Chapter 11 Trees

Chapter Summary

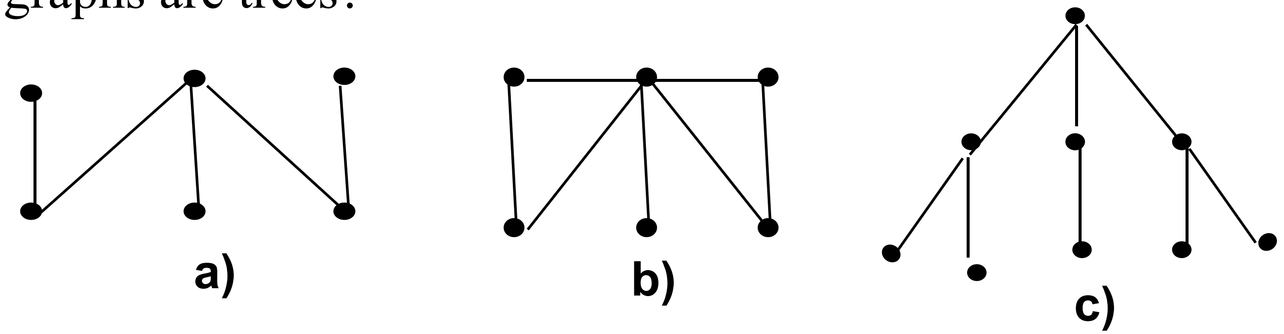
- Introduction to Trees
- Applications of Trees
- Tree Traversal
- Spanning Trees
- Minimum Spanning Trees

11.1 Introduction to Trees

Tree

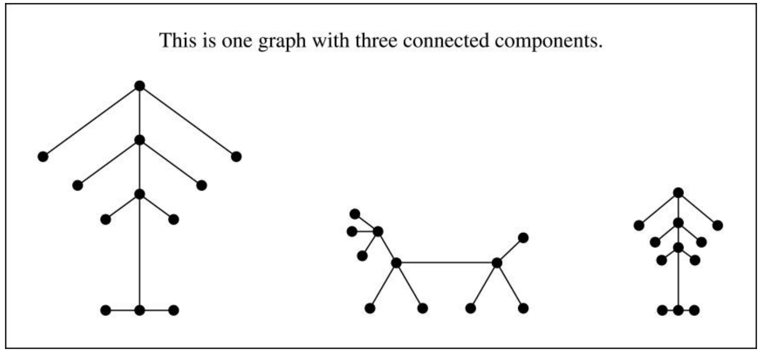
Definition: A *tree* is a connected undirected graph with no simple circuits.

〖Example 1〗 Which graphs are trees?



Note: Any tree must be a simple graph.

Definition: A *forest* is a graph that has no simple circuit, but is not connected. Each of the connected components in a forest is a tree.



【 Theorem 1 】 An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.

Proof:

(1) $\Rightarrow$

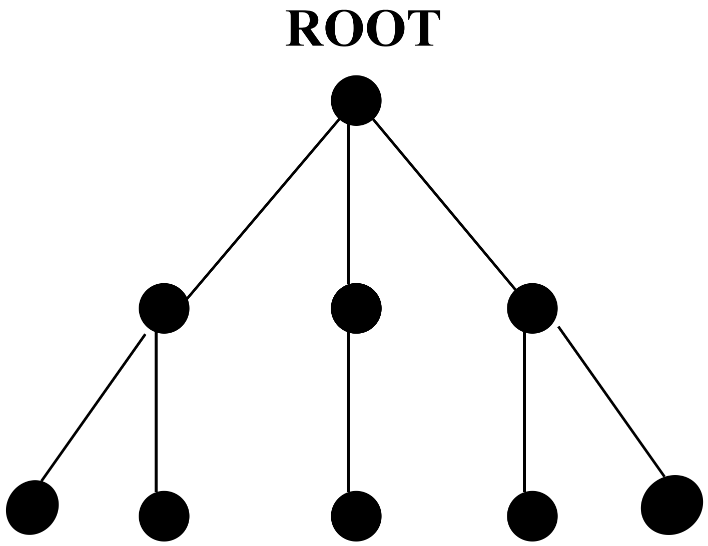
- ✓ **there is a simple path between any two of its vertices**
- ✓ **unique**

(2) $\Leftarrow$

- ✓ **connected**
- ✓ **no simple circuits**

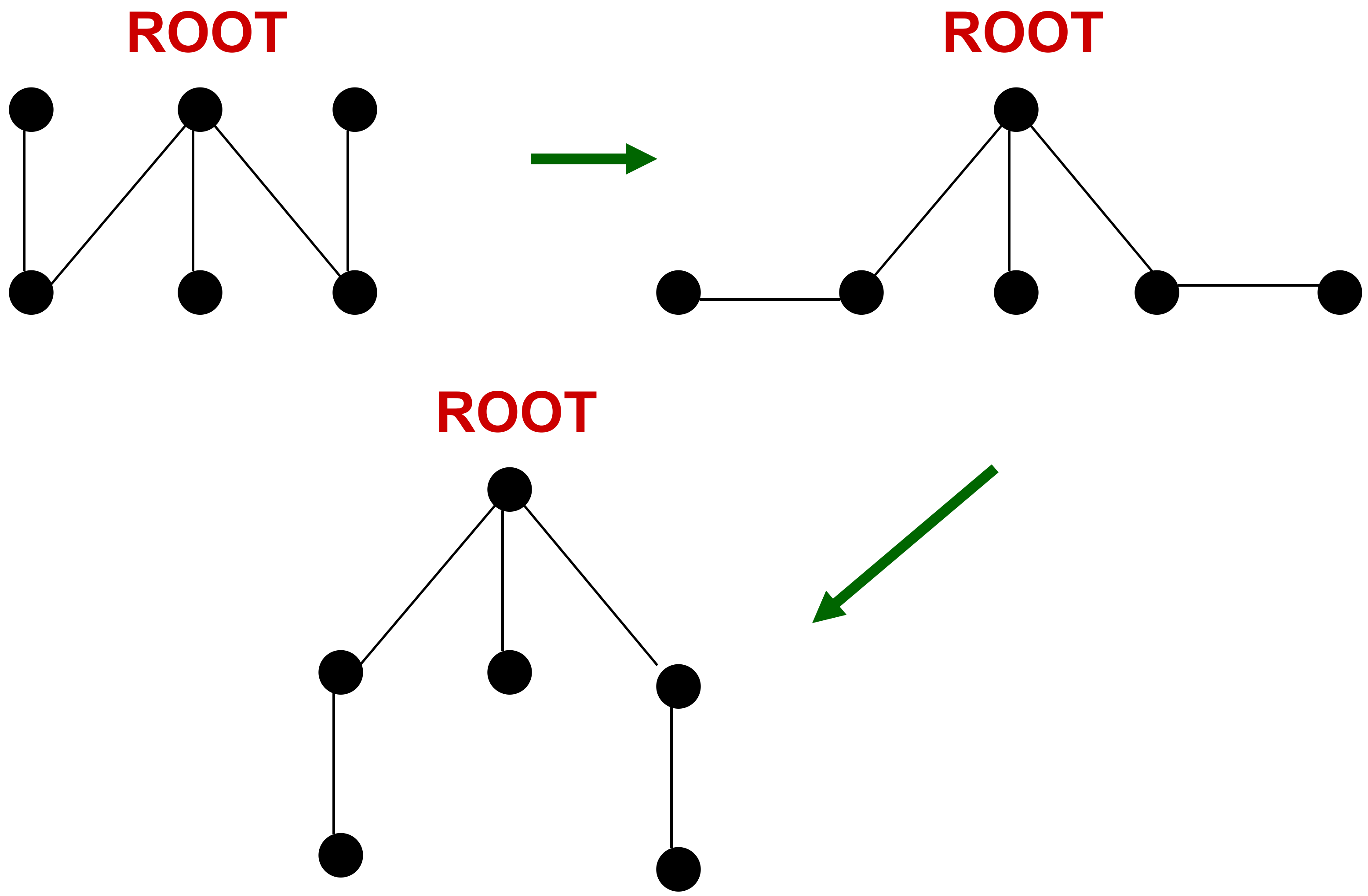
Root tree

Definition: A *rooted tree* is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

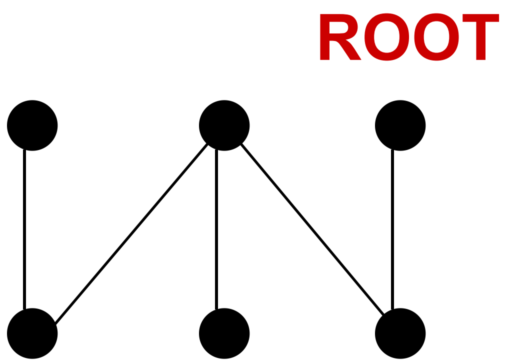


An unrooted tree is converted into different rooted trees when different vertices are chosen as the root.

