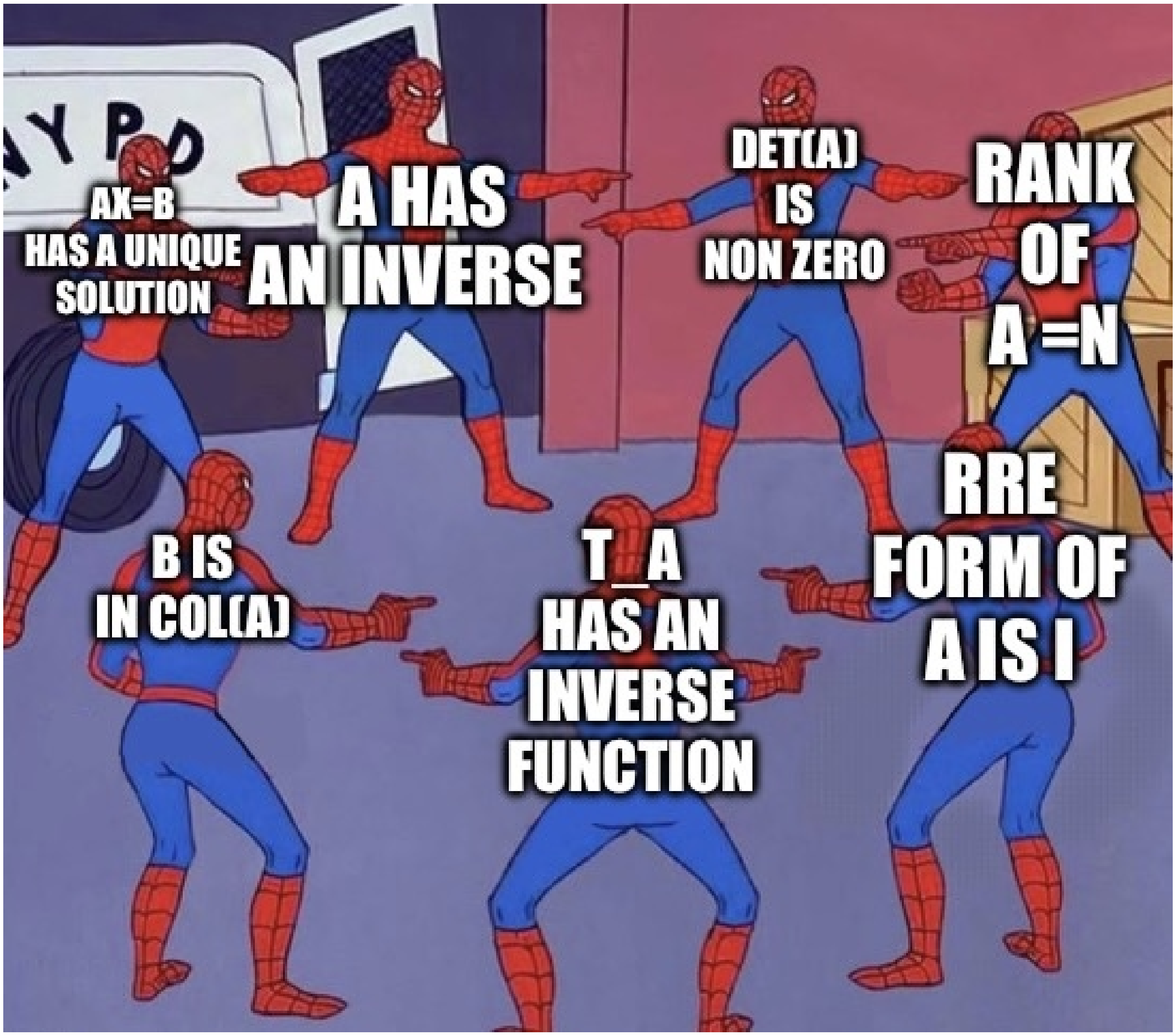
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MATH 3333 PRACTICE PROBLEM SET 2

