

Sample Answers to Tutorial Exercises, Week 3

19. (a)

$$\begin{aligned} & \neg(A \wedge \neg(B \wedge C)) \\ & \neg A \vee (B \wedge C) && \text{(push negation in)} \\ & (\neg A \vee B) \wedge (\neg A \vee C) && \text{(distribute } \vee \text{ over } \wedge) \end{aligned}$$

The result is now in reduced CNF.

(b)

$$\begin{aligned} & A \vee (\neg B \wedge (C \vee (\neg D \wedge \neg A))) \\ & (A \vee \neg B) \wedge (A \vee C \vee (\neg D \wedge \neg A)) && \text{(distribute } \vee \text{ over } \wedge) \\ & (A \vee \neg B) \wedge (A \vee C \vee \neg D) \wedge (A \vee C \vee \neg A) && \text{(distribute } \vee \text{ over } \wedge) \end{aligned}$$

The result is in CNF but not RCNF. To get RCNF we need to eliminate the last clause which is a tautology, and we end up with $(A \vee \neg B) \wedge (A \vee C \vee \neg D)$.

(c)

$$\begin{aligned} & (A \vee B) \Rightarrow (C \wedge D) \\ & \neg(A \vee B) \vee (C \wedge D) && \text{(unfold } \Rightarrow) \\ & (\neg A \wedge \neg B) \vee (C \wedge D) && \text{(de Morgan)} \\ & (\neg A \vee (C \wedge D)) \wedge (\neg B \vee (C \wedge D)) && \text{(distribute } \vee \text{ over } \wedge) \\ & (\neg A \vee C) \wedge (\neg A \vee D) \wedge (\neg B \vee C) \wedge (\neg B \vee D) && \text{(distribute } \wedge \text{ over } \vee) \end{aligned}$$

The result is in RCNF. We could have chosen different orders for the distributions.

(d)

$$\begin{aligned} & A \wedge (B \Rightarrow (A \Rightarrow B)) \\ & A \wedge (\neg B \vee \neg A \vee B) && \text{(unfold both occurrences of } \Rightarrow) \\ & A && \text{(rightmost clause is tautological: remove it)} \end{aligned}$$

20. Let us follow the method given in a lecture, except we do the double-negation elimination aggressively, as soon as opportunity arises:

$$\begin{aligned} & \neg((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)) \\ & \neg(\neg(B \vee \neg A) \vee \neg(B \vee A) \vee B) && \text{(unfold } \Rightarrow \text{ and eliminate double negation)} \\ & (B \vee \neg A) \wedge (B \vee A) \wedge \neg B && \text{(de Morgan for outermost neg; elim double neg)} \end{aligned}$$

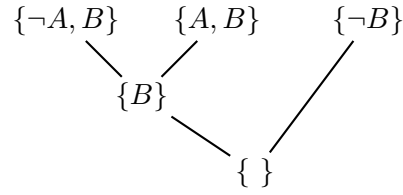
This is RCNF without further reductions.

We could also have used other transformations—sometimes this can shorten the process. For example, we could have rewritten the sub-expression $\neg B \Rightarrow \neg A$ as $A \Rightarrow B$ (the contraposition principle). You may want to check that this does not change the result.

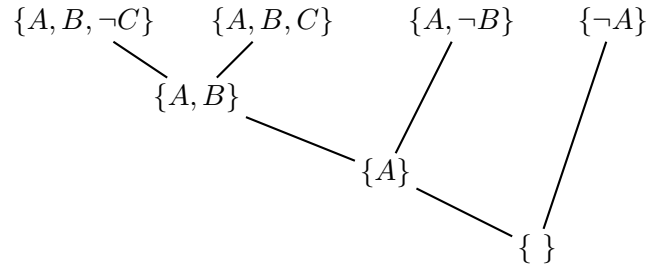
The resulting formula, written as a set of sets of literals:

$$\{\{\neg A, B\}, \{A, B\}, \{\neg B\}\}$$

We can now construct the refutation:



21. Here is a refutation:



From this we conclude that $(A \vee B \vee \neg C) \wedge \neg A \wedge (A \vee B \vee C) \wedge (A \vee \neg B)$ is unsatisfiable.

