Week 06: Graph Data Structures

Graph Definitions

Graphs

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Many applications require

- a collection of *items* (i.e. a set)
- *relationships*/connections between items

Examples:

- maps: items are cities, connections are roads
- web: items are pages, connections are hyperlinks

Collection types you're familiar with

- lists … linear sequence of items (week 3, COMP9021)
- trees … branched hierachy of items (COMP9021)

Graphs are more general … allow arbitrary connections

... Graphs

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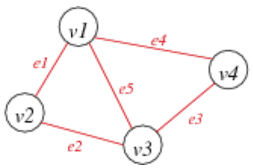
A *graph G = (V,E)*

- *V* is a set of *vertices*
- *E* is a set of *edges* (subset of *V×V*)

Example:



V = {v1, v2, v3, v4}

E = {e1, e2, e3, e4, e5}

... Graphs

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A real example: Australian road distances

Distance	Adelaide	Brisbane	Canberra	Darwin	Melbourne	Perth	Sydney
Adelaide	-	2055	1390	3051	732	2716	1605
Brisbane	2055	-	1291	3429	1671	4771	982
Canberra	1390	1291	-	4441	658	4106	309
Darwin	3051	3429	4441	-	3783	4049	4411
Melbourne	732	1671	658	3783	-	3448	873
Perth	2716	4771	4106	4049	3448	-	3972
Sydney	1605	982	309	4411	873	3972	-

Notes: vertices are cities, edges are distance between cities, symmetric

... Graphs

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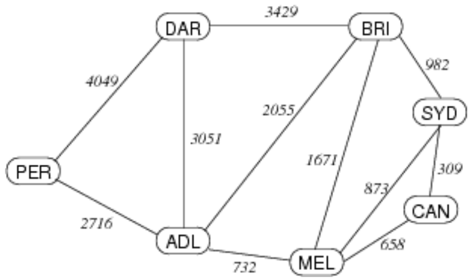
Alternative representation of above:



... Graphs

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Questions we might ask about a graph:

- is there a way to get from item A to item B?
- what is the best way to get from A to B?
- which items are connected?

Graph algorithms are generally more complex than tree/list ones:

- no implicit order of items
- graphs may contain cycles
- concrete erpresentation is less obvious
- algorithm complexity depends on connection complexity

Properties of Graphs

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Terminology: *|V|* and *|E|* (cardinality) normally written just as *V* and *E*.

A graph with *V* vertices has at most *V(V-1)/2* edges.

The ratio *E:V* can vary considerably.

- if *E* is closer to V^2, the graph is *dense*
- if *E* is closer to *V*, the graph is *sparse*
  - Example: web pages and hyperlinks

Knowing whether a graph is sparse or dense is important

- may affect choice of data structures to represent graph
- may affect choice of algorithms to process graph

Exercise #1: Number of Edges

The edges in a graph represent pairs of connected vertices. A graph with *V* has V^2 such pairs.

Consider *V = {1,2,3,4,5}* with all possible pairs:

E = { (1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), …, (4,5), (5,5) }

Why do we say that the maximum #edges is *V(V-1)/2*?

… because

- *(v,w)* and *(w,v)* denote the same edge (in an undirected graph)
- we do not consider loops *(v,v)*

Graph Terminology

For an edge *e* that connects vertices *v* and *w*

- *v* and *w* are *adjacent* (neighbours)
- *e* is *incident* on both *v* and *w*

Degree of a vertex *v*

- number of edges incident on *e*

Synonyms:

- vertex = node, edge = arc = link (Note: some people use arc for *directed* edges)

... Graph Terminology

Path: a sequence of vertices where

- each vertex has an edge to its predecessor

Cycle: a path where

- last vertex in path is same as first vertex in path

Length of path or cycle:

- #edges

Path: 1–2, 2–3, 3–4 *Cycle: 1–2, 2–3, 3–4, 4–1*

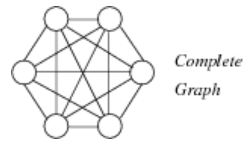
... Graph Terminology

Connected graph

- there is a *path* from each vertex to every other vertex
- if a graph is not connected, it has ≥2 *connected components*

Complete graph K_V

- there is an *edge* from each vertex to every other vertex
- in a complete graph, *E = V(V-1)/2*