Problem 1. Compute the cofactor matrix and the determinant of the matrix given below. If it is invertible, write down its inverse.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution:

$C_{11}(A) = (-1)^2 \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}$=1

$C_{12}(A) = (-1)^3 \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix}$= -3

$C_{13}(A) = (-1)^4 \begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix}$= -3

$C_{21}(A) = (-1)^3 \begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix}$= -1

$C_{22}(A) = (-1)^4 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$=1

$C_{23}(A) = (-1)^5 \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix}$=1

$C_{31}(A) = (-1)^4 \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix}$=-2

$C_{32}(A) = (-1)^5 \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$= 6

$C_{33}(A) = (-1)^6 \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$=4

By taking the cofactor expansion along row 1, we have $|A| = 1 \times 1 + (-1) \times (-3) + 2 \times (-3) = -2$

$cof(A) = \begin{bmatrix} 1 & -3 & -3 \\ -1 & 1 & 1 \\ -2 & 6 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(cof(A))^T = \frac{1}{-2}\begin{bmatrix} 1 & -1 & -2 \\ -3 & 1 & 6 \\ -3 & 1 & 4 \end{bmatrix}$

Problem 2. Determine whether following statements are true or false. If true, write down a short proof. If false, provide a counter example.

(a) $det(A + B) = det(A) + det(B)$

Solution: False. Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $|A| = 0 = |B|$ but $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has the determinant -1.

(b) If $det(A) = 0$ then A has two identical rows.

Solution: False. A counterexample would be $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(c) If R is the RRE form of A, then $det(a) = det(R)$

Solution: False. Some of the row operations can change the determinant. $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ has the determinant 2. But its RRE form $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has the determinant 1.

(d) If A is 2×2 then $det(7A) = 49det(A)$

Solution: True. As we saw in the class, for an $n \times n$ matrix A and a scalar k, $|kA| = k^n|A|$.

(e) If A has a row of all zeroes then so does $cof(A)$.

Solution: True. Let's say R_i of A has only zeroes. Then any sub matrix obtained by deleting any column and a row different from R_i will have a row of all zeroes, making the determinant of that sub matrix 0. Hence the cofactors along that whole row will be zero.

(f) Recall that the transpose of $cof(A)$ is called as the adjoint of A and denoted by $adj(A)$. If $adj(A) = A^{-1}$ then $det(A) = 1$ **Solution:** True. We showed in the class, $A.adj(A) = |A|I$. If $A^{-1} = adj(A)$ then $A.adj(A) = I$.

Hence we have $|A|I = I \Rightarrow \begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & 0\dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & |A| \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & 0\dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} \Rightarrow |A| = 1$

Problem 3. Suppose that the first row of a matrix A is $\begin{bmatrix}1 & -1 & 2\end{bmatrix}$ and the first row of the cofactor matrix of a A is $\begin{bmatrix}3 & 1 & 2\end{bmatrix}$

(a) Determine whether A is invertible.
Solution: $|A| = a_{11}C_{11}(A) + a_{12}C_{12}(A) + a_{13}C_{13}(A) = 1 \times 3 + (-1) \times 1 + 2 \times 2 = 6$. So A is invertible as $|A| \neq 0$

(b) Why cannot the second row the cofactor matrix be $\begin{bmatrix}0 & -1 & 2\end{bmatrix}$? **Solution:** We saw in the class that $a_{i1}C_{j1}(A)+a_{i2}C_{j2}(A)+\cdots+a_{in}C_{jn}(A)$ is equal to $|A|$ if $i=j$ and to 0 if $i \neq j$.

But if the second row is as given above, then
$a_{21}C_{11}(A) + a_{22}C_{12}(A) + a_{23}C_{13}(A) = 0 - 1 + 4 = 3 \neq 0$ which is not possible.

Problem 4. Let S and T be two linear transformations as described below.

