Backtracking

Rationale of the Backtracking Algorithms

A sure-fire way to find the answer to a problem is to **make a list of all candidate answers**, **examine each**, and following the examination of all or some of the candidates, declare the identified answer.

Backtracking enables us to **eliminate** the explicit examination of **a large subset** of the candidates while still guaranteeing that the answer will be found if the algorithm is run to termination.

Sure — if the list is finite and it is possible to identify the answer following the examinations. **AND**, if there are not too many candidates.

pruning

The **basic idea** is to start with a partial solution $(x_1, \ldots, x_i)$ where each $x_k \in S_k$ for $1 \le k \le i < n$. The... check if $(x_1, \ldots, x_i, x_{i+1})$ satisfies the constr... "**yes**", we **continue** to add the next x, **else** we delete the current x_{i+1} and try the next $x_{i+1} \in S_{i+1}$. If all possible $x_{i+1} \in S_{i+1}$ fail to satisfy the constraints, we **backtrack** to the previous partial solution $(x_1, \ldots, x_{i-1})$.

Eight Queens

Find a placement of 8 queens on an 8×8 chessboard such that no two queens attack.

Two queens are said to attack iff they are in the same row, column, diagonal, or antidiagonal of the chessboard.

Discussion 16:

Please draw 2 different solutions.

	1	2	3	4	5	6	7	8
1				Q				
2						Q		
3								Q
4		Q						
5							Q	
6	Q							
7			Q					
8					Q			

Q_i ::= queen in the *i*-th row

x_i ::= the column index in which Q_i is

S

This implies
that the solution must
be a permutation of 1, 2, ... , 8.
Thus the number of candidates
in the solution space
is reduced to 8!.

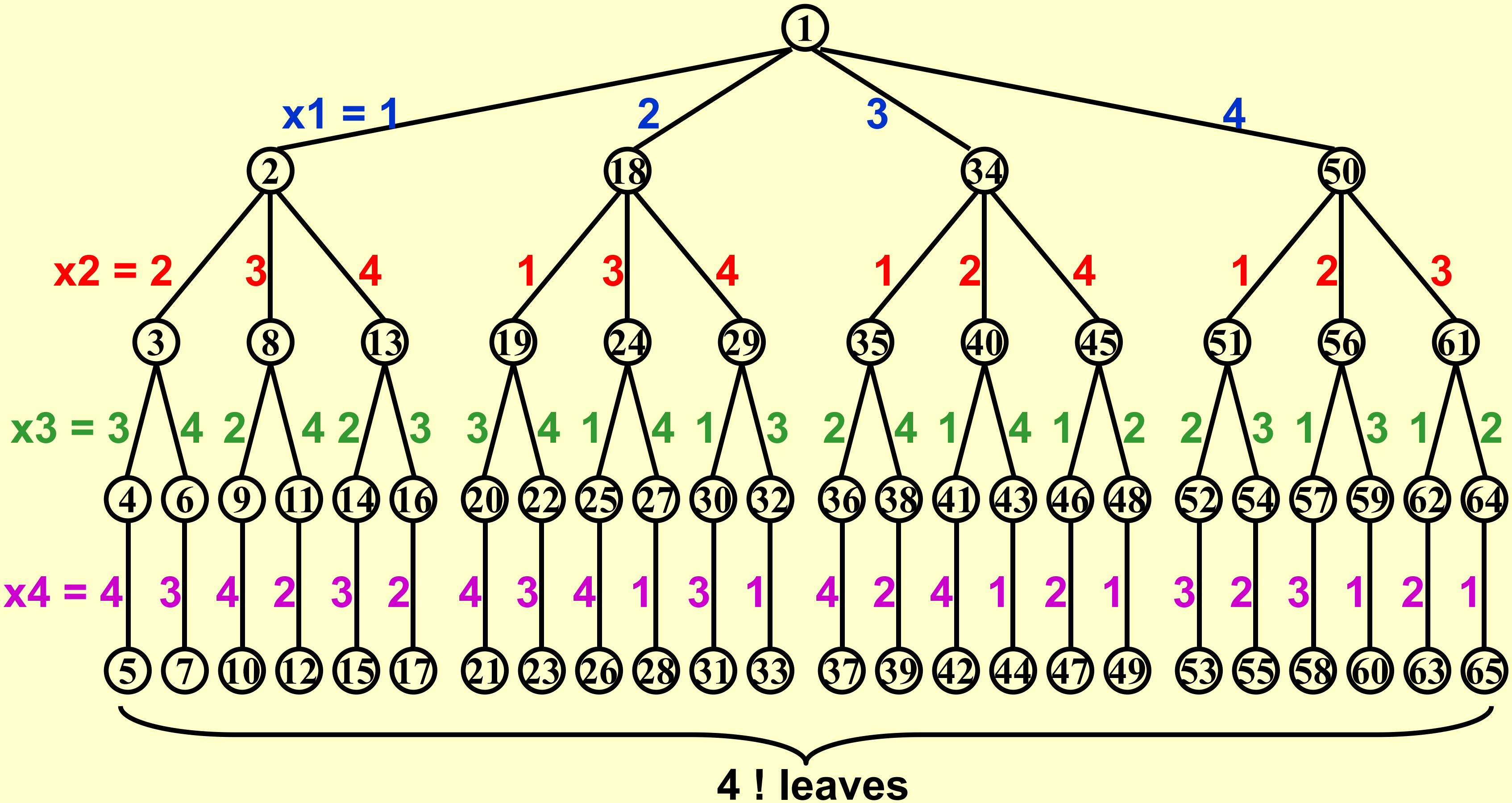
Constrains: ① $S_i = \{ 1,2,3,4,5,6,7,8 \}$ for $1 \le i \le 8$

