1.4 Predicates and Quantifiers

Section Summary

- Predicates
- Variables
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Negating Quantifiers
  - De Morgan's Laws for Quantifiers
- Translating English to Logic
- Logic Programming

Propositional Logic Not Enough

If we have:

“All men are mortal.”

“Socrates is a man.”

Does it follow that **“Socrates is mortal?”**

- Can’t be represented in propositional logic.
- **Need a language that talks about objects, their properties, and their relations.**

Introducing Predicate Logic

- Predicate logic uses the following new features:
  - Variables: x, y, z
  - Predicates: P, M
  - Quantifiers

Propositional Functions

Consider the statement "*x* > 0".

This can be represent by propositional function *P*(*x*),where *P* represents the property "*is greater than* 0" and *x* is the variable.

- **Propositional functions are a generalization of propositions.**
  - They contain variables and a predicate, e.g., $P(x)$
  - Variables can be replaced by elements from their *domain*.

Propositional Functions

- **Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the domain or bound by a quantifier** (as we will see later)**.**
- **The statement *P(x)* is said to be the value of the propositional function *P* at *x*.**
  - For example, let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:

    P(-3) is false.

    P(0) is false.

    P(3) is true.
  - Often the domain is denoted by U. So in this example U is the integers.

Examples of Propositional Functions

Let "$x + y = z$" be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

R(2,-1,5) **Solution: F**

R(3,4,7) **Solution: T**

R(x, 3, z) **Solution: Not a Proposition**

In general, a statement involving the n variables $x_1, x_2, \ldots, x_n$ can be denoted by $P(x_1, x_2, \ldots, x_n)$.

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \ldots, x_n)$ and P is called a n-ary predicate.

Compound Expressions

- **Connectives from propositional logic carry over to predicate logic.**
  - If $P(x)$ denotes "$x > 0$," find these truth values:

    $P(3) \vee P(-1)$

    $P(3) \wedge P(-1)$

    $P(3) \rightarrow P(-1)$

    $P(-1) \rightarrow P(3)$

- **Expressions with variables are not propositions and therefore do not have truth values.**
  - For example,

    $P(3) \wedge P(y)$

    $P(x) \rightarrow P(y)$

When used with quantifiers (to be introduced next), these expressions become propositions.

Preconditions and postconditions

- **Predicates are also used to establish the correctness of computer programs.**
  - ✓ **preconditions：the statements that describe valid input**
  - ✓ **postconditions：the conditions that the output should satisfy when the program has run**

Example，

Consider the following program, designed to interchange the values of two variables *x* and *y*.

temp := *x*

x* := *y

***y* := temp**

Find predicates that we can use as the precondition and the postcondition to verify the correctness of this program. Then explain how to use them to verify that for all valid input the program does what is intended.

Preconditions and postconditions

Example,

Consider the following program, designed to interchange the values of two variables x and y.

$$\begin{aligned}&\text{temp} := x\\&x := y\\&y := \text{temp}\end{aligned}$$

Find predicates that we can use as the precondition and the postcondition to verify the correctness of this program. Then explain how to use them to verify that for all valid input the program does what is intended.

***Solution*:**

the precondition: $P(x, y)$, where $P(x, y)$ is the statement "$x = a$ and $y = b$," where a and b are the values of x and y before we run the program.

the postcondition: $Q(x, y)$, where $Q(x, y)$ is the statement "$x = b$ and $y = a$."

To verify that the program always does what it is supposed to do.

suppose that the precondition $P(x, y)$ holds. After this program is run, the postcondition $Q(x, y)$ holds.

Quantifiers

- We need ***quantifiers*** to express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- **The two most important quantifiers are:**
  Universal Quantifier, "For all," symbol: $\forall$
  Existential Quantifier, "There exists," symbol: $\exists$
- We write as in $\forall x\, P(x)$ (*the universal quantification of P(x)*)and $\exists x\, P(x)$.
  $\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.
  $\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.
- The quantifiers are said to bind the variable *x* in these expressions.

