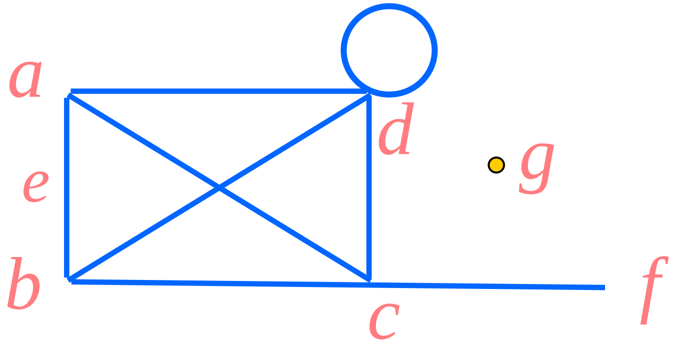
10.2 Graph Terminology and Special Types of Graphs

Basic Terminology

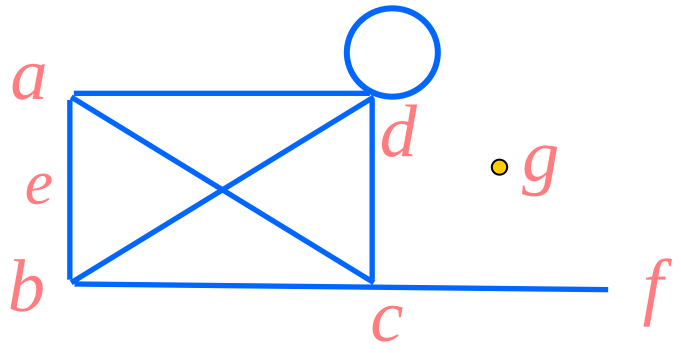


Undirected Graphs $G=(V, E)$

- Two vertices, u and v in an undirected graph G are called adjacent (or neighbors) in G, if {u, v} is an edge of G.
- An edge e connecting u and v is called incident with vertices u and v, or is said to connect u and v.
- The vertices u and v are called endpoints of edge {u, v}.
- Loop: an edge connects a vertex to itself.
- The neighborhood of v (N(v)): the set of all neighbors of a vertex v
- The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex

  Notation: deg(v)

  - If deg(v) = 0, v is called isolated.
  - If deg(v) = 1, v is called pendant.



Find the degree of all the vertices.

deg(a) =3 deg(6) = 3 deg(c) = 4 deg(d) = 5

deg(f) = 1 deg(g) = 0

TOTAL of degrees = 3 + 3 + 4 + 5 + 1 + 0 = 16

TOTAL NUMBER OF EDGES = 8

【Theorem 1】 The Handshaking Theorem

Let G = (V, E) be an undirected graph G with e edges. Then

$$\sum_{v \in V} \deg(v) = 2e$$

The sum of the degrees over all the vertices equals twice the number of edges.

Proof:

Each edge represents contributes twice to the degree count of all vertices.

Note:

This applies even if multiple edges and loops are present.

Questions:

1. The sum, over the set of people at a party, of the number of people a person has shaken hands with, is even?
2. How many edges are there in a graph with 10 vertices each of degree 6?
3. If a graph has 5 vertices, can each vertex have degree 3? 4?
   - The sum is $3 \cdot 5 = 15$ which is an odd number.

     Not possible.
   - The sum is $20 = 2\,|E|$ and $20/2 = 10$.

     May be possible.

【Theorem 2】 An undirected graph has an even number of vertices of odd degree.

Proof:

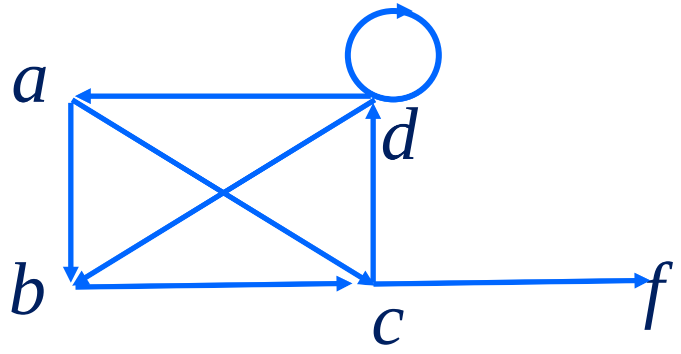
Let V_1, V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively.

$$\sum_{v \in V_1} d(v) + \sum_{v \in V_2} d(v) = 2m$$

Questions:

1. Is it possible to have a graph with 3 vertices each of which has degree 3?
2. Is it possible that a graph has a sequence of degrees (3,3,2,3) or (5,2,3,1,4)?
3. Show that among 9 factories,
   - It is impossible that each factory has business relation only with other three factories.
   - It is impossible that only four factories have business relation with factories with even number.
4. G is an nonempty simple graph, then there must exist vertices with same degrees.

Directed Graphs $G=(V, E)$



Let (u, v) be an edge in G. Then u is an initial vertex and is adjacent to v and v is a terminal vertex and is adjacent from u.

The in degree of a vertex v, denoted $deg^-(v)$ is the number of edges which terminate at v.

Similarly, the out degree of v, denoted $deg^+(v)$, is the number of edges which initiate at v.

underlying undirected graph

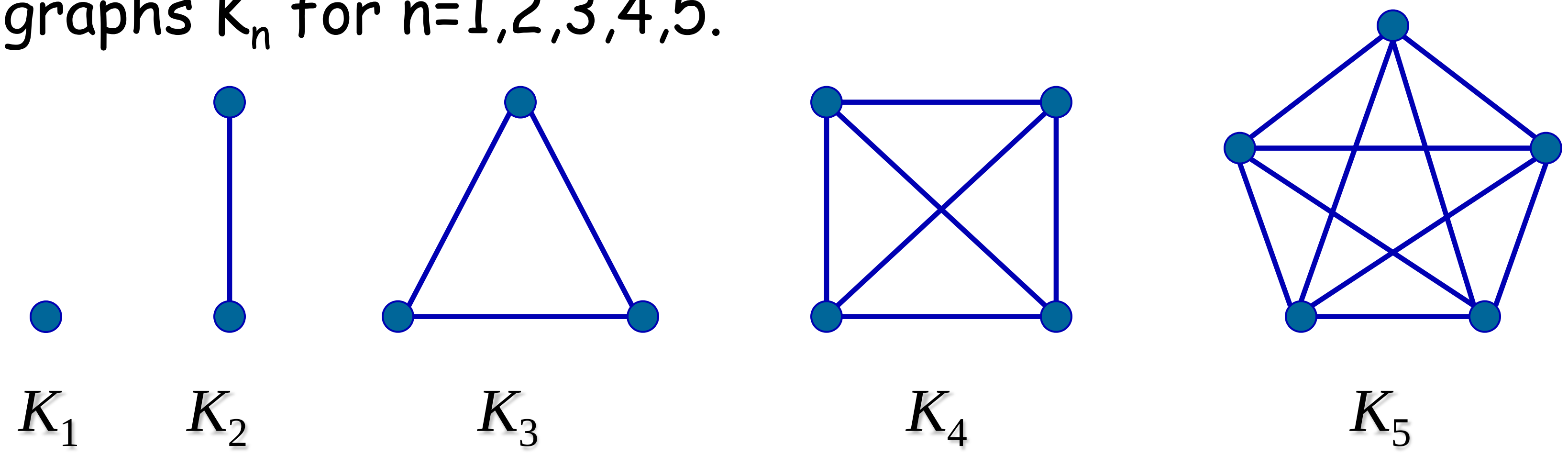
【Theorem 3】 Let G = (V, E) be a graph with direct edges. Then

