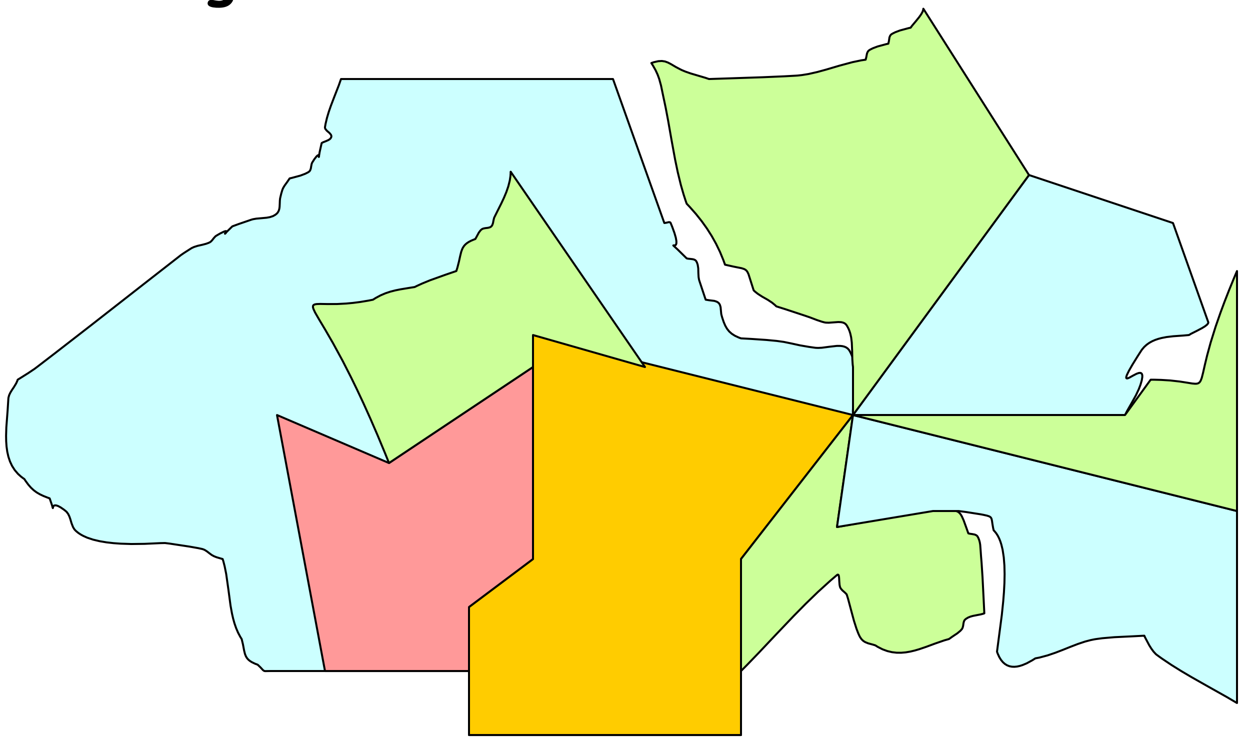
10.8 Graph Coloring

The problem related to the coloring of maps

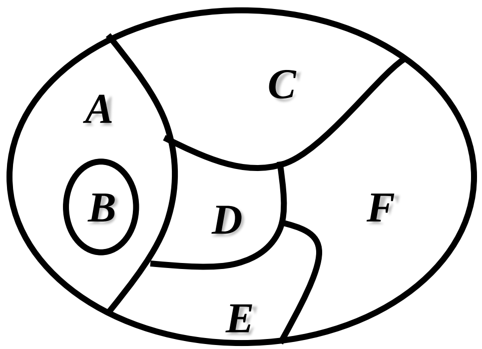
Determining the least number of colors that can be used to color a map so that adjacent regions never have the same color.

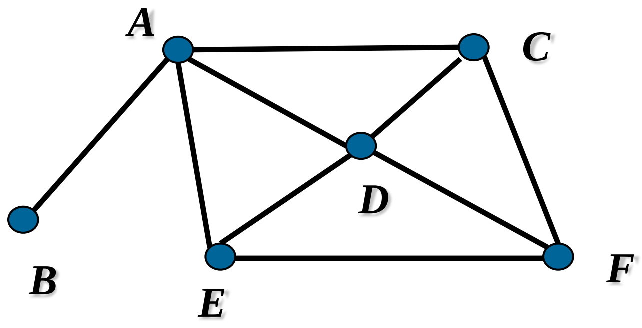


The problem of coloring a map, can be reduced to a graph-theoretic problem.

