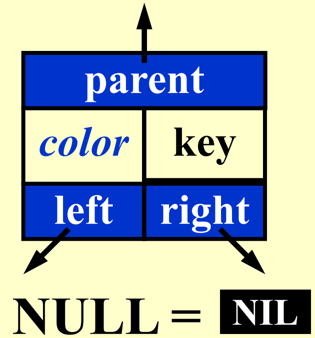
Red-Black Trees

and B+ Trees

Red-Black Trees



Target: Balanced binary search tree

【Definition】A red-black tree is a binary search tree that satisfies the following *red-black properties*:

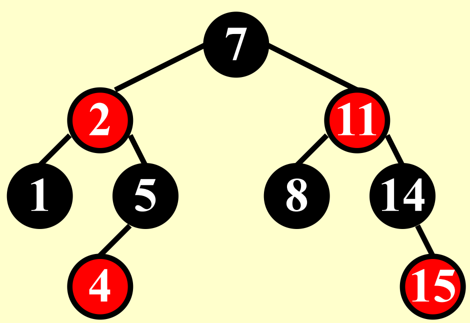
(1) Every node is either **red** or **black**.

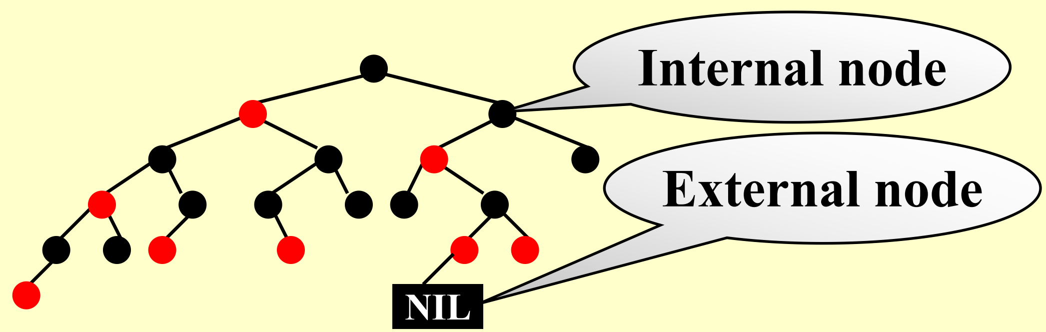
(2) The root is **black**.

(3) Every leaf (NIL) is **black**.

(4) If a node is **red**, then both its children are **black**.

(5) For each node, all simple paths from the node to descendant leaves contain the same number of **black** nodes.





【Definition】 The black-height of any node x, denoted by bh(x), is the number of **black** nodes on any simple path from x (x not incl ...) = bh(root).

Number of internal nodes in the subtree rooted at x

【Lemma】 A red ... has height at most $2\ln(N+1)$.

Proof: ① For any node x, $\text{sizeof}(x) \geq 2^{\text{bh}(x)} - 1$. Prove by induction.

If $h(x) = 0$, x is NULL $\Rightarrow$ $\text{sizeof}(x) = 2^0 - 1 = 0$ ✔

Suppose it is true for all x with $h(x) \leq k$.

For x with $h(x) = k + 1$, $\text{bh}(child) = \text{bh}(x)$ or $\text{bh}(x) - 1$

Since $h(child) \leq k$, $\text{sizeof}(child) \geq 2^{\text{bh}(child)} - 1 \geq 2^{\text{bh}(x)-1} - 1$

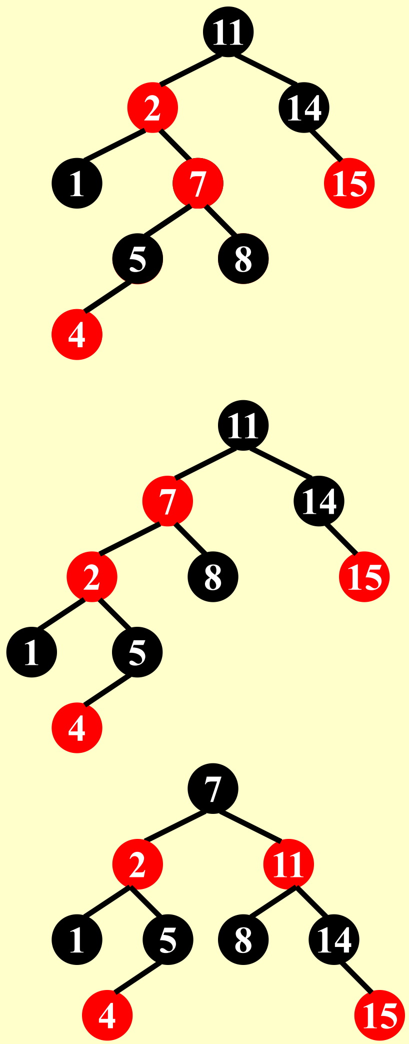
Hence $\text{sizeof}(x) = 1 + 2\text{sizeof}(child) \geq 2^{\text{bh}(x)} - 1$ ✔

