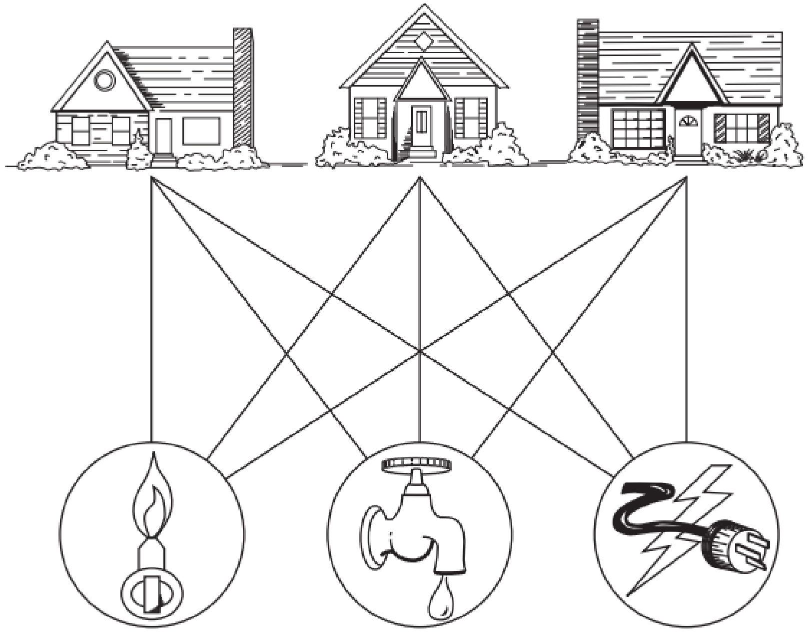
10.7 Planar Graphs

Why Planar Graph?

Is it possible to join these houses and utilities so that none of the connections cross?

Modeled using K3,3

Can K3,3 be drawn in the plane so that no two of its edges cross?



Three houses and three utilities

Application of Planar Graph

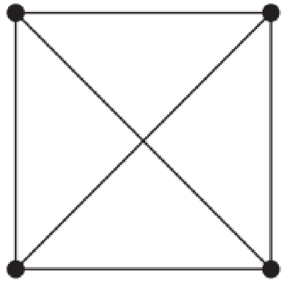
- Planarity of graphs plays an important role in the following domains:
  - ✓ The design of electronic circuits
  - ✓ The design of road networks

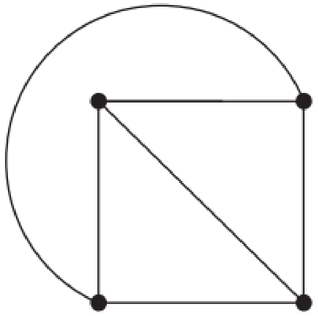
Definition of Planar Graph

【Definition】 A graph is called **planar** if it can be drawn in the plane without any edges crossing .

Such a drawing is called a **planar representation** of the graph.

Is K_4 planar?

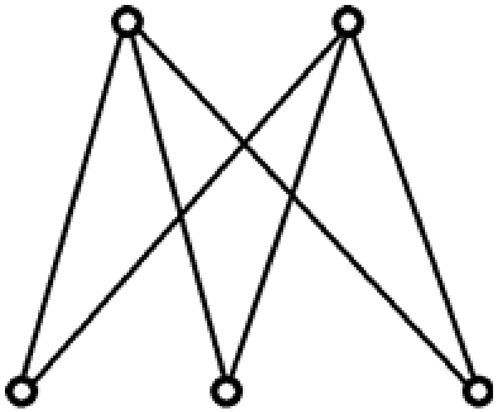


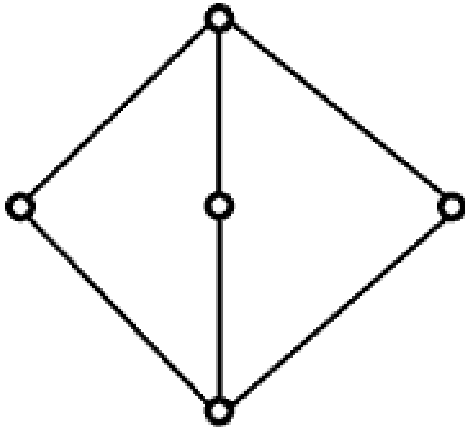


Note:

- A graph may be planar even if it is usually drawn with crossings.
- We can prove that a graph is planar by displaying a planar representation.

〖Example 1〗 Is $K_{2,3}$ planar?