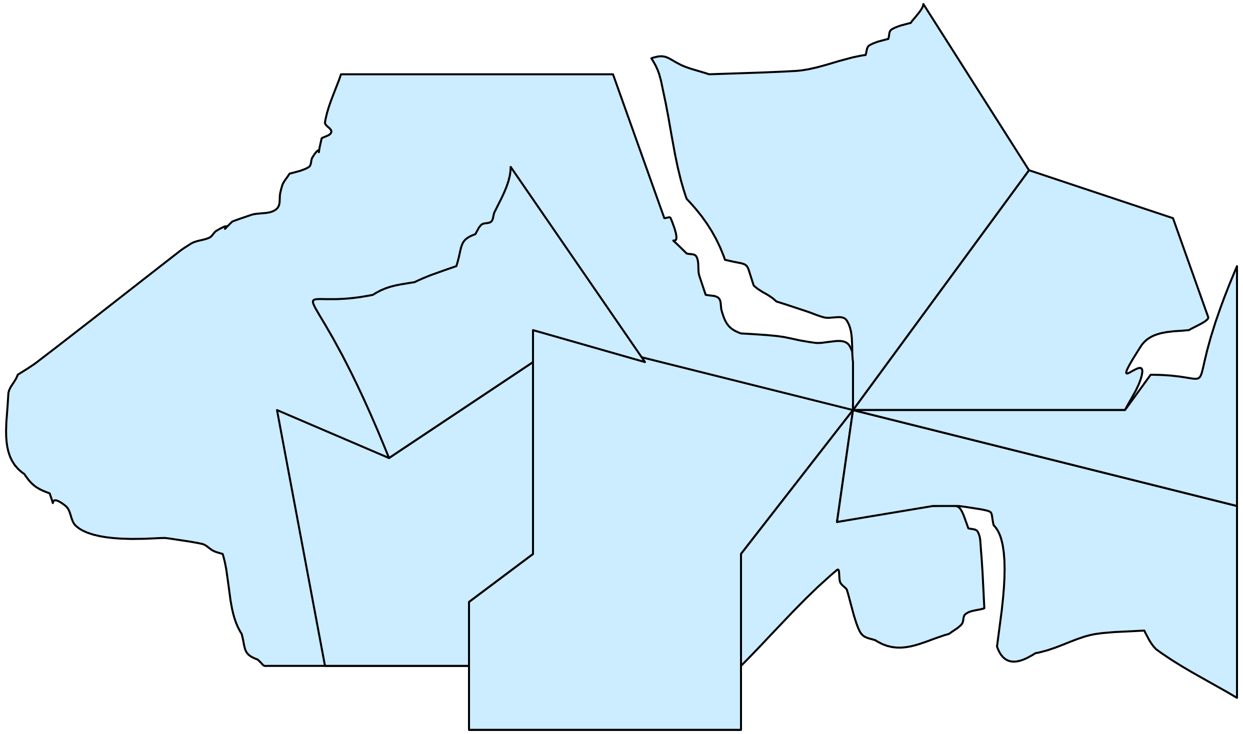
the dual graph of the map

- Each region of the map is represented by a vertex.
- Edge connect two vertices if the regions represented by these vertices have a common border.
- Two regions that touch at only one point are not considered adjacent.

