Chapter 2

The Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter Summary

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities
- Functions
  - Types of Functions
  - Operations on Functions
  - Computability
- Sequences and Summations
- Set Cardinality
  - Countable Sets
- Matrices

2.1 Sets

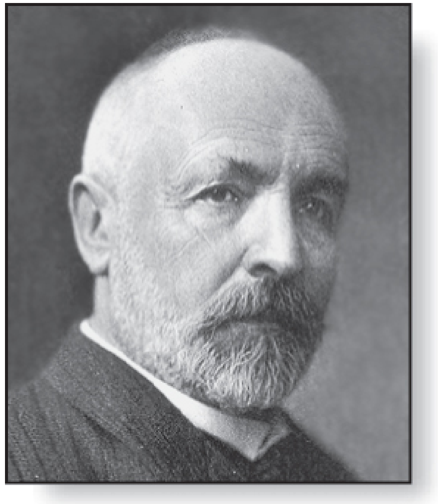
Brief introduction of set theory

- Set theory is the part of mathematics devoted to the study of set
- Set theory is the foundation of modern mathematics
  - ✓ 数学的大多数分支所研究的对象或者可以看做一种特定结构的集合，或者可以通过集合定义
  - ✓ 集合概念、方法已渗透到数学的几乎所有领域，是必不可少的数学工具和表达语言

Brief introduction of set theory

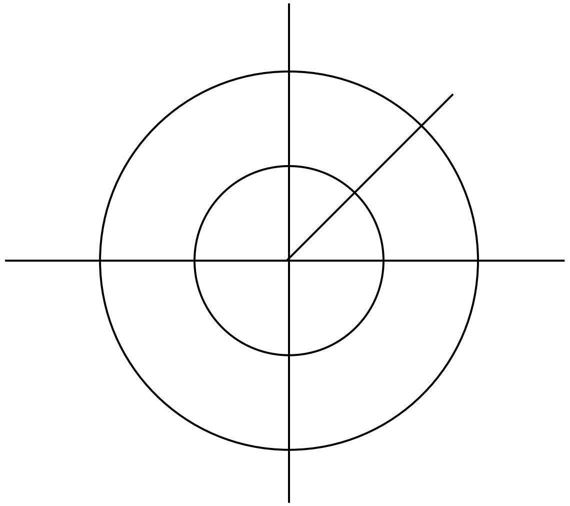
- **Learning set theory in discrete mathematics**
  - **Much of discrete mathematics is devoted to the study of discrete structures**
  - **Many important discrete structures are built using sets**
    - ✓ combinations
    - ✓ Relations
    - ✓ Graphs, etc.

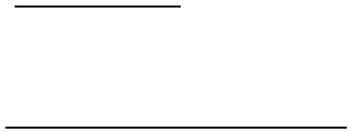
Brief introduction of set theory



G. Cantor **(1845-1918)**

- **German mathematician**
- **The founder of set theory (Naive set theory)**
- **Great achievement: research on infinite sets**

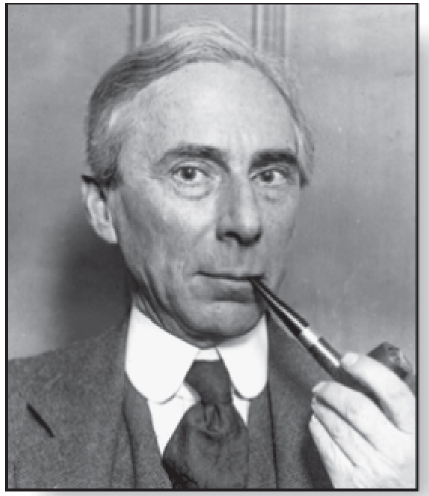




$$\mathbf{1\quad 2\quad 3\quad 4\quad \ldots\quad n\quad \ldots}$$

$$\mathbf{1\quad 2^2\quad 3^2\quad 4^2\ \ldots\ n^2\ \ldots}$$

Brief introduction of set theory



B. Russell (1872-1970)

- English Philosopher
- His most famous work "*Principia Mathematica*" (written with A. N. Whitehead)
- Russell won the Nobel Prize for literature in 1950.
- Russell Paradox

  In a town, the barber shaves all and only those who do not shave themselves. Does the barber shave himself?