② $\text{bh}(Tree) \geq h(Tree) / 2$?

Discussion 2: Please finish the proof.

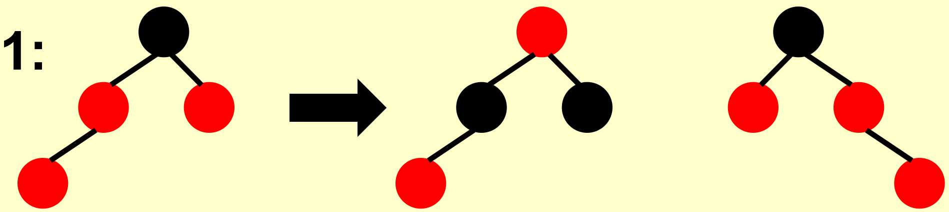
☞ Insert — can be done *iteratively*

Sketch of the idea: Insert & color red

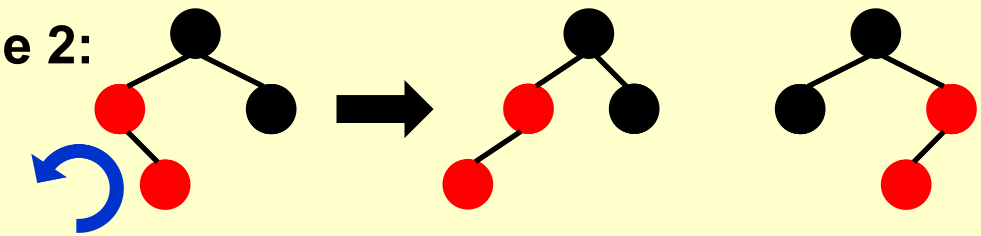


Symmetric

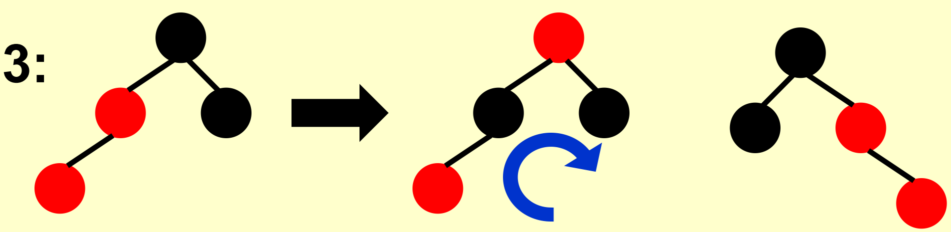
Case 1:

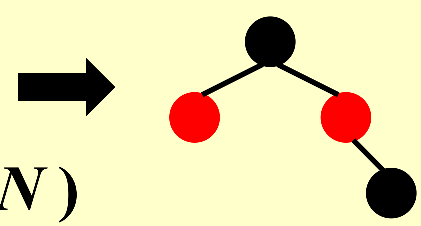


Case 2:



Case 3:





$$T = O(h) = O(\ln N)$$

☞ Delete

❖ **Delete a leaf node :** Reset its parent link to NIL.

❖ **Delete a degree 1 node :** Replace the node by its single child.

Adjust only if the node is black.

❖ **Delete a degree 2 node :**

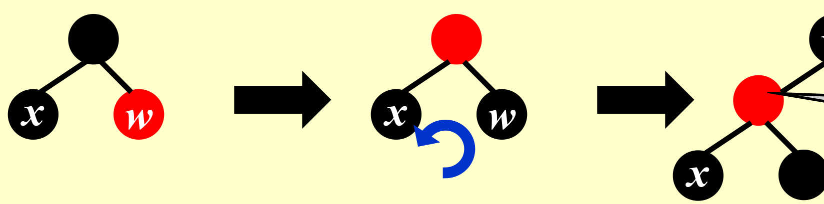
① Replace the node by the **largest** one [illegible] or the **smallest** one in its **right** subtree.

