Week 10: Search Tree Algorithms

Searching

An extremely common application in computing

- given a (large) collection of *items* and a *key* value
- find the item(s) in the collection containing that key
  - item = (key, val_1, val_2, …) (i.e. a structured data type)
  - key = value used to distinguish items (e.g. student ID)

Applications: Google, databases, ...

... Searching

Since searching is a very important/frequent operation,
many approaches have been developed to do it

Linear structures: arrays, linked lists, files

Arrays = random access. Lists, files = sequential access.

Cost of searching:

	Array	List	File
Unsorted	O(n) (linear scan)	O(n) (linear scan)	O(n) (linear scan)
Sorted	O(log n) (binary search)	O(n) (linear scan)	O(log n) (*seek*, *seek*>/$>, …)

- *O(n)* … linear scan   (search technique of last resort)
- *O(log n)* … binary search, *search trees*   (trees also have other uses)

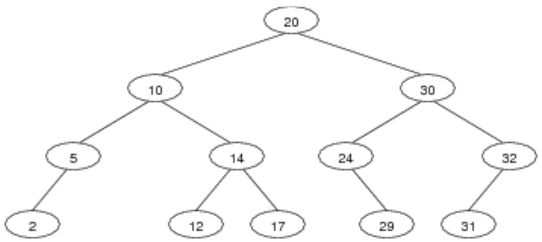
Also (cf. COMP9021): hash tables   (*O(1)*, but only under optimal conditions)

... Searching

Maintaining the order in sorted arrays and files is a costly operation.

Search trees are as efficient to search but more efficient to maintain.

Example: the following tree corresponds to the sorted array `[2,5,10,12,14,17,20,24,29,30,31,32]`:

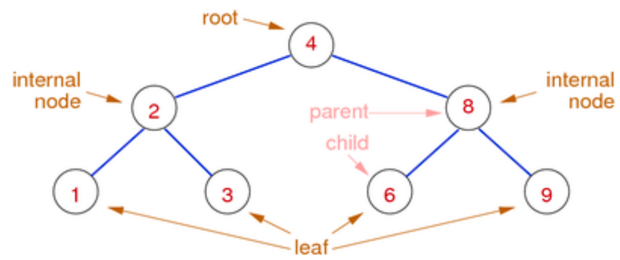


Tree Data Structures

Trees

Trees are connected graphs

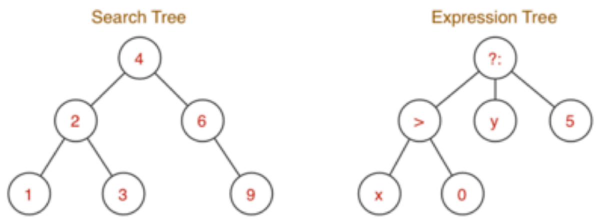
- consisting of nodes and edges (called *links*), with no cycles (no "up-links")
- each node contains a data value   (or key+data)
- each node has links to ≤ *k* other child nodes   (*k=2* below)



... Trees

Trees are used in many contexts, e.g.

- representing hierarchical data structures (e.g. expressions)
- efficient searching (e.g. sets, symbol tables, …)



... Trees

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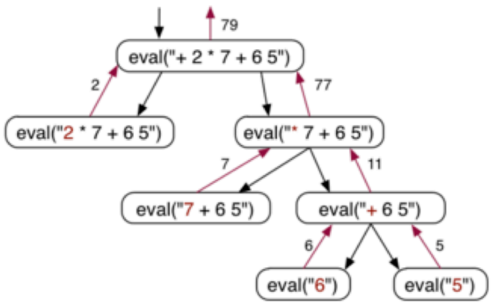
Trees can be used as a data structure, but also for illustration.

