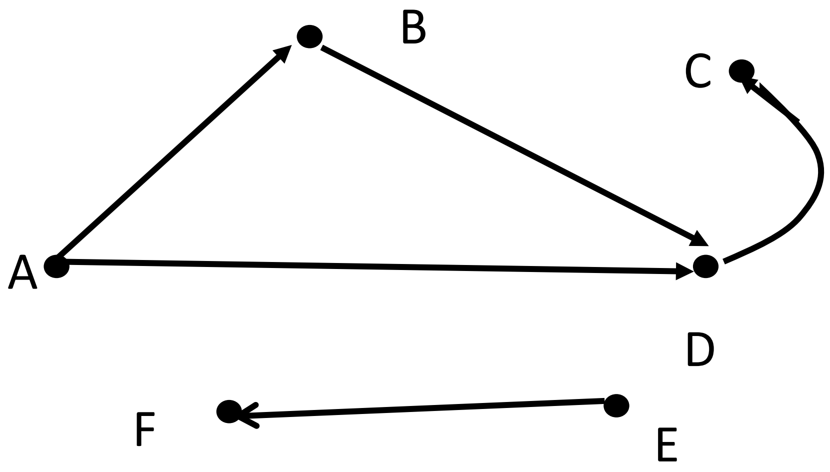
9.4
Closures of Relations

Why Closure of relation ?



R={(a, b)| There is a direct, one way telephone line from a to b}

- ***How can we determine if there is some link composed of one or more telephone lines from one center to another?***
- ***How to construct a relation that we can find all pairs of data centers that have a link?***

What Is Closures of Relations

【Definition】 The closure of a relation R with respect to property P is the relation S with property P containing R such that S is a subset of every relation with property P containing R.

The smallest relation with property P containing R

We will analyze

- Reflexive Closure
- Symmetric Closure
- Transitive Closure

Reflexive Closure

【 Theorem 】 Let R be a relation on A. The reflexive closure of R, denoted by $r(R)$, is $R \cup I_A$

The *diagonal relation* on A
$I_A = \{(x, x) \mid x \in A\}$

Proof:

① containing R

② is a reflexive relation

$$\forall x \in A, (x, x) \in I_A \subseteq R \cup I_A$$

③ is the smallest reflexive relation which contains R

Suppose that R' is a reflexive relation containing R, then

$$R \subseteq R', I_A \subseteq R' \Rightarrow r(R) = R \cup I_A \subseteq R'$$

【 **Corollary** 】 $R = R \cup I_A \Leftrightarrow$ **R is a reflexive relation**.

Proof:

① **The reflexive closure is a reflexive relation**

② **Since *R* is a reflexive relation,**

$I_A \subseteq R$

$\therefore R \cup I_A = R$

Question:

Given R, how to obtain its reflexive closure?

$$r(R) = R \cup I_A$$

- ✓ Add to *R* all ordered pairs of the form (a,a) with $a \in A$, not already in *R*
- ✓ Add loops to all vertices on the digraph representation of *R*.
- ✓ Put 1's on the diagonal of the connection matrix of *R*.

〖Example 1〗 $R=\{(a,b)\mid a<b, a,b\in Z\}$ **, What is *r*(*R*)?**

Solution:

$$r(R)=R\cup I_A$$

$$=\{(a,b)\mid a<b, a,b\in Z\}\cup\{(a,a)\mid a\in Z\}$$

$$=\{(a,b)\mid a\le b, a,b\in Z\}$$

Symmetric Closure

【 Theorem 】 Let R be a relation on A. The symmetric closure of R, denoted by $s(R)$, is $R \cup R^{-1}$

Proof:

