8.4 Generating Functions

Section Summary

- ✓Generating Functions
- ✓Useful Facts About Power Series
- ✓Counting Problems and Generating Functions
- ✓Solving Recurrence Relations Using Generating Functions
- ✓Proving Identities Using Generating Functions

Why should we study generating functions?

Generating functions are useful for manipulating sequences.

- ✓ **to solve many kinds of counting problems**

  For example, the problem of combination or permutation with constraints

- ✓ **to solve the recurrence relations**
- ✓ **to prove combinatorial identities**

Generating Functions

【Definition 1】 The generating function for the sequence $a_0, a_1, a_2, \ldots, a_k, \ldots$ of real numbers is the infinite series.

$$G(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

〖Example 1〗

(1) What is the generating function for the sequence 1, 1, 1, 1, ...?

$$G(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

(2) What is the generating function for the sequence 0, 1, 2, 3, 4, 5, ...?

$$G(x) = \sum_{k=0}^{\infty} k x^k$$

Generating Functions for Finite Sequences

The generating function for finite sequence of real numbers

$a_0, a_1, a_2, \ldots, a_n$ **is**

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

〖Example 2〗

(1) The finite sequence: 1,1,1. The generating function for this sequence is

$$G(x) = 1 + x + x^2 = \frac{1 - x^3}{1 - x}$$

(2) Let $a_k = C(m,k), k = 0,1,2,\ldots,m$ **. The generating function for this sequence is**

$$G(x) = C(m,0) + C(m,1)x + C(m,2)x^2 + \ldots + C(m,m)x^m = (1+x)^m$$

Useful Facts About Power Series

【Theorem 1】Let $f(x)=\sum_{k=0}^{\infty} a_k x^k, g(x)=\sum_{k=0}^{\infty} b_k x^k$. **Then**

(1) $f(x)+g(x)=\sum_{k=0}^{\infty}(a_k+b_k)x^k$

(2) $\alpha \cdot f(x)=\sum_{k=0}^{\infty}\alpha \cdot a_k x^k$ $\qquad \alpha \in R$

(3) $x \cdot f'(x)=\sum_{k=0}^{\infty} k \cdot a_k x^k$

(4) $f(\alpha x)=\sum_{k=0}^{\infty}\alpha^k \cdot a_k x^k$

(5) $f(x)g(x)=\sum_{k=0}^{\infty}(\sum_{j=0}^{k} a_j b_{k-j})x^k$

Proof:

$$\sum_{k=0}^{\infty} k a_k x^k = \sum_{k=0}^{\infty} a_k \cdot x \cdot k x^{k-1}$$

$$= x\sum_{k=0}^{\infty} a_k (x^k)'$$

$$= x(\sum_{k=0}^{\infty} a_k x^k)'$$

$$= xf'(x)$$

Useful facts about power series

Let $f(x)=\sum_{k=0}^{\infty} a_k x^k, g(x)=\sum_{k=0}^{\infty} b_k x^k$. **Then**

(1) $f(x)+g(x)=\sum_{k=0}^{\infty}(a_k+b_k)x^k$

(2) $\alpha \cdot f(x)=\sum_{k=0}^{\infty}\alpha \cdot a_k x^k \qquad \alpha \in R$

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(5) $f(x)g(x)=\sum_{k=0}^{\infty}(\sum_{j=0}^{k} a_j b_{k-j})x^k$

Proof:

$$\sum_{k=0}^{\infty}(\sum_{j=0}^{k} a_j b_{k-j})x^k$$

$$= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2$$

$$+\ldots+(\sum_{j=1}^{k} a_j b_{k-1})x^k + \ldots$$

$$= (a_0 + a_1x + a_2x^2 + \ldots)(b_0 + b_1x + b_2x^2 + \ldots)$$

$$= f(x)\cdot g(x)$$

- **Using the above properties, the generating functions of some sequence can be obtained easily.**

〖Example 3〗 (1) What is the generating function for the sequence 0,1,2,3,4,5,…?