$$\sum_{v\in V} d^+(v) = \sum_{v\in V} d^-(v) \quad = |E|$$

Some Special Simple Graphs

(1) Complete Graphs - K_n: the simple graph with
- n vertices
- exactly one edge between every pair of distinct vertices.

- The graphs K_n for n=1,2,3,4,5.



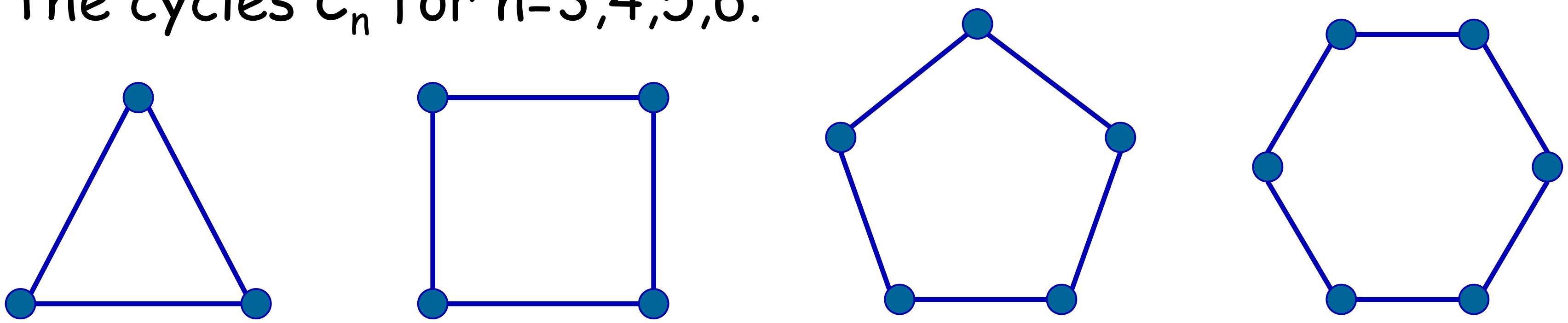
Qusetion: **The number of edges in K_n ?**

(2) Cycles C_n (n>2)

$C_n=(V,E)$, where $V=\{v_1,v_2,\ldots,v_n\}$,

$E=\{(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n),(v_n,v_1)\}$, $n \geq 3$

- The cycles C_n for n=3,4,5,6.

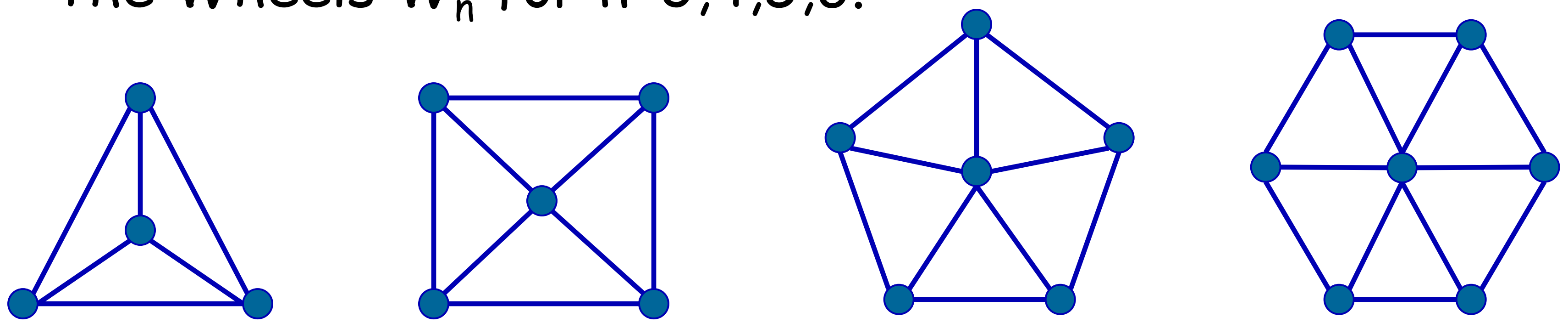


C_3 C_4 C_5 C_6

(3) Wheels W_n (n>2)

Add one additional vertex to the cycle C_n and add an edge from each vertex to the new vertex to produce W_n.

- The Wheels W_n for n=3,4,5,6.



W_3 W_4 W_5 W_6

(4) n-Cubes Q_n ($n>0$)

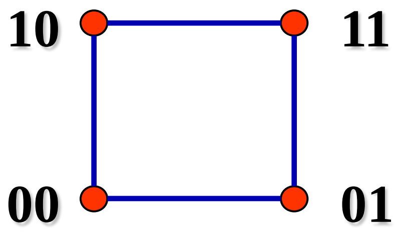
$Q_n = \langle V,E \rangle$ is the graph with 2^n vertices representing bit strings of length n, where

$V = \{ v \mid v = a_1 a_2 \ldots a_n,\ a_i = 0,1,\ i = 1,2,\ldots,n \}$

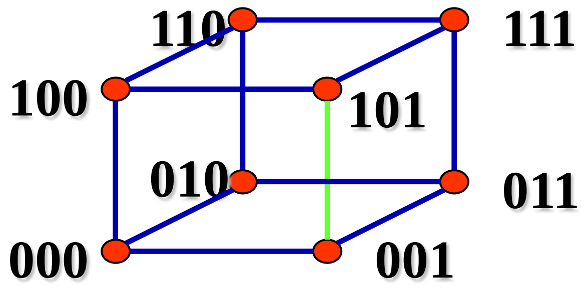
$E = \{(u,v) \mid u,v \in V \wedge u$ and v differ by one bit position$\}$.