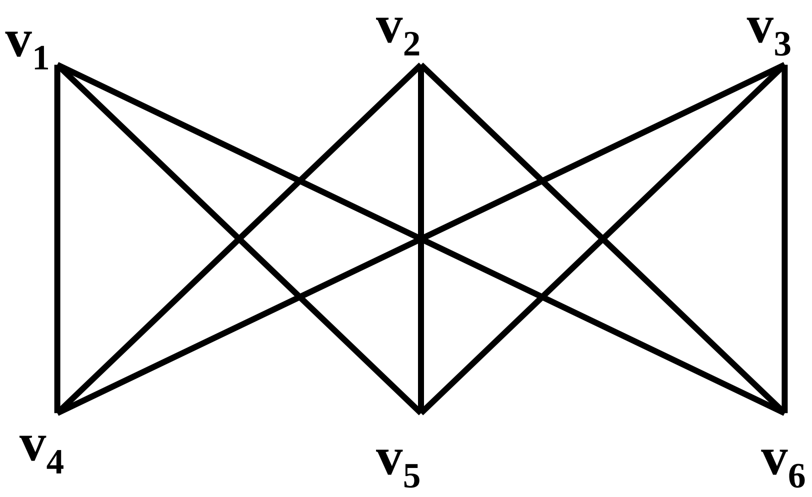


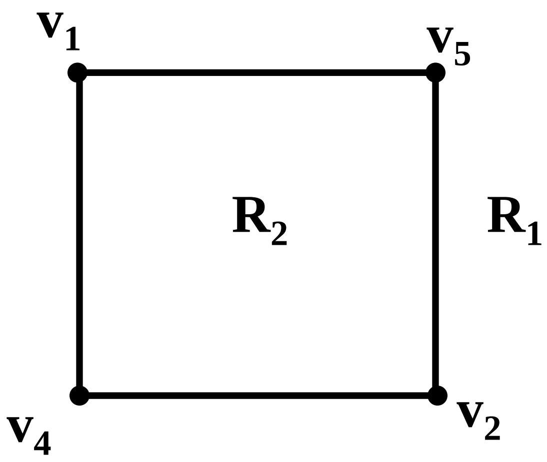
$K_{2,3}$ is planar

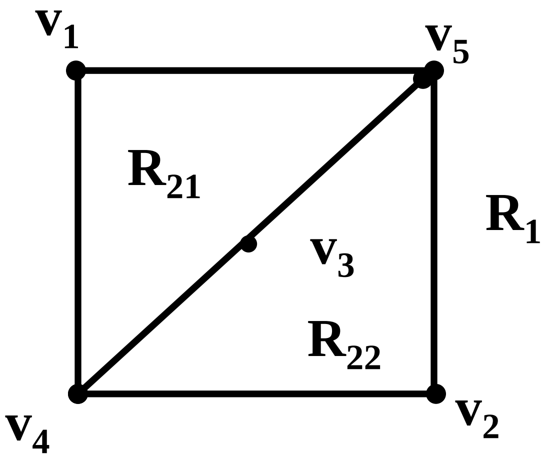
Note:

- Complete bipartite graphs $K_{2,n}(n \geq 1)$ are planar.
- Complete bipartite graphs $K_{1,n}$ are planar.

〖Example 2〗 Is $K_{3,3}$ planar?







$K_{3,3}$ is not planar

Euler's Formula

Some terminologies:

- **Region:** a part of the plane completely disconnected off from other parts of the plane by the edges of the graph.
  - Bounded region
  - Unbounded region

  Note: There is one unbounded region in a planar graph.

- **the boundary of region**
- **the Degree of Region R (Deg(*R*)):** the number of the edges which surround R, suppose *R* is a region of a connected planar simple graph
- **adjacent regions**: two regions with a common border
- If e is not a cut edge, then it must be the **common border** of two regions

〖Example 3〗 There are 4 regions in the right graph.

