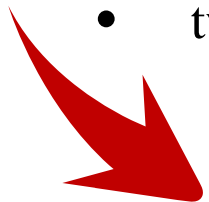


2.5 Cardinality of Sets

Introduction

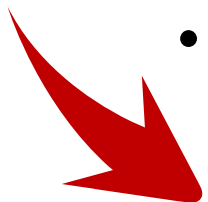
The cardinality of a finite set

- two sets have the same size or when one is bigger than the other



Extend to infinite set

- A way to measure the relative sizes of infinite sets
- Countable infinite sets and uncountable infinite sets



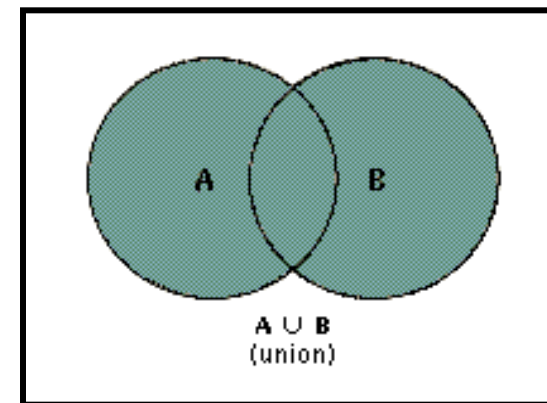
Applications

- Solve some interesting problems
- Computer science: explain why uncomputable functions exist

Review the cardinality of a finite set

- The cardinality of a finite set was defined to be the number of **distinct elements** in the set.
 - This definition tell us when two sets have the same size or when one is bigger than the other
- The cardinality of the union of two finite sets:
 - **The principle of Inclusion-exclusion**

$$|A \cup B| = |A| + |B| - |A \cap B|$$



The union of more finite sets?

For the union of three finite sets:

$$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |P \cap R| - |Q \cap R| + |P \cap Q \cap R|$$

Proof:

$$\text{Let } Q \cup R = S$$

$$\begin{aligned} |P \cup S| &= |P| + |S| - |P \cap S| \\ &= |P| + |Q \cup R| - |P \cap (Q \cup R)| \\ &= |P| + |Q \cup R| - |(P \cap Q) \cup (P \cap R)| \end{aligned}$$

For the union of n finite sets:

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{i=1}^n |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| + \dots + (-1)^{n-1} |A_1 \cap A_2 \cap \dots \cap A_n|$$

Extend to infinite sets

Some questions:

- 1) How to determine the cardinality of an infinite set?
- 2) Is the number of positive integers double the number of positive even integers?
- 3) Do the set of rational numbers and the set of real numbers in $(0,1)$ have same cardinality?
- 4) Do the set of real numbers in (a, b) and the set of real numbers in $(0,1)$ have same cardinality?
- 5) Are all of the cardinalities of infinite sets same?

Cardinality

Definition: The sets A and B **have the same cardinality** (denoted by $|A| = |B|$) iff there exists a one-to-one correspondence (bijection) from A to B .

Note: This provides a relative measure of the sizes of two sets, rather than a measure of the size of one particular set.

[[**Example 1**]] Let A be the set of 26 lowercase English alphabets. $B = \{1, 2, \dots, 26\}$. Then $|A| = |B|$.

[[**Example 2**]] The set of natural numbers is denoted by $N = \{1, 2, \dots, n, \dots\}$, $M = \{1, 2^2, \dots, n^2, \dots\}$. Then $|N| = |M|$.

【Example 3】 Let A be the set of real numbers between a and b ($a < b$), and B be the set of real numbers between 0 and 1. Show that $|A| = |B|$.

Proof:

Let f be a function from A to B .

$$\frac{x - a}{b - a} = \frac{y - 0}{1 - 0}$$

$$y = f(x) = \frac{x - a}{b - a}$$

Then y is a bijection from (a, b) to $(0, 1)$.