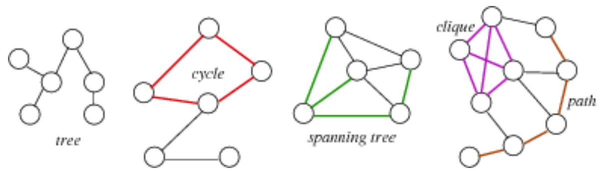
... Graph Terminology

Tree: connected (sub)graph with no cycles

Spanning tree: tree containing all vertices

Clique: complete subgraph

Consider the following single graph:



This graph has 25 vertices, 32 edges, and 4 connected components

Note: The entire graph has no spanning tree; what is shown in green is a spanning tree of the third connected component

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... Graph Terminology

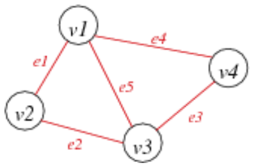
A *spanning tree* of connected graph $G = (V,E)$

- is a subgraph of G containing all of V
- and is a single tree (connected, no cycles)

A *spanning forest* of non-connected graph $G = (V,E)$

- is a subgraph of G containing all of V
- and is a set of trees (not connected, no cycles),
    - with one tree for each *connected component*

Exercise #2: Graph Terminology



V = {v1, v2, v3, v4}

E = {e1, e2, e3, e4, e5}

1. How many edges to remove to obtain a spanning tree?
2. How many different spanning trees?

---

1. 2
2. $\frac{5 \cdot 4}{2} - 2 = 8$ spanning trees (no spanning tree if we remove *{e1,e2}* or *{e3,e4}*)

... Graph Terminology

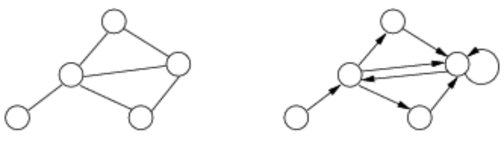
Undirected graph

- *edge(u,v) = edge(v,u)*, no self-loops (i.e. no *edge(v,v)*)

Directed graph

- *edge(u,v) ≠ edge(v,u)*, can have self-loops (i.e. *edge(v,v)*)

Examples:



Undirected graph *Directed graph*

... Graph Terminology

Other types of graphs ...

Weighted graph

- each edge has an associated value (weight)
- e.g. road map (weights on edges are distances between cities)

Multi-graph

- allow multiple edges between two vertices
- e.g. function call graph (`f()` calls `g()` in several places)

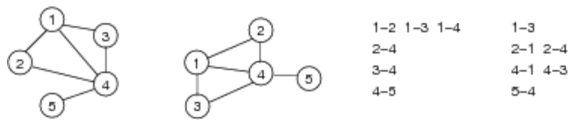
Graph Data Structures

Graph Representations

Defining graphs:

- need some way of identifying vertices
- could give diagram showing edges and vertices
- could give a list of edges

E.g. four representations of the same graph:



(a) *(b)* *(c)* *(d)*

... Graph Representations

We will discuss three different graph data structures:

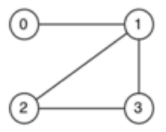
1. Array of edges
2. Adjacency matrix
3. Adjacency list

Array-of-edges Representation

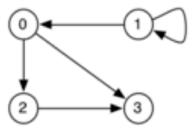
Edges are represented as an array of `Edge` values (= pairs of vertices)

- space efficient representation

- adding and deleting edges is slightly complex
- undirected: order of vertices in an `Edge` doesn't matter
- directed: order of vertices in an `Edge` encodes direction



[(0,1), (1,2), (1,3), (2,3)]



[(1,0), (1,1), (0,2), (0,3), (2,3)]

For simplicity, we always assume vertices to be numbered `0..V-1`

... Array-of-edges Representation

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Graph initialisation

```
newGraph(V):
|  Input  number of nodes V
|  Output new empty graph
|
|  g.nV = V   // #vertices (numbered 0..V-1)
|  g.nE = 0   // #edges
|  allocate enough memory for g.edges[]
|  return g
```

... Array-of-edges Representation

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Edge insertion

```
insertEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  i=0
|  while i<g.nE ∧ (v,w)≠g.edges[i] do
|     i=i+1
|  end while
|  if i=g.nE then       // (v,w) not found
|     g.edges[i]=(v,w)
|     g.nE=g.nE+1
|  end if
```