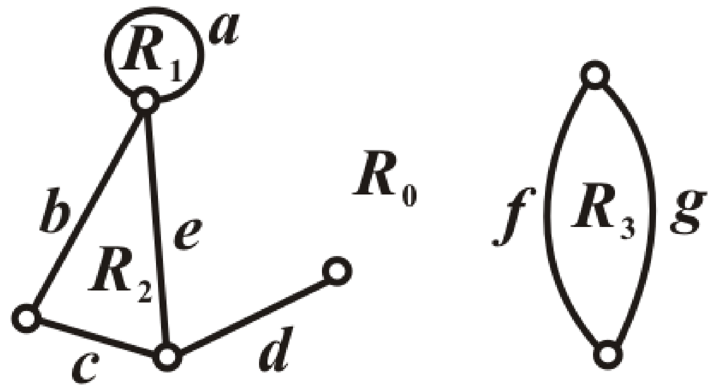
the boundary of region

R_1: a

R_2: bce

R_3: fg

R_0: $abcdde$, fg



$\deg(R_1) =$ 1

$\deg(R_2) =$ 3

$\deg(R_3) =$ 2

$\deg(R_0) =$ 8

〖Example 4〗 **The following graph is a planar representation of a graph.**

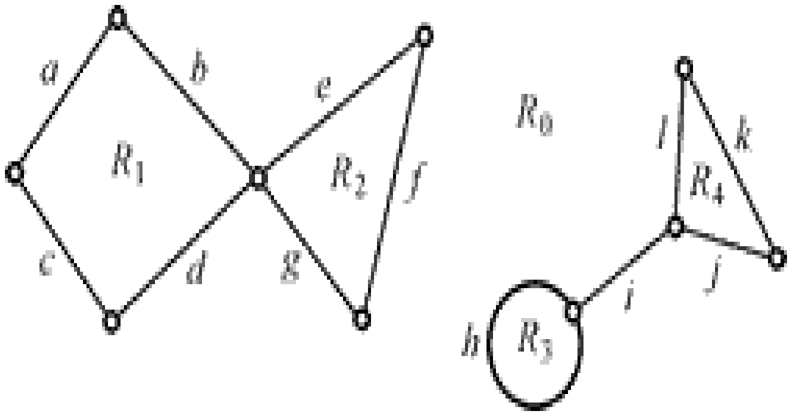
◆**There are 5 regions .**

◆**The boundaries of regions R_1, R_2, R_3 and R_4 are *abdc*, *efg*, *h*, *kjl*.**

deg(R_1)=4, deg(R_2)=3,

deg(R_3)=1, deg(R_4)=3

◆**The boundary of unbounded region R_0 is constructed by *abefgdc* and *kjihil*, deg(R_0)＝13。**



Note: The sum of the degrees of the regions is exactly twice the number of edges in the planar graph.

$$2e = \sum_{all\ \ region\ \ R} \deg(R)$$

【Theorem 1】 Euler's formula

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r=e-v+2$.

Proof:

First, we specify a planar representation of G. We will prove the theorem by constructing a sequence of subgraphs $G_1, G_2, \cdots, G_e = G$, successively adding an edge at each stage.

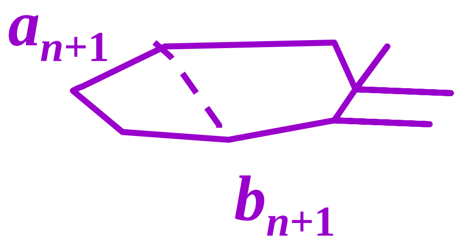
The constructing method: Arbitrarily pick one edge of G to obtain G_1. Obtain G_n from G_{n-1} by arbitrarily adding an edge that is, incident with a vertex already in G_{n-1}.

Let r_n, e_n, and v_n represent the number of regions, edges, and vertices of the planar representation of G_n induced by the planar representation of G, respectively.

(1) The relationship $r_1 = e_1 - v_1 + 2$ is true for G_1, since $e_1 = 1$, $v_1 = 2$, and $r_1 = 1$.

(2) Now assume that $r_n = e_n - v_n + 2$. Let $\{a_{n+1}, b_{n+1}\}$ be the edge that is added to G_n to obtain G_{n+1}.

◆ Both a_{n+1} and b_{n+1} are already in G_n.



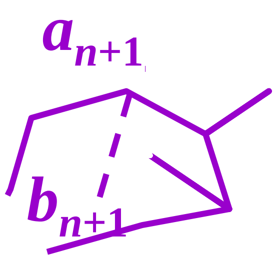
These two vertices must be on the boundary of a common region R, or else it would be impossible to add the edge $\{a_{n+1}, b_{n+1}\}$ to G_n without two edges crossing (and G_{n+1} is planar).

The addition of this new edge splits R into two regions.

Consequently, $r_{n+1} = r_n + 1$, $e_{n+1} = e_n + 1$, and $v_{n+1} = v_n$. Thus, $r_{n+1} = e_{n+1} - v_{n+1} + 2$.

- **One of the two vertices of the new edge is not already in G_n .**

Suppose that a_{n+1} is in G_n but that b_{n+1} is not.



