COMP30026 Models of Computation

Undecidable Languages

Harald Søndergaard

Lecture Week 11 Part 2

Semester 2, 2020

This Lecture is Being Recorded

An Undecidable Language

Now let us study undecidable problems/languages.

We start by showing that it is undecidable whether a Turing machine accepts a given input string. That is,

$$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$$

is undecidable.

The main difference from the case of A_{CFG}, for example, is that a Turing machine may fail to halt.

TM Acceptance Is Undecidable

Theorem:

$$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$$

is undecidable.

Proof: Assume (for contradiction) that A_{TM} is decided by a TM H:

$$H\langle M, w\rangle = \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

Using H we can construct a Turing machine D which decides whether a given machine M fails to accept its own encoding $\langle M\rangle$:

1. Input is $\langle M\rangle$, where M is some Turing machine.
2. Run H on $\langle M, \langle M\rangle\rangle$.
3. If H accepts, reject. If H rejects, accept.

TM Acceptance

In summary:

$$D(\langle M \rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ does not accept } \langle M \rangle \\ \textit{reject} & \text{if } M \text{ accepts } \langle M \rangle \end{cases}$$

But no machine can satisfy that specification!

Why? Because we obtain an absurdity when we investigate D's behaviour when we run it on its own encoding:

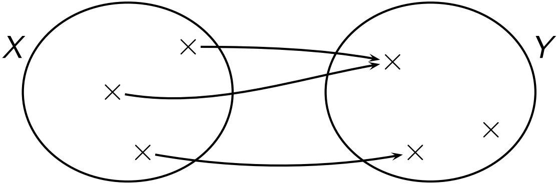
$$D(\langle D \rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D \rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D \rangle \end{cases}$$

Hence neither D nor H can exist.

Bijections Again

Recall that a function $f : X \to Y$ is

- surjective (or onto) iff $f[X] = Y$.
- injective (or one-to-one) iff $f(x) = f(y) \Rightarrow x = y$.
- bijective iff it is both surjective and injective.



Inverse Function

Given $f : X \to Y$, a function $g : Y \to X$ is its inverse iff $g \circ f = 1_X$ and $f \circ g = 1_Y$.

An inverse function, if it exists, is unique.

A function has an inverse iff it is bijective.

If $f : X \to Y$ is a bijection, we denote its inverse by $f^{-1} : Y \to X$.

Bijections and Enumerations

In Week 6 we looked at the bijection $d : \mathbb{Z} \to \mathbb{N}$ defined by

$$d(n) = \begin{cases} 2n - 1 & \text{if } n > 0 \\ -2n & \text{if } n \leq 0 \end{cases}$$

Its inverse function $e : \mathbb{N} \to \mathbb{Z}$ is

$$e(n) = \begin{cases} (n+1)/2 & \text{if } n \text{ is odd} \\ -n/2 & \text{if } n \text{ is even} \end{cases}$$

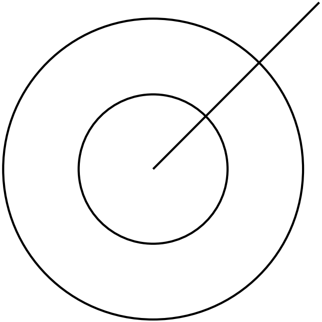
A bijection in $\mathbb{N} \to X$ gives us an enumeration of the set X.

e gives an enumeration of $\mathbb{Z}$, namely $0, 1, -1, 2, -2, 3, -3, 4, \ldots$

Galileo's Paradox

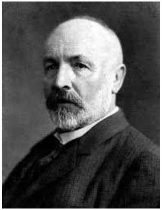
$\mathbb{N}$ and the set of perfect squares are in a one-to-one relation: f defined by $f(n) = n^2$ is a bijection.

$$\begin{array}{cccccccc} \{ & 0 & 1 & 2 & 3 & 4 & 5 & \ldots\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \{ & 0 & 1 & 4 & 9 & 16 & 25 & \ldots\} \end{array}$$



Galileo: The outer circle has twice as many points as the inner circle, as the ratio between the circumferences is 2. Yet considering radial lines shows a one-to-one relation.

Comparing Sizes of Sets: Cantor's Criterion



So what does 'equals' and 'less' mean for infinite cardinality?

How do we compare the "sizes" of infinite sets?

Cantor's criterion:

- $card(X) \leq card(Y)$ iff there is a total, injective $f : X \to Y$.
- $card(X) = card(Y)$ iff

$$card(X) \leq card(Y) \text{ and } card(Y) \leq card(X).$$

As a consequence, there are (infinitely) many degrees of infinity.

To Infinity and Beyond

X is countable iff $card(X) \leq card(\mathbb{N})$.

X is countably infinite iff $card(X) = card(\mathbb{N})$.

Examples: $\mathbb{Z}, \mathbb{N}^k$, and $\mathbb{N}^*$ (the set of all finite sequences of natural numbers) are all countably infinite.

Importantly, Σ^* is countable for all finite alphabets Σ, including the alphabet of printable characters on your keyboard.

$\mathcal{P}(\mathbb{N})$, $\mathbb{N} \to \mathbb{N}$, and $\mathbb{Z} \to \mathbb{Z}$ are uncountable, as can be shown by diagonalisation.

Diagonalisation Showing $\mathbb{Z} \to \mathbb{Z}$ Is Uncountable

Theorem: There is no bijection $h : \mathbb{N} \to (\mathbb{Z} \to \mathbb{Z})$.