Universal Quantifier

- $\forall x\, P(x)$ is read as "For all x, $P(x)$" or "For every x, $P(x)$"

Examples:

1) If $P(x)$ denotes "$x > 0$" and U is the integers, then $\forall x\, P(x)$ is false.
2) If $P(x)$ denotes "$x > 0$" and U is the positive integers, then $\forall x\, P(x)$ is true.
3) If $P(x)$ denotes "x is even" and U is the integers, then $\forall x\, P(x)$ is false.

The universal quantification of $P(x)$ create a proposition from a propositional function.

Existential Quantifier

- $\exists x\, P(x)$ is read as "For some x, $P(x)$", or as "There is an x such that $P(x)$," or "For at least one x, $P(x)$."

Examples:

1. If $P(x)$ denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Thinking about Quantifiers

- **When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.**
  - To evaluate $\forall x\, P(x)$ loop through all x in the domain.
    - If at every step P(x) is true, then $\forall x\, P(x)$ is true.
    - If at a step P(x) is false, then $\forall x\, P(x)$ is false and the loop terminates.
  - To evaluate $\exists x\, P(x)$ loop through all x in the domain.
    - If at some step, P(x) is true, then $\exists x\, P(x)$ is true and the loop terminates.
    - If the loop ends without finding an x for which P(x) is true, then $\exists x\, P(x)$ is false.
  - Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.

Statement	When true?	When false?
$\forall x\, P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.

- If U consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Uniqueness Quantifier

- **$\exists! x\ P(x)$ means that $P(x)$ is true for one and only one x in the universe of discourse.**

- This is commonly expressed in English in the following equivalent ways:
  - "There is a unique x such that $P(x)$."
  - "There is one and only one x such that $P(x)$"
- Examples:
  1. If $P(x)$ denotes "$x + 1 = 0$" and U is the integers, then $\exists! x\ P(x)$ is true.
  2. But if $P(x)$ denotes "$x > 0$," then $\exists! x\ P(x)$ is false.
- The uniqueness quantifier is not really needed as the restriction that there is a unique x such that $P(x)$ can be expressed as:

$$\exists x\ (P(x) \land \forall y\ (P(y) \rightarrow y = x))$$

Precedence of Quantifiers

◆ **The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.**

For example,

- $\forall x\, P(x) \lor Q(x)$ means $(\forall x\, P(x)) \lor Q(x)$
- $\forall x\, (P(x) \lor Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \lor Q(x)$ when they mean $\forall x\, (P(x) \lor Q(x))$.

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic:

"Every student in this class has taken a course in Java."

Solution:

First decide on the domain U.

Solution 1: If U is all students in this class, define a propositional function $J(x)$ denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Solution 2: But if U is all people, also define a propositional function $S(x)$ denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \land J(x))$ **is not correct. What does it mean?**

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic:

"Some student in this class has taken a course in Java."

Solution:

First decide on the domain U.

Solution 1: If U is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 2: But if U is all people, then translate as

$$\exists x\, (S(x) \land J(x))$$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Returning to the Socrates Example

- Introduce the propositional functions *Man(x)* denoting "*x* is a man" and *Mortal(x)* denoting "*x* is mortal."
- The two premises are: $\forall x(Man(x) \rightarrow Mortal(x))$

  $Man(Socrates)$
- The conclusion is: $Mortal(Socrates)$

Later we will show how to prove that the conclusion follows from the premises.

Equivalences in Predicate Logic

- **Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter**
  - ✓ which predicates are substituted into these statements and
  - ✓ which domain of discourse is used for the variables in these propositional functions

- The notation $S \equiv T$ indicates that S and T are logically equivalent.

  Example: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Negating Quantified Expressions

- Consider $\forall x\ J(x)$:

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in calculus" and the domain is students in your class.

- Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken calculus."

Symbolically $\neg\forall x\ J(x)$ and $\exists x\ \neg J(x)$ are equivalent.

Negating Quantified Expressions (cont)

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

Symbolically $\neg\exists x J(x)$ **and** $\forall x \neg J(x)$ **are equivalent.**

De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

These are important. You will use these.