... Array-of-edges Representation

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Edge removal

```
removeEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  i=0
|  while i<g.nE ∧ (v,w)≠g.edges[i] do
|     i=i+1
|  end while
|  if i<g.nE then                // (v,w) found
|     g.edges[i]=g.edges[g.nE-1] // replace by last edge in array
|     g.nE=g.nE-1
|  end if
```

Cost Analysis

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Storage cost: *O(E)*

Cost of operations:

- initialisation: *O(1)*
- insert edge: *O(E)* (assuming edge array has space)
- delete edge: *O(E)* (need to find edge in edge array)

If array is full on insert

- allocate space for a bigger array, copy edges across ⇒ still *O(E)*

If we maintain edges in order

- use binary search to find edge ⇒ *O(log E)*

Exercise #3: Array-of-edges Representation

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Assuming an array-of-edges representation …

Write an algorithm to output all edges of the graph

```
show(g):
|  Input graph g
|
|  for all i=0 to g.nE-1 do
|     print g.edges[i]
|  end for
```

Adjacency Matrix Representation

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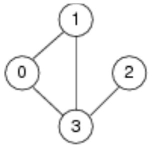
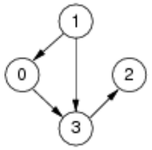
Edges represented by a $V \times V$ matrix



Undirected graph

A	0	1	2	3
0	0	1	0	1
1	1	0	0	1
2	0	0	0	1
3	1	1	1	0



Directed graph

A	0	1	2	3
0	0	0	0	1
1	1	0	0	1
2	0	0	0	0
3	0	0	1	0

... Adjacency Matrix Representation

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Advantages

- easily implemented as 2-dimensional array
- can represent graphs, digraphs and weighted graphs
  - graphs: symmetric boolean matrix
  - digraphs: non-symmetric boolean matrix
  - weighted: non-symmetric matrix of weight values

Disadvantages:

- if few edges (sparse) ⇒ memory-inefficient

... Adjacency Matrix Representation

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Graph initialisation

```
newGraph(V):
|  Input  number of nodes V
|  Output new empty graph
|
|  g.nV = V   // #vertices (numbered 0..V-1)
|  g.nE = 0   // #edges
|  allocate memory for g.edges[][]
|  for all i,j=0..V-1 do
|     g.edges[i][j]=0   // false
|  end for
|  return g
```

... Adjacency Matrix Representation

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Edge insertion

```
insertEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  if g.edges[v][w]=0 then  // (v,w) not in graph
|     g.edges[v][w]=1       // set to true
|     g.edges[w][v]=1
|     g.nE=g.nE+1
|  end if
```

... Adjacency Matrix Representation

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Edge removal

```
removeEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  if g.edges[v][w]≠0 then  // (v,w) in graph
|     g.edges[v][w]=0       // set to false
|     g.edges[w][v]=0
|     g.nE=g.nE-1
|  end if
```

Exercise #4: Show Graph

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Assuming an adjacency matrix representation …

Write an algorithm to output all edges of the graph (no duplicates!)

... Adjacency Matrix Representation

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```
show(g):
|  Input graph g
|
|  for all i=0 to g.nV-1 do
|  |  for all j=i+1 to g.nV-1 do
|  |     if g.edges[i][j]≠0 then
|  |        print i"—"j
|  |     end if
|  |  end for
|  end for
```

Exercise #5:

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Analyse storage cost and time complexity of adjacency matrix representation

Storage cost: $O(V^2)$

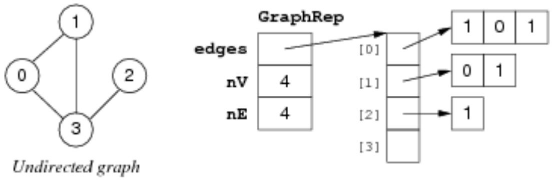
If the graph is sparse, most storage is wasted.

Cost of operations:

- initialisation: $O(V^2)$ (initialise $V \times V$ matrix)
- insert edge: $O(1)$ (set two cells in matrix)
- delete edge: $O(1)$ (unset two cells in matrix)

... Adjacency Matrix Representation

A storage optimisation: store only top-right part of matrix.



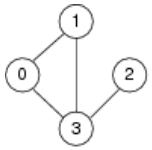
Undirected graph

New storage cost: $V\text{-}1$ int ptrs + $V(V+1)/2$ ints (but still $O(V^2)$)

Requires us to always use edges (v,w) such that $v < w$.

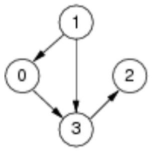
Adjacency List Representation

For each vertex, store linked list of adjacent vertices:



Undirected graph

A[0] = <1, 3>
A[1] = <0, 3>
A[2] = <3>
A[3] = <0, 1, 2>



Directed graph

A[0] = <3>
A[1] = <0, 3>
A[2] = < >
A[3] = <2>

... Adjacency List Representation

Advantages

- relatively easy to implement in languages like C
- can represent graphs and digraphs
- memory efficient if $E:V$ relatively small

Disadvantages:

- one graph has many possible representations
  (unless lists are ordered by same criterion e.g. ascending)

... Adjacency List Representation

Graph initialisation

