School of Computing and Information Systems
COMP30026 Models of Computation Tutorial Week 7

16–18 September 2020

The exercises

51. Let A, B, and C be sets. Show:

    (a) $A \not\subseteq B \Leftrightarrow A \setminus B \neq \emptyset$.

    (b) $A \cap B = A \setminus (A \setminus B)$.

    Hint: Use the formal (logical) definitions of the concepts involved.

52. Recall that the *symmetric difference* of sets A and B is the set $A \oplus B = (A \setminus B) \cup (B \setminus A)$. For each of the following set equations, give an equivalent equation that does not use $\oplus$. However, do not simply replace $\oplus$ by its definition; instead try to find the simplest equivalent equation.

    (a) $A \oplus B = A$

    (b) $A \oplus B = A \setminus B$

    (c) $A \oplus B = A \cup B$

    (d) $A \oplus B = A \cap B$

    (e) $A \oplus B = A^c$

53. Consider this statement: For all sets S and T, $S \times T = T \times S$ iff $S = T$.

    If the statement is true, prove it. Otherwise provide a counter-example.

54. Show that a relation R on A is transitive iff $R \circ R \subseteq R$. Then give an example of a transitive relation R for which $R \circ R = R$ fails to hold.

55. Relations are sets. To say that $R(x, y) \wedge S(x, y)$ holds is the same as saying that (x, y) is in the relation R and also in the relation S, that is, $(x, y) \in R \cap S$.

    Suppose R and S are reflexive relations on a set A. Then $\Delta_A \subseteq R$ and $\Delta_A \subseteq S$, so $\Delta_A \subseteq R \cap S$. That is, $R \cap S$ is also reflexive. We say that intersection *preserves* reflexivity. It is easy to see that union also preserves reflexivity. Similarly, if R is reflexive then so is R^{-1}, but the complement $A^2 \setminus R$ is clearly not. The following table lists these results. Complete the table, indicating which operations on relations preserve symmetry and transitivity.

Property	Reflexivity	Symmetry	Transitivity
preserved under $\cap$?	yes		
preserved under $\cup$?	yes		
preserved under inverse?	yes		
preserved under complement?	no		

56. Continuing from the previous question, now assume that R and S are equivalence relations. From your table's first two rows, determine whether $R \cap S$ necessarily is an equivalence relation, and whether $R \cup S$ is.

57. Suppose we know about functions $f : A \to B$ and $g : B \to A$ that $f(g(y)) = y$ for all $y \in B$. What, if anything, can be deduced about f and/or g being injective and/or surjective?

58. Suppose $h : X \to X$ satisfies $h \circ h \circ h = 1_X$. Show that h is a bijection. Also give a simple example of a set X and a function $h : X \to X$ such that $h \circ h \circ h = 1_X$, but h is not the identity function (hint: think paper-scissors-rock).

59. (Drill.) The *Cartesian product* of two sets A and B is defined $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$. That is, a pair whose first component comes from A and whose second component comes from B is an element of $A \times B$ (and no other pairs are). Recall that $\cap$ and $\cup$ are absorptive, commutative and associative. Does $\times$ have any of those properties?

60. (Drill.) Consider this conjecture: If a binary relation R on some set A is both symmetric and anti-symmetric then R is reflexive. Prove or disprove the conjecture.

61. (Drill.) Suppose A is a set of cardinality 42, that is, A has 42 elements. What, if anything, can we say about B's cardinality if we know that some function $f : A \to B$ is injective? What, if anything, can we say about B's cardinality if we know that some function $f : A \to B$ is surjective?

62. (Optional.) Let $\leq$ be a partial order on a set X. We say that a function $h : X \to X$ is:

    - *idempotent* iff $\forall x \in X \ (h(h(x)) = h(x))$
    - *monotone* iff $\forall x, y \in X \ (x \leq y \Rightarrow h(x) \leq h(y))$
    - *increasing* iff $\forall x \in X \ (x \leq h(x))$

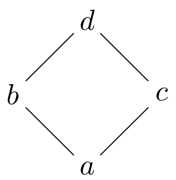
    Note that an idempotent function does all of its work "in one go"; repeated application will not change its result. A monotone function is one that respects order; if its input grows, its output must grow too (or stay the same).

    A function which is idempotent and monotone is a *closure operator*. If it is also increasing, we call it an *upper* closure operator. Closure operators are important and appear in many different contexts. We have met several—let $\mathcal{R}$ be the set of all binary relations. Then the functions *refl*, *symm*, and *trans*, in $\mathcal{R} \to \mathcal{R}$, producing a relation's reflexive, symmetric, and transitive closure, respectively, are all upper closure operators. Soon we will meet an "ϵ-closure" function that is part of the algorithm for turning a non-deterministic automaton into an equivalent deterministic automaton—yet another upper closure operator.

    Consider $D = \{a, b, c, d\}$ and the partial order $\leq$ on D, defined by

    $$x \leq y \text{ iff } x = a \vee x = y \vee y = d$$

    Below is the so-called Hasse diagram for D. A Hasse diagram provides a helpful way of depicting a partially ordered set. The nodes are the elements of the set, and the order is given by the edges: $x \leq y$ if and only if there is a path from x to y travelling upwards only, along edges (and the path can have length 0).

    Define eight functions $f_1, \ldots, f_8 : D \to D$, exhibiting all possible combinations of the three properties. That is, find some

    (a) f_1 which is idempotent, monotone, and increasing;
    (b) f_2 which is idempotent and monotone, but not increasing;
    (c) f_3 which is idempotent and increasing, but not monotone;
    (d) f_4 which is monotone and increasing, but not idempotent;
    (e) f_5 which is idempotent, but neither monotone nor increasing;
    (f) f_6 which is monotone, but neither idempotent nor increasing;
    (g) f_7 which is increasing, but neither idempotent nor monotone;
    (h) f_8 which is neither idempotent, monotone, nor increasing.