② $x_i \neq x_j$ if $i \neq j$ ③ $(x_i - x_j) / (i - j) \neq \pm 1$

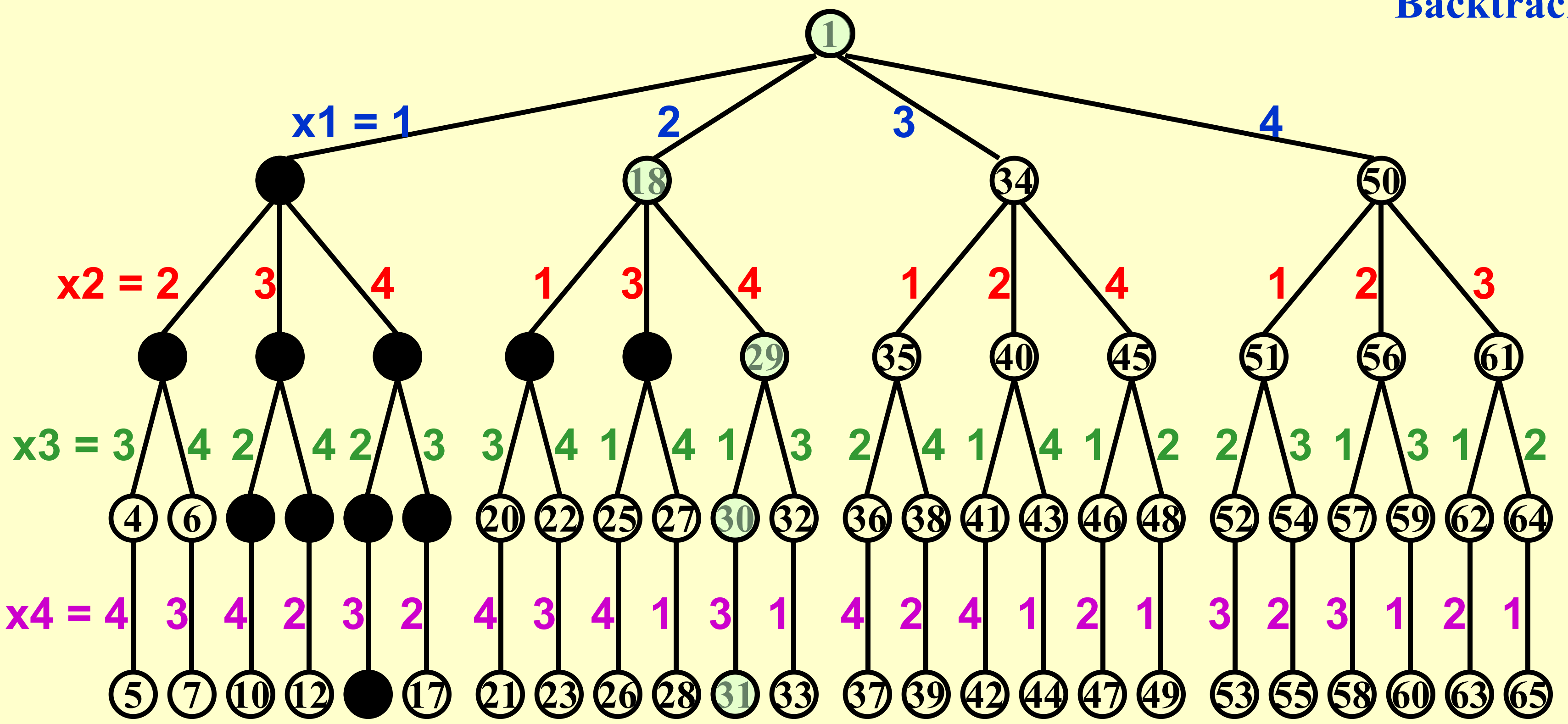
For the problem with *n* queens,
there are *n*! candidates
in the solution space.