  - Let $P(x) = x \notin x$ and $B = \{x \mid x \notin x\}$
  - $B \in B$–true or false?
  - $B \in B$ =) $B \notin B$, $B \notin B$ =) $B \in B$
  - Contradiction!

Some Concepts of set theory

【Definition】A set is an unordered collection of objects.

The objects in a set are called the elements, or members, of the set.

A set is said to contain its elements.

Note:

- **Uppercase letters are usually used to denote sets, and lowercase letters are usually used to denote elements of sets.**
- $a \in A$: **a is a member of A or a is an element of A**

  $a \notin A$: **a is not an element of A**

The description of set

One can describe a set by

1. Roster method: list all the members of a set, when this is possible (enumeration)

For example,

(1)The set of all odd positive integers less than 10.

{1,3,5,7,9}

(2) $S = \{a, b, c, d\}$

Brace notation with ellipses

For example,

$S = \{\ldots, -3, -2, -1\}$

The description of set

2. Use set builder notation (specification by predicates)

   The universal set U:

   ----The set contains all the objects under consideration.

   S= {x| P(x)}

   ---- S contains all the elements from U which make the predicate P true.

For example,

$\{x \mid x$ **is an odd positive integers less than 10**$\}$

〖Example 1〗 Some examples of set.

(1) N = { 0,1,2,3,…} the set of natural numbers

(2) Z = { x | x is an integer} the set of integers

(3) Z^+ = { 1,2,3,…} the set of positive integers

(4) Q= {p/q |$p \in$Z, $q \in$Z, and $q \neq 0$}, the set of rational numbers

(5) R = {x | x is a real number}

〖Example 2〗 The set {N, Z, Q, R}

Remark: The concept of a datatype, or type, in computer science is the name of a set, together with a set of operations that can be performed on objects from that set.

the empty set

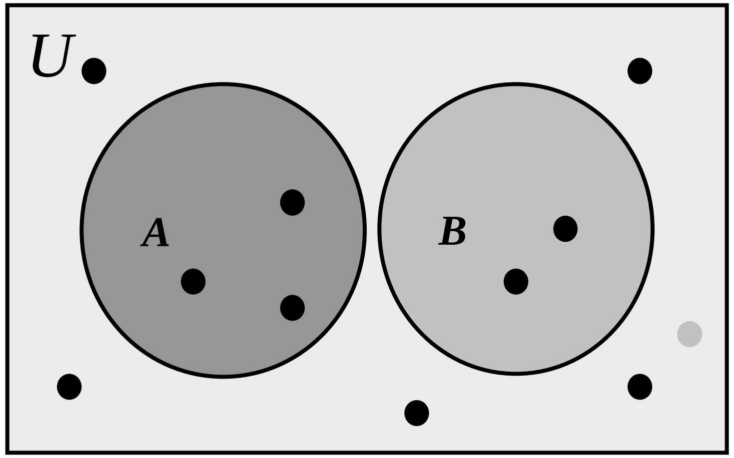
- **the set with no elements**
- **The *void* set, the *null* set**
- **Notation: ϕ, {}**

Singleton set

- **A set with one element.**
- **Note that ϕ and $\{\phi\}$ are different.**

Venn diagrams

- In Venn diagrams, the universal set *U* is represented by a rectangle.
- Inside this rectangle, circles or other geometrical figures are used to represent sets.
- Some points are used to represent the particular elements of the set.



Subsets

$A \subseteq B$ -- A is a subset of the set B.

every element of A is also an element of B.

$$A \subseteq B \Leftrightarrow \forall x\,(x \in A \rightarrow x \in B)$$

Problems:

For any set A,

(1) $\phi \subseteq A$?

Y. The empty set is a subset of any set.

(2) $A \subseteq A$?

Y.

Showing that A is a subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a subset of B To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.

