tag{1.114}$$

If $\{\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}}^i\}$ is a family of local trivializations of $\mathfrak{X} \in \Gamma(TP/G)$, and supposing that G is a matrix Lie group, we may combine the previous propositions to find the following relation between local trivializations over open sets $U_{ij} \neq \emptyset$:

$$\boldsymbol{X} \oplus \widetilde{\boldsymbol{\eta}}^i = \boldsymbol{X} \oplus (g_{ij} \widetilde{\boldsymbol{\eta}}^j g_{ij}^{-1} + g_{ij} d g_{ij}^{-1}(\boldsymbol{X})) \tag{1.115}$$

Example 1.5.16 (Local trivialization of representation of TP/G on $E = P \times V/G$)**.** Suppose that the open U_i trivializes E as $\beta_i : U_i \times V \to E|_{U_i}$. Let $\eta \in C^\infty(U_i, \mathfrak{g})$, $f \in C^\infty(U_i, V)$ be arbitrary. Then

$$(\overline{\boldsymbol{S_i(X \oplus \eta)}})_{\sigma_i(m)} = \sigma_{i,*}(X) + \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \sigma_i(m) \cdot \exp[-t\eta(m)] \in T_{\sigma_i(m)} P$$

and

$$\overline{\boldsymbol{\beta_i(f)}} : \sigma_i(m) g \mapsto g^{-1} f(m)) \in C_G^\infty(P, V).$$

Therefore:

$$\begin{aligned}
\beta_i^{-1}[\phi \circ S_i(X \oplus \eta)(\beta_i(f))](m) &= \overline{\boldsymbol{\phi \circ S_i(X \oplus \eta)}}_{\sigma_i(m)}(\overline{\boldsymbol{\beta_i(f)}}) \\
&= \sigma_{i,*}(X_m)(\overline{\boldsymbol{\beta_i(f)}}) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \overline{\boldsymbol{\beta_i(f)}}(\sigma_i(m) \cdot \exp[-t\eta(m)]) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sigma_i(\gamma(t))(\overline{\boldsymbol{\beta_i(f)}}) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp[t\eta(m)]f(m) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t)) + (\eta \cdot f)(m) \\
&= X_m(f) + (\eta \cdot f)(m),
\end{aligned}$$

where $\gamma : I \to U_i$ is a path on M such that $\gamma'(0) = X_m$. We conclude that the local trivialization $\phi_i : TU_i \times \mathfrak{g} \to \mathfrak{D}(U_i \times V)$ of the representation $\phi : TP/G \to P \times V/G$ is

$$\phi_i(X \oplus \eta)(f) = X(f) + \eta \cdot f, \tag{1.116}$$

i.e. it is the group induced representation (see example 1.85 and Definition 1.5.8).

1.5.4 Examples

We have seen 1.3.21 that to every principal bundle $G \to P \to M$ there is an associated a transitive Lie algebroid with Atiyah sequence $0 \to P \times \mathfrak{g}/G \to TP/G \to TM \to 0$. Furthermore, theorem 1.5.11 states that local trivializations of P and M induce local trivializations of the Lie algebroids $P \times \mathfrak{g}/G$ and TP/G. We will now give a description of some principal bundles over projective spaces isomorphic to spheres which will have a trivializations with only 2 charts for both the manifold and the principal bundle, each of which covers all but one point. This easily shows us that a version of the reconstruction theorem applies, at least for these Lie algebroids over spheres, where *the necessary maps to reconstruct the Lie algebroid are the transition maps α_j^i of 1.5.13 and χ_j^i of 1.5.14.* The α maps are the transition functions of the adjoint Lie algebroid $P \times \mathfrak{g}/G$, so our first task will be to understand the Lie algebras $\mathfrak{g}$ to then understand the trivializations and the transition functions.

The examples that will be developed throughout this document are those associated to the principal bundles $S^1 \to P^k \to S^2$ and $S^3 \to P^k \to S^4$ One of our examples, the Atiyah Lie algebroid associated to the complex Hopf bundle, has a simple structure that allows it to be pictured without using local trivializations, so we will develop in this "global" language mainly to acquire intuition on the manipulations that are made in this context.

Lie algebras and transition functions of adjoint algebroid

The Lie algebra of the Lie group S^1 is $i\mathbb{R} \subseteq \mathbb{C}$, which can be easily seen since every element of $g \in S^1$ may be written as $g = e^{ir}$ for some $r \in \mathbb{R}$. A basis of this algebra is simply $\{i\} \subseteq i\mathbb{R}$, with only one element and so the Lie bracket is trivially 0. S^1 can be canonically identified with $U(1)$, so its Lie algebra can be seen embedded in the associative matrix Lie algebra $iM_1(\mathbb{R}) \subseteq M_1(\mathbb{C})$, so the adjoint action of any $g \in U(1)$, $Ad_g : iM_1(\mathbb{R}) \to iM_1(\mathbb{R})$, is given by the matrix multiplication $(ir) \mapsto g(ir)g^{-1}$, but $M_1(\mathbb{C})$ is a commutative algebra, implying that

$$Ad_g = Id : i\mathbb{R} \to i\mathbb{R}$$

for any $g \in S^1$.

Hence, the first pasting map, the transition functions α_S^N of 1.5.13 for the Atiyah Lie algebroids $0 \to P^k \times i\mathbb{R}/S^2 \to TP^k/S^1 \to TS^2 \to 0$, $k \in \mathbb{Z}_{\geq 0}$, are

$$\alpha_S^N = Id = \alpha_N^S, \tag{1.117}$$

where this identity may refer to both the identity as LAB maps $\alpha_j^i : U_{SN} \times i\mathbb{R} \to U_{SN} \times i\mathbb{R}$, as well as elements of $C^\infty(U_{SN}, \mathfrak{gl}(i\mathbb{R}))$.

To understand the Lie algebra associated to our second family of principal bundles, we first study the space $Im\,\mathbb{H} := \{xi + yj + zk \mid (x, y, z) \in \mathbb{R}^3\} \subseteq \mathbb{H}$, whose elements we denote, by abuse of notation, $\vec{x} := xi + yj + zk \in Im\,\mathbb{H} \cong \mathbb{R}^3$. The product of two elements $\vec{x}, \vec{y}$ of this subspace of the associative normed algebra $\mathbb{H}$ is

$$\begin{aligned}\vec{x}\vec{y} &= \frac{1}{2}(\vec{x}\vec{y} - \vec{y}\vec{x}) + \frac{1}{2}(\vec{x}\vec{y} + \vec{y}\vec{x}) \\ &= \vec{x} \times \vec{y} + \vec{x} \cdot \vec{y} \in \mathbb{H}\end{aligned}$$

where the $\times$ and $\cdot$ operations are the cross product and dot product of the corresponding vectors in $\mathbb{R}^3$. The induced commutator is, then,

$$[\vec{x}, \vec{y}] = 2(\vec{x} \times \vec{y}) \in Im\,\mathbb{H}, \tag{1.118}$$

showing that $Im\,\mathbb{H}$ is a real Lie algebra, which has as basis the set $\{i, j, k\}$ that satisfy the commutation relations

$$[i, j] = 2k \qquad [j, k] = 2i \qquad [k, i] = 2i. \tag{1.119}$$

We now wish to see what is the (unique) simply connected Lie group $G \subseteq \mathbb{H}$ associated to the Lie algebra $Im\,\mathbb{H}$. Given an arbitrary $\vec{x} \in Im\,\mathbb{H}$, the exponential $e^{x^1 i + x^2 j + x^3 k}$ is unitary, so the Lie group G we search is a subgroup of $S(\mathbb{H})$. Now, recall that the real Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ of anti-hermitian matrices is generated by the basis $\{u_1, u_2, u_3\} \subseteq \mathfrak{su}(\mathfrak{2})$ where $u_i = i\sigma_i$, $i = 1, 2, 3$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.120}$$

where this basis satisfy the bracket in $\mathfrak{su}(2)$ is determined by the relations

$$[u_1, u_2] = 2u_3 \qquad [u_2, u_3] = 2u_1 \qquad [u_3, u_1] = 2u_2. \tag{1.121}$$

hence the following is an isomorphism of real Lie algebras

$$\begin{aligned} Im\,\mathbb{H} &\xrightarrow{\cong} \mathfrak{su}(2) \\ i &\longmapsto i\sigma_1 \\ j &\longmapsto i\sigma_2 \\ k &\longmapsto i\sigma_3 \\ \vec{x} &\longmapsto i\vec{x} \cdot \vec{\sigma} = \begin{pmatrix} ix^3 & x^2 + ix^1 \\ -x^2 + ix^1 & -ix^3 \end{pmatrix}. \end{aligned} \tag{1.122}$$

This algebra isomorphism induces the isomorphism of smooth Lie groups

$$\begin{aligned} G \subseteq S(\mathbb{H}) &\xrightarrow{\cong} SU(2) \cong S^3 \\ e^{x^1 i + x^2 j + x^3 k} &\longmapsto e^{i(x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3)}, \end{aligned} \tag{1.123}$$

but then the simply-connected Lie group G associated to the Lie algebra $Im\,\mathbb{H}$ is precisely $S(\mathbb{H}) \equiv S^3$. All of this allows us to conclude that we can naturally use $Im\,\mathbb{H}$ as the Lie algebra for S^3.

Finally, for every $e^{\vec{x}} = \cos|\vec{x}| + \hat{x}\sin|\vec{x}| \in S^3$ the adjoint action

$$\begin{aligned} Ad_{e^{\vec{x}}} : Im\,\mathbb{H} &\to Im\,\mathbb{H} \\ \vec{y} &\mapsto e^{\vec{x}}\,\vec{y}\,e^{-\vec{x}} = R_{\hat{x},|\vec{x}|}(\vec{y}) \end{aligned} \tag{1.124}$$

is the rotation operation of $|\vec{x}|$ radians in $\mathbb{R}^3$ with respect to the axis $\hat{x}$.

Transition algebra-valued forms

To complete the description of the Atiyah Lie algebroid associated to the principal bundles P^k from transition functions given the open cover $\{U_S, U_N\}$ we only need the Lie algebra-valued Maurer-Cartan forms $\chi_S^N = -\chi_N^S$ found in Proposition 1.5.14 to be $\chi_S^N = g_{NS}dg_{NS}^{-1} = -\chi_N^S$ since the structure groups S^3 and S^1 are isomorphic to the matrix Lie groups $SU(2)$ and $U(1)$, respectively.

For the Lie algebroids over S^2, with transition function $g_{NS} = e^{ik\theta} \in C^\infty(U_{SN}, S^1)$ it then follows that the form $\chi_S^N : \Gamma_{U_{SN}} \to C^\infty(U_{SN}, i\mathbb{R})$

$$\begin{aligned} \chi_S^N &= -ikd\theta, \\ \chi_N^S &= +ikd\theta. \end{aligned} \tag{1.125}$$

This allows us to conclude that the change of coordinate maps between local trivializations of the Atiyah Lie algebroid bundles over S^2 are

$$\begin{aligned} S_S^N(X \oplus \widetilde{\boldsymbol{\eta}}) &= X \oplus (\widetilde{\boldsymbol{\eta}} - ikd\theta(X)) \\ S_N^S(X \oplus \widetilde{\boldsymbol{\eta}}) &= X \oplus (\widetilde{\boldsymbol{\eta}} + ikd\theta(X)) \end{aligned} \tag{1.126}$$

In particular,

$$S_N^S(\partial_\theta) = \partial_\theta + ik$$

is the only element of the local frame $\{\partial_\phi, \partial_\theta, i\} \subseteq \Gamma(TU_{SN}) \oplus C^\infty(U_{SN})$ of the trivializing trivial Lie algebroid over U_{SN} that changes forms.

Let us now focus on the complex Hopf bundle momentarily. Notice that the local frame $\{\partial_\phi, \partial_{\xi^1}, \partial_{\xi^2}\} \in \Gamma_{\mathcal{U}_{SN}}(TS^3)$, *for the coordinates* ϕ, χ^1, χ^2 *on* S^3 *defined in* (1.51), *is made of* S^1*-invariant vector fields, hence they induce the following local frame of* TS^3/S^1*:*

$$\{\langle\partial_\phi\rangle, \langle\partial_{\xi^1}\rangle, \langle\partial_{\xi^2}\rangle\} \in \Gamma_{U_{SN}}(TS^3/S^1),$$

the notation of Proposition 1.3.6 has been used. Having the local frame allows to express a section of TS^3/S^1 with functions of S^2 as components. *A local section over* U_{SN} *may be expressed as*:

$$\mathfrak{X}^{\phi}(\phi,\theta)\,\langle\partial_\phi\rangle + \mathfrak{X}^{\xi^1}(\phi,\theta)\,\langle\partial_{\xi^1}\rangle + \mathfrak{X}^{\xi^2}(\phi,\theta)\,\langle\partial_{\xi^2}\rangle \in \Gamma_{U_{SN}}(TS^3/S^1)$$

for $\mathfrak{X}^{\phi}, \mathfrak{X}^{\xi^1}, \mathfrak{X}^{\xi^2} \in C^\infty(S^2)$; notice that this is a section of the Lie algebroid TS^3/S^1, and *not* of the trivial Lie algebroid $TS^2 \times i\mathbb{R}$ that we would have to use if we were to work with trivializations of the Lie algebroid. Since in our case the adjoint bundle is trivial with global frame $i \in C^\infty(S^2, i\mathbb{R}) \cong \Gamma_{U_{SN}}(S^3 \times i\mathbb{R}/S^1)$, *a global section of the adjoint bundle* $S^3 \times i\mathbb{R}/S^1$ *may be expressed as*

$$r(\phi,\theta)i \tag{1.127}$$

with $r(\phi,\theta) \in C^\infty(S^2)$, *hence the adjoint bundle of the Atiyah Lie algebroid of the complex Hopf bundle will be studied through the module* $C^\infty(S^2, i\mathbb{R})$. The precise isomorphism is:

$$\begin{aligned} C^\infty(S^2) &\mapsto C^\infty_{S^1}(S^3) \\ f &\mapsto f \circ \pi. \end{aligned} \tag{1.128}$$

We now give an intuitive description of the different maps that have been defined on the Atiyah Lie algebroid TS^3/S^1 associated to the complex Hopf bundle over $U_{SN} = U_S \cap U_N$ without using auxiliary trivializing Lie algebroids. We will ignore adding to each map an indication that it is the restriction of a map to U_{SN}.

First, let us understand the horizontal part of the Lie algebroid atlas defined for Atiyah Lie algebroids. The local sections of the complex Hopf bundle satisfy:

$$\sigma_S : U_{SN} \to S^3, \qquad E(\phi,\theta) \mapsto T(\phi,\theta,0) = (\cos\frac{\phi}{2}e^{i\theta}, \sin\frac{\phi}{2});$$
$$\sigma_N : U_{SN} \to S^3, \qquad E(\phi,\theta) \mapsto T(\phi,0,-\theta) = (\cos\frac{\phi}{2}, \sin\frac{\phi}{2}e^{-i\theta}).$$

It is then straightforward to calculate the pushforward maps:

$$\sigma_{S*} : \Gamma_{U_{SN}}(TS^2) \to \Gamma_{\mathcal{U}_{SN}}(TS^3), \qquad \partial_\phi \mapsto \partial_\phi, \qquad \partial_\theta \mapsto \partial_{\xi^1};$$

$$\sigma_{N*} : \Gamma_{U_{SN}}(TS^2) \to \Gamma(TS^3)_{\mathcal{U}_{SN}}, \qquad \partial_\phi \mapsto \partial_\phi, \qquad \partial_\theta \mapsto -\partial_{\xi^2}.$$

The resulting G-invariant vector fields induce the following trivializing Lie algebroid morphisms, and flat connections, $\nabla^{0,i}$ (Definition 1.5.10) $\Gamma_{U_{SN}}(TS^3/S^1)$:

$$\nabla^{0,S} : \Gamma_{U_{SN}}(TS^2) \to \Gamma_{U_{SN}}(TS^3/S^1), \quad \partial_\phi \mapsto \langle \partial_\phi \rangle, \quad \partial_\theta \mapsto \langle \partial_{\xi^1} \rangle; \tag{1.129}$$
$$\nabla^{0,N} : \Gamma_{U_{SN}}(TS^2) \to \Gamma_{U_{SN}}(TS^3/S^1), \quad \partial_\phi \mapsto \langle \partial_\phi \rangle, \quad \partial_\theta \mapsto -\langle \partial_{\xi^2} \rangle. \tag{1.130}$$

Hence, the following change of local trivialization applies

$$S_N^S(\partial_\phi) = \partial_\phi, \qquad S_N^S(\partial_\theta) = \partial_\theta + i, \qquad S_N^S(i) = i.$$

Now, for the vertical part of the Lie algebroid atlas of TS^3/S^1, first notice that the injection map $j : S^3 \times i\mathbb{R}/S^1 \to TS^3/S^1$ is determined by its action on the global frame $\{i\}$, so at any given point $E(\phi,\theta) \in S^2$:

$$\begin{aligned}
j(i) &= \left\langle \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T(\phi,\xi^1,\xi^2)e^{-it} \right\rangle \\
&= \left\langle \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T(\phi,\xi^1 - t,\xi^2 - t) \right\rangle \\
&= \left\langle T_{*,(\phi,\xi^1,\xi^2)}(0,-1,-1) \right\rangle \\
&= \left\langle T_{*,(\phi,\xi^1,\xi^2)}(0,-1,0) + T_{*,(\phi,\xi^1,\xi^2)}(0,0,-1) \right\rangle \\
&= \left\langle \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T(\phi,\xi^1 - t,\xi^2) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T(\phi,\xi^1,\xi^2 - t) \right\rangle \\
&= \left\langle -\partial_{\xi^1 | T(\phi,\xi^1,\xi^2)} - \partial_{\xi^2 | T(\phi,\xi^1,\xi^2)} \right\rangle \\
&= -\langle \partial_{\xi^1} \rangle_{|E(\phi,\theta)} - \langle \partial_{\xi^2} \rangle_{|E(\phi,\theta)}.
\end{aligned}$$

Hence, j as a map between sections satisfies:

$$j : C^\infty(U_{SN}, i\mathbb{R}) \to \Gamma_{U_{SN}}(TS^3/S^1) \tag{1.131}$$
$$if(\phi,\theta) \mapsto -f(\phi,\theta)(\langle \partial_{\xi^1} \rangle + \langle \partial_{\xi^2} \rangle). \tag{1.132}$$

The fact that the Lie algebra is commutative implied that the adjoint action of S^1 on $i\mathbb{R}$ was trivial, which also implies that the global section $i \in \Gamma(S^3 \times i\mathbb{R}/S^1)$ is S^1-invariant and, therefore, that the isomorphism $\Gamma_U(S^3 \times i\mathbb{R}/S^1) \cong C^\infty(U_{SN}, i\mathbb{R})$ is given simply by mapping the section i to the function i, hence that the LAB trivializing maps are trivial:

$$\psi_S = \psi_N = Id : C^\infty(U_{SN}, i\mathbb{R}) \to \Gamma_U(S^3 \times i\mathbb{R}/S^1).$$

Thus, the vertical trivialization maps $j\psi_S$ and $j\psi_N$ satisfy:

$$\begin{aligned} j\psi_S = j\psi_N : C^\infty(U_{SN}, i\mathbb{R}) &\to \Gamma_{U_{SN}}(TS^3/S^1) \\ if &\mapsto -f(\langle\partial_{\xi^1}\rangle + \langle\partial_{\xi^2}\rangle). \end{aligned} \tag{1.133}$$

We thus conclude that the trivializing Lie algebroid morphisms

$$S_S, S_N : \Gamma_{U_{SN}}(TS^2) \oplus C^\infty(U_{SN}, i\mathbb{R}) \to \Gamma(TS^3/S^1)$$

defined in 1.5.12 by the Lie algebroid atlas on TS^3/S^1 induced by the principal bundle atlas have the following formulas for an arbitrary $X^\phi(\phi,\theta)\partial_\phi + X^\theta(\phi,\theta)\partial_\theta + if(\phi,\theta) \in \Gamma_{U_{SN}}(TS^2) \oplus C^\infty(U_{SN}, i\mathbb{R})$:

$$S_S : X^\phi\partial_\phi + X^\theta\partial_\theta + if \mapsto X^\phi\left\langle\partial_\phi\right\rangle + X^\theta\left\langle\partial_{\xi^1}\right\rangle - f(\langle\partial_{\xi^1}\rangle + \langle\partial_{\xi^2}\rangle), \tag{1.134}$$

and

$$S_N : X^\phi\partial_\phi + X^\theta\partial_\theta + if \mapsto X^\phi\left\langle\partial_\phi\right\rangle - X^\theta\left\langle\partial_{\xi^2}\right\rangle - f(\langle\partial_{\xi^1}\rangle + \langle\partial_{\xi^2}\rangle). \tag{1.135}$$

Since we are using local frames on both sides, we may write

$$[S_S]^{(\langle\partial_\phi\rangle,\langle\partial_{\xi^1}\rangle,\langle\partial_{\xi^2}\rangle)}_{(\partial_\phi,\partial_\theta,i)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} = [S_S^{-1}]^{(\partial_\phi,\partial_\theta,i)}_{(\langle\partial_\phi\rangle,\langle\partial_{\xi^1}\rangle,\langle\partial_{\xi^2}\rangle)}. \tag{1.136}$$

This means, in particular, that

$$\begin{aligned} S_S(\partial_\phi) &= \left\langle\partial_\phi\right\rangle, & \left\langle\partial_\phi\right\rangle &= S_S(\partial_\phi), \\ S_S(\partial_\theta) &= \left\langle\partial_{\xi^1}\right\rangle, & \left\langle\partial_{\xi^1}\right\rangle &= S_S(\partial_\theta), \\ S_S(i) &= -\left\langle\partial_{\xi^1}\right\rangle - \left\langle\partial_\xi^2\right\rangle, & \left\langle\partial_{\xi^2}\right\rangle &= S_S(-\partial_\theta - i); \end{aligned}$$

and

$$\begin{aligned} S_N(\partial_\phi) &= \left\langle\partial_\phi\right\rangle, & \left\langle\partial_\phi\right\rangle &= S_N(\partial_\phi), \\ S_N(\partial_\theta) &= -\left\langle\partial_{\xi^2}\right\rangle, & \left\langle\partial_{\xi^1}\right\rangle &= S_N(\partial_\theta - i), \\ S_N(i) &= -\left\langle\partial_{\xi^1}\right\rangle - \left\langle\partial_\xi^2\right\rangle, & \left\langle\partial_{\xi^2}\right\rangle &= S_N(-\partial_\theta). \end{aligned}$$

Chapter 2

Differential Forms

In this chapter we define differential forms on general Lie algebroids, and study their structure. Their role in our goal to define gauge theories on a transitive Lie algebroid A will be two-fold: definition of various notions of connections on A, and definition of integration on A. Firstly, in Chapter 3 we define connections on A, A-connections on representation vector bundles E, and their curvatures, and all will have a formulation as forms on A with values in different representation vector bundles. Secondly, in Chapter 4 we define two notions of integration of differential forms on A, each one associated to a volume form on A. The combination of these concepts will give rise to the action function defining a gauge theory on A, as the integral of a sum of differential forms on A.

The theory for this chapter came mostly from [5] and [4]. In section 2.1, we complement this exposition with a clear framework to which the various differential forms observed in these articles are part of in order to correctly understand its properties and possible manipulations. Then, on section 2.2 some time is invested into acquiring a firm understanding of the spaces of forms on trivial Lie algebroids, since this enables a concrete study of the differential forms on transitive Lie algebroids from their trivializations. Finally, we apply the theory of this chapter to the two families of examples of transitive Lie algebroids developed throughout this book.

2.1 Spaces of differential forms and their Structure

Throughout this section, let A be a Lie algebroid over the manifold M with anchor a, not necessarily transitive.

Forms on A will be a generalization the concept of vector valued forms on a manifold, they will be multilinear vector bundle maps on A taking values on representation vector bundles E. Which representation space E is relevant will depend on the context, for example E will be an adjoint Lie algebroid of A when dealing with connections on A and its curvature, but it will be $End(E)$ when we talk about A-connections on E, a generalized version of covariant derivatives.

In order for the definitions of differential forms to be useful, we need to understand their structure and how to manipulate them. In general, the space of E-valued forms will have the structure of a differential complex, but when E is an algebra bundle additional structure will arise, making of the space of differential forms a differential graded algebra. We devote this section to laying out the details to make sense of these statements.

2.1.1 Module of differential forms

Definition 2.1.1. Let E be a vector bundle over M. Let us now define the following vector bundles:

- Let $\underline{Alt^0(A,E)} := E$.
- For $n \in \mathbb{Z}_{\geq 1}$, let $\underline{Alt^n(A,E)}$ be the natural embedding, as alternating maps, of the (hom-)bundle $Hom(\bigwedge^n A, E)$ into the vector bundle $Hom(\bigotimes^n A, E)$ (as defined according to [10]) over M. Its fiber at $m \in M$ is, then, the vector space of alternating linear transformations $\{\omega : \underbrace{A_m \otimes \cdots \otimes A_m}_{n \text{ times}} \to E_m \mid \omega \text{ is } \mathbb{R}\text{-linear and alternating}\}$.
- Let $\underline{Alt^\bullet(A,E)} := \bigoplus_{n=0}^{\infty} Alt^n(A,E)$.

Remark 2.1.2. A local section over the open $U \subseteq M$ of a hom-bundle $Hom(E,E')$, for E and E' vector bundles over M, is a vector bundle morphism : $E|_U \to E|_{U'}$ between vector bundles over U. Naturally, then, the elements of $\Gamma_U(Hom(E, E'))$ are equivalent to $C^\infty(U)$-linear maps : $\Gamma_U(E) \to \Gamma_U(E)$.

It is important to realize that a local section of a hom-bundle $Hom(E, E')$ need not have an extension to a full vector bundle morphism : $E \to E'$.

Notice that $Alt^\bullet(A, E)$ is indeed a vector bundle, with finite dimensional fibers, since, if $U \subseteq M$ is a trivializing neighborhood of both A and E with fibers V and W respectively, $Alt^\bullet(A, E)|_U \cong U \times Hom(V, \bigwedge^* W)$ is a trivial vector bundle.

Definition 2.1.3. Let E be a vector bundle over M on which there is a representation $\phi : A \to \mathfrak{D}(E)$ of A. Let $U \subseteq M$ be open:

- For $n \in \mathbb{Z}_{\geq 0}$, define the $C^\infty(U)$-module $\underline{\Omega^n_U(A, E)}$:$= \Gamma_U\, Alt^n(A, E)$. Its elements are called local E-valued n-forms on A over U. If $U = M$, simply write $\underline{\Omega^n(A, E)}$ and call its elements E-valued n-forms on A.
- Define the $C^\infty(U)$-module $\underline{\Omega^\bullet_U(A, E)} := \bigoplus_{n=0}^\infty \Omega^n_U(A, E)$. Its elements are simply called local E-valued forms on A over U. If $U = M$, simply write $\underline{\Omega^\bullet(A, E)}$ and call its elements E-valued forms on A.

Remark 2.1.4. Due to the bijective correspondence between $C^\infty(U)$-linear maps between local sections over U of vector bundles and vector bundle morphisms of the bundles restricted to U, $\Omega^n_U(A, E)$ may be considered the space of $C^\infty(U)$-multilinear antisymmetric maps from $\Gamma_U(A)^n$ to $\Gamma_U(E) = \Omega^0_U(A, E)$. We will refer as (local) E-valued n-forms on A to both

$$\omega : \underbrace{A|_U \otimes \cdots \otimes A|_U}_{n \text{ times}} \to E|_U, \qquad \text{alternating, } \mathbb{R}\text{-vector bundle map and} \tag{2.1}$$

$$\omega : \underbrace{\Gamma_U(A) \times \cdots \times \Gamma_U(A)}_{n \text{ times}} \to \Gamma_U(E), \qquad \text{alternating, } C^\infty(U)\text{-multilinear map.} \tag{2.2}$$

Example 2.1.5. On the trivial vector bundle $M \times \mathbb{R}$, the associated space of differential forms on A is denoted by $\Omega^\bullet(A)$, and it is called the space of scalar valued forms on A. Notice that the module $\Omega^n(A) = \Gamma \bigwedge^n A^*$.

This is the fundamental example of differential forms on an algebroid A. It has been used to define a cohomology theory on Lie algebroids and .

Example 2.1.6. Let $0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0$ be a transitive Lie algebroid sequence. The space $\Omega^\bullet(A, L)$ of L-valued forms on A will be an important example since (generalized) algebroid connections and their curvatures are particular cases of this type of differential forms on A.

Proposition 2.1.7. Let $U \subseteq M$ be open:

- $\mathit{Alt}^n(A, E)|_U$ and $E|_U \otimes \bigwedge^n A^*|_U$ are isomorphic vector bundles.
- $\Omega^n_U(A, E)$ *and* $\Gamma_U(E) \otimes \Omega^n_U(A)$ *are isomorphic* $C^\infty(U)$-modules.
- If U is trivializing neighborhood for both A and E, with typical fibers V and W respectively, then $\Omega^n_U(A, E) \cong C^\infty(U, V \otimes \bigwedge^\bullet W^*)$.

Proof. We may assume simply that $U = M$, since using an arbitrary U simply means that we are working on manifolds over U. The first part follows from $\mathit{Alt}^n(A, E) \cong \mathit{Hom}(\bigwedge^n A, E)$, by definition, and so $\mathit{Hom}(\bigwedge^n A, E) \cong E \otimes (\bigwedge^n A)^* \cong E \otimes \bigwedge^n A^*$, but $\bigwedge^n A^* = \Omega^n(A)$.

The second part follows from the first one, since the Γ functor on the category of vector bundles distributes over the tensor product.

Lastly, $\Omega^n_U(A, E) \cong C^\infty(U, V) \otimes_{C^\infty(U)} \Gamma(U \times \bigwedge^n W^*) \cong C^\infty(U, V) \otimes_{C^\infty(M)} C^\infty(U, \bigwedge^n W^*)$. $\square$

2.1.2 Differential Complexes

In the following, assume that E is a vector bundle over M on which there is a representation $\phi : A \to \mathfrak{D}(E)$ of the Lie algebroid A over M.

Definition 2.1.8. Let R be a commutative ring with unit, and let M be a R-module. A differential on M is a morphism $d : M \to M$ of R-modules such that $d \circ d = 0$; then, we call (M, d) a differential module.

If, furthermore, $M = \oplus_{n=0}^\infty M^n$ is a graded module over the graded ring $R = \oplus_{n=0}^\infty R^n$, i.e. a R-module such that $R^p \cdot M^q \subseteq M^{p+q}$, and $d(M^n) \subseteq M^{n+1}$ for all $n \in \mathbb{Z}_{\geq 0}$, then we call (M, d) a differential complex or a differential graded module. The elements in M^n are called homogeneous elements of degree n, for all $n \in \mathbb{Z}_{\geq 0}$, with the degree n of $\omega \in M^n$ denoted by $|\omega|$. A morphism of differential complexes over R between (M, d) and (M', d') is a graded R-module morphism such that $d' \circ \phi = \phi \circ d$.

Definition 2.1.9. Let E be a vector bundle over M on which there is a representation $\phi : A \to \mathfrak{D}(E)$ of A. Define the differential of E-valued forms on A by the Koszul formula:

$$\hat{d}_\phi : \Omega^\bullet(A, E) \to \Omega^{\bullet+1}(A, E)$$

$$\begin{aligned}(\hat{d}_\phi\omega)(\boldsymbol{\mathfrak{X}}_1, \ldots, \boldsymbol{\mathfrak{X}}_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}\phi(\boldsymbol{\mathfrak{X}}_i) \cdot \omega(\boldsymbol{\mathfrak{X}}_1, \cdots, \overset{\vee}{\boldsymbol{\mathfrak{X}}}_i, \cdots, \boldsymbol{\mathfrak{X}}_{p+1}) \\ &+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j}\omega([\boldsymbol{\mathfrak{X}}_i, \boldsymbol{\mathfrak{X}}_j], \boldsymbol{\mathfrak{X}}_1, \cdots, \overset{\vee}{\boldsymbol{\mathfrak{X}}}_i, \cdots, \overset{\vee}{\boldsymbol{\mathfrak{X}}}_j, \cdots, \boldsymbol{\mathfrak{X}}_{p+1})\end{aligned} \tag{2.3}$$

for any $p \in \mathbb{Z}_{\geq 0}$, p-form ω, and any $\boldsymbol{\mathfrak{X}}_1, \ldots, \boldsymbol{\mathfrak{X}}_{p+1} \in \Gamma(A)$.

Proposition 2.1.10. $(\Omega^\bullet(A, E), \hat{d}_\phi)$ is a differential complex over the ring $\mathbb{R}$.

Proof. Notice that, although $C^\infty(M)$ is also a commutative unit ring, $\hat{d}_\phi$ isn't $C^\infty(M)$ linear since the maps $\phi(\boldsymbol{\mathfrak{X}}_i) : \Gamma(E) \to \Gamma(E)$ may not be so; they are, however, $\mathbb{R}$-linear and this proves the $\mathbb{R}$-linearity of $\hat{d}_\phi$. That $\hat{d}_\phi$ raises the degree of the homogeneous elements is part of their definition.

