6.4 Generalized Permutations and Combinations

Section Summary

- ✓Permutations with Repetition
- ✓Combinations with Repetition
- ✓Permutations with Indistinguishable Objects
- ✓Distributing Objects into Boxes

Problems:

How to solve counting problems where elements may be used more than once?

For example, how many strings of length *n* can be formed from the uppercase letters of the English alphabet?

How to solve counting problems in which some elements are indistinguishable?

For example, how many different strings can be made form the letters in MISSISSIPPI, using all the letters?

Permutations With Repetition

r-permutation with repetition: ***r*-permutations of a set with repetition allowed.**

【Theorem 1】 The number of *r*-permutations of a set of *n* objects with repetition allowed is n^r.

Proof:

By the product rule.

Example:

How many strings of length *n* can be formed from the uppercase letters of the English alphabet?

Combination With Repetition

H_n^r

r-Combination with repetition: combinations with repetition of elements allowed

【 Theorem 2 】 There are $C(n-1+r, r)$ r-combination from a set with n elements when repetition of elements is allowed.

Proof:

① The n-1 bars are used to mark off n different cells, with the ith cell contains a star for each time the ith element of the set occurs in the combination.

For example, * * | * | | * | * * *

② Each r-combination of a set with n elements when repetition is allowed can be represented by a list of n-1 bars and r stars.

③ The number of such lists is $C(n-1+r, r)$.

Combinations with Repetition

〖Example 1〗 Suppose that a cake shop provides eight different kinds of cake. There are 12 cakes in one box. How many different boxes with cake?

$$H_8^{12}$$

Question:

How many different boxes with at least one of each kind?

$$H_8^{4}$$

〖Example 2〗How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 16$$

where $x_i (i = 1,2,3,4)$ **is nonnegative integer?**

Solution:

Since a solution of this equation corresponds to a way of selecting 16 items from a set with four element, such that x_1 items of type one, x_2 items of type two, x_3 items of type three, x_4 items of type four are chosen.

Hence the number of solutions is

$$H_4^{16} = C(4-1+16, 16) = C(19, 16)$$

$$= C(19, 3)$$

〖Example 3〗 **How many solutions are there to the equation**

$$x_1 + x_2 + x_3 + x_4 = 16$$

where $x_i (i = 1,2,3,4)$ **is nonnegative integer?**

Question:

(1) $x_i > 1, \text{ for } i = 1,2,3,4$ ⇨ $x_i \geq 2$

$$H_4^8 = C(4-1+8, 8) = C(11, 8) = C(11, 3)$$

(2) $x_1 + x_2 + x_3 + x_4 \leq 16$

We can introduce an auxiliary variable x_5 **so that**

$$x_1 + x_2 + x_3 + x_4 + x_5 = 16$$

$$H_5^{16} = C(5-1+16, 16) = C(20, 16) = C(20, 4)$$

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

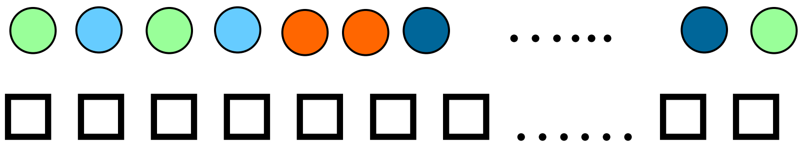
Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations of Sets With Indistinguishable Objects

n-Permutation with limited repetition

$$A = \{ n_1 \bullet a_1, n_2 \bullet a_2, \ldots, n_k \bullet a_k \}, \quad \text{where } n_1+n_2+\ldots+n_k = n$$

【 Theorem 3 】 The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1,…, and n_k indistinguishable objects of type k, is $n!/(n_1!n_2!\ldots n_k!)$



Proof:

$$C(n,n_1)\cdot C(n-n_1,n_2)\cdot\ldots\cdot C(n-n_1-n_2-\ldots-n_{k-1},n_k)$$

$$=\frac{n!}{n_1!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdot\cdots\cdot\frac{(n-n_1-n_2-\ldots-n_{k-1})!}{n_k!(n-n_1-n_2-\ldots-n_k)!}=\frac{n!}{n_1!n_2!\ldots n_k!}$$

〖Example 4〗There are 50 students in a class.