Adding this new edge does not produce any new regions, since b_{n+1} must be in a region that has a_{n+1} on its boundary.

Consequently, $r_{n+1}= r_n$. Moreover, $e_{n+1}= e_n +1$ and $v_{n+1}= v_n +1$.

Hence, $r_{n+1}= e_{n+1}-1- v_{n+1}+1 +2$.

Note:

1) The Euler's formula is **necessary condition**.

2) How about unconnected simple planar graph?

Suppose that a planar graph G has k connected components, e edges, and v vertices. Let r be the number of regions in a planar representation of G.

Then $r=e-v+k+1$.

【Corollary 1】 If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v-6$

Proof:

Suppose that a connected planar simple graph divides the plane into r regions, the degree of each region is at least 3.

Since $2e = \sum \deg(R_i) \geq 3r$, it imply $r \leq (2/3)e$

Using Euler's formula $e-v+2=r$, we obtain

$e-v+2 \leq (2/3)e$, this shows that $e \leq 3v-6$.

Note:

- For unconnected planar simple graph, $e \leq 3v-6$ is also holds.

Since for a component, $e_i \leq 3v_i - 6$

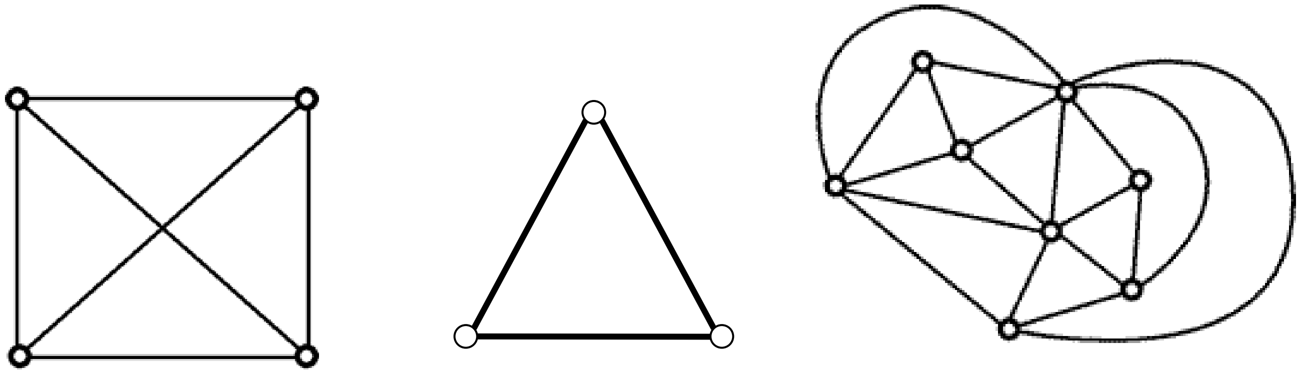
$$e = \sum e_i \leq \sum (3v_i - 6) < 3\sum v_i - 6 = 3v - 6$$

【 Corollary 2 】 If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3, then $e \leq 2v-4$.

Note:

- **Generally, if every region of a planar connected graph has at least k edges, then** $$e \leq \frac{(v-2)k}{k-2}$$

- A connected planar simple graph with $e=3v-6$?


(a) (b) (c)

【 Corollary 3 】 If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

Proof:

(1) G has one or two vertices

(2) G has at least three vertices

By Corollary 1 , we know that $e \le 3v-6$, so $2e \le 6v-12$

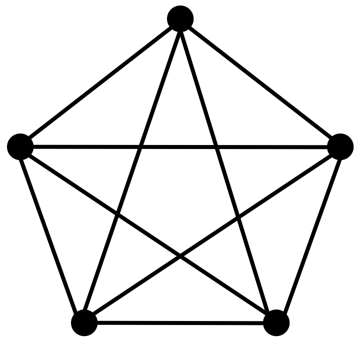
If the degree of every vertex were at least six, then

$2e \ge 6v$

〖Example 5〗 Show that k_5, $k_{3,3}$ are nonplanar.

Proof:

(1)

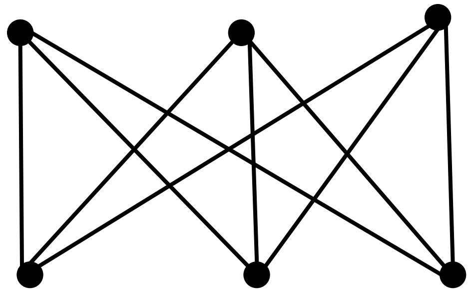


The graph k_5 has 5 vertices and 10 edges.