Equal

A = B: contain exactly the same elements

$$A = B \Leftrightarrow \forall x[(x \in A \to x \in B) \wedge (x \in B \to x \in A)]$$
$$\Leftrightarrow A \subseteq B \quad and \quad B \subseteq A$$

Problem: ϕ is unique?

〖Example 2〗 $S1 = \{a, b, c, d\}$, $S2 = \{a, d, b, c\}$, $S3 = \{b, c, a, d, d\}$.

$S1 = S2 = S3$?

Note:

- **The Order of elements does not matter.**
- **Repetition of elements does not matter.**

Showing Two Sets are Equal ***To show that A=B , show that A $\subseteq$ B and B $\subseteq$ A.***

Proper subset

$$\boldsymbol{A \subset B \Leftrightarrow \forall x\ (x \in A \rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)}$$

$$\Leftrightarrow A \subseteq B \quad and \quad A \neq B$$

The Size of a Set

【Definition】 Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S.

Notation:

$|S|$ ---- **the cardinality of** S

【Definition】 A set is said to be infinite if it is not finite.

〖Example 3〗

(1) Let A be the set of odd positive integers less than 10.

Then $|A| =$?

(2) $B = \{1, \{2, 3\}, \{4, 5\}, 6\}$ $|B| = 4$

(3) $C = \varnothing$ $|C| = 0$

(4) $D = \{ x \in N \mid x \leq 7000 \}$ $|D| = 7001$

(5) $E = \{ x \in N \mid x \geq 7000 \}$ E is infinite!

Problem: What is the cardinality of an infinite set?

Power Sets

〖Example 4〗 **Let** $A=\{a,b,c\}$, **find all subsets of the set *A*.**

Solution:

ϕ

$\{a\},\{b\},\{c\}$

$\{a, b\},\{b, c\},\{a, c\}$

$\{a, b, c\}$

Problem: How many subsets of the set *A* are there?

$$C_3^0+C_3^1+C_3^2+C_3^3=8$$

In general, for a set *A* with *n* elements, the number of subsets is

$$C_n^0+C_n^1+C_n^2+\ldots+C_n^n=\sum_{i=0}^{n}C_n^i=2^n$$

$$(x+y)^n=\sum_{i=0}^{n}C_n^i x^i y^{n-i}=2^n \qquad (\because x=y=1)$$

Power Sets

【**Definition** 】 Given a set S, the **power set** of S is the set of all subsets of the set S.

Notation:

$P(S)$ ---- the power set of S.

$$P(S) = \{x \mid x \subseteq S\}$$

Note:

(1) $|S|=n$ implies $|P(S)| = 2^n$

(2) S is finite and so is $P(S)$.

(3) $$\begin{cases} x \in P(S) \Rightarrow x \subseteq S \\ x \in S \Rightarrow \{x\} \in P(S) \\ S \in P(S) \end{cases}$$

〖Example 5〗 **What is the power set of the set** $\{\phi\} and \{\phi,\{\phi\}\}$ **?**

Solution:

(1) $S=\{\phi\}$

$$P(S)=\{\phi,\{\phi\}\}, \qquad |P(S)|=2$$

(2) $S=\{\phi,\{\phi\}\}$

$$P(S)=\{\phi,\{\phi\},\{\{\phi\}\},\{\phi,\{\phi\}\}\}, \qquad |P(S)|=4$$

〖Example 6〗 **Show that** $P(A) \in P(B) \Rightarrow A \in B$ **?**

Proof:

$$P(A) \in P(B) \Rightarrow P(A) \subseteq B$$

$$A \in P(A) \subseteq B \Rightarrow A \in B$$

Problem:

$$A \subseteq B \Rightarrow P(A) \subseteq P(B)?$$

Cartesian Products

【Definition】 The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, … , and a_n as its nth element.

$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \Leftrightarrow a_i = b_i \ (i=1,2,\ldots, n)$

In particular, 2-tuples are called ordered pairs.

Note:

If $x \neq y$, then $(x, y) \neq (y, x)$.