(1) How many ways to select 7 students to construct a leading group?

C(50,7)

(2) If two students are elected as a monitor and a vice monitor, then how many are there?

$$\frac{P(50,7)}{1!\,1!\,5!}$$

(3) If these 7 students are elected to have different tasks, then how many are there?

P(50,7)

〖Example 5〗

(1) How many bit strings of length 10？

2^{10}

(2) How many bit strings of length 10 are there that contain exactly two 0s, eight 1s?

10!/(2!8!)

〖Example 6〗 How many different strings can be made form the letters in MISSISSIPPI, using all the letters?

Solution:

$$A = \{1 \cdot M, 4 \cdot I, 4 \cdot S, 2 \cdot P\}$$

$$\frac{11!}{4!4!2!}$$

Distributing objects into boxes

◆ Many counting problems can be solved by counting the ways objects can be placed in boxes.

- The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).
- The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).

Distinguishable objects and distinguishable boxes

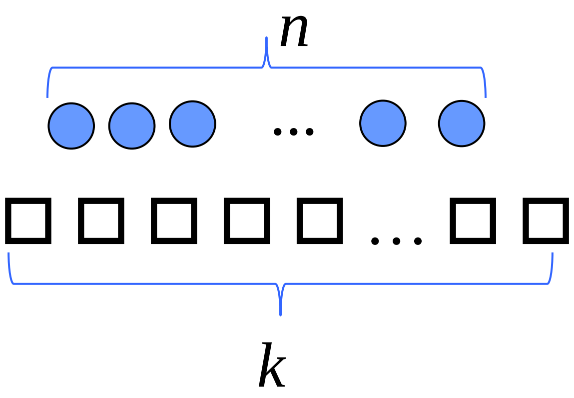
【Theorem 5】 The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are place into box i, i=1,2,…,k, equals

$$n!/(n_1!n_2!\ldots n_k!)$$

Question:

Why both the ways in theorem 3 and 5 are the same?

Analysis:



【Example 8】How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Solution:

It is typical problem that involves distributing distinguishable objects into distinguishable boxes.

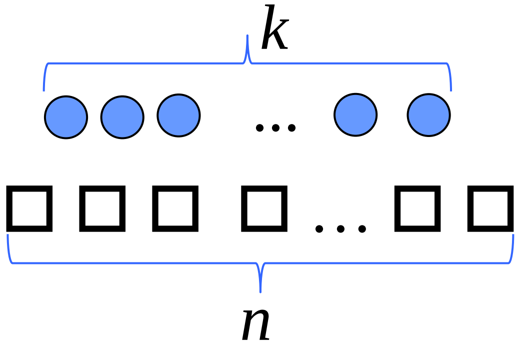
- The distinguishable objects are the 52 cards.
- The five distinguishable boxes are the hands of the four players and the rest of the deck.

$$\frac{52!}{5!\ 5!\ 5!\ 5!\ 32!}$$

Indistinguishable objects and distinguishable boxes

There are $C(n + k - 1, n - 1)$ ways to place k indistinguishable objects into n distinguishable boxes.

Proof based on one-to-one correspondence between *k*-combinations from a set with *n*-elements when repetition is allowed and the ways to place *k* indistinguishable objects into *n* distinguishable boxes.



Example: There are $C(8 + 10 - 1, 10)$ = C(17,10) = 19,448 ways to place 10 indistinguishable objects into 8 distinguishable boxes.