- The n-Cubes Q_n for n=1,2,3

Q_1



Q_2

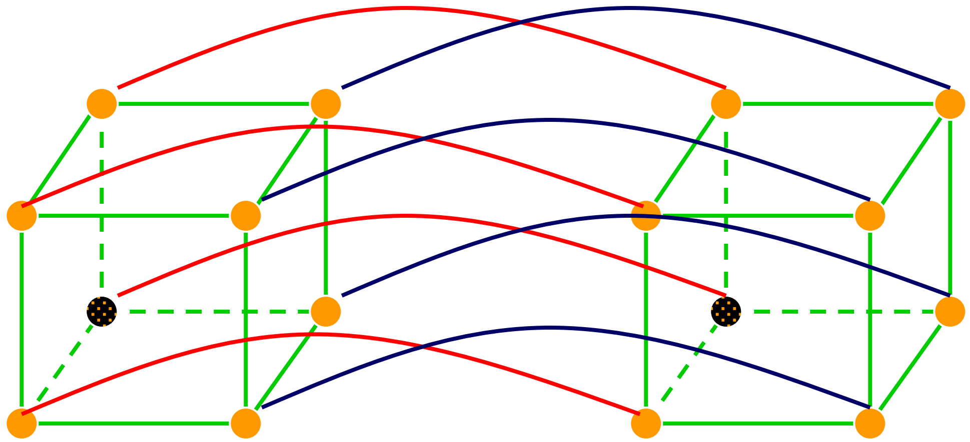


Q_3

Q_4 ?

Construct Q_{n+1} from Q_n :

- **making two copies of Q_n , prefacing the labels on the vertices with a 0 in one copy and with a 1 in the other copy**
- **adding edges connecting two vertices that have labels differing only in the first bit**

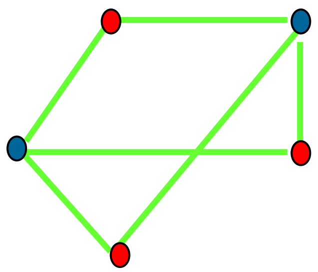


Q_4

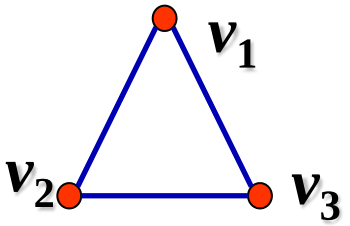
Bipartite Graphs

- A simple graph G is bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2.
- the pair $\{V_1, V_2\}$ is called a bipartition of the vertex V of G.

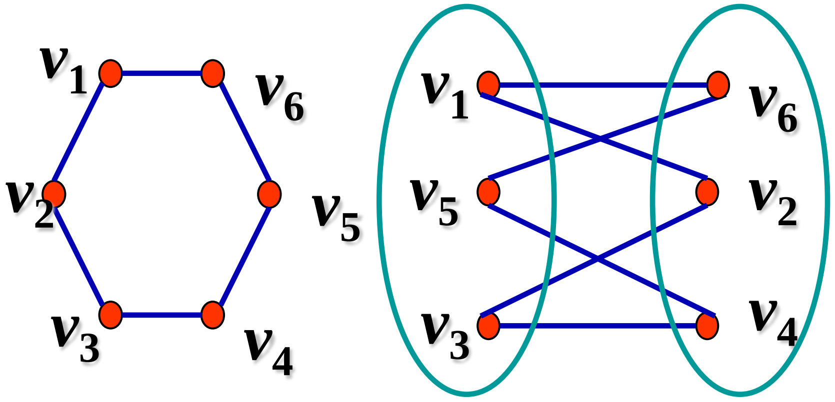
For example,



bipartite



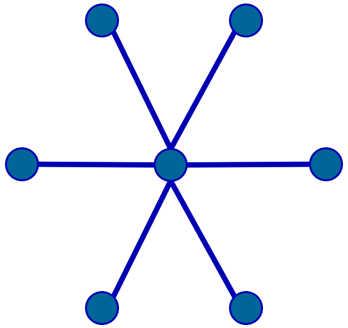
C_3 is not bipartite



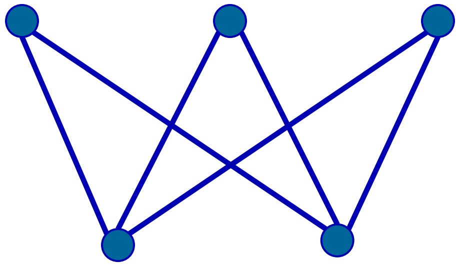
C_6 is bipartite

Bipartite Graphs

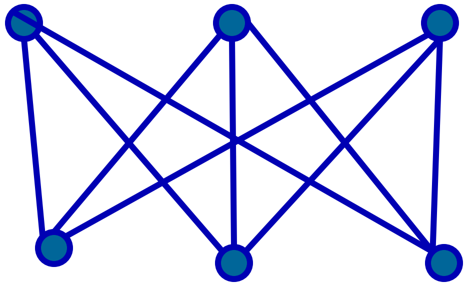
- The complete bipartite graph is the simple graph that has its vertex set partitioned into two subsets V_1 and V_2 with m and n vertices, respectively, and every vertex in V_1 is connected to every vertex in V_2, denoted by $K_{m,n}$, where $m = | V_1 |$ and $n = | V_2 |$.



$K_{1,n}$



$K_{3,2}$



$K_{3,3}$

【Theorem 4】 A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof:

(1) Suppose that $G=(V, E)$ is a bipartite simple graph. Then $V=V_1 \cup V_2$, where V_1, V_2 are disjoint sets and every edge in E connects a vertex in V_1 and a vertex in V_2.

(2) Suppose that it is possible to assign colors to the vertices of the graph using just two colors so that no two adjacent vertices are assigned the same color.

Regular graph

- A simply graph is called **regular** if every vertex of this graph has the same degree.
- A **regular graph** is called **n-regular** if every vertex in this graph has degree n.