Hence, $|A| = |B|$.

Cardinality

Definition: If there is a one-to-one function from A to B , the cardinality of A is less than or the same as cardinality of B ($|A| \leq |B|$).

When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.

Now we analysis infinite set...

Countable Sets

Definition: A set that is either **finite** or **has the same cardinality as the set of positive integers** is called **countable**.

A set that is not countable is called **uncountable**.

- The set of real numbers \mathbb{R} is an uncountable set.

When an infinite set is countable (**countably infinite**) its cardinality is \aleph_0 (where \aleph is aleph, the 1st letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality “aleph null.”

How to show that a set is countable?

- ◆ An infinite set is countable if and only if it is possible to **list the elements of the set in a sequence** (indexed by the positive integers).

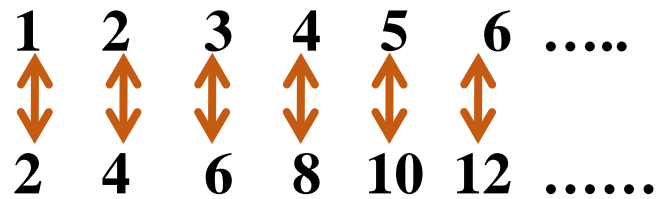
Reason:

A one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \dots, a_n, \dots$ where $a_1 = f(1)$, $a_2 = f(2), \dots, a_n = f(n), \dots$

How to show that a set is countable?

Example 4: Show that the set of positive even integers E is countable set.

Solution: Let $f(x) = 2x$.



Then f is a bijection from \mathbb{N} to E since f is both one-to-one and onto.

- To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$.
- To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$.

Note :

- ◆ E is countable infinite.
- ◆ E is a proper subset of \mathbb{Z}^+ ! But $|E| = |\mathbb{Z}^+|$.

Hilbert's Grand Hotel

- Invented by **David Hilbert**
- A paradox that shows that something impossible with finite sets may be possible with infinite sets.



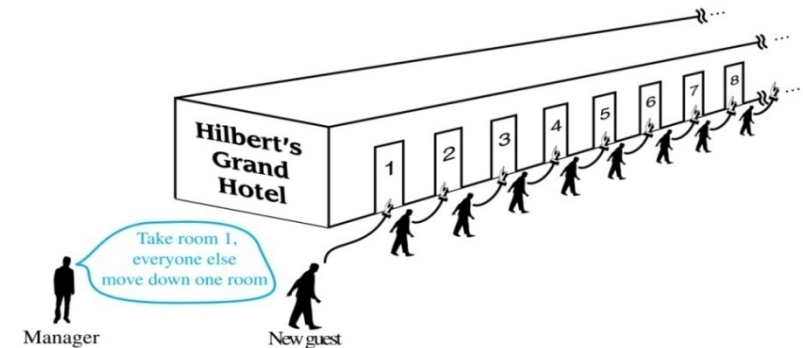
David Hilbert

The notion of the Grand Hotel: It has **countably infinite number of rooms**, each occupied by a guest.

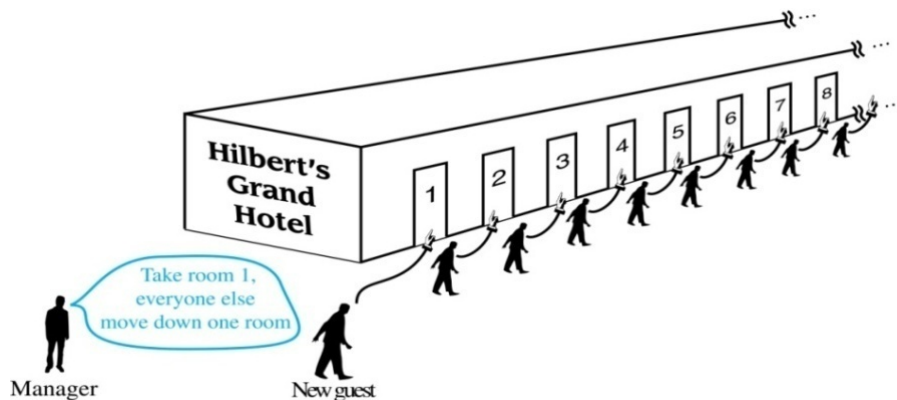
We can always accommodate a new guest at this hotel. How is this possible?

