8.5-8.6

Inclusion-Exclusion and Its Application

Recall

- The principle of Inclusion-exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- For the union of three finite sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

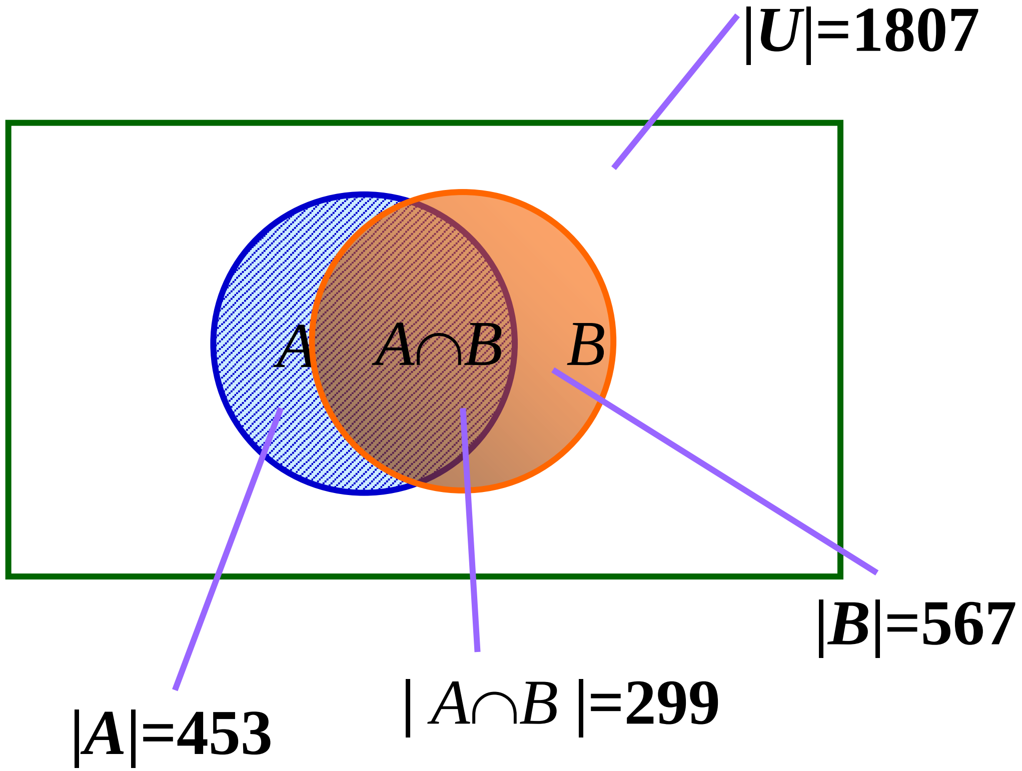
〖Example 1〗 Suppose that there are 1807 freshmen at your school. Of these, 453 are taking a course in computer science, 567 are taking a course in mathematics, and 299 are taking courses in both computer science and mathematics. How many are not taking a course either in computer science or in mathematics?

Solution:

To find the number of freshmen who are not taking a course in either mathematics or computer science, subtract the number that are taking a course in either of these subjects from the total number of freshmen.

$$|A \cup B| = |A| + |B| - |A \cap B| = 453 + 567 - 299 = 721$$

Consequently, there are 1807 − 721 = 1086 freshmen who are not taking a course in computer science or mathematics.



〖Example 2〗How many positive integers not exceeding 1000 that are not divisible by 5, 6 or 8?