```
newGraph(V):
|  Input   number of nodes V
|  Output  new empty graph
|
|  g.nV = V    // #vertices (numbered 0..V-1)
|  g.nE = 0    // #edges
|  allocate memory for g.edges[]
|  for all i=0..V-1 do
|     g.edges[i]=NULL    // empty list
|  end for
|  return g
```

... Adjacency List Representation

Edge insertion:

```
insertEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  if ¬inLL(g.edges[v],w) then   // (v,w) not in graph
|     insertLL(g.edges[v],w)
|     insertLL(g.edges[w],v)
|     g.nE=g.nE+1
|  end if
```

... Adjacency List Representation

Edge removal:

```
removeEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  if inLL(g.edges[v],w) then   // (v,w) in graph
|     deleteLL(g.edges[v],w)
|     deleteLL(g.edges[w],v)
|     g.nE=g.nE-1
|  end if
```

Exercise #6:

Analyse storage cost and time complexity of adjacency list representation

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Storage cost: *O(E)*

Cost of operations:

- initialisation: *O(V)* (initialise *V* lists)
- insert edge: *O(1)* (insert one vertex into list)
- delete edge: *O(E)* (need to find vertex in list)

If vertex lists are sorted

- insert requires search of list ⇒ *O(E)*
- delete always requires a search, regardless of list order

Comparison of Graph Representations

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	array of edges	adjacency matrix	adjacency list
space usage	*E*	V^2	*V+E*
initialise	*1*	V^2	*V*
insert edge	*E*	*1*	*1*
remove edge	*E*	*1*	*E*

Other operations:

	array of edges	adjacency matrix	adjacency list
disconnected(v)?	*E*	*V*	*1*
isPath(x,y)?	*E·log V*	V^2	*V+E*
copy graph	*E*	V^2	*V+E*
destroy graph	*1*	*V*	*V+E*

Graph Abstract Data Type

Graph ADT

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Data:

- set of edges, set of vertices

Operations:

- building: create graph, add edge
- deleting: remove edge, drop whole graph
- scanning: check if graph contains a given edge

Things to note:

- set of vertices is fixed when graph initialised
- we treat vertices as `int`s, but could be arbitrary `Item`s

... Graph ADT

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Graph ADT interface **`graph.h`**

```
// graph representation is hidden
typedef struct GraphRep *Graph;

// vertices denoted by integers 0..N-1
typedef int Vertex;

// edges are pairs of vertices (end-points)
typedef struct Edge { Vertex v; Vertex w; } Edge;

// operations on graphs
Graph newGraph(int V);                    // new graph with V vertices
void  insertEdge(Graph, Edge);
void  removeEdge(Graph, Edge);
bool  adjacent(Graph, Vertex, Vertex); /* is there an edge
                                          between two vertices */
void  freeGraph(Graph);
```

Graph ADT (Array of Edges)

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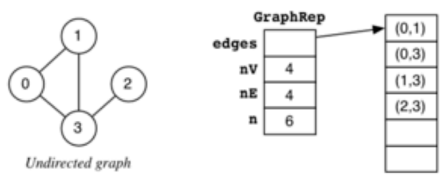
Implementation of `GraphRep` (array-of-edges representation)