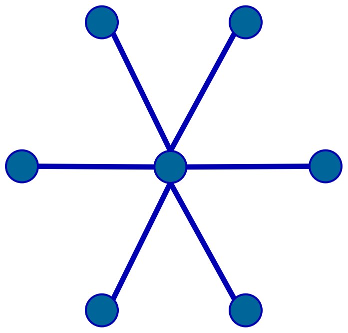
For example,

(1) k_n **is a (*n*-1)-regular.**

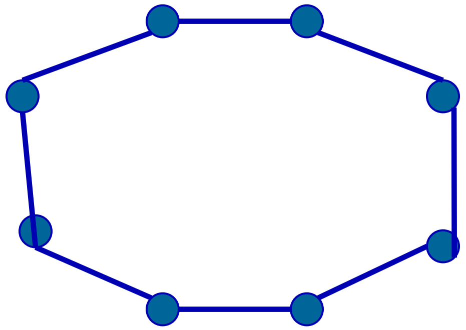
(2) For which values of *m* and *n* is $K_{m,n}$ regular?

Some applications of special types of graphs

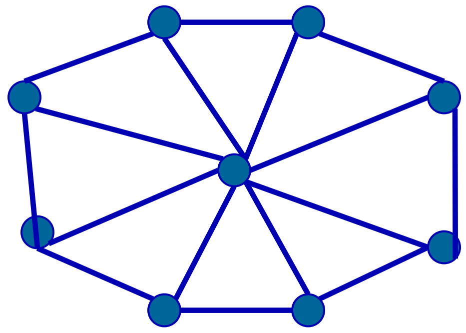
Local Area Networks.



Star topology



Ring topology



Hybrid topology

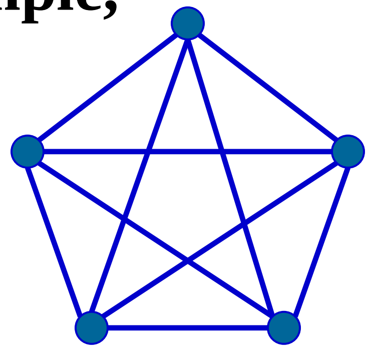
New Graphs From Old

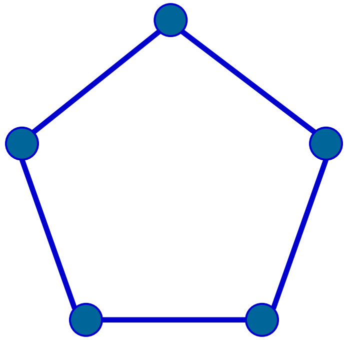
Subgraph

G= (V, E) , H= (W, F)

- ***H*** **is a** ***subgraph*** **of** ***G*** **if** $W \subseteq V, F \subseteq E$.
- **subgraph** ***H*** **is a** ***proper subgraph*** **of** ***G if H≠G*****.**
- ***H*** **is a** ***spanning subgraph*** **of** ***G*** **if** $W = V, F \subseteq E$.

For example,



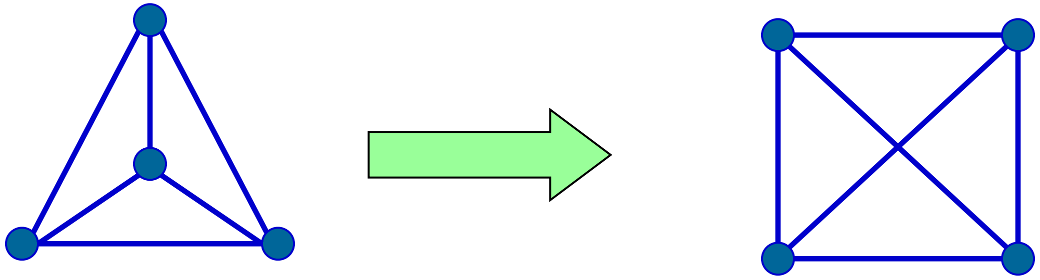


K_5

C_5 **is subgraph of** K_5

Question:

How many subgraphs with at least one vertex does W_3 have?

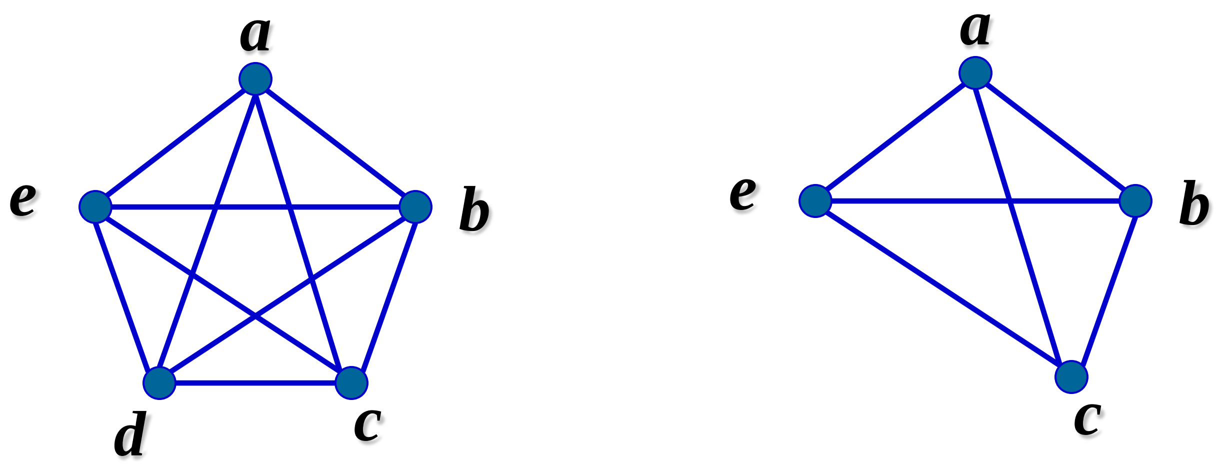


$$C(4,1) + C(4,2) \times 2 + C(4,3) \times 2^3 + C(4,4) \times 2^6$$

◆ Subgraph induced by a subset of V

Let G=(V,E) be a simple graph. The subgraph induced by a subset W of the vertex set V is the graph (W,F), where the edge set F contains an edge in E iff both endpoints of this edge are in W.

For example,



- **Removing edges of a graph**

  $G\text{-}e = (V, E\text{-}\{e\})$

- **Adding edges to a graph**

  $G\text{+}e = (V, E\cup\{e\})$

- **Edge contration**

  Remove an edge e with endpoints u and v, merge u and v into a new single vertex w, and for each edge with u or v as an endpoint replaces the edge with one with w as endpoint in place of u and v and with the same second endpoint.