〖Example 2〗Change an unrooted tree into a rooted tree.



Problem: **What if a different root is chosen?**



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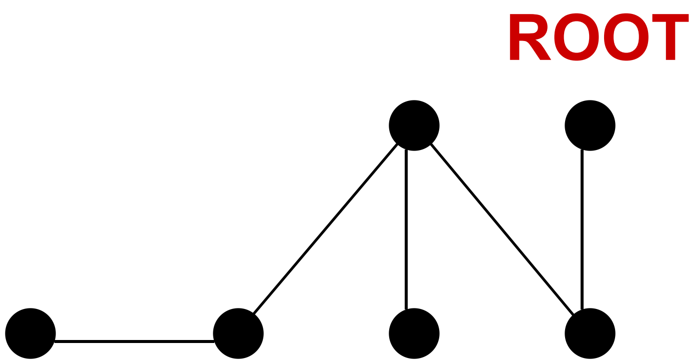

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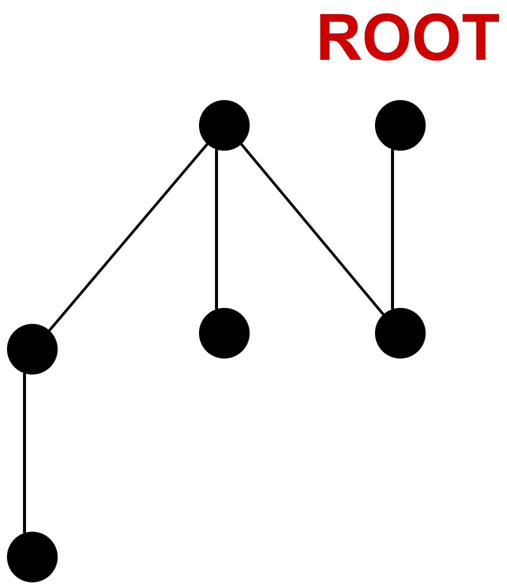

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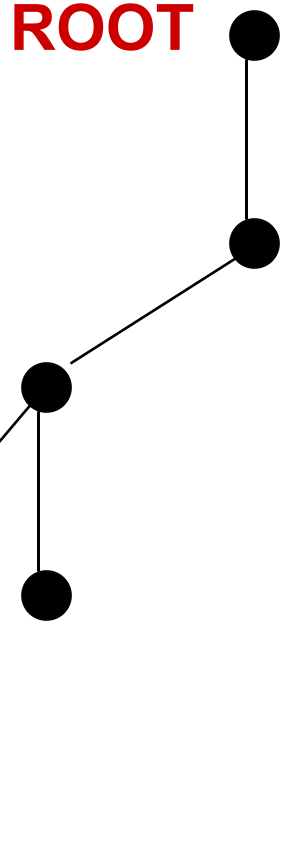
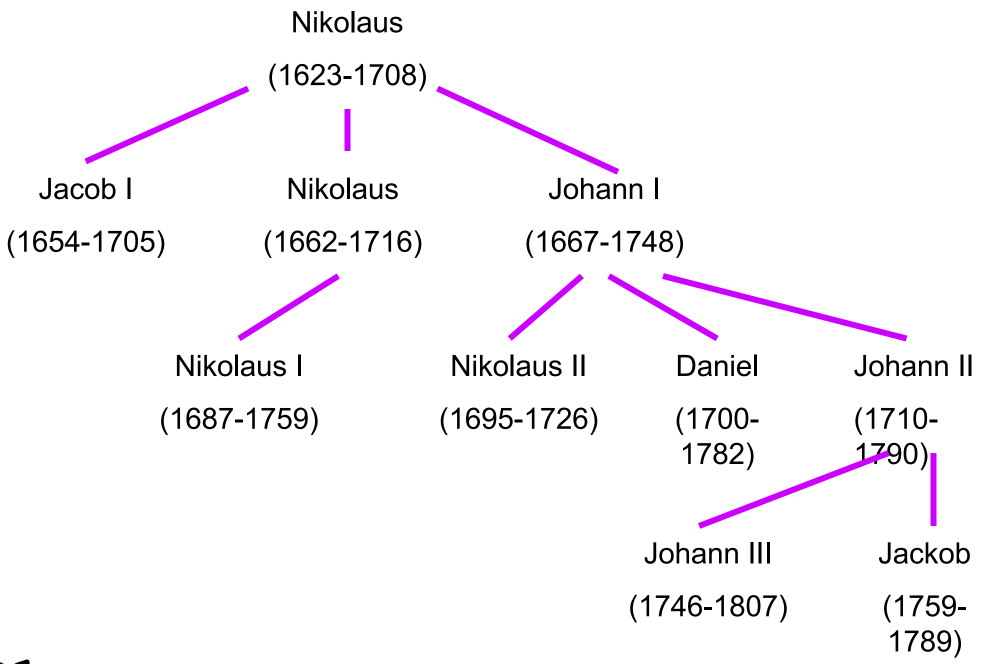


Rooted Tree Terminology

Terminology for rooted trees is a mix from botany and genealogy (such as this family tree of the Bernoulli family of mathematicians).

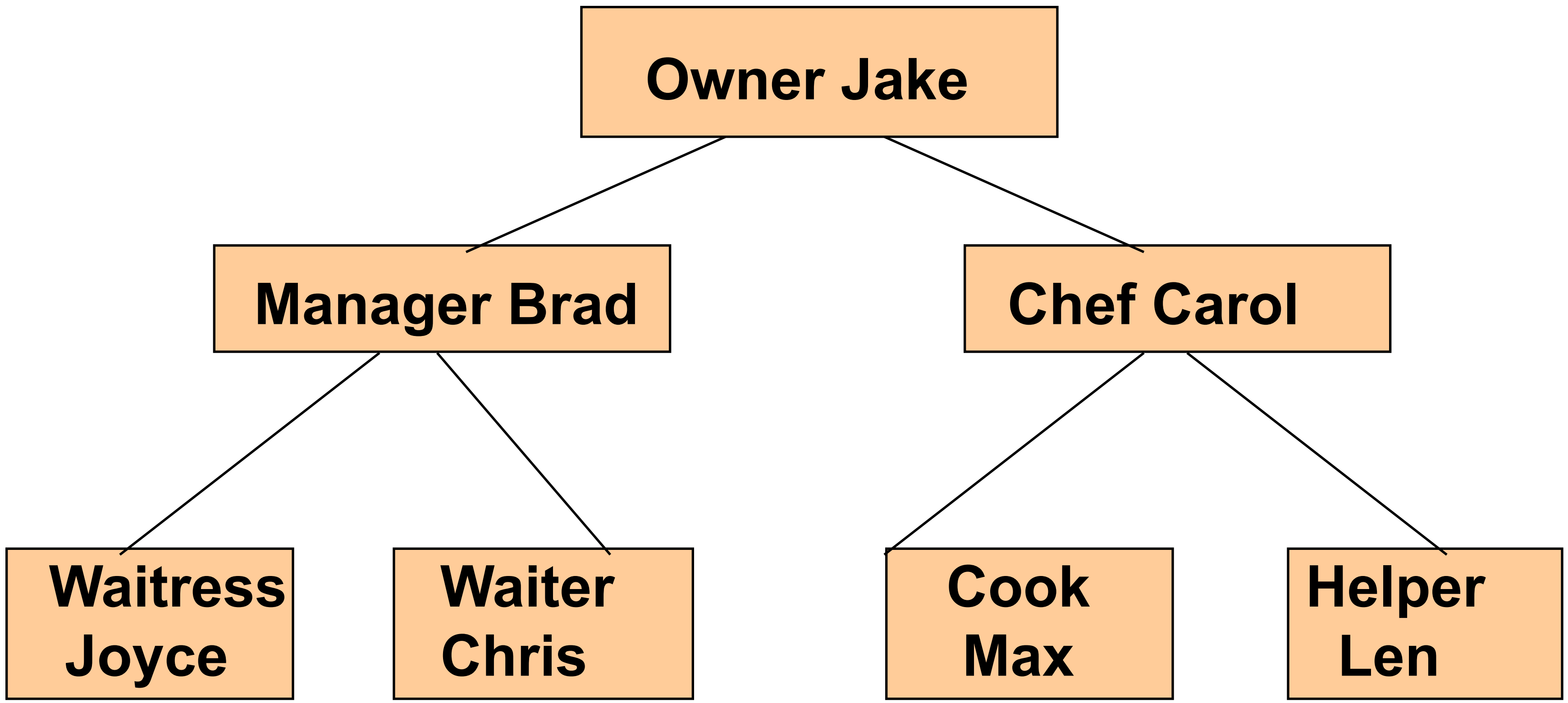
Rooted Tree Terminologies

- Parents VS. Children
- Siblings
- Ancestor VS. Descendants
- Root, leaf, and internal vertices
- Subtrees



