# Chapter 6 Counting

# Introduction

## Some problems:

- ✓ The time complexity analysis of a given algorithms?
- ✓ The number of passwords of a computer system?
- ✓ The number of different IPv4 addresses?

Counting problems arise throughout mathematics and computer science.

# Chapter Summary

- ✓The Basics of Counting
- ✓The Pigeonhole Principle
- ✓Permutations and Combinations
- ✓Binomial Coefficients and Identities
- ✓Generalized Permutations and Combinations
- ✓Generating Permutations and Combinations

# 6.1 The Basics of Counting

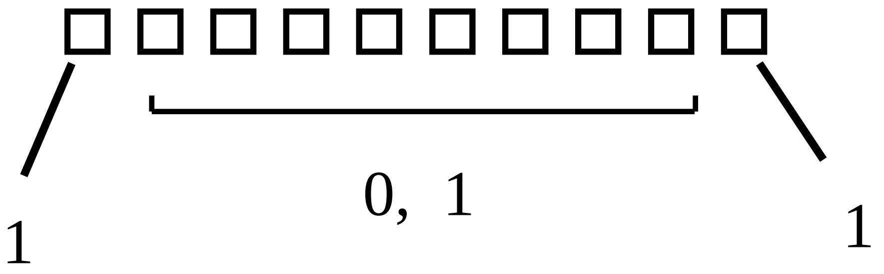
# Section Summary

- ✓The Product Rule
- ✓The Sum Rule
- ✓The Subtraction Rule
- ✓The Division Rule
- ✓Examples, Examples, and Examples
- ✓Tree Diagrams

# Basic Counting Principles: The Product Rule

**Suppose that a procedure can be broken down into two tasks. If there are $n_1$ ways to do the first task and $n_2$ ways to do the second after the first task has been done, then there are $n_1 n_2$ ways to complete the procedure.**

**〖Example 1〗 How many bit strings of length 10 begin and end with a 1?**

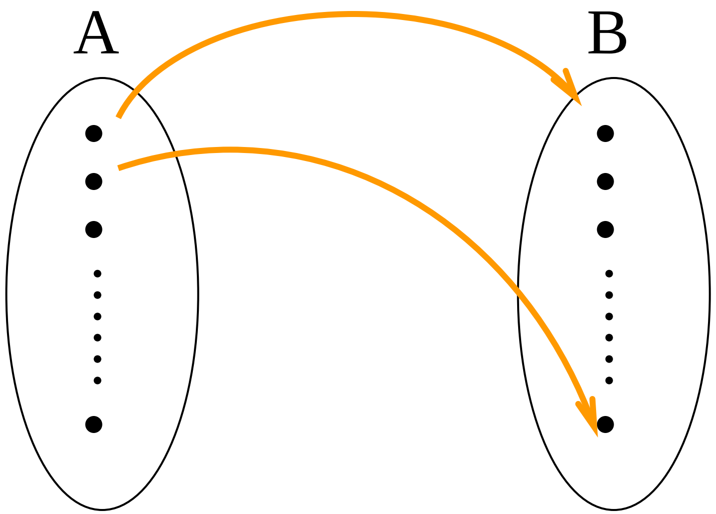



***Solution:***

**The product rule shows that there are a total of $2^8$ different bit strings of length 10 begin and end with a 1.**

# Counting Function

〖Example 2〗 **How many functions are there from a set with *m* elements to one with *n* elements?**