① containing R

② is a symmetric relation

$$(a,b) \in R \cup R^{-1} \Rightarrow \left\{ \begin{array}{ll} (a,b) \in R & \Rightarrow (b,a) \in R^{-1} \\ (a,b) \in R^{-1} & \Rightarrow (b,a) \in R \end{array} \right\}$$

$$\Rightarrow (b,a) \in R \cup R^{-1}$$

③ **is the smallest symmetric relation which containing *R***

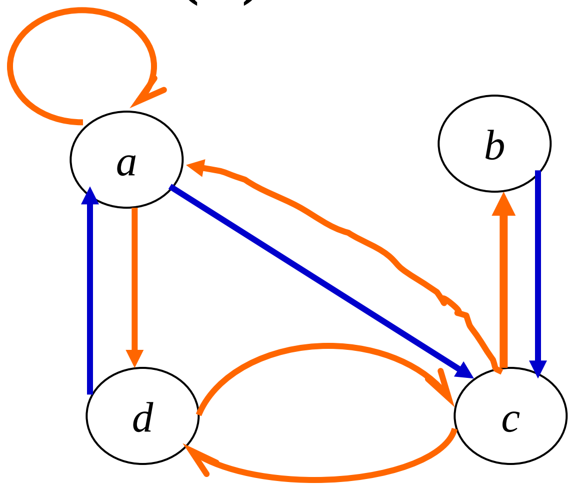
Suppose that *R'* is a symmetric relation containing *R*, then

If $(a,b) \in R \cup R^{-1}$

$$\Rightarrow \left\{ \begin{array}{l} \left. \begin{array}{c} (a,b) \in R \\ R \subseteq R' \end{array} \right\} \Rightarrow (a,b) \in R' \\ (a,b) \in R^{-1} \Rightarrow (b,a) \in R \quad \left. \begin{array}{r} \Rightarrow (b,a) \in R' \\ R' \text{ is a symmetric relation} \end{array} \right\} \Rightarrow (a,b) \in R' \end{array} \right\}$$

$$\Rightarrow R \cup R^{-1} \subseteq R'$$

〖Example 2〗 What is $s(R)$ of the following relation?



Note:

- **Add an edge from x to y whenever this edge is not already in directed graph but the edge from y to x is.**
- **Add all ordered pairs of the form (b,a) where (a,b) is in the relation, that are not already in R.**
- $M_{s(R)} = M_R \vee M_R^T$

【Corollary】 $R = R \cup R^{-1} \Leftrightarrow$ **R is a symmetric relation .**

Proof:

① **The symmetric closure is a symmetric relation**

② **Since *R* is a symmetric relation, it follows that**

$$(a,b) \in R \Rightarrow (b,a) \in R$$

$$\Rightarrow R^{-1} \subseteq R$$

$$\Rightarrow R^{-1} \cup R = R$$

The transitive closure of a relation *R*, *t*(*R*),

- the smallest transitive relation containing *R*.

How can we construct *t*(*R*)?

Can *t*(*R*) be produced by adding all the pairs of the form (a, c) where (a, b) and (b, c) are already in the relation?

〖Example 3〗 Suppose that $R=\{(1,2),(2,3),(3,1)\}$ **be a relation on the set** $A=\{1,2,3\}$

$$R'=\{(1,2),(2,3),(3,1),(1,3),(2,1),(3,2)\}$$

$$R'=t(R)\,?$$

***t*(*R*)=?**
How to compute?

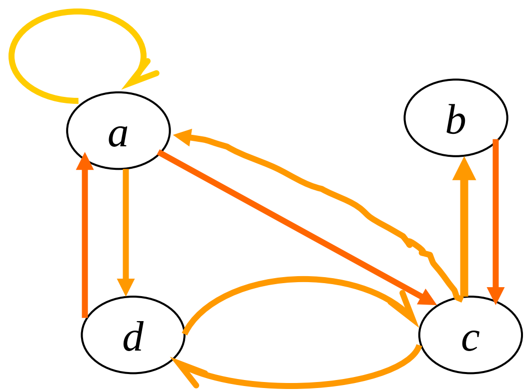
Terminologies:

A path of length n in a digraph G :

A sequence of edges $(x_0, x_1),(x_1, x_2), \ldots ,(x_{n-1}, x_n)$

Notation: $x_0, x_1, x_2, \ldots , x_{n-1}, x_n$.

Cycle or circuit : If $x_0 = x_n$



The term path also applies to relation.