However, the inequality $e \leqq 3v-6$ is not satisfied for this graph since $e=10$ and $3v-6=9$.

Therefore, k_5 is not planar.

(2)



$K_{3,3}$ has 6 vertices and 9 edges.

Since $K_{3,3}$ has no circuits of length 3 (this is easy to see since it is bipartite), Corollary 3 can be used .

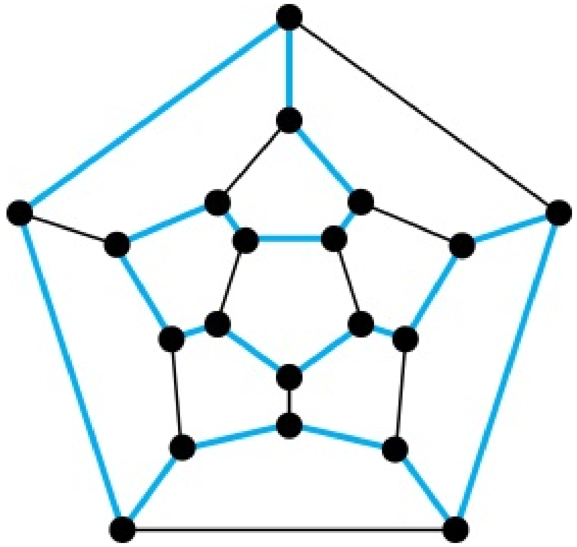
Since $e=9$ and $2v-4=8$, corollary 3 shows that $k_{3,3}$ is nonplanar.

〖Example 6〗 If G is a planar simple graph with vertices not exceeding 11, then G must exist vertices of degrees less than five.

〖Example 7〗 $K_n (n \geq 7)$ is not planar.

〖Example 8〗 The construction of Dodecahedron .

Solution:



Since the degree of every vertex is 3 and the degree of every region is 5. Then

$$2e = 3v$$

$$2e = 5r$$

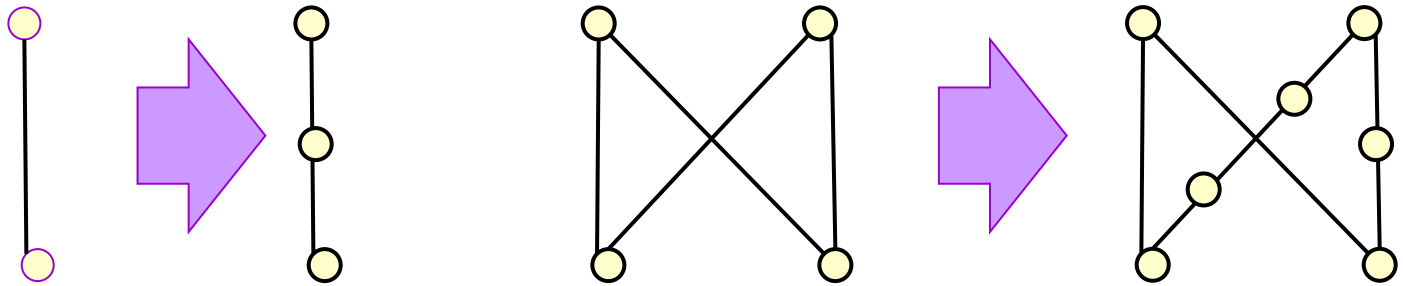
$$r = e - v + 2$$

It follows that v=20, e=30 and r=12.