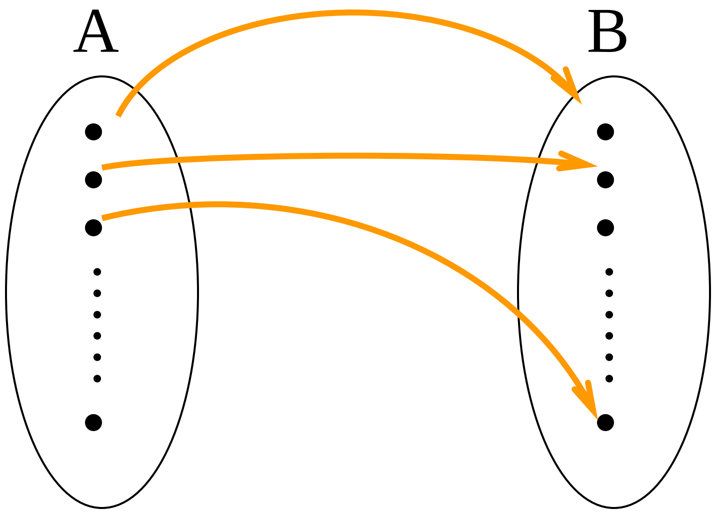



$f: A \to B$

*Solution:*

**By the product rule there are $n \cdot n \cdot \ldots \cdot n = n^m$ functions from a set with *m* elements to one with *n* elements.**

**Question:** How many one-to-one functions are there from a set with $m$ elements to one with $n$ elements?




- **Onto function?**
- **One to one correspondence function?**

***Solution:***

**(1)** $m > n$

**There are no one-to-one functions from a set with** $m$ **elements to one with** $n$ **elements.**

**(2)** $m \leq n$

$n(n-1)(n-2)\ldots(n-m+1)$

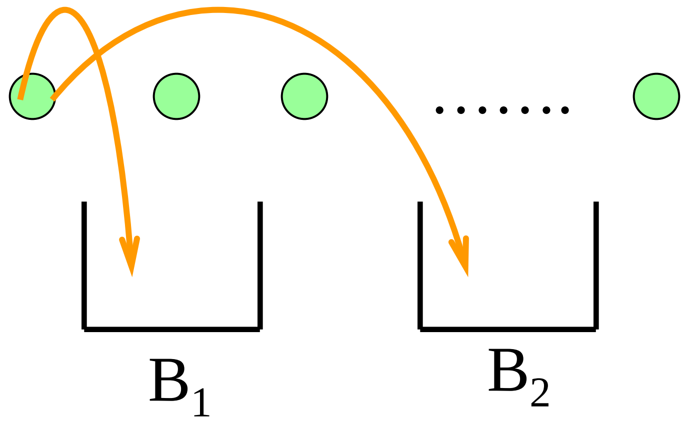
# Counting Subsets of a Finite Set

**〖Example 3〗 If $|A|=n$, then $|P(A)|=2^n$.**

**(1)** $C_n^0 + C_n^1 + C_n^2 + \ldots + C_n^n = \sum_{i=0}^{n} C_n^i = 2^n$

**(2) Mathematic Induction**

**(3)**




$$\mathbf{2 \times 2 \times \ldots \times 2 = 2^{n}}$$

**(4) Using the bit string representation of sets**

$$\underbrace{\square\ \square\ \square\ \ \ldots\ \ \square}_{0,\ 1}$$

# Product Rule in Terms of Sets

- If $A_1$, $A_2$, … , $A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.
  - The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in $A_1$, an element in $A_2$ , …, and an element in $A_m$.
  - By the product rule, it follows that:

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

# Basic Counting Principles: The Sum Rule

If a task can be done either in one of $n_1$ ways or in one of $n_2$ ways, where none of the set of $n_1$ ways is the same as any of the set of $n_2$ ways, then there are $n_1 + n_2$ ways to do the tasks.

**〖Example 4〗 Suppose statement labels in a programming language must be a single letter or a single decimal digit. Determine the number of statement labels.**

***Solution:***

**Since a label cannot be both at the same time,**

**the number of labels = the number of letters + the number of decimal digits**

**= 26 + 10 = 36.**

# The Sum Rule in terms of sets

- The sum rule can be phrased in terms of sets.

  $|A \cup B| = |A| + |B|$ as long as $A$ and $B$ are disjoint sets.

- Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

$$\textbf{when } A_i \cap A_j = \emptyset \textbf{ for all } i, j.$$

- The case where the sets have elements in common will be discussed when we consider the subtraction rule and taken up fully in Chapter 8.

# Counting Passwords

- Combining the sum and product rule allows us to solve more complex problems.

**Example 5**: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

**Solution**: Let $P$ be the total number of passwords, and let $P_6$, $P_7$, and $P_8$ be the passwords of length 6, 7, and 8.

- By the sum rule $P = P_6 + P_7 + P_8$.
- To find each of $P_6$, $P_7$, and $P_8$ , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

$$P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$$
$$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920.$$
$$P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880.$$

Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$

**〖Example 6〗 Counting the number of elements in *A*,**
***A* = {length 10 bit strings with 0-streak of length exactly 5}.**

***Solution:***

**Since the set *A* can be break up into the following case.**

$A_1 = \{000001{*}{*}{*}{*}\}$ (* is either 0 or 1)

$A_2 = \{1000001{*}{*}{*}\}$

$A_3 = \{{*}1000001{*}{*}\}$

$A_4 = \{{*}{*}1000001{*}\}$

$A_5 = \{{*}{*}{*}1000001\}$

$A_6 = \{{*}{*}{*}{*}100000\}$

Apply the sum rule:

$$|A| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| + |A_6|$$

〖Example 7〗Choose three different numbers from the integers between 1 to 300 such that the sum of the three integers can be divisible by 3. How many the ways are there?

*Solution:*

$A = \{x \mid 1 \le x \le 300,\ x(\text{mod } 3) = 1\}$

$B = \{x \mid 1 \le x \le 300,\ x(\text{mod } 3) = 2\}$

$C = \{x \mid 1 \le x \le 300,\ x(\text{mod } 3) = 0\}$

$|A| = |B| = |C| = 100$

(1) All of the three numbers are chosen form the set $A$ $C_{100}^{3}$

(2) All of the three numbers are chosen form the set $B$ $C_{100}^{3}$

(3) All of the three numbers are chosen form the set $C$ $C_{100}^{3}$

(4) Chose one number form the set A, B, C $C_{100}^{1} \times C_{100}^{1} \times C_{100}^{1}$

# Counting Internet Address

◆ **Version 4 of the Internet Protocol (IPv4) uses 32 bits.**

| Bit number | 0 | 1 | 2 | 3 | 4 | 8 | 16 | 24 | 31 |
|---|---|---|---|---|---|---|---|---|---|
| Class A | 0 | netid: 7-bit | | | | | hostid: 24-bit | | |
| Class B | 1 | 0 | netid: 14-bit | | | | | hostid : 16-bit | |
| Class C | 1 | 1 | 0 | netid: 21-bit | | | | | hostid : 8-bit |
| Class D | 1 | 1 | 1 | 0 | Multicast Address | | | | |
| Class E | 1 | 1 | 1 | 1 | 0 | Address | | | |

- **Class A Addresses:** used for the largest networks, a 0,followed by a 7-bit netid and a 24-bit hostid.
- **Class B Addresses:** used for the medium-sized networks, a 10,followed by a 14-bit netid and a 16-bit hostid.
- **Class C Addresses:** used for the smallest networks, a 110,followed by a 21-bit netid and a 8-bit hostid.
  - Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
  - 1111111 is not available as the netid of a Class A network.
  - Hostids consisting of all 0s and all 1s are not available in any network.