$T : \mathbb{R}^3 \to \mathbb{R}^3$ is the projection onto $yz-$plane. That is for any vector U in $\mathbb{R}^3$, $T(U)$ is the projection of U onto $yz-$plane.

$S : \mathbb{R}^3 \to \mathbb{R}^3$ is the function that scales vectors by 5. That is for any vector U in $\mathbb{R}^3$, $S(U) = 5U$.

(1) Write down a matrix A such that T is the transformation induced by A.
Solution: Let $U = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$. Then $T(U) = \begin{bmatrix}0\\x_2\\x_3\end{bmatrix}$
Hence $A = \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

(2) Find eigenvalues and associated eigenvectors of A. Can you geometrically interpret your answer ?
Solution: $A - \lambda I = \begin{bmatrix}-\lambda & 0 & 0\\0 & 1-\lambda & 0\\0 & 0 & 1-\lambda\end{bmatrix}$
Then $det(A - \lambda I) = -\lambda(1-\lambda)^2 = 0$. Then $\lambda = 0, 1$ are the eigenvalues of A

If $\lambda = 0$
$(A - \lambda I)X = 0 \Rightarrow AX = 0$
The augmented matrix is

$$\left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right]$$

By $R_2 \leftrightarrow R_1$ and then $R_3 \leftrightarrow R_2$ we obtain the RRE form

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Take $x_1 = s$. Then $x_2 = x_3 = 0$. So eigenvectors associated wih $\lambda = 0$ are the vectors of the form $X = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

If $\lambda = 1$
$(A - \lambda I)X = 0 \Rightarrow (A - I)X = 0$
The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By $R_1 \to -R_1$ we obtain the RRE form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Take $x_2 = s$ and $x_3 = t$. Then $x_1 = 0$. So eigenvectors associated wih $\lambda = 1$ are the vectors of the form $X = \begin{bmatrix} 0 \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
Most vectors change the direction when projected onto $yz-$plane. What are the vectors that don't change the direction or change the direction by 180 degrees ?

- If a vector U is on $yz-$plane , then its projection onto $yz-$plane is itself ; $AU = U$. So the vectors on $yz-$plane are eigenvectors of A associated to $\lambda = 1$. The answer we got for $\lambda = 1$ was $\begin{bmatrix} 0 \\ s \\ t \end{bmatrix}$ and these are precisely all the vectors on $yz-$plane.
- If a vector U lies on the $x-$ axis, its projection onto $yz-$ plane is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; $AU = 0 = 0U$. So the vectors on $x-$ axis are eigenvectors of A associated to $\lambda = 0$. The answer we got for $\lambda = 0$ was $s \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}$ and these are precisely all the vectors that lie on $x-$ axis.

(3) Write down a matrix B such that S is the transformation induced by B.

Solution: Let $U = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then $S(U) = \begin{bmatrix} 5x_1 \\ 5x_2 \\ 5x_3 \end{bmatrix}$

Hence $B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(4) Find eigenvalues and associated eigenvectors of B. Can you geometrically interpret your answer ?

Solution: $A - \lambda I = \begin{bmatrix} 5-\lambda & 0 & 0 \\ 0 & 5-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}$

Then $det(A - \lambda I) = \lambda(5-\lambda)^3 = 0$. Then $\lambda = 5$ is the only eigenvalue of B

If $\lambda = 5$

$(A - 5I)X = 0 \Rightarrow AX = 0$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So all three variables are free variables. Take $x_1 = r, X_2 = s$ and $x_3 = t$. So eigenvectors associated with $\lambda = 5$ are the vectors of the form $X = \begin{bmatrix} r \\ s \\ t \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

When multiplying by 5 (or by any scalar), no vector changes its direction. So any non-zero vector in $\mathbb{R}^3$ is an eigenvector of A and since $AU = 5U$, all the vectors in $\mathbb{R}^3$ are associated with the eigenvalue 5. Note that the answer we got for $\lambda = 5$ was $\begin{bmatrix} r \\ s \\ t \end{bmatrix}$ describes all the vectors in $\mathbb{R}^3$.