All what's left to chceck is that $\hat{d}_\phi \circ \hat{d}_\phi = 0$. The formula used to define $\hat{d}_\phi$ guarantees this as long as: the bracket applied to sections on A satisfies the Jacobi identity, and each $\phi(\boldsymbol{\mathfrak{X}}_i) : \Gamma(E) \to \Gamma(E)$ is a Lie algebra morphism [1]. $\square$

Suppose that $\omega \in \Omega^p(A, E)$ is the 0 section on $\Gamma_U(Alt^p(A, E))$, for $p \in \mathbb{Z}_{\geq 0}$ arbitrary. Then, equation 2.3 shows us that $(\hat{d}_\phi\omega)|_U = 0$ since every term on the left will be 0. Hence, the $\mathbb{R}$-linear map $\hat{d}_\phi : \Gamma\, Alt^\bullet(A, E) \to \Gamma\, Alt^\bullet(A, E)$ is a local operator. This fact implies that on every $U \subseteq M$ open, the differential $\hat{d}_\phi$ has a restriction $\hat{d}_\phi|_U : \Omega^\bullet_U(A, E) \to \Omega^\bullet_U(A, E)$ that satisfies, for every $\omega \in \Omega^\bullet(A, E)$,

$$\hat{d}_\phi|_U(\omega|_U) = (\hat{d}_\phi\omega)|_U. \tag{2.4}$$

Hence the next theorem follows.

Theorem 2.1.11. $(\Omega^\bullet_U(A,E), \hat{d}_\phi|_U) = (\Omega^\bullet(A|_U, E|_U), \hat{d}_\phi|_U)$ is a differential complex over the ring $\mathbb{R}$.

Remark 2.1.12. Thanks to this theorem, every result and definition that is stated for global forms can and will be extended to local forms over every $U \subseteq M$ open, since $(\Omega^\bullet_U(A,E), \hat{d}_\phi|_U)$ is simply the differential complex where the underlying manifold is U. Every local differential $\hat{d}_\phi|_U$ will usually be denoted simply by $\hat{d}_\phi$.

Example 2.1.13. Let $A = TM$ given a manifold M. Then $(\Omega^\bullet(TM), \wedge, \hat{d}_{TM})$ is precisely the usual differential graded algebra and module $(\Omega^\bullet(M), \wedge, d)$ of differential forms on M.

Letting $E = M \times V$ given a vector space V, on which TM is represented trivially by $i : TM \to \mathfrak{D}(E)$, $X \mapsto X$, the differential complex $(\Omega^\bullet(TM, M \times V), \hat{d}_i)$ is precisely the traditional complex of vector valued-forms on M $(\Omega^\bullet(M,V), d)$.

If, in addition, V is an algebra with multiplication $\bullet$, for example a Lie algebra $\mathfrak{g}$ with operation $[\cdot,\cdot]$, $(\Omega^\bullet(TM,E), \wedge^\bullet, \hat{d}_i)$ is exactly the traditional differential graded algebra of algebra valued-forms on M $(\Omega^\bullet(M,V), \bullet, d)$.

Example 2.1.14. Let A act by the trivial action $\phi^0 : A \to \mathfrak{D}(M \times \mathbb{R})$, i.e. considering $\Gamma(M \times \mathbb{R})$ as $C^\infty(M)$ and letting the representation be the anchor. The associated differential on $\Omega^\bullet(A)$ will be denoted by $\hat{d}_A$, making of $(\Omega^\bullet(A), \hat{d}_A)$ a differential complex. This space of differential forms will be a basic building block for other similar spaces.

Example 2.1.15. Let $0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0$ be a transitive Lie algebroid sequence. The differential complex associated to the adjoint action defined in 1.4.17, $ad : A \to \mathfrak{D}(L)$, $\mathfrak{X} \mapsto j^{-1}[\mathfrak{X}, \cdot]$, will be denoted by $(\Omega^\bullet(A,L), \hat{d})$. This differential complex will be used to define connection on A and their curvature.

Example 2.1.16. If $\phi : A \to \mathfrak{D}(E)$ is an arbitrary representation on E, then on $End(E)$ we have the representation $\widetilde{\phi} : A \to \mathfrak{D}(End(E))$ defined by $\widetilde{\phi}(\mathfrak{X})(T) = [\phi(\mathfrak{X}), T]$ for any $\mathfrak{X} \in A$ and $T \in End(E)$. The resulting differential graded complex is $(\Omega^\bullet(A, End(E)), \hat{d}_{\widetilde{\phi}})$. This differential complex will arise when we define a generalized notion of covariant derivatives on E called A-connection.

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So far we have seen that $(\Omega^\bullet(A,E),\hat{d}_\phi)$ is a differential complex over $\mathbb{R}$, but it will be so over the graded ring of scalar valued forms with the product defined below.

Definition 2.1.17. Define the $C^\infty(M)$-bilinear operation operation

$$\wedge : \Omega^\bullet(A) \times \Omega^\bullet(A,E) \to \Omega^\bullet(A,E)$$

for any $p,q \in \mathbb{Z}_{\geq 0}$, through the linear extension of the following map, defined on the homogeneous elements:

$$(\omega \wedge \eta)(\boldsymbol{\mathfrak{X}}_1,\ldots,\boldsymbol{\mathfrak{X}}_{p+q}) := \frac{1}{p!q!}\sum_{\sigma\in S_{p+q}}(-1)^\sigma \underbrace{\omega(\mathfrak{X}_{\sigma(1)},\cdots,\mathfrak{X}_{\sigma(p)})}_{\in\Gamma(M\times\mathbb{R})}\cdot\underbrace{\eta(\mathfrak{X}_{\sigma(p+1)},\cdots,\mathfrak{X}_{\sigma(p+q)})}_{\in\Gamma(L)} \tag{2.5}$$

for any scalar valued p-form ω, L-valued q-form η and any $\boldsymbol{\mathfrak{X}}_1,\ldots,\boldsymbol{\mathfrak{X}}_{p+q} \in \Gamma(A)$. With the same formula we also define

$$\wedge : \Omega^\bullet(A,E) \times \Omega^\bullet(A) \to \Omega^\bullet(A,E);$$

for ω and η as above, the two maps are related by the graded commutativity property:

$$\omega\wedge\eta = (-1)^{pq}\eta\wedge\omega. \tag{2.6}$$

This maps are called, collectively, the wedge product of scalar and E-valued forms on A. As usual, When one of the factors is a 0-form, i.e. a section of $\Gamma(E)$ or $\Gamma(A)$, the $\wedge$ symbol is usually not written.

Remark 2.1.18. Notice that the previous construction also applies for $E = M\times\mathbb{R}$ with the trivial representation a of A. This (associative) product makes $(\Omega^\bullet(A),\wedge)$ *a graded ring*; in fact, it makes it a graded algebra over the field $\mathbb{R}$.

Remark 2.1.19. If we consider forms to be vector bundle maps as in the first equation of remark 2.1.4, the $C^\infty(M)$-multilinearitiy of the wedge product implies that it is also a local operator, even a point-operator, and so it has a restriction over each U open, also denoted by $\wedge$, to local forms in such a way that:

$$(\omega\wedge\eta)|_U = \omega|_U \wedge \eta|_U$$

for all $\omega \in \Omega^\bullet(A)$ and $\eta \in \Omega^\bullet(A,E)$.

Let E be a vector bundle over M, and suppose that on the open $U \subseteq M$, $\{\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_d\} \subseteq \Gamma_U(E)$ and $\{\boldsymbol{\mathfrak{X}}_1, \dots, \boldsymbol{\mathfrak{X}}_N\} \subseteq \Gamma_U(A)$ are local frames. For this local frame of A, let the dual frame be $\{\alpha^1, \dots \alpha^N\} \in \Gamma_U(A^*) = \Omega^1_U(A)$, i.e. satisfying $\alpha_i(\boldsymbol{\mathfrak{X}}_j) = \delta_{ij} \in C^\infty(U)$. Then following facts reveal the local structure of the spaces of differential forms, from which we make computations and we can prove important algebraic properties of the differential:

- Each α^i, $i = 1, \dots, N$, is in fact a local 1-form in $\Omega^1_U(A) = \Gamma_U(A^*)$.
- Any product $\alpha^{i_1} \wedge \alpha^{i_p}$, $i_1, \dots, i_p = 1, \dots, N$, is an element of $\Omega^p_U(A)$.
- For any $f \in C^\infty(U)$,
$$f\, \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_p},$$
is also an element of $\Omega^p_U(A)$. In fact, any scalar valued p-form on A is a linear combination of differential forms of this type; this is easily seen by the analogous result on the fiber vector space of $A|_U$, and transferring it to $A|_U$ using the local trivialization of A given by the local frame.
- For any $\mu \in \Gamma_U(E) = \Omega^0_U(A, E)$,
$$\mu \wedge \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_p} \equiv \mu\, \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_p} \tag{2.7}$$
is an element of $\Omega^p_U(A, E)$, where the first $\wedge$ refers to the product between the scalar valued p-form $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_p}$ and the L-valued 0-form μ; we have used the convention of removing the $\wedge$ symbol if one of the factors is a 0-form. In fact, any scalar valued p-form on A is a linear combination of differential forms of this type.
- For $\mu \in \Omega^0_U(A, E)$,
$$\hat{d}_\phi \mu = \sum_{i=1}^N \phi(\boldsymbol{\mathfrak{X}}_i) \cdot \mu\, \alpha^i, \tag{2.8}$$
since
$$\hat{d}_\phi \mu(\boldsymbol{\mathfrak{X}}) = \phi(\boldsymbol{\mathfrak{X}}) \cdot \mu. \tag{2.9}$$
In particular, for $f \in C^\infty(U) = \Omega^0_U(A)$:
$$\hat{d}_A f = \sum_{i=1}^N X_i(f) \alpha^i, \tag{2.10}$$
where $X_i := a(\boldsymbol{\mathfrak{X}}_i)$ and a is the anchor of A.

- Let C^i_{jk} be the structure constants of the Lie algebra $\Gamma_U(A)$, i.e. $c^i_{jk} = \alpha^i([\mathfrak{X}_j, \mathfrak{X}_k]) \in C^\infty(U)$. Then, for any $i = 1, \ldots, N$,

$$\hat{d}_A \alpha^i = \sum_{1 \leq j < k \leq N} c^i_{kj} \alpha^j \wedge \alpha^k; \tag{2.11}$$

this follows from the calculation, for $j, k = 1, \ldots, N$,

$$\begin{aligned} \hat{d}_A \alpha^i(\mathfrak{X}_j, \mathfrak{X}_k) &= a(\mathfrak{X}_j) \cdot \alpha^i(\mathfrak{X}_k) - a(\mathfrak{X}_k) \cdot \alpha^i(\mathfrak{X}_j) - \alpha^i[\mathfrak{X}_j, \mathfrak{X}_k] \\ &= a(\mathfrak{X}_j) \cdot \delta^i_k - a(\mathfrak{X}_j) \cdot \delta^i_j - \alpha^i(\sum_l c^l_{jk} \mathfrak{X}_l) \\ &= -c^i_{jk}. \end{aligned}$$

These results can be used to prove the graded Leibniz rule stated in the following theorem:

Theorem 2.1.20. $(\Omega^\bullet(A, E), \hat{d}_\phi)$ *is a differential complex over the graded ring* $(\Omega^\bullet(A), \wedge)$. Furthermore, if ω and η are forms, ω is a homogeneous form, and one of ω or η is a scalar valued form and the other one is an E-valued form, then the following graded Leibniz rule is satisfied:

$$\hat{d}_\phi(\omega \wedge \eta) = (\hat{d}_{\phi_\omega} \omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge (\hat{d}_{\phi_\eta} \eta) \in \Omega^\bullet(A, E), \tag{2.12}$$

where we have denoted by ϕ_ω and ϕ_η the representations of A on the vector bundles on which ω and η take values, respectively.

2.1.3 Differential Graded Algebras

When the fiber of the representation vector bundle E have an algebra structure, the differential complex $(\Omega^\bullet(A, E), \wedge, \hat{d}_\phi)$ will also be, under some circumstances, a multiplication called the wedge product of E-valued forms 2.1.22. The additional structure gained by the differential complex will make it a differential graded algebra, as we will see in this subsection. When E is, furthermore, a LAB the differential graded algebra will have some additional properties that give rise to familiar properties for the different notions of connection and their curvatures soon to be defined.

Definition 2.1.21. Let $\mathcal{A} = \oplus_{n=0}^\infty \mathcal{A}^n$ be a graded (not necessarily associative) algebra over the field $\mathbb{R}$, and let $\cdot$ denote the product.

- An element $\omega \in \mathcal{A}^n$ is called a homogeneous element of degree n, for $n \in \mathbb{Z}_{\geq 0}$, and its degree is denoted by $|\omega|$.

- An antiderivation on $\mathcal{A}$ is an $\mathbb{R}$-linear map $d : \mathcal{A} \to \mathcal{A}$ such that, if ω is homogeneous,

$$d(\omega \bullet \eta) = (d\omega) \bullet \eta + (-1)^{|\omega|}\omega \bullet (d\eta) \tag{2.13}$$

- An antiderivation $d : \mathcal{A} \to \mathcal{A}$ is of degree $m \in \mathbb{Z}$ if, for all homogeneous elements $\omega \in \mathcal{A}$,

$$|d\omega| = |\omega| + m. \tag{2.14}$$

  If the antiderivation is of degree 1 or -1, it is called a differential on $\mathcal{A}$.

- Let $d : \mathcal{A} \to \mathcal{A}$ be a differential on A. Then the pair $(A, \bullet, d)$ is called a differential graded algebra if $d \circ d = 0$.

- Suppose that A is a graded Lie superalgebra, i.e. for all homogeneous elements $\omega, \eta, \beta \in A$ it satisfies the generalized anticommutativity property

$$\omega \bullet \eta = -(-1)^{|\omega||\eta|}\eta \bullet \omega, \tag{2.15}$$

  and the graded Jacobi identity

$$(-1)^{|\omega||\gamma|}\omega \bullet (\eta \bullet \gamma) + (-1)^{|\eta||\omega|}\eta \bullet (\gamma \bullet \omega) + (-1)^{|\beta||\eta|}\gamma \bullet (\omega \bullet \eta) = 0. \tag{2.16}$$

  If d is a differential on the graded Lie superalgebra A, then we say that $(A, \bullet, d)$ is a differential graded Lie algebra.

- Let $(\mathcal{A}, \bullet, d)$ and $(\mathcal{A}', \bullet', d')$ be differential graded algebras. A morphism of differential graded algebras is a graded algebra morphism $\phi : \mathcal{A} \to \mathcal{A}'$ such that $d' \circ \phi = \phi \circ d$.

Definition 2.1.22. Let E be an algebra bundle over M with fiberwise multiplication $\bullet$, i.e. a vector bundle such that each fiber is an algebra and such that there exists an atlas compatible with the algebra multiplication. Suppose that ϕ is a Lie algebroid representation of A on E, which, additionally, is compatible with the algebra product, i.e. it satisfies

$$\phi(\mathfrak{X})(\mu_1 \bullet \mu_2) = \phi(\mathfrak{X})(\mu_1) \bullet \mu_2 + \mu_1 \bullet \phi(\mathfrak{X})(\mu_2), \tag{2.17}$$

for all $\boldsymbol{\mathfrak{X}} in \Gamma(A)$ and all $\mu_1, \mu_2 \in \Gamma(E)$. Define on $\Omega^\bullet(A,E)$ the $C^\infty(M)$-bilinear operation, called the wedge product of algebra bundle-valued forms on A

$$\wedge^\bullet : \Omega^\bullet(A,E) \times \Omega^\bullet(A,E) \to \Omega^\bullet(A,E)$$

as the linear extension of the maps, for any $p, q \in \mathbb{Z}_{\geq 0}$

$$\wedge^\bullet : \Omega^p(A,E) \times \Omega^q(A,E) \to \Omega^{p+q}(A,E)$$

$$(\omega \wedge^\bullet \eta)(\boldsymbol{\mathfrak{X}}_1, \ldots, \boldsymbol{\mathfrak{X}}_{p+q}) := \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} (-1)^\sigma \omega(\mathfrak{X}_{\sigma(1)}, \cdots, \mathfrak{X}_{\sigma(p)}) \bullet \eta(\mathfrak{X}_{\sigma(p+1)}, \cdots, \mathfrak{X}_{\sigma(p+q)}). \tag{2.18}$$

for any p-form ω, q-form η and any $\boldsymbol{\mathfrak{X}}_1, \ldots, \boldsymbol{\mathfrak{X}}_{p+q} \in \Gamma(A)$.

Remark 2.1.23. Just as remarked in 2.1.19 for the wedge product of scalar and E-valued forms, the $C^\infty(M)$-bilinearity of $\wedge^\bullet$ implies that it is a point operator, and so it has a restriction, also denoted by $\wedge$, to local forms in such a way that, for all ω and η E-valued forms,

$$(\omega \wedge^\bullet \eta)|_U = \omega|_U \wedge^\bullet \eta|_U. \tag{2.19}$$

Theorem 2.1.24. Let E be an algebra bundle over M on which there is a representation $\phi : A \to \mathfrak{D}(E)$ of A which is compatible with the algebra product. *Then,* $(\Omega^\bullet(A,E), \wedge^\bullet, \hat{d}_\phi)$ *is a differential graded algebra over* $\mathbb{R}$.

Proof. The only missing piece is the graded Leibniz rule, and this can be proved using again the local formulas stated in section 2.1.2. It is important to notice that the compatibility condition between the representation and the product plays a role, as we can see from the calculation of $\hat{d}_\phi \mu$, when $\mu \in \Gamma(E)$ may be written as $\mu = \mu_1 \bullet \mu_2$ for some arbitrary $\mu_1, \mu_2 \in \Gamma(E)$, since then $\bullet$ coincides with $\wedge^\bullet$; let $\boldsymbol{\mathfrak{X}} \in \Gamma(A)$ be arbitrary, then:

$$\begin{aligned}
\phi(\mathfrak{X}) \cdot \mu &= \hat{d}_\phi \mu(\boldsymbol{\mathfrak{X}}) \\
&= \hat{d}_\phi(\mu_1 \bullet \mu_2)(\boldsymbol{\mathfrak{X}}) \\
&= \hat{d}_\phi(\mu_1 \wedge^\bullet \mu_2)(\boldsymbol{\mathfrak{X}})
\end{aligned}$$

$$
\begin{aligned}
&= \hat{d}_\phi(\mu_1)(\boldsymbol{\mathfrak{X}}) \wedge^\bullet \mu_2 + \mu_1 \wedge^\bullet \hat{d}_\phi(\mu_2)(\boldsymbol{\mathfrak{X}}) \\
&= \phi(\boldsymbol{\mathfrak{X}})(\mu_1) \bullet \mu_2 + \mu_1 \bullet \phi(\boldsymbol{\mathfrak{X}})(\mu_2).
\end{aligned}
$$

□

Example 2.1.25. The scalar valued forms on A are not only a differential complex over $\mathbb{R}$ and a graded algebra, but, in fact, $(\Omega^\bullet(A), \wedge, \hat{d}_A)$ *is a differential graded algebra*. This follows from noticing that the wedge product associated to the algebra multiplication of $\mathbb{R}$ coincides with the already defined wedge product of scalar and vector valued forms, and so the statement that $(\Omega^\bullet(A), \hat{d}_A)$ was a differential complex over $(\Omega^\bullet(A), \wedge)$, together with the graded Leibniz that accompanies it, is precisely the statement that $\Omega^\bullet(A)$ is a differential graded algebra; the compatibility condition on the representation is simply the traditional Lebniz rule of vector fields acting on functions of M.

In fact, $(\Omega^\bullet(A), \wedge, \hat{d}_A)$ is a differential graded-commutative algebra. The additional conditions is simply that $\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$ for all homogeneous forms α, β; this follows easily from $\mathbb{R}$ being commutative and the formula (2.5).

Theorem 2.1.26. Let the representation space E be a LAB, with a representation compatible with the Lie algebra bracket. Then $(\Omega^\bullet(A, E), \wedge^\bullet, \hat{d}_\phi)$ is a differential graded Lie algebra.

Proof. All we need to see is that $(\Omega^\bullet(A, E), \wedge^\bullet)$ is a Lie graded superalgebra. The generalized anticommutativity of $(\Omega^\bullet(A, E), \wedge^\bullet)$ is easily seen from the definition of the $\wedge^\bullet$ in equation (2.1.22); similarly, the graded Jacobi property can be verified through a manipulation of permutations in the same equation, applying the Jacobi rule of the field of Lie algebras on E. □

Example 2.1.27. Recall the differential complex $(\Omega^\bullet(A, L), \hat{d})$ of example 2.1.15 given a transitive Lie algebroid A. For any $\mathfrak{X} \in A_m$ and $\eta, \theta \in L_m$, $ad(\mathfrak{X})([\eta, \theta]) = [X, [j\eta, j\theta]]$ which, by the Jacobi identity of the Lie algebroid bracket, is equal to $ad(X)([\eta, \theta]) = [ad(\mathfrak{X})(\eta), \theta] + [\eta, ad(\mathfrak{X})\theta]$; we have thus proved that the representation ad is compatible with the product in the fibers of L. Therefore we may apply Theorem 2.1.26 to conclude that $(\Omega^\bullet(A, L), \wedge^{[,]}, \hat{d})$ is a differential graded Lie algebra.

Example 2.1.28. Recall the differential complex $(\Omega^\bullet(A, End(E), \hat{d}_{\widetilde{\phi}})$ of example 2.1.15 given a representation $\phi : A \to \mathfrak{D}(E)$ that induces the representation $\widetilde{\phi} : A \to \mathfrak{D}(End(E))$. Recall that $End(E)$ is the kernel of the anchor of the transitive Lie algebroid $\mathfrak{D}(E)$, meaning that $End(E)$ is a LAB, where the Lie bracket on the space of sections is nothing but the commutator of vector bundle endomorphisms of E. Hence, given $\mathfrak{X} \in A_m$ and $S, T \in End(E_m)$, the compatibilitiy condition $\widetilde{\phi}(\mathfrak{X})[S,T] = [\widetilde{\phi}(\mathfrak{X})(S), T] + [S, \widetilde{\phi}(\mathfrak{X})(T)]$ is simply the Jacobi identity of the commutator on $End(E_m)$. Thus, Theorem 2.1.26 implies that $(\Omega^\bullet(A, End(E)), \wedge^{[\cdot]}, \hat{d})$ is a differential graded Lie algebra.

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The following important theorem relates the notions of vector valued connections on principal bundles, that give rise to principal connections and their curvatures, and the theory of differential forms on the Atiyah Lie algebroid associated to the principal bundle. For the proofs refer to [6]; this follows essentially from the equivalence between G-invariant sections of TP and sections of TP/G stated in Theorem 1.3.9, and of the G-invariant sections of $P \times \mathfrak{g}$ with the sections of $P \times \mathfrak{g}/G$ shown in Theorem 1.3.16.

Theorem 2.1.29. Let $G \to P \to M$ be a principal bundle over M.

- The differential graded module and algebra $(\Omega^\bullet(TP/G), \wedge, \hat{d}_{TP/G})$ *is isomorphic to the subspace of* $(\Omega^\bullet(TP), \wedge, d)$ *associated to the* R_**-invariant forms on* P.

- Similarly, the differential graded module and algebra $(\Omega^\bullet(TP/G, P \times \mathfrak{g}/G), \hat{d})$ *is isomorphic to the subspace of* $(\Omega^\bullet(TP) \otimes \mathfrak{g}, \wedge, d)$ *associated to the* (R_*, Ad)*-equivariant* $\mathfrak{g}$*-valued forms on* P.

Example 2.1.30 (Complex Hopf Bundle)**.** Recall that $\{\langle\partial_\phi\rangle, \langle\partial_{\xi^1}\rangle, \langle\partial_{\xi^2}\rangle\}$ is a local frame of the Atiyah Lie algebroid associated to the complex Hopf principal bundle $S^1 \to S^3 \to S^2$. So, the building blocks of any local form over U_{SN} will be the dual frame, which we will denote by $\{\langle d\phi\rangle, \langle d\xi_1\rangle, \langle\xi_2\rangle\} \in \Omega^1(TS^3/S^1)$.

Since $L = S^3 \times i\mathbb{R}/S^1$ has the global frame $\{i\}$, any section of L can be written as $f\, i$ for some $f \in C^\infty(S^2)$. Hence, *any* $S^3 \times i\mathbb{R}/S^1$*-valued local form of* TS^3/S^1 *over* U_{SN} *can be written as a wedge product of the forms* $\{\langle d\phi\rangle, \langle d\xi_1\rangle, \langle\xi_2\rangle\}$*, with coefficients in* $C^\infty(S^2, i\mathbb{R})$.

An important local 1-form on TS^3/S^1 over U_{SN} is

$$\omega^{MP} := \frac{i}{2}(1+\cos\phi)\,\langle d\xi_1\rangle + \frac{i}{2}(1-\cos\phi)\,\langle d\xi_2\rangle \in \Omega^1_{U_{SN}}(TS^3/S^1, S^3 \times i\mathbb{R}/S^1). \tag{2.20}$$

This 1-form is associated to the presence of a magnetic monopole on 3-dimensional space.

Due to Theorem 2.1.29, the differential d of G-invariant forms on P corresponds to the differential $\hat{d}_{TS^3/S^1}$ under the identification of G-invariant sections of TS^3 with sections of TS^3/S^1 shown in proposition 1.3.6. Hence, $\hat{d}_{TS^3/S^1}\,\langle\xi_i\rangle = \langle d\xi_i\rangle$, and, since $d^2 = 0$,

$$\hat{d}_{TS^3/S^1}\langle d\xi_i\rangle = 0$$

for $i = 1, 2$; similarly

$$\hat{d}_{TS^3/S^1}\,\langle d\phi\rangle = 0.$$

Now, applying the graded Leibniz rule to the above 1-form considerint it a sum of products of $S^3 \times i\mathbb{R}/S^1$-valued 0-forms and scalar valued 1-forms, we conclude that

$$R^{MP} := \hat{d}\omega^{MP} = -\frac{i}{4}\sin\phi\,\langle d\phi\rangle \wedge (\langle d\xi_1\rangle - \langle d\xi_2\rangle) \in \Omega^2(TS^3/S^1, S^3 \times i\mathbb{R}/S^1). \tag{2.21}$$

2.2 Trivial Lie Algebroid

Our focus on this document is on transitive Lie algebroids, all of which are locally isomorphic to trivial Lie algebroids. This will allow us to think of global objects, in particular forms, as families of simple local objects. In this section we study the forms on trivial Lie algebroids, which will enable in the next section the study of differential forms on transitive Lie algebroids.

2.2.1 Scalar-valued forms

Let $A = TM \oplus (M \times \mathfrak{g})$ be a trivial Lie algebroid over the manifold M, with $\mathfrak{g}$ a Lie algebra.

In proposition 2.1.7 we used the natural distribution of the Γ functor to find several modules isomorphic to spaces of homogeneous differential forms on A. We now complement this study with our current knowledge on their structure as differential complexes to find a computationally useful point of view of the the differential forms on a trivial Lie algebroid.

The graded vector spaces $\Omega^\bullet(TM)$ and $C^\infty(M, \bigwedge^\bullet \mathfrak{g}^*)$ can be seen as subspaces of the space of scalar-valued forms $\Omega^\bullet(TM \times \mathfrak{g})$. Explicitly, for any $\alpha \in \Omega^r(TM)$ $r, s \in \mathbb{Z}_{\geq 0}$ and , the inclusions of this space into $\Omega^\bullet(TM \times \mathfrak{g})$ is the following:

$$\begin{aligned}\widetilde{\alpha} : (\Gamma(TM) \oplus C^\infty(M, \mathfrak{g})) \times \cdots (\Gamma(TM) \oplus C^\infty(M, \mathfrak{g})) &\to C^\infty(M) \\ (X_1 \oplus \widetilde{\eta}_1, \ldots, X_r \oplus \widetilde{\eta}_r) &\mapsto \alpha(X_1, \ldots, X_r);\end{aligned} \tag{2.22}$$

similarly, for any $\beta \in C^\infty(M, \bigwedge^s \mathfrak{g}^*)$:

$$\begin{aligned}\widetilde{\beta} : (\Gamma(TM) \oplus C^\infty(M, \mathfrak{g})) \times \cdots (\Gamma(TM) \oplus C^\infty(M, \mathfrak{g})) &\to C^\infty(M) \\ (X_1 \oplus \widetilde{\eta}_1, \ldots, X_s \oplus \widetilde{\eta}_s) &\mapsto \beta(\widetilde{\eta}_1, \ldots, \widetilde{\eta}_s).\end{aligned} \tag{2.23}$$

From now on the tilde notation used above to denote the image inside of $\Omega^\bullet(TM \times \mathfrak{g})$ of forms α and β as above will be omitted, unless clarity is necessary. Under this identifications, more can be said:

Proposition 2.2.1.

- $(C^\infty(M, \bigwedge^\bullet \mathfrak{g}^*), \wedge)$ is a graded subalgebra of the space of scalar-valued forms on $TM \times \mathfrak{g}$.
- $(\Omega^\bullet(TM), \wedge, d)$ is a differential graded subalgebra of $(\Omega^\bullet(TM \times \mathfrak{g}), \wedge, \hat{d}_{TM\times\mathfrak{g}})$. Furthermore, the restriction of the differential $d_{TM\times\mathfrak{g}}$ to $\Omega^\bullet(TM)$ coincides with the traditional differential d of forms on the manifold M.

Proof. The closure of this subspaces under the wedge product is clear from its formulaic definition (2.1.22). For the second part, let $\alpha \in \Omega^p(TM)$, then:

$$d\alpha(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} X_i(\alpha(X_1, \ldots, \overset{\vee}{X_i}, \ldots, X_{p+1})$$

$$+\sum_{1\le i\le j\le p+1}(-1)^{i+j}\alpha([X_i,X_j],X_1,\ldots,\overset{\vee}{\boldsymbol{X}}_i,\ldots,\overset{\vee}{X}_j,\ldots,X_{p+1})$$

$$=\sum_{i=1}^{p+1}(-1)^{i+1}a(X_i\oplus\eta_i)(\widetilde{\alpha}(X_1\oplus\eta_1,\ldots,X_i\overset{\vee}{\oplus}\eta_i,\ldots,X_{p+1}\oplus\eta_{p+1})$$

$$+\sum_{1\le i\le j\le p+1}(-1)^{i+j}\widetilde{\alpha}([X_i\oplus\eta_i,X_j\oplus\eta_j],X_1\oplus\eta_1,\ldots,\ldots,X_{p+1}\oplus\eta_{p+1})$$

$$=\hat{d}_{TM\times\mathfrak{g}}\widetilde{\alpha}((X_1\oplus\eta_1,\ldots,X_{p+1}\oplus\eta_{p+1});$$

where, for $i=1,\ldots,p{+}1$, $X_i\in\Gamma(TM)$ and $\eta_i\in C^\infty(M,\mathfrak{g})$ are arbitrary. □

Remark 2.2.2. Notice that the subalgebra $C^\infty(M,\bigwedge^\bullet\mathfrak{g}^*)$ *is not closed under the application of the differential.* We can see this, for example, taking any function $f\in C^\infty(M)$ which is an element of $C^\infty(M,\bigwedge^0\mathfrak{g}^*)$ as well as of $\Omega^0(TM)$, but $\hat{d}_{TM\times\mathfrak{g}}=df\in\Omega^1(TM)\nsubseteq C^\infty(M,\bigwedge^\bullet\mathfrak{g}^*)$, contrasting what is said in the guiding articles [5, 4, 3].

A formula that will be useful to use is the application of the differential to differential 1-forms $C^\infty(M,\bigwedge^1\mathfrak{g}^*)$ when they are constant functions, e.g. taake constant values in elements of a basis $\{\epsilon^a\}_{a=1,\ldots,n}$ of $\mathfrak{g}^*$ dual to a basis $\{E_a\}_{a=1,\ldots,n}$ of $\mathfrak{g}$. Let ϵ be arbitrary in $C^\infty(M,\bigwedge^1\mathfrak{g}^*)$, then

$$\begin{aligned}\hat{d}_{TM\times\mathfrak{g}}\epsilon(X\oplus\eta,&Y\oplus\theta)\\ &=a(X\oplus\eta)\epsilon(Y\oplus\theta)-a(Y\oplus\theta)\epsilon(X\oplus\eta)-\epsilon[X\oplus\eta,Y\oplus\theta]\\ &=X(\epsilon(\theta))-Y(\epsilon(\eta))-\epsilon(X(\theta)-Y(\eta)+[\eta,\theta])\\ &=X(\epsilon)(\theta)-Y(\epsilon)(\eta)-\epsilon[\eta,\theta]\\ &=-\epsilon[\eta,\theta],\end{aligned}\tag{2.24}$$

for arbitrary $X\oplus\eta,Y\oplus\theta\in\Gamma(TM\times\mathfrak{g})$; our assumption that $\epsilon:C^\infty(M,\mathfrak{g})\to C^\infty(M)$ is constant is only used to write the last equation (notice that the second to last equation shows, again, that the subalgebra $C^\infty(M,\bigwedge^\bullet\mathfrak{g}^*)$ is not closed under the differential). The last equation is $s(\epsilon)(\eta,\theta)$, where s is the Chevalley-Eilenbert differential on $\bigwedge^\bullet\mathfrak{g}^*$, extended naturally to functions taking values in this last space.

Suppose, in addition to previous assumptions, that on M there are $m\in\mathbb{Z}_{\ge0}$ global coordinates $\{x^\mu:M\to\mathbb{R}^m\}_{\mu=1,\ldots,m}$. Let $\{E_a\}_{a=1,\ldots,n}$ be a basis of $\mathfrak{g}$, with dual basis $\{\epsilon^a\}_{a=1,\ldots,n}\subseteq\mathfrak{g}^*$; also denote by E_a and ϵ^a the corresponding constant functions in $C^\infty(U_i, \mathfrak{g}) = \Omega^0(TU_i \times \mathfrak{g}, U_i \times \mathfrak{g})$ and $C^\infty(U_i, \mathfrak{g}^*) = \Omega^1(U_i \times \mathfrak{g})$. Then:

Proposition 2.2.3. Let ω be a scalar valued p-form on $TM \times \mathfrak{g}$, $p \in \mathbb{Z}_{\geq 0}$. Then, ω can be written as

$$\omega = \sum_{r+s=p} \omega_{\mu_1 \cdots \mu_r, a_1 \cdots a_s} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \wedge \epsilon^{a_1} \wedge \cdots \wedge \epsilon^{a_s}, \tag{2.25}$$

where each $\omega_{\mu_1 \cdots \mu_r, a_1 \cdots a_s} \in C^\infty(M)$, for $\mu_1 \cdots \mu_r = 1, \ldots, m,\ , a_1 \cdots a_s = 1, \ldots, n$.