$(x, y) = (u, v) \Rightarrow x = u \ \ and \ \ y = v$

The *Cartesian product* of A and B: $A \times B = \{(a, b) \mid a \in A, b \in B\}$

The *Cartesian product* of $A_1, A_2, \ldots, A_n$:

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \ \text{ for } i=1, 2, \ldots, n\}$.

〖Example 7〗 $A = \{a, b\}, B = \{0,1,2\}$ **,** $A \times B = ?\ \ B \times A = ?$

solution:

$$A \times B = \{(a,0),(a,1),(a,2),(b,0),(b,1),(b,2)\}$$

$$B \times A = \{(0,a),(1,a),(2,a),(0,b),(1,b),(2,b)\}$$

Note:

- **If |*A*|=*m*, |*B*|=*n*, Then** $|A \times B| = |B \times A| = mn$
- $A \times B \neq B \times A$
- $A \times \phi = \phi \times A = \phi$
- $(x, y) \in A \times B \Rightarrow x \in A \;\; and \;\; y \in B; (x, y) \notin A \times B \Rightarrow x \notin A \;\; or \;\; y \notin B$

Using Set Notation with Quantifiers

- **Restrict the domain of a quantified statement explicitly by making use of a particular notation.**

$$\forall x \in S(P(x)): \quad \forall x(x \in S \to p(x))$$

$$\exists x \in S(P(x)): \quad \exists x(x \in S \land p(x))$$

Problems: What do the following statements mean?

$\forall x \in \mathrm{R}\ (x^2 \geq 0)$ **?**

$\exists x \in \mathrm{Z}\ (x^2 = 1)$ **?**

Truth Sets of Quantifiers

Given a predicate P, and a domain D.

The **truth set** of P to be the set of elements x in D for which p(x) is true. namely,

The **truth set** of P: $P = \{x \mid p(x)\}$

Note:

$\forall x\, P(x)$ is true over the domain U if and only if the truth set of P is the set U.

$\exists x\, P(x)$ is true over the domain U if and only if the truth set of P is nonempty.

2.2 Set Operations

Section Summary

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
  - Symmetric difference
- More on Set Cardinality
- Set Identities
- Proving Identities
- Membership Tables

Two, or more, sets can be combined in many different ways:

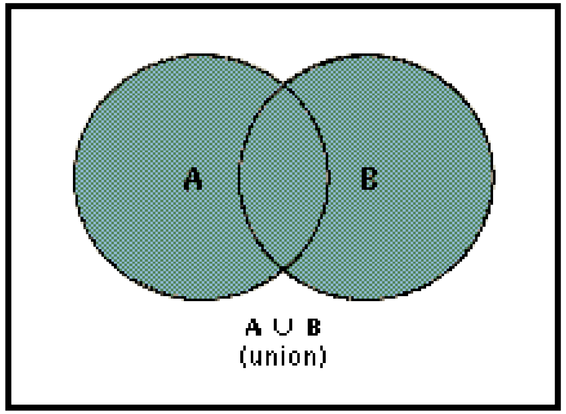
- Union
- Intersection
- Difference
- Symmetric difference

Union

$A \cup B$: The union of the sets A and B

the set that contains those elements that are either in A or in B, or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



Useful rules:

$(1) A \subseteq A \cup B, \qquad B \subseteq A \cup B$

$(2) A \subseteq C, B \subseteq C \Rightarrow A \cup B \subseteq C$

$(3) |A \cup B| \leq |A| + |B|$

$(4) A \cup B = B \Leftrightarrow A \subseteq B$

Intersection

A∩B: The intersection of sets A and B

the set containing those elements in both A and B.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Note: Two sets are called disjoint if their intersection is the empty set, namely $A \cap B = \varnothing$

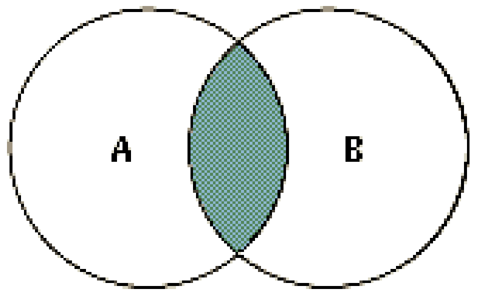
Useful rules:

$(1) A \cap B \subseteq A, \qquad A \cap B \subseteq B$

$(2) C \subseteq A, C \subseteq B \Rightarrow C \subseteq A \cap B$

$(3) |A \cap B| \leq |A|, |A \cap B| \leq |B|$

$(4) A \cap B = A \Leftrightarrow A \subseteq B$



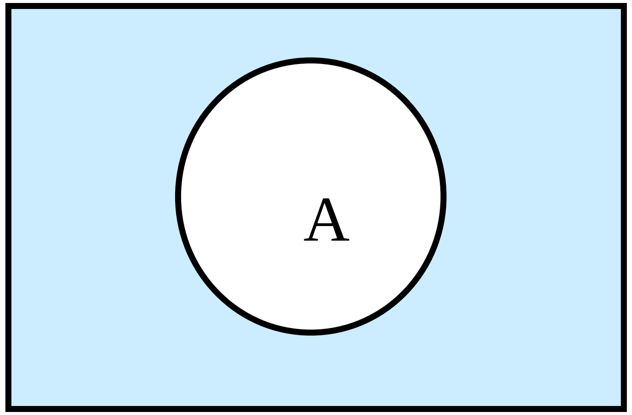
A ∩ B
(intersection)

Complement

Let U be universal set. The complement of the set A denoted by $\bar{A}$,is the complement of A with respect to U, namely, U – A.

(The complement of A is sometimes denoted by A^c.)

$$\bar{A} = \{x \mid x \notin A, x \in U\} \text{ or } \{x \mid \neg(x \in A)\}$$



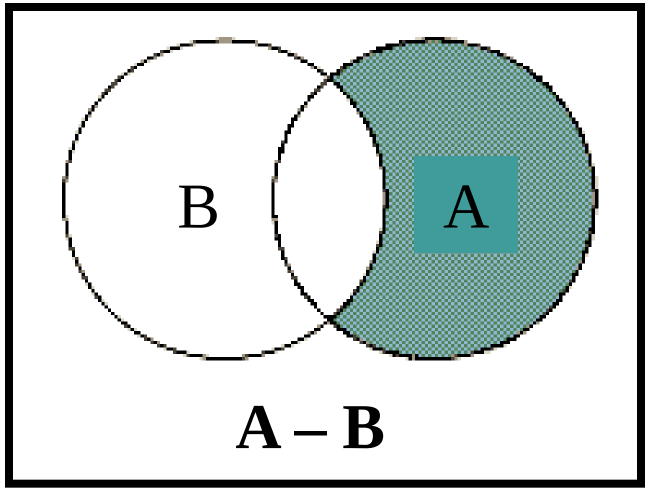
Difference

A- B: The difference of A and B

the set containing those elements that are in A but not in B.

Note: The difference of *A* and *B* is also called the complement of *B* with respect to *A*.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$
$$= A \cap \overline{B}$$

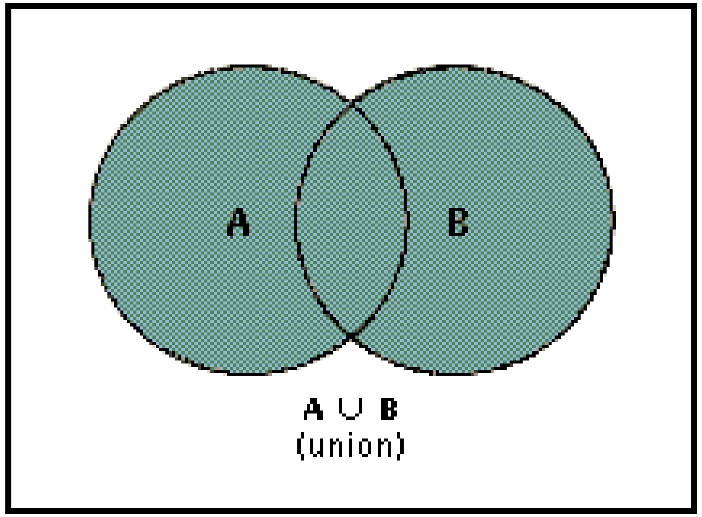


The Cardinality of a Union of Two Sets

- **The principle of Inclusion-exclusion**

$$\underbrace{|A \cup B| = |A| + |B|}_{} - |A \cap B|$$

counts each elements that is in *A* but not in *B* or in *B* but not in *A* exactly once, and each element that is in both *A* and *B* exactly twice



