

Selected Tutorial Solutions, Week 7

51. (a) We can negate both sides of the biimplication, so we just need to show:

$$A \subseteq B \Leftrightarrow A \setminus B = \emptyset$$

The left-hand side is, by definition: $\forall x(x \in A \Rightarrow x \in B)$. The right-hand side can be written: $\neg \exists y(y \in A \wedge y \notin B)$. Pushing the negation in, we get $\forall y(y \notin A \vee y \in B)$, or equivalently, $\forall y(y \in A \Rightarrow y \in B)$.

- (b) It is easier to look at the logical expressions. The left-hand side is $\{x \mid x \in A \wedge x \in B\}$. The right-hand side is

$$\begin{aligned} & \{x \mid x \in A \wedge x \notin A \setminus B\} \\ = & \{x \mid x \in A \wedge \neg(x \in A \wedge x \notin B)\} \\ = & \{x \mid x \in A \wedge (x \notin A \vee x \in B)\} \\ = & \{x \mid x \in A \wedge x \in B\} \end{aligned}$$

52. These are simpler expressions:

- (a) $A \oplus B = A$ is equivalent to $B = \emptyset$.
- (b) $A \oplus B = A \setminus B$ is equivalent to $B \subseteq A$.
- (c) $A \oplus B = A \cup B$ is equivalent to $A \cap B = \emptyset$.
- (d) $A \oplus B = A \cap B$ is equivalent to $A \cup B = \emptyset$.
- (e) $A \oplus B = A^c$ is equivalent to $B = X$, assuming a universal set X .

53. The statement is false, as we have, for example, $\{42\} \times \emptyset = \emptyset \times \{42\} = \emptyset$, but $\emptyset \neq \{42\}$.

54. Assume that R is transitive. Let (x, z) be in $R \circ R$. That means there is some y , such that $R(x, y)$ and $R(y, z)$ hold. By transitivity, $R(x, z)$ holds, so $(R \circ R) \subseteq R$.

Conversely, assume $(R \circ R) \subseteq R$. Consider x, y, z such that $R(x, y)$ and $R(y, z)$ hold. Clearly (x, z) is in $R \circ R$, and hence, by assumption, in R . But that means R is transitive.

As an example of a transitive relation for which $R \circ R = R$ does not hold, consider $<$ on \mathbb{Z} . It is transitive, but $< \circ <$ does not contain, say $(2, 3)$. Since $(2, 3)$ is in $<$, $<$ is different from $< \circ <$.

55. Here is the complete table:

Property	Reflexivity	Symmetry	Transitivity
preserved under \cap ?	yes	yes	yes
preserved under \cup ?	yes	yes	no
preserved under inverse?	yes	yes	yes
preserved under complement?	no	yes	no

To see how transitivity fails to be preserved under union, consider two relations on $\{a, b, c\}$, namely $R = \{(a, a), (a, b), (b, b)\}$ and $S = \{(c, a)\}$, both transitive. $R \cup S$ is not transitive, because in the union we have (c, a) and (a, b) , but not (c, b) . And R 's complement, $\{(a, c), (b, a), (b, c), (c, a), (c, b), (c, c)\}$ is not transitive either, as it contains, for example, (a, c) and (c, a) , but not (a, a) .

56. From the first row of the last question's table, it follows that, if R and S are equivalence relations, then so is their intersection. But their union may not be. As an example, take the reflexive, symmetric, transitive closures of R and S from the previous answer, to get these two equivalence relations:

$$R' = \{(a, a), (a, b), (b, a), (b, b), (c, c)\} \quad \text{and} \quad S' = \{(a, a), (a, c), (b, b), (c, a), (c, c)\}.$$

Their union fails to be transitive, as it contains (c, a) and (a, b) but not (c, b) .

57. From $f(g(y)) = y$ we conclude that g is injective. Namely, if B has cardinality 1 then g is trivially injective. Otherwise, consider $y, y' \in B$, with $y \neq y'$. Suppose $g(y) = g(y')$. Then, applying f to both, we have $y = f(g(y)) = f(g(y')) = y'$, contradicting $y \neq y'$. So we must have $g(y) \neq g(y')$, that is, g is injective.

Similarly we can show that f is surjective. To do this, we must show that for each $y \in B$ there is some $x \in A$ such that $f(x) = y$. But that is easy—that x is $g(y) \in A$.

58. We have: $h(h(h(x))) = x$ for all $x \in X$. First, let us show that h must be injective. If $h(x) = h(y)$, then, applying h twice on each side, we have $h(h(h(x))) = h(h(h(y)))$, whence $x = y$. So h is injective. Second, let us show that h must be surjective. Consider an arbitrary element $x \in X$. We have $x = h(h(h(x)))$, that is, h maps $h(h(x))$ to x . Since x was arbitrary, h is surjective.

For the counter-example, take $X = \{a, b, c\}$ and let h map a to b , b to c , and c to a . Then h is not the identity function on X , but $h \circ h \circ h$ is.

59. We certainly do not have $A \times A = A$. In fact, no member of A is a member of $A \times A$, and no member of $A \times A$ is a member of A . So \times is not absorptive.

Neither is it commutative. Let $A = \{0\}$ and $B = \{1\}$. Then $A \times B = \{(0, 1)\}$ while $B \times A = \{(1, 0)\}$, and those singleton sets are different, because the members are.

If we also define $C = \{2\}$ then $A \times (B \times C) = \{(0, (1, 2))\}$ while $(A \times B) \times C = \{((0, 1), 2)\}$. Again, these are different. However, it is not uncommon to identify both of $(0, (1, 2))$ and $((0, 1), 2)$ with the triple $(0, 1, 2)$ ("flattening" the nested pairings). If we agree to do that then \times is associative, and we can simply write $A \times B \times C$ for the set of triples.

60. The conjecture is false. For a counter-example, take A to be $\{0, 1\}$ and $R = \{(0, 0)\}$. Then R is symmetric, and also anti-symmetric, but R is not reflexive, as it does not include $(1, 1)$.
61. If f is injective then B has at least 42 elements. If f is surjective then B has at most 42 elements. (So if f is bijective, B has exactly 42 elements.)
62. Here are some functions that satisfy the requirements. We show $f_i(x)$ in the table's row x , column i :

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
a	a	a	b	b	b	a	c	b
b	b	a	b	d	b	a	b	a
c	c	a	c	d	c	a	d	d
d	d	a	d	d	c	c	d	c

Maybe you skipped this optional exercise; but you may still want to verify, for each of these eight functions, that it really does satisfy its specification.