Method: Take the problem of 4 queens as an example

Step 1: Construct a game tree

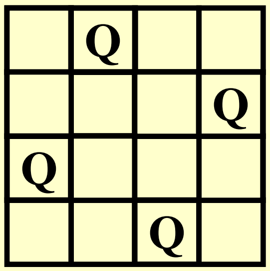


Each path from the root to a leaf defines an element of the solution space.



Step 2: Perform a depth-first search (post-order traversal) to examine the paths

(2, 4, 1, 3)



Note: No tree is actually constructed. The game tree is just an abstract concept.

The Turnpike Reconstruction Problem

Given N points on the x-axis with coordinates $x_1 < x_2 < \dots < x_N$. Assume that $x_1 = 0$. There are $N (N - 1) / 2$ distances between every pair of points.

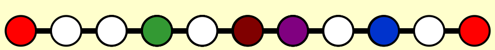
Given $N (N - 1) / 2$ distances. Reconstruct a point set from the distances.

〖Example〗 Given $D = \{ 1, 2, 2, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7, 8, 10 \}$

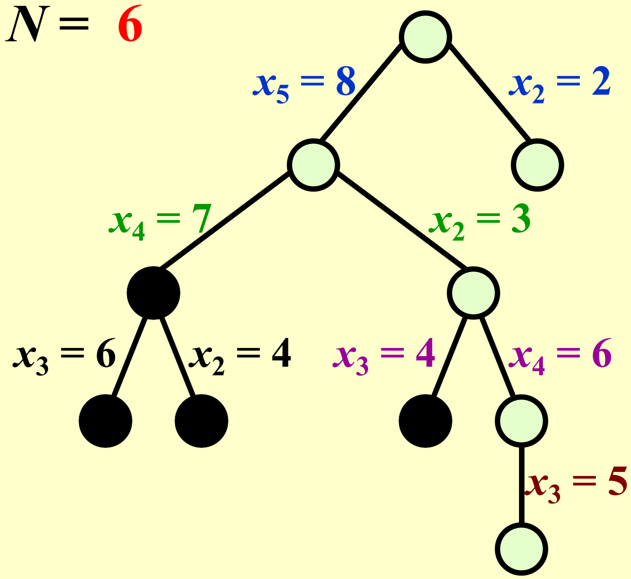
Step 1: $N (N - 1) / 2 = 15$ implies $N = 6$