Proof. On the fiber over $m \in M$ of $TM \times \mathfrak{g}$ it is clear that the set of forms $dx^{\mu_1}|_m \wedge \cdots \wedge dx^{\mu_r}|_m \wedge \epsilon^{a_1}|_m \wedge \cdots \wedge \epsilon^{a_s}|_m$ is a basis for $\Omega^\bullet_m(TM \times \mathfrak{g})$, hence $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \wedge \epsilon^{a_1} \wedge \cdots \wedge \epsilon^{a_s}$ is a local frame for $\Omega^\bullet(TM \times \mathfrak{g})$ and so any other element can be obtained as a linear combination of these forms with coefficients in $C^\infty(M)$. $\square$

2.2.2 Vector bundle-valued forms

Similar results to those obtained for scalar-valued forms on Lie algebroids can be applied to the spaces of vector valued forms to build an alternative and useful description. Let $E = M \times V$ be a trivial vector bundle over M with the vector space V as the fiber, and let $A = TM \times \mathfrak{g}$ be represented on E through $\phi : TM \times \mathfrak{g} \to \mathfrak{D}(M \times V)$; denote by $B \in \Omega^1(TM, M \times End(V))$ and let $\phi_L \in C^\infty(M, End(V))$ be the Maurer-Cartan form and Lie algebra endomorphism that trivialize the representation ϕ. One important example is the case $E = L = M \times \mathfrak{g}$, with the adjoint representation $\phi = ad$. Another example comes from taking V a representation vector space of G, hence V is also a representation space for $\mathfrak{g}$, and this induces naturally a representation ϕ. Notice that if E is not a trivial vector bundle, on each trivializing neighborhood U of E we fall under the case we are presently studying.

The graded vector subspaces $\Omega^\bullet(TM, M \times V) = \Omega^\bullet(TM) \otimes V$ and $C^\infty(M, \bigwedge^\bullet \mathfrak{g}^* \otimes V)$ may be seen as subspaces of the space $\Omega^\bullet(TM \times \mathfrak{g}, M \times V)$; this is done in a way entirely analogous to what was done for scalar-valued forms in equations (2.23) and (2.22). Furthermore:

Proposition 2.2.4. $(\Omega^\bullet(TM)\otimes V, \wedge)$ and $(C^\infty(M, \bigwedge^\bullet \mathfrak{g}^*\otimes V), \wedge)$ are graded subalgebras of $(\Omega^\bullet(TM\times\mathfrak{g}, M\times V), \wedge)$.

Remark 2.2.5. Neither $\Omega^\bullet(TM)\otimes V$ nor $C^\infty(M, \bigwedge^\bullet \mathfrak{g}^*\otimes V)$ are closed under the differential $\hat{d}_\phi$. We can see this by taking a vector valued function μ, which can be seen both as an element of $\Omega^0(TM)\otimes V$ and of $C^\infty(M, \bigwedge^0 \mathfrak{g}^*\otimes V)$; then, as implied by equation (2.9),

$$\hat{d}\mu(X\oplus\eta) = X(\mu)\oplus(B(X)\mu + \phi_L(\eta)\mu). \tag{2.26}$$

The previous equation further shows that under no representation ϕ is $C^\infty(M, \bigwedge^\bullet \mathfrak{g}^*\otimes V)$ closed under the differential, due to the first part of the left hand side, and that $\Omega^\bullet(TM)\otimes V$ is closed under $\hat{d}$ if and only if B and ϕ_L are 0, i.e. $\phi = a$ is the trivial representation. This contrasts what is said in the guiding articles [5, 4, 3].

Now, suppose that on M there are $m\in\mathbb{Z}_{\geq 0}$ global coordinates $\{x^\mu : M\to\mathbb{R}\}_{\mu=1,\dots,m}$. Let $\{E_a\}_{a=1,\dots,n}$ be a basis of $\mathfrak{g}$, with dual basis $\{\epsilon^a\}_{a=1,\dots,n}\subseteq\mathfrak{g}^*$; also denote by E_a and ϵ^a the corresponding constant functions in $C^\infty(U_i,\mathfrak{g}) = \Omega^0(TU_i\times\mathfrak{g}, U_i\times\mathfrak{g})$ and $C^\infty(U_i,\mathfrak{g}^*) = \Omega^1(U_i\times\mathfrak{g})$. Additionally, suppose that $\{e_u\}_{u=1,\dots,t}$ is a basis of V. The following in a straighforward generalization of proposition 2.2.3:

Proposition 2.2.6. Let ω be an E-valued p-form on $TM\times\mathfrak{g}$, $p\in\mathbb{Z}_{\geq 0}$. Then, ω can be written as

$$\omega = \sum_{r+s=p} \omega^\epsilon_{\mu_1\cdots\mu_r,a_1\cdots a_s}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}\wedge\epsilon^{a_1}\wedge\cdots\wedge\epsilon^{a_s}, \tag{2.27}$$

where each $\omega^\epsilon_{\mu_1\cdots\mu_r,a_1\cdots a_s}\in C^\infty(M,V)$, for $\mu_1\cdots\mu_r = 1,\dots,m,\ , a_1\cdots a_s = 1,\dots,n$. Viewing each e_c as a constant section of E, i.e. $e_c\in\Omega^0(TM\times\mathfrak{g}, E)$, we may also write

$$\omega = \sum_{r+s=p} (\omega^\epsilon)^u_{\mu_1\cdots\mu_r,a_1\cdots a_s}\; e_u\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_r}\wedge\epsilon^{a_1}\wedge\cdots\wedge\epsilon^{a_s}. \tag{2.28}$$

with each $(\omega^\epsilon)^u_{\mu_1\cdots\mu_r,a_1\cdots a_s}\in C^\infty(M,V)$, $u = 1,\dots,t$. The convention of ignoring the $\wedge$ symbol when one of the factors is a 0-form has been used in both formulas.

2.3 Local Description of differential forms on Transitive Lie Algebroids

Let $0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0$ be an Atiyah sequenece of the transitive Lie algebroid A over M. Let $\{(U_i, \psi_i : U_i \times \mathfrak{g} \to L|_{U_i}, \nabla^{0,i} : TU_i \to A|_{U_i})\}_{i\in I}$ be a Lie algebroid atlas for A and call $S_i : TU_i \times \mathfrak{g} \to A|_{U_i}$ the local trivialization over U_i.

Definition 2.3.1. Let $\omega \in \Omega^q(A)$.

- For each $i \in I$ define the local q-form $\omega_i := \omega \circ S_i \in \Omega^q(TU_i \times \mathfrak{g})$, called the local trivialization of ω over U_i, seen as a map

  $$\omega_i : (\Gamma_{U_i}(TM) \oplus C^\infty(U_i, \mathfrak{g})) \times \cdots \times (\Gamma_{U_i}(TM) \oplus C^\infty(U_i, \mathfrak{g})) \to C^\infty(U_i)$$

- A family of trivializations of ω is a set $\{\omega_i \in \Omega^q(TU_i \times \mathfrak{g})\}_{i\in I}$.

From now on let E be a vector bundle over M with typical fiber V, and suppose that each trivializing neighborhood U_i of A also trivializes E with trivialization map $\beta_i : U_i \times V \to E|_{U_i}$ over U_i. Let $\phi : A \to \mathfrak{D}(E)$ be a representation and let its trivialization ϕ_i over U_i be defined by the Maurer-Cartan form $B_i \in \Omega^1(TU_i, U_i \times End(V))$ and the Lie algebra endomorphisms $\phi_{L,i} \in C^\infty(U_i, End(V))$.

Definition 2.3.2. Let $\omega \in \Omega^q(A, E)$.

- For each $i \in I$ define the local q-form $\omega_i := \beta_i^{-1} \circ \omega \circ S_i \in \Omega^q(TU_i \times \mathfrak{g}, U_i \times V)$, called the local trivialization of ω over U_i, seen as a map

  $$\omega_i : (\Gamma_{U_i}(TM) \oplus C^\infty(U_i, \mathfrak{g})) \times \cdots (\Gamma_{U_i}(TM) \oplus C^\infty(U_i, \mathfrak{g})) \to C^\infty(U_i, V).$$

- The set $\{\omega_i \in \Omega^q(TU_i \times \mathfrak{g}, U_i \times V)\}_{i\in I}$ is called a family of trivializations of ω.

Remark 2.3.3. Due to proposition 2.2.3, if, additionally, there are local coordinates of M, $\{\vec{x}^i : U_i \to \mathbb{R}^m\}_{i\in I}$ on each U_i, the local trivializations of every form $\omega \in \Omega^p(A)$ satisfy

$$\omega_i = \sum_{r+s=p} (\omega_i^\epsilon)_{\mu_1\cdots\mu_r, a_1\cdots a_s}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \wedge \epsilon^{a_1} \wedge \cdots \wedge \epsilon^{a_s}, \qquad (2.29)$$

where each $(\omega_i^\epsilon)_{\mu_1\cdots\mu_r,a_1\cdots a_s} \in C^\infty(U_i, V)$, for $\mu_1 \cdots \mu_r = 1, \dots, m,\ , a_1 \cdots a_s = 1, \dots, n$.

Similarly for E-valued forms, viewing each e_c as a constant section of E, we can write

$$\omega_i = \sum_{r+s=p} (\omega_i^\epsilon)^c_{\mu_1\cdots\mu_r,a_1\cdots a_s}\ e_c\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \wedge \epsilon^{a_1} \wedge \cdots \wedge \epsilon^{a_s}. \tag{2.30}$$

with each $(\omega_i^\epsilon)^a_{\mu_1\cdots\mu_r,a_1\cdots a_s} \in C^\infty(M, V)$, $a = 1, \dots, n$.

Definition 2.3.4. Over each U_i use the local trivialization of the representation $\phi_i : TU_i \times \mathfrak{g} \to TU_i \times End(V)$ to define the differential $\hat{d}_\phi$ on $\Omega^\bullet(TU_i \times \mathfrak{g}, U_i \times V)$. Recall from section 1.5.2 that the local trivialization of the trivial representation is again the trivial representation, and of the adjoint representation is again the adjoint representation on the trivial Lie algebroid.

The following theorem allows us to use a family of trivialization of forms, i.e. form over trivial Lie algebroids, to understand forms on transitive Lie algebroids:

Theorem 2.3.5. Let U be a trivializing neighborhood of A. The map

$$\cdot_i : (\Omega^\bullet_U(A, E), \wedge, \hat{d}_{\phi_i}) \to (\Omega^\bullet(TU \oplus (U \times \mathfrak{g}), U \times V), \wedge, \hat{d}_\phi) \tag{2.31}$$

is an isomorphism of differential complexes, over the isomorphism of the graded rings $\Omega^\bullet_U(A)$ and $\Omega^\bullet(TU \oplus (U \times \mathfrak{g}))$. In particular

$$(\hat{d}\omega)_i = \hat{d}\omega_i. \tag{2.32}$$

Proof. Let U be an arbitrary U_i. Since $\Omega^\bullet_U(A) = \Omega^\bullet(A|_U) = \Gamma\, Alt^\bullet(A|_U)$ and $A|_U \cong TU \oplus (U \times \mathfrak{g})$, so $\bigwedge^p A|_U^*) \cong \bigwedge^p (TU \times \mathfrak{g})^*$; applying the Γ functor and proposition 2.1.7 we conclude that the degree p submodules $\Omega^p_U(A, E)$ and $\Omega^p(TU \times \mathfrak{g}, U \times V)$ of $\Omega^\bullet_U(A, E)$ and $\Omega^\bullet(TU \times \mathfrak{g}, U \times V)$, respectively, are isomorphic for all $p \in \mathbb{Z}_{\geq 0}$. The wedge product is defined using these degree p submodules with equation (2.18), so the graded algebras are indeed isomorphic.

All what's left to see is that the trivialization map respects the differential. Let $\omega \in \Omega^p(A, E)$, for some $p \in \mathbb{Z}_{\geq 0}$ and let $X_i \oplus \eta_i$, for $i = 1, \dots, p+1$ be arbitrary elements in $\Gamma(TU_i) \oplus C^\infty(U_i, \mathfrak{g})$, then:

$$
\begin{aligned}
&(\hat{d}_\phi\omega)_i(X_1 \oplus \eta_1, \dots, X_{p+1} \oplus \eta_{p+1}) \\
&\quad = \beta_i^{-1}[(\hat{d}_\phi\omega)_i(S_i(X_1 \oplus \eta_1), \dots, S_i(X_{p+1} \oplus \eta_{p+1}))] \\
&\quad = \sum_{k=1}^{p+1}(-1)^{k+1}\beta^{-1} \circ \phi(S_i(X_k \oplus \eta_k)) \cdot \omega(S_i(X_1 \oplus \eta_1), \cdots, S_i(X_{p+1} \oplus \eta_{p+1})) \\
&\quad + \sum_{1 \le k < j \le p+1}(-1)^{k+j}\beta_i^{-1} \circ \omega(S_i[X_k \oplus \eta_k, X_j \oplus \eta_j], \cdots, S_i(X_{p+1} \oplus \eta_{p+1})) \\
&\quad = \sum_{k=1}^{p+1}(-1)^{k+1}\phi_i(X_k \oplus \eta_k) \cdot \omega_i(X_1 \oplus \eta_1, \cdots, X_{p+1} \oplus \eta_{p+1}) \\
&\quad + \sum_{1 \le k < j \le p+1}(-1)^{k+j}\omega_i([X_k \oplus \eta_k, X_j \oplus \eta_j], X_1 \oplus \eta_1, \cdots, X_{p+1} \oplus \eta_{p+1}) \\
&\quad = \hat{d}_\phi\omega_i(X_1 \oplus \eta_1, \dots, X_{p+1} \oplus \eta_{p+1});
\end{aligned}
$$

notice that in the second to last line both the fact that the representation used on each trivial Lie algebroid is the trivialization of the representation on E, and that the trivialization S_i respects the Lie bracket of the algebroid. □

In particular, in two of our main examples: the differential graded algebra of scalar-valued differential forms $(\Omega^\bullet(A), \wedge, \hat{d}_A)$, and $(\Omega^\bullet(A, L), \wedge^{[,]}, \hat{d})$, a differential graded Lie algebra and a differential complex over the graded ring $(\Omega^\bullet(A), \wedge)$, their local trivializations as forms on trivial Lie algebroids are spaces of the same type for the same representation, since the local trivialization of the trivial and the adjoint representations are themselves.

Example 2.3.6 (TP^k/S^1 over S^2)**.** Based on section 1.5.3, for TP^k/S^1 we had determined in section 1.5.4 a Lie algebroid atlas for the Atiyah Lie algebroids associated to the principal bundles $S^1 \to P^k \to S^2$, coming from the local trivializations of the principal bundle P^k. The atlas was associated to the neighborhoods U_S and U_N that trivialize both P^k and S^2, hence remark 2.3.3 implies that we may use the following scalar valued 1-forms on the trivial vector bundles to build any (local) form on TP^k/S^1:

$$
\{dx^1, dx^2, Im\} \in \Omega^1(TU_S \times i\mathbb{R}), \quad \{dy^1, dy^2, Im\} \in \Omega^1(TU_N \times i\mathbb{R}), \tag{2.33}
$$

where Im is the restriction to the respective neighborhood of

$$
Im \in C^\infty(S^2, \bigwedge^1(i\mathbb{R})^*) \tag{2.34}
$$

dual to the global frame $i \in C^\infty(S^2, i\mathbb{R})$.

For example, any $S^2 \times i\mathbb{R}$ valued 1-form $\hat{\omega}$ has the local trivialization on the open set U_S

$$\hat{\omega}_S = (\hat{\omega}^\epsilon_S)_{1,}(\phi,\theta)dx^1 + (\hat{\omega}^\epsilon_S)_{2,}(\phi,\theta)dx^2 + (\hat{\omega}^\epsilon_S)_{,1}(\phi,\theta)Im \tag{2.35}$$

where $(\hat{\omega}^\epsilon_S)_{\mu,a} \in C^\infty(S^2)$, and (ϕ,θ) is a shortcut notation for $E(\phi,\theta) \in S^2$, where ϕ is the azimuthal angle and θ the polar angle in S^2; that the component functions are defined on all of S^2 guarantee that this local form can be extended to a global form on TP^k/S^1, as we will in example 2.4.1. To simplify notation, the 1-form (2.35) will usually be written instead as

$$\hat{\omega}_S = i\omega^\epsilon_{S;1}(\phi,\theta)dx^1 + i\omega^\epsilon_{S;2}(\phi,\theta)dx^2 + i\omega^\epsilon_{S;i}(\phi,\theta)Im. \tag{2.36}$$

Recall that, although strictly speaking the spherical coordinates (ϕ,θ) are not manifold coordinates on the neighborhood U_{SN}, they are so on two different charts that cover U_{SN}, therefore $\{\partial_\phi, \partial_\theta\} \subseteq \Gamma_{U_{SN}}(S^2)$ is a local frame of S^2 associated to coordinate functions. Hence, over U_{SN} we may once again apply remark 2.3.3 to write any form on TP^k/S^1 as the wedge product of the scalar valued local 1-forms

$$\{d\phi, d\theta, Im\} \subseteq \Omega^1_{U_{SN}}(TP^k/S^1).$$

From now on we will replace every instance of $\Omega_{U_S}, \Omega_{U_N}$ and $\Omega_{U_{SN}}$ by Ω_S, Ω_N and Ω_{SN}, respectively.

Thanks to Theorem 2.3.5 even if we don't know explicitly the form $\hat{\omega} \in \Omega^1(TP^k/S^1, P^k \times i\mathbb{R}/S^1)$, we can still find (the local trivializations of) $\hat{d}\hat{\omega}$. Furthermore, we may apply Theorem 2.1.20 along with the second part or proposition 2.2.1 to show that

$$\hat{d}\hat{\omega}_S = \hat{d}(i\omega^\epsilon_{S;\mu}) \wedge dx^\mu + i\omega^\epsilon_{S;\mu}d(dx^\mu) + \hat{d}(i\hat{\omega}^\epsilon_{S;i}) \wedge Im + i\omega^\epsilon_{S;i}dx^\mu \wedge \hat{d}_{TU_S \times i\mathbb{R}}Im;$$

notice that three distinct differentials are used in the previous equation. We may then apply the result obtained in (2.24) to $Im \in C^\infty(U_S, (i\mathbb{R})^*)$ to conclude that

$$\hat{d}_{TU_S \times i\mathbb{R}}Im = 0$$

because $i\mathbb{R}$ is commutative; also, we can apply (2.8) to conclude that

$$\hat{d}(i\omega^\epsilon_{S;\cdot}) = id\omega^\epsilon_{S;\cdot},$$

again because $i\mathbb{R}$ is commutative. Hence,

$$\hat{d}\hat{\omega}_S = id\omega^{\epsilon}_{S;\mu} \wedge dx^{\mu} + 0 + id\omega^{\epsilon}_{S;i} \wedge Im + 0$$

which equals

$$\hat{d}\omega_S = i(\partial_1\omega^{\epsilon}_{S;2} - \partial_2\omega^{\epsilon}_{S;1})dx^1 \wedge dx^2 + i\partial_1\omega^{\epsilon}_{S;i}dx^1 \wedge Im + i\partial_2\omega^{\epsilon}_{S;2}dx^2 \wedge Im. \quad (2.37)$$

Change between local trivialization of forms

For the rest of the section, let E be a vector bundle over M on which A is represented via ϕ; suppose that there are vector bundle trivializations $\beta_i : U_i \times V \to E|_{U_i}$ and $\beta_j : U_j \times V \to E|_{U_j}$ over the neighborhoods U_i and U_j, $U_{ij} = U_i \cap U_j \neq \emptyset$, that trivialize A, and denote by $\beta^i_j : U_{ij} \to Gl(V)$, $m \mapsto \beta_{i,m} \circ \beta^{-1}_{j,m}$ the transition map from the trivialization of E over U_j to the trivialization over U_i; notice that, for $E = L$, the transition functions β^i_j coincide with α^i_j coincides with the one given in section 1.5.1. Let $\omega \in \Omega^q(A, E)$ and let $\{\boldsymbol{\mathfrak{X}}_k\} \subseteq \Gamma(A)$ for $k = 1, \dots, q$. Let each $\boldsymbol{\mathfrak{X}}_k$ have a family of trivializations $\{\boldsymbol{X}_k \oplus \widetilde{\boldsymbol{\eta}}^i_k \in \Gamma(TU_i) \oplus C^{\infty}(U_i, \mathfrak{g})\}$, then, in $U_{ij} = U_i \cap U_j \neq \emptyset$,

$$\omega_i(X_1 \oplus \widetilde{\boldsymbol{\eta}}^i_1, \cdots, X_q \oplus \widetilde{\boldsymbol{\eta}}^i_q) = \alpha^i_j \circ \omega_j(X_1 \oplus \widetilde{\boldsymbol{\eta}}^j_1, \cdots, X_q \oplus \widetilde{\boldsymbol{\eta}}^j_q),$$

where we are denoting by β^i_j the map induced by the transition functions on functions, $\beta^i_j : C^{\infty}(U_{ij}, V) \to C^{\infty}(U_{ij}, V)$, $\beta^i_j(f)(m) := \beta^i_{j,m}(f(m))$ for any $f \in C^{\infty}(U_{ij}, V)$, which can be expressed as

$$\omega_i = \beta^i_j \circ \omega_j \circ S^j_i. \quad (2.38)$$

Definition 2.3.7.

- Let $\omega \in \Omega^q(A, E)$. Define

$$\begin{aligned} \hat{\beta}^i_j : \Omega^q_{U_{ij}}(TU_{ij} \times \mathfrak{g}, U_{ij} \times V) &\to \Omega^q_{U_{ij}}(TU_{ij} \times \mathfrak{g}, U_{ij} \times V) \\ \omega_j &\mapsto \omega_i = \beta^i_j \circ \omega_j \circ S^j_i \end{aligned} \quad (2.39)$$

- Let $\omega \in \Omega^q(A)$. Define

$$\begin{aligned} \hat{\alpha}^i_j : \Omega^q(TU_{ij} \times \mathfrak{g}) &\to \Omega^q(TU_{ij} \times \mathfrak{g}) \\ \omega_j &\mapsto \omega_i = \omega_j \circ S^j_i \end{aligned} \quad (2.40)$$

Notice that $\hat{\beta}^i_j$ and $\hat{\alpha}^i_j$ are morphisms between $C^\infty(U_{ij})$-modules.

Theorem 2.3.8. $\hat{\beta}^i_j : \Omega^q_{U_{ij}}(TU_{ij} \times \mathfrak{g}, U_{ij} \times V) \to \Omega^q_{U_{ij}}(TU_{ij} \times \mathfrak{g}, U_{ij} \times V)$ is an isomorphism of differential graded complex. Furthermore, $\hat{\alpha}^i_j : \Omega^q(TU_{ij} \times \mathfrak{g}) \to \Omega^q(TU_{ij} \times \mathfrak{g})$ is an isomorphism of differential graded algebras.

Proof. First notice that $\hat{\alpha}^i_j$ is a special case of β^i_j, so we may only prove everything for general E-valued forms. Recall that, since a differential d in a differential graded module is a local operator, a local version $\hat{d}|_U$ exists for all U open in M, and it is such that $(d\omega)|_U = d|_U\omega|_U$; we will omit the $|_U$ besides the differentials for ease of reading.

That the $\hat{\beta}^i_j$ commutes with the differential, follows from the following calculation, where we have temporarily denoted by $\hat{d}_{TLA}$ the differential on the trivial Lie algebroids, to distinguish it from the differential $\hat{d}$ on A; notice that the fact that this distinction is unnecessary is what this result shows:

$$\hat{d}_{TLA}\hat{\beta}^i_j(\omega_j|_{U_{ij}}) = \hat{\beta}^i_j(\hat{d}_{TLA}(\omega_i|_{U_{ij}})),$$

which follows from the following calculation:

$$\begin{aligned}
\hat{d}_{TLA}\hat{\beta}^i_j(\omega_j|_{U_{ij}}) &= \hat{d}_{TLA}(\omega_i|_{U_{ij}}) \\
&= (\hat{d}\omega)_i|_{U_{ij}} && \text{since } \cdot_i \text{ is isomorphism of diff. algebras} \\
&= \hat{\beta}^i_j((\hat{d}\omega)_j|_{U_{ij}}) \\
&= \hat{\beta}^i_j(\hat{d}_{TLA}(\omega_j|_{U_{ij}})).
\end{aligned}$$

The theorem will now be proven once we check that $\hat{\beta}^i_j$ respects the wedge product between scalar and E valued forms, since $E = M \times \mathbb{R}$ covers the second part of the theorem. Let $\theta \in \Omega^p_U(TU_{ij} \times \mathfrak{g})$ and $\gamma \in \Omega^q_U(TU_{ij} \times \mathfrak{g}, U_{ij} \times V)$, for some $p, q \in \mathbb{Z}_{\geq 0}$, then:

$$\begin{aligned}
&\hat{\beta}^i_j(\theta \wedge \gamma)(\boldsymbol{\mathfrak{X}}_1, \dots, \boldsymbol{\mathfrak{X}}_{p+q}) = \beta^i_j\left((\theta \wedge \gamma)(S^j_i(\boldsymbol{\mathfrak{X}}_1), \dots, S^j_i(\boldsymbol{\mathfrak{X}}_{p+q}))\right) \\
&= \beta^i_j\left(\sum_\sigma (-1)^\sigma \beta(S^j_i(\boldsymbol{\mathfrak{X}}_{\sigma(1)}), \dots, S^j_i(\boldsymbol{\mathfrak{X}}_{\sigma(p)})) \cdot \gamma(S^j_i(\boldsymbol{\mathfrak{X}}_{\sigma(p+1)}), \dots, S^j_i(\boldsymbol{\mathfrak{X}}_{\sigma(p+q)}))\right) \\
&= \sum_\sigma (-1)^\sigma \theta(S^j_i(\boldsymbol{\mathfrak{X}}_{\sigma(1)}), \dots, S^j_i(\boldsymbol{\mathfrak{X}}_{\sigma(p)})) \cdot \beta^i_j\left(\gamma(S^j_i(\boldsymbol{\mathfrak{X}}_{\sigma(p+1)}), \dots, S^j_i(\boldsymbol{\mathfrak{X}}_{\sigma(p+q)}))\right)
\end{aligned}$$

$$= \sum_{\sigma} (-1)^{\sigma} \hat{\beta}^i_j(\theta)(\boldsymbol{\mathfrak{X}}_1, \ldots, \boldsymbol{\mathfrak{X}}_{p+q})) \cdot \hat{\beta}^i_j(\gamma)(\boldsymbol{\mathfrak{X}}_{p+1}, \ldots, \boldsymbol{\mathfrak{X}}_{p+q}))$$
$$= (\hat{\beta}^i_j(\theta) \wedge \hat{\beta}^i_j(\gamma))(\boldsymbol{\mathfrak{X}}_1, \ldots, \boldsymbol{\mathfrak{X}}_{p+q}).$$

□

Remark 2.3.9. If we have the local trivialization with respect to $i \in I$ of a form on A written in terms of the decomposition into local 1-forms from remark 2.3.3, then we can use Theorem 2.3.8 to conclude that *to find its relation with the trivialization with respect to* $j \in I$ within U_{ij} if it isn't empty, *we need only know*

$$\hat{\alpha}^i_j(dx^\mu) \quad \mu = 1, \ldots, m \tag{2.41}$$
$$\hat{\alpha}^i_j(\epsilon^a) \quad a = 1, \ldots, n \tag{2.42}$$
$$\alpha^i_j(e_c) \quad c = 1, \ldots, h, \tag{2.43}$$

since $\hat{\alpha}^i_j$ respects the wedge product. Notice, however, that denoting by $\frac{\partial \vec{x}}{\partial \vec{y}}$ the Jacobian matrix of the coordinate transformation from x coordinates to y coordinates in U_{ij}, it is true that

$$\hat{\alpha}^i_j(dx^\mu) = dx^\mu \tag{2.44}$$
$$= \left(\frac{\partial \vec{x}}{\partial \vec{y}}\right)^{\mu}_{\nu} dy^\nu; \tag{2.45}$$

this follows from finding the decomposition of $\hat{\alpha}^i_j(dx^\mu)$ into the 1-forms by evaluating $\hat{\alpha}^i_j(dx^\mu)(X \oplus \eta) = dx^\mu(S^j_i(X \oplus \eta)) = dx^\mu(X \oplus \cdots) = dx^\mu(X) = X^\mu$, meaning that $\hat{\alpha}^i_j(dx^\mu) = dx^\mu$.

2.4 Examples

2.4.1 TP^k/S^1 over S^2

We already saw in example 2.3.6, the 1-form $Im \in \Omega^1(S^2 \times i\mathbb{R})$ of equation (2.34) can be used as a building block for all local forms over U_S or U_N together with the differentials of the coordinates on S^2.

Following remark 2.3.9, since S^2 is covered by the two neighborhoods U_S and U_N, the relation between the relation between the trivializations of any global form over these open sets may be deduced completely only from knowledge of $\hat{\alpha}^N_S(Im)$. To find this, recall from (1.126) that $S^S_N(X \oplus \eta) = X \oplus (\eta + ikd\theta(X))$ for any $X \oplus \eta \in \Gamma(TU_{SN} \times i\mathbb{R})$, and so $S^S_N(\partial_\phi) = \partial_\phi$, $S^S_N(\partial_\theta) = \partial_\theta + ik$ and $S^S_N(i) = i$. Hence,

$$\begin{aligned} \hat{\alpha}^N_S(Im)(\partial_\phi) &= 0 \\ \hat{\alpha}^N_S(Im)(\partial_\theta) &= k \\ \hat{\alpha}^N_S(Im)(i) &= 1, \end{aligned}$$

so

$$\hat{\alpha}^N_S(Im) = kd\theta + Im, \qquad \hat{\alpha}^S_N(Im) = -kd\theta + Im. \tag{2.46}$$

To be able to work with the coordinates $\vec{x}$ and $\vec{y}$ on S^2 defined in (1.52) and (1.54), we will now find $d\theta$ in terms of them, and viceversa. We can combine the multiple coordinates that we have defined over $U_{SN} \subseteq S^2$ to conclude that every point might be written as

$$\frac{(2x^1, 2x^2, ||\vec{x}||^2 - 1)}{||\vec{x}||^2 + 1} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = \frac{(2y^1, 2y^2, 1 - ||\vec{y}||^2)}{||\vec{y}||^2 + 1}. \tag{2.47}$$

Then, the polar coordinates in terms of the $\vec{x}$ and $\vec{y}$ coordinates satisfy:

$$\begin{aligned} \tan\theta &= \frac{x_2}{x_1} & \cot\frac{\phi}{2} &= ||\vec{x}|| \\ \tan\theta &= \frac{y_2}{y_1} & \tan\frac{\phi}{2} &= ||\vec{y}||; \end{aligned}$$

on the other hand

$$\begin{aligned} x^1 &= \cot\frac{\phi}{2}\cos\theta & y^1 &= \tan\frac{\phi}{2}\cos\theta \\ x^2 &= \cot\frac{\phi}{2}\sin\theta & y^2 &= \tan\frac{\phi}{2}\sin\theta. \end{aligned}$$

From the first set of equation we conclude that

$$\begin{aligned} d\theta &= \frac{-x^2dx^1 + x^1dx^2}{(x^1)^2 + (x^2)^2} \\ &= \frac{-y^2dy^1 + y^1dy^2}{(y^1)^2 + (y^2)^2}; \end{aligned} \tag{2.48}$$

from the second set we deduce that

$$\begin{aligned} dx^1 &= -\cos\theta\csc^2\frac{\phi}{2}d\phi - \sin\theta\cot\frac{\phi}{2}d\theta \\ dx^2 &= -\sin\theta\csc^2\frac{\phi}{2}d\phi + \cos\theta\cot\frac{\phi}{2}d\theta \end{aligned} \tag{2.49}$$

and

$$\begin{aligned} dy^1 &= \cos\theta\sec^2\frac{\phi}{2}d\phi - \sin\theta\tan\frac{\phi}{2}d\theta \\ dy^2 &= \sin\theta\sec^2\frac{\phi}{2}d\phi + \cos\theta\tan\frac{\phi}{2}d\theta \end{aligned} \tag{2.50}$$

Lastly, it will also be useful to change from $\vec{x}$ to $\vec{y}$ coordinates, so we will now calculate the differentials. Equation (1.48) implies that

$$x^\mu = \frac{y^\mu}{||\vec{y}||^2}, \qquad\qquad y^\nu = \frac{x^\nu}{||\vec{x}||^2}$$

for $\mu, \nu = 1, 2 = dim_{\mathbb{R}}(\mathbb{C})$. From here it immediately follows that:

$$\begin{aligned} dx^1 &= \frac{[-(y^1)^2 + (y^2)^2]dy^1 - 2y^1y^2dy^2}{[(-y^1)^2 + (y^2)^2]^2}, \\ dx^2 &= \frac{-2y^1y^2dy^1 + [(y^1)^2 - (y^2)^2]dy^2}{[(y^1)^2 + (y^2)^2]^2}; \end{aligned} \tag{2.51}$$

and

$$\begin{aligned} dy^1 &= \frac{[-(x^1)^2 + (x^2)^2]dx^1 - 2x^1x^2dx^2}{[(-x^1)^2 + (x^2)^2]^2}, \\ dy^2 &= \frac{-2x^1x^2dx^1 + [(x^1)^2 - (x^2)^2]dx^2}{[(x^1)^2 + (x^2)^2]^2}. \end{aligned} \tag{2.52}$$

Using the notation of example 2.3.6, let

$$\hat{\omega}_S = i\hat{\omega}^\epsilon_{S;1}(\phi,\theta)dx^1 + i\hat{\omega}^\epsilon_{S;2}(\phi,\theta)dx^2 + i\hat{\omega}^\epsilon_{S;i}(\phi,\theta)Im;$$

express the most general $L = C^\infty(U_S, i\mathbb{R})$-valued 1-form on the trivial Lie algebroid $TU_S \times i\mathbb{R}$, if we allow each $\omega^\epsilon_{S;\cdot}$ to be arbitrary functions in $C^\infty(U_S)$. However, notice that this local form may not be the local trivialization of a global form $\hat{\omega}$ on TP^k/S^2. We will now try to determine when $\hat{\omega}_S$ can be extended to a global form on the Atiyah Lie algebroid associated to $S^1 \to P^k \to S^2$.