- **Removing vertices from a graph**

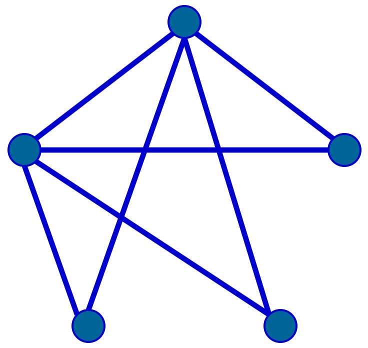
  G-v =(V-v, E'), where E' is the set of edges of G not incident to v

Graph Union

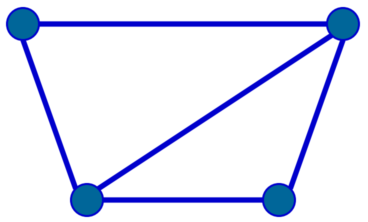
The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

Notation: $G_1 \cup G_2$

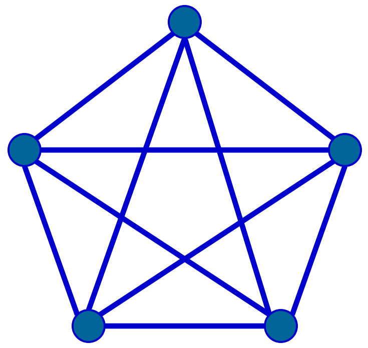
For example,



G_1



G_2



$G_1 \cup G_2 = K_5$

Homework:

SE: P. 665 5,21-25,41,53,60

EE: P. 699 5,21-25,41,55,62

10.3 Representing Graphs and Graph Isomorphism

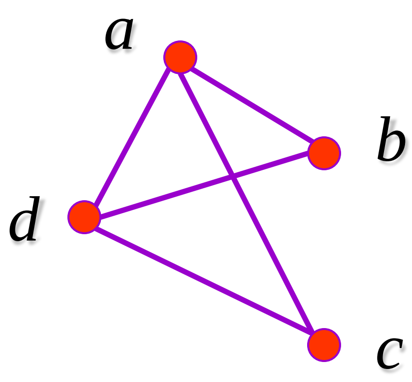
Representing Graphs

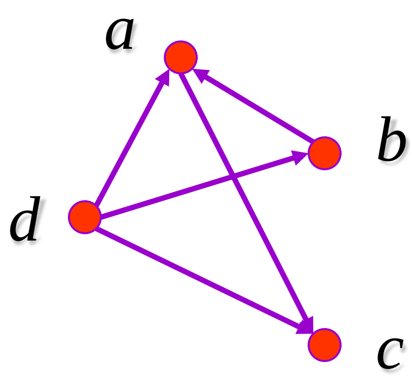
Common methods for representing graphs:

- Adjacency lists
- Adjacency matrices
- Incidence matrices
- …

Adjacency lists

- lists that specify the vertices that are adjacent to each vertex





vertex	Adjacent vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

Initial vertex	terminal vertices
a	c
b	a
c	
d	a, b, c

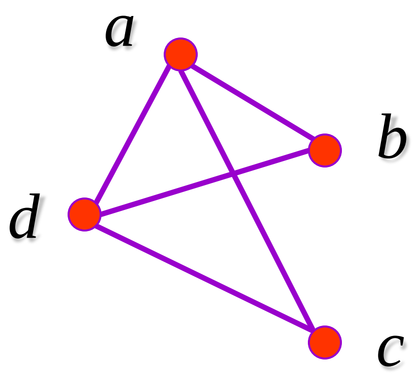
Adjacency Matrices

A simple graph $G = (V, E)$ with n vertices $(v_1, v_2, \ldots, v_n)$ can be represented by its adjacency matrix, A, where

$a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G,

$a_{ij} = 0$ otherwise.

Note: An adjacency matrix of a graph is based on the ordering chosen for the vertices.



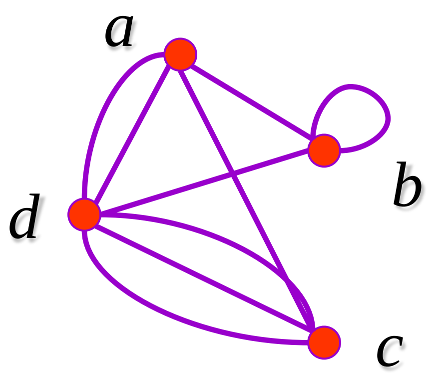
the adjacency matrix A_G based on the order of vertices a, b, c, d

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Note: Adjacency matrices of undirected graphs are always symmetric.

◆ The adjacency matrix of a multigraph or pseudograph

The (i, j)th entry of such a matrix equals the number of edges that are associated to $\{v_i, v_j\}$.



the adjacency matrix based on the order of vertices a, b, c, d

$$A_G = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

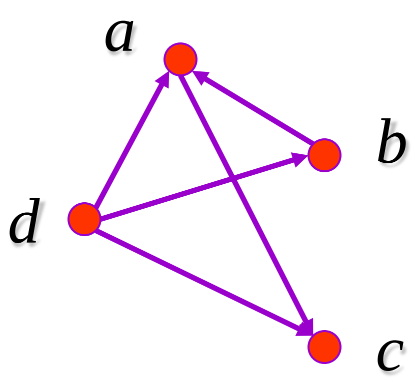
Note: **For undirected multigraph or pseudograph , adjacency matrices are symmetric.**

◆ The adjacency matrix of a directed graph

For directed graph $G = (V, E)$ with $|V| = n$, suppose that the vertices of G are listed in arbitrary order as $v_1, v_2, \ldots, v_n$, the adjacency matrix $A = [a_{ij}]$, where

$a_{ij} = 1$ if (v_i, v_j) is an edge of G,

$a_{ij} = 0$ otherwise.

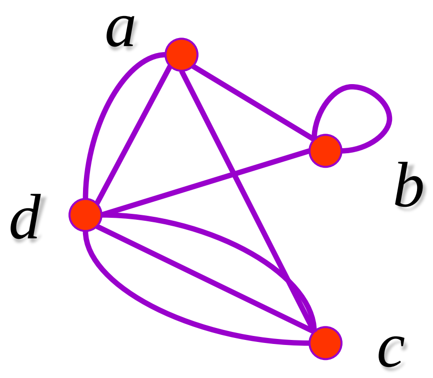


the adjacency matrix A_G based on the order of vertices a, b, c, d

$$A_G = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Question:

1. What is the sum of the entries in a row of the adjacency matrix for an undirected graph?



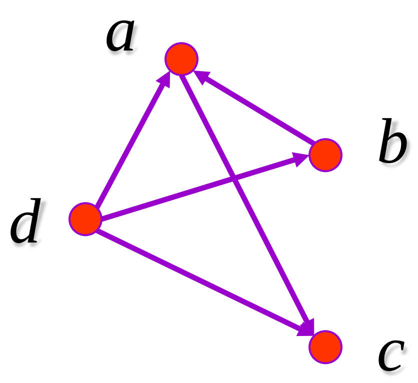
$$A_G = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

Question:

1. What is the sum of the entries in a row of the adjacency matrix for an undirected graph?

   The number of edges incident to the vertex i, which is the same as degree of i minus the number of loops at i.