The Bernoulli family of mathematicians

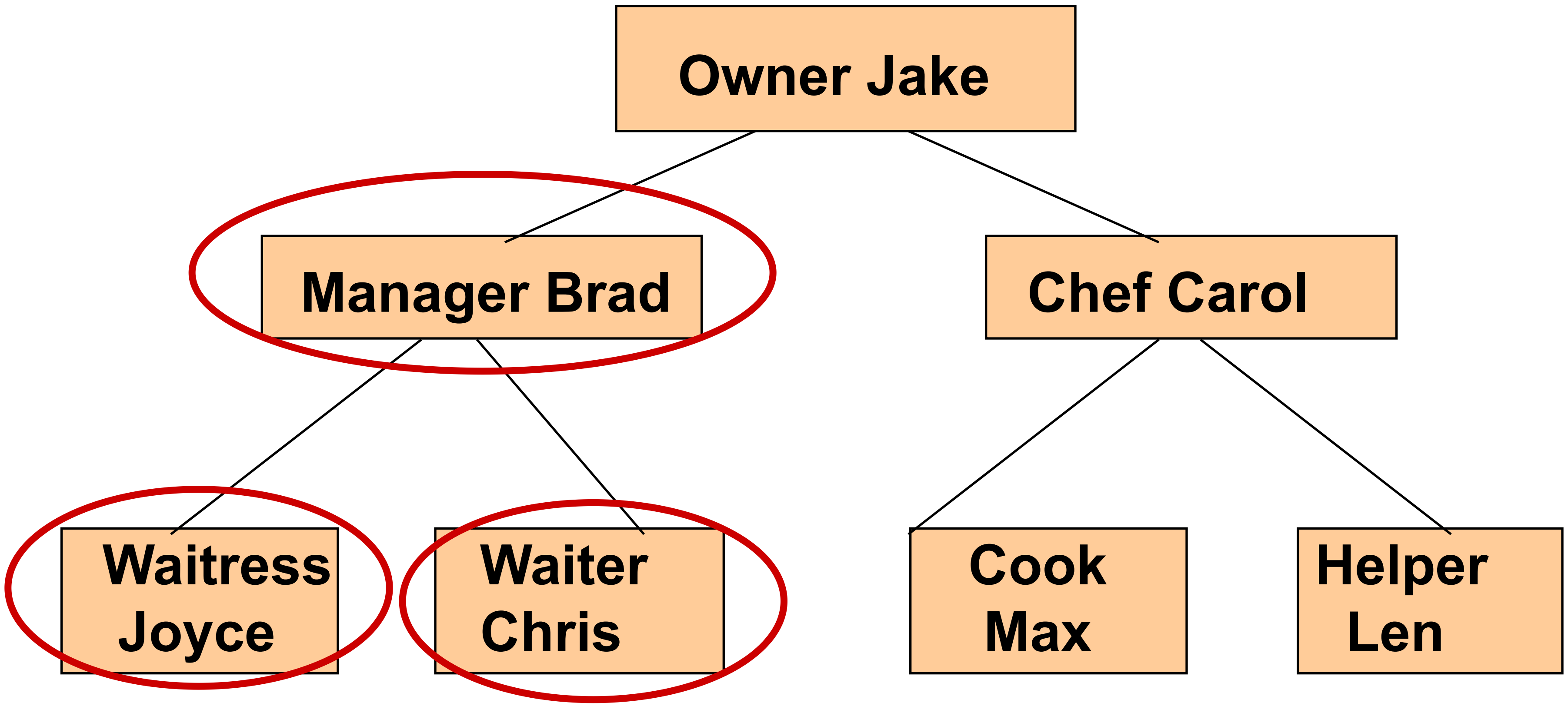
An Example: Jake's Pizza Shop Tree



Parent VS. Child

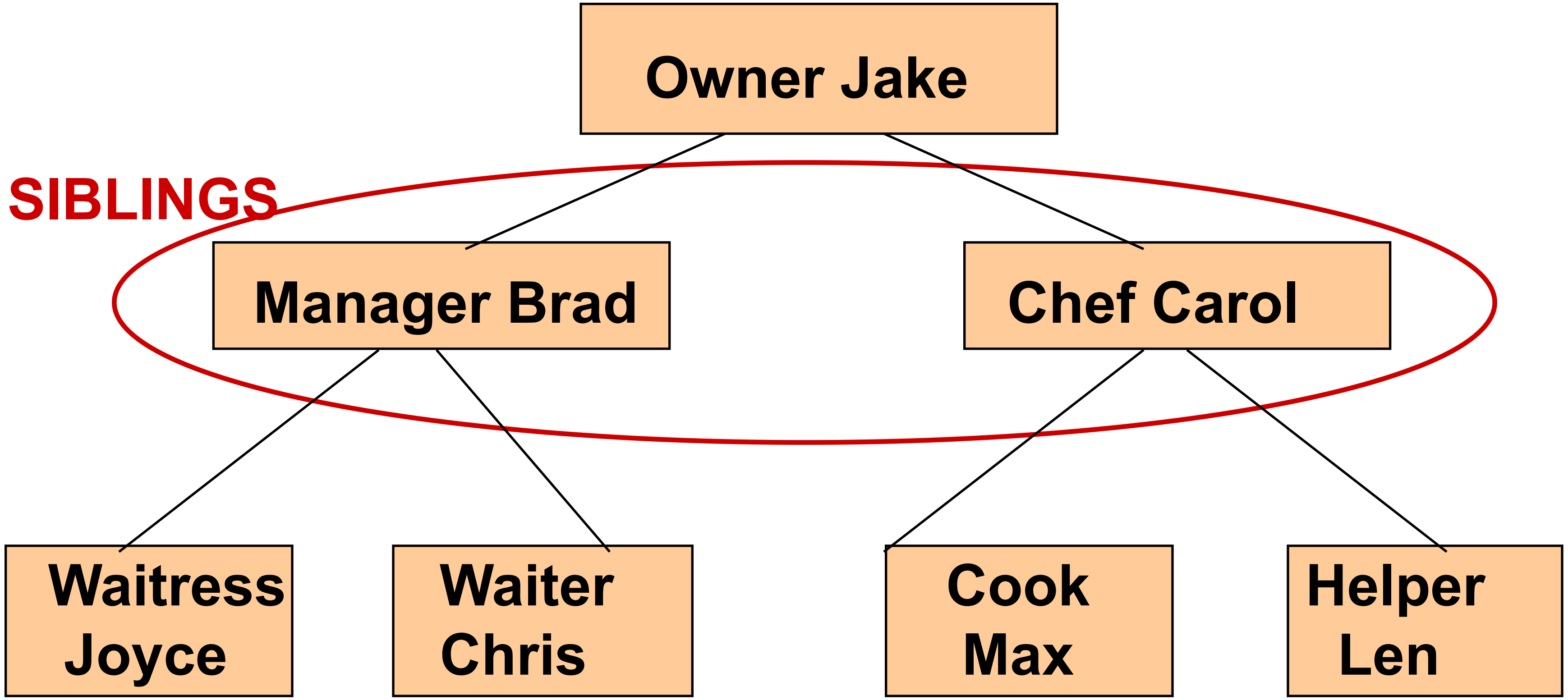
The parent of a non-root vertex v is the unique vertex u with a directed edge from u to v.

When *u* is the parent of *v*, *v* is called a *child* of *u*.