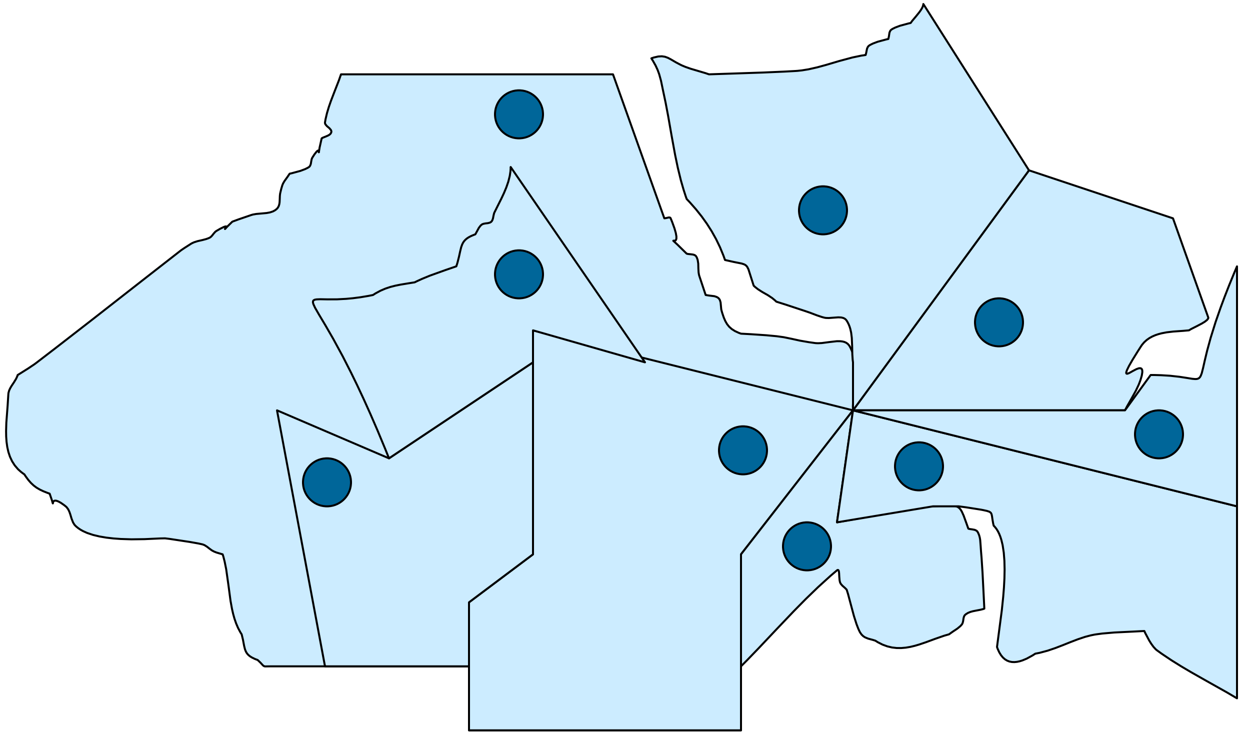




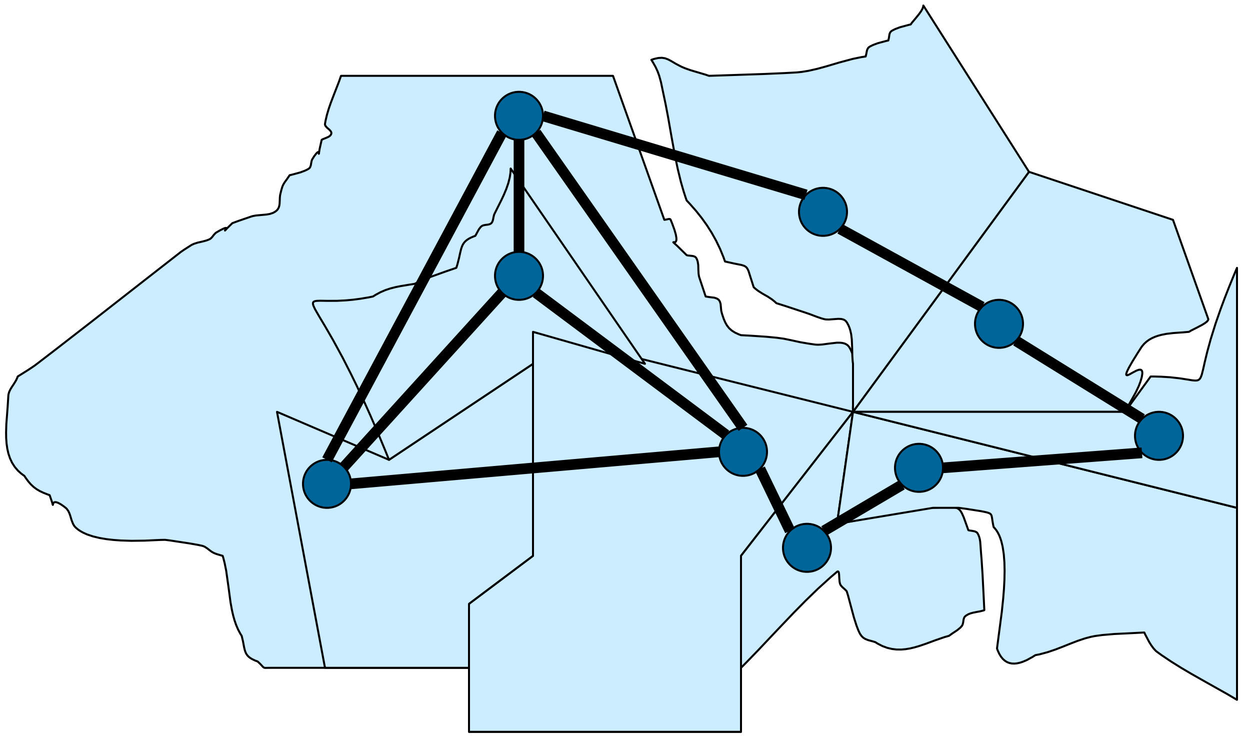
Coloring a map is equivalent to coloring the vertices of the dual graph so that no two adjacent vertices in this graph have the same color.



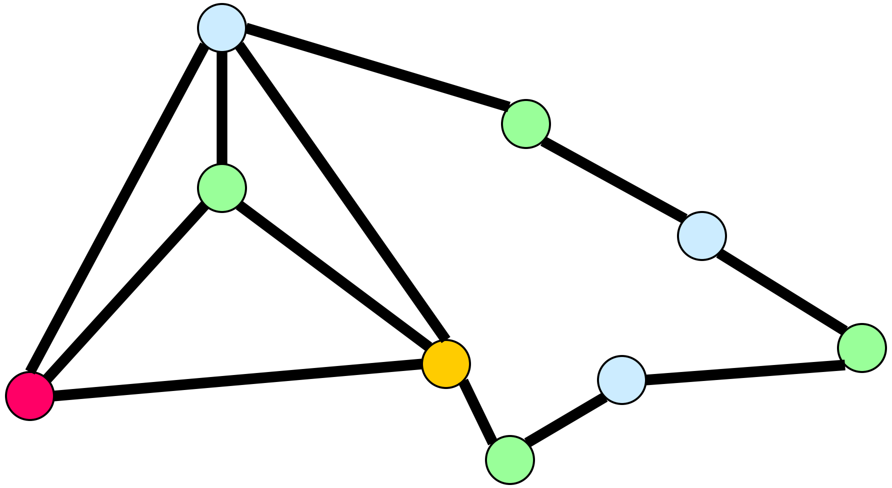
For each region introduce a vertex.