Solution:

$$b_k = k$$

$$G(x) = \sum_{k=0}^{\infty} kx^k$$

$$= x\left(\frac{1}{1-x}\right)'$$

$$= \frac{x}{(1-x)^2}$$

〖Example 3〗 (2) Suppose that the generating function of the sequence: $a_0, a_1, a_2, \ldots, a_n, \ldots$ **is $G(x)$. What is the generating function for the sequence** $b_k = \sum_{i=0}^{k} a_i$ **?**

Solution: $a_k \leftrightarrow G(x), \quad b_k \leftrightarrow F(x)$

$$c_k = 1$$

$$b_k = \sum_{i=0}^{k} a_i$$

$$= \sum_{i=0}^{k} a_i \times c_{k-i}$$

$$F(x) = G(x) \cdot \frac{1}{1-x}$$

For example:

$$1,1,1,\ldots \longleftrightarrow \frac{1}{1-x}$$

$$\Downarrow$$

$$1,2,3,4,\ldots,k+1,\ldots \longleftrightarrow \frac{1}{(1-x)^2}$$

〖Example 3〗 (3) What is the generating function for the sequence $a_k = k^2$?

Solution:

$$a_k = 1 \longleftrightarrow \frac{1}{1-x}$$

$$a_k = k \longleftrightarrow \frac{x}{(1-x)^2}$$

$$a_k = k^2 \longleftrightarrow x\left(\frac{x}{(1-x)^2}\right)' = \frac{x(1+x)}{(1-x)^3}$$

〖Example 3〗 (4) What is the generating function for the sequence $a_k = \sum_{i=1}^{k} i^2$?

Solution:

$$a_k = k^2 \longleftrightarrow x\left(\frac{x}{(1-x)^2}\right)' = \frac{x(1+x)}{(1-x)^3}$$

$$a_k = \sum_{i=1}^{k} i^2 \longleftrightarrow \frac{x(1+x)}{(1-x)^4}$$

【Example 4】 Let $f(x)=\dfrac{1}{1-4x^2}$. Find the coefficient $a_0, a_1, a_2, \ldots, a_n, \ldots$ in the expansion $f(x)=\sum_{k=0}^{\infty} a_k x^k$.

Solution:

$$f(x)=\frac{1}{1-4x^2} = \frac{1}{(1-2x)(1+2x)} = \frac{1}{2}\left(\frac{1}{1-2x}+\frac{1}{1+2x}\right)$$

$$2^k \qquad (-2)^k$$

$$\frac{1}{2}(2^k+(-2)^k) = \begin{cases} 2^k & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases}$$

The extended binomial coefficient

Recall:

$$\binom{m}{k} = C(m,k) = \frac{m!}{k!(m-k)!}$$

Where *m*, *k* are nonnegative integers, *k*≤*m*

【Definition 2】 Let *u* be a real number and *k* a nonnegative integer. Then the extended binomial coefficient is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\ldots(u-k+1)/k! & \text{if} \quad k>0 \\ 1 & \text{if} \quad k=0 \end{cases}$$

〖Example 5〗 (1) $\binom{1/2}{3} = ?$ (2) $\binom{-n}{k} = ?$

Solution:

$$(1)\binom{1/2}{3} = \frac{(1/2)(1/2-1)(1/2-2)}{3!} = 1/16$$

$$(2)\binom{-n}{k} = \frac{(-n)(-n-1)\dots(-n-k+1)}{k!}$$

$$= \frac{(-1)^k n(n+1)\dots(n+k-1)}{k!}$$

$$= (-1)^k C(n+k-1, k)$$

The extended Binomial Theorem

【Theorem 2】Let x be a real number with $|x|<1$ and let u be a real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

【Example 6】**Find** **the generating functions** **for**

$$(1+x)^{-n} \; and \; (1-x)^{-n}$$

where *n* is a positive integer, using the extended Binomial Theorem.