If $\hat{\omega}$ exists, we may apply $\hat{\alpha}_S^N$ to the trivialization $\hat{\omega}_S$ to obtain the local trivialization $\hat{\omega}_N$ over U_N. Hence:

$$\begin{aligned}\hat{\alpha}_S^N(\hat{\omega}_S|_{U_{SN}}) &= \hat{\alpha}_S^N(i\hat{\omega}^{\epsilon}_{S;1}|_{U_{SN}}dx^1 + i\hat{\omega}^{\epsilon}_{S;2}|_{U_{SN}}dx^2 + i\hat{\omega}^{\epsilon}_{S;i}|_{U_{SN}}Im)\\ &= i\hat{\omega}^{\epsilon}_{S;1}|_{U_{SN}}dx^1 + i\hat{\omega}^{\epsilon}_{S;2}|_{U_{SN}}dx^2 + i\hat{\omega}^{\epsilon}_{S;i}|_{U_{SN}}\hat{\alpha}_S^N(Im);\end{aligned}$$

we have applied remark 2.3.9 to the $C^\infty(U_{SN})$ map $\hat{\alpha}_S^N$ to reduce the calculation into knowing that $\hat{\alpha}_S^N(Im) = kd\theta + Im$.

From this calculation follows that, if the global form ω whose local trivialization is $\hat{\omega}_S$ exists, then

$$\omega_N|_{U_{SN}} = \hat{\omega}_S|_{U_{SN}} + k(i\hat{\omega}^{\epsilon}_{S;i}|_{U_{SN}})d\theta$$

which is equal to

$$\begin{aligned}\hat{\omega}_N|_{U_{NS}} &= \{\hat{\omega}^{\epsilon}_{S;1}[-(y^1)^2 + (y^2)^2] - 2\hat{\omega}^{\epsilon}_{S;2}y^1y^2 - k\hat{\omega}^{\epsilon}_{S;i}y^2||\vec{y}||^2\}\frac{idy^1}{||\vec{y}||^4}\\ &+ \{-2\hat{\omega}^{\epsilon}_{S;1}y^1y^2 + \hat{\omega}^{\epsilon}_{S;2}[(y^1)^2 - (y^2)^2] + k\hat{\omega}^{\epsilon}_{S;i}y^1||\vec{y}||^2\}\frac{idy^2}{||\vec{y}||^4}\\ &+ i\hat{\omega}^{\epsilon}_{S;1}Im.\end{aligned} \tag{2.53}$$

Thus, for $\hat{\omega}_N|_{U_{SN}}$ to be the restriction of a 1-form $\hat{\omega}_N$, and so for $\hat{\omega}_S$ and $\hat{\omega}_N$ to be local trivializations of a global form $\hat{\omega}$ on TP^k/S^2, $\hat{\omega}_N|_{U_{SN}}$ needs only a well defined limit at the point representing the north pole, i.e. at $\vec{y} = \vec{0}$. From the above equation we conclude that sufficient conditions are the following:

- The functions $\hat{\omega}^{\epsilon}_{S;1}(y^1, y^2)$ and $\hat{\omega}^{\epsilon}_{S;2}(y^1, y^2)$ have null 0th, 1st and 2nd derivatives at $\vec{y} = \vec{0}$;
- The function $\hat{\omega}^{\epsilon}_{S;i}(y^1, y^2)$ has null 0th and 1st derivatives at $\vec{y} = \vec{0}$ as a function of $\vec{y}$;

notice, however, that these are not necessary conditions since there would be an assymmetry between the value of the form at the north and south pole, because the previous conditions imply the form is null at the former while its value at the latter point is arbitrary.

In general, *a complete family of local trivializations of a form on TP^k/S^1, and hence the global form, may be found only from knowledge of one of its local trivializations over either of U_S or U_N*. To see this, notice that each of U_S and U_N covers all of S^2 except for one point, implying that once we know the local trivialization of a form ω on TP^k/S^1 over, say, U_S, we can translate its restriction over U_{SN} into the restriction over U_{SN} of the trivialization of ω over U_N using the $\hat{\alpha}_S^N$ map, which can then be extended by continuity to the complete trivialization of ω over U_N.

Lastly, let us notice that using equations (2.49), over U_{SN} we may also write the local trivialization of the most general 1-form on TP^k/S as:

$$\begin{aligned} \hat{\omega}_S &= if(\phi,\theta)d\phi + ig(\phi,\theta)d\theta + ih(\phi,\theta)Im, \\ \text{where } f(\phi,\theta) &= -\cos\theta\csc^2\frac{\phi}{2}\omega^\epsilon_{S;1}(\phi,\theta) - \sin\theta\csc^2\frac{\phi}{2}\omega^\epsilon_{S;2}(\phi,\theta), \\ g(\phi,\theta) &= \sin\theta\cot\frac{\phi}{2}\omega^\epsilon_{S;1}(\phi,\theta) + \cos\theta\cot\frac{\phi}{2}\omega^\epsilon_{S;2}(\phi,\theta), \\ h(\phi,\theta) &= \omega^\epsilon_{S;i}(\phi,\theta). \end{aligned} \tag{2.54}$$

2.4.2 TP^k/S^3 over S^4

Again, remark 2.3.9 implies that every form on the Atiyah Lie algebroid TP^k/S^3 associated to the principal bundle $S^3 \to P^k \to S^4$, whether scalar or vector bundle-valued, will have a local trivialization into wedge products of the scalar-valued 1-forms on the trivializing Lie algebroids

$$\{dx^\mu, Im, Jm, Km\}_{\mu=1,\dots,4} \in \Omega^1(TU_S \times Im\,\mathbb{H}), \tag{2.55}$$

$$\{dy^\mu, Im, Jm, Km\}_{\mu=1,\dots,4} \in \Omega^1(TU_N \times Im\,\mathbb{H}), \tag{2.56}$$

where $\{Im, Jm, Km\}$ are the restrictions to the respective neighborhood of the global forms $\{Im, Jm, Km\} \subseteq \Omega^1(S^4 \times Im\,\mathbb{H})$ dual to the global frame $\{i,j,k\} \in C^\infty(S^4, Im\,\mathbb{H})$.

Furthermore, every form can be determined from its local trivialization over U_S or U_N once, over U_{SN} the forms

$$\hat{\alpha}_S^N(Im), \qquad \hat{\alpha}_S^N(Jm), \qquad \hat{\alpha}_S^N(Km) \tag{2.57}$$

are determined.

Chapter 3

Connections and Curvature

In standard gauge theories [2, 8, 9], *matter fields* are sections of vector bundles E over spacetime M that have certain dynamics, determined by differential equations, and on which a structure group G acts fiberwise; these include the electron field, the neutrino fields and the quark fields, among others. These fields can be understood as G-equivariant functions taking value over the principal bundle P with structure group G to which E is associated due to the group action. The notion of a connection in standard gauge theories, also called a gauge potential, arises as a means to guarantee that the matter fields satisfy the same differential equations for its dynamics after a spacetime-varying action of the structure group is applied, under the justification that such a change of "inner" reference frame should not affect the dynamics. A connection form w on the principal bundle is then introduced and it is "minimally coupled" with the matter field, namely by changing the spacetime derivatives ∂_μ associated to spacetime coordinates x^μ, inducing covariant derivatives on E locally of the form $\partial_\mu + iE_b A^b_\mu$, where $\{iE_b\}$ is a basis of the Lie algebra of G and A^a real valued forms, called gauge fields, such that $iE_b A^b$ is the local pullback to spacetime of the Lie algebra-valued form w on P. The gauge fields include the electromagnetic 4-potential, the W and Z boson fields and the gluon fields.

In this chapter we introduce two notions of connection. In Section 3.1 we present the generalized notion of connection proposed in [5] and used in [3] to generalize connections on a principal bundle P to any transitive Lie algebroid A. In Section 3.3 we take the generalization of covariant derivatives on vector bundles E exposed in [5], called A-connections, and introduce them in a way that makes clear their analogy with traditional covariant derivatives, specially by introducing the concept of an A-connection on E produced by a connection on A (Definition 3.3.5) which is analogous to that of covariant derivative on a vector bundle induced by a connection on the corresponding principal bundle; the notion of A-connections is implicitly used in the article that postulates the formulation of gauge theories [3], but it is not mentioned throughout the document. The study of the local trivializations of representations and A-connections were introduced by us in order to have enough tools to study the examples seen throughout the document.

3.1 Connections on Transitive Lie Algebroids

Throughout this section consider A to be a transitive Lie algebroid over the manifold M, and $0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0$ be a Lie algebroid sequence of A.

3.1.1 Ordinary Transitive Lie Algebroid Connections

Definition 3.1.1. An ordinary connection on A is a right splitting $\nabla : TM \to A$ (i.e. a section of a) as vector bundles of the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "j"] & A \arrow[r, "a"] & TM \arrow[r] \arrow[l, bend left, "\nabla"] & 0. \end{tikzcd} \tag{3.1}$$

For any $X \in TM$, the application of the connection is written as $\nabla_X \in A$.

Theorem 3.1.2. Let $\nabla : TM \to A$ be an ordinary connection on A. There is a unique L-valued, surjective 1-form $\omega \in \Omega^1(A, L)$, called *the connection form*, such that

$$\nabla_X = \boldsymbol{\mathfrak{X}} + j \circ \omega(\boldsymbol{\mathfrak{X}}), \tag{3.2}$$

for all $\mathfrak{X} \in A$ with $X = a(\mathfrak{X})$. In particular, this form is *normalized*, meaning that

$$\omega \circ j = -1_L; \tag{3.3}$$

that is, $-\omega$ is a left splitting (i.e. a retract of j) of (3.1) as a short exact sequence of vector bundles. Conversely, equation (3.2) defines an equivalence between normalized L-valued one forms and ordinary connections.

Proof. For any $\mathfrak{X} \in A$, with $X = a(\mathfrak{X})$, $a(\nabla_X - \mathfrak{X}) \in ker(a)$, so we can define the one form

$$\omega(\mathfrak{X}) := j^{-1}(\nabla_X - \mathfrak{X}); \tag{3.4}$$

since the Lie algebra bundle morphism j is injective, this ω is the only vector bundle morphism that satisfies (3.2).

For any $l \in L$, $\omega(j(l)) = j^{-1}(-j(l)) = -j \circ j^{-1}(l) = -1$, proving the first equation, and therefore the surjectivity of ω.

Now, to see that equation (3.2) produces a well defined connection from the normalized 1-form ω, let $\mathfrak{X}_1, \mathfrak{X}_2 \in A$ be such that $a(\mathfrak{X}_1) = a(\mathfrak{X}_2) = X$. Since $a(\mathfrak{X}_2 - \mathfrak{X}_1) = 0$, let $l \in L$ be such that $j(l) = \mathfrak{X}_2 - \mathfrak{X}_1$; in fact, since ω is normalized, $l = -\omega(\mathfrak{X}_2 - \mathfrak{X}_1)$. Then ∇_X is well defined since

$$\begin{aligned}
\mathfrak{X}_2 + j \circ \omega(\mathfrak{X}_2) &= \mathfrak{X}_1 + j(l) + j \circ \omega(\mathfrak{X}_1 + j(l)) \\
&= \mathfrak{X}_1 + j \circ \omega(\mathfrak{X}_1) + j(l) + j \circ w \circ j(l) \\
&= \mathfrak{X}_1 + j \circ \omega(\mathfrak{X}_1).
\end{aligned}$$

□

Theorem 3.1.3. On any transitive Lie algebroid there is at least one ordinary connection.

Proof. This is simply a restatement of the (standard) fact that the short exact sequence (3.1) in the category of vector bundles is split, which is equivalent to the statement that

$$A \cong TM \oplus L \tag{3.5}$$

as vector bundles, where $\nabla : TM \to A$ is then an embedding and $\omega : A \to L$ a projection. To prove the last equation we follow a standard partition of unity argument: let $\{U_\alpha\}$ be an open cover that trivializes TM, L and A simultaneously; then, on each $U \in \{U_\alpha\}$, there is a vector bundle isomorphism $f_\alpha : A|_U \to TM|_U \oplus L|U$. Now take a partition of unity $\{\rho_\alpha : U_\alpha \to [0,1]\}$ subordinate to it; then $\sum_\alpha \rho_\alpha f_\alpha : A \to TM \oplus L$ is a vector bundle isomorphism.

□

Example 3.1.4. The following example shows how *the generalized notion of an ordinary connection on a transitive Lie algebroid coincides with that* *of connection on a principal bundle, when the transitive Lie algebroid in question is the Atiyah Lie algebroid associated to the principal bundle.* Let $G \to P \xrightarrow{\pi} M$ be a principal bundle over the manifold M with structure Lie group G, with $\mathfrak{g} = Lie(G)$. On a principal bundle the notion of connection has multiple equivalent definitions, including:

- A G-invariant subbundle HP of TP (under the pushforward of the right action R of G on P), called the *horizontal subbundle*, such that $TP = HP \oplus VP$ where $VP = ker(\pi_*)$.
- A G-equivariant Lie algebra valued form $w : TP \to P \times \mathfrak{g}$ such that $w(\eta^*) = \eta$ where η^* is the fundamental vertical vector field associated to $\eta \in \mathfrak{g}$.

These two definitions are related to each other by $HP = ker(w)$. From the first definition it follows that for each $p \in P$ there is a section or π_*, called *the horizontal lift* $\widetilde{\nabla}^p : T_{\pi(p)}M \to H_pP \subseteq T_pP$ with the property that $\widetilde{\nabla}^{pg}_X = R_{g,*}(\widetilde{\nabla}^p_X) \in T_{pg}P$ for all $g \in G$ and $X \in T_{\pi(p)}M$, and, hence, a G-equivariant *horizontal projection* onto HP

$$\begin{aligned} H : T_pP &\to T_pP \\ \mathcal{X} &\mapsto \widetilde{\nabla}^p_{\pi_*(\mathcal{X})} \in HP. \end{aligned} \tag{3.6}$$

From the second definition we can define the G-equivariant *vertical projection* onto VP

$$\begin{aligned} V : T_pP &\to T_pP \\ \mathcal{X} &\mapsto w(\mathcal{X})^*_p \in VP. \end{aligned} \tag{3.7}$$

Both H and V are indeed projections since $H^2 = H$ and $V^2 = V$, and so any any $\mathcal{X} \in T_pP$ with $\pi_*(\mathcal{X}) = X$ can be decomposed as

$$\mathcal{X} = \widetilde{\nabla}^p_X \oplus w(\mathcal{X})^*_p \in HP \oplus VP. \tag{3.8}$$

The G-invariance of the horizontal lift implies that it induces a well defined vector bundle map

$$\begin{aligned} \nabla : TM &\to TP/G \\ X &\mapsto \left\langle \widetilde{\nabla}^p_X \right\rangle = \langle H(\mathcal{X}) \rangle , \end{aligned} \tag{3.9}$$

where p is any element in the fiber of $\pi(X) \in M$ and $\mathcal{X}$ is any tangent vector of P such that $\pi_*(\mathcal{X}) = x$; furthermore, ∇ is a section of the anchor π_*^G of the Atiyah Lie algebroid TP/G, meaning that ∇ is an ordinary connection on the Atiyah Lie algebroid associated to the principal bundle P. Notice that ∇ can be seen as a horizontal lift where the horizontal subbundle of TP/G is the class of HP under the group action; furthermore, this horizontal lift is simpler to use since it is globally defined as a vector bundle morphism, and no specification of a $p \in P$ is needed. Similary, the G-equivariance of w implies that it induces a well defined element of $\Omega^1(TP/G, P \times \mathfrak{g}/G)$:

$$\begin{aligned} \omega : TP/G &\to P \times \mathfrak{g}/G \\ \langle \mathcal{X} \rangle &\mapsto \langle w(\mathcal{X}) \rangle \, ; \end{aligned} \tag{3.10}$$

the normalization condition of ω is then simply a restatement of the condition $w(\eta^*) = \eta$ where $\eta_p^* = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} p \cdot \exp(t\eta) = -\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} p \cdot \exp(-t\eta)$ for all $\eta \in \mathfrak{g}$. From the decomposition (3.8) of every $\mathcal{X} \in T_pP$ it follows that

$$\begin{aligned} \widetilde{\nabla}_X - j \circ \omega(\langle \mathcal{X} \rangle) &= \left\langle \widetilde{\nabla}_X^p \right\rangle - j \, \langle w(\mathcal{X}) \rangle \\ &= \left\langle \widetilde{\nabla}_X^p \right\rangle - \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \exp(-tw(\mathcal{X})) \right\rangle, && \text{definition of } j \\ &= \left\langle \widetilde{\nabla}_X^p + w(\mathcal{X})_p^* \right\rangle \\ &= \langle \mathcal{X} \rangle && \text{decomposition (3.8);} \end{aligned}$$

hence, the ordinary connection ∇ on TP/G is associated, as specified by Theorem 3.1.2, to the normalized 1-form ω.

Example 3.1.5. Let E be a vector bundle over M. We will now see how the notion of connection on E, also called a covariant derivative on E, coincides with that of ordinary connection on the transitive Lie algebroid $\mathfrak{D}(E)$ with associated sequence $0 \to End(E) \to \mathfrak{D}(E) \to TM \to 0$. A connection on the vector bundle E is an $\mathbb{K}$-linear map

$$\nabla^E : \Gamma(E) \to \Omega^1(TM, E) \tag{3.11}$$

such that, for all $X \in TM$, $\mu \in \Gamma(E)$ and $f \in C^\infty(M)$, the following Leibniz rule is satisfied:

$$\nabla_X^E \mu(f\mu) = X(f)\mu + f\nabla_X^E \mu. \tag{3.12}$$

Notice that we may reorder the different components entering the connection to rewrite ∇ as a vector bundle morphism

$$\begin{aligned} \nabla^E : TM &\to \mathfrak{D}(E) \\ X &\mapsto \nabla_X, \end{aligned} \tag{3.13}$$

since the Leibniz rule is what defines the elements of $\Gamma(\mathfrak{D}(E))$ within the $\mathbb{K}$-linear maps $\Gamma(E) \to \Gamma(E)$; furthermore, this Leibniz rule shows that $a(\nabla_X) = X$, where a is the anchor of $\mathfrak{D}(E)$. The same arguments show that such an anchor preserving vector bundle morphism : $TM \to \mathfrak{D}(E)$ defines a vector bundle connection of E. Hence, *ordinary connections of the transitive Lie algebroid $\mathfrak{D}(E)$ and vector bundle connections of E are the same notion.*

Now, let $G \to P \to M$ be a principal bundle, and let us use the notation of the previous example. Suppose that E is associated to P and that it has the vector space V as typical fiber. A connection on the principal bundle P induces a connection on $E = P \times V/G$ as follows: let $X \in T_pM$ and let $\widetilde{\mu} \in C^\infty_G(P, V)$ be the G-equivariant function associated to the section $\mu \in \Gamma(E)$; then define the connection ∇^E by

$$\widetilde{\nabla^E_X(\mu)} : p \mapsto \widetilde{\nabla}^p_X(\widetilde{\mu}) \in C^\infty_G(P, V). \tag{3.14}$$

If $\mathcal{X}$ is any element in T_pP such that $\pi_*(\mathcal{X}) = X$ it is then true that

$$\begin{aligned} \nabla^E_X : \mu \mapsto & \underline{H(\mathcal{X})(\widetilde{\mu})} \\ &= \underline{(\mathcal{X} - V(\mathcal{X}))(\widetilde{\mu})} \end{aligned} \tag{3.15}$$

where the underline is the $C^\infty(M)$-module isomorphism : $C^\infty_G(P, V) \to \Gamma(E)$, inverse to the tilde map.

This connection on E has a simple rewriting in the language of Lie algebroids. First recall that the map $\mu \mapsto \underline{\mathcal{X}(\widetilde{\mu})}$ is precisely $\phi(\langle \mathcal{X} \rangle) \in \mathfrak{D}(E)$ where

$$\phi : TP/G \to \mathfrak{D}(E) \tag{3.16}$$

is the Lie algebroid representation of TP/G in E defined in 1.4.21. Hence, we may rewrite the first equation in (3.15) as the composition

$$\nabla^E : TM \xrightarrow{\nabla} A \xrightarrow{\phi} \mathfrak{D}(E) \qquad\qquad \nabla^E_X = \phi(\nabla_X) \tag{3.17}$$

since $\langle H(\mathcal{X})\rangle \equiv \nabla_{\pi_G^*\langle \mathcal{X}\rangle}$. Notice, however, that other covariant derivatives on E may be induced by an ordinary connection on TP/G is another Lie algebroid representation ϕ is used. *In conclusion, the covariant derivative on vector bundles associated to principal bundles induced by principal bundle connections is an example of a $\mathfrak{D}(E)$ ordinary connection induced by a Lie algebroid representation of TP/G on E and an ordinary connection on TP/G.*

Definition 3.1.6. Let E be a vector bundle over M and let $\phi : A \to \mathfrak{D}(E)$ be a representation of A. Given an ordinary connection $\nabla : TM \to A$ on A, $\nabla^E : TM \to \mathfrak{D}(E)$, $\nabla^E := \phi \circ \nabla$ is an ordinary connection on $\mathfrak{D}(E)$ called the produced connection on E induced by ϕ and ∇.

Using the notation of the previous definition, and recalling that for a representation ϕ of a transitive Lie algebroid sequence $L \to A \to M$, $\phi \circ j$ is denoted by ϕ_L, the equation $\nabla_X = \mathfrak{X} - j \circ \omega(\mathfrak{X})$ relating ∇ and its associated normalized connection 1-form ω allows us to rewrite the produced connection on E as

$$\nabla^E_X = \phi(\mathfrak{X}) - \phi_L \circ \omega(\mathfrak{X}) \tag{3.18}$$

for any $\mathfrak{X} \in TP/G$ such that $\pi^G_*(\mathfrak{X}) = X$. Then,

$$\phi_L \circ \omega \in \Omega^1(A, End(E)) \tag{3.19}$$

is a 1-form associated to the ordinary connection ∇^E; notice that the equation (3.18) is well defined for X precisely because ω is normalized, since then the right hand sides vanishes when $\mathfrak{X} \in Im(j)$. We will now relax the normalization condition on transitive Lie algebroids when defining connections, in which case the "covariant derivatives" that will be induced on vector bundles will be taken in any direction $\mathfrak{X} \in A$; these are called A-connections on E and will be defined in Section 3.3.

3.1.2 Generalized Connections

Definition 3.1.7. Given a transitive Lie algebroid sequence $L \hookrightarrow A \twoheadrightarrow TM$ for the transitive Lie algebroid A, any L-valued 1-form $\hat{\omega} \in \Omega^1(A, L)$ is called a (generalized) connection form on A. Given $\hat{\omega}$, define the reduced kernel endomorphism $\tau \in End(L) = \Omega^1(L, L)$ (L represented on itself through the bracket, i.e. the restriction to L of the adjoint representation $ad : A \to \mathfrak{D}(L))$ by

$$\tau := \omega \circ j + 1_L. \tag{3.20}$$

Notice that, although changing the chosen adjoint Lie algebroid L changes the strict definition of what the connection forms on A are, since every adjoint Lie algebroid is isomorphic to $ker(a)$ the distinction is not important, it is just a matter of choice of which adjoint Lie algebroid to work with. Also notice that a connection form on A is ordinary, i.e. $\hat{\omega}$ is normalized, if and only if $\tau = 0$.

Proposition 3.1.8. Let $\hat{\omega}$ be a connection form on A. Then

$$\hat{\nabla}_{\mathfrak{X}} := \mathfrak{X} + \hat{\omega} \circ j(\mathfrak{X}) \tag{3.21}$$

is an anchor preserving vector bundle morphism. Conversely, any such map defines a generalized connection form $\hat{\omega}$ related to $\hat{\nabla}$ by the above equation. Furthermore, $\hat{\nabla}$ is associated to an ordinary connection on A if and only if $\hat{\nabla} \circ j = 0$, in which case $\nabla_X := \hat{\nabla}_{\mathfrak{X}} + j \circ \hat{\omega}(\mathfrak{X})$ is well defined and an ordinary connection.

Proof. The equivalence between $\hat{\omega}$ and $\hat{\nabla}$ is clear from equation (3.21), since j, being injective, has a left inverse.

If $\hat{\omega}$ is an ordinary connection form, then $\hat{\omega} \circ j = -1_L$, hence, for any $l \in L$ $\hat{\nabla}(j(l)) = j(l) + j \circ \hat{\omega} \circ j(l) = j(l) + j(-l) = 0$. Conversely, if $\hat{\theta} \circ j = 0$, then for any $\mathfrak{X}_1, \mathfrak{X}_2 \in A$ such that $a(\mathfrak{X}_1) = a(\mathfrak{X}_2) = X$ it is true that $\hat{\nabla}_{\mathfrak{X}_1} = \hat{\nabla}_{\mathfrak{X}_2}$ and so the map $\nabla_X := \hat{\nabla}_{\mathfrak{X}_1}$ is well defined for all $X \in TM$; since $a(\nabla_X) = a(\hat{\nabla}_{\mathfrak{X}_1}) = X$, ∇ is an ordinary connection on A. $\square$

Remark 3.1.9. Just as in the case of an Atiyah Lie algebroid shown in Example 3.1.4, an ordinary connection ∇ on A is equivalent to the definition of a subbundle $Im(\nabla)$ on A, called the horizontal subbundle, that is isomorphic to TM and that defines an isomorphism $A \cong TM \oplus L$; in that case the associated $\hat{\nabla} : A \to A$ is the projection onto this subbundle and $\hat{\nabla}^2 = \hat{\nabla}$. With the generalized connections on A such an equivalent definition in terms of a horizontal subbundle is lost. However, a formula analogous to (3.18) will still allow the definition of "covariant derivatives" on representation vector bundles E as we will see in Section 3.3; furthermore, the additional directions will give rise to a coupling of the matter fields, i.e. sections of E, with new fields that enable the appearance of new mass terms in the Lagrangian of a gauge theory based on transitive Lie algebroids.

If we were to desire for $\hat{\nabla}$ to be a projection onto a subbundle, i.e. that $\hat{\nabla}^2 = \hat{\nabla}$, from equation (3.18) it follows that a necessary and sufficient condition is

$$\omega \circ j|_{Im(\hat{\omega})} = -1_{Im(\hat{\omega})}. \tag{3.22}$$

For example, $\hat{\omega} = 0$ is a generalized connection form on A and $\hat{\nabla} = 1_A$, so $\hat{\nabla}^2 = \hat{\nabla}$ trivially, and the induced "horizontal" subbundle of A is A itself.

The following proposition allows us to obtain an ordinary connection from a generalized connection, if we posses a "background" ordinary connection.

Proposition 3.1.10. Let $\hat{\omega}$ be a connection form on A with reduced kernel endomorphism τ, and let $\widetilde{\omega}$ be an ordinary connection form on A. Then,

$$\omega = \hat{\omega} + \tau \circ \widetilde{\omega} \tag{3.23}$$

is an ordinary connection form on A.

Proof. We see that ω is normalized from:

$$\begin{aligned} \omega \circ j &= \hat{\omega} \circ j + \tau \circ \widetilde{\omega} \circ j \\ &= (\tau - 1_L) + \tau \circ (-1_L) \\ &= -1_L; \end{aligned}$$

the second line follows from the definition of τ and the normalization of $\widetilde{\omega}$. □

3.1.3 Curvature

Definition 3.1.11. Let $\hat{\omega}$ be a connection form on A. The curvature form of $\hat{\omega}$ is the L-valued form $\hat{R}$

$$\hat{R} := \hat{d}\hat{\omega} + \frac{1}{2}\hat{\omega} \wedge^{[,]} \hat{\omega} \in \Omega^2(A, L), \tag{3.24}$$

where $\wedge^{[,]}$ is the wedge product in the Lie-graded differential Lie algebra $(\Omega^\bullet(A, L), \wedge^{[,]}, \hat{d})$ defined in 2.1.22; i.e. for any $\mathfrak{X}, \mathfrak{Y} \in \Gamma(A)$

$$\hat{R}(\mathfrak{X}, \mathfrak{Y}) = \hat{d}\hat{\omega}(\mathfrak{X}, \mathfrak{Y}) + [\hat{\omega}(\mathfrak{X}), \hat{\omega}(\mathfrak{Y})]. \tag{3.25}$$

The algebraic curvature of the reduced kernel endormorphism $\tau \in End(L)$ of $\hat{\omega}$ is

$$R_\tau(\eta, \theta) := [\tau(\eta), \tau(\theta)] - \tau[\eta, \theta]. \tag{3.26}$$

Proposition 3.1.12. Let $\hat{\omega}$ be a connection form on A with reduced kernel endomorphism τ. Then the curvature 2-form of $\hat{\omega}$ can be expressed as

$$\hat{R}(\mathfrak{X}, \mathfrak{Y}) = j^{-1}([\hat{\nabla}_\mathfrak{X}, \hat{\nabla}_\mathfrak{Y}] - [\hat{\nabla}_\mathfrak{X}, \hat{\nabla}_\mathfrak{Y}]), \tag{3.27}$$

for all $\mathfrak{X}, \mathfrak{Y} \in \Gamma(A)$.

Remark 3.1.13. From the previous proposition we see that the connection induced by $\hat{\omega}$ is flat, i.e. has 0 curvature, if and only if the endomorphism $\hat{\nabla} : A \to A$ is a Lie algebra bundle morphism, i.e. $\hat{\nabla}$ is a Lie algebroid morphism. Similarly, τ is flat, i.e. its algebraic curvature is 0 if and only if the endomorphism $\tau : L \to L$ is a Lie algebra bundle morphism.

If $\hat{\omega}$ is an ordinary connection form, then $j^*\hat{R} = 0$ since $\hat{\nabla} \circ j = 0$, and so it the curvature 2-form $\hat{R}$ defines an element $R \in \Omega^2(TM, L)$ called the curvature of the ordinary connection on A. When $\hat{\omega}$ is an ordinary connection on $A = TP/G$ for some $G \to P \to M$ principal bundle, then $R \in \Omega^2(TM, P \times \mathfrak{g}/G)$ is the form that corresponds to the basic [1] 2-form on P that is usually defined to be the curvature of the connection on P; when $\hat{\omega}$ is an ordinary connection on $A = \mathfrak{D}(E)$ for some vector bundle E over M, $R \in \Omega^2(TM, End(E))$ is precisely what is called the curvature of the corresponding connection on E.

Proposition 3.1.14. Let $\hat{\omega}$ be a connection form on A. Then the curvature form $\hat{R} \in \Omega^2(A, L)$ satisfies the Bianchi identity:

$$\hat{d}R + \hat{\omega} \wedge^{[,]} \hat{R} = 0. \tag{3.28}$$

Proof. Recall from Example 2.1.27 that $(\Omega^\bullet(A, L), \wedge^{[,]}, \hat{d})$ is a differential graded Lie algebra, hence:

$$\hat{d}\hat{R} + \hat{\omega} \wedge^{[,]} \hat{R} = \hat{d}(\hat{d}\hat{\omega} + \frac{1}{2}\hat{\omega} \wedge^{[,]} \hat{\omega}) + \hat{\omega} \wedge^{[,]} (\hat{d}\hat{\omega} + \frac{1}{2}\hat{\omega} \wedge^{[,]} \hat{\omega})$$

[1] A G-equivariant and Horizontal form on P. The G-equivariance allows to quotient both the initial and target space by G, the horizontality allows to replace TP/G by TM. Recall that vector fields of associated vector bundles are basic 0-forms on P.

$$= (0 + \frac{1}{2}\hat{d}\hat{\omega} \wedge^{[,]} \hat{\omega} - \frac{1}{2}\hat{\omega} \wedge^{[,]} \hat{d}\hat{\omega}) + (\hat{\omega} \wedge^{[,]} \hat{d}\hat{\omega} + \frac{1}{2}\hat{\omega} \wedge^{[,]} (\hat{\omega} \wedge^{[,]} \hat{\omega}))$$
$$= (-\hat{\omega} \wedge^{[,]} \hat{d}\hat{\omega}) + (\hat{\omega} \wedge^{[,]} \hat{d}\hat{\omega} + 0)$$
$$= 0.$$

The second line follows from applying the graded Leibniz rule (2.13) satisfies by $\hat{d}$; the third line follows from applying the anticommutativity (2.15) in the first parenthesis, and the graded Jacobi identity (2.16) to the second parenthesis. $\square$

Definition 3.1.15. Let $\hat{\omega}$ be an connection form on A with reduced kernel endomorphism $\tau \in End(L)$, let $\widetilde{\omega}$ be an ordinary connection form, call it a <u>background connection</u>, and let $\omega = \hat{\omega} + \tau \circ \widetilde{\omega}$ be the induced ordinary connection form. For $\widetilde{\omega}$ and ω let $\widetilde{\nabla}$ and $\nabla : TM \to A$ be the associated connections and $\widetilde{R}, R \in \Omega^2(TM, L)$ be the curvatures of $\widetilde{\omega}$ and ω.

