Leftist Heaps and Skew Heaps

Leftist Heaps

Target : Speed up merging in O(N).

Heap: Structure Property + Order Property

> **Discussion 5:** How fast can we merge two heaps if we simply use the original heap structure?

Leftist Heap:

Order Property – the same

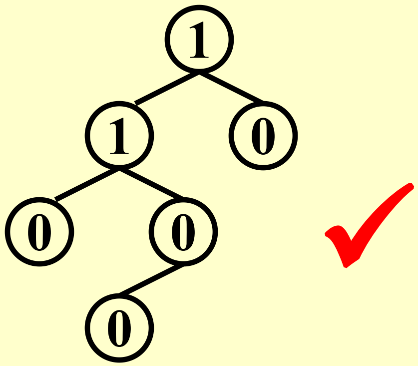
Structure Property – binary tree, but *unbalanced*

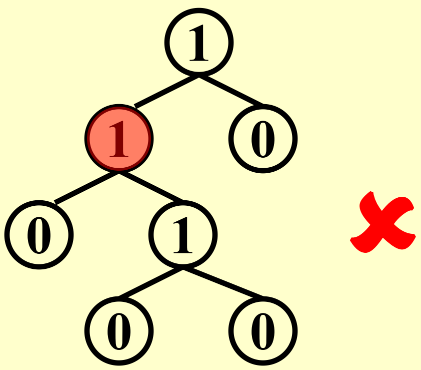
【Definition】The **null path length**, **Npl**(X), of any node X is the length of the shortest path from X to a node without two children. Define Npl(NULL) = –1.

Note:

$$\text{Npl}(X) = \min \{ \text{Npl}(C) + 1 \text{ for all } C \text{ as children of } X \}$$

【Definition】The **leftist heap property** is that for every node X in the heap, the null path length of the **left** child is **at least as large as** that of the **right** child.





The tree is biased to get deep toward the *left*.

【Theorem】A leftist tree with r nodes on the right path must have at least $2^r – 1$ nodes.

Proof: By induction on p. 162.

Discussion 6: How long is the right path of a leftist tree of *N* nodes? What does this conclusion mean to us?

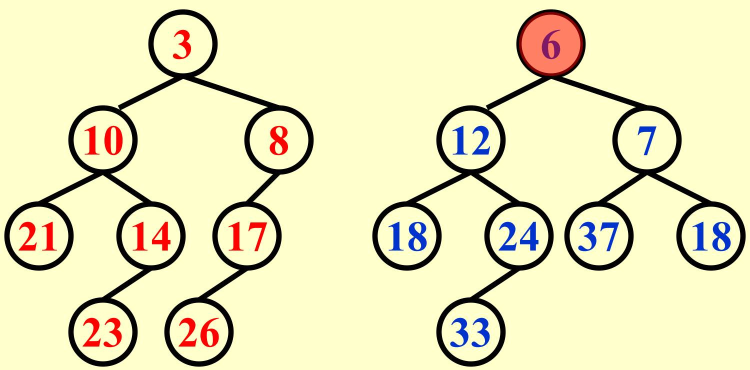
Trouble makers: Insert and Merge

Note: Insertion is merely a special case of merging.

☞ **Declaration:**

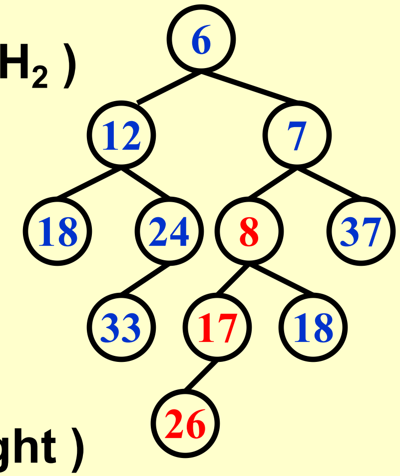
```
struct TreeNode
{
    ElementType     Element;
    PriorityQueue   Left;
    PriorityQueue   Right;
    int             Npl;
} ;
```