KURATOWSKI'S THEOREM

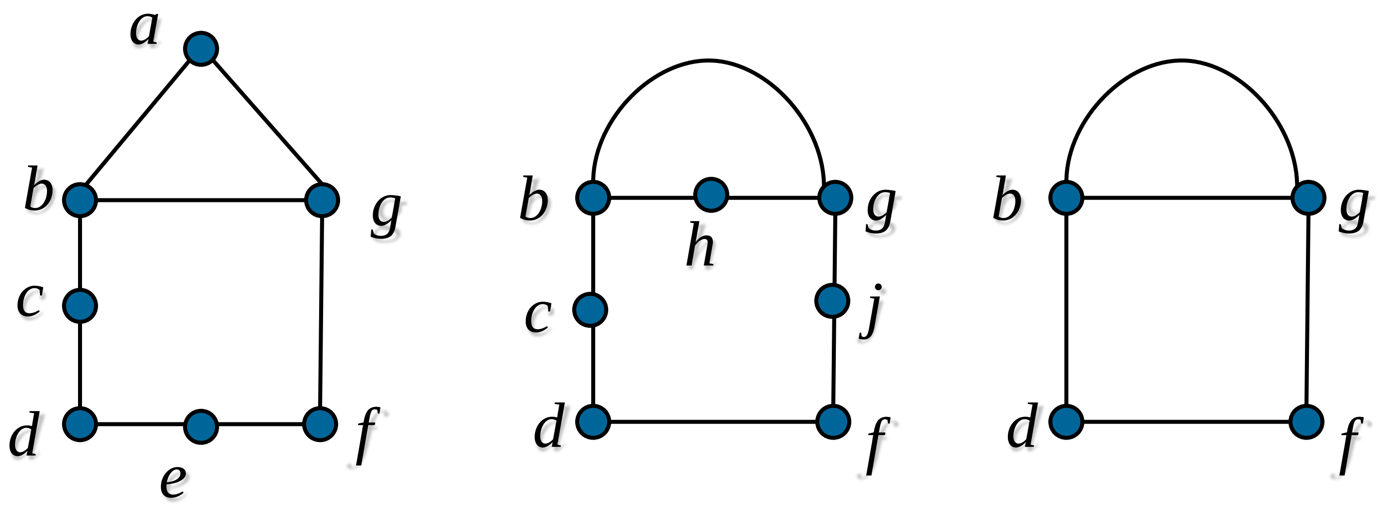
Terminologies:

- **Elementary subdivision:** If a graph is planar, so will be any graph obtained by removing an edge {u, v} and adding a new vertex w together with edges {u,w} and {w,v}.

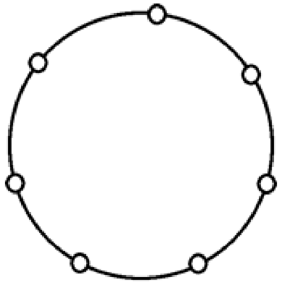


- **Homeomorphic:** the graph $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivision.

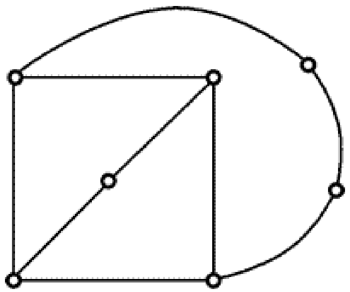
Examples of homeomorphic graphs



These three graphs are homeomorphic



(a)



(b)

(a) is homeomorphic to K_3

(b) is homeomorphic to K_4

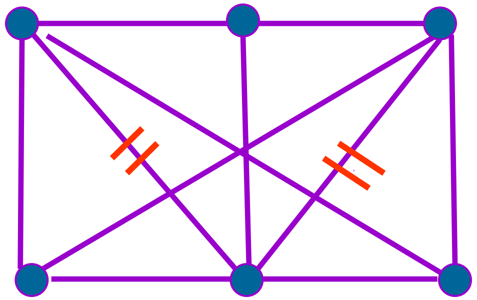
【Theorem 2】 A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

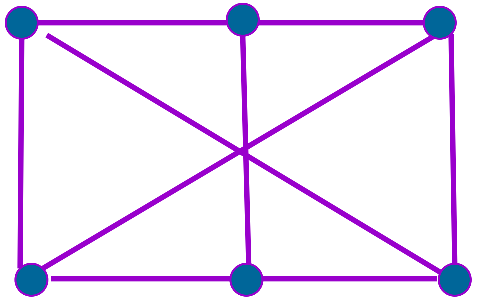
Proof:

- ✓ **It is clear that a graph containing a subgraph homeomorphic to $K_{3,3}$ or K_5 is nonplanar.**
- ✓ **Every nonplanar graph contains a subgraph homeomorphic to $K_{3,3}$ or K_5**

〖Example 9〗 **Determine whether the following graphs are planar.**

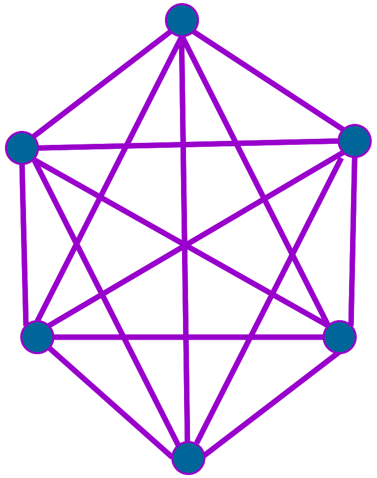
(1)





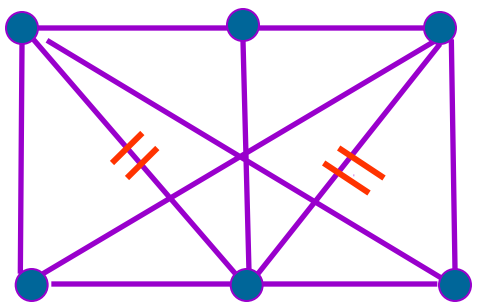
This graph is not planar.