- Define $\hat{F} = R - \tau \circ \widetilde{R} \in \Omega^2(TM, L)$. Notice that $a^*\hat{F} \in \Omega^2(A, L)$.
- Let $\mathcal{D}\tau \in \Omega^1(TM, End(L))$ defined by

$$(\mathcal{D}_X\tau)(\eta) := [\nabla_X, \tau(\eta)] - \tau[\widetilde{\nabla}_X, \eta]. \tag{3.29}$$

  Now, define

$$[a^*\mathcal{D}\tau, \widetilde{\omega}](\mathfrak{X}, \mathfrak{Y}) := (\mathcal{D}_{a(\mathfrak{X})})(\widetilde{\omega}\mathfrak{Y}) - (\mathcal{D}_{a(\mathfrak{Y})})(\widetilde{\omega}\mathfrak{X}). \tag{3.30}$$

  This is an element of $\Omega^2(A, L)$.

The ordinary connection ω allows us to write any $\mathfrak{X} \in A$ as $\mathfrak{X} = \phi(\nabla_{a(\mathfrak{X})}) - \phi_L \circ \omega(\mathfrak{X})$. This decomposition on the elements of A induces the following decomposition of the curvature form [3]:

Theorem 3.1.16. Using the notation of the previous definition, the curvature form of the connection form $\hat{\omega}$ may be decomposed as

$$\hat{R} = a^*\hat{F} - [a^*\mathcal{D}\tau, \widetilde{\omega}] + \widetilde{\omega}^* R_\tau. \tag{3.31}$$

This is called the <u>decomposition of the curvature form of $\hat{\omega}$ with respect to the background connection $\widetilde{\omega}$</u>.

3.2 Local Trivialization of Generalized Connection Forms

3.2.1 Connection Forms

Throughout this section consider A to be a transitive Lie algebroid over the manifold M, and $0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0$ be a Lie algebroid sequence of A; suppose all the vector bundles on this section have vector spaces over $\mathbb{K}$ as typical fibers, where $\mathbb{K}$ is one of $\mathbb{R}$ or $\mathbb{C}$. Also, suppose that $\{\psi^i, \nabla^{0,i}\}_{iinI}$ is a transitive Lie algebroid atlas of A. Additionally, let $\hat{\omega} \in \Omega^1(A, L)$ be a connection for on A with reduced kernel endomorphism $\tau \in End(L)$; then, the local trivialization of $\hat{\omega}$ over U_i is $\hat{\omega}_i \in \Omega^1()$. As usual with local trivializations, we prefer to adopt the module perspective.

Theorem 3.2.1. Over any U_i, $i \in I$, the local trivialization of the connection form $\hat{\omega}$ may be decomposed as the following element of $\Omega^1(TU_i, U_i \times \mathfrak{g}) \oplus_{C^\infty(U_i)} \Omega^1(U_i \times \mathfrak{g}, U_i \times \mathfrak{g})$:

$$\hat{\omega}_i = A_i \oplus (-\epsilon + \tau_i) \tag{3.32}$$

where

- $\tau_i := (\psi^i)^{-1} \circ \tau \circ \psi^i \in C^\infty(U_i, End(\mathfrak{g})) \cong \Omega^1(U_i \times \mathfrak{g}, U_i \times \mathfrak{g})$ is called <u>the local trivialization of τ</u> over U_i.
- $\epsilon = Id_{C^\infty(U_i, U_i \times \mathfrak{g})} \in \Omega^1(U_i \times \mathfrak{g}, U_i \times \mathfrak{g})$.
- $A_i = \psi_i^{-1} \circ \hat{\omega} \circ \nabla^{0,i} \in \Omega^1(TU_i, U_i \times \mathfrak{g})$ is called <u>the local tangent connection form of $\hat{\omega}$</u>.

Furthermore, $A_i - \epsilon$ is an ordinary connection form on the trivial Lie algebroid $TU_i \times \mathfrak{g}$, and ω_i is a generalized connection form on $TU_i \times \mathfrak{g}$.

We may also write

$$\hat{\omega}_i = A_i - \epsilon + \tau_i, \tag{3.33}$$

understanding A as $A \circ pr_1$, where $pr_1 : \Gamma(TU_i) \oplus C^\infty(U_i, \mathfrak{g}) \to \Gamma(TU_i)$ is the projection to the first coordinate, and understanding ϵ, τ_i as the ones from the propositions precomposed with pr_2.

Proof.

$$\begin{aligned}\hat{\omega}(X \oplus \eta) &= \psi_i^{-1} \circ \hat{\omega}(\nabla^{0,i}_X + j\psi_i(\eta))\\ &= \psi_i^{-1} \circ \hat{\omega}(\nabla^{0,i}_X) + \psi_i^{-1} \circ \hat{\omega} \circ j(\psi_i(\eta))\\ &= A_i(X) + \psi_i^{-1} \circ (\tau - 1_L)(\psi_i(\eta))\\ &= A_i(X) - \eta + \psi_i^{-1} \circ \tau \circ psi_i(\eta)\\ &= A_i(X) - \epsilon(\eta) + \tau_i(\eta).\end{aligned}$$

That $A_i - \epsilon$ is an ordinary connection follows simply from the fact that, for every $\eta \in C^\infty(U_i, \mathfrak{g})$, $(A_i - \epsilon)(\eta) = -\epsilon(\eta) = -\eta$. The last statement is a rewriting of $\hat{\omega}_i$ being a 1-form. □

Proposition 3.2.2. Let $\hat{\omega}$ be a connection form on A with local trivialization $\hat{\omega} = A_i - \epsilon + \tau_i$, decomposed as proposed in Theorem 3.2.1, on each U_i, $i \in I$. Then, for all $X \in \Gamma(TM)$ and all U_i, U_j such that $U_{ij} = U_i \cap U_j \neq \emptyset$,

- The local tangent connection form A_i has the following transformation law:
$$A_j(X) = \alpha^i_j(A_i(X)) + \chi^j_i(X). \tag{3.34}$$

- The local trivialization τ_i of τ has the following transformation law:
$$\tau_j(\eta) = \tau_i(\eta) - \tau_j \circ \chi^j_i(X). \tag{3.35}$$

Notice that if $\hat{\omega}$ is an ordinary connection form, a family of local trivializations $\{\omega_i \in \Omega^1(TU_i \times \mathfrak{g}, U_i \times \mathfrak{g})\}_{i\in I}$ of $\hat{\omega}$ is completely specified by its associated local tangent connection forms $\{A_i \in \Omega^1(TU_i, U_i \times \mathfrak{g})\}$, since then $\hat{\omega}_i = A_i - \epsilon$. Furthermore, notice that if A is an Atiyah Lie algebroid associated to a principal bundle, the transformation law of the A_i's is precisely that of the local connection 1-forms of a principal connection, hence the family $\{A_i\}_{i\in I}$ where the A_i's transform as indicated in the previous proposition does specify a principal connection of P. In general, *a family* $\{A_i, \tau_i\}_{i\in I}$ *with the transformation laws indicated above define a connection form on the transitive Lie algebroid* A.

Definition 3.2.3. Let $\hat{\nabla} \in End(A)$ be the anchor preserving endomorphism associated to the connection form $\hat{\omega}$. The local trivialization $\hat{\nabla}^i$ of $\hat{\nabla}$ over U_i is the map

$$\hat{\nabla}^i = S_i^{-1} \circ \hat{\nabla} \circ S_i \quad \in End(TU_i \times \mathfrak{g}). \tag{3.36}$$

If $\hat{\omega}$ is normalized, the local trivialization ∇^i over U_i of the associated map $\nabla : TM \to A$ is

$$\nabla^i = S_i^{-1} \circ \nabla \quad : TU_i \to TU_i \times \mathfrak{g}. \tag{3.37}$$

For any $X \oplus \eta \in \Gamma(TU_i) \oplus C^\infty(U_i, \mathfrak{g})$, it follows from the definitions that

$$\begin{aligned} \hat{\nabla}^i_{X\oplus\eta} &= X \oplus \eta + \hat{\omega}_i(X \oplus \eta) \\ &= X \oplus (A_i(X) - \tau(\eta)). \end{aligned} \tag{3.38}$$

If $\hat{\omega}$ were an ordinary connection form, then $\tau = 0$ and the local trivialization of $\nabla : TM \to A$ satisfies, for any $X \in \Gamma(TU_i)$:

$$\begin{aligned} \nabla^i_X &= X \oplus A_i(X) \\ &= \hat{\nabla}^i_{X\oplus\eta}, \end{aligned} \tag{3.39}$$

for all $\eta \in C^\infty(U_i, \mathfrak{g})$.

Let $\{E_a\}_{a=1,\dots,n}$ be a basis for $\mathfrak{g}$. Let us also denote by E_a, $a = 1, \dots, n$, the corresponding constant functions in $C^\infty(U_i, \mathfrak{g}) = \Omega^0(TU_i \times \mathfrak{g}, U_i \times \mathfrak{g})$ for all $i \in I$. Recall our convention of not writing the wedge product symbol if one of the factors is a a 0-form, like E_a.

Definition 3.2.4. Let $a, b \in 1, \dots, n$. The components of A_i, ϵ and τ_i with respect to the basis $\{E_a\}_{a=1,\dots,n}$ of $\mathfrak{g}$ are, respectively:

- $(A_i)^a \in \Omega^1(TU_i)$ be such that

  $$A_i = \sum_{a=1}^n (A_i)^a E_a,$$

  and if a set of coordinates $x^\mu : U_i \to \mathbb{R}$ on M, $\mu = 1, \dots, m$, is clear from the context, we will denote by

  $$(A_i)^a_\mu \equiv (A_i)^a(\partial_{x^\mu}) \in C^\infty(U_i); \tag{3.40}$$

- $\epsilon^a \in \Omega^1(U_i \times \mathfrak{g})$ such that
$$\epsilon = \sum_{a=1}^{n} \epsilon^a E_a,$$
i.e. each ϵ^a can be seen as the constant function in $C^\infty(U_i, \mathfrak{g}^*)$ used in Theorem 2.2.6, taking the constant value $\epsilon^a \in \mathfrak{g}^*$ that satisfies $\epsilon^a(E_b) = \delta^a_b$;

- $(\tau_i)^a_b \in C^\infty(U_i)$ such that
$$\tau_i = \sum_{a,b=1}^{n} (\tau_i)^a_b E_b \epsilon^b,$$
i.e. $\tau_i(E_b) = \sum_{a=1}^{n} (\tau_i)^a_b E_a$.

3.2.2 Mixed Local Basis

In this section let $\omega \in \Omega(A, L)$ be an ordinary connection form with local trivializations $\omega_i = A_i - \epsilon$ over each U_i, and associated ordinary connection $\nabla : TM \to A$.

Let $\{E_a\}_{a=1,\ldots,n}$ be a basis for $\mathfrak{g}$ with dual basis $\{\epsilon^a\}_{a=1,\ldots,n}$. Notice that, for any $i, j \in I$, $\hat{\alpha}^i_j(\epsilon^a)(X) = \epsilon^a(s^j_i(X)) = \epsilon^a(\chi^j_i(X)) \neq 0$; hence, $\hat{\alpha}^i_j(\epsilon^a)$ is a linear combination of forms that must include forms outside $\{\epsilon^b\}_{b=1,\ldots,n}$. In this section we develop new 1-forms that will allow the decomposition of any form on A into linear combination of products of 1-forms, where each product of 1-form transforms into one of the same kind under $\hat{\alpha}^i_j$.

Definition 3.2.5. Let $a = 1, \ldots, n$. The local 1-forms,
$$\mathfrak{a}^a_i := (A_i)^a - \epsilon^a \in \Omega^1(TU_i \times \mathfrak{g}) \tag{3.41}$$
are called the mixed local basis of the inner part of $\Omega^1(TU_i \times \mathfrak{g}, U_i \times \mathfrak{g})$ with respect to the ordinary connection form ω and the basis $\{E_a\}_{a=1,\ldots,n}$ of $\mathfrak{g}$.

For the rest of the section suppose that in U_i the tangent bundle is trivialized with coordinates $x^\mu : U_i \to \mathbb{R}$, $\mu = 1, \ldots, m$, with associated local vector fields $\partial_\mu \equiv \partial_{x^\mu}$. Denote by $\nabla_\mu := \nabla_{\partial_{x^\mu}} \in \Gamma(TU_i \times \mathfrak{g})$. Notice that, from the trivialization of ∇ equation (3.39) it follows that
$$\nabla^i_\mu = \partial_\mu \oplus A_i(\partial_\mu). \tag{3.42}$$

Proposition 3.2.6. The set

$$\{\nabla^i_\mu, -E_a\}_{\mu=1,\dots,m;a=1,\dots,n} \subseteq \Gamma(TU_i \times \mathfrak{g}) \tag{3.43}$$

is a global frame of the transitive Lie algebroid $TU_i \times \mathfrak{g}$ dual to the the frame

$$\{dx^\mu, \mathfrak{a}^a\}_{\mu=1,\dots,m;a=1,\dots,n} \subseteq \Gamma((TU_i \times \mathfrak{g})^*) = \Omega^1(TU_i \times \mathfrak{g}). \tag{3.44}$$

Proof. The linear independence of $\{\nabla^i_\mu, -E_a\}$ follows from that, if we try to write a ∇^i_μ as a linear combination $\sum_{\nu\neq\mu} c^\nu \nabla^i_\nu + \sum_a d^a E_a$, applying the anchor reduces the argument to the fact that $\{\partial_\mu\}$ is linearly independent; similarly, writing E_a as a linear combination $\sum_\mu c^\mu \nabla^i_\nu + \sum_{b\neq a} d^b E_b$ and applying ϵ reduces the argument to $\{E_b\}$ being linearly independent.

Now to see that they are indeed dual to each other. Let $\mu = 1,\dots,m$, $a = 1,\dots,n$, and recall that $A - \epsilon = \mathfrak{a}^a E_a$ is an ordinary connection form, hence normalized:

$$\begin{aligned}
dx^\mu(\partial_\nu \oplus (A_i(\partial_\nu))) &= dx^\mu(\partial_\nu) = \delta^\mu_\nu \\
dx^\mu(0 \oplus (-E_a)) &= 0 \\
\mathfrak{a}^a(\partial_\mu + A_i(\partial_\mu)) &= A(\partial_\mu) + \mathfrak{a}^a(+ \oplus A_i(\partial_\mu)) \\
&= A(\partial_\mu) - A(\partial_\mu) = 0 \\
\mathfrak{a}^a(0 \oplus (-E_b)) &= -\epsilon^a(-E_b) = \delta^a_b.
\end{aligned}$$

Since $\{dx^\mu, \mathfrak{a}^a\}$ is dual to the global frame $\{\nabla^i_\mu, -E_a\}$, then it must be a global frame of the dual bundle. $\square$

The following proposition now follows, giving an alternative decomposition of forms to the one given in Theorem 2.3.3:

Proposition 3.2.7. Let E be a vector bundle over M that trivializes over U_i as $U_i \times V$, and let $\{e_u\}_{r=1,\dots,t}$ be a basis of the vector space V, and let A be represented on E. Let β be an E-valued p-form on A, $p \in \mathbb{Z}_{\geq 0}$. Viewing each e_u as a constant section of E, i.e. $e_u \in \Omega^0(TM \times \mathfrak{g}, E)$, each local trivialization of β may be decomposed as follows

$$\beta_i = \sum_{r+s=p} (\beta_i)^u_{\mu_1\cdots\mu_r,a_1\cdots a_s}\, e_u\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \wedge \mathfrak{a}^{a_1}_i \wedge \cdots \wedge \mathfrak{a}^{a_s}_i. \tag{3.45}$$

with each $(\beta_i)^a_{\mu_1\cdots\mu_r,a_1\cdots a_s} \in C^\infty(M,V)$, $a = 1,\dots,n$, are called the scalar components of β_i with respect to the local mixed basis. The convention of ignoring the $\wedge$ symbol when one of the factors is a 0-form has been used.

Let the matrix valued function G^i_j on $U_{ij} \neq \emptyset$

$$G^{i\,b}_{j\,a} = \epsilon^b \circ \alpha^i_j(E_a) \tag{3.46}$$

be the matrix representation of the transition function α^i_j from U_j to U_i of L with respect to the basis $\{E_a\}_{a=1,\ldots,n}$ of $\mathfrak{g}$.

Theorem 3.2.8. Let $\mu_1, \ldots, \mu_r \in \{1, \ldots, m\}$ and $a_1, \ldots, a_s \in \{1, \ldots n\}$. Then the following homogeneous transformation law under change of trivializations is satisfied:

$$\begin{aligned}\hat{\alpha}^i_j(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \wedge \mathfrak{a}^{a_1}_j \wedge \cdots \wedge \mathfrak{a}^{a_s}_j) = \\ G^{j\,a_1}_{i\,b_1} \cdots G^{j\,a_1}_{i\,b_1} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \wedge \mathfrak{a}^{b_1}_i \wedge \cdots \wedge \mathfrak{a}^{b_s}_i;\end{aligned}$$

where Einstein's summation convention has been used. In particular, let E be a vector bundle over M that trivializes over U_i and U_j as $U_i \times V$ and $U_j \times V$, $U_{ij} \neq \emptyset$ and with transition function from U_j to U_i $\alpha^i_j : U_{ij} \to End(V)$; let β be an E-valued p-form on A, $p \in \mathbb{Z}_{\geq 0}$; for E-valued forms on A, its components with respect to the mixed local basis transform as follows

$$(\beta_i)_{\mu_1 \cdots \mu_r a_1 \ldots a_s} = G^{j\,b_1}_{i\,a_1} \cdots G^{j\,b_s}_{i\,a_s} \alpha^i_j((\beta_j)_{\mu_1 \cdots \mu_r b_1 \ldots b_s}) \tag{3.47}$$

Proof. Since $\hat{\alpha}^i_j$ is a morphism of differential graded modules over the ring of scalar valued forms on $TU_{ij} \times \mathfrak{g}$, it respects the wedge product and so the proof reduces to demonstrating that

$$\hat{\alpha}^j_i(\mathfrak{a}^a_i) = G^{i\,b}_{j\,a} \mathfrak{a}^b_j. \tag{3.48}$$

Since $\mathfrak{a}_i = \mathfrak{a}^a_i E_a$ and $\mathfrak{a}_j$ are local trivializations of the connection form ω, we know that $\hat{\alpha}^i_j(\mathfrak{a}_j) = \mathfrak{a}_i$, but

$$\begin{aligned}\hat{\alpha}^i_j(\mathfrak{a}_j) &= \alpha^i_j \circ \mathfrak{a}_j \circ s^i_j \\ &= \alpha^i_j[(\mathfrak{a}^b_j \circ s^j_i)E_b] \\ &= G^{i\,a}_{j\,b}(\mathfrak{a}^b_j \circ s^j_i)E_a \\ &= \mathfrak{a}_i;\end{aligned}$$

applying $\hat{\alpha}^j_i = (\hat{\alpha}^i_j)^{-1}$ on both sides gives the desired equation. $\square$

3.2.3 Curvature Form

Let $\hat{R} \in \Omega^2(A, L)$ be the curvature form of the connection form $\hat{\omega}$, and recall the maps defined in 3.1.15 for the background connection form $\widetilde{\omega}$ and the induced ordinary connection form ω. Let $\{E_a\}_{a=1,\dots,n}$ be a basis of $\mathfrak{g}$.

Definition 3.2.9.

- Over U_i, <u>the local trivialization of the algebraic curvature of τ</u>, the reduced kernel endomorphism of $\hat{\omega}$, is $C^\infty(U_i)$-multilinear antisymmetric map

$$(R_\tau)_i = \psi_i^{-1} \circ \psi_i^* R_\tau : C^\infty(U_i, \mathfrak{g}) \times C^\infty(U_i, \mathfrak{g}) \to C^\infty(U_i, \mathfrak{g}).$$

  Denote by $(W_i)^c_{ab} \in C^\infty(U_i)$ the functions that satisfy

$$(R_\tau)_i(E_a, E_b) = (W_i)^c_{ab} E_c;$$

  call them <u>the components of R_τ over U_i with respect to the basis $\{E_a\}_{a=1,\dots n}$</u>.

- Over U_i, <u>the local trivialization of $\mathcal{D}\tau$</u> is the element

$$\begin{aligned}(\mathcal{D}\tau_i) : \Gamma(TU_i) &\to C^\infty(U_i, End(\mathfrak{g}^*))\\ X &\mapsto \psi_i^{-1} \circ \psi_i^*(\mathcal{D}_{\nabla^{0,i}_X}\tau)\end{aligned}$$

  induced by the local trivialization of A over U_i. Let $x^\mu : U_i \to \mathbb{R}$, $\mu = 1, \dots, m$ be coordinates on U_i and denote by $(\mathcal{D}\tau_i)^b_{\mu,a} \in C^\infty(U_i)$ the functions that satisfy

$$(\mathcal{D}_{\partial_\mu}\tau_i)(E_a) = (\mathcal{D}\tau_i)^b_{\mu,a} E_b. \tag{3.49}$$

Theorem 3.2.10. Let $U_i \subseteq M$ be an open on which $x^\mu : U_i \to \mathbb{R}$ are coordinates and the transitive Lie algebroid A trivializes. Let $\widetilde{\omega} \in \Omega^1(A, L)$ be an ordinary connection form on A with local trivialization $\mathfrak{a}_i$ over U_i, and call it a background connection on A. Finally, for a (generalized) connection form $\hat{\omega}$ on A with reduced kernel endomorphism τ, let $\omega = \hat{\omega} + \tau \circ \widetilde{\omega}$ be the ordinary connection induced by $\hat{\omega}$ with respect to the background connection $\widetilde{\omega}$, let $\hat{R} \in \Omega^2(A, L)$ be its curvature 2-form, and $\hat{F} \in \Omega^2(TM, L)$, $\mathcal{D}\tau \in \Omega^1(TM, End(L))$ and $R_\tau \in \Omega^2(L, L)$ be defined as in Definition 3.1.15. Then, the local trivialization of $\hat{R}$ has the following decomposition in terms of the mixed local basis of the inner part of $\Omega^1(TU_i \times \mathfrak{g})$ with respect to the basis $\{E_a\}_{a=1,\ldots,n}$ of $\mathfrak{g}$:

$$\hat{R}_i = (\hat{F}_i)^a_{\mu\nu} E_a dx^\mu \wedge dx^\nu + (\mathcal{D}\tau_i)^b_{\mu,a} E_b dx^\mu \wedge (\mathfrak{a}_i)^a + (W_i)^c_{ab} E_c (\mathfrak{a}_i)^a \wedge (\mathfrak{a}_i)^b$$

where $a, b, c \in \{1, \ldots a\}$, $\mu, \nu \in \{1, \ldots, m\}$ and where Einstein's summation convention for the sum over repeated super- and sub-indices is used.

Proof. What we want to see is that the local trivialization over U_i of each of the terms on the decomposition 3.1.16 coincides with each of the stated terms.

That $a^* F_i = \hat{F}^a_{\mu\nu} E_a dx^\mu \wedge dx^\nu$ is clear since $\hat{F}$ is a 2-form on TM and a is the anchor.

To see that $[a^*\mathcal{D}, \widetilde{\omega}]_i = (\mathcal{D}\tau_i)^b_{\mu,a} dx^\mu \wedge (\mathfrak{a}_i)^a$, first recall that $\{\widetilde{\nabla}^i_\mu, -E_a\} \subseteq \Gamma(TU_i \times \mathfrak{g})$, where $\widetilde{\nabla}^i$ is the local trivialization of the connection $\widetilde{\nabla} : TM \to A$ associated to $\widetilde{\omega}$, is the global frame of $TU_i \times \mathfrak{g}$ dual to $\{dx^\mu, \mathfrak{a}^a\}$. Then, the result follows from the following three calculations:

$$\begin{aligned}
[a^*\mathcal{D}, \widetilde{\omega}]_i(\widetilde{\nabla}^i_\mu, \widetilde{\nabla}^i_\nu) &= (\mathcal{D}_{\partial_\mu}\tau_i)(\widetilde{\omega}_i(\widetilde{\nabla}^i_\mu)) - (\mathcal{D}_{\partial_\nu}\tau_i)(\widetilde{\omega}_i(\widetilde{\nabla}^i_\nu)) \\
&= (\mathcal{D}_{\partial_\mu}\tau_i)(0) - (\mathcal{D}_{\partial_\nu}\tau_i)(0) = 0;
\end{aligned}$$

$$\begin{aligned}
[a^*\mathcal{D}, \widetilde{\omega}]_i(\widetilde{\nabla}^i_\mu, -E_a) &= (\mathcal{D}_{\partial_\mu}\tau_i)(\widetilde{\omega}_i(-E_a)) - (\mathcal{D}_0\tau_i)(\widetilde{\omega}_i(\widetilde{\nabla}^i_\mu)) \\
&= (\mathcal{D}\tau_i)^b_{\mu,a} E_b;
\end{aligned}$$

$$[a^*\mathcal{D}, \widetilde{\omega}]_i(-E_a, -E_b) = (\mathcal{D}_0\tau_i)(\widetilde{\omega}_i(-E_b)) - (\mathcal{D}_0\tau_i)(\widetilde{\omega}_i(-E_a)) = 0.$$

Finally, that $(\widetilde{\omega}^* R_\tau)_i = (W_i)^c_{ab} E_c (\mathfrak{a}_i)^a \wedge (\mathfrak{a}_i)^b$ follows from:

$$\begin{aligned}
(\widetilde{\omega}^* R_\tau)_i(\widetilde{\nabla}^i_\mu, \widetilde{\nabla}^i_\nu) &= (R_\tau)_i(\widetilde{\omega}_i(\widetilde{\nabla}^i_\mu), \widetilde{\omega}_i(\widetilde{\nabla}^i_\nu)) \\
&= (R_\tau)_i(0, 0) = 0;
\end{aligned}$$

$$\begin{aligned}
(\widetilde{\omega}^* R_\tau)_i(\widetilde{\nabla}^i_\mu, -E_a) &= (R_\tau)_i(\widetilde{\omega}_i(\widetilde{\nabla}^i_\mu), \widetilde{\omega}_i(-E_a)) \\
&= (R_\tau)_i(0, E_a) = 0;
\end{aligned}$$

$$\begin{aligned}
(\widetilde{\omega}^* R_\tau)_i(-E_a, -E_b) &= (R_\tau)_i(\widetilde{\omega}_i(-E_a), \widetilde{\omega}_i(-E_b)) \\
&= (R_\tau)_i(E_a, E_b) \\
&= (W_i)^c_{ab} E_c.
\end{aligned}$$

□

Denoting by $C^c_{ab} = \epsilon^c([E_a, E_b])$ the structure constants of $\mathfrak{g}$ in the basis $\{E_a\}_{a=1,\ldots,n}$, it follows directly from the definition of $\hat{F}$ and R_τ, in 3.1.15, that

$$(\hat{F}_i)^a_{\mu,\nu} = F^a_{\mu,\nu} - (\tau_i)^a_b \widetilde{F}^b_{\mu,\nu}, \tag{3.50}$$

and

$$(W_i)^c_{ab} = (\tau_i)^b_a (\tau_i)^e_b C^c_{de} - C^d_{ab} (\tau_i)^c_d. \tag{3.51}$$

Similarly, from the definition of $\mathcal{D}\tau$ if follows that

$$\begin{aligned}
(\mathcal{D}\tau_i)^b_{\mu,a} &= \epsilon^b((\mathcal{D}_{\partial_\mu}\tau)(E_a)) \\
&= \epsilon^b([\partial_\mu + A_i(\partial_\mu), \tau_i(E_a)] - \tau_i[\partial_\mu + \widetilde{A}_i(\partial_\mu), E_a]) \\
&= \epsilon^b([\partial_\mu + (A_i)^c_\mu E_c, (\tau_i)^d_a E_d] - \tau_i[\partial_\mu + (\widetilde{A}_i)^d_\mu E_d, E_a]) \qquad (3.52) \\
&= \epsilon^b(\partial_\mu((\tau_i)^b_a E_b) + (A_i)^c_\mu \tau^d_a [E_c, E_d] - (\widetilde{A}_i)^d_\mu \tau_i [E_d, E_a]) \\
&= \partial_\mu(\tau_i)^b_a + (A_i)^c_\mu (\tau_i)^d_c C^b_{cd} - (\widetilde{A}_i)^d_\mu (\tau_i)^b_c C^c_{da}.
\end{aligned}$$

3.3 A-connections

In this section, let A be any Lie algebroid over a manifold M, not necessarily transitive. A-connections will be yet another generalizations of connections, in this case of vector bundle connections or covariant derivatives, on which the directions in which we take the derivatives are not only the tangent directions of TM, but the "generalized directions" of A. Additionally, let E be a vector bundle over M.

Definition 3.3.1. An A-connection on E is an anchor preserving vector bundle map

$$\hat{\nabla}^E : A \to \mathfrak{D}(E). \tag{3.53}$$

The curvature of $\hat{\nabla}^E$ is the $End(E)$-valued 2-form:

$$\hat{R}^E(\mathfrak{X}, \mathfrak{Y}) := j^{-1}([\hat{\nabla}^E_{\mathfrak{X}}, \hat{\nabla}^E_{\mathfrak{Y}}] - \hat{\nabla}^E_{[\mathfrak{X},\mathfrak{Y}]}). \tag{3.54}$$

Example 3.3.2. On any vector bundle E over M, an TM-connection is simply an ordinary connection on E, which also coincides with a connection on the vector bundle E as seen in Example 3.1.4.

Example 3.3.3 (All TLA-connections on a trivial vector bundle)**.** Let $A = TM \times \mathfrak{g}$ be a trivial Lie algebroid and $E = M \times V$ a vector bundle. An A-connection $\hat{\nabla}^E : A \to \mathfrak{D}(E)$ is an anchor preserving vector bundle morphisms, therefore it has the decomposition

$$\hat{\nabla}^E_{X\oplus\eta} = X \oplus (C(X) + \nu(\eta)), \tag{3.55}$$

for all $X \oplus \eta \in TM \times \mathfrak{g}$, where $C : TM \to M \times \mathfrak{gl}(V))$ and and $\nu : M \times \mathfrak{g} \to M \times \mathfrak{gl}(V)$ are vector bundle morphisms. In their algebraic versions, i.e. seen as maps between the modules of sections, $C : \Gamma(TM) \to C^\infty(M, \mathfrak{g})$ and $\nu : C^\infty(M, \mathfrak{g}) \to C^\infty(M, \mathfrak{gl}(V))$ are $C^\infty(M)$-linear maps, or ν can also be seen as a function $\nu \in C^\infty(U_i, Hom(\mathfrak{g}, \mathfrak{gl}(V)))$.

Example 3.3.4. Let A be a transitive Lie algebroid, and suppose that $\hat{\nabla}^E$ is an A-connection on E. If $\hat{\nabla} : A \to A$ is the anchor preserving vector bundle endomorphism associated to a connection $\hat{\omega}$ on A, then, the composition:

$$\hat{\nabla}^{E,\hat{\omega}} : A \xrightarrow{\hat{\nabla}} A \xrightarrow{\hat{\nabla}^E} \mathfrak{D}(E) \tag{3.56}$$

is an anchor preserving vector bundle morphism, i.e. it is an A-connection on E.

An important subfamily of examples that sits at the core of this document is the following. From now on we will assume that $0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0$ is a transitive Lie algebroid sequence.

Definition 3.3.5. Let $\hat{\omega}$ be a connection 1-form on A with associated vector bundle endomorphism $\hat{\nabla} : A \to A$. Let $\phi : A \to \mathfrak{D}(E)$ be a representation of A on E. Then, the A-connection on E produced by the connection $\hat{\omega}$ is the composition

$$\hat{\nabla}^{E,\hat{\omega}} : A \xrightarrow{\hat{\nabla}} A \xrightarrow{\phi} \mathfrak{D}(E). \tag{3.57}$$

Theorem 3.3.6. Let $\phi : A \to \mathfrak{D}(E)$ be a representation. Then there is a bijective correspondence between A-connections $\hat{\nabla}^E$ on E and 1-forms $\hat{\omega}^E \in$

$\Omega^1(A, End(E))$, using the representation of Example 2.1.28, given by the relation:

$$\hat{\nabla}^E_{\mathfrak{X}} = \phi(\mathfrak{X}) + j \circ \hat{\omega}^E(\mathfrak{X}). \tag{3.58}$$

Furthermore, the curvature of $\hat{\nabla}^E$ has the formula:

$$\hat{R}^E = \hat{d}_E \hat{\omega}^E + \frac{1}{2}\hat{\omega}^E \wedge^{[,]} \hat{\omega}^E, \tag{3.59}$$

and it satisfies the Bianchi identity:

$$\hat{d}_E \hat{R}^E + \hat{\omega}^E \wedge^{[,]} \hat{R}^E = 0. \tag{3.60}$$

Proof. Given an A-connection $\hat{\nabla}^E$, $a(\hat{\nabla}^E_{\mathfrak{X}} - \phi(\mathfrak{X})) = a(\mathfrak{X}) - a(\mathfrak{X}) = 0$ for all $\mathfrak{X} \in A$, so there is an element $\hat{\omega}^E(\mathfrak{X}) \in End(E)$ (recall that $End(E) \xrightarrow[hook]{j} \mathfrak{D}(E) \twoheadrightarrow TM$ is a Lie algebroid sequence for the transitive Lie algebroid $\mathfrak{D}(E)$). That $\hat{\omega}^E : A \to End(E)$ is a vector bundle morphism, and hence a form, follows from the linearity of $\hat{\nabla}^E - \phi$.