E.g. showing evaluation of a prefix arithmetic expression



... Trees

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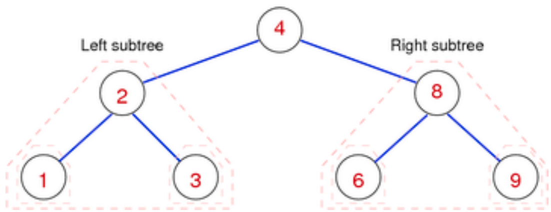
Binary trees (*k=2* children per node) can be defined recursively, as follows:

A *binary tree* is either

- empty (contains no nodes)
- consists of a *node*, with two *subtrees*
  - node contains a value
  - left and right subtrees are *binary trees*



... Trees

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Other special kinds of tree

- *m-ary tree*: each internal node has exactly *m* children
- *Ordered tree*: all left values < root, all right values > root
- *Balanced tree*: has ≅minimal height for a given number of nodes
- *Degenerate tree*: has ≅maximal height for a given number of nodes

Search Trees

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Binary Search Trees

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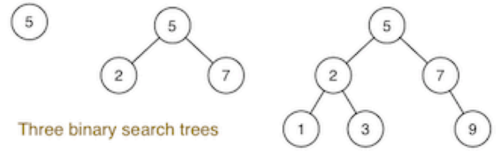
Binary search trees (or *BSTs*) have the characteristic properties

- each node is the root of 0, 1 or 2 subtrees
- all values in any left subtree are less than root
- all values in any right subtree are greater than root
- these properties applies over all nodes in the tree

(perfectly) *balanced trees* have the properties

- #nodes in left subtree = #nodes in right subtree
- this property applies over all nodes in the tree



... Binary Search Trees

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Operations on BSTs:

- *insert*(Tree,Item) … add new item to tree via key
- *delete*(Tree,Key) … remove item with specified key from tree
- *search*(Tree,Key) … find item containing key in tree
- plus, "bookkeeping" … new(), free(), show(), …

Notes:

- nodes contain `Items`; we just show `Item.key`
- keys are unique (not technically necessary)

... Binary Search Trees

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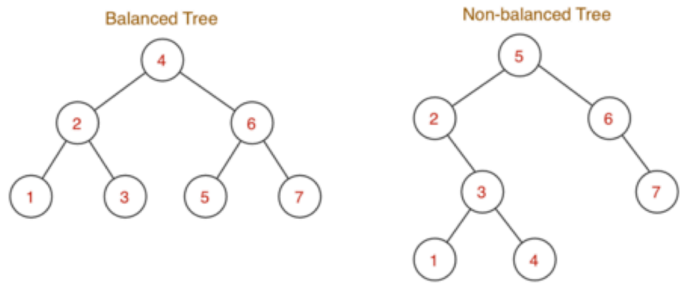
Examples of binary search trees:



Shape of tree is determined by order of insertion.

... Binary Search Trees

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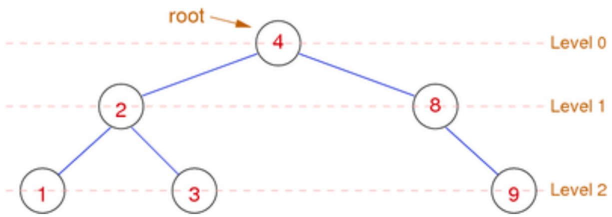
Level of node = path length from root to node

Height (or: *depth*) of tree = max path length from root to leaf



Height-balanced tree: ∀ nodes: height(left subtree) = height(right subtree)

Time complexity of tree algorithms is typically *O(height)*

Exercise #1: Insertion into BSTs

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For each of the sequences below

- start from an initially empty binary search tree
- show tree resulting from inserting values in order given

(a) 4 2 6 5 1 7 3

(b) 6 5 2 3 4 7 1

(c) 1 2 3 4 5 6 7

Assume new values are always inserted as new leaf nodes

(a) the balanced tree from 3 slides ago (height = 2)

(b) the non-balanced tree from 3 slides ago (height = 3)

(c) a fully degenerate tree of height 6

Representing BSTs

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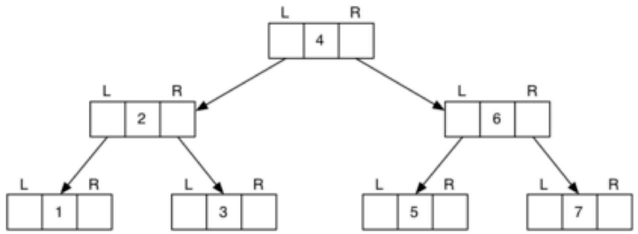
Binary trees are typically represented by node structures

- containing a value, and pointers to child nodes

Most tree algorithms move *down* the tree.
If upward movement needed, add a pointer to parent.



