Week 7 - Sets and Relations

Billy Price

Definition A set, A, is a collection defined by a predicate, which decides for each object, x, whether $x \in A$ or $x \notin A$

Set Comprehension Given a predicate, $P(x)$, $\{x \mid P(x)\}$ is the set where

$a \in \{x \mid P(x)\}$ if and only if $P(a)$

Small Sets

Empty Set: $\phi := \{x \mid f\}$

Singleton Set: $\{a\} := \{x \mid x = a\}$

Finite Set: $\{a, b, c\} := \{x \mid x = a \vee x = b \vee x = c\}$

Set Construction Given sets A, B, the following define new sets

Intersection $A \cap B := \{x \mid x \in A \wedge x \in B\}$

Union $A \cup B := \{x \mid x \in A \vee x \in B\}$

Difference $A \setminus B := \{x \mid x \in A \wedge x \notin B\}$

Symmetric Diff. $A \oplus B := \{x \mid x \in A \oplus x \in B\}$

Product $A \times B := \{(x, y) \mid x \in A \wedge y \in B\}$

Powerset $\mathcal{P}(A) := \{Y \mid Y \subseteq A\}$

Complement $A^c := X \setminus A$

universal

Equality $A = B$ means $\forall x (x \in A \Leftrightarrow x \in B)$

Subset $A \subseteq B$ means $\forall x (x \in A \Rightarrow x \in B)$

Relations A relation between two sets, A, B, is a subset of $A \times B$

If $R \subseteq A \times B$ and $(a, b) \in R$, we think of a & b as related by R, and write $R(a, b)$

Examples

$A \times B$ is a relation—the full relation from A to B.

$\emptyset$ is a relation.

$\Delta_A = \{(x, x) \mid x \in A\}$ is a relation on A—the identity relation.

If R is a relation from A to B then $R^{-1} = \{(b, a) \mid (a, b) \in R\}$ is a relation from B to A, called the inverse of R.

Clearly $(R^{-1})^{-1} = R$.

Since relations are sets, all the set operations, such as $\cap$ and $\cup$, are applicable to relations.

Relations on one Set

Let A be a non-empty set and let R be a relation on A.

R is reflexive iff $R(x, x)$ for all x in A.

R is irreflexive iff $R(x, x)$ holds for no x in A.

R is symmetric iff $R(x, y) \Rightarrow R(y, x)$ for all x, y in A.

R is asymmetric iff $R(x, y) \Rightarrow \neg R(y, x)$ for all x, y in A.

R is antisymmetric iff $R(x, y) \wedge R(y, x) \Rightarrow x = y$ for all x, y in A.

R is transitive iff $R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$ for all x, y, z in A.

A **function** $f : A \to B$ is a relation $f \subseteq A \times B$ such that for all $x \in A$ there is a <u>unique</u> $y \in B$ with $(x, y) \in f$

we write $f(x) = y$ instead of $(x, y) \in f$

An **equivalence** relation is reflexive, symmetric and transitive

Not mentioned Here composition, surjectivity, injectivity, bijections, closures,

Useful Set Identities

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Do not memorise these without understanding why they are true

Double complement: $A = (A^c)^c$

De Morgan: $(A \cap B)^c = A^c \cup B^c$
$(A \cup B)^c = A^c \cap B^c$

Absorption: $A \cap A = A$
$A \cup A = A$

Duality: $X^c = \emptyset$ and $\emptyset^c = X$

Identity: $A \cup \emptyset = A$ and $A \cap X = A$

Commutativity: $A \cap B = B \cap A$
$A \cup B = B \cup A$

Dominance: $A \cap \emptyset = \emptyset$ and $A \cup X = X$

Associativity: $A \cap (B \cap C) = (A \cap B) \cap C$
$A \cup (B \cup C) = (A \cup B) \cup C$

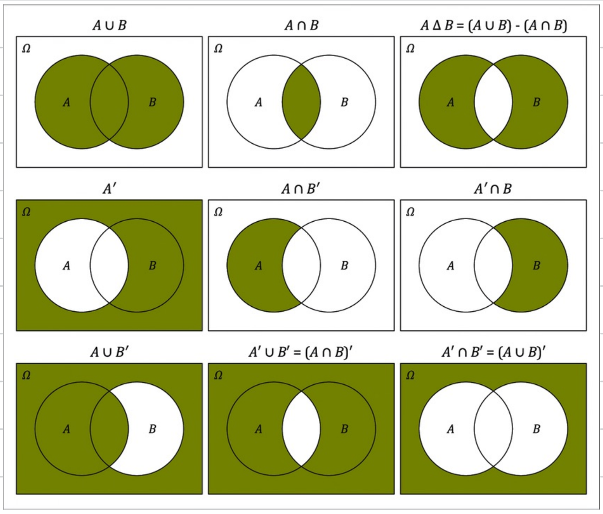
Complementation: $A \cap A^c = \emptyset$ and $A \cup A^c = X$

Subset characterisation: $A \subseteq B \equiv A = A \cap B \equiv B = A \cup B$

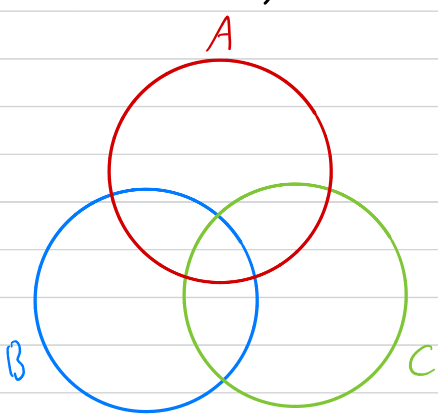
Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Contraposition: $A^c \subseteq B^c \equiv B \subseteq A$
$A \subseteq B^c \equiv B \subseteq A^c$
$A^c \subseteq B \equiv B^c \subseteq A$

You've already seen these rules in propositional logic, just replace $\cap$ with $\wedge$, $\cup$ with $\vee$, $\emptyset$ with f, ^{c} with $\neg$



Use this to visualize Distributivity



Exercise

Normalise the following sets (write them as either ϕ or a list of the elements) and group together equal sets.

(a) ϕ (b) $\{\phi\}$ (c) $\phi \cup \{\phi\}$ (d) $\phi \cap \{\phi\}$

(e) $\{\phi\} \cap \{\phi\}$ (f) $\phi \cup \phi$ (g) $\phi \cap \phi$ (h) $\phi \times \{\phi\}$

(i) $\{\phi\} \times \{\phi\}$ (j) $\mathcal{P}(\phi)$ (k) $\mathcal{P}(\{\phi\})$

Hint: Write (a) & (b) with the set comprehension form, i.e. $\{x \mid \_\_\_\}$ to clarify the difference.