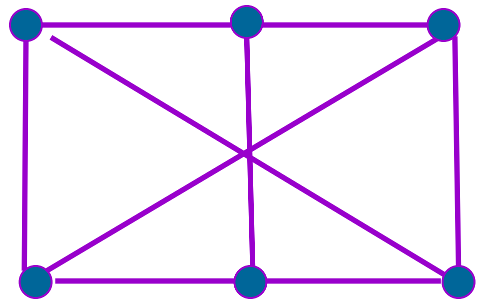
(2)



〖Example 9〗 **Determine whether the following graphs are planar.**

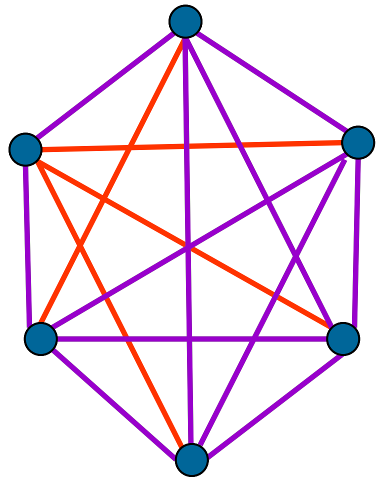
(1)

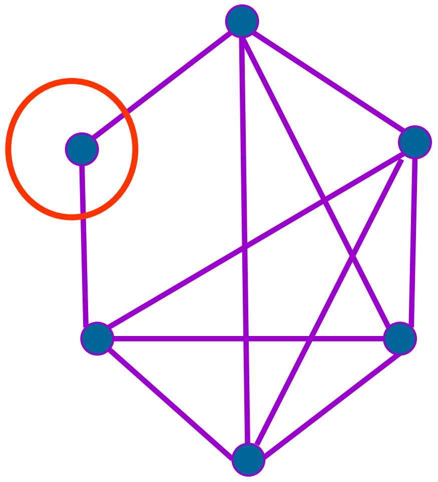




This graph is not planar.

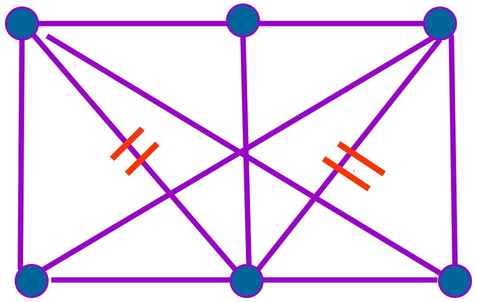
(2)

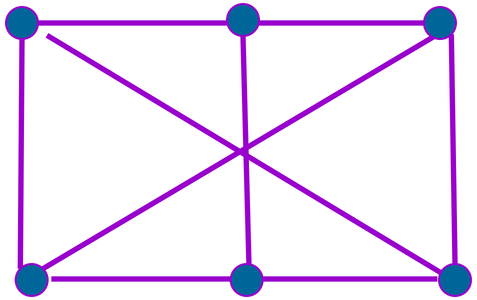




〖Example 9〗 **Determine whether the following graphs are planar.**

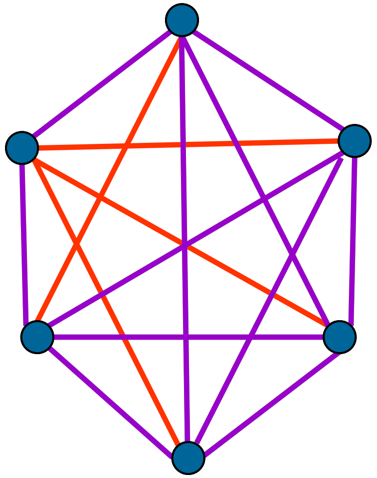
(1)

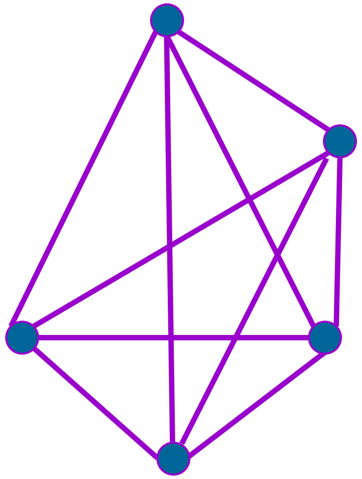




This graph is not planar.

