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CS-4413 || Homework 4

2/11/2025

1. Illustrate the operation of Build-Max-Heap on array: $A = \{7, 19, 16, 10, 22, 3, 6, 33, 21\}$

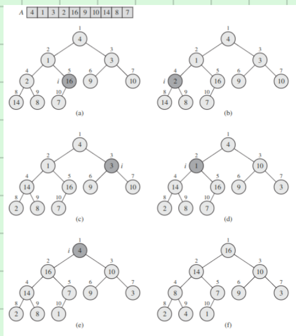
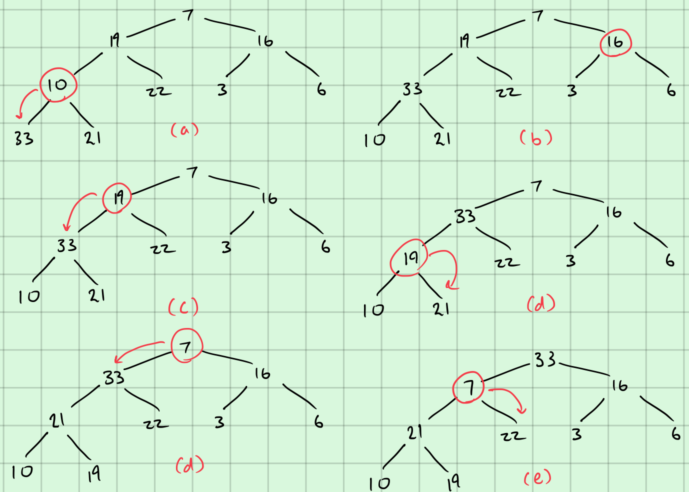
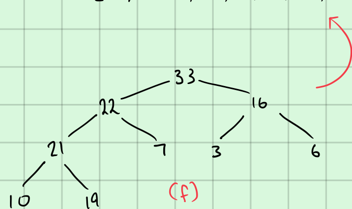


Figure 6.3 The operation of BUILD-MAX-HEAP, showing the data structure before the call to MAX-HEAPIFY in line 3 of BUILD-MAX-HEAP. **(a)** A 10-element input array A and the binary tree it represents. The figure shows that the loop index i refers to node 5 before the call MAX-HEAPIFY(A, i). **(b)** The data structure that results. The loop index i for the next iteration refers to node 4. **(c)–(e)** Subsequent iterations of the **for** loop in BUILD-MAX-HEAP. Observe that whenever MAX-HEAPIFY is called on a node, the two subtrees of that node are both max-heaps. **(f)** The max-heap after BUILD-MAX-HEAP finishes.



Final max heap array:
$A = \{33, 22, 16, 21, 7, 3, 6, 10, 19\}$



2. Illustrate the operation of Max-Heapify(A,3) on the array $A = \{27, 17, 3, 16, 13, 10, 1, 5, 7, 12, 4, 8, 9, 0\}$

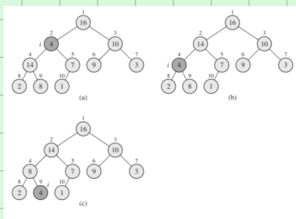
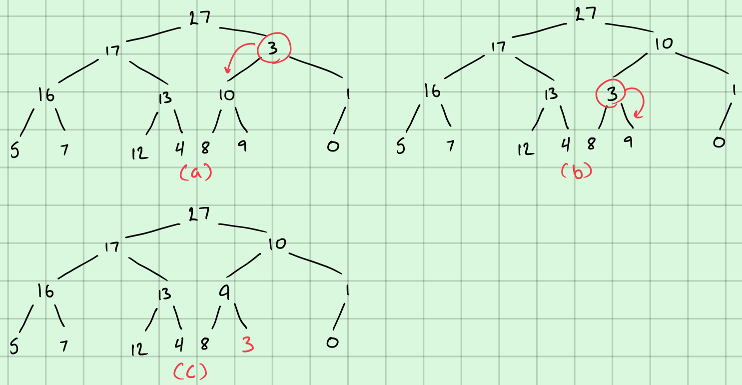