For each pair of regions with a positive-length common border introduce an edge.



Coloring regions is equivalent to coloring vertices of dual graph.

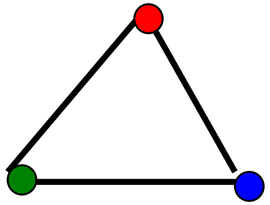


The chromatic numbers of a graph

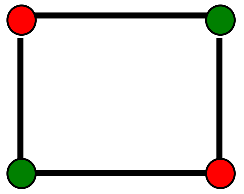
Terminologies:

Coloring: the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

chromatic number $\chi(G)$: the least number of colors needed for a coloring of this graph



$\chi(G)=3$



$\chi(G)=2$

Question: How to show that the chromatic numbers of a graph is *n*?

- ✓ Show that the graph can be colored with *n* colors.

  Method: constructing such a coloring.

- ✓ Show that the graph cannot be colored using fewer than *n* colors.

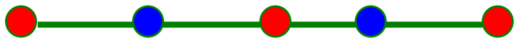
The chromatic numbers of some simple graphs

(1) The graph G contains only some isolated vertices.

$$\chi(G)=1$$

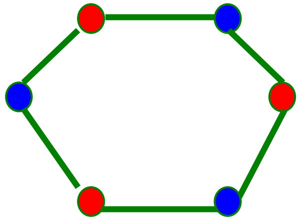
(2) The graph G is a path containing no circuit.

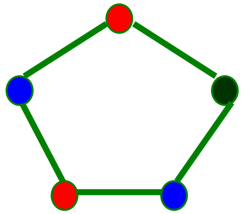
$$\chi(G)=2$$



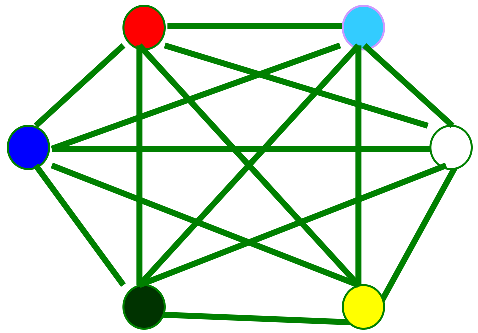
(3) $C_n(n\geq 3)$

$$\begin{cases} \chi(C_n)=2 & \textbf{if } \boldsymbol{n} \textbf{ is even} \\ \chi(G)=3 & \textbf{if } \boldsymbol{n} \textbf{ is odd} \end{cases}$$





(4) K_n

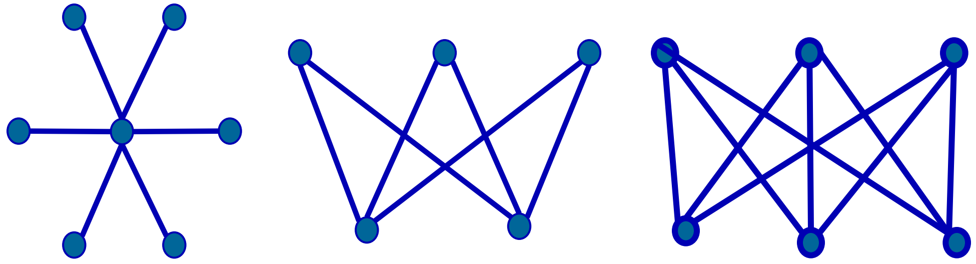


$$\chi(K_n) = n$$

$$\chi(K_n - e) = n - 1$$

(5) $K_{m,n}$

$$\chi(K_{m,n}) = 2$$



$k_{1,n}$ $K_{3,2}$ $K_{3,3}$

Note:

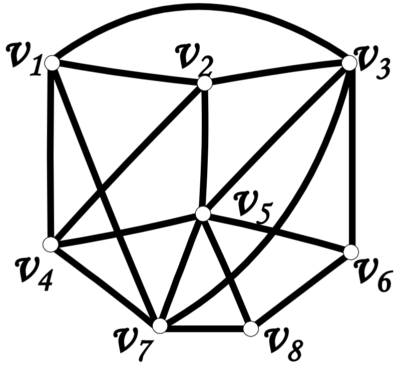
A simple graph with a chromatic number of 2 is bipartite.

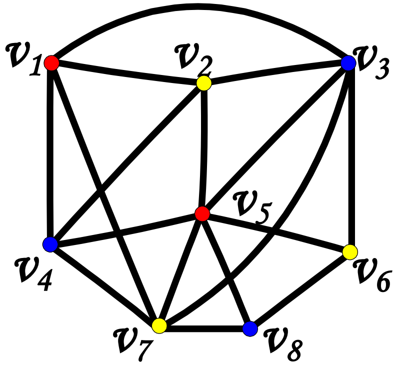
A connected bipartite graph has a chromatic number of 2.