(2)

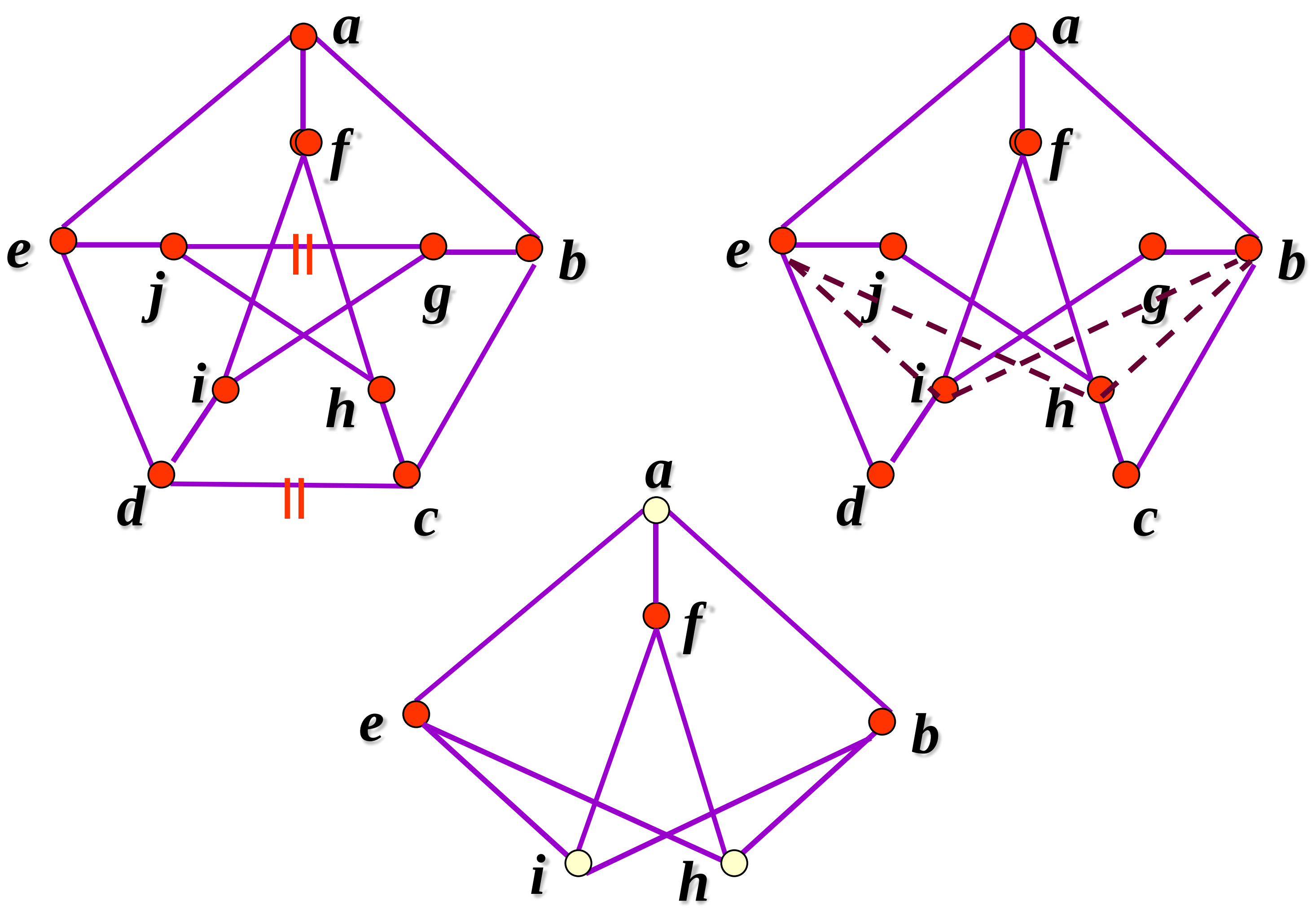




K_6 is not planar.

〖Example 9〗 **Determine whether the following graphs are planar.**

(3)



The Petersen graph is not planar.

Homework:

SE：P. 725 1, 7, 20, 22, 23, 25

EE：P. 760 1, 7, 20, 22, 23, 25