Figure 6.2 The action of MAX-HEAPIFY$(A, 2)$, where $A.heap\text{-}size = 10$. **(a)** The initial configuration, with $A[2]$ at node $i = 2$ violating the max-heap property since it is not larger than both children. The max-heap property is restored for node 2 in **(b)** by exchanging $A[2]$ with $A[4]$, which destroys the max-heap property for node 4. The recursive call MAX-HEAPIFY$(A, 4)$ now has $i = 4$. After swapping $A[4]$ with $A[9]$, as shown in **(c)**, node 4 is fixed up, and the recursive call MAX-HEAPIFY$(A, 9)$ yields no further change to the data structure.



3. Derive and solve the recurrence for the average case of Quick Sort.

worst case: $O(n)^2$

Average/expected: $O(n\log n)$

$$E[T(n)] = \frac{2}{n}\sum_{q=2}^{n-1} E[T(q)] + \Theta(n)$$

$$T(n) = \frac{2}{n}\sum_{k=0}^{n-1} T(k) + O(n)$$

$$T(n) = \frac{2}{n}\sum_{k=0}^{n-1} O(k\log k) + O(n)$$

$$\sum_{k=0}^{n-1} k\log k = \frac{n^2\log n}{2} \rightarrow T(n) = \frac{2}{n}\cdot\frac{n^2\log n}{2} + O(n) = O(n\log n)$$

4. Solve the following recurrences using method of substitution:

a) $T(n) = 2T(n/4) + 1,\ T(1) = 1$

$$T(n) = 2T(n/4) + 1$$
$$T(n/4) = 2T(n/16) + 1$$
$$T(n/16) = 2T(n/64) + 1 \quad > T(1) = 1$$
$$T(n) = 2(2T(n/16) + 1) + 1 = 4T(n/16) + 2 + 1$$
$$= 4(2T(n/64) + 1) + 3 = 8T(n/64) + 4 + 3$$
$$= 8T(n/64) + 7$$

recursion ends when $\frac{n}{4^k} = 1$ or $k = \log_4 n$

$$T(n) = 2^k T(n/4^k) + \sum_{i=0}^{k-1} 2^i$$
$$= 2^{\log_4 n} T(1) + \sum_{i=0}^{\log_4 n - 1} 2^i$$
$$= n^{\log_4 2}\cdot 1 + (2^{\log_4 n} - 1)$$
$$= n^{1/2} + (n^{1/2} - 1) = 2n^{1/2} - 1$$
$$T(n) = 2\sqrt{n} - 1$$

b) $T(n) = 2T(n/4) + n^2,\ T(1) = 1$

$$T(n/4) = 2T(n/16) + (n/4)^2$$
$$T(n/16) = 2T(n/64) + (n/16)^2$$
$$\ldots\ T(1) = 1$$
$$T(n) = 2(2T(n/16) + (n/4)^2) + n^2$$
$$= 4T(n/16) + 2(n/4)^2 + n^2$$
$$= 4(2T(n/64) + (n/16)^2) + n^2/8 + n^2$$
$$= 8T(n/64) + 4(n/16)^2 + n^2/8 + n^2$$
$$= 8T(n/64) + n^2/64 + n^2/8 + n^2$$
$$\Downarrow$$
$$T(n) = 2^k T(n/4^k) + n^2\sum_{i=0}^{k-1} (2/16)^i$$

rec. ends when $\frac{n}{4^k} = 1 \Rightarrow k = \log_4 n$

$$T(n) = 2^{\log_4 n} T(1) + n^2 \sum_{i=0}^{\log_4 n - 1} (1/8)^i$$
$$= n^{\log_4 2}\cdot 1 + n^2\cdot\frac{1 - (1/8)^{\log_4 n}}{1 - 1/8}$$
$$= n^{1/2} + n^2\cdot\frac{1 - n^{-\log_4 8}}{7/8}$$
$$= \sqrt{n} + 8/7\, n^2(1 - n^{-3/2})$$
$$\therefore T(n) = \sqrt{n} + \frac{8}{7}n^2 - \frac{8}{7}n^{1/2}$$