Distinguishable objects and indistinguishable boxes

- ✓ **counting the ways to place *n* distinguishable objects into *k* indistinguishable boxes**

〖Example 9〗How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

Solution: **We represent the four employees by A,B,C,D.**

(1) All four are put into one office: 1 ways

{A,B,C,D}

(2) Three are put into one office and a fourth is put into a second office: 4 ways

{{A,B,C},{D}}; {{A,B,D}, {C}}; {{A,C,D}, {B}}; {{B,C,D}, {A}}

(3) Two are put into one office and two put into a second office:3 ways

{{A,B},{C,D}}; {{A,D},{B,C}}; {{A,C},{B,D}}

(4) Two are put into one office, and one each put into the other two office:6 ways

{{A,B},{C},{D}}; {{A,C},{B},{D}}; {{A,D},{B},{C}}; {{B,C},{A},{D}};
{{B,D},{A},{C}}; {{C,D},{A},{B}}

Therefore， there are 1+4+3+6=14 ways to put four different employees into three indistinguishable offices.

〖Example 9〗How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

Solution: **Another way:**

Look at the number of offices into which we put employees.

1) There are 6 ways to put four different employees into three indistinguishable offices so that no office is empty.
2) There are 7 ways to put four different employees into two indistinguishable offices so that no office is empty.
3) There are 1 ways to put four different employees into one offices so that it is not empty.

Problem: distribute n distinguishable objects into j indistinguishable boxes so that no boxes is empty

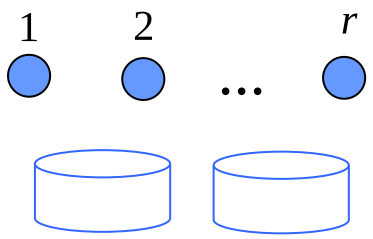
Stirling numbers of the second kind

Notation: $S(n,j)$

the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no boxes is empty

(1) $S(r,1)=S(r,r)=1 \quad (r\geq 1)$

(2) $S(r,2)=2^{r-1}-1$



(3) $S(r,r-1)=C(r,2)$

(4) $S(r+1,n)=S(r,n-1)+nS(r,n)$

Note:

1. *S(n,j)* is the number of ways to partition the set with *n* elements into *j* nonempty and disjoint subsets.
2. *S(n,j)j!* is the number of ways to distribute *n* distinguishable objects into *j* distinguishable boxes so that no boxes is empty
   - the number of onto functions from a set with *n* elements to a set with *j* elements

$$S(n,j)\,j! = \left(\sum_{i=0}^{j-1} (-1)^i C_j^i (j-i)^n\right)$$

3. the number of ways to place n distinguishable objects into k indistinguishable boxes

$$\sum_{j=1}^{k} S(n,j) = \sum_{j=1}^{k} \left(\left(\sum_{i=0}^{j-1} (-1)^i C_j^i (j-i)^n\right) / j!\right)$$

〖Example 9〗 How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

Solution: **Another way:**

Look at the number of offices into which we put employees.

1) **There are 6 ways to put four different employees into three indistinguishable offices so that no office is empty.**
2) **There are 7 ways to put four different employees into two indistinguishable offices so that no office is empty.**
3) **There are 1 ways to put four different employees into one offices so that it is not empty.**

$$S(4,3)+S(4,2)+S(4,1)=6+7+1=14$$

Indistinguishable objects and indistinguishable boxes

- ✓ counting the ways to distribute indistinguishable objects into indistinguishable boxes

〖Example 10〗How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

Solution:

We can enumerate all ways to pack the books. For each ways to pack the books, we will list the number of books in the box with the largest of books, followed by the number of books in each box containing at least one book, in order of decreasing number of books in a box.

The ways we can pack the books are

6

5,1　　4,2　　4,1,1

3,3　　3,2,1　　3,1,1,1

2,2,2　　2,2,1,1

- The number of ways of distributing n indistinguishable objects into k indistinguishable boxes equals $p_k(n)$, the number of ways to write n as the sum of at most k positive integers in increasing order.
- No simple closed formula exists for this number.

6.6 Generating Permutations and Combinations

Generating Permutations

Suppose that a salesperson must visit six cities. In which order should these cities be visited to minimize total travel time?

Problem: List the permutations of any set of *n* elements.

How?