## 〖Example 8〗How many different IPv4 addresses are available for computers on the Internet?

| Bit number | 0 | 1 | 2 | 3 | 4 | 8 | 16 | 24 | 31 |
|---|---|---|---|---|---|---|---|---|---|
| Class A | 0 | netid: 7-bit | | | | | hostid: 24-bit | | |
| Class B | 1 | 0 | netid: 14-bit | | | | | hostid : 16-bit | |
| Class C | 1 | 1 | 0 | netid: 21-bit | | | | | hostid : 8-bit |
| Class D | 1 | 1 | 1 | 0 | Multicast Address | | | | |
| Class E | 1 | 1 | 1 | 1 | 0 | Address | | | |

Class A: $(2^7-1)\times(2^{24}-2) = 127\cdot 16{,}777{,}214 = 2{,}130{,}706{,}178.$

Class B: $2^{14}\times(2^{16}-2) = 16{,}384\cdot 16{,}534 = 1{,}073{,}709{,}056.$

Class C: $2^{21}\times(2^8-2) = 2{,}097{,}152\cdot 254 = 532{,}676{,}608$

-------------------------------------------------------------------

3,737,091,842

**Not Enough Today !!**

**The newer IPv6 protocol solves the problem of too few addresses.**

# Basic Counting Principles: The Subtraction Rule

**If a task can be done in either $n_1$ ways or $n_2$ ways, then the number of ways to do the task in $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.**

**Note:**

**The subtraction rule is also known as the principle of inclusion-exclusion**

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

**〖Example 9〗 How many positive integers not exceeding 100 are divisible by neither 4 nor 6?**

***Solution:***

$U = \{ 1, 2, \ldots , 100 \}$

$A = \{ x \mid x \in Z^+,\ 1 \le x \le 100,\ \ 4 \mid x \}$

$B = \{ x \mid x \in Z^+,\ 1 \le x \le 100,\ \ 6 \mid x \}$

$$|\overline{A} \cap \overline{B}| = |\overline{A \cup B}| = |U| - |A \cup B| = |U| - (|A| + |B| - |A \cap B|)$$

**〖Example 10〗How many bit strings of length eight either start with a 1 bit or end with the two bits 00?**

***Solution:***

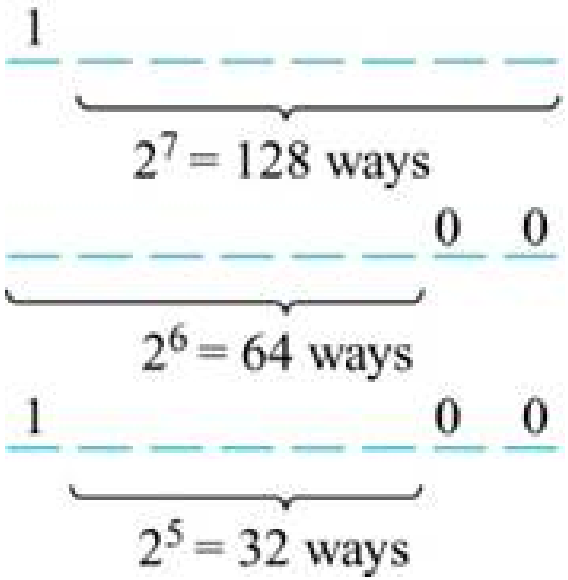
Use the subtraction rule.

Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$

Number of bit strings of length eight that end with bits 00: $2^6 = 64$

Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is $128 + 64 - 32 = 160$.



# Basic Counting Principles: Division Rule

**Division Rule**: There are $n/d$ ways to do a task if it can be done using a procedure that can be carried out in $n$ ways, and for every way $w$, exactly $d$ of the $n$ ways correspond to way $w$.

- Restated in terms of sets: If the finite set $A$ is the union of $n$ pairwise disjoint subsets each with $d$ elements, then $n = |A|/d$.
- In terms of functions: If $f$ is a function from $A$ to B, where both are finite sets, and for every value $y \in B$ there are exactly $d$ values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$.