   For a directed graph?



$$A_G = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Question:

1. What is the sum of the entries in a row of the adjacency matrix for an undirected graph?

   The number of edges incident to the vertex i, which is the same as degree of i minus the number of loops at i.

   For a directed graph?

   $deg^{+}(v_i)$

2. What is the sum of the entries in a column of the adjacency matrix for an undirected graph?

   The number of edges incident to the vertex i, which is the same as degree of i minus the number of loops at i.

   For a directed graph?

   $deg^{-}(v_i)$

Question:

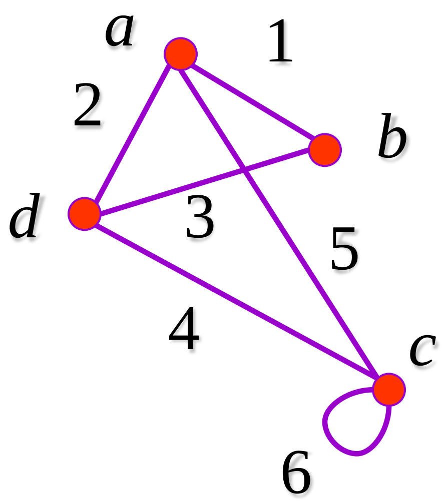
Adjacency lists or adjacency matrices ?

Incidence matrices

$G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}, E = \{e_1, e_2, \ldots, e_m\}$.

The incidence matrix with respect to this ordering of V and E is $n\times m$ matrix $\mathbf{M} = [m_{ij}]_{n\times m}$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$



$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

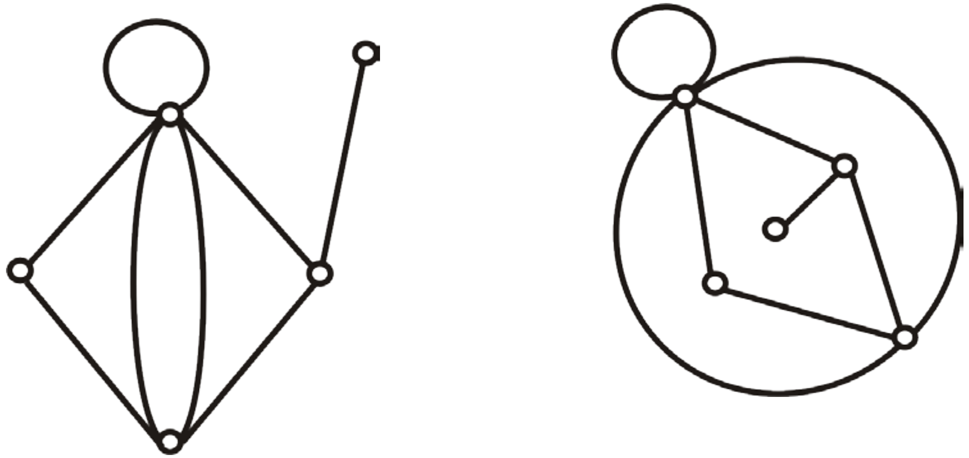
Note:

Incidence matrices of undirected graphs contain two 1s per column for edges connecting two vertices and one 1 per column for loops.

Isomorphism Of Graphs

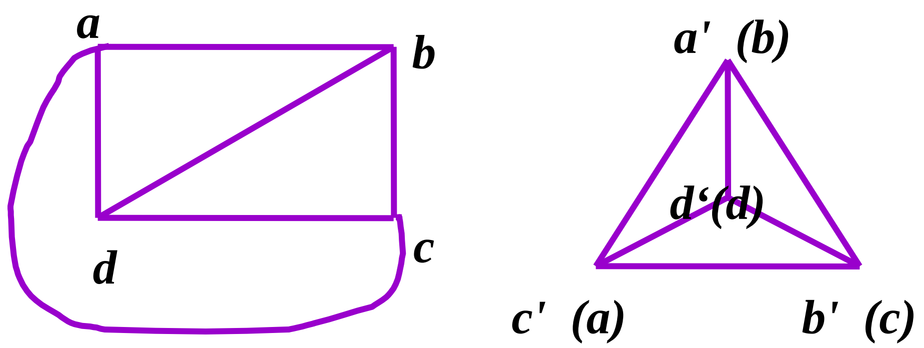
The problem of isomorphism of graphs?

It is possible that two graphs are the same although these two graphs looks very different.



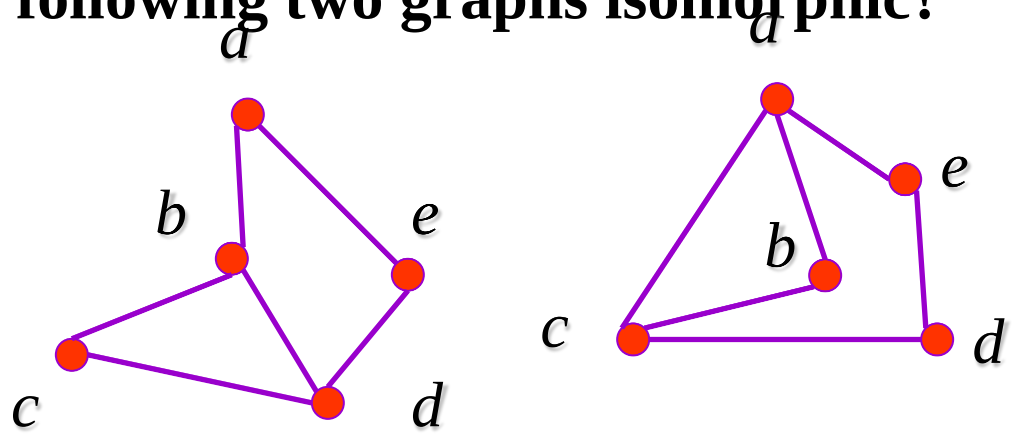
The definition of isomorphism of graphs?

- Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a 1-1 and onto function f (f is called an **isomorphism**) from V_1 to V_2 such that for all a and b in V_1, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.
- In other words, when two simple graphs are isomorphic, there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship.



How to determine?

〖Example 〗 Are the following two graphs isomorphic?



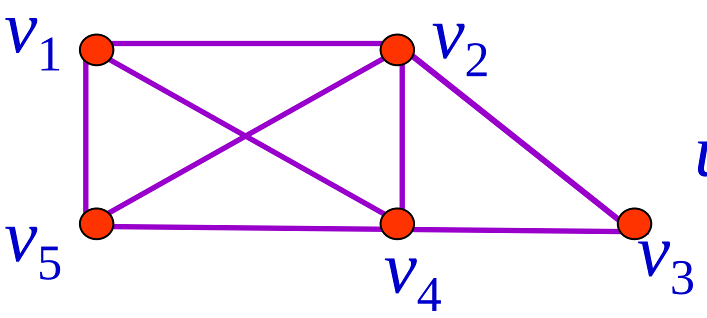
Solution:

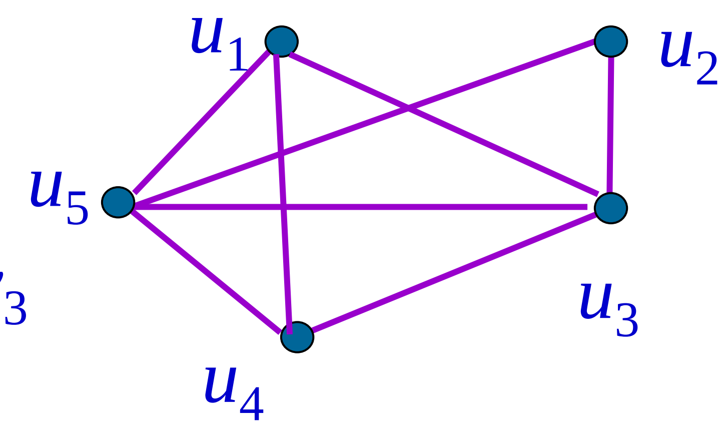
They are isomorphic, because they can be arranged to look identical.

You can see this if in the right graph you move vertex *b* to the left of the edge {*a*, *c*}. Then the isomorphism *f* from the left to the right graph is:

$f(a) = e$, $f(b) = a$,
$f(c) = b$, $f(d) = c$, $f(e) = d$.

〖Example 〗 **Show that the following two graphs are isomorphic.**





Proof:

- **Try to find an isomorphism *f***
- **Show that *f* preserves adjacency relation**

- The adjacency matrix of a graph *G* is the same as the adjacency matrix of another graph *H*, when rows and columns are labeled to correspond to the images under *f* of the vertices in *G*

$$A_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \\ u_4 \\ u_5 \\ u_2 \\ u_3 \\ u_1 \end{array} \begin{array}{c} \begin{array}{ccccc} u_4 & u_5 & u_2 & u_3 & u_1 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

- It is usually difficult to find an isomorphism f since there are $n!$ possible 1-1 correspondence between the two vertex sets with n vertices.
- some properties (called invariants) in the graphs may be used *to* show that they are not isomorphic.

Graph invariant

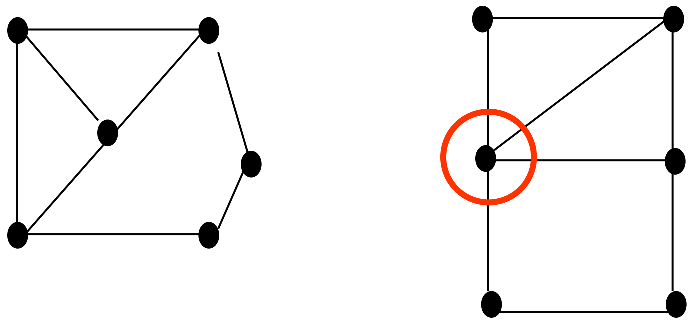
-- A property preserved by isomorphism of graphs.

Important invariants in isomorphic graphs:

- the number of vertices
- the number of edges
- the degrees of corresponding vertices
- if one is bipartite, the other must be
- if one is complete, the other must be
- if one is a wheel, the other must be

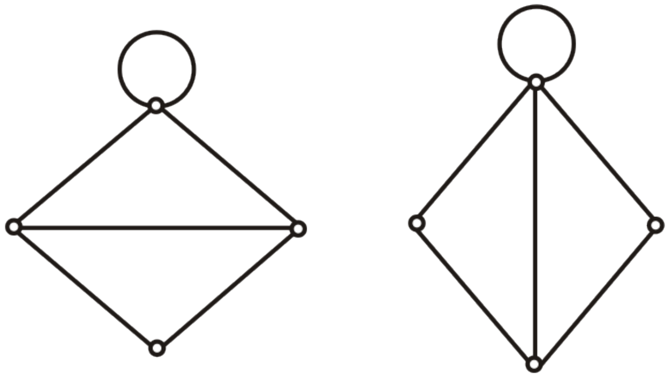
etc.

〖Example 7〗 Determine whether the given pair of graphs is isomorphic?

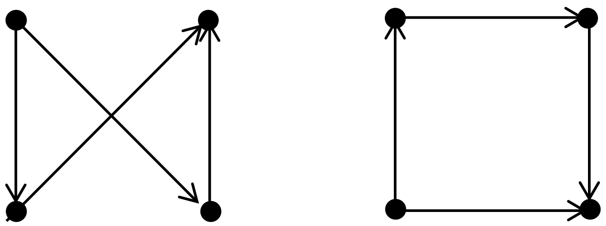


✘

✘



✘



✘

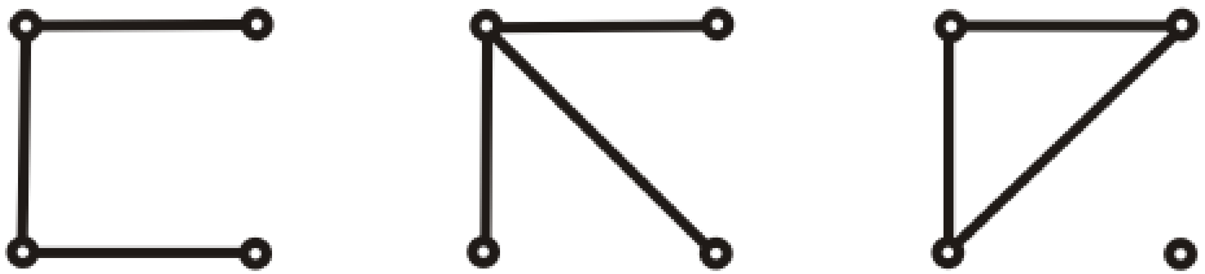
〖Example 〗 Draw all nonisomorphic undirected simple graph with four vertices and three edges.

