Common Integral Inequalities

lhz7269874

July 2025

1 Cauchy-Schwarz Inequality

If $f, g \in L^2[a,b]$,then

$$(\int_a^b f(x)g(x)dx)^2 \leq (\int_a^b f^2(x)dx)(\int_a^b g^2(x)dx).$$

The equality holds if and only if $f = \lambda g$ almost everywhere.
Proof:$0 \leq \int_a^b (f(x)-tg(x))^2 dx = t^2 \int_a^b g^2(x)dx - 2t\int_a^b f(x)g(x)dx + \int_a^b f^2(x)dx$.This is a quadratic equation with one unknown and its $\triangle \leq 0$,which indicates what we want to prove.

2 Hölder's Inequality

Assume that $f \in L^p[a,b], g \in L^q[a,b], p > 1, q > 1, and \frac{1}{p} + \frac{1}{q} = 1$,then

$$\int_a^b |f(x)g(x)|dx \leq (\int_a^b |f(x)|^p dx)^{\frac{1}{p}} (\int_a^b |g(x)|^q dx)^{\frac{1}{q}}$$

The equality holds if and only if $|f|^p = |g|^q$ almost everywhere.
Proof:let $F = (\int_a^b |f(x)|^p dx)^{\frac{1}{p}}, G = (\int_a^b |g(x)|^q dx)^{\frac{1}{q}}$,if F=0 or G=0,then f=0 a.e. or g=0 a.e. and obviously the left side is 0.So we assume that $F \neq 0$ and $G \neq 0$.Let $u(x) = \frac{|f(x)|}{F}, v(x) = \frac{|g(x)|}{G}$.then $\int_a^b u^p(x)dx = 1, \int_a^b v^q(x)dx = 1$.
Now we put u(x) and v(x) into Young Inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, a, b \geq 0$$

and integrate both sides:

$$\int_a^b u(x)v(x)dx \leq \int_a^b (\frac{u^p(x)}{p} + \frac{v^q(x)}{q})dx = 1/p + 1/q = 1$$

Restore the variables and the proof finished.

3 Mincowski Inequality

Assume that $f, g \in L^p(X, \mu), 1 \leq p < \infty$,then

$$\int\limits_X |f+g|^p \leq \int\limits_X |f|^p + \int\limits_X |g|^p$$

Proof:We will use the Hölder's inequality:

$$\int |uv| \leq (\int |u|^p)^{\frac{1}{p}} (\int |v|^p)^{\frac{1}{q}}, \frac{1}{p} + \frac{1}{q} = 1, p, q > 1$$

$$\int |f+g|^p = \int |f+g||f+g|^{p-1} \leq \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \leq (\int |f|^p)^{\frac{1}{p}} (\int |f+g|^{(p-1)q})^{\frac{1}{q}} + (\int |g|^p)^{\frac{1}{p}} (\int |f$$

As pq=p+q,(p-1)q=p,we will find

$$\int |f+g|^p \leq (\int |f|^p)^{\frac{1}{p}} (\int |f+g|^p)^{\frac{1}{q}} + (\int |g|^p)^{\frac{1}{p}} (\int |f+g|^p)^{\frac{1}{q}} = [(\int |f|^p)^{\frac{1}{p}} + (\int |g|^p)^{\frac{1}{p}})](\int |f+g|^p)^{\frac{1}{q}}$$

Divide both sides by$(\int |f+g|^{(p-1)q})^{\frac{1}{q}}$ and proof done.