☞ **Merge (recursive version):**

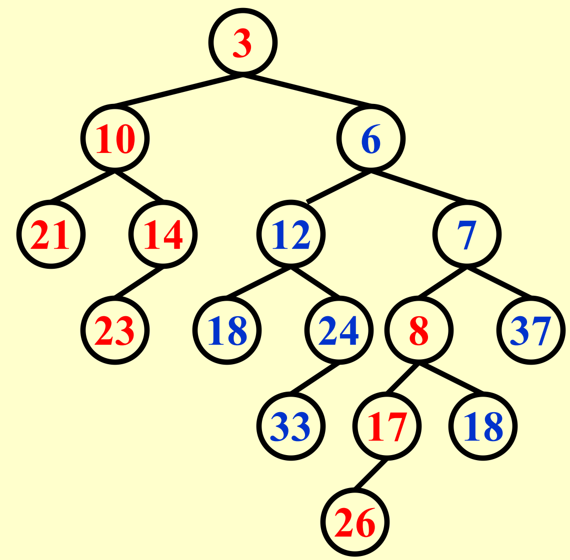


H_1 H_2

Step 1: Merge(H_1->Right, H_2)



Step 2: Attach(H_2, H_1->Right)



Step 3: Swap(H_1->Right, H_1->Left) if necessary

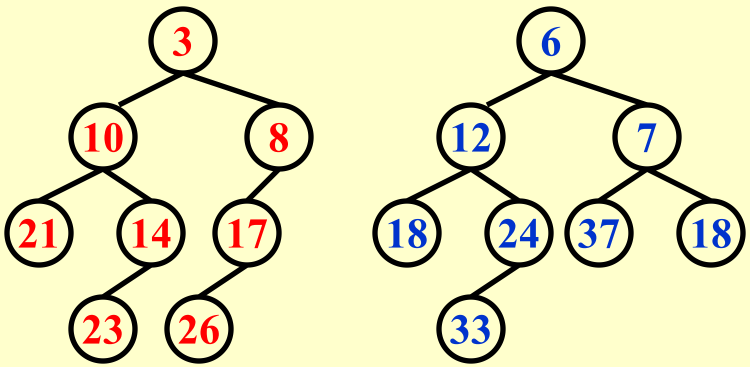
```
PriorityQueue  Merge ( PriorityQueue H1, PriorityQueue H2 )
{
    if ( H1 == NULL )   return H2;
    if ( H2 == NULL )   return H1;
    if ( H1->Element < H2->Element )  return Merge1( H1, H2 );
    else return Merge1( H2, H1 );
}
```

```
static PriorityQueue
Merge1( PriorityQueue H1, PriorityQueue H2 )
{
    if ( H1->Left == NULL )     /* single node */
            H1->Left = H2;      /* H1->Right is already NULL
                                   and H1->Npl is already 0 */
    else {
            H1->Right = Merge( H1->Right, H2 );     /* Step 1 & 2 */
            if ( H1->Left->Npl < H1->Right->Npl )
                        SwapChildren( H1 );         /* Step 3 */
            H1->Npl = H1->Right->Npl + 1;
    } /* end else */
    return H1;
}
```

$T_p = O(\log N)$

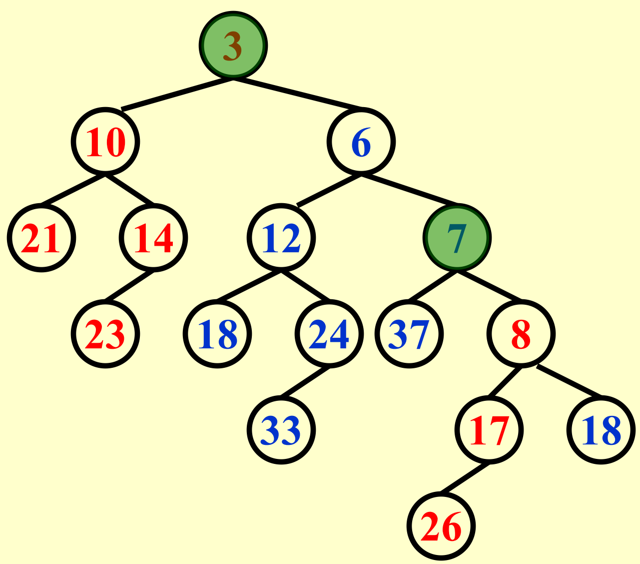
What if *Npl* is NOT updated?