Sibling

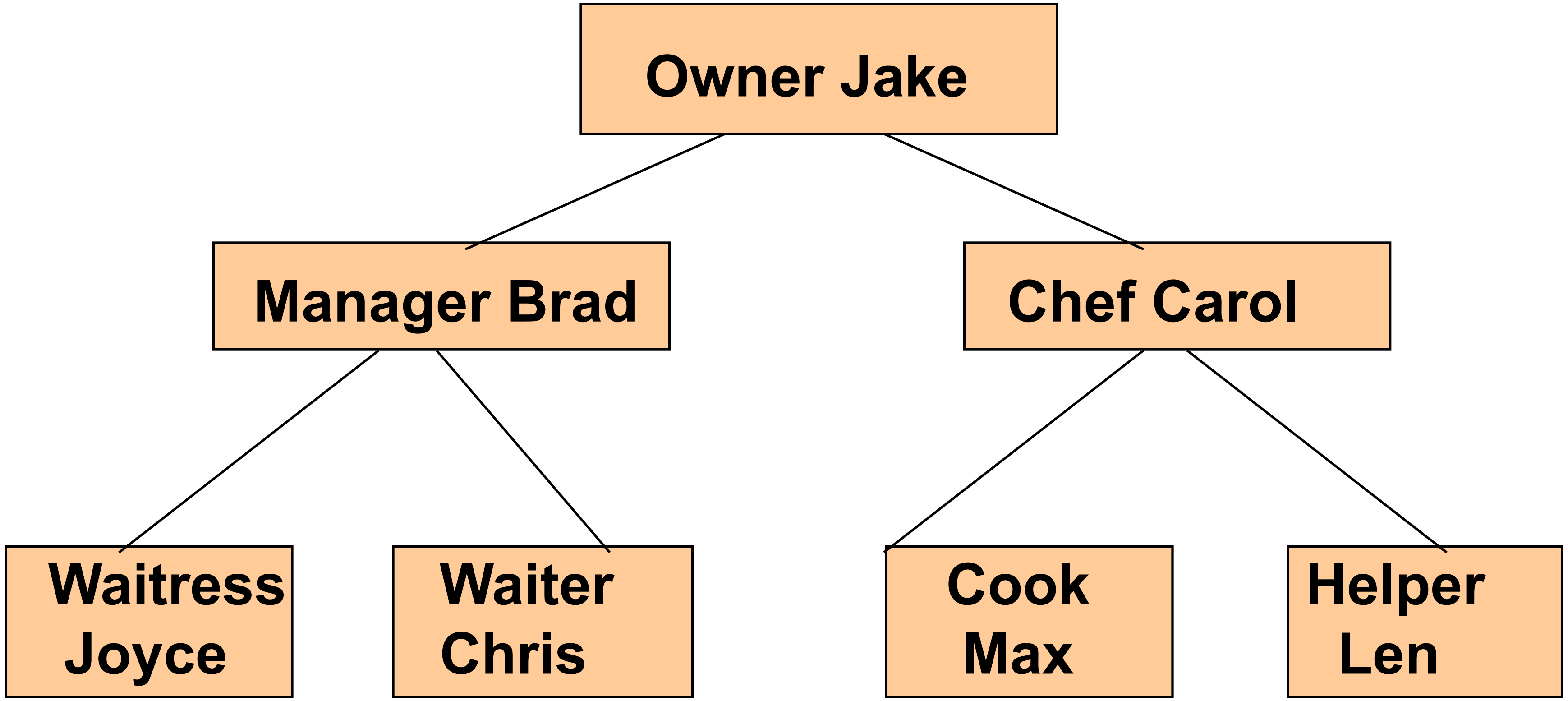
Vertices with the same parent are called siblings .



Ancestors VS. Descendants

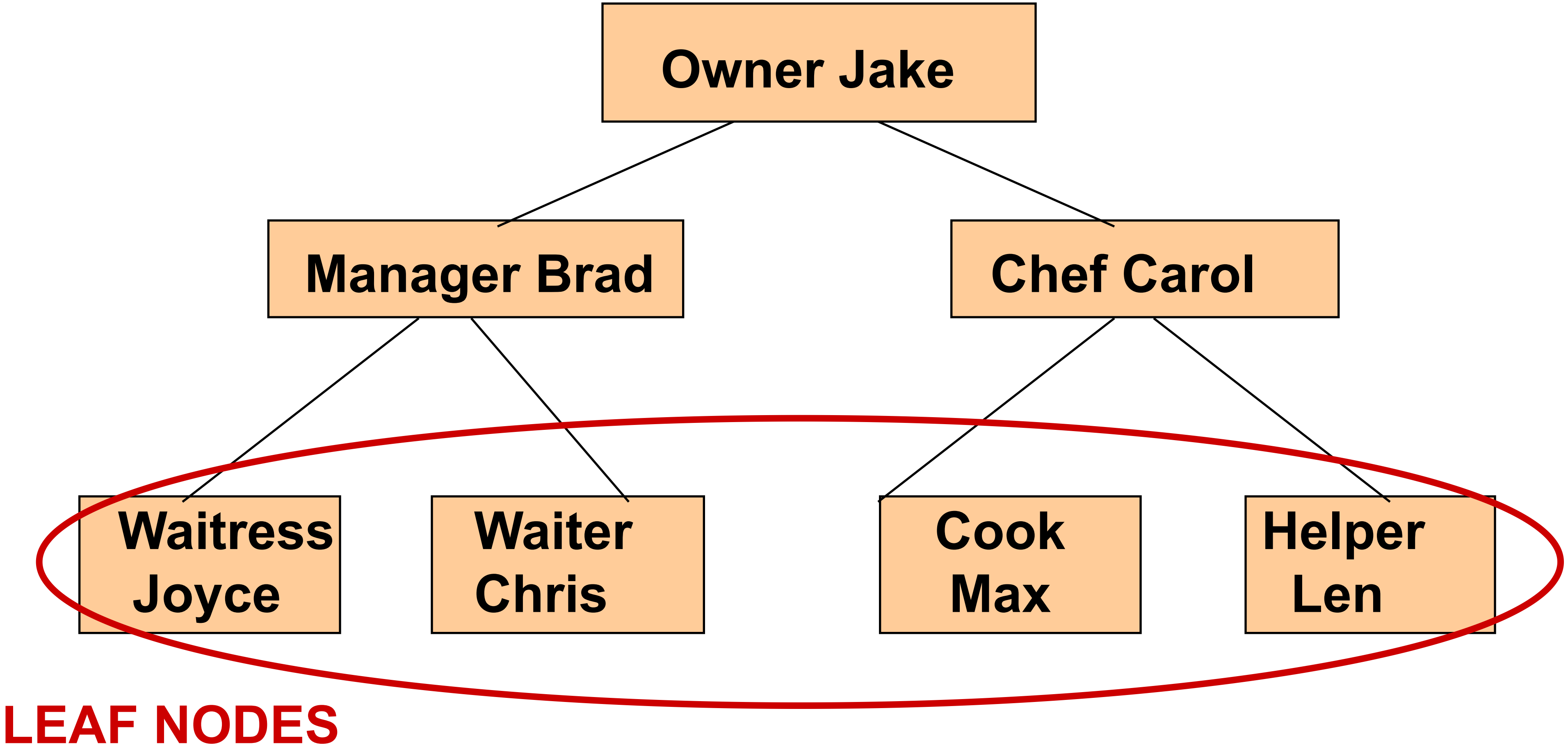
The ancestors of a non-root vertex are all the vertices in the path from root to this vertex.

The descendants of vertex v are all the vertices that have v as an ancestor.



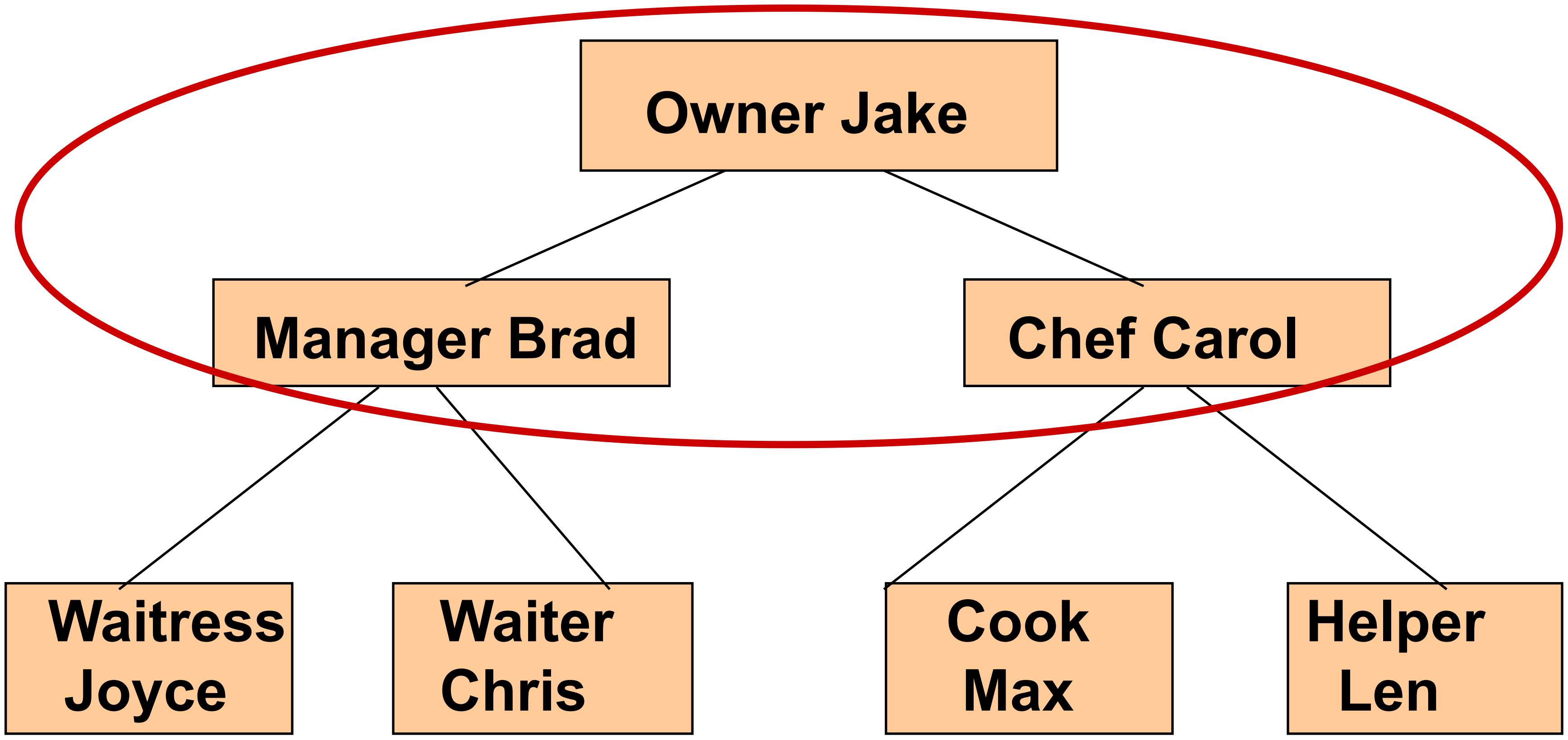
Leaf

A vertex is called a **leaf** if it has no children.