Explanation:

- Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on.
 - When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room n to Room $n + 1$, for all positive integers n .
 - This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.
- ◆ When there are finitely many rooms in a hotel, the notion that all rooms are occupied is equivalent to the notion that no new guests can be accommodated.
- ◆ This equivalence no longer holds when there are infinitely many rooms.



Hilbert's Grand Hotel



Hilbert's Grand Hotel can also accommodate

- ✓ A finite group of new guests
- ✓ A countably infinite number of new guests
- ✓ A countable number of new guests, all the guests on a countably infinite number of buses where each bus contains a countably infinite number of guests.

How to show that a set is countable?

Example 5: Show that the set of integers \mathbb{Z} is countable.

Solution: Can list in a sequence:

$$0, -1, 1, -2, +2, \dots$$

Or can define a bijection from \mathbb{N} to \mathbb{Z} :

$$f(i) = \begin{cases} 2|i| & i < 0 \\ 1 & i = 0 \\ 2i + 1 & i > 0 \end{cases}$$

f is a bijection from \mathbb{Z} to \mathbb{Z}^+ .

Hence, \mathbb{Z} is countable infinite set.



The Positive Rational Numbers are Countable

Example 6: Show that the positive rational numbers are countable.

■ $\forall x \in \mathbb{Q}^+, x = q/p, p, q \in \mathbb{N}, q \neq 0, \mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$

■ Let $S = \{ (p, q) \mid p, q \in \mathbb{N} \} = \mathbb{N} \times \mathbb{N}$.

■
$$\left. \begin{array}{l} |\mathbb{Q}^+| \leq |S| \\ |S| = |\mathbb{N}| \\ |\mathbb{N}| \leq |\mathbb{Q}^+| \end{array} \right\} \Rightarrow |\mathbb{Q}^+| = |\mathbb{N}|$$

SCHRÖDER-BERNSTEIN THEOREM:

If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

In other words, if there are one-to-one functions f from A to B and g from B to A , then there is a one-to-one correspondence between A and B .