Proof: Assume h exists. Then

$$h(0), h(1), h(2), \ldots, h(n), \ldots$$

contains every function in $\mathbb{Z} \to \mathbb{Z}$, without duplicates.

Now construct $f : \mathbb{Z} \to \mathbb{Z}$ as follows:

$$f(n) = h(n)(n) + 1$$

Then $f \neq h(n)$ for all n, so we have a contradiction.

Why This Is Called Diagonalisation

Here is some hypothetical listing of all the functions $h(0), h(1), \ldots$ that make up $\mathbb{Z} \rightarrow \mathbb{Z}$:

	0	1	2	3	4	5	...
h(0)	19	3	42	0	7	9	...
h(1)	42	42	42	42	42	42	...
h(2)	42	43	44	45	46	47	...
h(3)	6	93	17	84	6	93	...
h(4)	45	18	-8	-5	63	-9	...
⋮							

Why This Is Called Diagonalisation

Here is some hypothetical listing of all the functions $h(0), h(1), \ldots$ that make up $\mathbb{Z} \to \mathbb{Z}$:

	0	1	2	3	4	5	...
h(0)	19	3	42	0	7	9	...
h(1)	42	42	42	42	42	42	...
h(2)	42	43	44	45	46	47	...
h(3)	6	93	17	84	6	93	...
h(4)	45	18	-8	-5	63	-9	...
⋮							

f is defined in such a way that it cannot possibly be in the listing:

	0	1	2	3	4	5	...
f	20	43	45	85	64	...	...

Algorithms vs Functions

Consider the set of algorithms that realise functions $f : \mathbb{Z} \to \mathbb{Z}$.

How large is that set?

It is infinite, but we can enumerate it. It is contained in Σ^*, where Σ is the set of (printable) characters on my keyboard and as we have seen, that set is countable.

So there cannot be any more, say, Haskell functions, of type `Integer -> Integer` than there are integers. Namely, each Haskell function is represented finitely, as a finite sequence of symbols from a finite alphabet.

Algorithms vs Functions

However, we saw that $\mathbb{Z} \to \mathbb{Z}$ is not countable.

In other words, there are number-theoretic functions (in fact, lots of them) that do not have a corresponding algorithm.

So are there any "important" functions that are not computable?

As it turns out, yes, very much so!

Problems that Have No Algorithmic Solution

Some undecidable problems:

- Are two given CFGs equivalent?
- Are there strings that a given CFG cannot generate?
- Is a given CFG unambiguous?
- Will a given Python program halt for all input?
- Will it halt on input 42?
- Will a given Java program ever throw a certain exception?

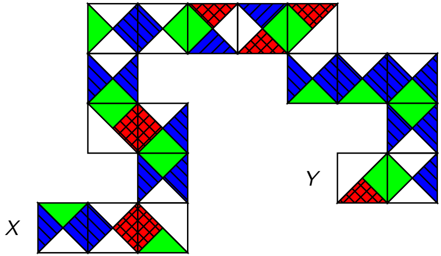
Next week we will explore some other undecidable problems.

Domino Snakes

Consider a finite set of types of tiles ⊠.

There are infinitely many tiles of each type.

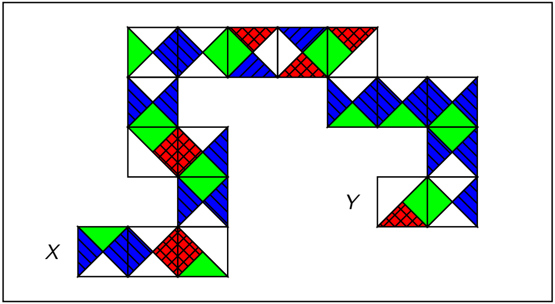
Can points X and Y in the plane be connected?



The (unconstrained) problem is decidable.

Domino Snakes

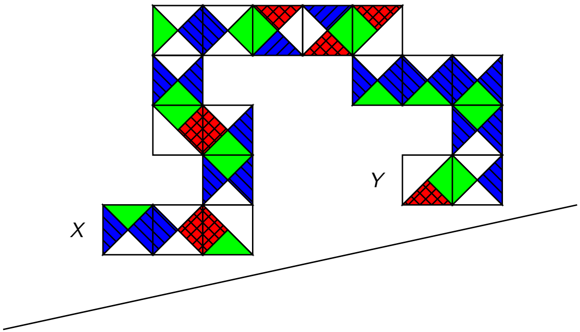
Can X and Y be connected?



In finite segment of plane: also decidable.

Domino Snakes

Can points X and Y be connected?



In half-plane: Undecidable!

Intuition is sometimes a poor guide to decidability.

Busy Beavers (Not Examinable)

We have made a video available (optional viewing) with an interesting example of an uncomputable function. It involves some fascinating Turing machines called busy beavers, and it includes a proof of the undecidability of Turing machine halting-on-empty-input.

There are also a couple of other (optional) Turing machine related videos.

You'll find them in the Week 11 Canvas module (where we will also leave the slides from the busy beaver video).

Coming Up in the Next Two Weeks

Next week we will use the technique of reduction to find a bunch of interesting undecidable problems.

We will also have a review of what we achieved this semester.

We plan to run the practice exam on Tuesday 3 November from 15:00 to 18:30, on Grok.

We also plan to have a Zoom catch-up mega-tute and feedback session on Friday 6 November (using this lecture Zoom link).

Make sure you know when the actual exam is.