Step 2: $x_1 = 0$ and $x_6 = 10$

Step 3: find the next largest distance and check



$(0, 3, 5, 6, 8, 10)$



```
bool Reconstruct ( DistType X[ ], DistSet D, int N, int left, int right )
{ /* X[1]...X[left-1] and X[right+1]...X[N] are solved */
    bool Found = false;
    if ( Is_Empty( D ) )
        return true; /* solved */
    D_max = Find_Max( D );
    /* option 1：X[right] = D_max */
    /* check if |D_max-X[i]|∈D is true for all X[i]'s that have been solved */
    OK = Check( D_max, N, left, right ); /* pruning */
    if ( OK ) { /* add X[right] and update D */
        X[right] = D_max;
        for ( i=1; i<left; i++ )  Delete( |X[right]-X[i]|, D);
        for ( i=right+1; i<=N; i++ )  Delete( |X[right]-X[i]|, D);
        Found = Reconstruct ( X, D, N, left, right-1 );
        if ( !Found ) { /* if does not work, undo */
            for ( i=1; i<left; i++ )  Insert( |X[right]-X[i]|, D);
            for ( i=right+1; i<=N; i++ )  Insert( |X[right]-X[i]|, D);
        }
    }
    /* finish checking option 1 */
```

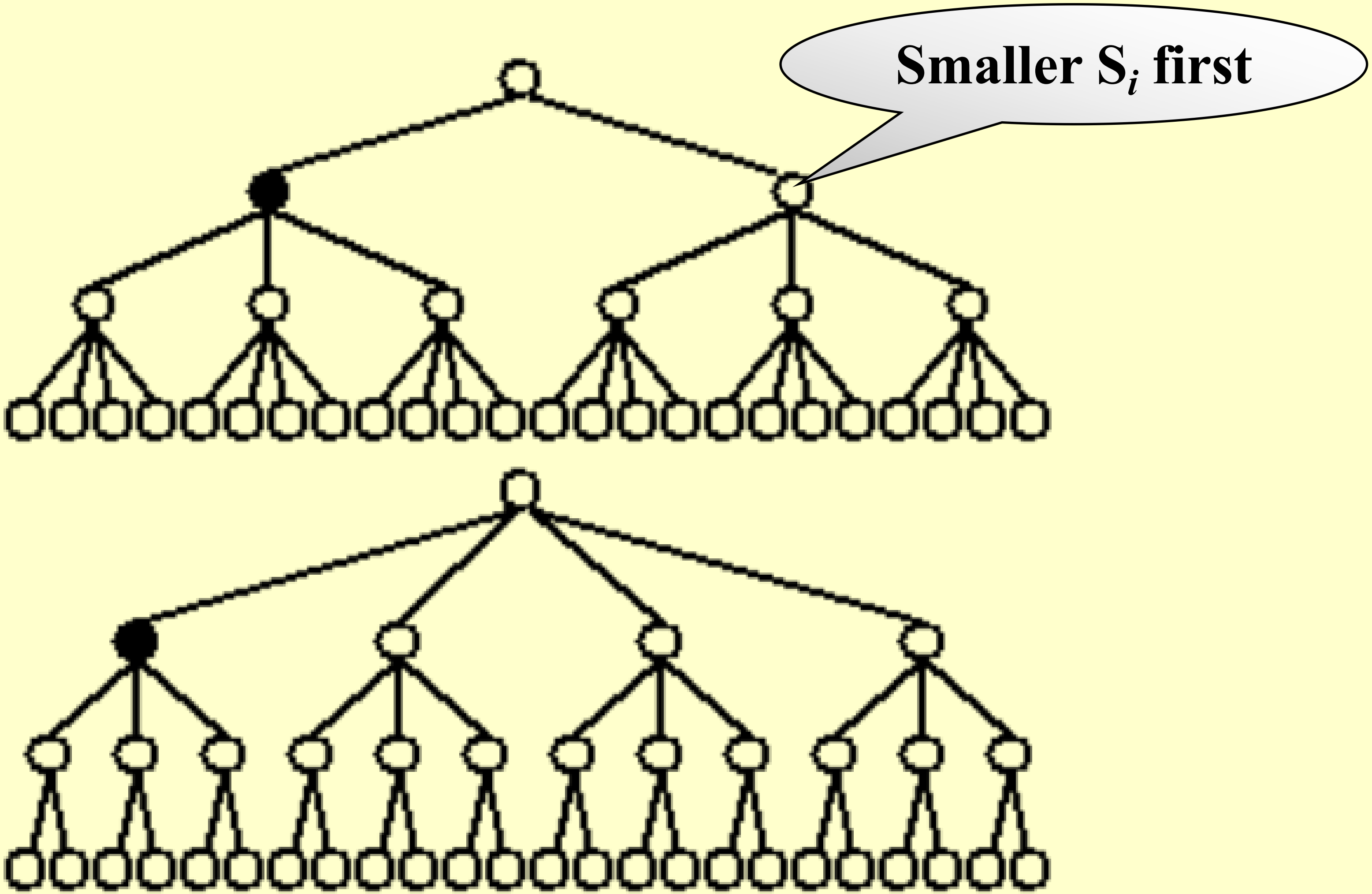
```
    if ( !Found ) { /* if option 1 does not work */
        /* option 2: X[left] = X[N]-D_max */
        OK = Check( X[N]-D_max, N, left, right );
        if ( OK ) {
            X[left] = X[N] – D_max;
            for ( i=1; i<left; i++ )  Delete( |X[left]-X[i]|, D);
            for ( i=right+1; i<=N; i++ )  Delete( |X[left]-X[i]|, D);
            Found = Reconstruct (X, D, N, left+1, right );
            if ( !Found ) {
                for ( i=1; i<left; i++ ) Insert( |X[left]-X[i]|, D);
                for ( i=right+1; i<=N; i++ ) Insert( |X[left]-X[i]|, D);
            }
        }
        /* finish checking option 2 */
    } /* finish checking all the options */

    return Found;
}
```