The Cardinality of a Union of Two Sets

- **The principle of inclusion-exclusion is an important techniques used in enumeration.**

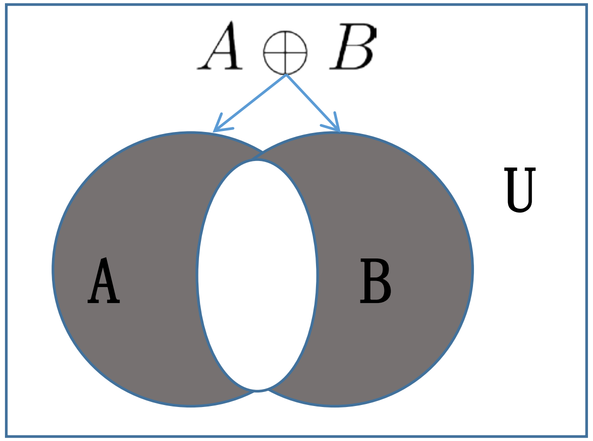
- **Example: Let *A* be the math majors in your class and *B* be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.**

- Note:
- We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Symmetric difference

$A \oplus B$: **The symmetric difference of A and B**

$$A \oplus B = (A \cup B) - (A \cap B)$$



Example:

U = {0,1,2,3,4,5,6,7,8,9,10}

A = {1,2,3,4,5} B ={4,5,6,7,8}

What is $A \oplus B$?

Solution: {1,2,3,6,7,8}

Set Identities

Set Identities	
Identity	Name
$A \cup \varnothing = A,\ A \cap U = A$	Identity laws
$A \cup U = U,\ A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A,\ A \cap A = A$	Idempotent laws
$\overline{\overline{A}} = A$	Complementation law
$A \cup B = B \cup A,\ A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan’s laws

Proving Set Identities

Different ways to prove set identities:

① Subset method

② Membership Tables

③ Apply existing identities

Proof of First De Morgan Law

〖Example 1〗 **Show that** $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ **?**

solution:

(1) ***A* = *B* iff *A* ⊆ *B* and *B* ⊆ *A***

❖ $\overline{(A \cup B)} \subseteq \overline{A} \cap \overline{B}$

Suppose that $x \in \overline{(A \cup B)}$.

It follows that $x \notin A \cup B$.

This implies that $x \notin A$ and $x \notin B$.

Hence, $x \in \overline{A}$ and $x \in \overline{B}$.

Thus $x \in \overline{A} \cap \overline{B}$.

❖ $\overline{A} \cap \overline{B} \subseteq \overline{(A \cup B)}$

Set-Builder Notation: First De Morgan Law

〖**Example 1**〗 **Show that** $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ **?**

solution:

$$\overline{(A \cup B)} = \{x \mid x \notin A \cup B\}$$
$$= \{x \mid \neg(x \in A \cup B)\}$$
$$= \{x \mid \neg(x \in A \vee x \in B)\}$$
$$= \{x \mid \neg(x \in A) \wedge \neg(x \in B)\}$$
$$= \{x \mid x \notin A \wedge x \notin B\}$$
$$= \{x \mid x \in \overline{A} \wedge x \in \overline{B}\}$$
$$= \{x \mid x \in \overline{A} \cap \overline{B}\}$$
$$= \overline{A} \cap \overline{B}$$

Membership Table

Verify that elements in the same combination of sets belong to both the sets in the identity

〖Example 2〗 Let *A*, *B*, and *C* be sets. Show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

solution:

A	B	C	B∪C	A∩(B∪C)	A ∩ B	A ∩ C	(A ∩ B) ∪(A ∩ C)
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