Solution:

By the extended Binomial Theorem , it follows that

$$(1+x)^{-n}$$

$$=\sum_{k=0}^{\infty}\binom{-n}{k}x^k$$

$$=\sum_{k=0}^{\infty}(-1)^k C(n+k-1,k)x^k$$

$$(1-x)^{-n}$$

$$=\sum_{k=0}^{\infty}\binom{-n}{k}(-x)^k$$

$$=\sum_{k=0}^{\infty}(-1)^k C(n+k-1,k)(-1)^k x^k$$

$$=\sum_{k=0}^{\infty}C(n+k-1,k)x^k$$

Some Common Used Generating Functions

Sequence	**Generating function**
(1) $C(n,k)$	$\sum_{k=0}^{\infty} C(n,k)x^k = (1+x)^n$
(2) $C(n,k)a^k$	$(1+ax)^n$
(3) $1,1,\ldots,1$	$1+x+x^2+\ldots+x^n = \dfrac{1-x^{n+1}}{1-x}$
(4) $1,1,1,\ldots$	$\dfrac{1}{1-x}$
(5) a^k	$\dfrac{1}{1-ax}$
(6) $k+1$	$\dfrac{1}{(1-x)^2}$

Some Common Used Generating Functions

Sequence	Generating function
(7) $C(n+k-1,k)$	$(1-x)^{-n}$
(8) $(-1)^k C(n+k-1,k)$	$(1+x)^{-n}$
(9) $C(n+k-1,k)a^k$	$(1-ax)^{-n}$
(10) $\frac{1}{k!}$	e^x
(11) $\frac{(-1)^{k+1}}{k}$	$\ln(1+\mathrm{x})$

Some Common Used Generating Functions

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

Counting Problems and Generating Functions

〖Example 7〗Find the number of solutions of $e_1+e_2+e_3=17$ **where** e_1,e_2,e_3 **are nonnegative integers with** $2\le e_1\le 5, 3\le e_2\le 6, and\ 4\le e_3\le 7$

Solution:

$$e_1+e_2+e_3=17$$

(1) $e_i\ge 0$ $\Rightarrow$ $H_3^{17}=C(3-1+17,17)$

(2) $e_1\ge 10$ $\Rightarrow$ $e_1+e_2+e_3=7(e_i\ge 0)$

$\Rightarrow$ $H_3^{7}=C(3-1+7,7)$

(3) $2\le e_1\le 5, 3\le e_2\le 6, and\ 4\le e_3\le 7$?

The generating function for this counting problem is

$$G(x)=(x^2+x^3+x^4+x^5)(x^3+x^4+x^5+x^6)(x^4+x^5+x^6+x^7)$$

The number of solutions is the coefficient of x^{17} in the expansion of $G(x)$.

〖Example 8〗Use generating functions to find the number of *r*-combinations from a set with *n* elements when repetition of elements is allowed.

Solution:

Since there are *n* elements in the set, each can be selected zero times, one times and so on. It follows that

$$G(x) = (1 + x + x^2 + x^3 + \ldots)^n = \left(\frac{1}{1-x}\right)^n = \frac{1}{(1-x)^n}$$

the number of *r*-combinations from a set with *n* elements when repetition of elements is allowed, is the coefficient a_r of x^r in the expansion of $G(x)$. Since

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1, k) x^k$$

Then the coefficient a_r equals $C(n+r-1,r)$

【Example 9】**Suppose that there are 2*r* red balls, 2*r* blue balls, and 2*r* white balls. How many ways to select 3*r* balls from these balls?**

Solution:

How to find a_{3r} ?

$$G(x)=(1+x+x^2+\ldots+x^{2r})^3$$

The coefficient a_{3r} of x^{3r} in the expansion of $G(x)$ is the solution of this problem.

$$G(x)=(1+x+x^2+\ldots+x^{2r})^3=\left(\frac{1-x^{2r+1}}{1-x}\right)^3=\frac{1-3x^{2r+1}+3x^{4r+2}-x^{6r+3}}{(1-x)^3}$$

$$F(x)=\frac{1}{(1-x)^3}=(1+x+x^2+\ldots)^3$$

The coefficient of term x^i in $F(x)$ is $H_3^i=C_{3+i-1}^i=C_{i+2}^i$

$$2r+1+y=3r \qquad \therefore y=r-1$$

The coefficient of term x^{r-1} in $F(x)$ is C_{r+1}^{r-1}

$$\therefore a_{3r}=C_{3r+2}^{3r}-3C_{r+1}^{r-1}$$

〖Example 10〗 **Determine the number of ways to insert tokens worth \$1,\$2 and \$5 into a vending machine to pay for an item that costs *r* dollars in both the case when the order in which the tokens are inserted does not matter and when the order does matter.**

Solution:

(1) The order in which the tokens are inserted does not matter

$$G(x)=(1+x+x^2+x^3+\ldots)(1+x^2+x^4+x^6+\ldots)(1+x^5+x^{10}+x^{15}+\ldots)$$

The coefficient of x^r in the expansion of *G(x)* is the solution of this problem.