c) $T(n) = 3T(n/2) + 5n,\ T(1) = 1$

$$T(n/2) = 3T(n/4) + 5(n/2)$$
$$T(n/4) = 3T(n/8) + 5(n/4)$$
$$\ldots\ T(1) = 1$$
$$T(n) = 3(3T(n/4) + \frac{5n}{2}) + 5n$$
$$= 9T(n/4) + \frac{15n}{2} + 5n$$
$$= 9(3T(n/8) + \frac{5n}{4}) + \frac{15n}{2} + 5n$$
$$= 27T(n/8) + \frac{45n}{4} + \frac{15n}{2} + 5n$$
$$\ldots$$
$$T(n) = 3^k T(n/2^k) + 5n\sum_{i=0}^{k-1} (3/2)^i$$

$\frac{n}{2^k} = 1 \Rightarrow k = \log_2 n$

$$T(n) = 3^{\log_2 n} T(1) + 5n\sum_{i=0}^{\log_2 n - 1} \left(\frac{3}{2}\right)^i$$
$$= n^{\log_2 3}\cdot 1 + 5n\,\frac{(3/2)^{\log_2 n} - 1}{3/2 - 1}$$
$$= n^{\log_2 3} + 5n\,\frac{n^{\log_2 3/2} - 1}{1/2}$$
$$= n^{\log_2 3} + 10n(n^{\log_2 3/2} - 1)$$
$$= n^{\log_2 3} + 10n\left(\frac{n^{\log_2 3}}{n} - 1\right)$$
$$= n^{\log_2 3} + 10n^{\log_2 3} - 10n$$
$$= 11n^{\log_2 3} - 10n$$
$$\therefore T(n) = 11n^{\log_2 3} - 10n$$

d) $T(n) = 2T(n/2) + (n-1),\ T(1) = 1$

$$T(n/2) = 2T(n/4) + (n/2 - 1)$$
$$T(n/4) = 2T(n/8) + (n/4 - 1)$$
$$\ldots$$
$$T(1) = 1$$
$$T(n) = 2(2T(n/4) + (n/2 - 1)) + (n - 1)$$
$$= 4T(n/4) + 2(n/2 - 1)) + (n - 1)$$
$$= 4T(n/4) + n - 2 + n - 1 = 4T(n/4) + 2n - 3$$
$$= 4(2T(n/8) + (n/4 - 1)) + 2n - 3$$
$$= 8T(n/8) + 4(n/4 - 1) + 2n - 3$$
$$= 8T(n/8) + n - 4 + 2n - 3 = 8T(n/8) + 3n - 7$$
$$\ldots$$
$$T(n) = 2^k T(n/2^k) + \sum_{i=0}^{k-1} (2^i n - (2^{i+1} - 1))$$

end $n/2^k = 1 \Rightarrow k = \log_2 n$

$$T(n) = 2^{\log_2 n} T(1) + \sum_{i=0}^{\log_2 n - 1} (2^i n - (2^{i+1} - 1))$$

$n^{\log_2 2}$

$$= n\cdot 1 + n\cdot\sum_{i=0}^{\log_2 n - 1} 2^i - \sum_{i=0}^{\log_2 n - 1} (2^{i+1} - 1)$$
$$= n + n(2^{\log_2 n} - 1) - (2(2^{\log_2 n} - 1) - \log_2 n)$$
$$= n + n^2 - n - 2n + 2 + \log_2 n$$
$$= n^2 - 2n + 2 + \log_2 n$$
$$\therefore T(n) = n^2 - 2n + 2 + \log_2 n$$