... Representing BSTs

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Typical data structures for trees …

```
// a Tree is represented by a pointer to its root node
typedef struct Node *Tree;

// a Node contains its data, plus left and right subtrees
typedef struct Node {
   int  data;
   Tree left, right;
} Node;

// some macros that we will use frequently
#define data(tree)  ((tree)->data)
#define left(tree)  ((tree)->left)
#define right(tree) ((tree)->right)
```

We ignore items ⇒ `data` in `Node` is just a key

... Representing BSTs

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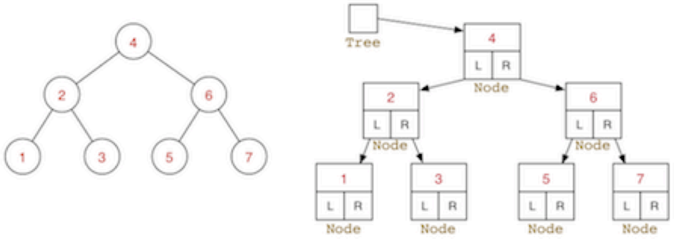
Abstract data vs concrete data ...



Tree Algorithms

Searching in BSTs

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Most tree algorithms are best described recursively:

```
TreeSearch(tree,item):
|  Input  tree, item
|  Output true if item found in tree, false otherwise
|
|  if tree is empty then
|     return false
|  else if item < data(tree) then
|     return TreeSearch(left(tree),item)
|  else if item > data(tree) then
|     return TreeSearch(right(tree),item)
|  else         // found
|    return true
|  end if
```

Insertion into BSTs

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Insert an item into appropriate subtree:

```
insertAtLeaf(tree,item):
|  Input  tree, item
|  Output tree with item inserted
|
|  if tree is empty then
|     return new node containing item
|  else if item < data(tree) then
|     return insertAtLeaf(left(tree),item)
|  else if item > data(tree) then
|     return insertAtLeaf(right(tree),item)
|  else
|     return tree    // avoid duplicates
|  end if
```

Tree Traversal

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Iteration (traversal) on ...

- `Lists` ... visit each value, from first to last
- `Graphs` ... visit each vertex, order determined by DFS/BFS/...

For binary `Trees`, several well-defined visiting orders exist:

- *preorder* (NLR) ... visit root, then left subtree, then right subtree
- *inorder* (LNR) ... visit left subtree, then root, then right subtree
- *postorder* (LRN) ... visit left subtree, then right subtree, then root
- *level-order* ... visit root, then all its children, then all their children

... Tree Traversal

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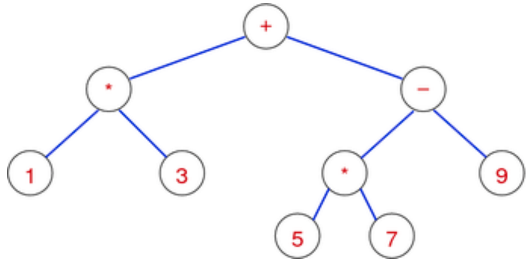
Consider "visiting" an expression tree like:



NLR: + * 1 3 - * 5 7 9 (prefix-order: useful for building tree)
LNR: 1 * 3 + 5 * 7 - 9 (infix-order: "natural" order)
LRN: 1 3 * 5 7 * 9 - + (postfix-order: useful for evaluation)
Level: + * - 1 3 * 9 5 7 (level-order: useful for printing tree)

Exercise #2: Tree Traversal

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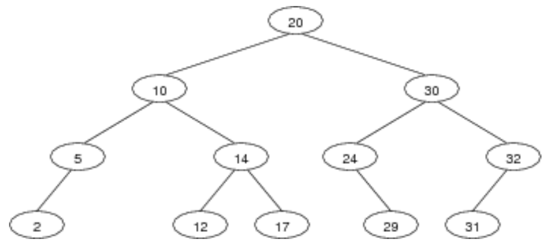
Show NLR, LNR, LRN traversals for the following tree:



NLR (preorder): 20 10 5 2 14 12 17 30 24 29 32 31

LNR (inorder): 2 5 10 12 14 17 20 24 29 30 31 32

LRN (postorder): 2 5 12 17 14 10 29 24 31 32 30 20

Exercise #3: Non-recursive traversals

Write a non-recursive *preorder* traversal algorithm.