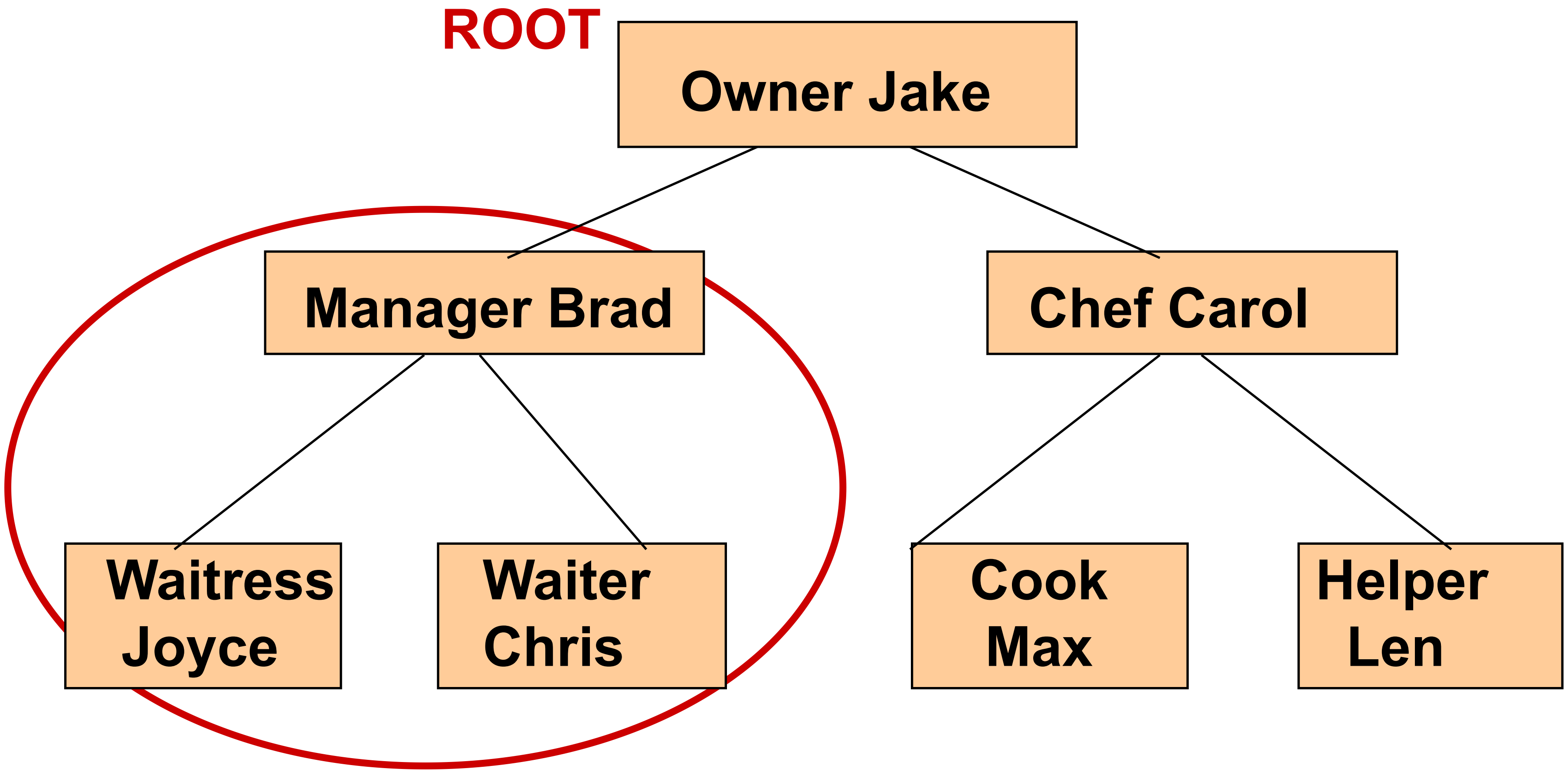
Internal Vertex

A vertex that have children is called an **internal vertex**.



Subtree

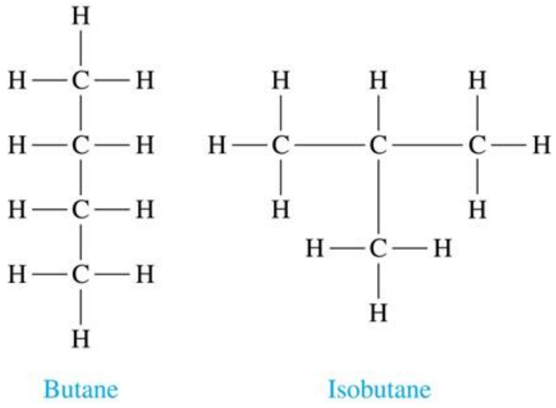
The subtree at vertex v is the subgraph of the tree consisting of vertex v and its descendants and all edges incident to those descendants.

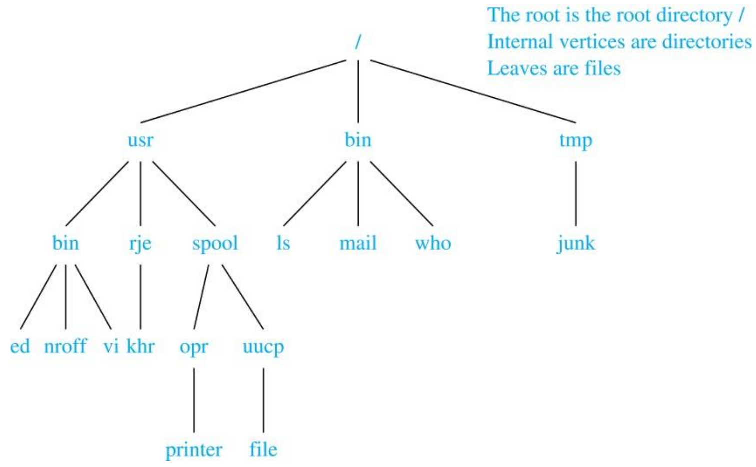


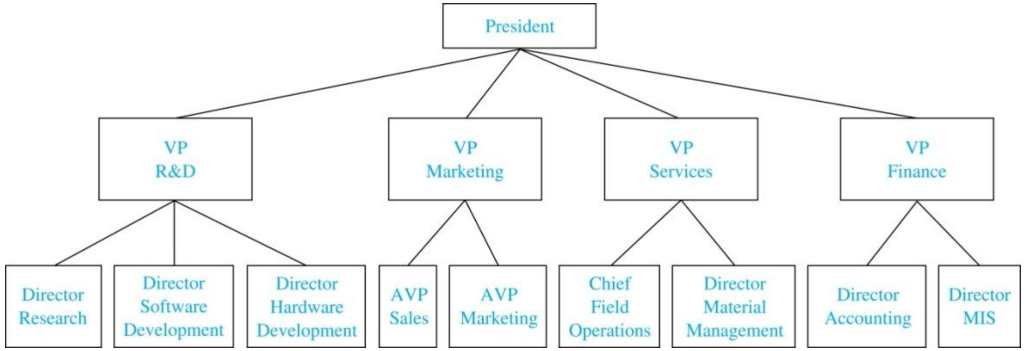
Trees as Models

Trees are used as models in computer science, chemistry, geology, botany, psychology, and many other areas.