Conversely, given a 1-form $\hat{\omega}^E(A, End(E))$, for all $\mathfrak{X} \in A$, $\phi(\mathfrak{X})$ and $\jmath \circ \hat{\omega}^E(\mathfrak{X})$ are elements of $\mathfrak{D}(E)$, and so is their sum, so $\hat{\nabla}^E := \phi + j \circ \hat{\omega}^E : A \to \mathfrak{D}(E)$ is a vector bundle morphism; Since $a(j \circ \hat{\omega}^E(\mathfrak{X})) = 0$, $\hat{\nabla}^E$ is anchor preserving.

Now, let $\mathfrak{X}, \mathfrak{Y} \in A$ be arbitrary:

$$\begin{aligned}
j \circ \hat{R}^E(\mathfrak{X}, \mathfrak{Y}) &= [\hat{\nabla}^E_{\mathfrak{X}}, \hat{\nabla}^E_{\mathfrak{Y}}] - \hat{\nabla}^E_{[\mathfrak{x},\mathfrak{Y}]} \\
&= [\mathfrak{X} + j \circ \hat{\omega}^E(\mathfrak{X}), \mathfrak{Y} + j \circ \hat{\omega}^E(\mathfrak{Y})] - ([\mathfrak{X}, \mathfrak{Y}] + j \circ \hat{\omega}^E[\mathfrak{X}, \mathfrak{Y}]) \\
&= [j \circ \hat{\omega}^E(\mathfrak{X}), \mathfrak{Y}] + [\mathfrak{X}, j \circ \hat{\omega}^E(Y)] + [j \circ \hat{\omega}^E(\mathfrak{X}), j \circ \hat{\omega}^E(\mathfrak{Y})] - j \circ \hat{\omega}^E[\mathfrak{X}, \mathfrak{Y}] \\
&= [\mathfrak{X}, j \circ \hat{\omega}^E(\mathfrak{Y})] - [\mathfrak{Y}, j \circ \hat{\omega}^E(\mathfrak{X})] - j \circ \hat{\omega}^E[\mathfrak{X}, \mathfrak{Y}] + j[\hat{\omega}(\mathfrak{X}), \hat{\omega}(\mathfrak{Y})] \\
&= j(\hat{d}_E \omega^E(\mathfrak{X}, \mathfrak{Y}) + [\hat{\omega}^E(\mathfrak{X}), \hat{\omega}^E(\mathfrak{Y})]) \\
&= j \circ (\hat{d}_E \hat{\omega}^E + \frac{1}{2}\hat{\omega}^E \wedge^{[,]} \hat{\omega}^E).
\end{aligned}$$

That the curvature 2-form satisfies the Bianchi identity follows from the previous formula for $\hat{R}^E$ and from the fact that $(\Omega^\bullet(A, End(E)), \wedge^{[,]}, \hat{d}_E)$ is a differential graded Lie algebra that follows from Theorem 2.1.26, so a calculation identical to that followed for the curvature 2-form of a connection on a transitive Lie algebroid, in the proof of 3.1.14, can be applied. □

For an A-connection $\hat{\nabla}^{E,\hat{\omega}}$ produced by the connection 1-form $\hat{\omega} \in \Omega^1(A, L)$,

$$\begin{aligned}\hat{\nabla}^{E,\hat{\omega}}_{\mathfrak{X}} &= \phi(\hat{\nabla}_{\mathfrak{X}}) \\ &= \phi(\mathfrak{X} + j \circ \hat{\omega}(\mathfrak{X})) \\ &= \phi(\mathfrak{X}) + j \circ \phi_L \circ \hat{\omega}(\mathfrak{X})\end{aligned} \tag{3.61}$$

for all $\mathfrak{X} \in A$. This means that *the* $End(E)$*-valued* 1*-form associated to the* A*-connection produced by* $\hat{\omega}$ *is* $\phi_L \circ \hat{\omega}$.

Definition 3.3.7. Let $\hat{\nabla}^E$ be an A-connection on E, for A with Lie algebroid atlas over $\{U_i\}_{i\in I}$, and let E have local trivialization $\beta_i : U_i \times V \to E|_{U_i}$ over U_i. The local trivialization over U_i of the A-connection $\hat{\nabla}^E$ is the $TU_i \times \mathfrak{g}$-connection on $U_i \times V$ defined by

$$\hat{\nabla}^{E,i}_{X\oplus\eta} f := \beta_i^{-1}(\hat{\nabla}^E_{S_i(X\oplus\eta)}\beta_i(f)) \tag{3.62}$$

for all $f \in C^\infty(U_i, V)$.

Following Example 3.3.3, the local trivialization of an A-connection over a set U_i that trivializes both A and E has a decomposition

$$\hat{\nabla}^{E,i}_{X\oplus\eta} = X \oplus (C_i(X) + \nu_i(\eta)), \tag{3.63}$$

for all $X \oplus \eta \in TU_i \times \mathfrak{g}$, where $C_i : TM \to M \times \mathfrak{gl}(V))$ and and $\nu_i : M \times \mathfrak{g} \to M \times \mathfrak{gl}(V)$ are vector bundle morphisms. If, furthermore, the vector bundle E is locally trivialized by each U_i in the open cover $\{U_i\}_{i\in I}$ over which there is a Lie algebroid atlas of A, then an A-connection $\hat{\nabla}^E : A \to \mathfrak{D}(E)$ is encoded in a family of trivializations $\{(C_i, \nu_i)\}_{i\in I}$.

In the case that the A-connection is produced by a connection $\hat{\omega}$ locally trivialized over U_i as $\hat{\omega}_i = A_i - \epsilon + \tau_i$, given a representation $\phi : A \to \mathfrak{D}(E)$ locally trivialized over U_i as $\phi_i = a \oplus (B_i + \phi_{L,i})$, we can easily check that

$$C_i = B_i + \phi_L \circ A_i, \qquad \nu_i = -\phi_{L,i} \circ \tau_i,$$

therefore *the trivializiation of an* A*-connection on* E *produced by a connection on* A *has the following formula*:

$$\hat{\nabla}^{E,i}_{X\oplus\eta} = X \oplus [B_i(X) + \phi_{L,i} \circ A_i(X) - \phi_{L,i} \circ \tau_i(\eta)]. \tag{3.64}$$

We can be even more explicit. If we have a basis $\{E_a\}_a$ with $a, b = 1, \dots, n$ of $\mathfrak{g}$ dual to $\{\epsilon^a\}_a$, $\{e_u\}_u$ with $u, v = 1, \dots h$ of V dual to $\{\widetilde{e}^u\}$, and coordinates $\{x^\mu : U_i \to \mathbb{R}\}_\mu$ with $\mu, \nu = 1, \dots m$ of M we can define the following derivations over $U_i \times V$:

$$\begin{aligned} \hat{\nabla}^{E,i}_\mu := \hat{\nabla}^{E_i}_{\partial_\mu \oplus 0} &= \partial_\mu + (B_i)_\mu + A^b_\mu \phi_L(E_b) \\ &= \partial_\mu \oplus (B_i)^u_{\mu,v} e_u \widetilde{e}^v + (\phi_{L,i})^u_{b,v} (A_i)^b_\mu e_u \widetilde{e}^v \end{aligned} \tag{3.65}$$

and

$$\begin{aligned} \hat{\nabla}^{E,i}_a := \hat{\nabla}^{E_i}_{0 \oplus E_a} &= -\tau^b_a \phi_{L,i}(E_b) \\ &= -(\phi_{L,i})^u_{b,v} \tau^b_a e_u \widetilde{e}^v \end{aligned} \tag{3.66}$$

Example 3.3.8 (Trivialization of A-connection produced by connection on A given a group induced representation)**.** Let U_i be a neighborhood on which the representation $\phi : A \to \mathfrak{D}(E)$ trivializes as the G-induced representation of $TU_i \times \mathfrak{g}$ on $U_i \times V$, where G is a Lie group with $\mathfrak{g}$ as Lie algebra, and V is the typical fiber of E over U_i. Let $\hat{\omega}$ be a connection form on the transitive Lie algebroid A with local trivialization $\hat{\omega}_i = A_i \oplus (-\epsilon + \tau)$. The A-connection $\hat{\nabla}^E$ on E produced by $\hat{\omega}$ trivializes over U_i as:

$$\begin{aligned} \hat{\nabla}^{E,i}_{X \oplus \eta} f &= \phi_i(\hat{\nabla}^i_{X \oplus \eta}) f \\ &= \phi_i(X \oplus (A_i(X) - \tau(\eta))) f \\ &= X(f) + (A_i(X) - \tau(\eta)) \cdot f. \end{aligned}$$

The group induced A-connections given a representation π are associated to the following derivations:

$$\begin{aligned} \hat{\nabla}^{E,i}_\mu &= \partial_\mu \oplus (A_i)^b_\mu \pi_V(E_b) \\ &= \partial_\mu \oplus + \pi(E_b)^u_v (A_i)^b_\mu e_u \widetilde{e}^v, \end{aligned} \tag{3.67}$$

and

$$\begin{aligned} \hat{\nabla}^{E,i}_a := \hat{\nabla}^{E_i}_{0 \oplus E_a} &= -\tau^b_a \pi(E_b) \\ &= -\pi(E_b)^u_v \tau^b_a e_u \widetilde{e}^v. \end{aligned} \tag{3.68}$$

3.4 Examples

3.4.1 TP^k/S^1 over S^2

In Example 2.3.6 we saw that the most general form, and hence the most general connection form on TP^k/S^1, has the local trivialization over U_S:

$$\begin{aligned}\hat{\omega}_S &= i\hat{\omega}^{\epsilon}_{S;1}(\phi,\theta)dx^1 + i\hat{\omega}^{\epsilon}_{S;2}(\phi,\theta)dx^2 + i\hat{\omega}^{\epsilon}_{S;i}(\phi,\theta)Im \\ &= i\hat{f}(\phi,\theta)d\phi + i\hat{g}(\phi,\theta)d\theta + i\hat{h}(\phi,\theta)Im\end{aligned}$$

where $\hat{\omega}^{\epsilon}_{S;\cdot}, \hat{f}, \hat{g}, \hat{h} \in C^{\infty}(S^2)$ with some restrictions, as explained in that example; recall from Example 2.4.1 that this suffices to determine completely the form over all of S^2 since U_S covers all but one point of S^2.

Since i is a global section of TP^k/S^1, the reduced kernel endomorphism $\tau \in End(S)$ is determined from a single function $\widetilde{\tau} \in C^{\infty}(S^2)$ such that $\tau = i\widetilde{\tau}$, and so the local components of τ are:

$$(\tau_S)^1_1 = \widetilde{\tau}|_{U_S} \qquad\qquad (\tau_N)^1_1 = \widetilde{\tau}|_{U_N}. \tag{3.69}$$

Decomposing ω_S as $A_S - Im + \tau_S$ and $\omega_N = A_N - Im - \tau_N$, we see that the $\widetilde{\tau}$ is related to its associated connection form by:

$$\widetilde{\tau} = \hat{h} + 1 = \hat{\omega}_{S;i} + 1. \tag{3.70}$$

Finally, the local trivializations of the anchor preserving vector bundle morphisms associated to the connections forms (see 3.2.3) are associated, according to equation (3.42) to the following $TU_S \times i\mathbb{R}$ elements:

$$\begin{aligned}\hat{\nabla}_\mu &= \partial_\mu \oplus i\omega^{\epsilon}_{S;\mu} \\ \hat{\nabla}_i &= -i\widetilde{\tau},\end{aligned} \tag{3.71}$$

for $\mu = 1, 2$; these elements will pass through the representations to give generalized covariant derivatives.

The Lie algebra $i\mathbb{R}$ is commutative, so the curvature of the connection form $\hat{\omega}$ is simply

$$\hat{R} = \hat{d}\hat{\omega}, \tag{3.72}$$

which we computed in Example 2.3.6, in equation (2.37), and that can be easily seen to be equal to to be local trivialized as

$$\begin{aligned}\hat{R}_S &= \hat{d}\hat{\omega}_S \\ &= i(\partial_1\hat{\omega}^{\epsilon}_{S;2} - \partial_2\hat{\omega}^{\epsilon}_{S;1})dx^1 \wedge dx^2 + i\partial_1\widetilde{\tau}dx^1 \wedge Im + i\partial_2\widetilde{\tau}dx^2 \wedge Im;\end{aligned} \tag{3.73}$$

on U_{SN} with polar coordinates we also have

$$\begin{aligned}\hat{R}_{SN} &= \hat{d}\omega_{SN} \\ &= i(\partial_\phi g - \partial_\theta f)d\phi \wedge d\theta + i\partial_\phi\widetilde{\tau}d\phi \wedge Im + i\partial_\theta\widetilde{\tau}d\theta \wedge Im;\end{aligned} \tag{3.74}$$

notice that if $\tau = 0$, then $\hat{\omega}$ is an ordinary connection on TP^k/S^1 and the last 2 terms of the trivialiations of the curvature form disappear, making it possible to define an equivalent 2-form $R \in \Omega^2(TS^2, P^k \times i\mathbb{R}/S^1)$.

Suppose we are given a background (ordinary) connection (in the next chapter we'll see how they come to be) with local trivialization:

$$\begin{aligned}\widetilde{\omega}_S &= i\widetilde{\omega}^{\epsilon}_{S;1}dx^1 + i\widetilde{\omega}^{\epsilon}_{S;2}dx^2 - iIm \\ &= i\widetilde{f}d\phi + i\widetilde{g}d\theta - iIm\end{aligned} \tag{3.75}$$

then the ordinary connection $\omega = \hat{\omega} + \tau \circ \widetilde{\omega}$ induced by $\widetilde{\omega}$ (see 3.1.10) is

$$\begin{aligned}\omega_S &= i(\hat{\omega}^{\epsilon}_{S;1} + \widetilde{\tau}\widetilde{\omega}^{\epsilon}_{S;1})dx^1 + i(\hat{\omega}^{\epsilon}_{S;2} + \widetilde{\tau}\widetilde{\omega}^{\epsilon}_{S;2})dx^2 - iIm \\ &= i(\hat{f} + \widetilde{\tau}\widetilde{f})d\phi + i(\hat{g} + \widetilde{\tau}\widetilde{g})d\theta - iIm,\end{aligned} \tag{3.76}$$

and this connection form on $TU_S \times i\mathbb{R}$ is associated to the ordinary connection $\nabla : TU_S \to TU_S \times i\mathbb{R}$ (see 3.2.3) determined by the Lie algebroid elements determined by local frame $\{\partial_{x^\mu} \equiv \partial_\mu\} \subseteq \Gamma_{U_S}(TS^2)$:

$$\nabla_\mu = \partial_\mu \oplus i(\hat{\omega}^{\epsilon}_{S;\mu} + \widetilde{\tau}\widetilde{\omega}^{\epsilon}_{S;\mu}); \tag{3.77}$$

over U_{SN} in polar coordinates we may also write

$$\begin{aligned}\nabla_{\partial_\phi} &= \partial_\phi \oplus i(\hat{f} + \widetilde{\tau}\widetilde{f}) \\ \nabla_{\partial_\theta} &= \partial_\theta \oplus i(\hat{g} + \widetilde{\tau}\widetilde{g}).\end{aligned} \tag{3.78}$$

Now let's understand the decomposition of the curvature form $\hat{R}$. Using equations (3.50), (3.52) and (3.51) we can quickly conclude that, with respect to the basis $\{i\}_{a=1}$ of $i\mathbb{R}$ and with respect to the $\vec{x}$ coordinates over U_S, the following relations hold:

$$\begin{aligned}
C^c_{ab} &= \epsilon^c[E_a, E_b] = 0,\\
\tau^1_1 &= \widetilde{\tau},\\
W^c_{ab} &= 0,\\
(\mathcal{D}\tau)^1_{1,1} &= \partial_1\widetilde{\tau},\\
(\mathcal{D}\tau)^1_{2,1} &= \partial_2\widetilde{\tau},\\
\widetilde{F}^1_{12} &= -\widetilde{F}^1_{21} = \partial_1\widetilde{\omega}^\epsilon_{S;2} - \partial_2\widetilde{\omega}^\epsilon_{S;1},\\
F^1_{12} &= -\widetilde{F}^1_{21} = \partial_1[\hat{\omega}^\epsilon_{S;2} + \widetilde{\tau}\widetilde{\omega}^\epsilon_{S;2}] - \partial_2[\hat{\omega}^\epsilon_{S;1} + \widetilde{\tau}\widetilde{\omega}^\epsilon_{S;1}],\\
\hat{F}^1_{12} &= -\hat{F}^1_{21} = \partial_1\hat{\omega}^\epsilon_{S;2} - \partial_2\hat{\omega}^\epsilon_{S;1} + (\partial_1\widetilde{\tau})\widetilde{\omega}^\epsilon_{S;2} - (\partial_2\widetilde{\tau})\widetilde{\omega}^\epsilon_{S;1};
\end{aligned} \tag{3.79}$$

we can readily check that the decomposition of the curvature given in Theorem 3.2.10 using the above components does indeed produce the formula (3.74) that we previously obtained. Over U_{SN} analogous formulas are satisfied with respect to the polar coordinates (ϕ, θ).

3.4.2 TP^k/S^3 over S^4

For reason entirely analogous to those use for the Atiyah Lie algebroids TP^k/S^1, and continuing the example in Section 2.4.2, we know that the most general connection $\hat{\omega} \in \Omega^1(TP^k/S^3, P^k \times Im\,\mathbb{H}/S^3)$ on TP^k/S^4 has the local trivialization over U_S

$$\hat{\omega}_S = i\omega^1_S + j\omega^2_S + k\omega^3_S - iIm - jJm - kKm + \tau_S, \tag{3.80}$$

where $\omega^a_S = (\omega_S)^a_\mu dx^\mu$, $a = 1,2,3$, are restrictions to U_S of scalar valued 1-forms in $\Omega^1(TS^7)$, hence the components of $\hat{\omega}$ over U_S with respect to the basis i,j,k of $Im\,\mathbb{H}$ are actually functions $(\omega_S)^i_\mu \in C^\infty(S^2)$ defined over all of S^2 with some restrictions (analogous to those of Section 2.4.1 that allow its extension to a global section on TP^k/S^4; similarly, τ_S is any function $C^\infty(S^2, \mathfrak{gl}(Im\,\mathbb{H}))$, made up of 9 different matrix $(\tau_S)^a_b$ elements, or it can also be seen as a collection of 3 different Lie algebra valued fields

$$(\tau_S)_b = \tau_S(E_b) \in C^\infty(S^2, Im\,\mathbb{H}),$$

for $\{E_b\}_{b=1,2,3} = \{i,j,k\}$ basis of the Lie algebra.

If we use the matrix algebra $\mathfrak{su}(2)$ as the representation of the Lie algebra, then:

$$\hat{\omega}_S = \begin{pmatrix} i\hat{\omega}_S^3 & \hat{\omega}_S^2 + i\hat{\omega}_S^1 \\ \hat{\omega}_S^2 - i\hat{\omega}_S^1 & -i\hat{\omega}_S^3 \end{pmatrix} - i\sigma_1 Im - i\sigma_2 Jm - i\sigma_3 Km + \tau_S \tag{3.81}$$

where $\hat{\omega}_S^\mu \in \Omega^1(S^2)$, for $\mu = 1, 2, 3$ and $\tau_S \in C^\infty(S^2, \mathfrak{gl}(\mathfrak{su}(2)))$ and the set $\{Im, Jm, Km\}$ is now the ordered basis of $\mathfrak{su}(2)^*$ dual to $\{i\sigma_1, i\sigma_2, i\sigma_3\}$.

Chapter 4

Integration Theory

The formulation of a gauge theory on a transitive Lie algebroid A will be made through the action functional $S[\cdot,\cdot]$ of the theory, which will be the integral over A of a certain form, found by the application of the product of forms induced by metrics on both the adjoint Lie algebroid of A and on an arbitrary representation vector bundle. This chapter is essentially a rewriting of the exposition found on these topics on [4] and [3]. Metrics on representation vector bundles and the product they induce on vector bundle-valued differential forms is studied in Section 4.1. Then, metrics on transitive Lie algebroids and their equivalence with metrics on both the base space and the adjoint algebroid together with a horizontal subbundle as studied in Section 4.2. In Section 4.3 the concepts of inner integration and integration over A are studied. Finally, in Section 4.4 metrics on A are seen to induce inverse metrics and Hodge-$*$ operators on differential forms, which combine with the integration on A to define an inner product of homogeneous forms on A. The content of Section 4.5 was developed by us to continue progressing the examples developed throughout the document.

Throughout this chapter let $0 \to L \xrightarrow{j} A \xrightarrow{a} TM$ be a transitive Lie algebroid for the transitive Lie algebroid A over the manifold M. Let $\{(U_i, \psi_i : U_i \times \mathfrak{g} \to L|_{U_i}, \nabla^{0,i} : TU_i \to A|_{U_i})\}_{i\in I}$ be a Lie algebroid atlas for A, where $\mathfrak{g}$ is a Lie algebra with basis $\{E_a\}_{a=1,\ldots,n}$ and associated dual basis $\{\epsilon^a\}_{a=1,\ldots,n}$. Also, let E be a vector bundle over M on which A is represented by $\phi : A \to \mathfrak{D}(E)$, with vertical component $\phi_L : L \to End(E)$. Suppose that E is trivialized over each U_i, $i \in I$, by the vector bundle isomorphisms

$\beta_i : U_i \times V \to E|_U$, where V is a vector space with basis $\{e_u\}_{u=1,\dots,t}$ and associated dual basis $\{\widetilde{e}_u\}_{u=1,\dots,t}$.

4.1 Degenerate Metrics on Representation Vector Bundles

Definition 4.1.1. Let E be any vector bundle over M. A degenerate metric on E on E is a symmetric vector bundle map $h^E : E \otimes E \to M \times \mathbb{R}$. If h^E is non-degenerate, i.e. if $h^E(\mu, \mu) = 0$, only if $\mu = 0$, then h^E is called a metric on E.

In the case that $E = L$ is an adjoint Lie algebroid of A, we also use the names degenerate inner metric on A and inner metric on A, respectively.

In contrast to the notations in [4, 3], we have called "degenerate metric" what they call "metric" to distinguish the possibly degenerate and non-degenerate cases. Notice that a (degenerate) metric on E is equivalent to a $C^\infty(M)$-linear map $h^E : \Gamma(E)\otimes\Gamma(E) \to C^\infty(M)$ which is a field of symmetric bilinear forms if h^E is degenerate, and a field of metrics in the non-degenerate case. We now return to consider E as a representation vector bundle of A.

For the rest of the chapter h^E will be a degenerate metric on E and h will denote a degenerate inner metric on A.

Definition 4.1.2. Over the trivializing neighborhood U_i, the components of h^E with respect to the basis $\{e_u\}$, with $u, v = 1, \dots,$ are the functions $h^{E,i}_{u,v} := h^E|_{U_i}(\beta_i(e_u), \beta_i(e_v)) \in C^\infty(U_i)$, where e_u and e_v are the corresponding constant V-valued functions on U_i.

In particular, h_{ab} are the components of h with respect to the basis $\{E_a\}$, $a, b = 1, \dots, n$. Notice that, in contrast to the components of forms, the components of a (degenerate) metric are symmetric on the indices.

Proposition 4.1.3. Let $H^{iu}_{j\,v} = \widetilde{e}^u(\beta^i_j(e_v))$, for $i, j \in I$ and $u, v \in \{1, \dots, t\}$ be the matrix representation of the transition function of E from U_j to U_i. Then,

$$h^{E,j}_{u_1 u_2} = H^{iv_1}_{j\,u_1} H^{iv_2}_{j\,u_2} h^{E,i}_{v_1 v_2} \tag{4.1}$$

for $u_1, u_2, v_1, v_2 \in \{1, \dots, t\}$.

Proof.

$$\begin{aligned} h^{E,j}_{u_1u_2} &= h^E(\beta_j(e_{u_1}), \beta_j(e_{u_2})) \\ &= h^E(\beta_i \circ \beta^i_j(e_{u_1}), \beta_i \circ \beta^i_j(e_{u_2}) \\ &= h^E(H^{i\,v_1}_{j\,u_1}\beta_i(e_{u_2}), H^{i\,v_1}_{j\,u_1}\beta_i(e_{u_2})) \\ &= H^{i\,v_1}_{j\,u_1}H^{i\,v_2}_{j\,u_2}h^{E,i}_{v_1v_2}. \end{aligned}$$

□

Recall that the transition functions of L are denoted by α^i_j and its matrix representation with respect to $\{E_a\}_{a=1,\ldots,n}$ is denoted by $G^{i\,a}_{j\,b}$.

Definition 4.1.4. Given a degenerate metric h^E on the representation vector bundle E of A, let us induce the map (also denoted by h^E):

$$h^E : \Omega^p(A, E) \otimes \Omega^q(A, E) \to \Omega^{p+q}(A) \tag{4.2}$$

for all $p, q \in \mathbb{Z}_{\geq 0}$, to be the skew-symmetric $C^\infty(M)$-linear map defined by

$$h^E(\omega, \eta)(\mathfrak{X}_1, \ldots, \mathfrak{X}_{p+q}) := \frac{1}{p!q!} \sum_{\sigma \in Sym_{p+q}} h^E\left(\omega(\mathfrak{X}_{\sigma(1)}, \ldots, \mathfrak{X}_{\sigma(p)}), \eta(\mathfrak{X}_{\sigma(p+1)}, \ldots, \mathfrak{X}_{\sigma_{p+q}})\right), \tag{4.3}$$

for all $\omega \in \Omega^p(A, E)$, $\eta \in \Omega^q(A, E)$, and $\mathfrak{X}_1, \ldots, \mathfrak{X}_{p+q} \in A$.

Proposition 4.1.5. Let $\alpha, \beta \in \Omega^\bullet_{U_i}(A)$, and let $\mu, \nu \in \Gamma_{U_i}(E) = \Omega^0_{U_i}(A, E)$. Then

$$h^E(\mu\alpha, \nu\beta) = h^E(\mu, \nu)\, \alpha \wedge \beta. \tag{4.4}$$

Proof. This is a direct consequence of the bilinearity of h^E and how the formula for h^E applied to forms is a generalization of the formula for the wedge product.

$$\begin{aligned} &h^E(\mu\alpha, \nu\beta)(\mathfrak{X}_1, \ldots, \mathfrak{X}_{p+q}) \\ &\quad = \frac{1}{p!q!} \sum_{\sigma \in Sym_{p+q}} h^E\left(\mu\omega(\mathfrak{X}_{\sigma(1)}, \ldots, \mathfrak{X}_{\sigma(p)}), \nu\eta(\mathfrak{X}_{\sigma(p+1)}, \ldots, \mathfrak{X}_{\sigma_{p+q}})\right) \\ &\quad = \frac{1}{p!q!} \sum_{\sigma \in Sym_{p+q}} h^E(\mu, \nu)\, \omega(\mathfrak{X}_{\sigma(1)}, \ldots, \mathfrak{X}_{\sigma(p)})\eta(\mathfrak{X}_{\sigma(p+1)}, \ldots, \mathfrak{X}_{\sigma_{p+q}}) \end{aligned}$$

$$= h^E(\mu,\nu)\,(\omega \wedge \eta)(\mathfrak{X}_1,\ldots,\mathfrak{X}_{p+q}).$$

$\square$

The previous proposition may be applied to the components of E-valued forms on each U_i with respect to the basis $\{e_u\}$, since then $\omega = \omega^u e_u$ and $\eta = \eta^v e_v$ with ω^u and η^v are scalar valued forms; hence

$$h^E|_{U_i}(\omega,\eta) = h^{E,i}_{uv}\,\omega^u \wedge \eta^v \tag{4.5}$$

The following definition will be important for the important property of a gauge theory: the "'gauge invariance" of the action functional.

Definition 4.1.6. A degenerate metric h^E on E is called compatible with the representation if

$$h^E(\phi_L(\eta)\mu_1,\mu_2) + h^E(\mu_1,\phi_L(\eta)\mu_2) = 0,$$

for all $\eta \in L_m$ and $\mu_1,\mu_2 \in E_m$ for all $m \in M$.

In the case that E is the adjoint Lie algebroid L, we also say that h is a Killing degenerate metric.

Since the representation of A on L is is the adjoint representation, a Killing inner (degenerate) metric h satisfies, for all $m \in M$, $\eta,\theta,\gamma \in L_m$:

$$h([\eta,\theta],\gamma) = h(\theta,[\eta,\gamma]) = 0. \tag{4.6}$$

4.2 Degenerate Metrics on Transitive Lie Algebroids

Definition 4.2.1. Let $\hat{g}$ be a degenerate metric on A. The inner part of $\hat{g}$ is the degenerate inner metric

$$h := j^* g. \tag{4.7}$$

If h is non-degenerate, $\hat{g}$ is called an inner non-degenerate metric on A.

Let g be a (degenerate) metric on TM, i.e. a (degenerate) metric on M, then we may define a degenerate metric $\hat{g}$ on A by

$$\hat{g} = a^* g; \tag{4.8}$$

this is a degenerate metric on A since its inner part is null.

Definition 4.2.2. Let $\nabla : TM \to A$ be an ordinary connection on A with induced connection form $\mathfrak{a} \in \Omega^1(A, L)$. Then:

- Given an inner degenerate metric h on A, the degenerate metric induced by ∇ is

  $$\hat{g} := \mathfrak{a}^* h; \tag{4.9}$$

  this metric has h as inner part.

- Given a degenerate metric $\hat{g}$ on A, the tangent degenerate metric induced by ∇ is the degenerate metric g on TM:

  $$g := \nabla^* \hat{g}. \tag{4.10}$$

Theorem 4.2.3. An inner non-degenerate metric $\hat{g}$ on A is equivalent to a triple (g, h, ∇) through the equation

$$\hat{g}(\mathfrak{X}, \mathfrak{Y}) = g(a(\mathfrak{X}), a(\mathfrak{Y})) + h(\mathfrak{a}(\mathfrak{X}), \mathfrak{a}(\mathfrak{Y})); \tag{4.11}$$

where g is a degenerate metric on TM, h is an inner non-degenerate metric on A and ∇ is an ordinary connection on A with associated connection 1-form $\mathfrak{a}$. In particular, h is the inner part of $\hat{g}$ and g is the tangent degenerate metric on TM induced by the connection.

Proof. For a given triple (g, h, ∇), the corresponding degenerate metric is precisely $\hat{g} = a^* g + \mathfrak{a}^* h$.

For the converse, let $\hat{g}$ be an inner non-degenerate metric on A. Then, let h be the inner part of $\hat{g}$. We now need to see that an inner non-degenerate metric induces an ordinary connection ∇ for which the vertical bundle of A and the horizontal bundle induced by the connection are orthogonal, i.e. a connection such that

$$\hat{g}(\nabla_X, j(\eta)) = 0, \qquad \text{for all } X \in TM \text{ and } \eta \in L; \tag{4.12}$$

then, g will simply be the tangent degenerate metric induced by $\hat{g}$. To prove this an argument adapted from the proof of Riesz representation theorem can be followed [3]. $\square$

Suppose that on U_i there is are coordinates $x^\mu : U_i \to \mathbb{R}$ for M, with $\mu, \nu \in \{1, \dots, m\}$. Let $\hat{g}$ be an inner non-degenerate metric on A associated to the triple (g, h, ∇). Then, with respect to the local frame $\{\nabla_\mu, -E_a\}$ of $TU_i \times \mathfrak{g}$, dual to $\{dx^\mu, \mathfrak{a}^a\}$, the following is true for the following scalar valued functions, called the local components of $\hat{g}$ with respect to the coordinates x^μ and the basis $\{E_a\}$ of $\mathfrak{g}$, where $\mu, \nu = 1, \dots, m$ and $a, b = 1, \dots, n$:

$$\begin{aligned} \hat{g}^i_{\mu\nu,} &:= \hat{g}^i(\nabla_\mu, \nabla_\nu) = g_{\mu\nu}, \\ \hat{g}^i_{\mu,a} &:= \hat{g}^i(\nabla_\mu, -E_a) = 0, \\ \hat{g}^i_{,ab} &:= \hat{g}^i(-E_a, -E_b) = h_{ab}, \end{aligned}$$

where $g^i := S^{i*}\hat{g}|_{U_i}$ is the local trivialization of $\hat{g}$ over U_i, i.e.

$$\hat{g}^i = g^i_{\mu\nu} dx^\mu \otimes dx^\nu + h^i_{ab} \mathfrak{a}^a \otimes \mathfrak{a}^b; \tag{4.13}$$

and so its local matrix representation with respect to the same elements, i.e. with respect to $\{\nabla_\mu, -E_a\}$, is the following symmetric matrix:

$$(\hat{g}^i) = \begin{pmatrix} (g^i_{\mu\nu})_{\mu,\nu=1,\dots,m} & 0 \\ 0 & (h^i_{ab})_{a,b=1,\dots,n} \end{pmatrix}. \tag{4.14}$$

4.3 Inner Integration and Integration

Definition 4.3.1. The transitive Lie algebroid A is inner orientable if the adjoint Lie algebroid L is an orientable vector bundle.

Since the matrices $G^{i\,a}_{j\,b}$ represent the transition functions of L, where $a, b \in \{1, \dots, n\}$ and $i, j \in I$ such that $U_{ij} \neq \emptyset$, A is inner orientable if and only if $det(G^i_j) \geq 0$.