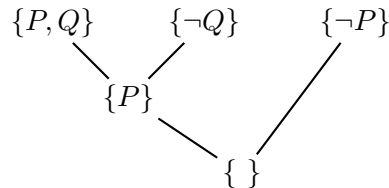
22. (a) $(P \vee Q) \Rightarrow (Q \vee P)$. First negate the formula (why?), to get $\neg((P \vee Q) \Rightarrow (Q \vee P))$. Then we can use the usual techniques to convert the negated proposition to RCNF. Here is a useful shortcut, combining \Rightarrow -elimination with one of de Morgan's laws:

$$\neg(A \Rightarrow B) \equiv A \wedge \neg B.$$

So:

$$\begin{aligned} & \neg((P \vee Q) \Rightarrow (Q \vee P)) \\ & (P \vee Q) \wedge \neg(Q \vee P) \quad (\text{shortcut}) \\ & (P \vee Q) \wedge \neg Q \wedge \neg P \quad (\text{de Morgan}) \end{aligned}$$

The result allows for a straight-forward refutation:



- (b) $(\neg P \Rightarrow P) \Rightarrow P$. Again, first negate the formula, to get $\neg((\neg P \Rightarrow P) \Rightarrow P)$. Then turn the result into RCNF:

$$\begin{aligned} & \neg((\neg P \Rightarrow P) \Rightarrow P) \\ & (\neg P \Rightarrow P) \wedge \neg P \quad (\text{shortcut from above}) \\ & (\neg \neg P \vee P) \wedge \neg P \quad (\text{unfold } \Rightarrow) \\ & (P \vee P) \wedge \neg P \quad (\text{eliminate double negation}) \\ & P \wedge \neg P \quad (\vee\text{-absorption}) \end{aligned}$$

The resolution proof is immediate; we will leave it out.

- (c) $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$. Again, negate the formula, to get $\neg(((P \Rightarrow Q) \Rightarrow P) \Rightarrow P)$. Then turn the result into RCNF:

$$\begin{aligned}
& \neg(((P \Rightarrow Q) \Rightarrow P) \Rightarrow P) \\
& ((P \Rightarrow Q) \Rightarrow P) \wedge \neg P && \text{(shortcut, outermost } \Rightarrow \text{)} \\
& ((\neg P \vee Q) \Rightarrow P) \wedge \neg P && \text{(unfold } \Rightarrow \text{)} \\
& (\neg(\neg P \vee Q) \vee P) \wedge \neg P && \text{(unfold } \Rightarrow \text{)} \\
& ((\neg\neg P \wedge \neg Q) \vee P) \wedge \neg P && \text{(de Morgan)} \\
& ((P \wedge \neg Q) \vee P) \wedge \neg P && \text{(double negation)} \\
& (P \vee P) \wedge (\neg Q \vee P) \wedge \neg P && \text{(distribution)} \\
& P \wedge (\neg Q \vee P) \wedge \neg P && \text{(absorption)}
\end{aligned}$$

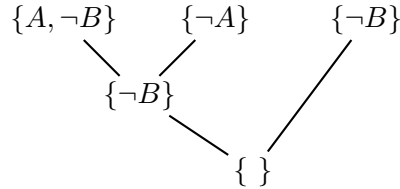
Again this gives an immediate refutation: just resolve $\{P\}$ against $\{\neg P\}$.

- (d) $P \Leftrightarrow ((P \Rightarrow Q) \Rightarrow P)$. Negating the formula, we get $P \oplus ((P \Rightarrow Q) \Rightarrow P)$. Let us turn the resulting formula into RCNF:

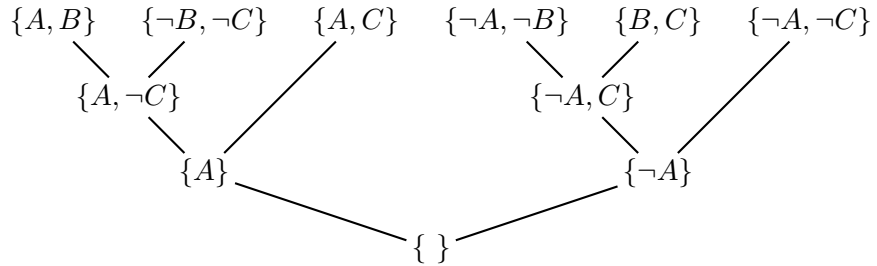
$$\begin{aligned}
& P \oplus ((P \Rightarrow Q) \Rightarrow P) \\
& (P \vee ((P \Rightarrow Q) \Rightarrow P)) \wedge (\neg P \vee \neg((P \Rightarrow Q) \Rightarrow P)) && \text{(eliminate } \oplus \text{)} \\
& (P \vee ((P \Rightarrow Q) \Rightarrow P)) \wedge (\neg P \vee ((P \Rightarrow Q) \wedge \neg P)) && \text{(shortcut from above)} \\
& (P \vee (\neg(\neg P \vee Q) \vee P)) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) && \text{(} \Rightarrow \text{-elimination)} \\
& (P \vee (\neg\neg P \wedge \neg Q) \vee P) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) && \text{(de Morgan)} \\
& (P \vee (P \wedge \neg Q) \vee P) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) && \text{(double negation)} \\
& P \wedge (P \vee \neg Q) \wedge (\neg P \vee ((\neg P \vee Q) \wedge \neg P)) && \text{(} \vee \text{-absorption, distribution)} \\
& P \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge \neg P && \text{(} \vee \text{-absorption, distribution)}
\end{aligned}$$

Once again, now just resolve $\{P\}$ against $\{\neg P\}$.

23. (a) $\{\{A, B\}, \{\neg A, \neg B\}, \{\neg A, B\}\}$ stands for the formula $(A \vee B) \wedge (\neg A \vee \neg B) \wedge (\neg A \vee B)$. This is satisfiable by $\{A \mapsto \mathbf{f}, B \mapsto \mathbf{t}\}$.
- (b) $\{\{A, \neg B\}, \{\neg A\}, \{B\}\}$ stands for $(A \vee \neg B) \wedge \neg A \wedge B$. A refutation is easy:



- (c) $\{\{A\}, \emptyset\}$ stands for $A \wedge \mathbf{f}$, which is clearly not satisfiable.
- (d) We have $\{\{A, B\}, \{\neg A, \neg B\}, \{B, C\}, \{\neg B, \neg C\}, \{A, C\}, \{\neg A, \neg C\}\}$. This set is not satisfiable, as a proof by resolution shows:



24. Let us give names to the propositions:

- C : Ann clears 2 meters
- F : Ann gets the flu
- K : The selectors are sympathetic
- S : Ann is selected
- T : Ann trains hard

The four assumptions then become:

- (a) $C \Rightarrow S$
- (b) $T \Rightarrow (F \Rightarrow K)$
- (c) $(T \wedge \neg F) \Rightarrow C$
- (d) $K \Rightarrow S$

It is easy to see that S is not a logical consequence of these, as we can give all five variables the value *false*, and all the assumptions will thereby be true.

To see that $T \Rightarrow S$ is a logical consequence of the assumptions, we can negate it, obtaining $T \wedge \neg S$. Then, translating everything to clausal form, we can use resolution to derive an empty clause.

Alternatively, note that $T \Rightarrow (F \Rightarrow K)$ is equivalent to $(T \wedge F) \Rightarrow K$. Since also $K \Rightarrow S$, we have $(T \wedge F) \Rightarrow S$. Similarly, $(T \wedge \neg F) \Rightarrow C$ together with $C \Rightarrow S$ gives us $(T \wedge \neg F) \Rightarrow S$.

But from $(T \wedge F) \Rightarrow S$ and $(T \wedge \neg F) \Rightarrow S$ we get $T \Rightarrow S$. (You may want to check that by massaging the conjunction of the two formulas.)

25. Let us give names to the propositions:

- A : The commissioner apologises
- F : The commissioner can attend the function
- R : The commissioner resigns

The four statements then become

- (a) $F \Rightarrow (A \wedge R)$
- (b) $(R \wedge A) \Rightarrow F$
- (c) $R \Rightarrow F$
- (d) $F \Rightarrow A$

The first translation may not be obvious. But to say “ X does not happen unless Y happens” is the same as saying “it is not possible to have X happen and at the same time Y does not happen.” That is, $\neg(X \wedge \neg Y)$, which is equivalent to $X \Rightarrow Y$. Note that (a) entails (d) and (c) entails (b).