- **Trees were introduced by the mathematician Cayley in 1857 in his work counting the number of isomers of saturated hydrocarbons. The two isomers of butane are shown at the right.**
- **The organization of a computer file system into directories, subdirectories, and files is naturally represented as a tree.**
- **Trees are used to represent the structure of organizations.**





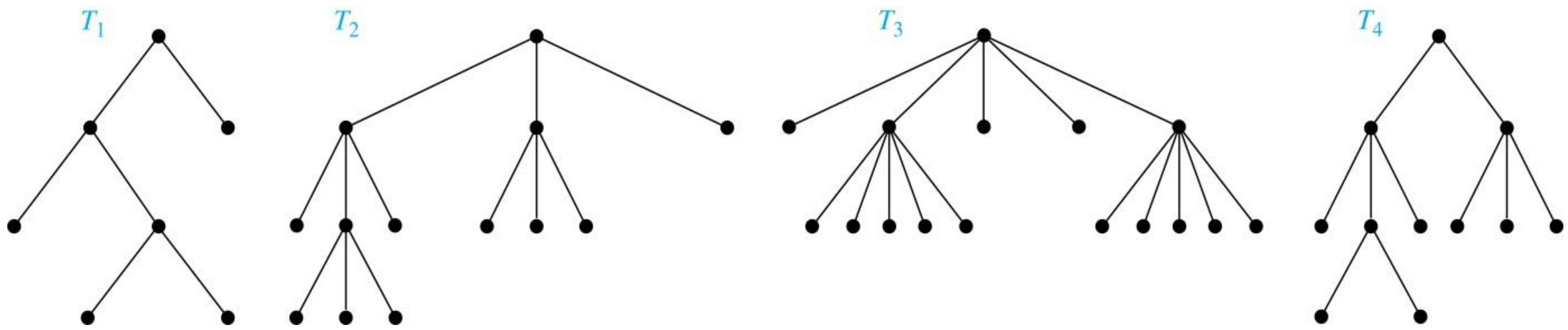


m-ary Rooted Trees

Definition : A rooted tree is called a **m-ary tree** if every internal vertex has no more than m children.

It is a **binary tree** if m = 2.

The tree is called a **full m-ary tree** if every internal vertex has exactly m children.



Ordered rooted tree

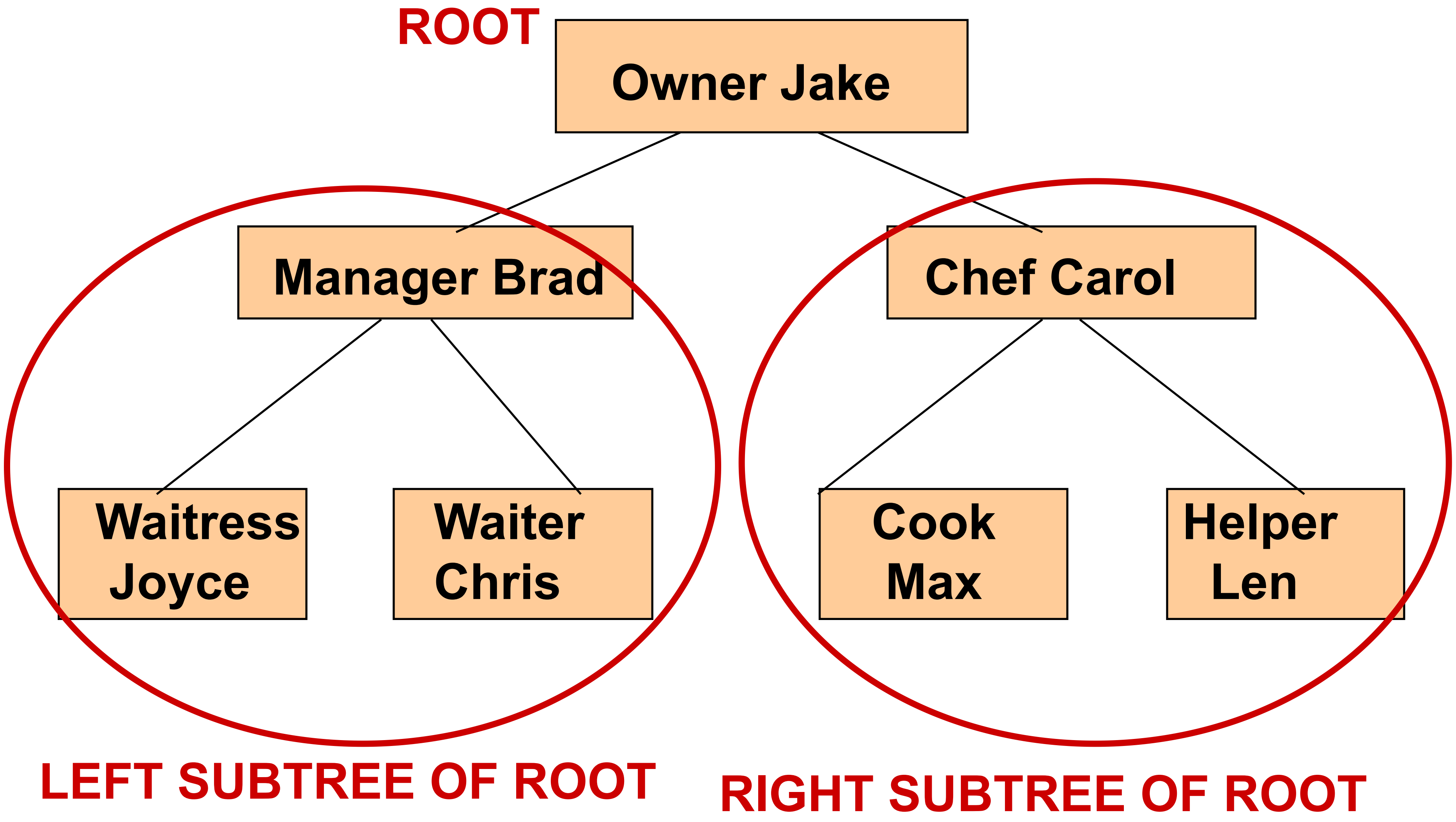
Definition: An ordered rooted tree is a rooted tree where the children of each internal vertex are ordered.

- We draw ordered rooted trees so that the children of each internal vertex are shown in order from left to right.

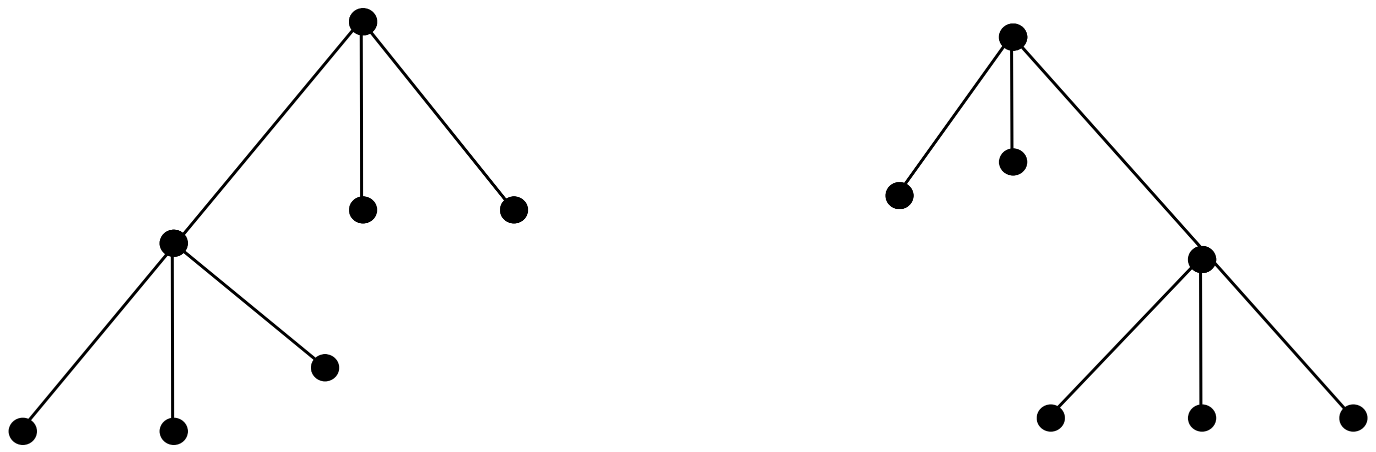
In an ordered binary tree, the two possible children of a vertex are called the left child and the right child, if they exist.

The tree rooted at the left child is called the left subtree, and that rooted at the right child is called the right subtree.

Left Subtree and right subtree



IS these two trees isomorphic?



Properties of Trees

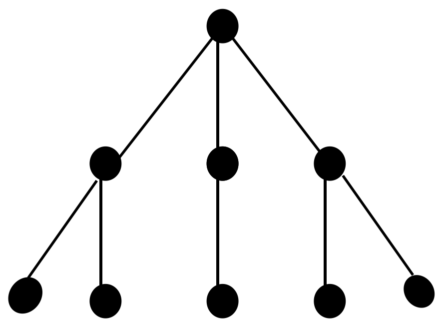
【 Theorem 2】 A tree with n vertices has n-1 edges.

