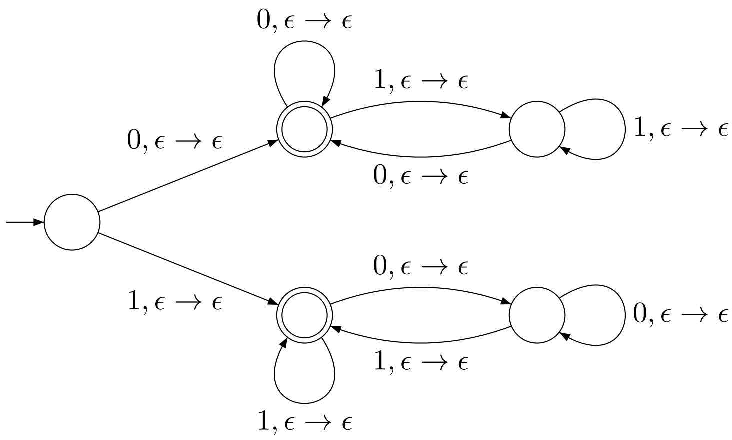
The University of Melbourne
School of Computing and Information Systems
COMP30026 Models of Computation

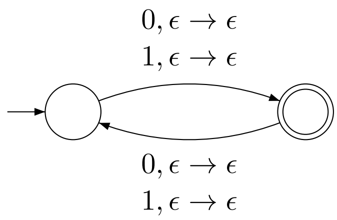
Selected Tutorial Solutions, Week 11

90. Here are PDAs for the languages from Exercise 77.

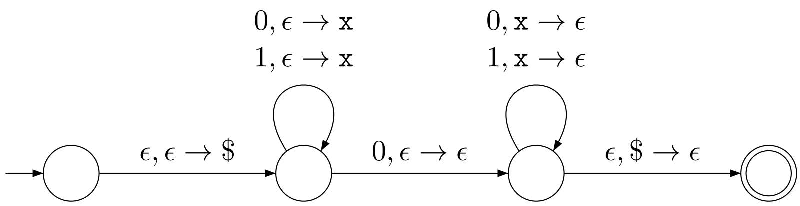
(a) The language is regular, so a PDA will not need to use its stack at all. The result is very similar to a DFA seen in a lecture:



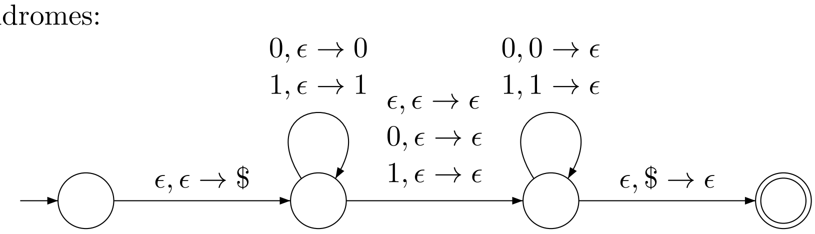
(b) Again, the language is regular—too easy:



(c) Odd-length strings with 0 in the middle:



(d) Palindromes:



91. The first two PDAs were both progressive and deterministic. The last two were neither.

92. For the case $v \neq \epsilon$ we define

$$\delta((q_p, q_d), v, x) = \{ \ ((r_p, r_d), y) \mid (r_p, y) \in \delta_P(q_p, v, x) \wedge r_d = \delta_D(q_d, v) \ \}$$

But we must also allow transitions that don't consume input, so:

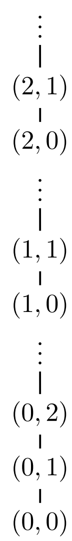
$$\delta((q_p, q_d), \epsilon, x) = \{ \ ((r_p, q_d), y) \mid (r_p, y) \in \delta_P(q_p, \epsilon, x) \ \}$$

93. (a) Assume that A is context-free and let p be the pumping length. Consider the string $\mathsf{a}^p\mathsf{b}^p\mathsf{a}^p\mathsf{b}^p \in A$. The pumping lemma tells us that the string can be written $uvxyz$, with v and y not both empty, and with $|vxy| \leq p$, such that $uv^ixy^iz \in A$ for all i. Clearly, if one (or both) of v and y contains an a as well as a b then pumping up will lead to a string that is not in A, as the result will have more than two substrings ab. So each of v and y must contain as only, or bs only, unless it is empty. If neither v nor y contains a b then both must come from the same a^i segment (the first or the last), because $|vxy| \leq p$; and then, when we pump up, that segment alone grows, while the other a^i segment is untouched. Similarly, if neither v nor y contains an a. So in these cases the result of pumping is not in A. The only remaining cases are when v is from the first a^i segment and y is from the first b^j segment, or v is from the first b^j segment and y is from the second a^i segment, or v is from the second a^i segment and y is from the second b^j segment. In each case, pumping up will take the string outside A. We conclude that A is not context-free.

(b) Here is a context-free grammar for B:

$$\begin{array}{lcl} S & \rightarrow & T \mid \mathsf{a}\ S\ \mathsf{b} \\ T & \rightarrow & \epsilon \mid \mathsf{b}\ T\ \mathsf{a} \end{array}$$

(c) If we pick the obvious candidate string $\mathsf{a}^p\mathsf{b}^p\mathsf{a}^p\mathsf{b}^p \in B$, we fail to get a contradiction with the pumping lemma. Namely we might have $v = \mathsf{b}^k$ and $y = \mathsf{a}^k$ $(0 < k \leq p/2)$ where v is a substring of the first b^j segment and y is a substring of the second a^j segment. In that case, the result of pumping (up or down) is in B, and we don't get a contradiction.



94. (b) is not well-founded, as we can have infinite strictly decreasing sequences in $\mathbb{Q}$, such as $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. But (a), (c) and (d) are all well-founded. For (d) it may help to look at the Hasse diagram for $\mathbb{N} \times \mathbb{N}$ ordered by $\prec$ (shown here in the margin).

95. We can describe the contents of the bag with a triple $(w, b, r) \in \mathbb{N}^3$. The triple stands for w white, b blue, and r red marbles being present in the bag. We claim that each round decreases the content of the bag, according to the lexicographic ordering of triples. Namely, look at what happens in one round, assuming the current state is described by (w, b, r):

- In case (b) we will be left with either $(w-1, b, r-1)$, $(w, b-1, r-1)$, or $(w, b, r-2)$, depending on whether the red marble's companion is white, blue, or red, respectively.
- In case (c) we are left with $(w-1, b+5, r)$.
- In case (d) one marble is blue and the other is white or blue. We are left with either $(w-1, b-1, r+10)$ or $(w, b-2, r+10)$, according as the blue marble's companion is white or blue, respectively.

Each of those six triples is strictly smaller than (w, b, r) in the lexicographic ordering. Since this ordering is well-founded on $\mathbb{N}^3$, the process must halt after a finite number of rounds.