Algorithm for coloring simple graphs

P.734

Example, Construct a coloring of the following graph





Solution:

① **List the vertices in order of decreasing degree**

$v_5, v_3, v_7, v_1, v_2, v_4, v_6, v_8$

② **Assign color 1: v_5, v_1**

③ **Assign color 2: v_3, v_4, v_8**

④ **Assign color 3: v_7, v_2, v_6**

【Theorem 1】 The Four Color Theorem
The chromatic number of a planar graph is no greater than four.

- Any planar map of regions can be depicted using 4 colors so that no two regions that share a positive-length border have the same color.
- The four color theorem was originally proposed as a conjecture in 1852.
- Proof by Haken and Appel used exhaustive computer search in 1976.
- The four color theorem applies only to planar graphs. Nonplanar graphs can have arbitrarily large chromatic numbers.

Applications of graph colorings

(1) Scheduling Exams

How can the exams at a university be scheduled so that no student has two exams at the same time?

Solution:

This scheduling problem can be solved using a graph model, with vertices representing courses and with an edge between two vertices if there is a common student in the courses they represent.

Each time slot for a final exam is represented by a different color.

A scheduling of the exams corresponds to a coloring of the associated graph.

For example, Suppose want to schedule some final exams for CS courses with following call numbers:

1007, 3137, 3157, 3203, 3261, 4115, 4118, 4156

Suppose also that there are no common students in the following pairs of courses because of prerequisites:

1007-3137

1007-3157, 3137-3157

1007-3203

1007-3261, 3137-3261, 3203-3261

1007-4115, 3137-4115, 3203-4115, 3261-4115

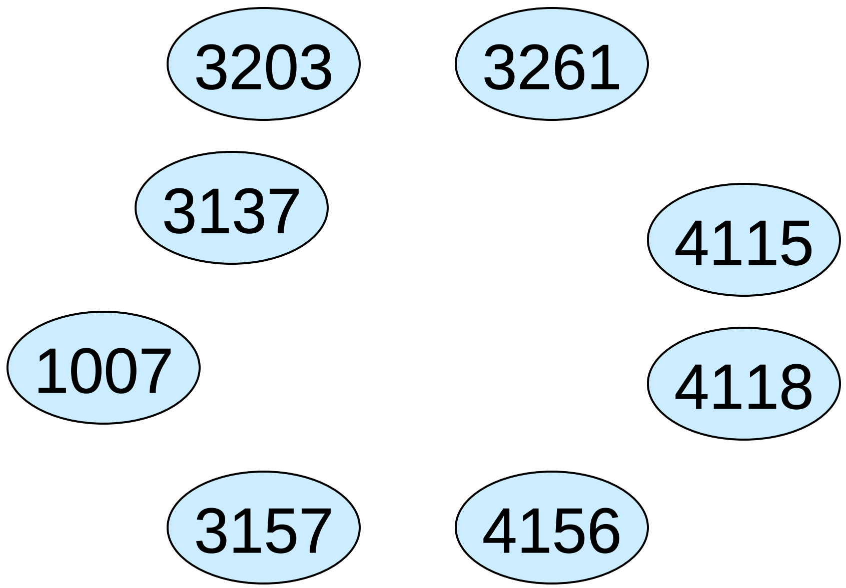
1007-4118, 3137-4118

1007-4156, 3137-4156, 3157-4156

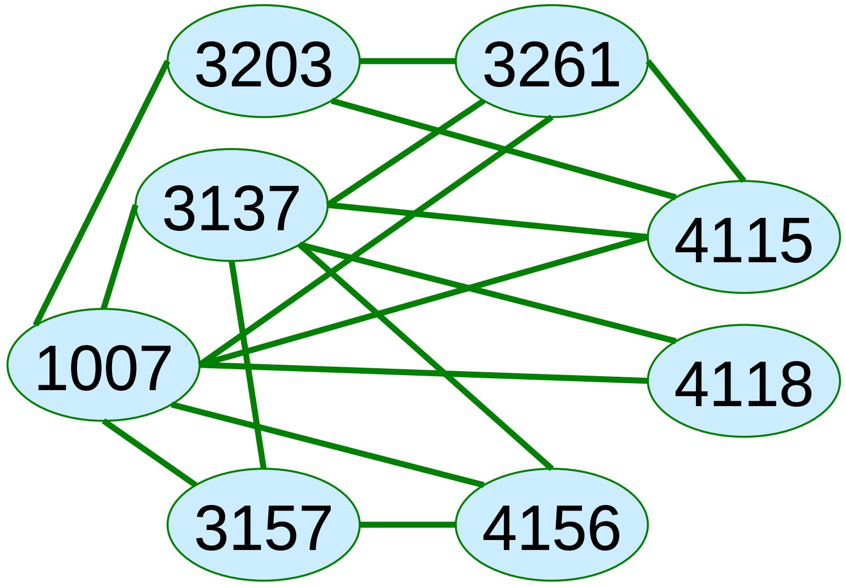
How many exam slots are necessary to schedule exams?

Turn this into a graph coloring problem.