Proof (1):

Choose the vertex r as the root of the tree.

We set up a one-to-one correspondence between the edges and the vertices other than r by associating the terminal vertex of an edge to that edge.

For example,



Since there are n-1 vertices other than r, there are n-1 edges in the tree.

Properties of Trees

【 Theorem 2】 **A tree with *n* vertices has *n*-1 edges.**

Proof (2):

$$T=(V,E),|V|=n,|E|=e \qquad \Rightarrow \qquad e=n-1$$

Any tree must be planar and connected. Then

$$r=e-n+2$$

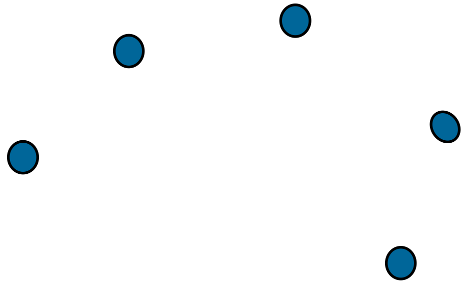
Any tree have no circuits. Then

$$r=1$$

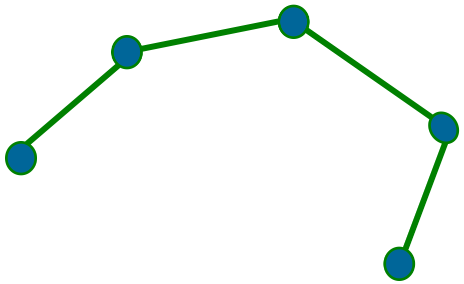
It follows that, $e=n-1$

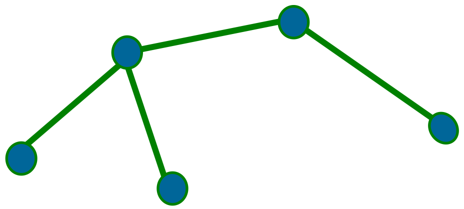
〖Example 3〗(1) How many nonisomorphic unrooted trees are there with n vertices if n=5 ?

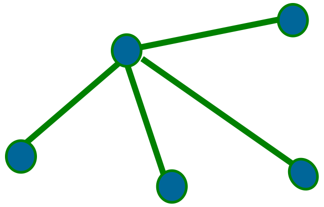
Solution:



A tree must be connected and have no simple circuits, and have 4 edges.

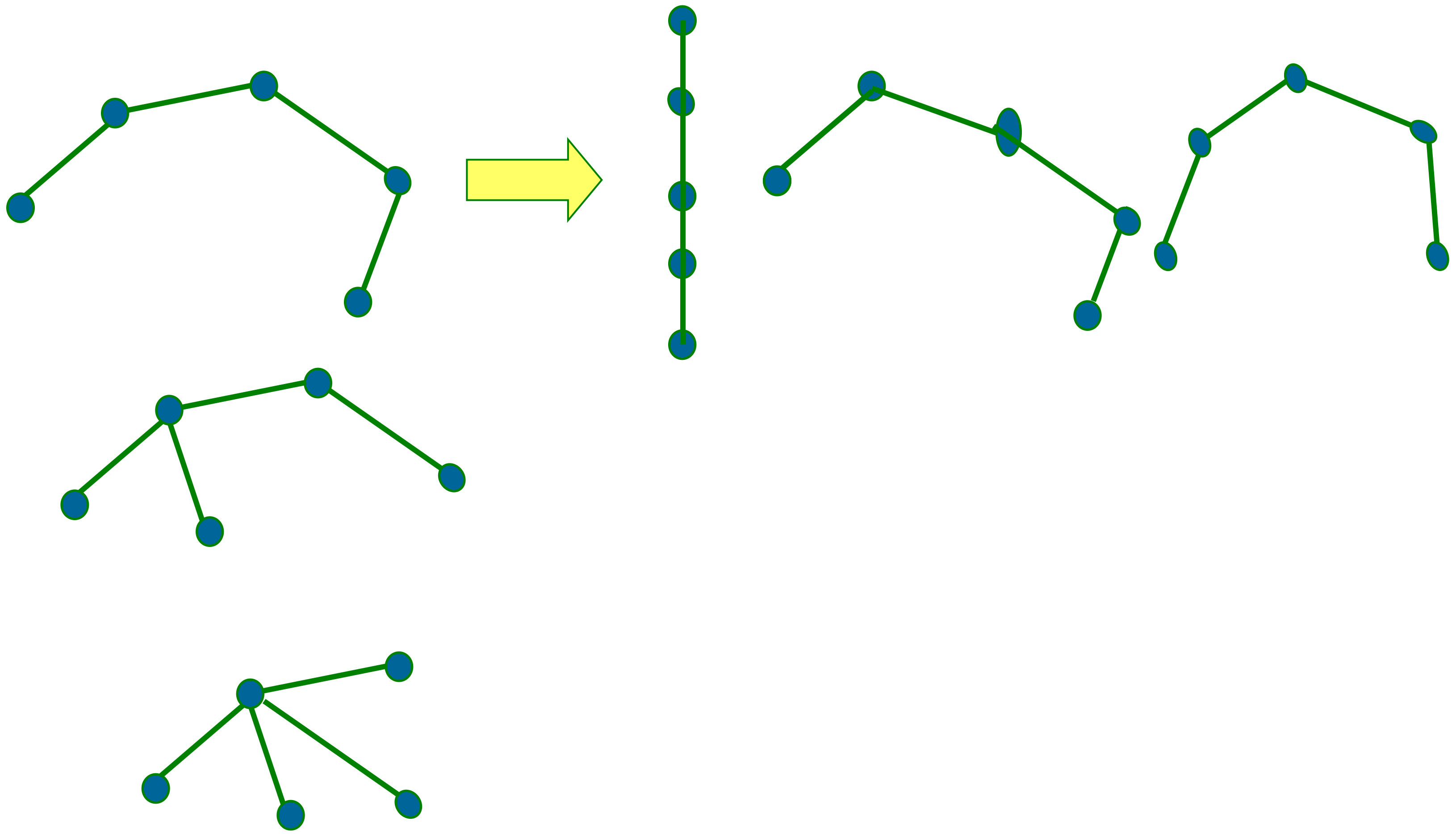






〖Example 3〗(2) How many **nonisomorphic rooted trees** are there with n vertices if n=5 ?

Solution:



〖Example 4〗A tree has two vertices of degree 2, one vertex of degree 3, three vertices of degree 4. How many leafs does this tree has?

Solution:

Suppose that there are x leafs.

$$v = 2+1+3+x$$

$$e = \frac{1}{2}\,(2\times2 + 1\times3 + 3\times4 + x \times1) = v - 1$$

$$x = 9$$

Question:

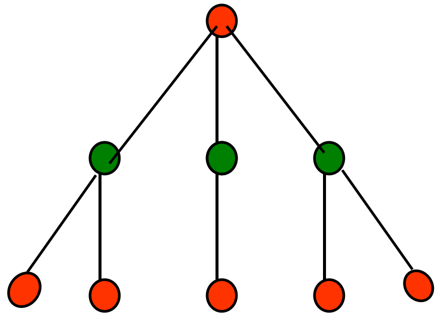
Every tree is a bipartite?

Yes.

Every tree can be colored using two colors.

Method:

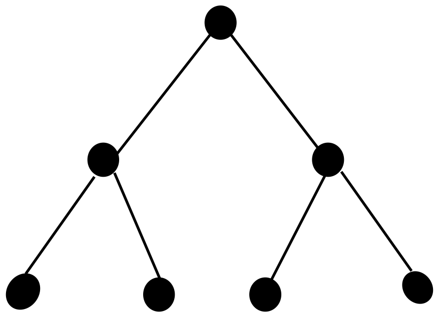
We choose a root and color it red. Then we color all the vertices at odd levels blue and all the vertices at even levels red.



Counting Vertices in Full m-Ary Trees

【 Theorem 3】 A full m-ary tree with i internal vertices contains $n=mi+1$ vertices.

Proof :



Every vertex, except the root, is the child of an internal vertex.

Since each of the i internal vertices has m children, there are mi vertices in the tree other than the root.

Therefore, the tree contains $n=mi+1$ vertices.

Counting Vertices in Full m-Ary Trees (continued)

【Theorem 4】 A full m-ary tree with

- n vertices has $i=(n-1)/m$ internal vertices and $l=[(m-1)n+1]/m$ leaves
- i internal vertices has $n=mi+1$ vertices and $l=(m-1)i+1$ leaves
- l leaves has $n=(ml-1)/(m-1)$ vertices and $i=(l-1)/(m-1)$ internal vertices

Proof:

$$n = mi + 1$$
$$n = i + l$$

Note:

For a full binary tree, $l = i + 1$, $e = v - 1$.