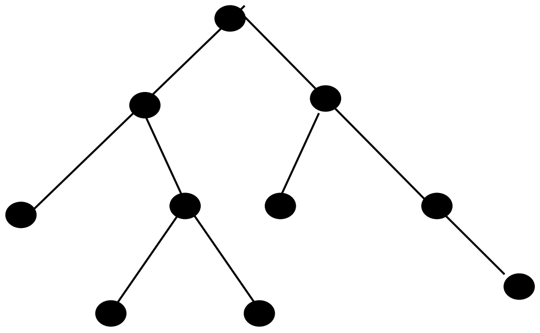
**Example 11**: How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

**Solution**: Number the seats around the table from 1 to 4 proceeding clockwise. There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4. Thus there are 4! = 24 ways to order the four people. But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating.

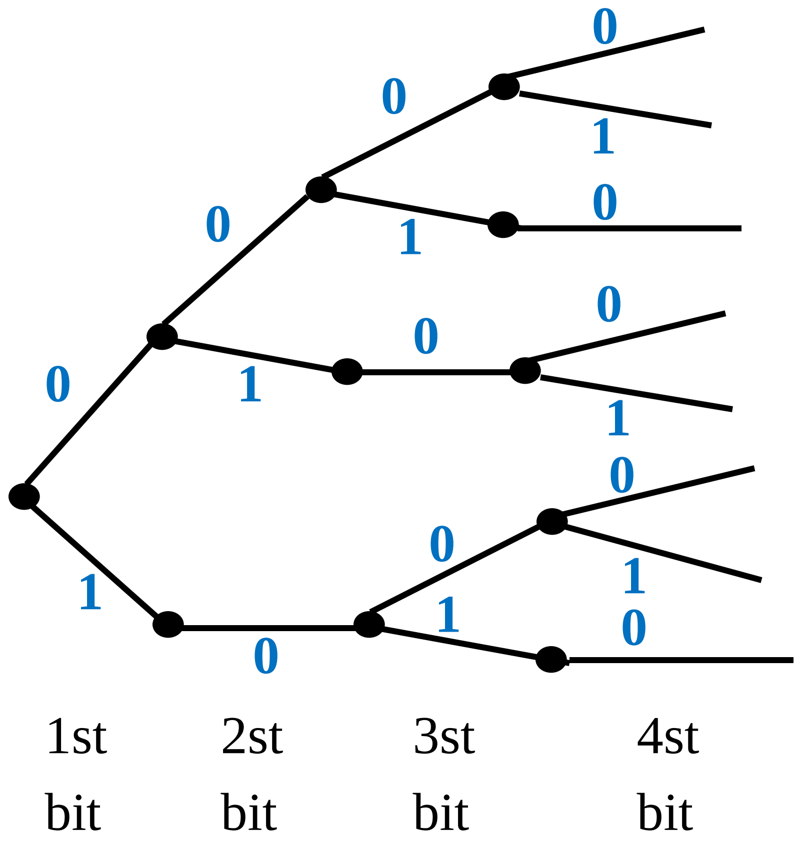
Therefore, by the division rule, there are 24/4 = 6 different seating arrangements.

# Tree Diagrams

- **Tree Diagrams:** We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.



〖Example 12〗How many bit strings of length four do not have two consecutive 1s?