Assume that you have a stack ADT available.

```
showBSTreePreorder(t):
|  Input tree t
|
|  push t onto new stack S
|  while stack is not empty do
|  |  t=pop(S)
|  |  print data(t)
|  |  if left(t) is not empty then
|  |     push left(t) onto S
|  |  end if
|  |  if right(t) is not empty then
|  |     push right(t) onto S
|  |  end if
|  end while
```


Joining Two Trees

An auxiliary tree operation …

Tree operations so far have involved just one tree.

An operation on two trees: `t = joinTrees(t1,t2)`

- Pre-conditions:
  - takes two BSTs; returns a single BST
  - $\max(key(t_1)) < \min(key(t_2))$
- Post-conditions:
  - result is a BST (i.e. fully ordered)
  - containing all items from t_1 and t_2

... Joining Two Trees

Method for performing tree-join:

- find the min node in the right subtree (t_2)
- replace min node by its right subtree
- elevate min node to be new root of both trees

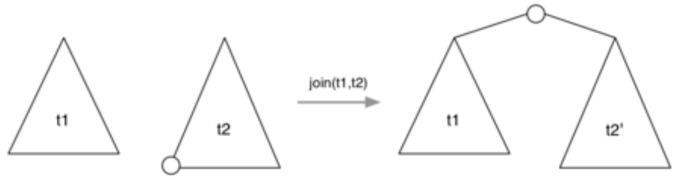
Advantage: doesn't increase height of tree significantly

$x \le height(t) \le x+1$, where $x = \max(height(t_1),height(t_2))$

Variation: choose deeper subtree; take root from there.

... Joining Two Trees

Joining two trees:



Note: t2' may be less deep than t2

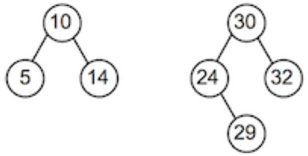
... Joining Two Trees

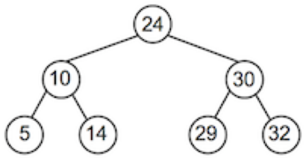
Implementation of tree-join:

```
joinTrees(t1,t2):
|  Input  trees t1,t2
|  Output t1 and t2 joined together
|
|  if t1 is empty then return t1
|  else if t2 is empty then return t2
|  else
|  |  curr=t2, parent=NULL
|  |  while left(curr) is not empty do     // find min element in t2
|  |     parent=curr
|  |     curr=left(curr)
|  |  end while
|  |  if parent≠NULL then
|  |     left(parent)=right(curr)  // unlink min element from parent
|  |     right(curr)=t2
|  |  end if
|  |  left(curr)=t1
|  |  return curr                  // curr is new root
|  end if
```

Exercise #4: Joining Two Trees

Join the trees





Deletion from BSTs

Insertion into a binary search tree is easy.

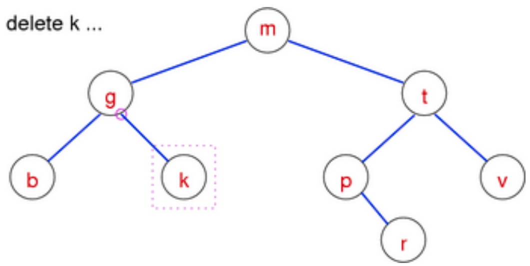
Deletion from a binary search tree is harder.