② Delete the replacing node from the subtree.

Keep the color

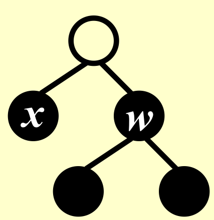
Must *add 1 black* to the path of the replacing node.

Case 1:

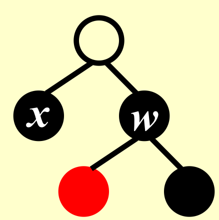


Still not solved

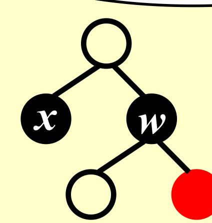
Case 2:



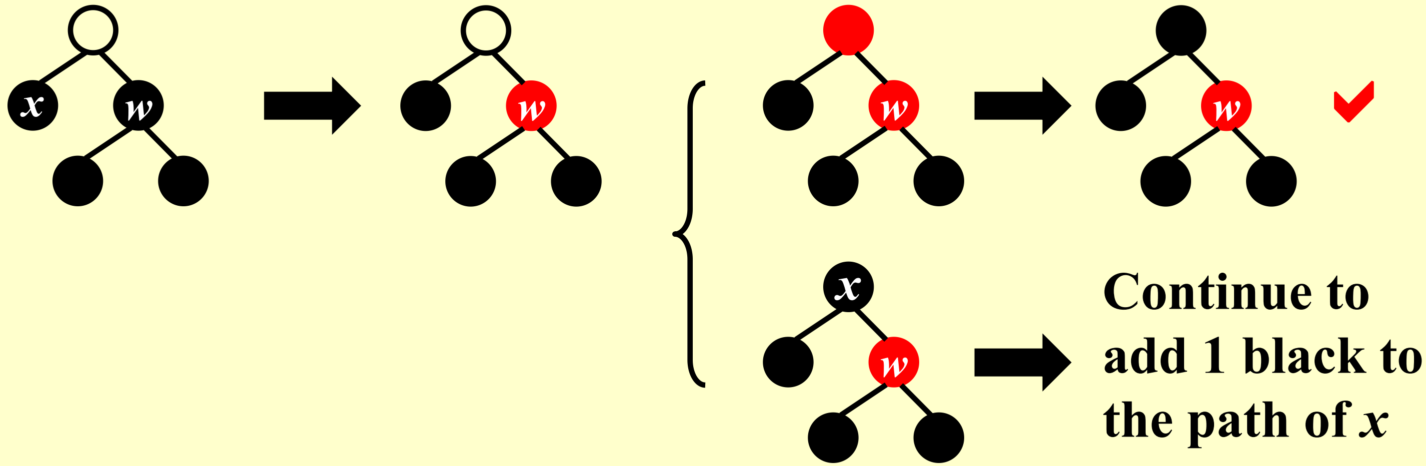
Case 3:



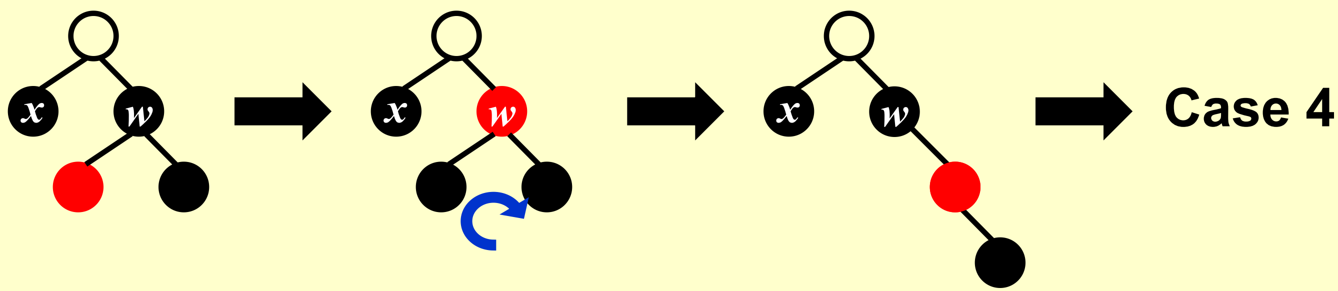
Case 4:



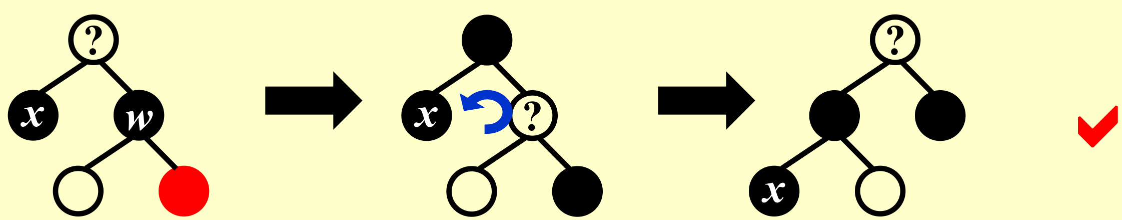
Case 2:



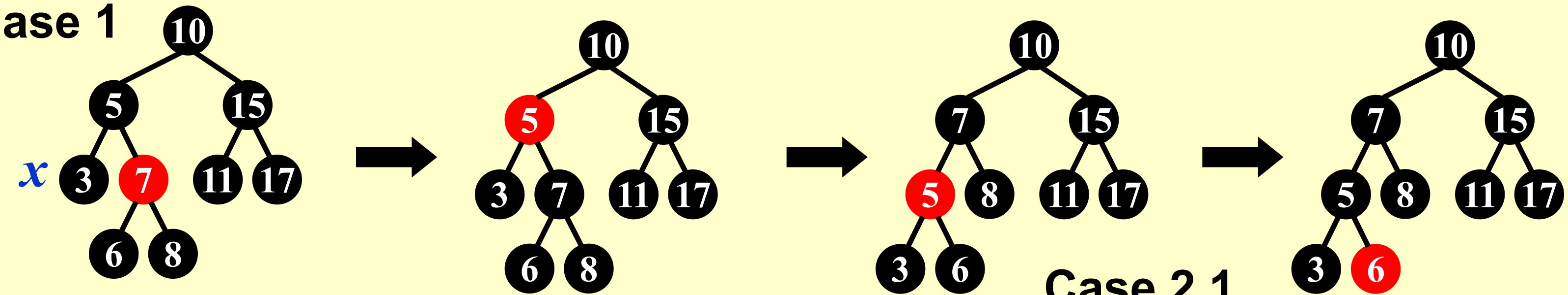
Case 3:



Case 4:

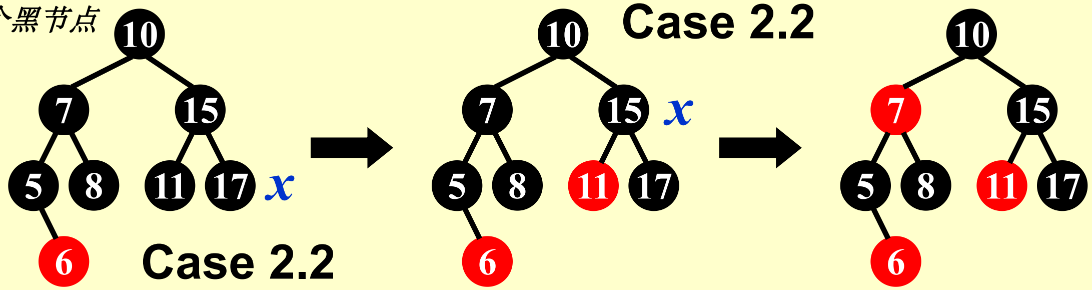


Case 1



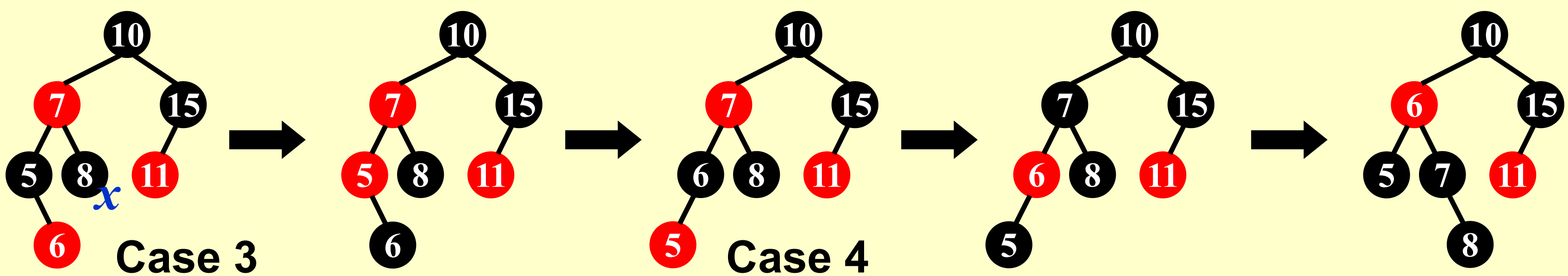
Case 2.1

Case 2.2. Continue to add 1 black to the path of *x*, *补充说明：要删17，将11染红。如果15是根节点，这样x的兄弟已经平衡（同时减一个黑的），否则，15这个分支需要增加1个黑节点*



Case 2.2

Case 2.2



Case 3

Case 4

Number of *rotations*

	AVL	Red-Black Tree
Insertion	≤ 2	≤ 2
Deletion	$O(\log N)$	≤ 3

B+ Trees

【Definition】A B+ tree of order M is a tree with the following structural properties:

(1) The root is either a leaf or has between 2 and M children.

(2) All nonleaf nodes (except the root) have between ⌈M/2⌉ and M children.

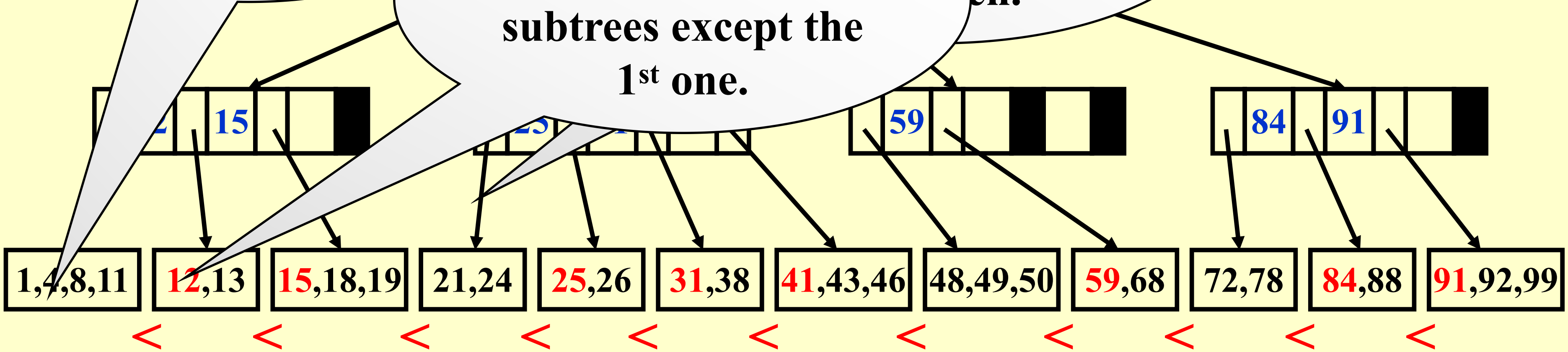
(3) All leaves are at the same depth.

leaf also has between ⌈M/2⌉ and M keys.