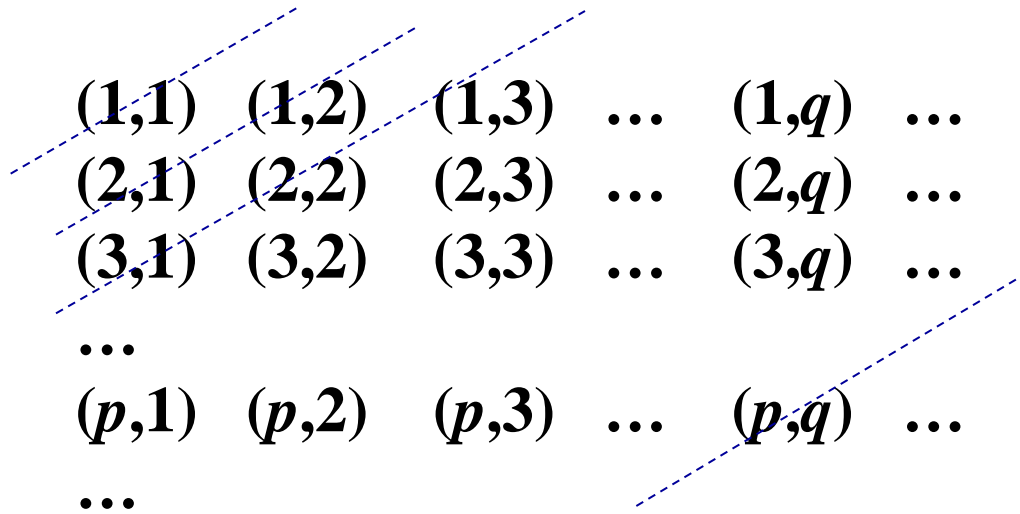
$$(1) \quad |Q^+| \leq |S|$$

Suppose that $\frac{q}{p} \in Q^+$

$\frac{q}{p} \rightarrow (p, q)$ is a injective

$$\therefore |Q^+| \leq |S|$$

$$(2) \quad |S| = |N|$$



$$\begin{array}{cccccc}
 (1,1) & (1,2) & (1,3) & \dots & (1,q) & \dots \\
 (2,1) & (2,2) & (2,3) & \dots & (2,q) & \dots \\
 (3,1) & (3,2) & (3,3) & \dots & (3,q) & \dots \\
 \dots & & & & & \\
 (p,1) & (p,2) & (p,3) & \dots & (p,q) & \dots \\
 \dots & & & & &
 \end{array}$$

$$1 + 2 + \dots + (p + q - 2) = \frac{(p + q - 2)(p + q - 1)}{2}$$

$$n = \frac{1}{2}(p + q - 2)(p + q - 1) + p$$

Note: $|N \times N| = |N|$

$$(3) \quad |N| \leq |Q^+|$$

Since $N \subseteq Q^+$

Therefore, $|N| \leq |Q^+|$

Note :

- ◆ The set of all rational numbers Q , positive and negative, is countable infinite.
- ◆ The set of rational numbers and the set of natural numbers have same cardinality, namely $|Q| = |N|$.

Strings

Example 7: Show that the set of finite strings S over a finite alphabet A is countably infinite.

Assume an alphabetical ordering of symbols in A

Solution:

Show that the strings can be listed in a sequence.

- First list all the strings of length 0 in alphabetical order.
- Then all the strings of length 1 in lexicographic (as in a dictionary) order.
- Then all the strings of length 2 in lexicographic order.
- And so on.

This implies a bijection from \mathbb{N} to S and hence it is a countably infinite set.



The set of all Java programs is countable.

Example 8: Show that the set of all Java programs is countable.

Solution:

Let S be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:

Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program.)

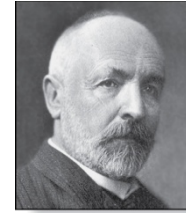
If the compiler says YES, this is a syntactically correct Java program, we add the program to the list.

We move on to the next string.

In this way we construct an implied bijection from \mathbb{N} to the set of Java programs. Hence, the set of Java programs is countable.

Uncountable Sets

Georg Cantor
(1845-1918)



【Theorem】 The set of real numbers between 0 and 1 is uncountable.

Proof:

- use an important proof method known as the **Cantor diagonalization argument**.

$$A = \{x \mid x \in (0,1) \wedge x \in R\}$$

$$\left. \begin{array}{l} (1) \mid N \mid \leq \mid A \mid \\ (2) \mid N \mid \neq \mid A \mid \end{array} \right\} \longrightarrow \mid N \mid < \mid A \mid$$

$$(1) \mid N \mid \leq \mid A \mid$$

$$A = \{x \mid x \in (0,1) \wedge x \in R\}$$

$$B = \left\{ \frac{1}{n+1} \mid n \in N \right\}$$

$$\mid B \mid = \mid N \mid \quad B \subseteq A$$

$$\therefore \mid N \mid \leq \mid A \mid$$

(2) $|N| \neq |A|$

Let $A = \{r_1, r_2, r_3, \dots, r_n, \dots\}$

Represent each real number in the list using *its decimal expansion*.

THE LIST....

$$r_1 = 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16} \dots$$

$$r_2 = 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26} \dots$$

$$r_3 = 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36} \dots$$

...

where $d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Now construct the number $x = 0.x_1x_2x_3x_4x_5x_6x_7 \dots$

$$x_i = 4 \text{ if } d_{ii} \neq 4$$

$$x_i = 5 \text{ if } d_{ii} = 4$$

Then x is not equal to any number in the list.