Four cases to consider …

- empty tree … new tree is also empty
- zero subtrees … unlink node from parent
- one subtree … replace by child
- two subtrees … replace by successor, join two subtrees

... Deletion from BSTs

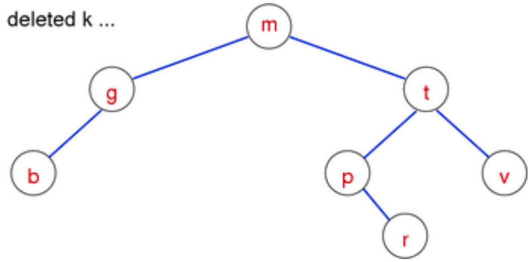
Case 2: item to be deleted is a leaf (zero subtrees)



Just delete the item

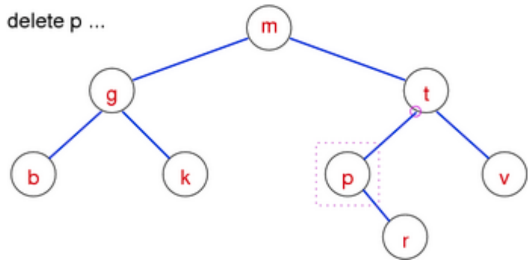
... Deletion from BSTs

Case 2: item to be deleted is a leaf (zero subtrees)



... Deletion from BSTs

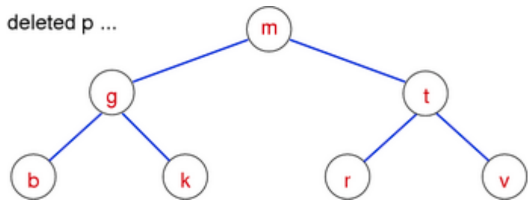
Case 3: item to be deleted has one subtree



Replace the item by its only subtree

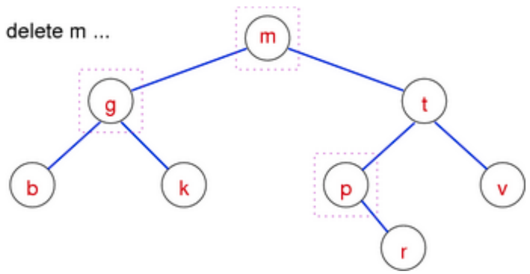
... Deletion from BSTs

Case 3: item to be deleted has one subtree



... Deletion from BSTs

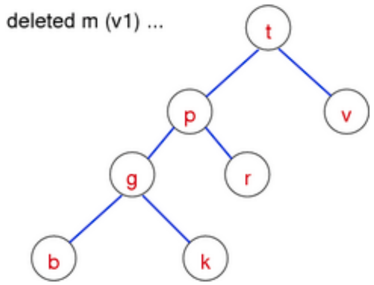
Case 4: item to be deleted has two subtrees



Version 1: right child becomes new root, attach left subtree to min element of right subtree

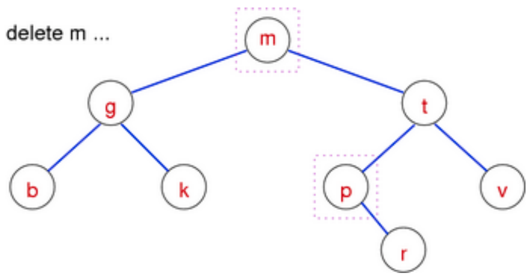
... Deletion from BSTs

Case 4: item to be deleted has two subtrees



... Deletion from BSTs

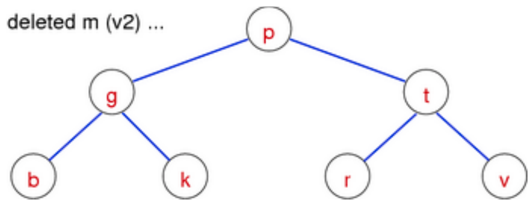
Case 4: item to be deleted has two subtrees



Version 2: *join* left and right subtree

... Deletion from BSTs

Case 4: item to be deleted has two subtrees



... Deletion from BSTs

Pseudocode (version 2):

```
TreeDelete(t,item):
|  Input  tree t, item
|  Output t with item deleted
|
|  if t is not empty then           // nothing to do if tree is empty
|  |  if item < data(t) then        // delete item in left subtree
|  |     left(t)=TreeDelete(left(t),item)
|  |  else if item > data(t) then   // delete item in left subtree
|  |     right(t)=TreeDelete(right(t),item)
|  |  else                          // node 't' must be deleted
|  |  |  if left(t) and right(t) are empty then
|  |  |     new=empty tree                    // 0 children
|  |  |  else if left(t) is empty then
|  |  |     new=right(t)                      // 1 child
|  |  |  else if right(t) is empty then
|  |  |     new=left(t)                       // 1 child
|  |  |  else
|  |  |     new=joinTrees(left(t),right(t))   // 2 children
|  |  |  end if
|  |  |  free memory allocated for t
|  |  |  t=new
|  |  end if
|  end if
|  return t
```