```
typedef struct GraphRep {
   Edge *edges; // array of edges
   int   nV;    // #vertices (numbered 0..nV-1)
   int   nE;    // #edges
   int   n;     // size of edge array
} GraphRep;
```



... Graph ADT (Array of Edges)

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Implementation of graph initialisation (array-of-edges representation)

```
Graph newGraph(int V) {
   assert(V >= 0);
   Graph g = malloc(sizeof(GraphRep));    assert(g != NULL);

   g->nV = V; g->nE = 0;
   // allocate enough memory for edges
   g->n = Enough;
   g->edges = malloc(g->n*sizeof(Edge));  assert(g->edges != NULL);

   return g;
}
```

How much is enough? … No more than *V(V-1)/2* … Much less in practice (sparse graph)

... Graph ADT (Array of Edges)

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Implementation of edge insertion/removal (array-of-edges representation)

```
// check if two edges are equal
bool eq(Edge e1, Edge e2) {
   return ( (e1.v == e2.v && e1.w == e2.w)
            || (e1.v == e2.w && e1.w == e2.v) );
}

void insertEdge(Graph g, Edge e) {
   // ensure that g exists and array of edges isn't full
   assert(g != NULL && g->nE < g->n);
   int i = 0;
   while (i < g->nE && !eq(e,g->edges[i]))
      i++;
   if (i == g->nE)                         // edge e not found
      g->edges[g->nE++] = e;
}

void removeEdge(Graph g, Edge e) {
   assert(g != NULL);                      // ensure that g exists
   int i = 0;
   while (i < g->nE && !eq(e,g->edges[i]))
      i++;
   if (i < g->nE)                          // edge e found
      g->edges[i] = g->edges[--g->nE];
}
```

Exercise #7: Checking Neighbours (i)

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Assuming an array-of-edges representation …

Implement a function to check whether two vertices are directly connected by an edge

`bool adjacent(Graph g, Vertex x, Vertex y) { … }`

```
bool adjacent(Graph g, Vertex x, Vertex y) {
   assert(g != NULL);
   Edge e;
   e.v = x; e.w = y;
   int i = 0;
   while (i < g->nE) {
      if (eq(e,g->edges[i]))  // edge found
         return true;
      i++;
   }
   return false;              // edge not found
}
```

Graph ADT (Adjacency Matrix)

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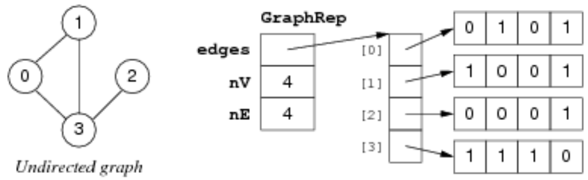
Implementation of `GraphRep` (adjacency-matrix representation)

```
typedef struct GraphRep {
   int  **edges; // adjacency matrix
   int    nV;    // #vertices
   int    nE;    // #edges
} GraphRep;
```



... Graph ADT (Adjacency Matrix)

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Implementation of graph initialisation (adjacency-matrix representation)

```
Graph newGraph(int V) {
   assert(V >= 0);
   int i;

   Graph g = malloc(sizeof(GraphRep));      assert(g != NULL);
   g->nV = V;  g->nE = 0;

   // allocate memory for each row
   g->edges = malloc(V * sizeof(int *));    assert(g->edges != NULL);
   // allocate memory for each column and initialise with 0
   for (i = 0; i < V; i++) {
      g->edges[i] = calloc(V, sizeof(int)); assert(g->edges[i] != NULL);
   }
   return g;
}
```

standard library function **`calloc(size_t nelems, size_t nbytes)`**

- allocates a memory block of size `nelems*nbytes`
- and sets all bytes in that block to *zero*

... Graph ADT (Adjacency Matrix)

Implementation of edge insertion/removal (adjacency-matrix representation)