〖Example 3〗 Show that $(A-B)\cup B = A\cup B$ **?**

solution:

$$(A-B)\cup B=(A\cap\overline{B})\cup B$$

$$=(A\cup B)\cap(\overline{B}\cup B)$$

$$=(A\cup B)\cap U$$

$$=A\cup B$$

Methods of Proving Set Identities

TABLE 3 Methods of Proving Set Identities.	
Description	***Method***
Subset method	Show that each side of the identity is a subset of the other side.
Membership table	For each possible combination of the atomic sets, show that an element in exactly these atomic sets must either belong to both sides or belong to neither side
Apply existing identities	Start with one side, transform it into the other side using a sequence of steps by applying an established identity.

〖Example 4〗 Simplify the following set.

$$((A\cup B\cup C)\cap(A\cup B)-((A\cup(B-C))\cap A)$$

solution:

$$((A\cup B\cup C)\cap(A\cup B)-((A\cup(B-C))\cap A)$$

$$=(A\cup B)-A$$

$$=(A\cup B)\cap\overline{A}$$

$$=(A\cap\overline{A})\cup(B\cap\overline{A})$$

$$=\phi\cup(B\cap\overline{A})$$

$$=B-A$$

Generalized Unions and Intersections

- Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

  We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

Contains those elements that are members of at least one set in the collection

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

Contains those elements that are members of all the set in the collection

These are **well defined**, since union and intersection are associative

Example, For $i = 1,2,\ldots,$ let $A_i = \{i, i + 1, i + 2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots..\} = A_n$$

Generalized Unions and Intersections

- the union or intersection of the infinite family of sets $A_1, A_2, \ldots, A_n, \ldots$

$$A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots = \bigcup_{i=1}^{\infty} A_i.$$

$$A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots = \bigcap_{i=1}^{\infty} A_i.$$

More generally, when I is a set, the notations $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ are used to denote the intersection and union of the sets A_i for $i \in I$, respectively. Note that we have $\bigcap_{i \in I} A_i = \{x \mid \forall i \in I\, (x \in A_i)\}$ and $\bigcup_{i \in I} A_i = \{x \mid \exists i \in I\, (x \in A_i)\}$.

Computer Representation of Set

Using bit strings to represent sets.

Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used).

(1) Specify an arbitrary ordering of the elements of *U*,

for instance $a_1, a_2, \ldots, a_n$

(2) Represent a subset *A* of *U* with the bit string of length *n*

Using bit strings to represent sets, it is easy to find complements of sets and union, intersection, and differences of sets.

〖Example 5〗 Let U ={1, 2, 3, 4, 5,6,7,8,9}, A={1, 2, 3, 4, 5}, B= (1, 3, 5, 7, 9). Use bit strings to find the difference of A and B.

solution:

(1) The bit string for the set A: 11 1110 000

The bit string for the set B: 10 1010 101

(2) $A - B = A \cap \overline{B}$

Questions:

Write programs.

Gives subsets A and B of a set with n elements, use bit strings to find $\overline{A}, A \cup B, A \cap B, A - B, A \oplus B$

Homework:

SE: P.125 11, 24, 25, 37, 39

P.136 24, 40, 48, 61

EE: P.132 13, 26, 27, 39,41

P.145 26, 46, 54, 67, 71(a)

The **Jaccard similarity** $J(A, B)$ of the finite sets A and B is $J(A, B) = |A \cap B|/|A \cup B|$, with $J(\emptyset, \emptyset) = 1$. The **Jaccard distance** $d_J(A, B)$ between A and B equals $d_J(A, B) = 1 - J(A, B)$.

71. Find $J(A, B)$ and $d_J(A, B)$ for these pairs of sets.

a) $A = \{1, 3, 5\}, B = \{2, 4, 6\}$

b) $A = \{1, 2, 3, 4\}, B = \{3, 4, 5, 6\}$

c) $A = \{1, 2, 3, 4, 5, 6\}, B = \{1, 2, 3, 4, 5, 6\}$

d) $A = \{1\}, B = \{1, 2, 3, 4, 5, 6\}$