A Template

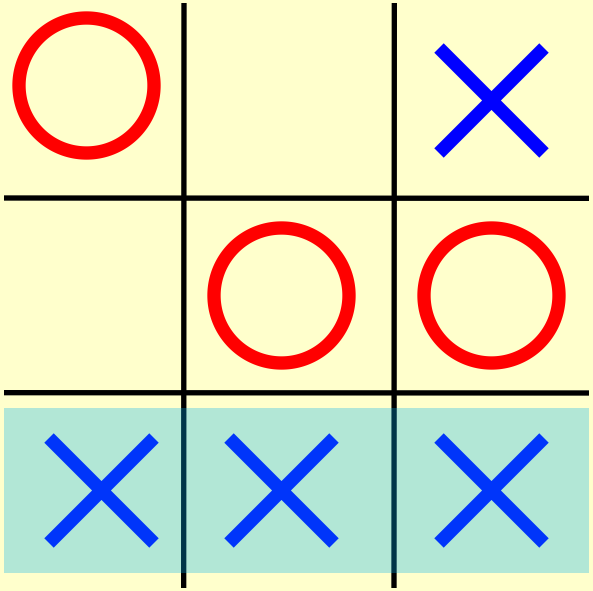
```
bool Backtracking ( int i )
{   Found = false;
    if ( i > N )
        return true; /* solved with (x1, …, xN) */
    for ( each xi ∈ Si ) {
        /* check if satisfies the restriction R */
        OK = Check((x1, …, xi) , R ); /* pruning */
        if ( OK ) {
            Count xi in;
            Found = Backtracking( i+1 );
            if ( !Found )
                Undo( i ); /* recover to (x1, …, xi-1) */
        }
        if ( Found ) break;
    }
    return Found;
}
```