There is a path of length n from a to b in R

$\exists a, x_1, x_2, \ldots , x_{n-1}, b$ such that $(a, x_1) \in R$, $(x_1, x_2) \in R$, …, $(x_{n-1}, b) \in R$

【 Theorem 】 Let R be a relation on A. There is a path of length n from a to b if and only if $(a,b) \in R^n$

Proof:

① **Inductive basis**

An edge from a to b is a path of length 1 which is in $R^1 = R$. Hence the assertion is true for $n = 1$.

② **Inductive step**

There is a path of length $n+1$ from a to b if and only if there is an x in A such that there is a path of length 1 from a to x and a path of length n from x to b.

From the Induction Hypothesis,

$$(a,x) \in R \qquad (x,b) \in R^n$$

$$(a,b) \in R^n \circ R = R^{n+1}$$

Connectivity Relation

【Definition】 The connectivity relation denoted by R^*, is the set of ordered pairs (a, b) such that there is a path (in R) from a to b:

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

〖Example 3〗 Let R be the relation on the set of all people in the world that contains (a, b) if a has met b. What is R^n , where n is a positive integer greater than one? What is R^*?

R^* contains (a, b) if there is a sequence of people , starting with a and ending with b, such that each person in the sequence has met the next person in the sequence.

(You, xidada)$\in R^*$?

Interesting conjecture: Almost every pair of people in the world are linked by a small chain of people, perhaps containing just five or fewer people.

【 Theorem 】 $t(R) = R^*$.

Proof:

① **containing** R

② **is a transitive relation**

Suppose (a, c) **and** (c, b) **are in** R^*. **Show that** (a, b) **is in** R^*.

By the definition of R^*,

$$(a,c),(c,b) \in R^*$$

$$\Rightarrow \exists i, j \quad (a,c) \in R^i, (c,b) \in R^j$$

$$\Rightarrow (a,b) \in R^{i+j} \subseteq R^*$$

③ is the smallest transitive relation which contains R

Now suppose that S is any transitive relation which contains R. We must show S contains R^* to show R^* is the smallest relation.

Since S is transitive, S^n also is transitive and $S^n \subseteq S$.

Furthermore, since $S^* = \bigcup_{k=1}^{\infty} S^k$

and $S^n \subseteq S$, it follows that $S^* \subseteq S$

If $R \subseteq S$, then $R^* \subseteq S^*$, because any path in R is also a path in S.

Note: 1. $R = t(R)$ $\Leftrightarrow$ R is transitive.

2. In fact, we need only consider paths of length n or less.

【 Theorem 】 If $|A| = n$, then any path of length > n must contain a cycle.

Proof:

If we write down a list of more than *n* vertices representing a path in *R*, some vertex must appear at least twice in the list (by the Pigeon Hole Principle).

$$a = x_0,\ x_1,\ x_2, \ldots,\ x_{i-1},\ x_i,\ x_{i+1}, \ldots,\ x_{j-1},\ x_j,\ x_{j+1},\ x_{j+2}, \ldots, x_m = b \qquad m > n$$

【Theorem】 If $|A|=n$, R is a relation on A, then

$$\exists k, k \leq n, R^* = R \cup R^2 \cup \ldots \cup R^k$$

【Corollary】 If $|A|=n$, then $t(R) = R^* = R \cup R^2 \cup \ldots \cup R^n$

【Corollary】 Let $\mathbf{M}_R$ be the zero-one matrix of the relation R on a set with n elements. The zero-one matrix of the transitive closure is

$$\mathbf{M}_{t(R)} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \ldots \vee \mathbf{M}_R^{[n]}$$

A Procedure for computing t(R)

$$\mathbf{M}_{t(R)} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \cdots \vee \mathbf{M}_R^{[n]}$$

$A = \mathbf{M}_R;$

$B = A;$

for (*i*=2; *i*<=n; *i*++)
{

$A = A \cdot \mathbf{M}_R;$

$B := B \vee A;$

}

The complexity of algorithm:

$$n^2(2n-1)(n-1) + (n-1)n^2 = 2n^3(n-1) = O(n^4)$$

Warshall's Algorithm

The interior vertices of a path:

$$x_0, \quad x_1, \quad x_2, \quad \ldots \quad , \quad x_{n-1}, \quad x_n$$

Warshall's algorithm is based on the construction of a sequence of zero-one matrices, such as