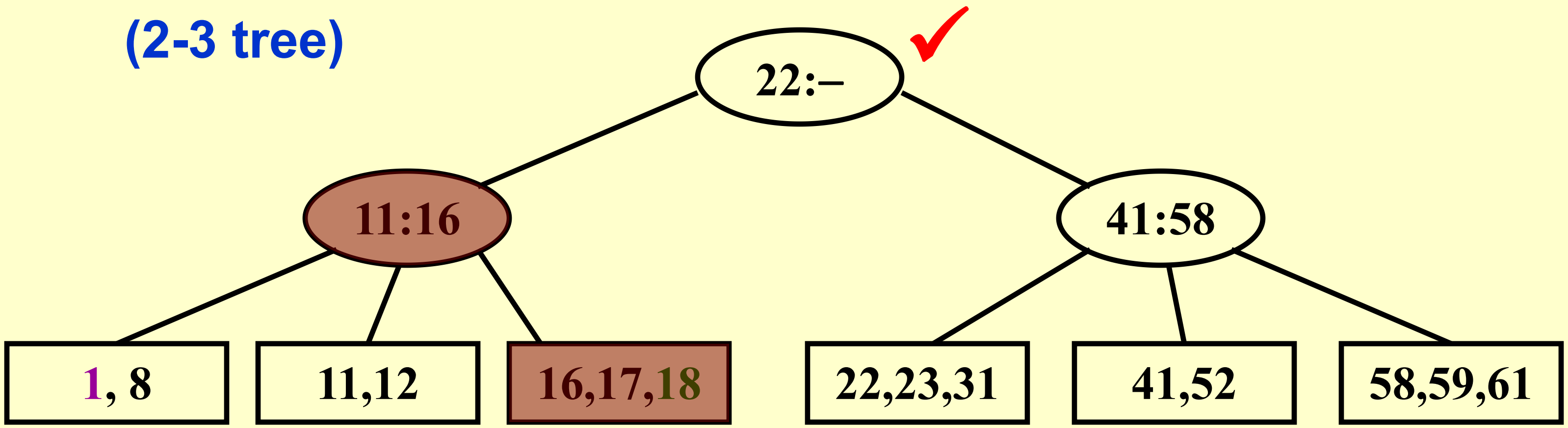
All the actual data are stored at the leaves.

Each interior node contains M pointers to the children.

And $M - 1$ smallest key values in the subtrees except the 1st one.

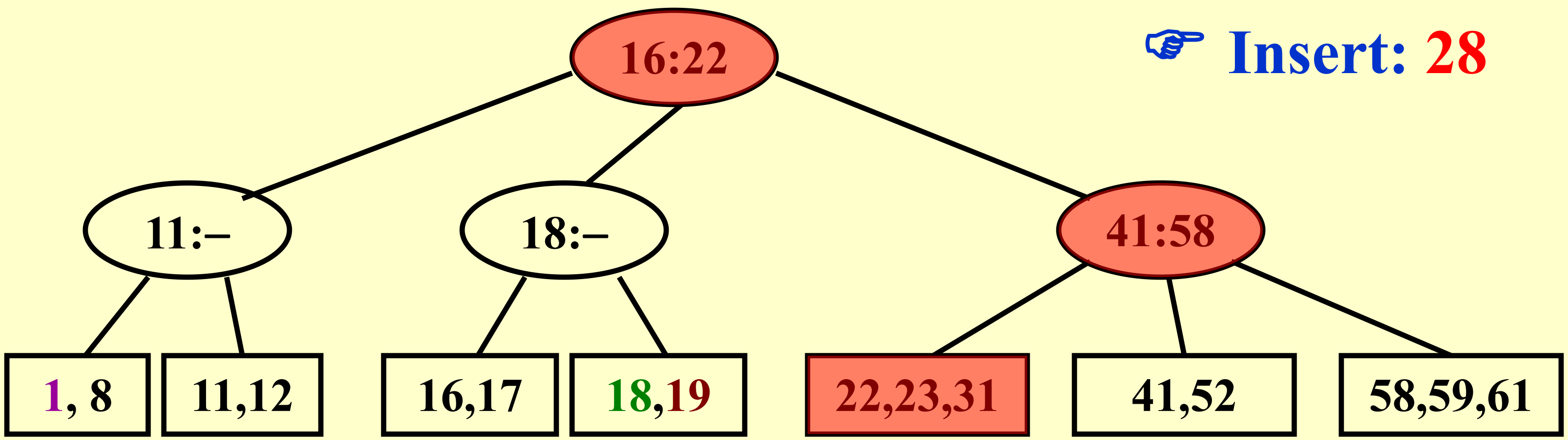


A B+ tree of order 3
(2-3 tree)