More Logical Equivalences

$$\forall x(A(x) \wedge B(x)) \equiv \forall x A(x) \wedge \forall x B(x)$$
$$\exists x(A(x) \vee B(x)) \equiv \exists x A(x) \vee \exists x B(x)$$

Note:

$$\forall x(A(x) \vee B(x)) \not\Leftrightarrow \forall x A(x) \vee \forall x B(x)$$
$$\exists x(A(x) \wedge B(x)) \not\Leftrightarrow \exists x A(x) \wedge \exists x B(x)$$

For example,

***U*：the set of real numbers，*Q*(*x*)：*x* is a rational number，*F*(*x*)：*x* is an irrational number**

More Logical Equivalences

$$\forall x(A(x) \land B(x)) \equiv \forall xA(x) \land \forall xB(x)$$
$$\exists x(A(x) \lor B(x)) \equiv \exists xA(x) \lor \exists xB(x)$$

Note:

$$\forall x(A(x) \lor B(x)) \not\Leftrightarrow \forall xA(x) \lor \forall xB(x)$$
$$\exists x(A(x) \land B(x)) \not\Leftrightarrow \exists xA(x) \land \exists xB(x)$$

$$\exists x(A(x) \land B(x)) \Rightarrow \exists xA(x) \land \exists xB(x)$$
$$\forall xA(x) \lor \forall xB(x) \Rightarrow \forall x(A(x) \lor B(x))$$

More Logical Equivalences

x is not occurring in P.

$$
\begin{array}{llll}
(1) & \forall x A(x) \vee P & \equiv & \forall x (A(x) \vee P) \\
(2) & \forall x A(x) \wedge P & \equiv & \forall x (A(x) \wedge P) \\
(3) & \exists x A(x) \vee P & \equiv & \exists x (A(x) \vee P) \\
(4) & \exists x A(x) \wedge P & \equiv & \exists x (A(x) \wedge P)
\end{array}
$$

More Logical Equivalences

x is not occurring in P and B.

(1)	$\forall x A(x) \vee P$	$\equiv$	$\forall x (A(x) \vee P)$
(2)	$\forall x A(x) \wedge P$	$\equiv$	$\forall x (A(x) \wedge P)$
(3)	$\exists x A(x) \vee P$	$\equiv$	$\exists x (A(x) \vee P)$
(4)	$\exists x A(x) \wedge P$	$\equiv$	$\exists x (A(x) \wedge P)$
(5)	$\forall x (B \to A(x))$	$\equiv$	$B \to \forall x A(x)$
(6)	$\exists x (B \to A(x))$	$\equiv$	$B \to \exists x A(x)$
(7)	$\forall x (A(x) \to B)$	$\equiv$	$\exists x A(x) \to B$
(8)	$\exists x (A(x) \to B)$	$\equiv$	$\forall x A(x) \to B$

More Logical Equivalences

x is not occurring in P and B.

(1) $\forall x A(x) \vee P \equiv \forall x (A(x) \vee P)$

(2) $\forall x A(x) \wedge P \equiv \forall x (A(x) \wedge P)$

(3) $\exists x A(x) \vee P \equiv \exists x (A(x) \vee P)$

(4) $\exists x A(x) \wedge P \equiv \exists x (A(x) \wedge P)$

(5) $\forall x (B \rightarrow A(x)) \equiv B \rightarrow \forall x A(x)$

Proof:

$$\begin{aligned}\forall x (B \rightarrow A(x)) &\equiv \forall x (\neg B \vee A(x)) \\ &\equiv \neg B \vee \forall x A(x) \\ &\equiv B \rightarrow \forall x A(x)\end{aligned}$$

Translation from English to Logic

Examples:

① "Some student in this class has visited Mexico."

Solution:

Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

$$\exists x\ (S(x) \land M(x))$$

② "Every student in this class has visited Canada or Mexico."

Solution: Add $C(x)$ denoting "x has visited Canada."

$$\forall x\ (S(x) \rightarrow (M(x) \lor C(x)))$$

Some Fun with Translating from English into Logical Expressions

- U = {lions, mammals(哺乳动物), carnivorous animals(肉食动物)}
  - $L(x)$: x is a lion
  - $M(x)$: x is a mammal
  - $C(x)$: x is a carnivorous animal

Translate "Everything is a lion"

Solution: $\forall x\, L(x)$

Translation (cont)

- U = {lions, mammals(哺乳动物), carnivorous animals(肉食动物)}

  $L(x)$: x is a lion

  $M(x)$: x is a mammal

  $C(x)$: x is a carnivorous animal

“Nothing is a mammal.”