Solution:

U: the set of positive integers not exceeding 1000

A: the set of positive integers not exceeding 1000 that are divisible by 5,

B: the set of positive integers not exceeding 1000 that are divisible by 6,

C: the set of positive integers not exceeding 1000 that are divisible by 8.

$$\left|\overline{A}\cap\overline{B}\cap\overline{C}\right| = |U| - |A\cup B\cup C|$$

$$= |U| - (|A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

$$= 1000 - \left(\left\lfloor\frac{1000}{5}\right\rfloor + \left\lfloor\frac{1000}{6}\right\rfloor + \left\lfloor\frac{1000}{8}\right\rfloor - \left\lfloor\frac{1000}{5\times 6}\right\rfloor - \left\lfloor\frac{1000}{6\times 4}\right\rfloor - \left\lfloor\frac{1000}{5\times 8}\right\rfloor + \left\lfloor\frac{1000}{5\times 6\times 4}\right\rfloor\right)$$

$$= 600$$

〖Example 3〗**How many permutations of the 26 letters of the English alphabet do not contain any of the strings *fish*, *rat* or *bird*?**

Solution:

***U*: the set of permutations of the 26 letters**

***A*: the set of permutations of the 26 letters containing *fish*,**

***B*: the set of permutations of the 26 letters containing *rat*,**

***C*: the set of permutations of the 26 letters containing *bird*.**

$$\left|\overline{A}\cap\overline{B}\cap\overline{C}\right| = |U| - |A\cup B\cup C|$$

$$= |U| - (|A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

$$= 26! - (23! + 24! + 23! - 21! - 0 - 0 - 0)$$

The Principle of inclusion-exclusion

- **The formula for the number of elements in the union of *n* finite sets:**

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

1. There are $2^n - 1$ terms in this formula.
2. How to prove?
An element in the union is counted exactly once by the right-hand side of the equation.

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Proof:

Suppose that *a* is an element of exactly *r* of the sets $A_1, A_2, \ldots, A_n$ **where** $1 \le r \le n$ **.**

This element is counted $C(r,1)$ **times by** $\sum_{i=1}^{n} |A_i|$ **.**

This element is counted $C(r,2)$ **times by** $\sum_{1 \le i < j \le n} |A_i \cap A_j|$ **.**

…

Thus, it is counted exactly

$$C(r,1) - C(r,2) + C(r,3) - \ldots + (-1)^{r-1} C(r,r) = 1$$

Why ? **Since** $(-1+1)^r = 0$

An alternative form of inclusion-exclusion

✓ to solve problems that ask for the number of elements in a set that have none of *n* properties.

$$P_1, P_2, \ldots, P_n$$

Let A_i be the subset containing the elements that have property P_i.

$N(P_1 P_2 \ldots P_k)$:The number of elements with all properties $P_1, P_2, \ldots, P_k$.

It follows that $N(P_1 P_2 \ldots P_k) = \left| A_1 \cap A_2 \cap \ldots \cap A_k \right|$

$N(P_1' P_2' \ldots P_n')$:The number of elements with none of the properties $P_1, P_2, \ldots, P_n$.

From the inclusion-exclusion principle, we see that

$$N(P_1' P_2' \ldots P_n') = N - \left| A_1 \cup A_2 \ldots \cup A_n \right| = N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j) + \ldots + (-1)^n N(P_1 P_2 \ldots P_n)$$

〖Example 4〗How many solutions does $x_1 + x_2 + x_3 = 13$ have, where x_i are nonnegative integers with $x_i < 6, i = 1,2,3$?

Solution:

Let a solution has property P_1 is $x_1 \geq 6$, property P_2 is $x_2 \geq 6$, property P_3 is $x_3 \geq 6$.

The number of solutions is

$$N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3)$$

$C(3-1+13,13)$

$N(P_i) = C(3-1+7,7)$

$N(P_iP_j) = C(3-1+1,1)$

$N(P_1P_2P_3) = 0$

The Sieve of Eratoshenes

〖Example 5〗Find the number of primes not exceeding a specified positive integer. Take 100 for example.