- ✓ **Any set with *n* elements can be placed in one-to-one correspondence with the set {1, 2, …, *n*}**
- ✓ **Generate the permutation of the *n* smallest positive integers, and then replace these integers with the corresponding elements.**

Introduction: lexicographic (or dictionary) ordering for permutation

What is lexicographic ordering for Permutations

The permutation $a_1a_2\ldots a_n$ precedes the permutation of $b_1b_2\ldots b_n$, if for some k, with $1 \leq k \leq n$, $a_1 = b_1$, $a_2 = b_2$, …, $a_{k-1} = b_{k-1}$, and $a_k < b_k$.

For example:

123465 precedes 124635

Algorithm for Generating Permutations

Algorithm of producing the $n!$ permutations of the integers 1, 2, …, n

- **Begin with the smallest permutation in lexicographic order, namely 1, 2, 3,4,…, n .**
- **Produce the next largest permutation.**
- **Continue until all $n!$ permutations have been found.**

Generating the next largest Permutations

Given permutation $a_1a_2 \ldots a_n$, find the next largest permutation in increasing order:

(1) Find the integers

$$a_j, a_{j+1} \text{ with } \; a_j < a_{j+1} \text{ and } \; a_{j+1} > a_{j+2} > \ldots > a_n$$

(2) Put in the *j*th position the least integer among

$a_{j+1}, a_{j+2}, \ldots, a_n$ **that is greater than** a_j

(3) List in increasing order the rest of the integers

$$a_j, a_{j+1}, \ldots, a_n$$

Question: This algorithm produce the next largest Permutation in lexicographic order?

- What is the next largest permutation in lexicographic order after 124653?

The next largest permutation of 124653 in lexicographic order is 125346

- Generate the permutation of the integers 1, 2, 3 in lexicographic order.

123 → 132 → 213 → 231 → 312 → 321

Question: The algorithm can produce all Permutations in lexicographic order?

Generating Combinations

Problem 1:

Generate all combinations of the elements of a finite set .

How?

- ✓ **A combination is just a subset. ⇒ We need to list all subsets of the finite set.**
- ✓ **Use bit strings of length n to represent a subset of a set with n elements. ⇒ We need to list all bit strings of length n.**
- ✓ **The 2^n bit strings can be listed in order of their increasing size as integers in their binary expansions.**

Algorithm of Producing All Bit Strings of length n

- ❖ **Start with the bit string 000…00,with *n* zeros.**
- ❖ **Then, successively find the next largest expansion until the bit string 111…11 is obtained.**

The method to find the next largest binary expansion:

Locate the first position from the right that is not a 1, then changing all the 1s to the right of this position to 0s and making this first 0 a 1.

For example:

1000110011 → 1000110100

Problem 2:

Generate all *r*-combinations of the set {1, 2, …, *n*}

The algorithm for generating the *r*-combination of the set {1, 2, …, *n*}

(1) $S_1 = \{1, 2, \ldots, r\}$

(2) If $S_i = \{a_1, a_2, \ldots, a_r\}, 1 \le i \le C_n^r - 1$ has found, then the next combination can be obtained using the following rules.

First, locate the last element a_i in the sequence such that $a_i \ne n - r + i$. Then replace a_i with $a_i + 1$ and a_j with $a_i + j - i + 1$, for $j = i+1, i+2, \ldots, r$.

- S_i ={2,3,5,6,9,10} is given. Find S_{i+1} .

$$S_{i+1} = \{2,3,5,7,8,9\}$$

- List all the 2-combination of {1,2,3,4,5}?

$\{1, 2\} \longrightarrow \{1, 3\} \longrightarrow \{1, 4\} \longrightarrow$

$\{1, 5\} \longrightarrow \{2, 3\} \longrightarrow \{2, 4\} \longrightarrow$

$\{2, 5\} \longrightarrow \{3, 4\} \longrightarrow \{3, 5\} \longrightarrow$

$\{4, 5\}$

Homework:

SE: P. 432 10, 16, 32, 42, 48, 50, 63

EE: P. 454 10, 16, 34, 42, 50,52， 65