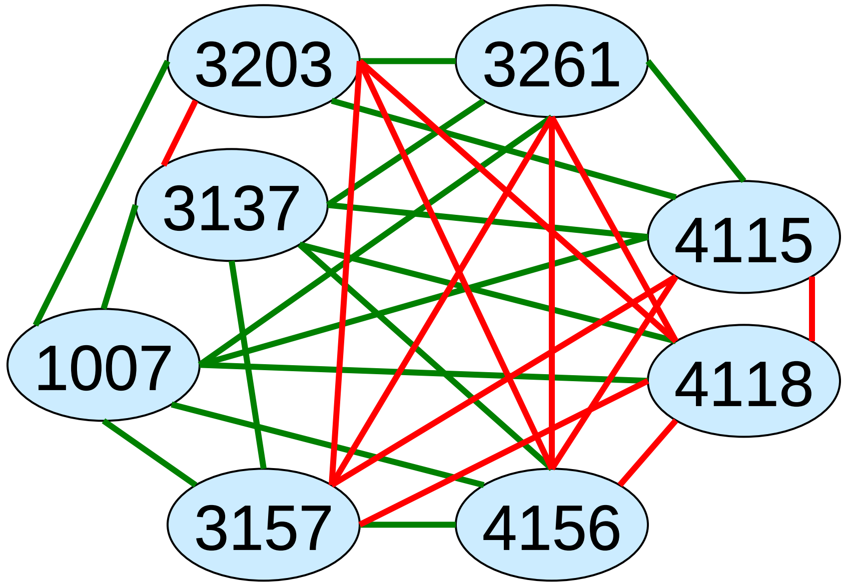
-- Vertices are courses, and edges are courses which *cannot* be scheduled simultaneously because of possible students in common:



One way to do this is to put edges down where students mutually excluded…



…and then compute the complementary graph:



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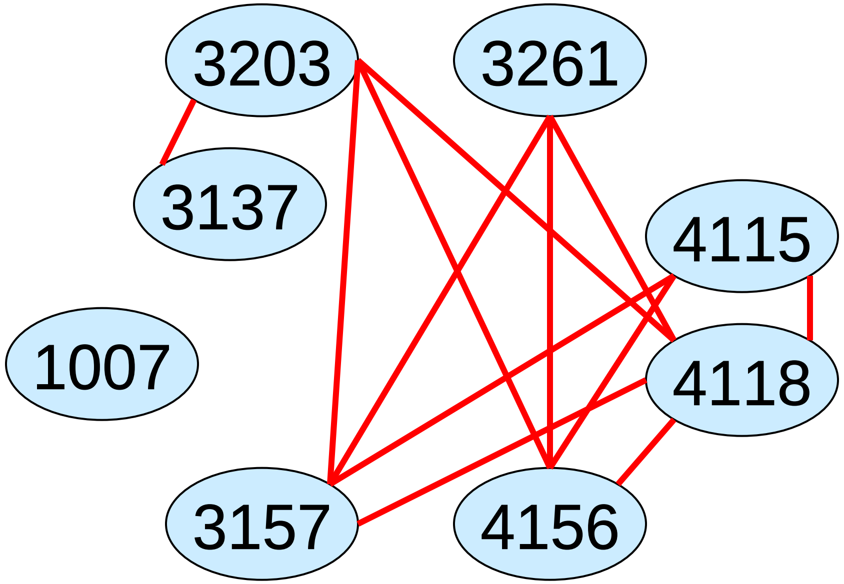
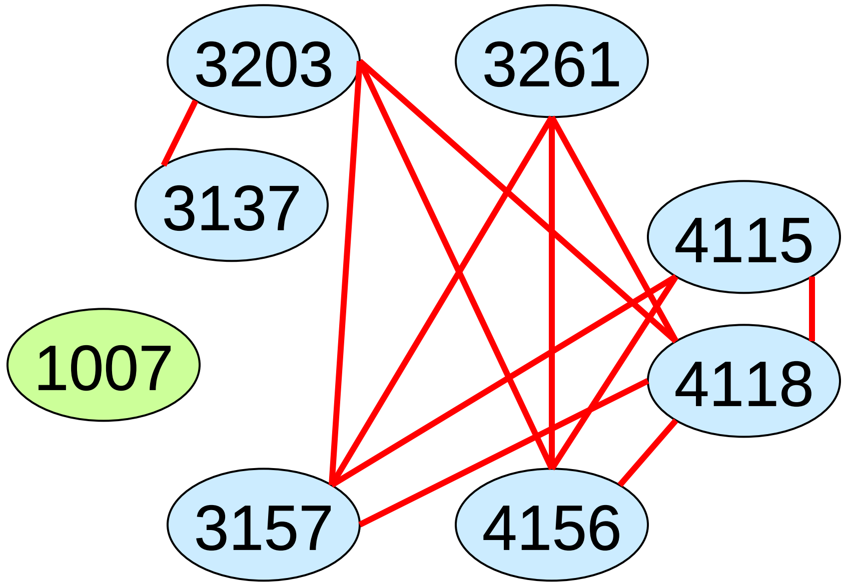
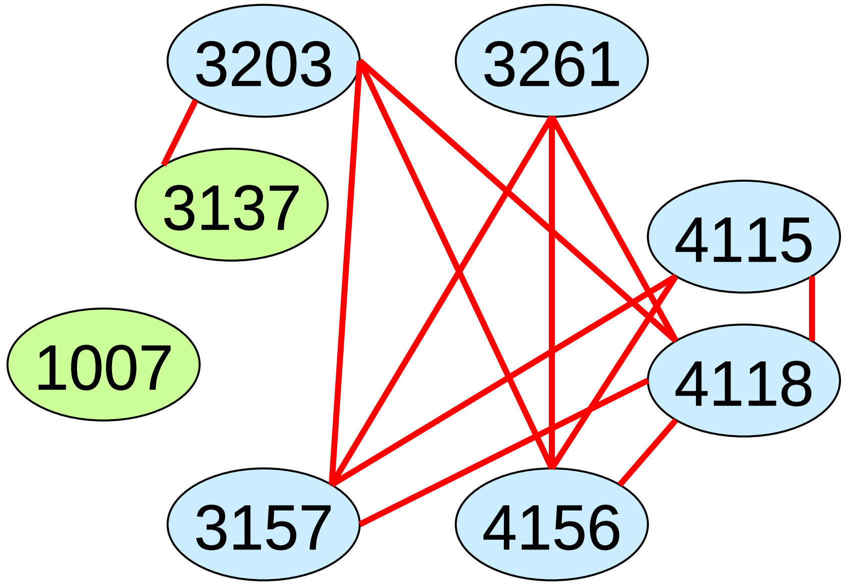