Hence, no such list can exist and hence the interval $(0,1)$ is uncountable .

【Theorem】 The set of real numbers is uncountable.

Proof:

Let $f(x)=\tan(x)$.

$f(x)$ is a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $R = (-\infty, +\infty)$.

$$\left|(-\frac{\pi}{2}, \frac{\pi}{2})\right| = |(0,1)| \quad \therefore |R| = |(0,1)|$$

Note:

- Any set with an uncountable subset is uncountable.
- $|R| = \aleph$
- It is said to have the **cardinality of the continuum, c.**

[[**Example 9**]] Show that $|(0,1)| = |[0,1]|$.

Proof:

$$A = [0,1] = \{x \mid x \in R, 0 \leq x \leq 1\}$$

$$B = (0,1) = \{x \mid x \in R, 0 < x < 1\}$$

$$(1) B \subseteq A \Rightarrow |B| \leq |A|$$

$$(2) \text{ Let } g(x) = \frac{1}{2}x + \frac{1}{4}, x \in [0,1]$$

Hence, $g(x)$ is a bijection from $[0,1]$ to $[1/4,3/4]$.

Thus $|A| \leq |B|$

Results about cardinality

- 1) No infinite set has a smaller cardinality than a countable set.
- 2) The union of two countable sets is countable.

Proof:

Suppose that A and B are both countable sets. Without loss of generality, we can assume that A and B are disjoint.

Case 1: A and B are finite.

Case 2: A is infinite and B is finite.

Case 3: A and B are both countably infinite.

We can list their elements as $a_1, a_2, a_3, \dots, a_n, \dots$ and $b_1, b_2, b_3, \dots, b_n, \dots$, respectively. By alternating terms of these two sequences, we can list the elements of $A \cup B$ in the infinite sequence $a_1, b_1, a_2, b_2, a_3, b_3, \dots, a_n, b_n, \dots$. This means that $A \cup B$ is countably infinite.

Results about cardinality

- 1) No infinite set has a smaller cardinality than a countable set.
- 2) The union of two countable sets is countable.
- 3) The union of finite number of countable sets is countable.
- 4) The union of a countable number of countable sets is countable.

Uncomputable Function--An important application in CS

【**Definition 4**】 A function is **computable** if there is a computer program in some programming language that finds the values of this function. If a function is not computable we say it is **uncomputable**.

Question: Show that there are functions that are not computable.

- Show that the set of all computer programs in any particular programming language is countable.
- There are uncountably many different function from a particular countably infinite set to itself.
 - Exercise 38 shows that the set of functions from the set of positive integers to itself is uncountable.



The Continuum Hypothesis

- ◆ The cardinality of the power set of an arbitrary set has a greater cardinality than the original arbitrary set.
- ◆ The power set of \mathbb{Z}^+ and the set of real numbers \mathbb{R} have the same cardinality.

$$|\mathcal{P}(\mathbb{Z}^+)| = |\mathbb{R}| = c$$

The Continuum Hypothesis

The continuum hypothesis (CH) asserts that there is no cardinal number a such that $\aleph_0 < a < \aleph$.

Homework:

SE: P.176 4, 7, 9, 29, 37, 38

EE: P.186 4, 7, 9, 29, 37, 38

2.6 *Matrices*

Introduction

- ◆ Matrices are useful **discrete structures** that can be used in many ways.
 - Describe certain types of functions known as linear transformations
 - Express relationships between elements in sets
 - Express which vertices of a graph are connected by edges
 - Transportation systems
 - Communication networks

Section Summary

- ◆ Definition of a Matrix
- ◆ Matrix Arithmetic
- ◆ Transposes and Powers of Arithmetic
- ◆ Zero-One matrices