6.3
Permutations and Combinations

Section Summary

- ✓Permutations
- ✓Combinations
- ✓Combinatorial Proofs

Permutation

permutation **: an ordered arrangement of the elements of a set**
r-permutation **: an ordered arrangement of *r* elements of a set**

【Theorem 1】 The number of *r*-permutations of a set with *n* distinct elements is

$$P(n, r)=n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Proof: **Using the product rule.**
n **choices for the first element, (*n* – 1) for the second one, (*n* – 2) for the third one…**

Note :

- **$P(n,0) = 1$, since there is only one way to order zero elements.**
- **$P(n, n)=n!$**

Solving Counting Problems by Counting Permutations

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

***r*-combination:** an unordered selection of *r* elements of a set

Note: An *r*-combination is simply a subset of a set with *r* elements.

***C*(*n*, *r*): the number of *r*-combination of a set with *n* element**

$$C(n,r)=\binom{n}{r}$$

Binomial coefficient

$$=\frac{n!}{r!(n-r)!}$$

【Theorem 2】 The number of *r*-combination of a set with *n* elements, where *n* is a positive integer and *r* is an integer with $0 \le r \le n$, equals

$$n(n-1)(n-2)\ldots(n-r+1)/r!$$

〖Example 1〗 **A soccer club has 8 female and 7 male members. For today's match, how many possible configurations are there?**

(1) The coach wants to have 6 female and 5 male players on the grass.

(2) The coach wants to have 11 players with at most 5 male players on the grass.

Solution:

(1) $C(8, 6) \cdot C(7, 5)$

$= 8!/(6!\cdot 2!) \cdot 7!/(5!\cdot 2!)$

$= 28\cdot 21$

$= 588$

(2) $C(8，6)C(7，5)+C(8，7)C(7，4)+C(8，8)\ C(7，3)$

【 Corollary 1 】 Combination Corollary

Let *n* and *r* be nonnegative integers with *r* ≤ *n*. Then

$C(n, r) = C(n, n-r)$

Proof:

(1) From Theorem 2, it follows that

$$C(n,r)=\frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r)=\frac{n!}{(n-r)![n-(n-r)]!}=\frac{n!}{(n-r)!r!}.$$

Hence, $C(n, r) = C(n, n - r)$.

【 Corollary 1 】 **Combination Corollary**

Let n and r be nonnegative integers with $r \le n$. Then

$$C(n, r) = C(n, n-r)$$

Proof:

(2) Using **Combinatorial Proof**

A combinatorial proof of an identity:

- **double counting proofs**

  uses counting arguments to prove that both sides of the identity count the same objects but in different ways.

- **bijective proofs**

  show that there is a bijection between the sets of objects counted by the two sides of the identity.

Combinatorial Proofs

- Here are two combinatorial proofs that

$$C(n, r) = C(n, n - r)$$

when r and n are nonnegative integers with $r < n$:

  - *Bijective Proof*: Suppose that S is a set with n elements. The function that maps a subset A of S to $\bar{A}$ is a bijection between the subsets of S with r elements and the subsets with $n - r$ elements. Since there is a bijection between the two sets, they must have the same number of elements.
  - *Double Counting Proof*: By definition the number of subsets of S with r elements is $C(n, r)$. Each subset A of S can also be described by specifying which elements are not in A, i.e., those which are in $\bar{A}$. Since the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements.

6.4
Binomial Coefficients

Section Summary

- ✓The Binomial Theorem
- ✓Pascal's Identity and Triangle
- ✓Other Identities Involving Binomial Coefficients

Powers of Binomial Expressions

Definition: A ***binomial* expression** is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

- **We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.**
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)\,(x + y)\,(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, $x\,y^2$, y^3 arise. The question is what are the coefficients?
  - To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
  - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
  - To obtain xy^2, an x must be chosen from of the sums and a y from the other two . There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
  - To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.
- **We have used a counting argument to show that** $(x + y)^3 = x^3 + 3x^2y + 3x\,y^2 + y^3$.
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

The Binomial Theorem

【 Theorem 1 】 The Binomial Theorem

Let x and y be varaibles, and let n be a nonnegative integer. Then $$(x+y)^n=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

Proof:

We use combinatorial reasoning .

The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0,1,2,\ldots,n$. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$.

Using the Binomial Theorem

〖Example 1〗 What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Solution:

We view the expression as $(2x+(-3y))^{25}$.

By the binomial theorem

$$(2x+(-3y))^{25}=\sum_{j=0}^{25}\binom{25}{j}(2x)^{25-j}(-3y)^{j}$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13}(2)^{12}(-3)^{13}=-\frac{25!}{13!12!}2^{12}3^{13}$$

Corollaries for the Binomial Theorem

Let *n* be a nonnegative integer. Then

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$$

$$\sum_{k=0}^{n}2^k\binom{n}{k} = 3^n$$

Proof (*using binomial theorem*): With $x = 1$ and $y = 1$, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n}\binom{n}{k}1^k 1^{(n-k)} = \sum_{k=0}^{n}\binom{n}{k}.$$

Proof (*combinatorial*): Consider the subsets of a set with n elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, …, and $\binom{n}{n}$ with n elements. Therefore the total is $\sum_{k=0}^{n}\binom{n}{k}.$

Since, we know that a set with n elements has 2^n subsets, we conclude: $\sum_{k=0}^{n}\binom{n}{k} = 2^n.$

PASCAL'S Identity

【 Theorem 2 】 PASCAL'S Identity

Let n and k be positive integers with $k \le n$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof: $A = \{x, a_1, a_2, \ldots, a_n\}$

the basis of *Pascal's triangle*

We construct subsets of size k from a set with $n + 1$ elements.

The total will include

- all of the subsets from the set of size n which do not contain the element x $C(n, k)$,

  plus

- the subsets of size k - 1 with the element x added

  $C(n, k\text{-}1)$.

Pascal's triangle

The *n*th row in the triangle consists of the binomial coefficients $\binom{n}{k}$,

***k* = 0,1,….,*n*.**

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\cdots
\end{array}
$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\cdots
\end{array}
$$

(b)

By Pascal’s identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

Other Identity Involving Binomial Coefficients

【 **Theorem 3** 】 **Vandermonde's Identity**

Let *m*, *n* and *r* be nonnegative integer with *r* not exceeding either *m* or *n*. Then

$$\binom{m+n}{r}=\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$$

Proof:

A and B are two disjoint sets.

$|A|=m$, $|B|=n$,

$C(m+n, r)$ ---- the number of ways to pick *r* elements from $A\cup B$

Another way to pick *r* element from $A\cup B$ is to pick *r*-*k* elements from *A* and then *k* elements from *B*, where $0\le k\le r$

【 Corollary 4 】If n is a nonnegative integer. Then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Proof:

We use Vandermonde's Identity with $m=r=n$ to obtain

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

【 Theorem 4 】Let *n* and *r* be nonnegative integer with $r \le n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Proof:

The left-hand side counts the bit strings of length $n+1$ containing $r+1$ 1s.

We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ ones.

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

Homework:

SE:

P.413 20,28, 30

P.421 10,24,27

EE:

P.413 20,30, 32

P.444 14,28,31