When different S_i's have different sizes



Games – *how did AlphaGo win*

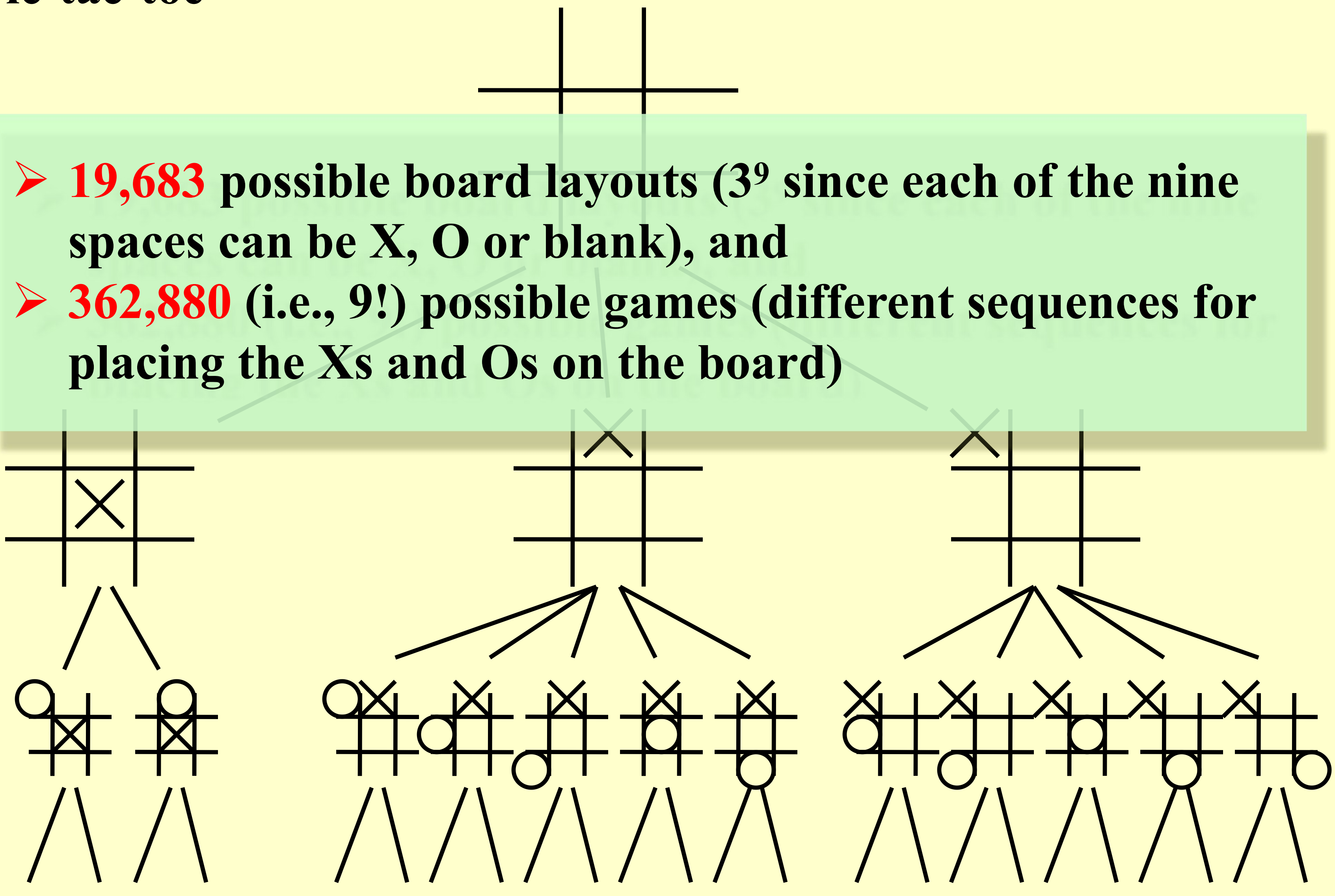
Tic-tac-toe



The player who succeeds in placing three of their marks in a horizontal, vertical, or diagonal row wins the game.

Tic-tac-toe

- **19,683 possible board layouts (3^9 since each of the nine spaces can be X, O or blank), and**
- **362,880 (i.e., 9!) possible games (different sequences for placing the Xs and Os on the board)**

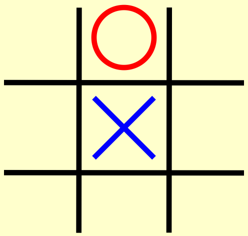


Tic-tac-toe: Minimax Strategy

Use an evaluation function to quantify the "goodness" of a position. For example:

$$f(P) = W_{Computer} - W_{Human}$$

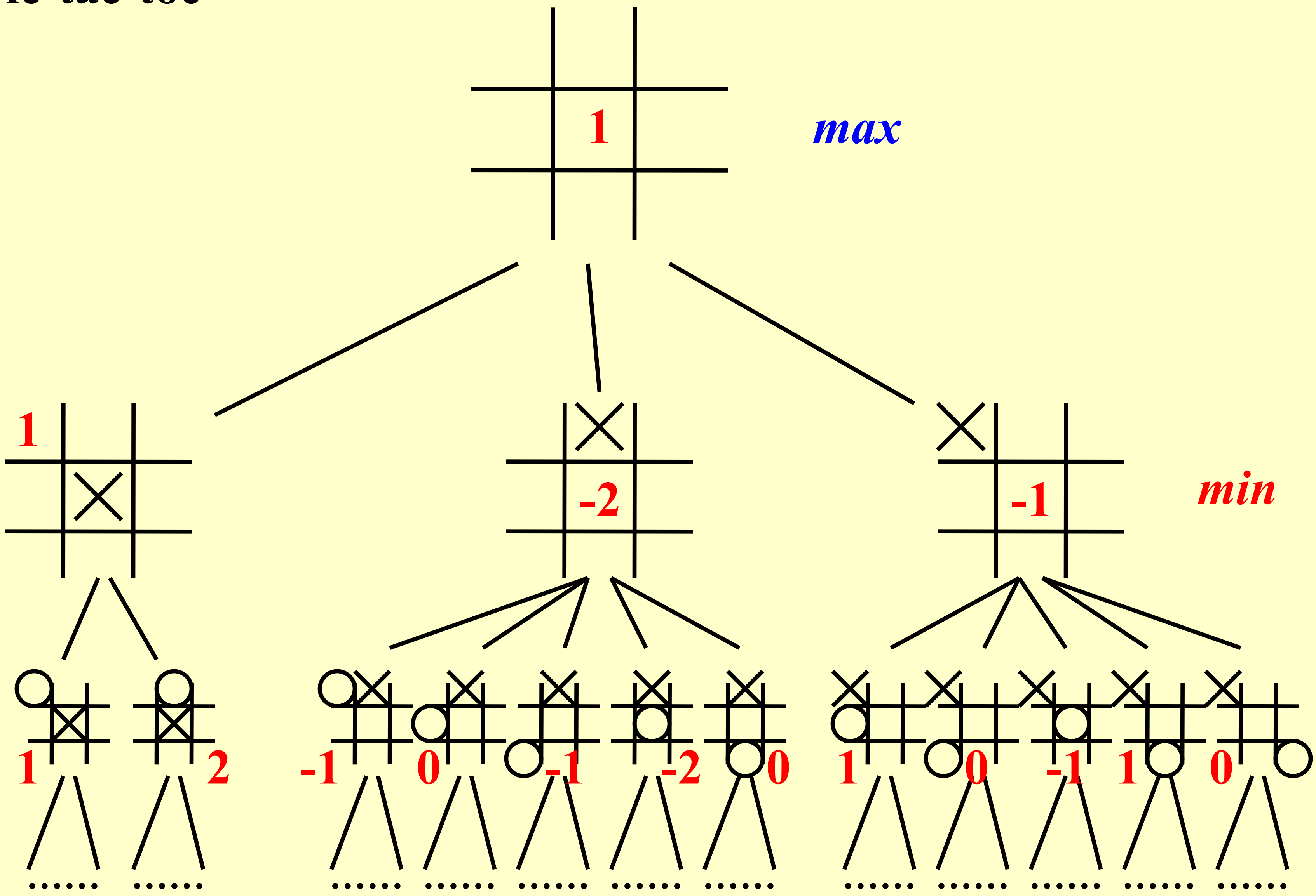
where *W* is the number of potential wins at position *P*.