Balanced BSTs

Balanced Binary Search Trees

Goal: build binary search trees which have

- minimum height ⇒ minimum worst case search cost

Perfectly balanced tree with *N* nodes has

- abs(#nodes(LeftSubtree) - #nodes(RightSubtree)) < 2, for every node
- height of log_2N ⇒ worst case search $O(log\ N)$

Three *strategies* to improving worst case search in BSTs:

- *randomise* — reduce chance of worst-case scenario occuring
- *amortise* — do more work at insertion to make search faster
- *optimise* — implement all operations with performance bounds

---

Operations for Rebalancing

To assist with rebalancing, we consider new operations:

Left rotation

- move right child to root; rearrange links to retain order

Right rotation

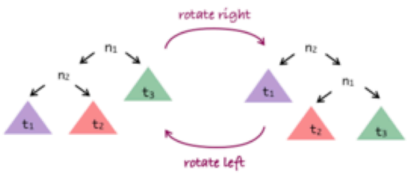
- move left child to root; rearrange links to retain order

Insertion at root

- each new item is added as the new root node

Tree Rotation

In tree below: $t_1 < n_2 < t_2 < n_1 < t_3$



... Tree Rotation

Method for rotating tree T right:

- N_1 is current root; N_2 is root of N_1's left subtree
- N_1 gets new left subtree, which is N_2's right subtree
- N_1 becomes root of N_2's new right subtree
- N_2 becomes new root

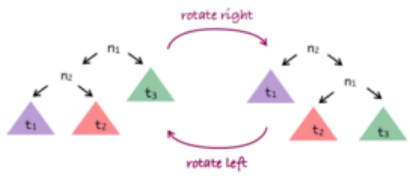
Left rotation: swap left/right in the above.

Cost of tree rotation: *O(1)*

... Tree Rotation

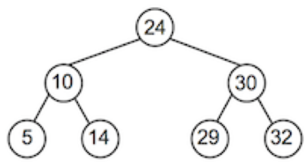
Algorithm for right rotation:

```
rotateRight(n1):
|  Input  tree n1
|  Output n1 rotated to the right
|
|  if n1 is empty ∨ left(n1) is empty then
|     return n1
|  end if
|  n2=left(n1)
|  left(n1)=right(n2)
|  right(n2)=n1
|  return n2
```

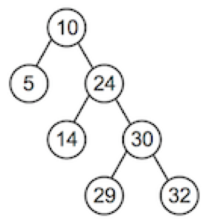


Exercise #5: Tree Rotation

Consider the tree `t`:

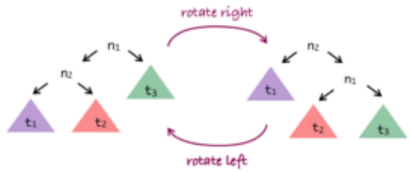


Show the result of `rotateRight(t)`



Exercise #6: Tree Rotation

Write the algorithm for left rotation




```
rotateLeft(n2):
|  Input  tree n2
|  Output n2 rotated to the left
|
|  if n2 is empty ∨ right(n2) is empty then
|     return n2
|  end if
|  n1=right(n2)
|  right(n2)=left(n1)
|  left(n1)=n2
|  return n1
```

Insertion at Root

Previous description of BSTs inserted at leaves.

Different approach: insert new item at root.