☞ **Merge (iterative version):**

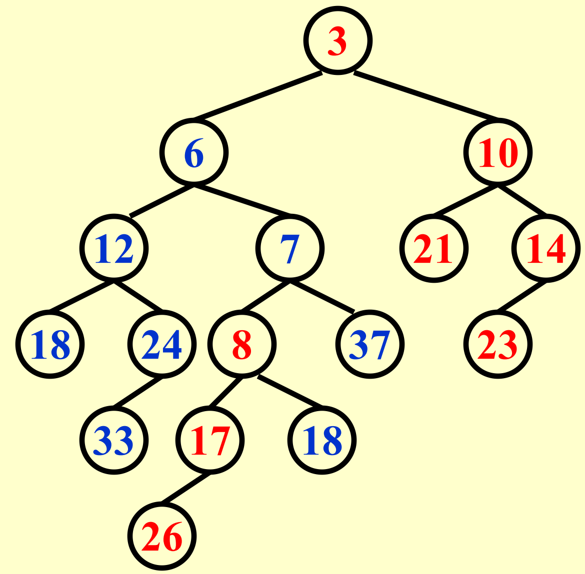


H_1 H_2

***Step 1*: Sort the right paths without changing their left children**



***Step 2*: Swap children if necessary**



☞ **DeleteMin:**

***Step 1*: Delete the root**

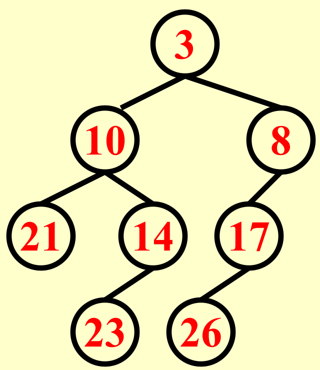
***Step 2*: Merge the two subtrees**

$T_p = O(\log N)$

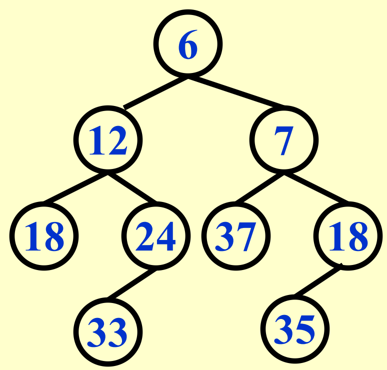
Skew Heaps -- a simple version of the leftist heaps

Target : Any M consecutive operations take at most $O(M \log N)$ time.

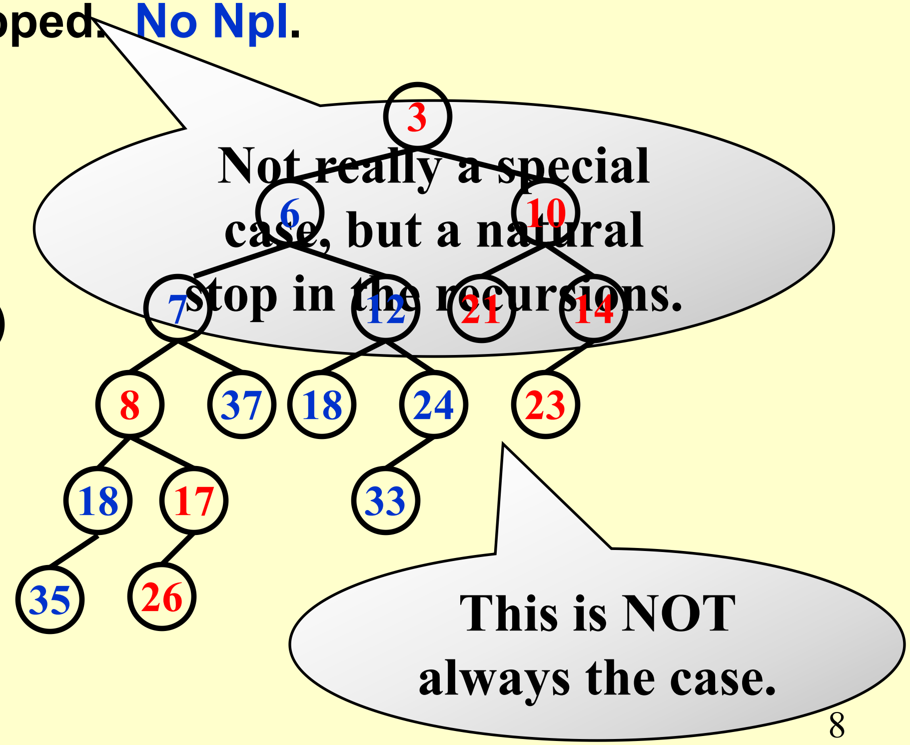
☞ **Merge:** Always swap the left and right children except that the largest of all the nodes on the right paths does not have its children swapped. No Npl.