Solution: $\neg\exists x\, M(x)$

What is this equivalent to?

Solution: $\forall x\, \neg M(x)$

Translation (cont)

- U = {lions, mammals(哺乳动物), carnivorous animals(肉食动物)}

  $L(x)$: x is a lion

  $M(x)$: x is a mammal

  $C(x)$: x is a carnivorous animal

“All lions are mammals.”

Solution: $\forall x\, (L(x) \rightarrow M(x))$

Translation (cont)

- U = {lions, mammals(哺乳动物), carnivorous animals(肉食动物)}

  $L(x)$: x is a lion

  $M(x)$: x is a mammal

  $C(x)$: x is a carnivorous animal

“Some mammals are carnivorous animals.”

Solution: $\boldsymbol{\exists x\,(M(x) \land C(x))}$

Translation (cont)

- U = {lions, mammals(哺乳动物), carnivorous animals(肉食动物)}

  $L(x)$: x is a lion

  $M(x)$: x is a mammal

  $C(x)$: x is a carnivorous animal

"No mammal is a carnivorous animal."

Solution: $\neg\exists \boldsymbol{x}\ (\boldsymbol{M}(\boldsymbol{x}) \land \boldsymbol{C}(\boldsymbol{x}))$

What is this equivalent to?

Solution: $\forall \boldsymbol{x}\ (\neg\boldsymbol{M}(\boldsymbol{x}) \lor \neg\boldsymbol{C}(\boldsymbol{x}))$

$$\equiv \forall \boldsymbol{x}\ (\boldsymbol{M}(\boldsymbol{x}) \rightarrow \neg\boldsymbol{C}(\boldsymbol{x}))$$

Translation (cont)

- U = {lions, mammals(哺乳动物), carnivorous animals(肉食动物)}

  L(x): *x* is a lion

  M(x): *x* is a mammal

  C(x): x is a carnivorous animal

"If any lion is a mammal then it is also a carnivorous animal."

Solution: $\forall x\ ((L(x) \land M(x)) \rightarrow C(x))$

System Specification Example

- Predicate logic can be used for representing system specification also.
- For example, translate into predicate logic:

“Every mail message larger than one megabyte will be compressed.”

- Decide on predicates and domains for the variables:
  - ➢ Let $L(m, y)$ be “Mail message m is larger than y megabytes.”, where the variable x has the domain of all mail messages and the variable y is a positive real number
  - ➢ Let $C(m)$ denote “Mail message m will be compressed.”
- Now we have:

$$\forall m(L(m,1) \rightarrow C(m))$$

System Specification Example

- Another example,

  "If a user is active, at least one network link will be available."

- Decide on predicates and domains for the variables:
  - Let $A(u)$ represent "User u is active." where the variable u has the domain of all users
  - Let $S(n, x)$ represent "Network link n is state x." where n has the domain of all network links and x has the domain of all possible states for a network link.

- Now we have:

$$\exists u\, A(u) \rightarrow \exists n\, S(n, available)$$

Lewis Carroll Example

Charles Lutwidge Dodgson
(AKA Lewis Caroll)
(1832-1898)

An argument

1. "All lions are fierce."
2. "Some lions do not drink coffee."
3. "Some fierce creatures do not drink coffee."

The first two are called *premises* and the third is called the *conclusion*.

One way to translate these statements to predicate logic：

Let $p(\boldsymbol{x})$, $\boldsymbol{q}(\boldsymbol{x})$, and $\boldsymbol{r}(\boldsymbol{x})$ be the propositional functions "$\boldsymbol{x}$ is a lion," "$\boldsymbol{x}$ is fierce," and "$\boldsymbol{x}$ drinks coffee," respectively. Domain of $\boldsymbol{x}$: All creatures.

1. $\forall x\ (p(x) \rightarrow q(x))$
2. $\exists x\ (p(x) \land \neg r(x))$
3. $\exists x\ (q(x) \land \neg r(x))$

Later we will see how to prove that the conclusion follows from the premises.

Homework:

SE： P.53 16, 24, 34, 51, 62

EE： P.57 16, 24, 34, 53, 64