Solution:

- ✧ **A composite integer is divisible by a prime not exceeding its square root.**
  - — **Composite integer not exceeding 100 must have a prime factor not exceeding 10.**
  - — **Since the only primes less than 10 are 2,3,5,7, the primes not exceeding 100 are these four primes and the positive integers greater than 1 and not exceeding 100 that are divisible by none of 2,3,5,7.**

P_1: the property that an integer is divisible by 2

P_2: the property that an integer is divisible by 3

P_3: the property that an integer is divisible by 5

P_4: the property that an integer is divisible by 7

The number of primes not exceeding positive integer 100 is

$$4 + N(P_1'P_2'P_3'P_4')$$

$$= 4 + N - N(P_1) - N(P_2) - N(P_3) - N(P_4) + N(P_1P_2) + N(P_1P_3) + N(P_1P_4)$$
$$+ N(P_2P_3) + N(P_2P_4) + N(P_3P_4) - N(P_1P_2P_3) - N(P_1P_2P_4) - N(P_1P_3P_4) - N(P_2P_3P_4) + N(P_1P_2P_3P_4)$$

$$= 25$$

99

$\lfloor 100/2 \rfloor$

$\lfloor 100/(2\times 3) \rfloor$

$\lfloor 100/(2\times 3 \times 5) \rfloor$

$\lfloor 100/(2\times 3 \times 5 \times 7) \rfloor$

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The sieve of Eratoshenes -1

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The number of onto functions

Theorem 1: Let *m* and *n* be positive integers with $m \geq n$. Then, there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \ldots + (-1)^{n-1}C(n,n-1)\cdot 1^m$$

onto functions from a set with *m* elements to a set with *n* elements.

Proof:

$$A = \{a_1, a_2, \ldots, a_m\} \qquad B = \{b_1, b_2, \ldots, b_n\}$$

Let P_i be the property that b_i is not in the range of the function, respectively.

Note that a function is onto if and only if it has none of the properties $P_i (i = 1,2,\ldots,n)$ **.**

By the principle of inclusion-exclusion, it follows that the number of onto functions is

$$N(P_1'P_2'\ldots P_n') = \underbrace{N}_{n^m} - \underbrace{\sum_{1\le i\le n} N(P_i)}_{C(n,1)(n-1)^m} + \underbrace{\sum_{1\le i<j\le n} N(P_iP_j)}_{C(n,2)(n-2)^m} - \ldots + \underbrace{(-1)^n N(P_1P_2\ldots P_n)}_{(-1)^n N(P_1P_2\ldots P_n)=0}$$

Problem:

S(m,n) : the number of ways to distribute m distinguishable objects into *n* indistinguishable boxes so that no boxes is empty

the number of ways to partition the set with *m* elements into *n* nonempty and disjoint subsets.

S(m,n) n!: the number of onto functions from a set with *m* elements to a set with *n* elements

Application:

- Assign *m* different jobs to *n* different employees if every employee is assigned at least one job.
- Distribute *m* different toys to *n* different children such that each child gets at least one toy.

Derangements

Definition: A *derangement* is a permutation of objects that leaves no object in the original position.

Example:

The permutation of 21453 is a derangement of 12345 because no number is left in its original position. But 21543 is not a derangement of 12345, because 4 is in its original position.

Derangements

Theorem 2: The number of derangements of a set with *n* elements is

$$D_n = n![1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}]$$

Proof:

Let a permutation have property P_i if it fixes element *i*.

The number of derangements is the number of permutation having none of the properties P_i for i=1, 2, …, n, namely

$$D_n = N(P_1'P_2'\ldots P_n')$$

$$= N - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) + \ldots + (-1)^n N(P_1P_2\ldots P_n)$$

$$= n! - C(n,1)(n-1)! + C(n,2)(n-2)! - C(n,3)(n-3)! + \ldots + (-1)^n \times C(n,n)(n-n)!$$

$$= n! - \frac{n!}{1!(n-1)!}\times(n-1)! + \frac{n!}{2!(n-2)!}\times(n-2)! - \frac{n!}{3!(n-3)!}\times(n-3)! + \ldots + (-1)^n \frac{n!}{n!(n-n)!}\times(n-n)!$$

$$= n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!})$$

Homework:

SE: P. 557 7, 12

P. 564 6, 11, 16

EE: P. 584 7, 14

P. 591 6, 11, 16