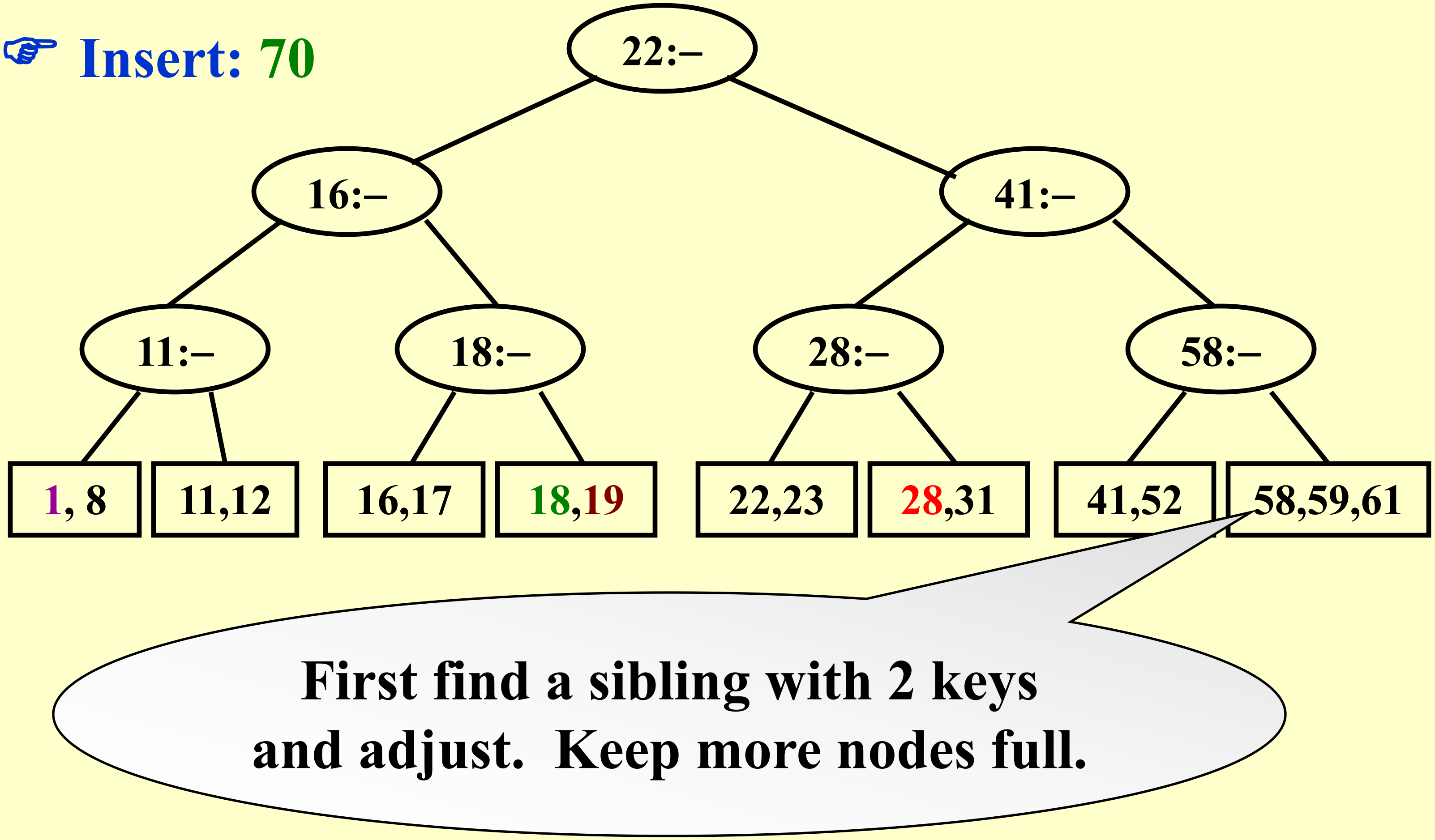


☞ Find: 52 ☞ Insert: 18 ☞ Insert: 1 ☞ Insert: 19

☞ Insert: 28



☞ Insert: 70



☞ **Deletion** is similar to insertion except that the root is removed when it loses two children.

For a general B+ tree of order **M**

```
Btree  Insert ( ElementType X,  Btree T )
{
    Search from root to leaf for X and find the proper leaf node;
    Insert X;
    while ( this node has M+1 keys ) {
        split it into 2 nodes with ⌈(M+1)/2⌉ and ⌊(M+1)/2⌋ keys, respectively;
        if (this node is the root)
            create a new root with two children;
        check its parent;
    }
}
```

$T = O(M)$

$T(M, N) = O(\,(M/\log M)\log N\,)$

Depth$(M, N) =\ O(\lceil \log_{\lceil M/2 \rceil} N \rceil)$

$T_{Find}(M, N) = O(\log N)$

Note: The best choice of M is **3** or **4**.

Reference:

Introduction to Algorithms, 3rd Edition: Ch.13，p. 308-338； Ch.18，p. 484-504； ***Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein. The MIT Press. 2009***