For the rest of the section, let A be inner orientable, and let $\hat{g}$ be an inner non-degenerate metric on A with (g, h, ∇) its associated triple. In [4] it has been checked that:

Proposition 4.3.2. For $i, j \in I$ such that $U_{ij \neq \emptyset}$,

$$\hat{\alpha}^i_j(\sqrt{|h^j|}\mathfrak{a}^1_j \wedge \dots \wedge \mathfrak{a}^n_j) = \sqrt{|h^i|}\mathfrak{a}^1_i \wedge \dots \wedge \mathfrak{a}^n_i \in \Omega^n(TU_{ij} \times \mathfrak{g}),$$

where $\mathfrak{a} = \mathfrak{a}^a_i E_a$ is the connection form associated to ∇, and $|h^i| \in C^\infty(U_i)$ denotes the absolute value of the determinant of h over U_i, which is simply the absolute value of the determinant of the matrix (h_{ab}).

Thanks to the previous proposition, the following form is well defined globally:

Definition 4.3.3. The global form $\omega_{h,\mathfrak{a}} \in \Omega^n(A)$ whose local trivializations with respect to the connection ∇ and the basis $\{E_a\}$ over each U_i are

$$(\omega_{h,\mathfrak{a}})_i = (-1)^n \sqrt{|h^i|}\mathfrak{a}_i^1 \wedge \cdots \wedge \mathfrak{a}_i^n$$

is called the inner volume form of A with respect to h and ∇ and, recall, $n = dim(\mathfrak{g})$.

Theorem 4.3.4. Suppose that $\beta \in \Omega^\bullet(A, E)$, and let its decomposition with respect to the connection ∇ and the basis $\{E_a\}$ over U_i be

$$\beta_i = \beta_i^M \wedge \omega_{h,\mathfrak{a}} + \beta^R,$$

where β^R is the sum of the terms which do not have $\mathfrak{a}_i^1 \wedge \cdots \wedge \mathfrak{a}_i^n$ as a factor. Then, $\beta_i^M \in \Omega_{U_i}(A, E)$ is the local trivialization over U_i of a globally defined form β^M.

Furthermore, β_i^M is also the term of the decomposition of β over U_i that multiplies the maximum inner degree term $\sqrt{|h_i|}\epsilon^1 \wedge \cdots \wedge \epsilon^n$, hence β^M only depends on h.

Proof. Thanks to the homogeneous transformation rules 4.3.2, for all $j \in I$ such that $U_{ij} \neq \emptyset$, $\hat{\alpha}_j^i(\beta^R)$ does not have any term with $\mathfrak{a}_j^1 \wedge \cdots \wedge \mathfrak{a}_j^n$ as factor, and so, since $\hat{\alpha}_j^i$ respects the wedge product, Proposition 4.3.2 implies that

$$\hat{\alpha}_i^j(\beta_i^M) = \beta_j^M. \tag{4.15}$$

The last part of the theorem is follows simply from decomposing each $\mathfrak{a}_i^a \in \Omega^1(TU_i \times \mathfrak{g})$ as $A_i^a - \epsilon^a$, showing that the term that accompanies $\epsilon^1 \wedge \cdots \wedge^n$ in the decomposition of β with respect to the local frame $\{dx^\mu, \epsilon^a\}$, if it exists, is precisely the same term that accompanies $(-1)\mathfrak{a}_i^1 \wedge \cdots \wedge \mathfrak{a}_i^n$ in the decomposition of β with respect to the frame $\{dx^\mu, \mathfrak{a}^a\}$, where $x^\mu : U_i \to \mathbb{R}$, $mu = 1, \ldots, m$ are any (auxiliary) coordinates defined in perhaps smaller open subsets of U_i that cover it; since $\hat{\alpha}_j^i(dx^\mu) = dx^\mu$, it does not matter that the coordinates x^μ are not defined globally. $\square$

Definition 4.3.5. On the inner orientable transitive Lie algebroid A with inner non-degenerate metric $\hat{g}$, define the inner integration of forms on A as the operation:

$$\int_{inner} : \Omega^\bullet(A) \to \Omega^\bullet(TM) \qquad\qquad \beta \mapsto \beta^M, \tag{4.16}$$

where we used the notation of Proposition 4.3.4.

The previous definition can be extended with the same formula to integration of E-valued forms, but we only state this definition since it suffices for our purposes of defining the action functional of a gauge theory.

Definition 4.3.6. The transitive Lie algebroid A is called orientable if it is inner orientable and the base manifold M is orientable.

Definition 4.3.7. Let A be an orientable transitive Lie algebroid with inner non-degenerate metric $\hat{g}$. Define the integration of scalar valued forms on A as the operation

$$\int_A : \Omega^\bullet(A) \to \mathbb{K} \qquad\qquad \beta \mapsto \int_M \int_{inner} \beta, \tag{4.17}$$

applied to forms for which the corresponding integration over M is well defined.

Remark 4.3.8. Notice that the integration on A only requires an inner metric h on A (to define $\int_{inner}$ and an orientable base manifold (to define $\int_M$), neither g nor the connection ∇ influence the integration of forms on A.

However, to define an action functional we will require one more ingredient: the Hodge-$*$ operator of forms, which will require the full complexity of $\hat{g}$.

4.4 Inverse Metrics and Hodge-$*$ Operator

A metric on A enables, through the inverse metrics and Hodge-$*$ operator, the definition of the norm of a homogeneous E-valued form on A of any degree, of which the gauge action functional and the matter action functional will be examples.

For the rest of this chapter, let the transitive Lie algebroid A be orientable and let $\hat{g}$ be a metric on A (in particular, non-degenerate) associated to the triple (g, h, ∇), where g and h are (non-degenerate) metrics and ∇ is an ordinary connection associated to the connection form $\mathfrak{a} \in \Omega^1(A, L)$. Additionally, let h^E be a metric on the representation vector bundle E. Finally, suppose that on each U_i there are local coordinates $x^\mu : U_i \to \mathbb{R}$, where $\mu = 1, \ldots, m$.

Definition 4.4.1. The volume form ω^{Vol} on A with respect to the metric $\hat{g}$ is the never zero top form $\Omega^{m+n}(A)$ with local trivialization over each U_i:

$$\begin{aligned}\omega_i^{Vol} &= (-1)^n \sqrt{|g_i|}\sqrt{|h_i|}dx^1 \wedge \cdots \wedge dx^m \wedge \mathfrak{a}_i^1 \wedge \cdots \wedge \mathfrak{a}_i^n \\ &= \sqrt{|g_i|}dx^1 \wedge \cdots \wedge dx^m \wedge \omega_{h,\mathfrak{a}}.\end{aligned} \tag{4.18}$$

Notice that ω^{Vol} is well defined because it transforms correctly between local trivializations of A and M, since both $\sqrt{|g_i|}dx^1 \wedge \cdots \wedge dx^m$ and $\omega_{h,\mathfrak{a}}$ transform well.

Inverse Metrics

The non-degeneracy of $\hat{g}$ implies that, over each U_i, the matrix representation of $\hat{g}^i$ with respect to the local frame $\{\nabla^i_\mu, -E_a\}_{\mu=1,\ldots,m;\, a=1,\ldots,n}$ is invertible, with inverse:

$$(\hat{g}_i^{-1}) = \begin{pmatrix} (g_i^{\mu\nu})_{\mu,\nu=1,\ldots,m} & 0 \\ 0 & (h_i^{ab})_{a,b=1,\ldots,n} \end{pmatrix}, \tag{4.19}$$

where the matrix $(g_i^{\mu\nu})_{\mu,\nu=1,\ldots,m}$ is the inverse of the matrix of the matrix representation $(g^i_{\mu\nu})_{\mu,\nu=1,\ldots,m}$ of the induced metric on TU_i and similarly for $(h_i^{ab})_{a,b=1,\ldots,n}$. We can use this to define a metric on the spaces of homogeneous forms of a given degree just as in the traditional $A = TM$ case.

Definition 4.4.2. The inverse metric of $\hat{g}$ is the symmetric, non-degenerate, $C^\infty(M)$-linear (family of) map

$$\hat{g}^{-1} : \Omega^p(A) \otimes \Omega^p(A) \to C^\infty(M, \mathbb{R}), \tag{4.20}$$

for all $p, q \in \mathbb{Z}_{\geq 0}$, such that its local trivialization $\hat{g}_i^{-1}$ over U_i, i.e. the metric of $TU_i \times \mathfrak{g}$ defined by $\hat{g}^{-1}|_{U_i}(\omega, \eta) = \hat{g}_i^{-1}(\omega_i, \eta_i)$ for all $\omega, \eta \in \Omega^p(A)$, satisfies:

$$\hat{g}_i^{-1}(\alpha, \beta) = \sum_{r+s=p} \frac{1}{p!} g_i^{\mu_1 \nu_1} \cdots g_i^{\mu_r \nu_r} h_i^{a_1 b_1} \cdots h_i^{a_s b_s} \alpha_{\mu_1 \ldots \mu_r a_1 \ldots a_s} \beta_{\nu_1 \ldots \nu_r b_1 \ldots b_s},$$

for all $\alpha, \beta \in \Omega^p(TU_i \times \mathfrak{g})$ and using their decomposition with respect to the local mixed basis induced by the connection ∇.

Using the notation

$$\beta^{\mu_1 \ldots \mu_r a_1 \ldots a_s} := g_i^{\mu_1 \nu_1} \cdots g^{\mu_r \nu_r} h^{a_1 b_1} \cdots h_i^{a_s b_s} \beta_{\nu_1 \ldots \nu_r b_1 \ldots b_s}, \tag{4.21}$$

which defines an element $\beta^{\#}$ in the dual module of $\Omega^p(A)$, we may also write, then,

$$\hat{g}_i^{-1}(\alpha, \beta) = \sum_{r+s=p} \frac{1}{p!} \alpha_{\mu_1 \ldots \mu_r a_1 \ldots a_s} \beta^{\mu_1 \ldots \mu_r a_1 \ldots a_s}. \tag{4.22}$$

That the inverse metric is a well defined global element follows from the transformation laws of the components of a form, given in theorem 3.2.8, and the transformation law of the inverse metric,

$$h_i^{a_1 b_1} = G^{i\,a_1}_{j\,a'_1} G^{i\,b_1}_{j\,b'_1} h_j^{a'_1 b'_1} \tag{4.23}$$

that follows from 4.1.3, which is inverse to that of the components; concretely, for $\omega, \eta \in \Omega^p(A)$, over $U_{ij} \neq \emptyset$ it is satisfies that

$$\begin{aligned}
\hat{g}_j^{-1}(\omega_i, \eta_i) &= \sum_{r+s=p} \frac{1}{p!} g_i^{\mu_1 \nu_1} \cdots g_i^{\mu_r \nu_r} h_i^{a_1 b_1} \cdots h_i^{a_s b_s} (\omega_i)_{\mu_1 \ldots \mu_r a_1 \ldots a_s} (\eta_i)_{\nu_1 \ldots \nu_r b_1 \ldots b_s} \\
&= \sum_{r+s=p} \frac{1}{p!} g_j^{\mu_1 \nu_1} \cdots g_j^{\mu_r \nu_r} (G^{i\,a_1}_{j\,a'_1} G^{i\,b_1}_{j\,b'_1} h_j^{a'_1 b'_1}) \cdots (G^{i\,a_s}_{j\,a'_s} G^{i\,b_s}_{j\,b'_s} h_j^{a'_s b'_s}) \times \\
&\quad G^{j\,a_1}_{i\,a'_1} \cdots G^{i\,a_s}_{j\,a'_s} (\omega_j)_{\mu_1 \ldots \mu_r a'_1 \ldots a'_s} G^{j\,b_1}_{i\,b'_1} \cdots G^{i\,b_s}_{j\,b'_s} (\eta_j)_{\nu_1 \ldots \nu_r b'_1 \ldots b'_s} \\
&= \sum_{r+s=p} \frac{1}{p!} g_j^{\mu_1 \nu_1} \cdots g_j^{\mu_r \nu_r} h_j^{a_1 b_1} \cdots h_j^{a_s b_s} (\omega_j)_{\mu_1 \ldots \mu_r a_1 \ldots a_s} (\eta_j)_{\nu_1 \ldots \nu_r b_1 \ldots b_s} \\
&= \hat{g}_j^{-1}(\omega_j, \eta_j).
\end{aligned}$$

Similarly, letting h^E be a metric on the representation vector bundle E, with typical fiber generated by the basis $\{e_u\}_{u=1,\ldots,t}$, we can define an "inverse" metric for E-valued forms:

Definition 4.4.3. The inverse metric $\hat{g}_{h^E}^{-1}$ on E-valued forms induced by h^E is the symmetric, non-degenerate, $C^\infty(M)$-linear map

$$\hat{g}_{h^E}^{-1} : \Omega^p(A,E) \otimes \Omega^p(A,E) \to C^\infty(M,\mathbb{R}), \tag{4.24}$$

with local trivialization g_{i,h^E}^{-1} over U_i defined by

$$\hat{g}_{h^E,i}^{-1}(\alpha_i,\beta_i) := \hat{g}_i^{-1}(\alpha_i^u,\beta_i^v)h_{uv}^E. \tag{4.25}$$

Using the notation

$$\beta_v^{\mu_1\ldots\mu_r a_1\ldots a_s} := h_{uv}^E g_i^{\mu_1\nu_1}\cdots g^{\mu_r\nu_r}h^{a_1b_1}\cdots h_i^{a_sb_s}\beta^u_{\nu_1\ldots\nu_r b_1\ldots b_s}, \tag{4.26}$$

we may also write, then,

$$\hat{g}_{h^E,i}^{-1}(\alpha,\beta) = \sum_{r+s=p}\frac{1}{p!}\alpha^u_{\mu_1\ldots\mu_r a_1\ldots a_s}\beta_u^{\mu_1\ldots\mu_r a_1\ldots a_s}. \tag{4.27}$$

Hodge-$*$ Operator

Definition 4.4.4. The Hodge-$*$ operator is the $C^\infty(M)$-linear (family of) operator

$$\begin{aligned} * : \Omega^p(A,E) &\to \Omega^{m+n-p}(A,E)\\ \alpha &\mapsto *\alpha \end{aligned}$$

for all $p \in \mathbb{Z}_{\geq 0}$, where the element $*\alpha \in \Omega^{m+n-p}(A,E)$ is defined to be $*\alpha := \alpha\omega^{Vol}$ if $p=0$, or, otherwise, by the local trivializations:

$$\begin{aligned}
(*\alpha)_i & \\
&= \sum_{r+s=p}(-1)^{s(m-r)+n}\frac{1}{r!s!(m+n-p)!}\sqrt{|h_i|}\sqrt{|g_i|}(\alpha_i)_{\mu_1\ldots\mu_r a_1\ldots a_s}\epsilon_{\nu_1\ldots\nu_m}\epsilon_{b_1\ldots b_n}\\
&\quad \times g^{\mu_1\nu_1}\cdots g^{\mu_r\nu_r}h^{a_1b_1}\cdots h^{a_sb_s}dx^{\nu_{r+1}}\wedge\cdots\wedge dx^{\nu_m}\wedge\mathfrak{a}^{b_{s+1}}\wedge\cdots\wedge\mathfrak{a}^{b_n}\\
&= \sum_{r+s=p}(-1)^{s(m-r)+n}\frac{1}{r!s!(m+n-p)!}\sqrt{|h_i|}\sqrt{|g_i|}(\alpha_i)^{\nu_1\ldots\nu_r b_1\ldots b_s}\epsilon_{\nu_1\ldots\nu_m}\epsilon_{b_1\ldots b_n}\\
&\quad \times dx^{\nu_{r+1}}\wedge\cdots\wedge dx^{\nu_m}\wedge\mathfrak{a}^{b_{s+1}}\wedge\cdots\wedge\mathfrak{a}^{b_n}
\end{aligned} \tag{4.28}$$

where ϵ are the totally antisymmetric Levi-Civita symbols.

That the previous local trivializations do define a global form $*\alpha$ follows easily from the multiplication of the inverse (G^i_j) matrices that appear when applying $\hat{\alpha}^i_j$, for $U_{ij} \neq \emptyset$, due to the transformations of $\sqrt{|h_i|}$, of the components of the inverse inner metric, of the components of α and of the subfactors of the inner volume element. Notice that $(*\alpha)_i = *(\alpha_i)$ by definition, so we may simply write $*\alpha_i$; also, notice that with respect to the basis $\{e_u\}$ of the typical fiber V of E, $*\alpha_i = (*\alpha^u_i)e_u$.

Proposition 4.4.5. The Hodge-$*$ operator is such that the following identity is satisfied:

$$h_E(\alpha, *\beta) = \hat{g}^{-1}_{h^E}(\alpha, \beta)\omega^{Vol}, \tag{4.29}$$

for all $\alpha, \beta \in \Omega^p(A, E)$ and all $p \in \mathbb{Z}_{\geq 0}$.

Proof. This can be proven through a combinatorial argument once we expand

$$\alpha^u_i = \sum_{r+s} \alpha_{\nu_1 \dots b_s} e_u dx^{\nu_1} \wedge \cdots \wedge \mathfrak{a}^{b_s}_i$$

and $*\beta_i$ as in the second equation of Definition 4.4.4, replace these formulas in $h^i_E(\alpha_i, *\beta_i) = (h^E_i)_{uv}\alpha^u_i \wedge *\beta^v_i$, expanding the wedge product and reordering the dx's and $\mathfrak{a}$'s in order to isolate ω^{Vol}. $\square$

Definition 4.4.6. The scalar product on each $\Omega^p(A, E)$ induced by the metrics $\hat{g}$ on A and h^E on E in the orientable transitive Lie algebroid A, for all $p \in \mathbb{Z}_{\geq 0}$, is defined by

$$(\alpha, \beta) := \int_A h^E(\alpha, *\beta).$$

When $E = L$, h^E is assumed to be h, the inner part of $\hat{g}$.

We now have all the necessary ingredients to define the action functional of a gauge theory, which will be studied in next chapter.

4.5 Examples

An inner metric h on a transitive Lie algebroid A defines an inner integration, which in turn defines an integration over A. An inner non-degenerate metric

$\hat{g}$ on A determines an inner product on homogeneous forms on A, and it is equivalent to an inner metric h, a degenerate metric g on the base manifold M, and an ordinary connection $\widetilde{\nabla} : TM \to A$ on A that determines a subbundle that we may call the horizontal subbundle of A. When formulating a gauge theory we will think of g as being given by the nature of the space or spacetime that the base manifold M is. Hence, the freedom to define the inner product and norm of differential forms on A comes from the freedom on the choice of inner metric h, and of the connection $\widetilde{\nabla}$ which will be called the "background connection" in next chapter.

For the inner metric h, in next chapter we will see that we need h to be Killing, and since the typical fiber of the adjoint Lie algebroids is a Lie algebra, the natural symmetric form on them, called the Killing form, deserves some consideration. Given a Lie algebra $\mathfrak{g}$ over the field $\mathbb{K}$, the adjoint operator $ad : \mathfrak{g} \to \mathfrak{g}$ is defined by $ad(\eta)(\theta) := [\eta, \theta]$, and it induces the Killing form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined as

$$K(\eta, \theta) := trace(ad(\eta) \circ ad(\theta)), \tag{4.30}$$

for all $\eta, \theta \in \mathfrak{g}$. Let G be a group with Lie algebra $\mathfrak{g}$, then the Killing form has the following properties:

- It is symmetric and bilinear.
- It is invariant under automorphisms of $\mathfrak{g}$, i.e. if $A \in Aut(\mathfrak{g})$, then:

  $$K(A(\eta), A(\theta)) = K(\eta, \theta);$$

  in particular, the adjoint operator for any Lie algebra element is a Lie algebra isomorphism.
- It is an invariant form, i.e.

  $$K(ad(\gamma)(\eta), \theta) + K(\eta, ad(\gamma)(\theta) = 0,$$

  for all $\gamma, \eta, \theta \in \mathfrak{g}$.
- If G is compact, then $\mathfrak{g}$ is negative semi-definite.
- The Lie algebra $\mathfrak{g}$ is semisimple [1] if and only if K is non-degenerate.

[1] A semisimple Lie algebra is a direct sum of simple Lie algebras. A simple Lie algebra is a nonabelian Lie algebra with no nonzero proper ideals.

- If $\mathfrak{g}$ is a simple Lie algebra, then the only invariant symmetric bilinear forms on $\mathfrak{g}$ are the scalar multiples of K.

The previous properties of the Killing form can be used to establish all the Killing metrics in $P \times \mathfrak{g}/G$ when G is a compact group and $\mathfrak{g}$ is simple, and P is a principal bundle with structure group G.

Definition 4.5.1. Let $G \to P \to M$ be a principal bundle. Let the Killing form K on $\mathfrak{g}$ induce the degenerate metric on $P \times \mathfrak{g}/G$ defined by:

$$K(\langle p, \eta\rangle, \langle p, \theta\rangle) := K(\eta, \theta),$$

for all $\eta, \theta \in \mathfrak{g}$ and $p \in P$. In addition, for any $c \in C^\infty(M)$ define the degenerate metric cK on $P \times \mathfrak{g}/G$ by

$$cK(\langle p, \eta\rangle, \langle p, \theta\rangle) := c(m)K(\langle p, \eta\rangle, \langle p, \theta\rangle),$$

where $m \in M$ is the projection of $p \in P$.

Notice that the degenerate inner metric K on $P \times \mathfrak{g}/G$ is well defined thanks to the automorphism invariance of the Killing form. Recall that $P\times\mathfrak{g}/G$ is a LAB (Definition 1.1.6), and notice that if $\psi_i : U_i\times\mathfrak{g} \to P\times\mathfrak{g}/G|_{U_i}$ is a LAB trivialization map, the degenerate metric cK on $P \times \mathfrak{g}/G$ is locally trivialized by:

$$\begin{aligned} cK_i : (U_i \times \mathfrak{g}) \otimes (U_i \times \mathfrak{g}) &\to \mathbb{R} \\ ((m, \eta), (m, \theta)) &\mapsto f(m)K(\eta, \theta). \end{aligned}$$

If $\mathfrak{g}$ is semisimple, K is then a Killing inner metric on TP/G, (Definitions 4.1.1 and 4.1.6), and if $c \in C^\infty(M)$ is either positive or negative, then cK are also Killing inner metrics on TP/G. Finally, if $\mathfrak{g}$ is, furthermore, a simple, then the uniqueness property of the invariant symmetric forms on $\mathfrak{g}$ and the comment on the trivializations of the metrics imply that

Proposition 4.5.2. Let $G \to P \to M$ be a principal bundle, and suppose that $\mathfrak{g}$ is a simple Lie algebra. Then the only Killing inner metrics on the transitive Lie algebroid TP/G are of the form cK for $c \in C^\infty(M)$ either always positive or always negative.

We will now define inner metrics and background connections on the Atiyah Lie algebroids we have been using so far.

4.5.1 TP^k/S^1 over S^2

On the base manifold S^2 a natural metric is the round metric g^R which, in spherical coordinates, is given by:

$$g^R = d\phi^2 + \sin^2\phi d\theta^2.$$

The Lie algebra $i\mathbb{R}$ is not simple since it is commutative, but the fact that it is 1-dimensional allows us to see that the only symmetric bilinear forms on it are constant multiples of the multiplication in $i\mathbb{R}$. Thus, every degenerate metric h on $P^k \times i\mathbb{R}/S^1$ is determined by a function $c \in C^\infty(S^2)$ by:

$$h^c(\langle m, ia\rangle, \langle m, ib\rangle) := c(m)ab; \tag{4.31}$$

this amounts to saying that

$$h^c(i,i) = c,$$

where i is the global section $i \in \Gamma(P^k \times i\mathbb{R}/S^1)$.

The fact that the degenerate inner metric h is Killing is trivial since $i\mathbb{R}$ is commutative. Hence

Proposition 4.5.3. All Killing inner metrics on TP^k/S^1 are of the form h^c (4.36) for c either always positive of always negative.

The last freedom in the choice of a metric on TP^k/S^1 comes from the background (ordinary) connection $\widetilde{\nabla}$, associated to a connection form $\widetilde{\omega}$, also denoted by $\mathfrak{a}$. In Section 3.4.1 we say that such a background connection must have the following local trivialization over U_S and over U_{SN}, respectively:

$$\begin{aligned}\widetilde{\omega}_S &= i\widetilde{\omega}^\epsilon_{S;1}dx^1 + i\widetilde{\omega}^\epsilon_{S;2}dx^2 - iIm\\ &= i\widetilde{f}d\phi + i\widetilde{g}d\theta - iIm\end{aligned}$$

where $\widetilde{\omega}^\epsilon_{S;\mu}, \widetilde{f}, \widetilde{g}$ are functions over S^2 with some restrictions along the lines as those specified in Section 2.4.1.

With this definitions, over U_S the inner volume form has the local trivialization

$$\omega^S_{h^c,\mathfrak{a}} = -\sqrt{|c|}(\widetilde{\omega}^\epsilon_{S;1}dx^1 + \widetilde{\omega}^\epsilon_{S;2}dx^2 - Im); \tag{4.32}$$

and the volume form is

$$\omega^{Vol} = \sqrt{|c|}\sin\phi\, dx^1 \wedge dx^2 \wedge Im, \tag{4.33}$$

where $\sin\phi = \frac{2||\vec{x}||^2}{1+||\vec{x}||^2}$. The local trivializations over U_N may be determined with the results of Section 2.4.1.

Let us now examine the metrics for representation vector bundles. The irreducible complex representations of S^1 are all on $\mathbb{C}$ and are indexed by $h \in \mathbb{Z}$:

$$\pi^h(e^{ir})(z) := e^{ihr}z,$$

for all $z \in \mathbb{C}$. Then, the group representation π^h induces the Lie algebra representation π^h of $i\mathbb{R}$:

$$\pi^h(ir)(z) = irh\, z. \tag{4.34}$$

Each of these group representations scales up to Lie algebroid a representation $\phi^{k,h} : TP^k/S^1 \to E^k$, for $k \in \mathbb{Z}$, on the vector bundle E^k locally trivial on $U = U_S$ and U_N as $U \times \mathbb{R}^2$ with transition function $\beta_S^N = e^{ik\theta}$, producing the a group induced representation (Definition 1.5.8); notice that TP^k/S^1 is represented on $E^{k'}$ if $k = k'$, since that's when E^k is a vector bundle associated to P^k. By operating with direct sums and tensor products of these vector bundles we get new representation vector bundles of TP^k/S^1

The typical fiber of E^k is $\mathbb{R}^2$ and on it we have the natural metric of $\mathbb{R}^2$, which can be made in to a metric on $U_S \times \mathbb{R}^2$ and of $U_N \times \mathbb{R}^2$. Since the transition functions e^{ik} of E^k are in fact rotations of the fiber $\mathbb{R}^2$, hence respect this metric, the metrics on $U_S \times \mathbb{R}^2$ and $U_N \times \mathbb{R}^2$ extend to a metric h^E of E^k. A more general metric on E can be multiplying h^E by a function $c^E \in C^\infty(S^2)$ everywhere positive or everywhere negative. Let us call this metric

$$c^E h^E : E^k \otimes E^k \to S^2 \times \mathbb{R}. \tag{4.35}$$

The metric $c^E h^E$ is in fact compatible with the representation. To see this, let $\vec{x}, \vec{y} \in \mathbb{R}^2$ be arbitrary, let R denote the counterclockwise rotation by $\pi/2$, and let $ia \in i\mathbb{R}$ be arbitrary, then:

$$\begin{aligned}
h_S^E(\pi^h(ia)\vec{x}, \vec{y}) + h_S^E(\vec{x}, \pi^h(ia)\vec{y}) &= ah[h_S^E(R\vec{x}, \vec{y}) + h^E(\vec{x}, R\vec{y})] \\
&= ah[h_S^E(\vec{x}, -R\vec{y}) + h^E(\vec{x}, R\vec{y})] \\
&= ah[h_S^E(\vec{x}, 0)] \\
&= 0;
\end{aligned}$$

the previous calculation done for the trivialization of the metric h^E over U_S, since is being done pointwise, extends to show the compatibility of the metric $c^E h^E$ with the representations $\phi^{k,h} : TP^k/S^1 \to E^k$. This compatibility with the metric, is an important property that must be satisfied in order to define gauge invariant matter actions, as we will see in next chapter.

4.5.2 TP^k/S^3 over S^4

This example has not been fully studied, so we will make only the following observations:

- If we were working only over $U_S \cong \mathbb{R}^4$, we could use the Minkowski metric $g = -(dx^1)^2 - (dx^2)^2 - (dx^1)^3 + (dx^4)^2$ for the base manifold, but it can not be extended as is into a metric of S^4, as can be seen evaluating at $\vec{y} = \vec{0}$ the equations (2.52), that are very much analogous to the equations that we have on S^4 when changing coordinates from $\vec{x}$ to $\vec{y}$.

- Sice S^3 is a compact Lie group and $Im\,\mathbb{H}$ is simple, the Killing form of $Im\,\mathbb{H}$ is unique and proportional to

$$h(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y},$$

  where $\vec{x} \in Im\,\mathbb{H}$ and $\cdot$ denotes the dot product of the corresponding vectors in $\mathbb{R}^3$. This means, by Proposition 4.5.2 that the most general Killing inner metric on TP^k/S^3 (and also on the Atiyah Lie algebroids associated to the principal bundles over $\mathbb{R}^4 \cong U_S$ with fiber S^3) has the form:

$$h^c(\langle m, \vec{x}\rangle, \langle m, \vec{y}\rangle) := c(m)\, \vec{x} \cdot \vec{y}, \tag{4.36}$$

  where $c \in C^\infty(S^4)$ (or over $\mathbb{R}^4$) is either always positive or always negative.

- A background connection $\widetilde{\nabla}$ has is associated to a connection form $\widetilde{\omega}$ with local trivialization over U_S given by the formulas

$$\widetilde{\omega}_S = i\widetilde{\omega}_S^1 + j\widetilde{\omega}_S^2 + k\widetilde{\omega}_S^3 - iIm - jJm - kKm, \tag{4.37}$$

if the Lie algebra is taken to be $Im\,\mathbb{H}$, with $\omega_S^a = (\omega_S)^a_\mu dx^\mu$, $a = 1, 2, 3$ restrictions to U_S of 1-forms in $\Omega^1(TS^7)$; alternatively

$$\widetilde{\omega}_S = \begin{pmatrix} i\widetilde{\omega}_S^3 & \widetilde{\omega}_S^2 + i\widetilde{\omega}_S^1 \\ \widetilde{\omega}_S^2 - i\widetilde{\omega}_S^1 & -i\widetilde{\omega}_S^3 \end{pmatrix} - i\sigma_1 Im - i\sigma_2 Jm - i\sigma_3 Km, \qquad (4.38)$$

where $\hat{\omega}_S^\mu \in \Omega^1(S^2)$.

Chapter 5

Gauge Theory in Transitive Lie Algebroids

We have finally introduced all the necessary ingredients to define gauge theories. The traditional gauge theories arise from imposing that the dynamics of a matter field, i.e. a section of a vector bundle, be left invariant under space dependent changes of "internal reference frame" associated to the action of a structure group G, or gauge transformations as defined in the next section, and this is achieved by the introduction of a connection on an associated principal bundle, also called a gauge potential, with certain dynamics. Fournel, et al. in [3] propose the formulation of gauge theories on transitive Lie algebroids from a Lagrangian approach through the action functional $\mathcal{S}$ of the theory, which decompomposes into the matter action $\mathcal{S}_{matter}$ and the gauge action $\mathcal{S}_{matter}$, and where the gauge invariance of the theory is the one induced by the definitions on gauge transformations in [5]. In Section 5.1 we briefly review the notion of gauge transformation from traditional gauge theories, which motivates the generalized definition of (infinitesimal) gauge transformation introduced in [5] that is reviewed and clearly defined in Section 5.2. In Section 5.3 the gauge invariant gauge action and Lagrangian given a transitive Lie algebroid with a metric is defined. The matter gauge action is then defined in Section 5.4 and the local point of view of A-connections that was studied by us in this document is then used to give a formula for the Lagrangian analogous to equation (5.15) introduced in [3] for the gauge action.

The complete framework for the formulation involves an orientable transitive Lie algebroid A, over a base manifold M, equipped with a metric $\hat{g} \equiv (g, h, \widetilde{\nabla})$ with adjoint Lie algebroid L on which h is a Killing metric, and a vector bundle E on which there is a representation $\phi : A \to \mathfrak{D}(E)$ and a metric h^E compatible with ϕ.

5.1 Traditional Gauge Transformations

Traditional gauge theories start with a principal bundle P over a manifold M with structure group G, where $\mathfrak{g}$ is its Lie algebra, and an associated vector bundle E with typical fiber V. The gauge group $\mathcal{G}(P) \ni f$ of P is the group of (vertical) automorphisms of P, that respect the group action; the group $C^\infty_G(P, G) \ni g$ of G-invariant functions, where G acts on itself through the adjoint action, is (anti)isomorphic to the $\mathcal{G}(P)$, via the relation $f(p) = R_{g(p)}$. $f \in \mathcal{G}(P)$ acts on vector valued forms on P through the pullback, this is called a (finite) gauge transformation; in particular, it acts on G-invariant forms of $\Omega^0(TP, P \times V)$, i.e. on $\Gamma(E)$, the space of matter fields; on the subset of $\Omega^1(TP, P \times \mathfrak{g})$ made of principal connection forms w, producing once again a connection; and on their curvatures, basic forms in $\Omega^2(TP, O \times \mathfrak{g})$, where the curvature of f^*w is f^*R where R is the curvature of w.