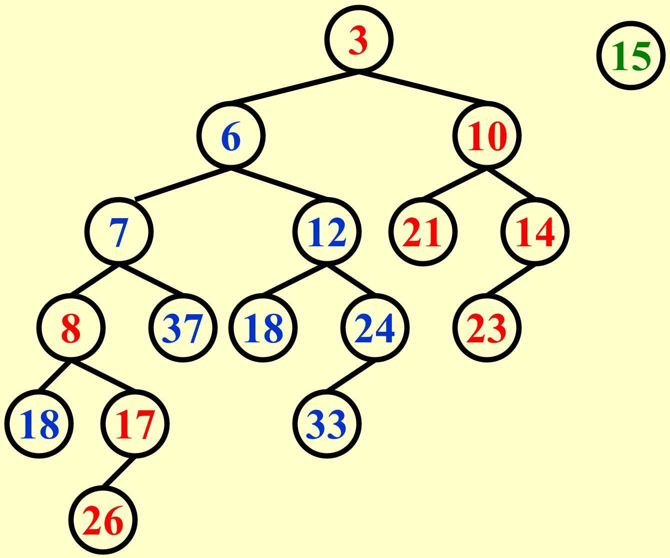
H_1

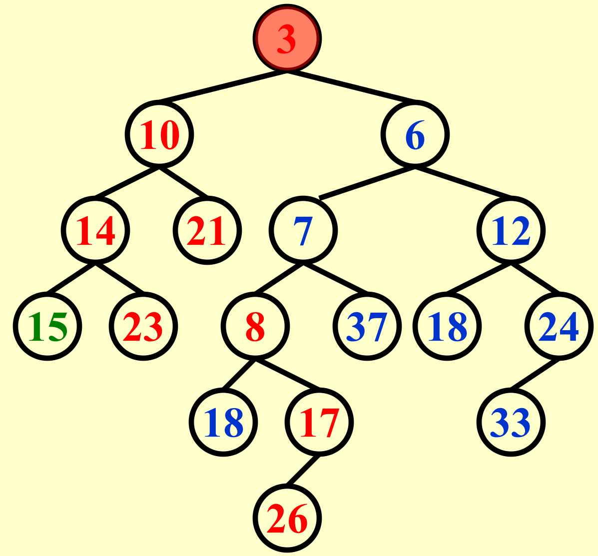


H_2

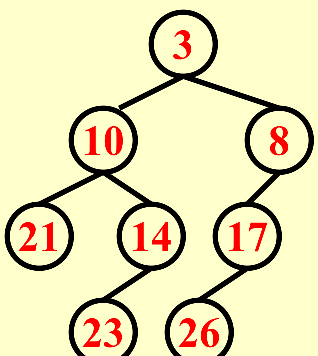


【Example】Insert 15

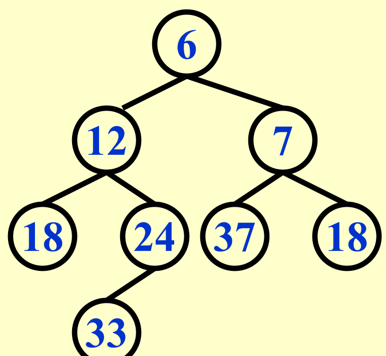




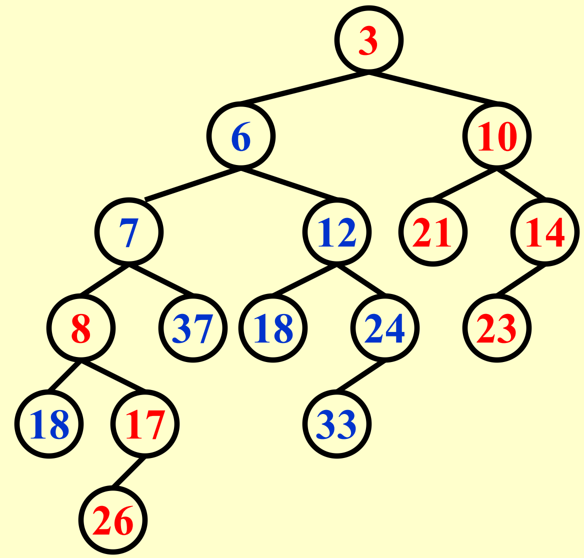
☞ **Merge (iterative version):**



H_1



H_2



Note:

☞ Skew heaps have the advantage that no extra space is required to maintain path lengths and no tests are required to determine when to swap children.