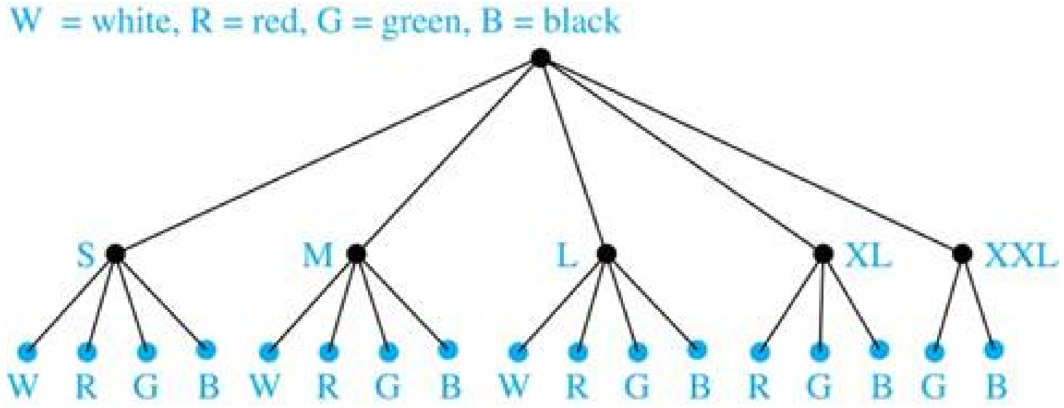



*Solution:*

**There are eight bit strings of length four without two consecutive 1s.**

〖**Example 13**〗 Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?

Solution: Draw the tree diagram.




The store must stock 17 T-shirts.

# 6.2 The Pigeonhole Principle

# Section Summary

- ✓The Pigeonhole Principle
- ✓The Generalized Pigeonhole Principle

# Some interesting facts:

① **Among any 102 students in a class, , there must be at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points.**

② **Among any group of 367 people, there must be at least two with the same birthday.**

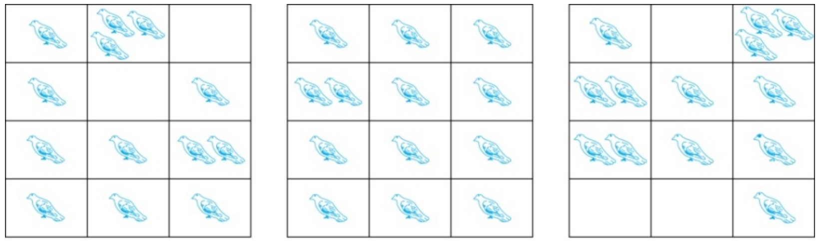
③ **Among 100 people there are at least 9 who were born in the same month.**

④ **During 11 weeks football games will be held at least 1 game a day, but at most 12 games be arranged each week. There must be a period of some number of consecutive days during which exactly 21 games must be played.**

**These facts can be proved by the pigeonhole principle.**

# The Pigeonhole Principle

**If a flock of 13 pigeons roosts in a set of 12 pigeonholes, one of the pigeonholes must have more than 1 pigeon.**



(a) (b) (c)

**【Theorem 1】 The Pigeonhole Principle**

**If $k$ is a positive integer and $k+1$ or more objects are placed into $k$ boxes, then there is at least one box containing two or more of the objects.**

**Proof: We use a proof by contraposition. Suppose none of the $k$ boxes has more than one object. Then the total number of objects would be at most $k$. This contradicts the statement that we have $k + 1$ objects.**

**It is also called**

**Dirichlet Drawer Principle**

# Some interesting facts

① **Among any 102 students in a class, , there must be at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points.**

② **Among any group of 367 people, there must be at lea… two with the same birthday.**

③ Among 100 … ple there are at least 9 who were b…

④ During 11 w… football games will be held at le… most 1… eek. There mu… numbe… which exactly 21 games must be played.

**Pigeons : the 102 students**

**Pigeonholes: 101 possible scores**

**Pigeons : the 367 people**

**Pigeonholes: 366 days**

# Other examples of the pigeonhole principle

- **Among any group of 11 integers, there are two integers $a$ and $b$ such that $10 \mid a-b$.**

  **Pigeons : 11 integers**

  **Pigeonholes: the remainder when $a$ and $b$ is divided by 10**

- **In a party of 2 or more people, there are 2 people with the same number of friends in the party. (Assuming you can't be your own friend and that friendship is mutual.)**

**Pigeons : the $n$ people (with $n > 1$).**

**Pigeonholes: the possible number of friends, i.e.**

**the set $\{0, 1, 2, 3, \ldots, n-1\}$**

# Other examples of the pigeonhole principle

- **Show that for every integer *n* there is a multiple of *n* that has only 0s and 1s in its decimal expansion.**

*Solution:*

Let $n$ be a positive integer.