```
// check if vertex is valid in a graph
bool validV(Graph g, Vertex v) {
   return (g != NULL && v >= 0 && v < g->nV);
}

void insertEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));

   if (!g->edges[e.v][e.w]) {  // edge e not in graph
      g->edges[e.v][e.w] = 1;
      g->edges[e.w][e.v] = 1;
      g->nE++;
   }
}

void removeEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));

   if (g->edges[e.v][e.w]) {   // edge e in graph
      g->edges[e.v][e.w] = 0;
      g->edges[e.w][e.v] = 0;
      g->nE--;
   }
}
```

Exercise #8: Checking Neighbours (ii)

Assuming an adjacency-matrix representation …

Implement a function to check whether two vertices are directly connected by an edge

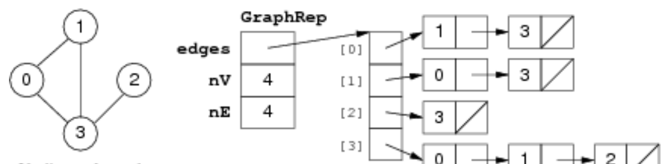
`bool adjacent(Graph g, Vertex x, Vertex y) { … }`

```
bool adjacent(Graph g, Vertex x, Vertex y) {
   assert(g != NULL && validV(g,x) && validV(g,y));

   return (g->edges[x][y] != 0);
}
```

Graph ADT (Adjacency List)

Implementation of `GraphRep` (adjacency-list representation)

```
typedef struct GraphRep {
   Node **edges;  // array of lists
   int    nV;     // #vertices
   int    nE;     // #edges
} GraphRep;

typedef struct Node {
   Vertex       v;
   struct Node *next;
} Node;
```



Undirected graph

... Graph ADT (Adjacency List)

Implementation of graph initialisation (adjacency-list representation)

```
Graph newGraph(int V) {
   assert(V >= 0);
   int i;

   Graph g = malloc(sizeof(GraphRep));     assert(g != NULL);
   g->nV = V;  g->nE = 0;

   // allocate memory for array of lists
   g->edges = malloc(V * sizeof(Node *));  assert(g->edges != NULL);
   for (i = 0; i < V; i++)
      g->edges[i] = NULL;

   return g;
}
```

... Graph ADT (Adjacency List)

Implementation of edge insertion/removal (adjacency-list representation)

```
void insertEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));

   if (!inLL(g->edges[e.v], e.w)) {    // edge e not in graph
      g->edges[e.v] = insertLL(g->edges[e.v], e.w);
      g->edges[e.w] = insertLL(g->edges[e.w], e.v);
      g->nE++;
   }
}
```

```c
void removeEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));

   if (inLL(g->edges[e.v], e.w)) {     // edge e in graph
      g->edges[e.v] = deleteLL(g->edges[e.v], e.w);
      g->edges[e.w] = deleteLL(g->edges[e.w], e.v);
      g->nE--;
   }
}
```

`inLL`, `insertLL`, `deleteLL` are standard linked list operations (as discussed in week 3)

Exercise #9: Checking Neighbours (iii)

Assuming an adjancency list representation …

Implement a function to check whether two vertices are directly connected by an edge

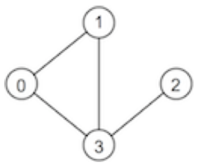
`bool adjacent(Graph g, Vertex x, Vertex y) { … }`

```c
bool adjacent(Graph g, Vertex x, Vertex y) {
   assert(g != NULL && validV(g,x));

   return inLL(g->edges[x], y);
}
```

Exercise #10: Graph ADT Client

Write a program that uses the graph ADT to

- build the graph depicted below
- print all the nodes that are incident to vertex `1` in ascending order



```c
#include <stdio.h>
#include "Graph.h"

#define NODES 4
#define NODE_OF_INTEREST 1

int main(void) {
   Graph g = newGraph(NODES);

   Edge e;
   e.v = 0; e.w = 1; insertEdge(g,e);
   e.v = 0; e.w = 3; insertEdge(g,e);
   e.v = 1; e.w = 3; insertEdge(g,e);
   e.v = 3; e.w = 2; insertEdge(g,e);

   int v;
   for (v = 0; v < NODES; v++) {
      if (adjacent(g, v, NODE_OF_INTEREST))
         printf("%d\n", v);
   }

   freeGraph(g);
   return 0;
}
```

Summary

- Graph terminology
  - vertices, edges, vertex degree, connected graph, tree
  - path, cycle, clique, spanning tree, spanning forest
- Graph representations
  - array of edges
  - adjacency matrix
  - adjacency lists

- Suggested reading:
  - Sedgewick, Ch.17.1-17.5