Potential disadvantages:

- large-scale rearrangement of tree for each insert

Potential advantages:

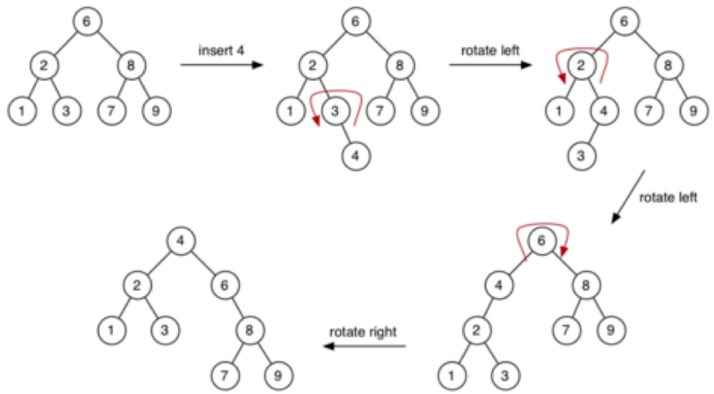
- recently-inserted items are close to root
- low cost if recent items more likely to be searched

... Insertion at Root

Method for inserting at root:

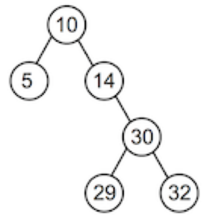
- base case:
  - tree is empty; make new node and make it root
- recursive case:
  - insert new node as root of appropriate subtree
  - lift new node to root by rotation

... Insertion at Root

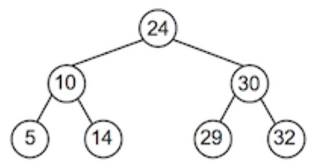


Exercise #7: Insertion at Root

Consider the tree `t`:



Show the result of `insertAtRoot(t,24)`



... Insertion at Root

Analysis of insertion-at-root:

- same complexity as for insertion-at-leaf: *O(height)*
- tendency to be balanced, but no balance guarantee
- benefit comes in searching
  - for some applications, search favours recently-added items
  - insertion-at-root ensures these are close to root
- could even consider "move to root when found"
  - effectively provides "self-tuning" search tree

Rebalancing Trees

An approach to balanced trees:

- insert into leaves as for simple BST
- periodically, rebalance the tree

Question: how frequently/when/how to rebalance?

```
NewTreeInsert(tree,item):
|  Input  tree, item
|  Output tree with item randomly inserted
|
|  t=insertAtLeaf(tree,item)
|  if #nodes(t) mod k = 0 then
|     t=rebalance(t)
|  end if
|  return t
```

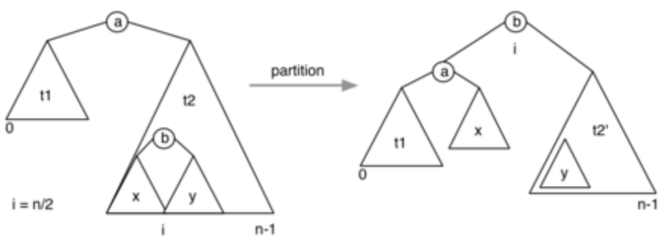
E.g. rebalance after every 20 insertions ⇒ choose *k=20*

Note: To do this efficiently we would need to change tree data structure and basic operations:

```
typedef struct Node {
   int  data;
   int  nnodes;      // #nodes in my tree
   Tree left, right; // subtrees
} Node;
```

... Rebalancing Trees

How to rebalance a BST?   Move median item to root.



... Rebalancing Trees

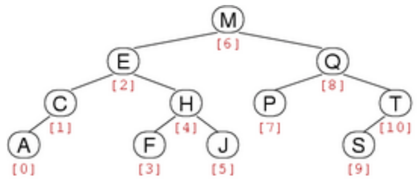
Implementation of rebalance:

```
rebalance(t):
|  Input  tree t with n nodes
|  Output t rebalanced
|
|  if n≥3 then
|  |  t=partition(t,⌊n/2⌋)          // put node with median key at root
|  |  left(t)=rebalance(left(t))   // then rebalance each subtree
|  |  right(t)=rebalance(right(t))
|  end if
|  return t
```

... Rebalancing Trees

New operation on trees:

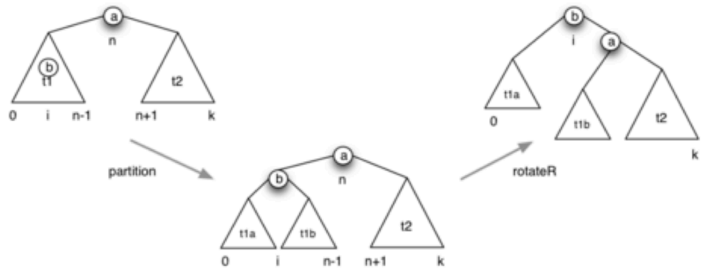
- **partition(tree,i)**: re-arrange tree so that element with index *i* becomes root



For tree with *N* nodes, indices are *0 .. N-1*

... Rebalancing Trees

Partition: moves *i* th node to root



... Rebalancing Trees

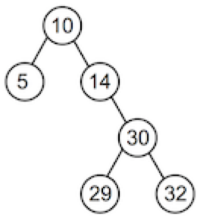
Implementation of partition operation:

```
partition(tree,i):
|  Input   tree with n nodes, index i
|  Output  tree with i^th item moved to the root
|
|  m=#nodes(left(tree))
|  if i < m then
|     left(tree)=partition(left(tree),i)
|     tree=rotateRight(tree)
|  else if i > m then
|     right(tree)=partition(right(tree),i-m-1)
|     tree=rotateLeft(tree)
|  end if
|  return tree
```

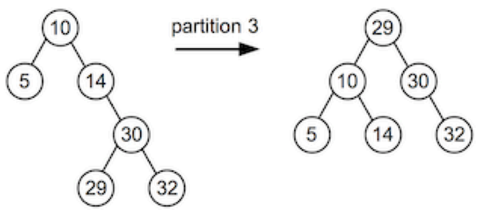
Note: size(tree) = n, size(left(tree)) = m, size(right(tree)) = n-m-1 (why -1?)

Exercise #8: Partition

Consider the tree `t`:



Show the result of `partition(t,3)`



... Rebalancing Trees

Analysis of rebalancing: visits every node ⇒ $O(N)$

Cost means not feasible to rebalance after each insertion.

When to rebalance? … Some possibilities:

- after every k insertions
- whenever "imbalance" exceeds threshold

Either way, we tolerate worse search performance for periods of time.

Does it solve the problem? … Not completely ⇒ Solution: real balanced trees (next week)

Application of BSTs: Sets

Trees provide efficient search.

Sets require efficient search

- to find where to insert/delete
- to test for set membership

Logical to implement a set ADT via `BSTree`

... Application of BSTs: Sets

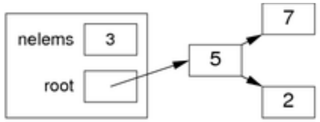
Assuming we have `Tree` implementation

- which precludes duplicate key values
- which implements

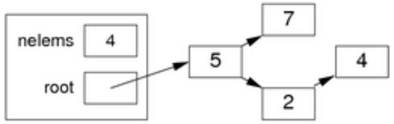
then `Set` implementation is

- `SetInsert(Set,Item)` ≡ `TreeInsert(Tree,Item)`
- `SetDelete(Set,Item)` ≡ `TreeDelete(Tree,Item.Key)`
- `SetMember(Set,Item)` ≡ `TreeSearch(Tree,Item.Key)`

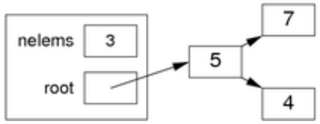
... Application of BSTs: Sets



After SetInsert(s,4):



After SetDelete(s,2):



... Application of BSTs: Sets

Concrete representation:

```
#include <BSTree.h>

typedef struct SetRep {
   int   nelems;
   Tree  root;
} SetRep;

Set newSet() {
   Set S = malloc(sizeof(SetRep));
   assert(S != NULL);
   S->nelems = 0;
   S->root = newTree();
   return S;
}
```

Summary

- Binary search tree (BST) data structure
- BST insertion and deletion
- Other tree operations
  - tree rotation
  - tree partition
  - joining trees
- Suggested reading:
  - Sedgewick, Ch.12.5-12.6,12.8-12.9

Produced: 3 Oct 2017