Consider the $n + 1$ integers 1, 11, 111, …., 11…1 (where the last has $n + 1$ 1s). There are $n$ possible remainders when an integer is divided by $n$.

By the pigeonhole principle, when each of the $n + 1$ integers is divided by $n$, at least two must have the same remainder.

Subtract the smaller from the larger and the result is a multiple of $n$ that has only 0s and 1s in its decimal expansion.

Use the pigeonhole principle to prove some results about functions.

**If $f$ is a function for $A$ to $B$, where $A$ and $B$ are finite sets with $|A| > |B|$, then there are elements $a_1, a_2$ in $A$ ($a_1 \neq a_2$) such that $f(a_1) = f(a_2)$.**

***Proof:***

$$\forall a_1, a_2 \in A, \quad a_1 \neq a_2,$$

If $f(a_1) \neq f(a_2)$, then $|A| = |f(A)| \leq |B|$.

***Corollary* 1** ***A*** **function *f* from a set with *k*+1 or more elements to a set with *k* elements is not one-to-one.**

# The Generalized Pigeonhole Principle

【Theorem 2】The Generalized Pigeonhole Principle　If $N$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

*Proof:*

Suppose that none of the boxes contains more than $\lceil N/k \rceil$ -1 objects. Then, the total number of objects is at most

$$k(\lceil N/k \rceil - 1) < k((N/k+1) - 1) = N$$

**Problem:** The minimum number of objects such that at least $r$ of these objects must be in one of $k$ boxes when these objects are distributed among the boxes?

$$N = k(r - 1) + 1$$

# The Generalized Pigeonhole Principle

**Phrased in terms of functions:**

$f: A \rightarrow B$, If $\left\lceil \frac{|A|}{|B|} \right\rceil = i$, then there must exist elements $a_1, a_2, \ldots, a_i \in A$ such that

$$f(a_1) = f(a_2) = \ldots = f(a_i) = b \in B$$

# Some interesting facts

① Among any 102 students in a class, , there must be at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points.

$\lceil 100/12 \rceil = 9$

② Among any group of 367 people, there must be at least two with the same birthday.

③ **Among 100 people there are at least 9 who were born in the same month.**

④

**Problem:**

**The minimum number of persons needed to ensure that at least 9 who were born in the same month?**

〖Example 1〗 **What is the least number of area codes needed to guarantee that the 25 million phones in a state have distinct 10-digit telephone numbers?**

$$\underbrace{\square\,\square\,\square}_{\substack{\text{area code}\\ \text{NXX}}} \;—\; \underbrace{\square\,\square\,\square}_{\substack{\text{office code}\\ \text{NXX}}} \;—\; \underbrace{\square\,\square\,\square\,\square}_{\substack{\text{station code}\\ \text{XXXX}}}$$

**N: 2 - 9**

**X: 0 - 9**

**(1) The number of phone numbers of the form NXX-XXXX**

$8 \times 10^6 = 8{,}000{,}000$

**(2) By the generalized pigeonhole principle, among 25 million phones, at least $\lceil 25/8 \rceil = 4$ of them must have identical numbers.**

**Hence, at least 4 area codes are required to.**

〖Example 2〗 **A bowl contains 10 red balls and 10 blue balls. One selects balls at random without looking at them.**

**How many balls must he select to be sure of having at least three balls of the same color?**

***Solution:***

**pigeonholes: red, blue color     pigeon:balls**

**By the generalized pigeonhole principle,**

$\lceil 5/2 \rceil = 3$

**or**

$N = 2 * (3-1) + 1$

# Some elegant application

**〖Example 3〗 Show that among any $n$+1 positive integers not exceeding 2$n$ there must be an integer that divides one of the other integers.**