〖Example 5〗 A *chain letter* starts when a person sends a letter to five others. Each person who receives the letter either sends it to five other people who have never received it or does not send it to anyone. Suppose that 10000 person send out the letter before the chain ends and that no one receives more then one letter. How many people receive the letter, and how many do not send it out?

Solution:

The chain letter can be represented using a full 5-ary tree.

$$i = 10000$$

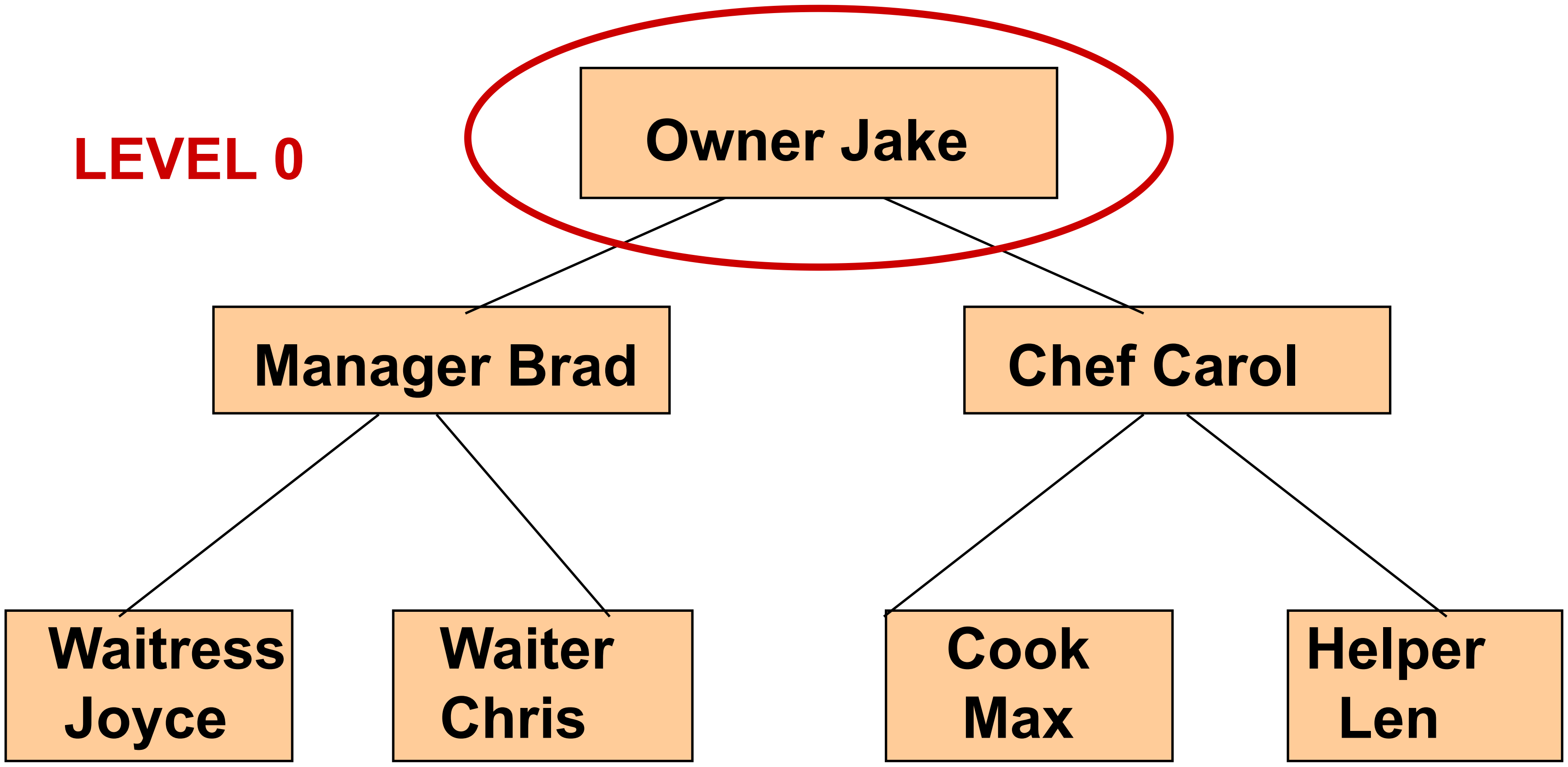
$$n = 5i + 1$$
$$n = i + l$$

$$l = 40001$$
$$n\text{-}1 = 50000$$

Level of vertices and height of trees

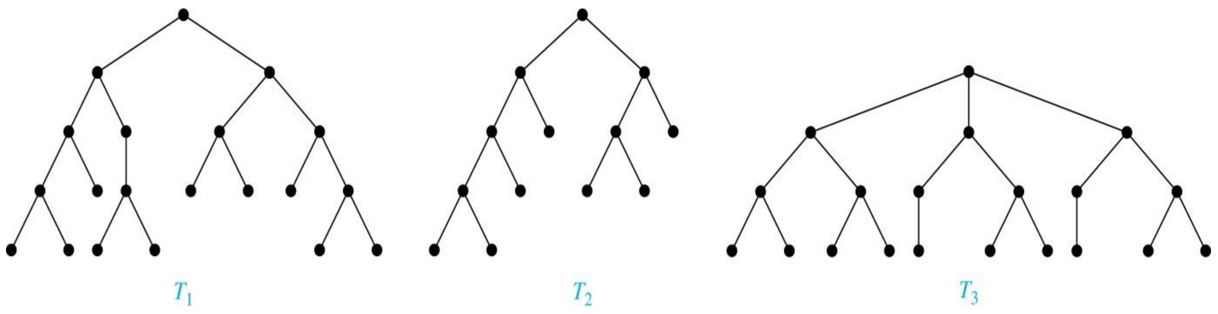
The level of vertex v in a rooted tree is the length of the unique path from the root to v.

The height of a rooted tree is the maximum of the levels of its vertices.



Balanced m-Ary Trees

A rooted m-ary tree of height h is called balanced if all its leaves are at levels h or h-1.



T_1 and T_3 are balanced.
T_2 is not because it has leaves at levels 2, 3, and 4.

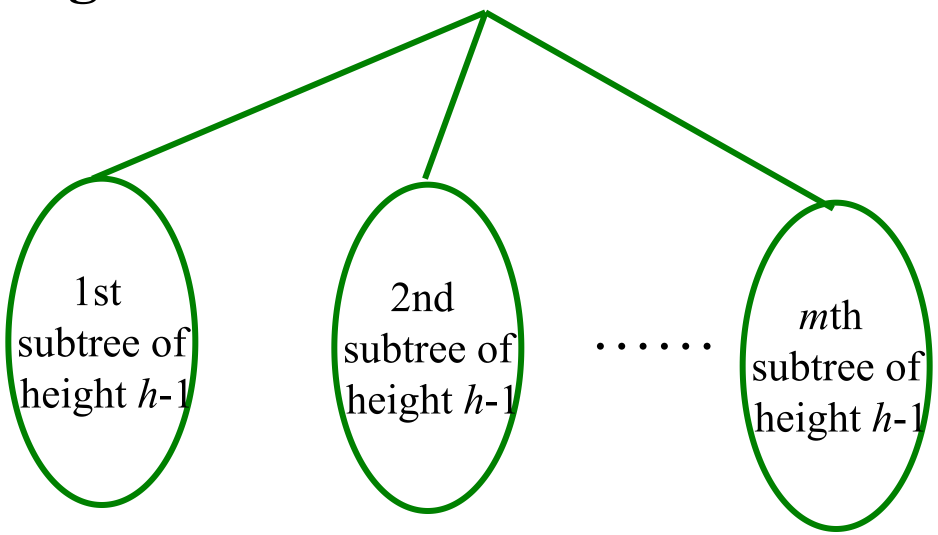
The Bound for the Number of Leaves in an m-Ary Tree

【Theorem 5】 There are at most m^h leaves in an m-ary tree of height h.

Proof:

(1) h=1

(2) Assume that the result is true for all m-ary tree of height less than h. Let T be an m-ary tree of height h.



【 Corallary 】 If an m-ary tree of height h has l leaves, then

$$h \geq \lceil \log_m l \rceil .$$

If the m-ary tree is full and balanced, then

$$h = \lceil \log_m l \rceil .$$

Proof:

(1) $l \leq m^h$

(2) Since the tree is balanced. Then each leaf is at level h or h-1, and since the height is h, there is at least one leaf at level h. It follows that,

$$\left.\begin{aligned} m^{h-1} &< l \\ l &\leq m^h \end{aligned}\right\} \Rightarrow \quad h-1 < \log_m l \leq h$$

Homework:

SE: P. 755   12, 20, 21, 28

EE: P. 792   12, 20, 21, 28