Solution:

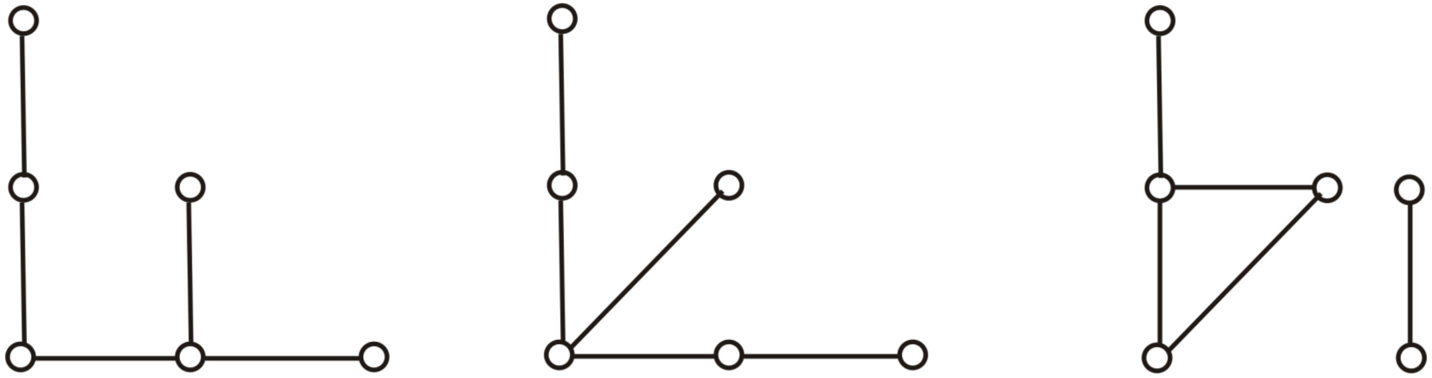
By the handshaking theorem, the sum of the degrees over four vertices is 6.

The maximal degree is 3, and the number of vertices with odd degrees must even. So there are 3 possible sequence of degrees:

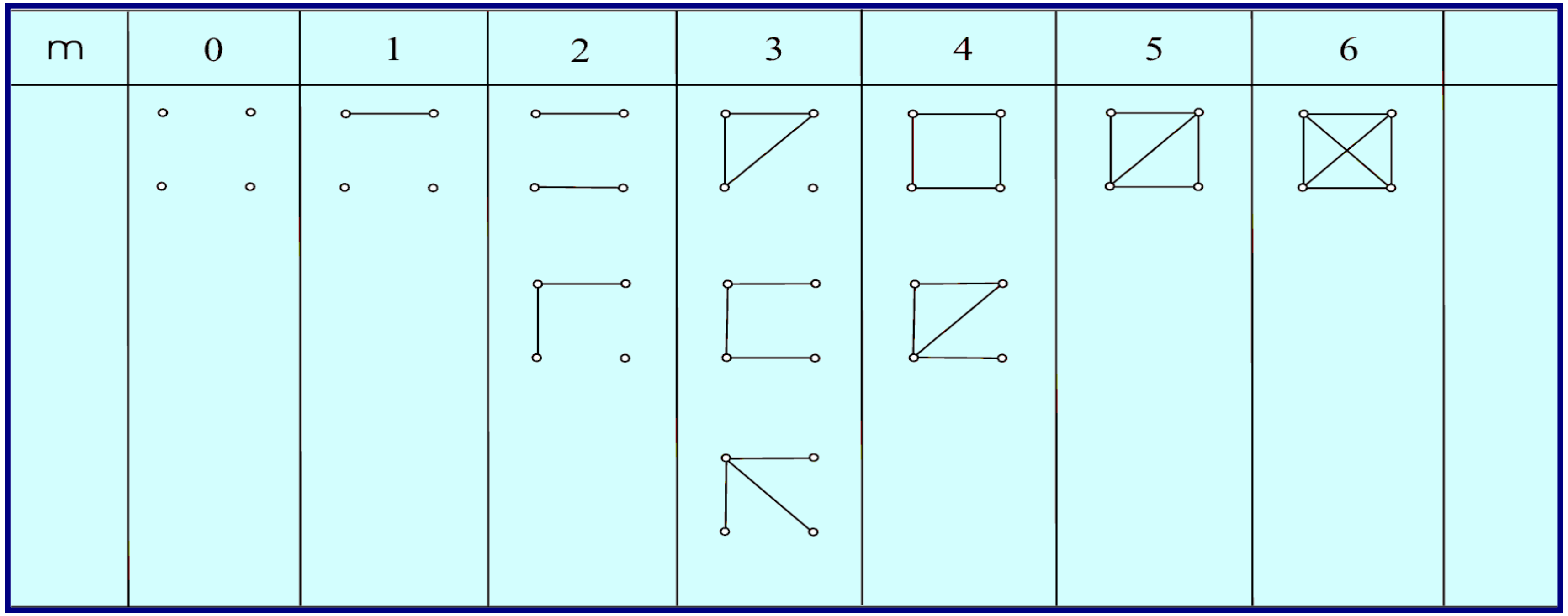
(1) 1,1,2,2; (2)1,1,1,3; (3)0,2,2,2.



〖Example 〗 Draw 3 nonisomorphic undirected simple graph with the sequence of degrees 1,1,1,2,2,3.



〖Example 〗 Draw all nonisomorphic spanning subgraphs of K_4.



Application of Graph Isomorphism

The question whether graphs are isomorphic plays an important role in applications of graph theory.

For example,

- Chemists use molecular graphs to model chemical compounds. Vertices represent atoms and edges represent chemical bonds. When a new compound is synthesized, a database of molecular graphs is checked to determine whether the graph representing the new compound is isomorphic to the graph of a compound that this already known.
- Electronic circuits are modeled as graphs in which the vertices represent components and the edges represent connections between them. Graph isomorphism is the basis for
  - the verification that a particular layout of a circuit corresponds to the design's original schematics.
  - determining whether a chip from one vendor includes the intellectual property of another vendor.

Homework:

SE: P. 675 8, 15, 17, 34-37

EE: P. 710 8, 15, 17, 38-41