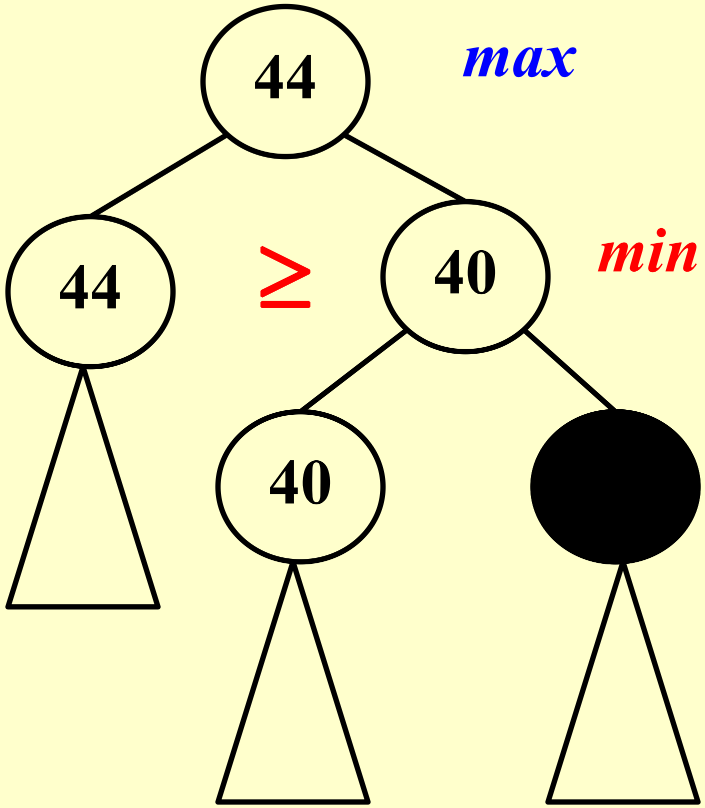
$$f(P) = 6 - 4 = 2$$

The human is trying to *minimize* the value of the position *P*, while the computer is trying to *maximize* it.

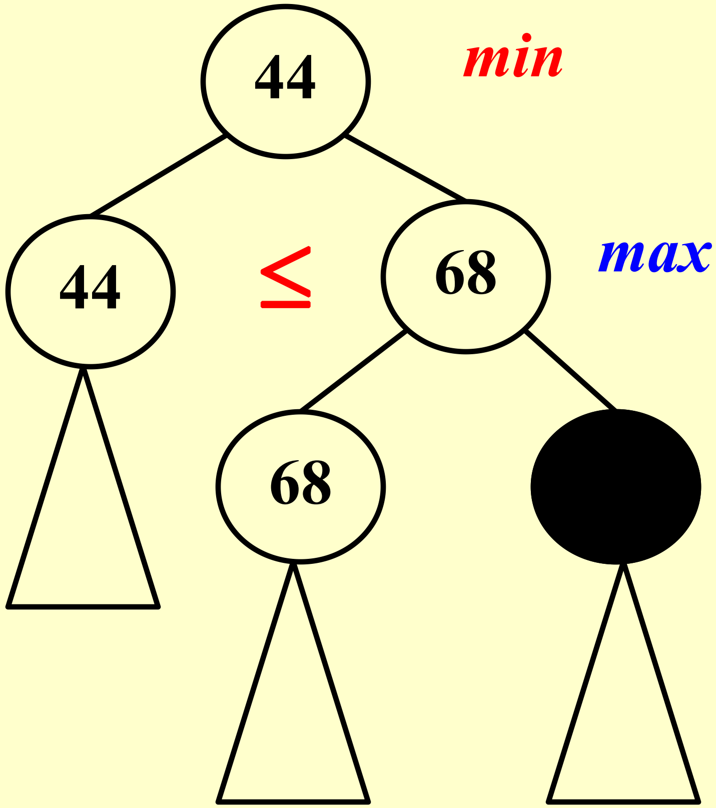
Tic-tac-toe



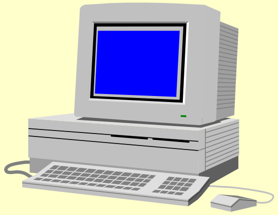
α pruning



β pruning



α-β pruning: when both techniques are combined. In practice, it limits the searching to only $O(\sqrt{N})$ nodes, where N is the size of the full game tree.



Research Project 3

To Buy or Not to Buy (30)

Eva would like to make a string of beads with her favorite colors so she went to a small shop to buy some beads. There were many colorful strings of beads. However the owner of the shop would only sell the strings in whole pieces. Hence in some cases Eva might have to buy several strings to get all the beads she needs. With a hundred strings in the shop, Eva needs your help to tell her whether or not she can get all the beads she needs with the least number of extra beads she has to pay for.

Detailed requirements can be downloaded from

https://pintia.cn/

Reference:

Data Structure and Algorithm Analysis in C (2nd Edition)：
Ch.10，p.403-414；***M.A.Weiss著、陈越改编，人民邮件出版社，2005***