$$W_0, \quad W_1, \quad W_2, \quad \ldots \quad , W_n$$

$$W_0 = M_R$$

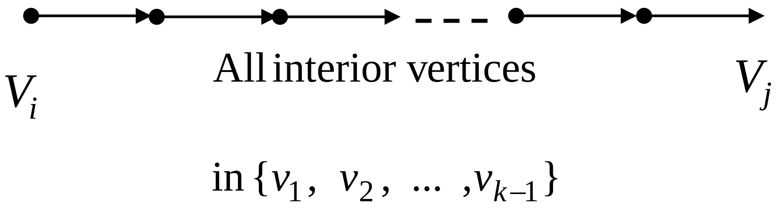
$$W_k = [w_{ij}^{(k)}]$$

$$w_{ij}^{(k)} = \begin{cases} 1 & \text{If there is a path from } V_i \text{ to } V_j \text{ such that all the interior vertices of this path} \\ & \text{are in the set } \{V_1, V_2, \ldots, V_k\} \\ 0 & \text{otherwise} \end{cases}$$

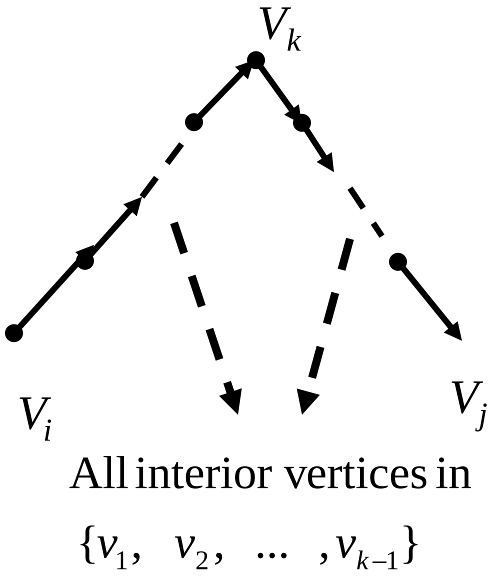
$$W_n = M_{t(R)}$$

$$w_{ij}^{(k)} = w_{ij}^{(k-1)} \vee (w_{ik}^{(k-1)} \wedge w_{kj}^{(k-1)})$$ **(P.606 LEMMA 2)**

Case 1



Case 2



Warshall's Algorithm

```
W = [w_ij]_{n×n};
for (k=1; k <=n; k ++)
{
        for (i=1; i <=n; i ++)
        {
                for (j=1; j <=n; j ++)
                    w_ij = w_ij ∨ (w_ik ∧ w_kj);
        }
}
```

$$W = [w_{ij}]_{n\times n};$$

$$w_{ij} = w_{ij} \vee (w_{ik} \wedge w_{kj});$$

```
if ( w_ik = 1 )
{
   for (j=1; j <=n; j ++)
         w_ij = w_ij ∨ w_kj;
}
```

The complexity of algorithm: $2n^3$

〖Example 3〗 Let

$$A=\{1,2,3,4,5\},\ R=\{(1,1),(1,2),(2,4),(3,5),(4,2)\},\ t(R)=?$$

Solution:

$$M=\begin{bmatrix}1&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&0&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{k=1}\begin{bmatrix}1&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&0&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{k=2}\begin{bmatrix}1&1&0&1&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&1&0\\0&0&0&0&0\end{bmatrix}$$

$$\xrightarrow{k=3}\begin{bmatrix}1&1&0&1&0\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&1&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{k=4}\begin{bmatrix}1&1&0&1&0\\0&1&0&1&0\\0&0&0&0&1\\0&1&0&1&0\\0&0&0&0&0\end{bmatrix}\xrightarrow{k=5}\begin{bmatrix}1&1&0&1&0\\0&1&0&1&0\\0&0&0&0&1\\0&1&0&1&0\\0&0&0&0&0\end{bmatrix}$$

Question:

How to find the smallest relation containing R that is

reflexive and transitive?

symmetric and transitive?

reflexive, symmetric and transitive?

Homework:

SE：P. 606 2,6,9(6),11(6),20,28(a),29

EE：P. 637 2,6,9(6),11(6),20,28(a),29