***Solution:***

**Let $n$+1 positive integers be** $a_1, a_2, \ldots, a_{n+1} (1 \le a_i \le 2n)$

**Write $a_i$($i$=1,2,…,$n$+1) as** $2^{k_i} q_i$ **, where $k_i$ is a nonnegative integer and $q_i$ is odd positive integers less than 2$n$.**

**Since there are only $n$ odd positive integers less than 2$n$, by the pigeonhole principle it follows that there exist integers $i$ and $j$ such that $q_i$=$q_j$=$q$,**

**then** $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$

**It follows that if $a_i < a_j$, then $a_i \mid a_j$, while if $a_j < a_i$, then $a_j \mid a_i$.**

**〖Example 4〗During 11 weeks football games will be held at least 1 game a day, but at most 12 games be arranged each week. Show that there must be a period of some number of consecutive days during which exactly 21 games must be played.**

***Solution:***

$x_i$ : the number of football games helded on the $i$th day

$a_i = \sum_{k=1}^{i} x_k$ $\qquad 1 \le a_1 < a_2 \ldots < a_{77} \le 12 \times 11 = 132$

$c_i = a_i + 21$ $\qquad c_1 < c_2 < \ldots < c_{77} \le 132 + 21 = 153$

$A = \{a_1, a_2, \ldots a_{77}, c_1, c_2, \ldots c_{77}\}$ $\qquad B = \{1, 2, \ldots, 153\}$

$\exists i \neq j \quad such \quad that \quad a_i = c_j$

$a_i = a_j + 21$

$a_i - a_j = x_i + x_{i-1} + \ldots + x_{j+1} = 21$

**〖Example 5〗 Suppose that there are *n* arbitrary integers $x_1, x_2, \ldots, x_n$ . Show that there exist some consecutive integers in this list such that the sum of these integers is the multiple of *n*.**

***Solution:***

$$a_i = \sum_{k=1}^{i} x_k \, (i = 1, 2, \ldots, n)$$

**(1)** $\exists k \ (n \mid a_k)$

**(2)** $\neg \exists k \ (n \mid a_k)$

**〖Example 6〗Every sequence of $n^2+1$ distinct integers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.**

***Proof:***

**For example, $n$=2**

**1,2,3,4,5 4,8,3,6,1; 1,4,5,3,2**

**Let the sequence be** $a_1, a_2, \ldots, a_{n^2+1}$

**Associate** $(x_k, y_k)$ **to the term** $a_k$ **, where** $x_k$ **is the length of the longest increasing subsequence starting at** $a_k$ **,** $y_k$ **…**

**Suppose that there is no increasing or decreasing subsequence of length $n$+1.Then**

$1 \le x_k \le n \qquad 1 \le y_k \le n$

**Hence there are** $n \times n = n^2$ **pairs (** $x_k, y_k$ **),**

**Since there are** $n^2+1$ $a_k$ **, By the pigeonhole principle, it follows that there exist terms** $a_i, a_j$ $(1 \le i < j \le n^2+1)$ **such that** $(x_i, y_i) = (x_j, j_j)$

**Since** $a_i \neq a_j$

**It follows that**

**(1)** $a_i < a_j$

**(2)** $a_i > a_j$

**〖Example 7〗 Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.**

***Proof:***

**Let the six people be** $a_1, a_2, a_3, a_4, a_5, a_6$

$A = \{a_2, a_3, a_4, a_5, a_6\}$

$B = \{b_1, b_2\}$, where $b_1$ represents that one is a friend of $a_1$,...

**By the generalized pigeonhole principle, of the five people in *A*, there are either three or more who are friends of $a_1$, or three or more who are enemies of $a_1$.**

**(1)** Suppose that $a_i, a_j, a_k$ are freinds of $a_1$

**(2)** Suppose that $a_i, a_j, a_k$ are enemies of $a_1$

**Homework:**

SE: P.396 3, 12, 22, 37, 50, 64

P.405 9,10,41,42,44

EE: P.396 3, 12, 22, 37, 52, 66

P.426 11,12,43,44,46