☞ It is an open problem to determine precisely the expected right path length of both leftist and skew heaps.

Amortized Analysis for Skew Heaps

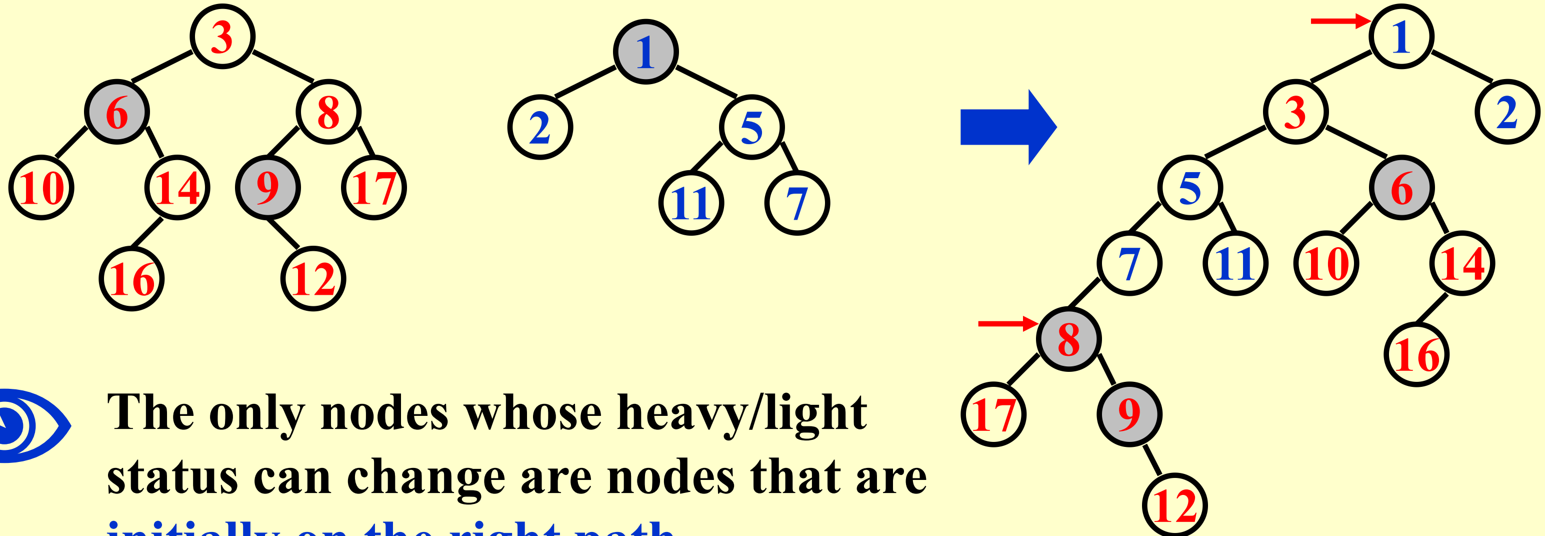
Insert & Delete are just Merge

$T_{amortized} = O(\log N)$?

D_i = the root of the resulting tree

$\Phi(D_i)$ = number of ***heavy*** nodes

【Definition】A node *p* is *heavy* if the number of descendants of *p*'s right subtree is at least half of the number of descendants of *p*, and *light* otherwise. Note that the number of descendants of a node includes the node itself.



The only nodes whose heavy/light status can change are nodes that are initially on the right path.

$H_i : l_i + h_i \; (i = 1, 2)$ ➡ $T_{worst} = l_1 + h_1 + l_2 + h_2$

Along the right path

Before merge: $\Phi_i = h_1 + h_2 + h$ $\quad T_{amortized} = T_{worst} + \Phi_{i+1} - \Phi_i$

After merge: $\Phi_{i+1} \le l_1 + l_2 + h$ $\quad \le 2 (l_1 + l_2)$

$l = O(\log N)$ ➡ $T_{amortized} = O(\log N)$

Reference:

Data Structure and Algorithm Analysis in C (2nd Edition)： Ch.5，p.161-169； Ch.11，p.435-437；*M.A.Weiss著、陈越改编，人民邮件出版社，2005*