The infinite dimensional Lie algebra $C^\infty_G(P, \mathfrak{g})$, isomorphic to the sections of $P \times \mathfrak{g}/G$, is called the gauge Lie algebra of P, since there is an exponential map $\mathrm{Exp} : C^\infty_G(P, \mathfrak{g}) \to C^\infty_G(G, G)$, and hence an exponential map $\exp : C^\infty_G(P, \mathfrak{g}) \to \mathcal{G}(P)$. This allows to define the infinitesimal gauge action of elements of $\eta \in C^\infty_G(P, \mathfrak{g})$ on forms α on P as

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\eta)\alpha. \tag{5.1}$$

In particular, from the fact that if $f = R_g \in \mathcal{G}(P)$, then for $\mathcal{X} \in T_pP$ the pushforward is $f_*\mathcal{X} = (L^{-1}_{g(p)*} g_*(\mathcal{X}))^*_{f(p)} + R_{g(p)*}(X)$, we can easily conclude that the infinitesimal gauge action of η on a matter field μ, on a connection w and on a curvature form R is, respectively:

$$-\eta \cdot \mu, \qquad d\eta + w \wedge \eta, \qquad -\eta \wedge R; \tag{5.2}$$

the wedge product used is the one on the differential graded Lie algebra of $\mathfrak{g}$-valued forms, which, for the last equation, is nothing more than the action of $\mathfrak{g}$ on $\mathfrak{g}$ induced by the G-action, i.e. the Lie bracket, just as $\eta\cdot$ is the action of $\mathfrak{g}$ on V induced by that of G.

Throughout this chapter we will be dealing with forms on differential graded Lie algebras, and we will use the symbol $\wedge$ instead of $\wedge^{[,]}$ to denote the multiplication within this algebra, unless confusion may arise.

5.2 Gauge Transformations associated to Transitive Lie Algebroids

For this framework of gauge theories on transitive Lie algebroids, Lazzarini, et al. in [5] define the gauge algebra in such a way that it coincides with the traditional definition in the case that A is the Atiyah Lie algebroid associated to a principal bundle: as the space of sections of an adjoint Lie algebroid.

Let $0 \to L \xrightarrow{j} A \xrightarrow{a} TM \to 0$ be a Lie algebroid sequence for the transitive Lie algebroid A.

Definition 5.2.1. The gauge algebra of A is the (infinite dimensional) Lie algebra $\Gamma(L)$.

Definition 5.2.2. Let $\hat{\omega} \in \Omega^1(A, L)$ be a connection for on A, and let $\eta \in \Gamma(L)$. The infinitesimal gauge action on $\hat{\omega}$ with respect to η is

$$\hat{d}\eta + \hat{\omega} \wedge \eta \tag{5.3}$$

and the infinitesimal gauge transformation of $\hat{\omega}$ is the connection

$$\hat{\omega}^\eta := \hat{\omega} + \hat{d}\eta + \hat{\omega} \wedge \eta. \tag{5.4}$$

The infinitesimal gauge action on the curvature with respect to η is defined to be

$$\hat{R} \wedge \eta \tag{5.5}$$

and its infinitesimal gauge transformation is

$$\hat{R}^\eta := \hat{R} + \hat{R} \wedge \eta \tag{5.6}$$

The curvature of $\hat{\omega}^\eta$ is

$$\hat{R} + \hat{d}\hat{\omega} \wedge \eta + \frac{1}{2}(\hat{\omega} \wedge \hat{\omega}) \wedge \eta + [\frac{1}{2}\hat{d}\eta \wedge \hat{d}\eta + \hat{d}\eta \wedge (\hat{\omega} \wedge \eta) + \frac{1}{2}(\hat{\omega} \wedge \eta) \wedge (\hat{\omega} \wedge \eta)],$$

meaning that it is equal to $\hat{R}^\eta$ plus <u>second order terms in η</u>, i.e. for arbitrary $\mathfrak{X}, \mathfrak{Y} \in A$, and ignoring the immersion j,

$$\begin{aligned}&\frac{1}{2}\hat{d}\eta \wedge \hat{d}\eta + \hat{d}\eta \wedge (\hat{\omega} \wedge \eta) + \frac{1}{2}(\hat{\omega} \wedge \eta) \wedge (\hat{\omega} \wedge \eta)](\mathfrak{X}, \mathfrak{Y})\\ &= [[\mathfrak{X}, \eta], [\mathfrak{Y}, \eta]] + [[\mathfrak{X}, \eta], [\widetilde{\omega}(\mathfrak{Y}), \eta]] + [[\mathfrak{Y}, \eta], [\widetilde{\omega}(\mathfrak{X}), \eta]] + [[\widetilde{\omega}(X), \eta], [\widetilde{\omega}(Y), \eta]];\end{aligned}$$

these are all terms that involve twice the product with η, hence we call them second order terms in η.

Proposition 5.2.3. The algebroid morphism corresponding to the infinitesimal gauge transformation $\hat{\omega}^\eta$ of a connection form $\hat{\omega}$ with respect to $\eta \in \Gamma(L)$ is

$$\hat{\nabla}^\eta := \hat{\nabla} + [\hat{\nabla}, j\eta]; \tag{5.7}$$

the reduced kernel endomorphism corresponding to $\hat{\omega}^\eta$ is

$$\tau^\eta := \tau + [\tau, \eta]. \tag{5.8}$$

These functions will be called <u>the infinitesimal gauge transformations</u> of $\hat{\nabla}$ and τ, respectively, with respect to η.

Proof. $\hat{\nabla}^\eta$ is defined to be, for every $\mathfrak{X} \in A$

$$\begin{aligned}\hat{\nabla}^\eta_{\mathfrak{X}} &= \mathfrak{X} + j\hat{\omega}^\eta(\mathfrak{X})\\ &= \mathfrak{X} + j\hat{\omega}(\mathfrak{X}) + j \circ (\hat{d}\eta + \hat{\omega} \wedge \eta)(\mathfrak{X})\\ &= \hat{\nabla}_{\mathfrak{X}} + [\mathfrak{X}, j\eta] + [j\hat{\omega}(\mathfrak{X}), j\eta]\\ &= \hat{\nabla}_{\mathfrak{X}} + [\hat{\nabla}_{\mathfrak{X}}, \eta].\end{aligned}$$

Similarly, given any $\theta \in L$,

$$\begin{aligned}\tau^\eta(\theta) &= \hat{\omega}^\eta \circ j(\theta) + id_L(\theta)\\ &= (\hat{\omega} \circ j + id_L)(\theta) + \hat{d}\eta(j\theta) + \hat{\omega} \wedge \eta(j\theta)\\ &= \tau + [\theta, \eta] + [\hat{\omega} \circ j(\theta), \eta]\\ &= \tau + [Id_L(\theta) + \hat{\omega} \circ j(\theta), \eta]\\ &= (\tau + [\tau, \eta])(\theta);\end{aligned}$$

the desired results follow from this calculations. $\square$

Throughout the rest of the chapter, ϕ will denote a representation of A on E, with vertical part $\phi_L : L \to End(E)$.

Definition 5.2.4. Let $\mu \in \Gamma(E)$, $\hat{\nabla}^E : A \to \mathfrak{D}(E)$ be an A-connection on E, and $\eta \in \Gamma(L)$. Then, the infinitesimal gauge transformation of μ with respect to η is the section

$$\mu^\eta := \mu - \phi_L(\eta)\mu; \tag{5.9}$$

the infinitesimal gauge transformation of $\hat{\nabla}^E$ is the A-connection

$$\hat{\nabla}^{E,\eta} := \hat{\nabla}^E + [\hat{\nabla}^E, j\phi_L(\eta)]; \tag{5.10}$$

the infinitesimal gauge transformation of the curvature $\hat{R}^E \in \Omega^2(A, End(L))$ is

$$\hat{R}^E \wedge \eta. \tag{5.11}$$

Following a calculation identical to that in the first part of the proof of Proposition 5.2.3, but in the inverser order, we conclude that, with respect to any representation $\phi : A \to \mathfrak{D}(E)$, the $End(E)$-valued form associated to $\hat{\nabla}^{E,\eta}$ is

$$\hat{\omega}^{E,\eta} := \hat{\omega}^E + \hat{d}_E(\phi_L(\eta)) + \hat{\omega}^E \wedge (\phi_L(\eta)). \tag{5.12}$$

Proposition 5.2.5. Given a connection form $\hat{\omega} \in \Omega^1(A, L)$ and an element $\nabla \in \Gamma(L)$, let $\hat{\nabla}^E$ the A be the A-connection produced by $\hat{\omega}$ on E with respect to ϕ. Then the infinitesimal transformation of $\hat{\nabla}^E$ coincides with the A-connection produced by $\hat{\omega}^\eta$.

Proof. The A-connection produced by $\hat{\omega}^\eta$ is

$$\begin{aligned}
\phi(\hat{\nabla}^\eta) &= \phi(\hat{\nabla} + [\nabla, j\eta]) \\
&= \phi(\hat{\nabla}) + [\phi(\hat{\nabla}), \phi_L(\eta)] \\
&= \hat{\nabla}^E + [\hat{\nabla}^E, \phi_L(\eta) \\
&= \hat{\nabla}^{E,\eta}.
\end{aligned}$$

□

5.3 Gauge Action Functional

Let A be an orientable transitive Lie algebroid over a base manifold M equipped with a metric $\hat{g} \equiv (g, h, \widetilde{\nabla})$ with adjoint Lie algebroid L on which h is a Killing metric, g is a metric on M and $\widetilde{\nabla}$ is an ordinary connection on A, called the background connection, with associated 1-form $\widetilde{\omega}$. This assumptions will be carried throughout the rest of the chapter.

Definition 5.3.1. The gauge Lagrangian density given a connection form $\hat{\omega} \in \Omega^1(A, L)$ with curvature $\hat{R}$ is defined as

$$\mathcal{L}_{gauge}[\hat{\omega}] := \int_{inner} h(\hat{R}, *\hat{R}); \tag{5.13}$$

the gauge action functional is defined as

$$\mathcal{S}_{gauge}[\hat{\omega}] := (\hat{R}, \hat{R}), \tag{5.14}$$

i.e. as the integral over M of the Lagrangian density.

Lemma 5.3.2. Let $\eta \in \Gamma(L)$. For any connection with curvature form $\hat{R}$, the form $h(\hat{R}, \hat{R})$ is invariant under infinitesimal gauge transformations up to first order terms in η.

Proof. Since η is a 0-form, it can be shown that $*(\eta \wedge \hat{R}) = \eta \wedge *\hat{R}$; also, since h is a Killing metric, $h(\eta \wedge \hat{R}, \alpha) + h(\hat{R}, \eta \wedge \alpha) = 0$ for any L-valued form α, hence

$$\begin{aligned} h(\hat{R}^\eta, \hat{R}^\eta) &= h(\hat{R} - \eta \wedge \hat{R}, *(\hat{R} - \eta \wedge \hat{R})) \\ &= h(\hat{R}, *\hat{R}) - [h(\eta \wedge \hat{R}, *\hat{R}) + h(\hat{R}, \eta \wedge *\hat{R})] + h(\eta \wedge \hat{R}, \eta \wedge *\hat{R}) \\ &= h(\hat{R}, *\hat{R}) + h(\eta \wedge \hat{R}, \eta \wedge *\hat{R}), \end{aligned}$$

since h is yet another product of forms, which locally looks like in equation (4.5), the last term is of second order in η, and so the statement is proven. $\square$

Proposition 5.3.3. The gauge Lagrangian density $\mathcal{L}_{gauge}$ is invariant under infinitesimal gauge transformations of the connection $\hat{\omega}$ with respect to any $\eta \in \Gamma(L)$, up to first order terms in η.

Proof. This is a corollary of Lemma 5.3.2, since $\mathcal{L}_{gauge} = \int_{inner} h(\hat{R}, *\hat{R})$ is equal to the factor of $h(\hat{R}, *\hat{R})$ that accompanies the inner volume form. $\square$

Due to Proposition 4.4.5 for the Hodge-*, and the decomposition of the curvature given in Theorem 3.2.10, in a trivializing neighborhoood U_i of A and M the Lagrangian density applied to the connection $\hat{\omega}$ with local decomposition $\hat{\omega}_i = \hat{A}_i - \epsilon + \tau_i$ is (supressing the subscript i for the metrics):

$$\begin{aligned}\mathcal{L}_{gauge}[\hat{A}_i, \tau_i] =& [g_i^{\mu_1\nu_1} g^{\mu_2\nu_2} h_{cd} (\hat{F}_i)^c_{\mu_1\mu_2} (\hat{F}_i)^d_{\nu_1\nu_2} + g^{\mu\nu} h^{ab} h_{cd} (\mathcal{D}\tau_i)^c_{\mu,a} (\mathcal{D}\tau_i)^d_{\nu,b} + \\ & h^{a_1b_1} g^{a_2b_2} h_{cd} (W_i)^c_{a_1a_2} (W_i)^d_{b_1b_2}] \sqrt{|g_i|} dx^1 \wedge \cdots \wedge dx^m;\end{aligned} \tag{5.15}$$

equivalently, using the notation introduced in (4.26):

$$\begin{aligned}\mathcal{L}_{gauge}[\hat{A}_i, \tau_i] = \Big[& (\hat{F}_i)^c_{\mu\nu} (\hat{F}_i)^{\mu\nu}_c + (\mathcal{D}\tau)^c_{\mu,a} (\mathcal{D}\tau)^{\mu,a}_c + \\ & (W_i)^c_{ab} (W_i)^{ab}_c \Big] \sqrt{|g_i|} dx^1 \wedge \cdots \wedge dx^m.\end{aligned} \tag{5.16}$$

This shows that the gauge action itself can be decomposed as

$$S_{gauge}[\hat{\omega}] = (a^*\hat{F}, a^*\hat{F}) + ([a^*\mathcal{D}\tau, \widetilde{\omega}], [a^*\mathcal{D}\tau, \widetilde{\omega}]) + (\widetilde{\omega}^* R_\tau, \widetilde{\omega}^* R_\tau). \tag{5.17}$$

When $\hat{\omega}$ is an ordinary connection form on A, $a^*\hat{F} = \hat{R}$ and so $\hat{F} \in \Omega^2(TM, L)$ is the traditional curvature of the ordinary connection, also called the field strength; since $\tau = 0$, the second and third term of the action functional annihilate and the left term coincides, in the case that A is an Atiyah Lie algebroid associated to a principal bundle, with the traditional action functional of the gauge potential.

Recall that in a trivializing neighborhood U_i of A on which there are coordinates $\{x^\mu\}_{\mu=1,\dots,m}$, with respect to a basis $\{E_a\}_{a=1,\dots,n}$ of $\mathfrak{g}$ dual $\{e^a\}_a$, with structure constants $C^a_{bc} E_a = [E_b, E_c]$, we have formulas (3.50), (3.51) and (3.52):

$$\begin{aligned}(\hat{F}_i)^a_{\mu\nu} &= R^a_{\mu\nu} - \tau^a_b \widetilde{R}^b_{\mu\nu}, \\ (\mathcal{D}\tau_i)^b_{\mu,a} &= \partial_\mu (\tau_i)^b_a + (A_i)^c_\mu (\tau_i)^d_c C^b_{cd} - (\widetilde{A}_i)^d_\mu C^c_{da} (\tau_i)^b_c, \\ (W_i)^c_{ab} &= (\tau_i)^b_a (\tau_i)^e_b C^c_{de} - C^d_{ab} (\tau_i)^c_d.\end{aligned} \tag{5.18}$$

So far no method to extremize the action to extract the equations of motion of the system has been studied in this document, but from its resemblance to the traditional Yang-Mills gauge theories, specially when looking at a local trivialization of the Lagrangian density, we might expect the following very rough interpretation of the decomposition (5.17) of the action functional:

1. The first term describes the dynamics of the connection.
2. The second term contains both the kinetic energy of the tau fields, and a coupling of the A and $\widetilde{A}$ fields with the τ fields in terms quadratic in the A fields, which give rise to mass terms for these A fields in the Lagrangian.
3. The third term is the potential term for the τ fields.

However, notice that new $\widetilde{A}_i$ fields appear in the first and second term due to the background ordinary connection that emerges as a consequence of a choice of metric on the algebroid, and their contribution to the action disappears whenever $\hat{\omega}$ is an ordinary connection. Also, the contribution of τ to the first term disappears if the background connection is flat, leaving the first term devoid of contributions by τ, only as a term that resembles the kinetic term of the "gauge fields" A in the traditional gauge theories. Also notice that $\hat{F}$ is not the "curvature" of the global object defined by the trivializations $\hat{A}_i$, nor do the τ contribution to the first term disappear when expanded, meaning that there is no clear choice for what the "gauge fields" should be; however, the full contribution of τ does disappear in some cases on the first term, leaving again a kinetic term for the "gauge fields" $\hat{A}$.

5.4 Matter Action Functional

Now, in addition to the assumptions on A stated at the beginning of the previous section for the matter part of the gauge theory, let E be a vector bundle E on which there is a representation $\phi : A \to \mathfrak{D}(E)$ and a metric h^E compatible with ϕ. The sections of E will be called matter fields of the gauge theory.

Definition 5.4.1. Given a connection form $\hat{\omega} \in \Omega^1(A, L)$ which produces the A-connection $\hat{\nabla}^E$, and a matter field μ, the matter Lagrangian density is defined as

$$\mathcal{L}_{matter}[\mu, \hat{\omega}] := \int_{inner} h^E(\hat{\nabla}^E\mu, *\hat{\nabla}^E\mu); \tag{5.19}$$

where $\hat{\nabla}^E\mu$ is an elements of $\Omega^1(A, E)$; the gauge action functional is defined as

$$\mathcal{S}_{gauge}[\hat{\omega}] := (\hat{\nabla}^E\mu, \hat{\nabla}^E\mu). \tag{5.20}$$

Lemma 5.4.2. Let $\eta \in \Gamma(L)$. Given $\hat{\omega} \in \Omega^1(A, L)$ which induces the A-connection $\hat{\nabla}^E$, and given any $\mu \in \Gamma(E)$, $h^E(\hat{\nabla}^E\mu, \hat{\nabla}^E\mu)$ is invariant under infinitesimal gauge transformations up to first order terms in $\phi_L(\eta)$ (where composition of endomorphisms of E plays the role of multiplication).

Proof. Recall from Proposition 5.2.5 that acting with η on $\hat{\omega}$ equates to acting with $\phi_L(\eta)$ on $\hat{\nabla}^E$, so, conserving only first order terms in $\phi_L(\eta)$, the infinitesimal gauge transformation on $\hat{\nabla}^E\mu$ is:

$$\begin{aligned}\hat{\nabla}^{E,\eta}\mu^\eta &= (\hat{\nabla}^E + [\hat{\nabla}^E, \phi_L(\eta)])(\mu - \phi_L(\eta)\mu) \\ &= \hat{\nabla}^E\mu + [\hat{\nabla}^E, \phi_L(\eta)]\mu - \hat{\nabla}^E(\phi_L(\eta)\mu) \\ &= \hat{\nabla}^E\mu + \phi_L \circ \hat{\nabla}^E\mu.\end{aligned} \tag{5.21}$$

Hence, conserving only first order terms in $\phi_L(\eta)$:

$$\begin{aligned}h^E((\hat{\nabla}^E\mu)^\eta, *\,(\hat{\nabla}^E\mu)^\eta) &= h^E(\hat{\nabla}^E\mu + \phi_L \circ \hat{\nabla}^E\mu, *(\hat{\nabla}^E\mu + \phi_L \circ \hat{\nabla}^E\mu)) \\ &= h^E(\hat{\nabla}^E\mu, \hat{\nabla}^E\mu) + h^E(\phi_L(\eta) \circ \hat{\nabla}^E\mu, *\hat{\nabla}^E\mu) \\ &+ h^E(\hat{\nabla}^E\mu, \phi_L(\eta) \circ *\hat{\nabla}^E\mu) \\ &= h^E(\hat{\nabla}^E\mu, \hat{\nabla}^E\mu);\end{aligned}$$

the last step follows from the fact that h^E is compatible with ϕ. □

From this lemma it follows that:

Proposition 5.4.3. The matter Lagrangian density $\mathcal{L}_{matter}$ is invariant under infinitesimal gauge transformations of the connection $\hat{\omega}$ with respect to any $\eta \in \Gamma(L)$, up to first order terms in $\phi_L(\eta)$.

Proposition 5.4.4. Let $\hat{\omega} \in \Omega^1(A, L)$ be a connection form in A that produces the A-connection $\hat{\nabla}^E$, and let $\nabla : TM \to A$ be the connection associated to the ordinary connection form $\omega = \hat{\omega} + \tau \circ \widetilde{\omega}$. For any $\mu \in \Gamma(E)$:

$$\hat{\nabla}^E \mu = a^* \phi(\nabla)\mu - (\phi_L(\tau)\mu) \circ \widetilde{\omega}, \tag{5.22}$$

i.e. for any $\mathfrak{X} \in A$, $\hat{\nabla}^E_{\mathfrak{X}} \mu = \phi(\hat{\nabla}_{a(\mathfrak{X})})\mu - \phi_L \circ \tau \circ \widetilde{\omega}(\mathfrak{X})\mu$.

Proof. This decomposition is induced by the ordinary connection $\omega = \hat{\omega} + \tau \circ \widetilde{\omega}$, which allows to write any $\mathfrak{X} \in A$ as $\mathfrak{X} = \phi(\nabla_{a(\mathfrak{X})}) - \phi_L \circ \omega(\mathfrak{X})$. Then

$$\begin{aligned} \hat{\nabla}^E_{\mathfrak{X}} &= \phi(\mathfrak{X}) + \phi_L \circ \hat{\omega}(\mathfrak{X}) \\ &= \phi(\nabla_{a(\mathfrak{X})}) - \phi_L \circ \omega(\mathfrak{X}) + \phi_L \circ (\omega - \tau \circ \widetilde{\omega})(\mathfrak{X}) \\ &= \phi(\nabla_{a(\mathfrak{X})}) - \phi_L \circ \tau \circ \widetilde{\omega}(\mathfrak{X}). \end{aligned}$$

□

Let $\{(U_i, \psi_i : U_i \times \mathfrak{g} \to L|_{U_i}, \nabla^{0,i} : TU_i \to A|_{U_i})\}_{i \in I}$ be a Lie algebroid atlas for A, where $\mathfrak{g}$ is a Lie algebra with basis $\{E_a\}_{a=1,\dots,n}$ and associated dual basis $\{\epsilon^a\}$. Assume that $\{e_u\}_{u=1,\dots,t}$ is a basis of the fiber V of E, and that the matter field $\mu \in \Gamma(E)$ is locally realized as $(\mu_i)^u e_u$ with $\mu_i \in C^\infty(U_i, V)$ for each $i \in I$; also suppose that there are coordinates $\{x^\mu : U_i \to \mathbb{R}\}$over $U_i \subseteq M$. Recall the decomposition (3.64) of a produced A-connection; then it can be easily seen that the decomposition of $\hat{\nabla}^E \mu$ of Proposition 5.4.4 is such that the local trivializations over U_i of each term are:

$$\begin{aligned} a^* \phi(\nabla)\mu_i &= [\partial_\nu (\mu_i)^u + (B_i)^u_{\nu,v} (\mu_i)^v + (\phi_{L,i})^u_{b,v} (A_i)^b_\nu (\mu_i)^v] e_u dx^\nu, \\ (\phi_L(\tau)\mu) \circ \widetilde{\omega}_i &= [(\phi_{L,i})^u_{b,v} \tau^b_a (\mu_i)^v] e_u \mathfrak{a}^a_i, \end{aligned} \tag{5.23}$$

where $B_i \in \Omega^1(TU_i, U_i \times End(V))$ is the Maurer-Cartan form that, together with $\phi_{L,i} \in C^\infty(U_i, End(V))$, trivializes the representation ϕ (recall that $B_i = 0$ and $\phi_{L,i} = \pi$ is the group representation on V for the group induced representations of Example 1.5.16); and $\mathfrak{a}^a_i$ is the component with respect to $E_a \in \mathfrak{g}$ of the background connection $\widetilde{\omega}$. Thus, thanks to Proposition 4.4.5 about the Hodge-* operator, the matter Lagrangian density has the formula

$$\begin{aligned} \mathcal{L}_{matter} = [&g^{\nu_1\nu_2} h^E_{u_1u_2} (a^*\phi(\nabla)\mu_i)^{u_1}_{\nu_1} (a^*\phi(\nabla)\mu_i)^{u_2}_{\nu_2} \\ &+ h^{a_1a_2} h^E_{u_1u_2} ((\phi_L(\tau)\mu) \circ \widetilde{\omega}_i)^{u_1}_{a_1} ((\phi_L(\tau)\mu) \circ \widetilde{\omega}_i)^{u_2}_{a_2}] \sqrt{|g_i|} dx^1 \wedge \dots \wedge dx^m; \end{aligned} \tag{5.24}$$

equivalently, using the notation introduced in (4.26):

$$\mathcal{L}_{matter} = [(a^*\phi(\nabla)\mu_i)^u_\nu (a^*\phi(\nabla)\mu_i)^\nu_u \\ + ((\phi_L(\tau)\mu)\circ\widetilde{\omega}_i)^u_a((\phi_L(\tau)\mu)\circ\widetilde{\omega}_i)^a_u]\sqrt{|g_i|}dx^1\wedge\cdots\wedge dx^m, \quad (5.25)$$

and the matter gauge functional has the decomposition

$$\mathcal{S}_{matter}[\mu,\hat{\omega}] = (a^*\phi(\nabla)\mu, a^*\phi(\nabla)\mu) + ((\phi_L(\tau)\mu)\circ\widetilde{\omega}, (\phi_L(\tau)\mu)\circ\widetilde{\omega}). \quad (5.26)$$

When A is the Atiyah Lie algebroid associated to a principal bundle, if the representation is the group induced representation and the connection $\hat{\omega}$ is ordinary, we are left only with the first term, which then coincides with the matter action of a Yang-Mills gauge theory, containing the kinetic term of the matter fields and the interaction term from the coupling of the gauge fields $A = \hat{A}$ with the matter field. In general, if the representation is the group induced one the first term represents a minimal coupling of the matter fields with the fields A associated to the ordinary connection $\omega = \hat{\omega} + \tau\circ\widetilde{\omega}$, and the second term of the action functional comes from the extra dimensions in which the A-connection allows "covariant derivatives" to be taken, and in the Lagrangian density it becomes a quadratic term for the matter field, i.e. a "mass terms".

5.5 Examples

5.5.1 TP^k over S^1

In Section 4.5.2 we introduced metrics on $P^k\times i\mathbb{R}/S^1$ and representations vector bundles E^k that were compatible with the representations $ad : TP^k/S^1 \to \mathfrak{D}(P^k\times i\mathbb{R}/S^1)$ and $\phi^{h,k} : TP^k/S^1 \to \mathfrak{D}(E)$, respectively, where $h\in\mathbb{Z}$. Recall that the representation $\phi^{h,k}$ is the group induced representation associated to the representation (4.34) of S^1 on $\mathbb{R}^2$. Let us suppose that we prove the Lie algebroid A with the metric induced by a general metric

$$g = g_{\mu\nu}dx^\mu dx^\nu$$

on S^2, for example the round metric, together with the Killing inner metric h^c (4.36) for some nowhere zero function $c \in C^\infty(S^2)$, and a background (ordinary) connection $\widetilde{\nabla} : TS^2 \to TP^k/S^1$ associated to the connection form $\widetilde{\omega}$ with local trivialization over U_S given by:

$$\begin{aligned}\widetilde{\omega}_S &= i\widetilde{\omega}^{\epsilon}_{S;1}dx^1 + i\widetilde{\omega}^{\epsilon}_{S;2}dx^2 - iIm \\ &= i\widetilde{f}d\phi + i\widetilde{g}d\theta - iIm\end{aligned}$$

where $\widetilde{\omega}^{\epsilon}_{S;\mu}, \widetilde{f}, \widetilde{g}$ are functions over S^2 with some restrictions along the lines as those specified in Section 2.4.1.

Given an arbitrary connection form $\hat{\omega}$ associated to the reduced kernel endomorphism τ, the trivialization of its curvature form over U_S was found in (3.74), and using equations (3.79) we deduce that the decomposition of $\hat{R}_S$ resulting from the application of Theorem 3.2.10 is

$$\begin{aligned}\hat{R}_S = i(\partial_1\hat{\omega}^{\epsilon}_{S;2} - \partial_2\hat{\omega}^{\epsilon}_{S;1} + i\partial_1\widetilde{\tau}\widetilde{\omega}_{S;2} - i\partial_2\widetilde{\tau}\widetilde{\omega}_{S;1})dx^1 \wedge dx^2 - \\ i\partial_1\widetilde{\tau}dx^1 \wedge \widetilde{\omega}^1_S - i\partial_2\widetilde{\tau}dx^2 \wedge \widetilde{\omega}^1_S, \quad (5.27)\end{aligned}$$

with $\widetilde{\omega}_S = i\widetilde{\omega}^1_S$. Hence, the gauge Lagrangian (5.15) locally trivialized over U_S, with respect to the basis $\{i\}$ of $i\mathbb{R}$ and the coordinates $\vec{x}$, is:

$$\begin{aligned}\mathcal{L}_{gauge}[\hat{\omega}_S, \widetilde{\tau}] = [4c(g^{12}_S)^2(\partial_1\hat{\omega}^{\epsilon}_{S;2} - \partial_2\hat{\omega}^{\epsilon}_{S;1} + i\partial_1\widetilde{\tau}\widetilde{\omega}_{S;2} - i\partial_2\widetilde{\tau}\widetilde{\omega}_{S;1})^2 + \\ g^{11}_S c^2(\partial_1\widetilde{\tau})^2 + 2g^{12}_S c^2(\partial_1\widetilde{\tau})(\partial_2\widetilde{\tau}) + g^{22}_S c^2(\partial_2\widetilde{\tau})^2 + 0]\sqrt{|g_S|}dx^1 \wedge dx^2. \quad (5.28)\end{aligned}$$

Notice that in the first term there are both quadratic, i.e. kinetic, terms on the $\hat{\omega}^{\epsilon}_{S;\cdot}$ fields from the connection, as well as coupling terms between the $\widetilde{\tau}$ field and the $\widetilde{\omega}^{\epsilon}_{S;\cdot}$ of the background connection. The remaining terms are kinetic terms for the $\widetilde{\tau}$ field, and no potential term for it appears since $W = 0$.

Let us now find an expression for the matter action given the representation $\phi^{h,k} : TP^k/S^1 \to E^k$ and a vector field $\psi \in E^k$ on which there is a metric $c^E h^E$ as in (4.35) for some nowhere zero function $c^E \in C^\infty(S^2)$. Recall from (3.76) that the trivialization A_S of the induced ordinary connection $\omega = \hat{\omega} + \tau \circ \widetilde{\omega}$ is

$$A_S = i(\hat{\omega}^{\epsilon}_{S;1} + \widetilde{\tau}\widetilde{\omega}^{\epsilon}_{S;1})dx^1 + i(\hat{\omega}^{\epsilon}_{S;2} + \widetilde{\tau}\widetilde{\omega}^{\epsilon}_{S;2})dx^2.$$

Hence, equation (5.24) becomes:

$$\mathcal{L}_{matter}[\hat{\omega}_S, \widetilde{\tau}, \psi_i] = [g_S^{\mu\nu} c^E(\partial_\mu \psi_S + ih\omega_{S;\mu}\psi_S) \cdot (\partial_\nu \psi_S + ih\omega_{S;\nu}\psi_S) \\ + cc^E(i\widetilde{\tau}h\psi_S)^2]\sqrt{|g_S|}dx^1 \wedge dx^2, \quad (5.29)$$

here $\psi_S \in C^\infty(U_S, \mathbb{C})$ is the trivialization over U_S of the vector field $\psi \in \Gamma(E^k)$, and the dot product and square of such functions refers to the euclidean inner product of $\mathbb{R}^2$-valued functions; note that in the last ecuation $h \in \mathbb{Z}$ is a label for the representation of S^1, it does not denote the inner metric.

For this family of examples, even though the Lie algebra is commutative, in the matter Lagrangian we obtained a quadratic coupling of the $\widetilde{\tau}$ field with both the $\widetilde{\omega}_{S;\cdot}$ field and the matter field ψ. These are additions to the familiar terms of minimal coupling between the ω_S fields with the matter field, as well and the kinetic terms for the matter field. Together with the kinetic terms for $\widetilde{\tau}$ that appear as a result of the gauge Lagrangian, this method has produced interesting new simple terms even though we worked with a commutative Lie algebra, which caused many terms to disappear.

5.5.2 TP^k/S^3 over S^4

Notice that, since we have used an open cover of two sets for the base manifold S^2, and each of its elements U_S and U_N cover all but one point of S^2, hence working on a local trivialization over U_S, for example, is enough to determine the complete action $\mathcal{S} = \mathcal{S}_{gauge} + \mathcal{S}_{matter}$ and the Lagrangian density of a gauge theory on the Atiyah Lie algebroids TP^k/S^3.

Throughout the document we have determined the necessary components to formulate the gauge action of a gauge theory on TP^k/S^3, since the general and ordinary connection forms and Killing inner metrics were studied in Sections 3.4.2 and 4.5.2. Taking into account the relation (1.121) the structure constants for the Lie algebra, for $a, b, c \in \{1, 2, 3\}$, are $C^a_{bc} = \epsilon_{abc}$ if the basis $\{i, j, k\}$ of $Im\,\mathbb{H}$ is used or, equivalently, if the basis $\{i\sigma_1, i\sigma_2, i\sigma_3\}$ of $\mathfrak{su}(2)$. Knowing this we can readily use formulas (5.18) to replace them in (5.15) to obtain the gauge Lagrangian density.

Similarly, once representation vector bundles of TP^k/S^3 are found, along with metrics invariant under the representation, formulas (5.23) can immediately be used and replaced in (5.24) to obtain the matter Lagrangian; the local approach for the concepts of representation and A-connections was introduced in this document precisely to reduce the application of the present formalism to a straightforward formulaic process. Furthermore, the procedure introduced by us to generate representation vector bundles based on a group action (see Definition 1.5.8 and the previous discussion), generates plenty of representations of TP^k/S^3 with known local trivializations; however, invariant metrics on them were not studied for the present document.

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