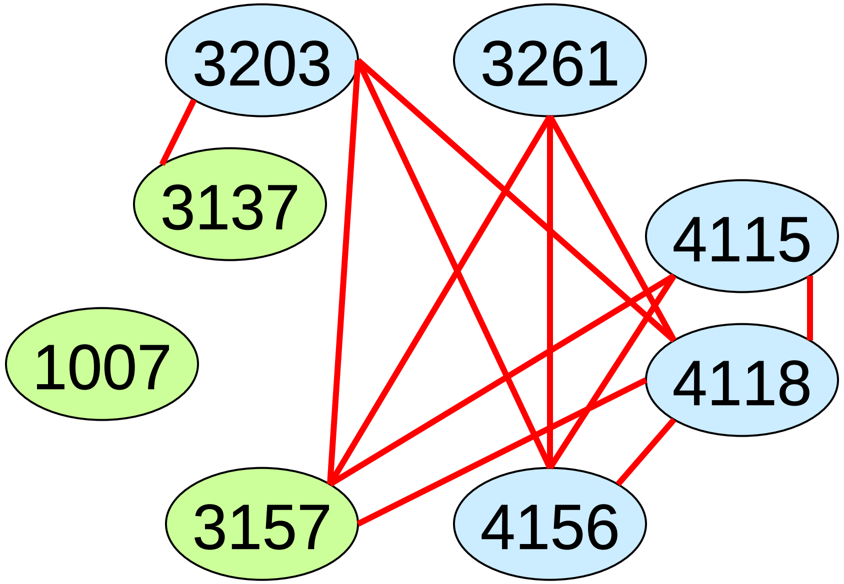
Coloring …



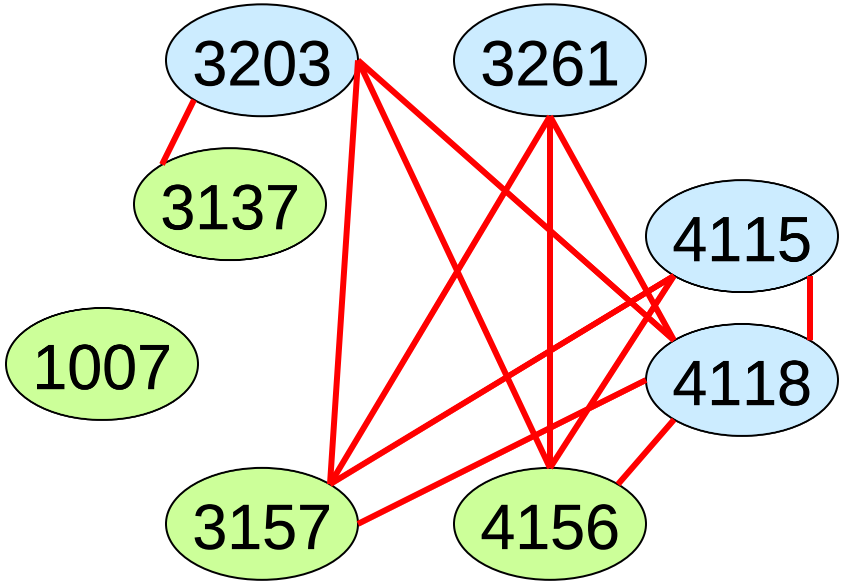
Coloring …



Coloring …



Coloring …



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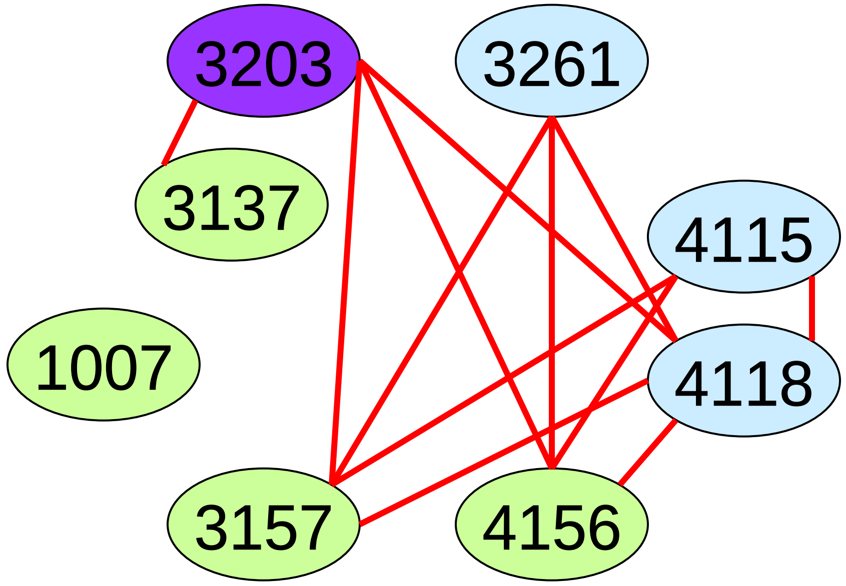

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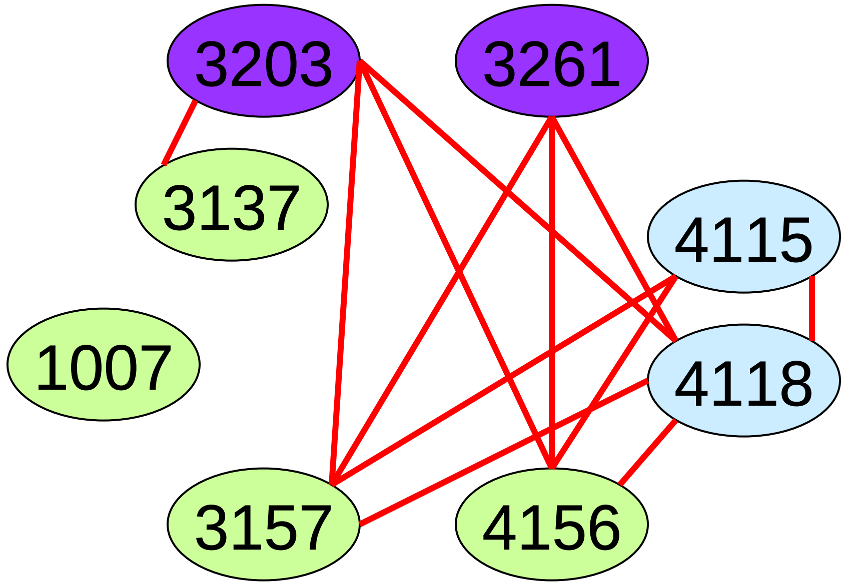
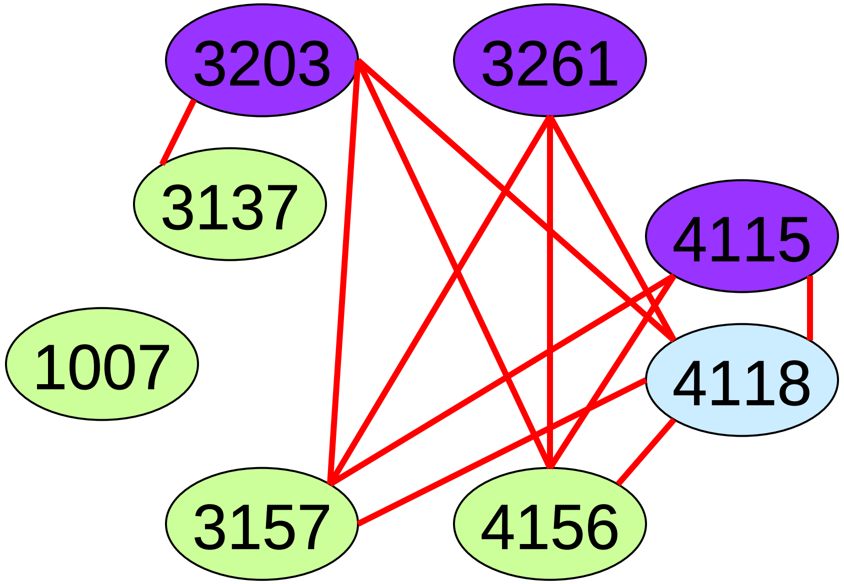


Coloring …



Applications of graph colorings

(2) Set up natural habitats of animal in a zoo

Solution:

Let the vertices of a graph be the animals.

Draw an edge between two vertices if the animals they represent cannot be in the same habitat because of their eating habits.

A coloring of this graph gives an assignment of habitats.

Homework:

SE：P. 732   3, 8, 9, 10, 17

EE：P. 768   3, 8, 9, 10, 17