(2) The order in which the tokens are inserted does matter

- **The number of ways to insert exactly *n* tokens to produce a total of *r*\$ is the coefficient of x^r in** $(x+x^2+x^5)^n$
- **Since any number of tokens may be inserted, the number of ways to produce *r*\$ using \$1,\$2 and \$5 tokens, is the coefficient of x^r in**

$$1+(x+x^2+x^5)+(x+x^2+x^5)^2+\ldots=\frac{1}{1-(x+x^2+x^5)}$$

Use Generating Function To Solve Recurrence Relations

The Method:

(1) Use the recurrence relation find the generating function of this sequence.

(2) $G(x) \Rightarrow a_n$

〖Example 11〗 Use generating functions to solve the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2} + 4^n + 6$ **with initial conditions** $a_0 = 20, a_1 = 60$ **.**

Solution:

$$a_n = 2a_{n-1} + 3a_{n-2} + 4^n + 6 \qquad \times x^n$$

$$a_n x^n = 2a_{n-1}x^n + 3a_{n-2}x^n + 4^n x^n + 6x^n$$

$$\sum_{n=2}^{\infty} a_n x^n = 2\sum_{n=2}^{\infty} a_{n-1}x^n + 3\sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=2}^{\infty} 4^n x^n + 6\sum_{n=2}^{\infty} x^n$$

$$G(x) - a_0 - a_1 x \qquad 2x\sum_{n=1}^{\infty} a_n x^n \qquad 3x^2\sum_{n=0}^{\infty} a_n x^n \qquad \frac{1}{1-4x} - 1 - 4x \qquad 6\left(\frac{1}{1-x} - 1 - x\right)$$

$$2x(G(x) - a_0) \qquad 3x^2 G(x)$$

$$(1-2x-3x^2)G(x)=\frac{20-80x+2x^2+40x^3}{(1-4x)(1-x)}$$

$$G(x)=\frac{20-80x+2x^2+40x^3}{(1-4x)(1-x)(1+x)(1-3x)}$$

$$=\frac{16/5}{1-4x}+\frac{-3/2}{1-x}+\frac{31/20}{1+x}+\frac{67/4}{1-3x}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{16}{5}\times 4^n \quad -\frac{3}{2}\times 1^n \quad \frac{31}{20}\times(-1)^n \quad \frac{67}{4}\times 3^n$$

$$a_n=\frac{16}{5}\times 4^n-\frac{2}{3}+\frac{31}{20}\times(-1)^n+\frac{67}{4}\times 3^n$$

Proving Identities Via Generating Functions

【**Example 12**】 **Use generating function to prove Pascal's identity** $C(n,r)=C(n-1,r)+C(n-1,r-1)$ **when *n* and *r* are positive integers with** $r<n$ **.**

Proof:

$$G(x)=(1+x)^n=\sum_{r=0}^{n}C(n,r)x^r$$

$$(1+x)^n=(1+x)(1+x)^{n-1}=(1+x)^{n-1}+x(1+x)^{n-1}$$

$$\sum_{r=0}^{n}C(n,r)x^r=\sum_{r=0}^{n-1}C(n-1,r)x^r+\sum_{r=0}^{n-1}C(n-1,r)x^{r+1}$$

$$=\sum_{r=0}^{n-1}C(n-1,r)x^r+\sum_{r=1}^{n}C(n-1,r-1)x^r$$

$$1+\sum_{r=1}^{n-1}C(n,r)x^r+x^n$$

$$=1+\sum_{r=1}^{n-1}C(n-1,r)x^r+\sum_{r=1}^{n-1}C(n-1,r-1)x^r+x^n$$

$$\sum_{r=1}^{n-1}\underline{C(n,r)}x^r=\sum_{r=1}^{n-1}[\underline{C(n-1,r)+C(n-1,r-1)}]x^r$$

Homework:

SE： P. 549 6,16, 24, 30, 34,49

EE： P. 575 6,16, 24, 32, 36, 51