(5) Find a matrix C such that $T \circ S$ is the transformation induced by C.

Solution: $C = AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(6) Find the eigenvalues and associated eigenvectors of C. Can you geometrically interpret your answer ? **Solution:** $C - \lambda I = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 5-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}$

Then $det(A - \lambda I) = -\lambda(5-\lambda)^2 = 0$. Then $\lambda = 0, 5$ are the eigenvalues of A

If $\lambda = 0$
$(C - \lambda I)X = 0 \Rightarrow CX = 0$
The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

First apply the row operation $R_2 \leftrightarrow R_1$ and then $R_3 \leftrightarrow R_2$ and finally divide both first and second rows by 5 to obtain the RRE form

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Take $x_1 = s$. Then $x_2 = x_3 = 0$. So eigenvectors associated with $\lambda = 0$ are the vectors of the form $X = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

If $\lambda = 5$
$(A - \lambda I)X = 0 \Rightarrow (A - 5I)X = 0$
The augmented matrix is

$$\left[\begin{array}{ccc|c} -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By $R_1 \to -1/5R_1$ we obtain the RRE form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Take $x_2 = s$ and $x_3 = t$. Then $x_1 = 0$. So eigenvectors associated with $\lambda = 5$ are the vectors of the form $X = \begin{bmatrix} 0 \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

We know no non zero vector changes direction when multiplied by B. Then the vectors that don't change the direction when multiplied by AB are same as the vectors that don't change the direction when multiplied by A. We can confirm this by observing that the eigenvectors of AB we computed in this part are same as the eigenvectors of A we computed in part (2).

Problem 5. Let A, B and C be 4×4 matrices such that $det(A) = -2, det(B) = 7$ and $det(C) = 5$. Compute the determinants of following matrices.

- $BC^2(A^{-1})^2(C^2)^T$

  Solution: $det(BC^2(A^{-1})^2(C^2)^T) = det(B)(det(C))^2\left(\frac{1}{det(A)}\right)^2(det(C))^2$

  $= 7 \times 5^2 \times \left(\frac{1}{-2}\right)^2 \times 5^2 = 1093.75$

- $E_1B^2(A^T)^3E_2$ where E_1 and E_2 are the elementary matrices obtained by performing the row operations $R_2 \to R_2 + 2R_1$ and $R_3 \to -3R_3$ on I respectively.

  Solution: The row operation $R_2 \to R_2 + 2R_1$ does not change the determinant. Hence $det(E_1) = det(I) = 1$. The row operation $R_3 \to -3R_3$ multiplies the determinant by a factor of -3. Hence, $det(E_2) = -3det(I) = -3$.

  $det(E_1B^2(A^T)^3E_2) = det(E_1)(det(B))^2(det(A))^3det(E_2) = 1 \times 7^2 \times (-2)^3 \times (-3) = 1176$

Problem 6. Let $A = \begin{bmatrix} x & -2 & x^2 \\ y & -2 & y^2 \\ 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -x & 2 & -x^2 \\ 4y & -8 & 4y^2 \\ x-y+1 & 2 & x^2-y^2+3 \end{bmatrix}$.

If $|A| = 6$ compute $|B|$.

Solution: We need to send A through a sequence of row operations till we obtain B and keep track on the changes in the determinant under each row operation.
We start with $R_3 \to R_3 + R_1$ on A.

$$\begin{bmatrix} x & -2 & x^2 \\ y & -2 & y^2 \\ x+1 & -2+2 & x^2+3 \end{bmatrix}$$

This row operation does not change the determinant. So the determinant of the above matrix is same as the determinant of A which is 6.

Then $R_3 \to R_3 - R_2$ yields

$$\begin{bmatrix} x & -2 & x^2 \\ y & -2 & y^2 \\ x-y+1 & (-2+2)+2 & x^2-y^2+3 \end{bmatrix}$$

This row operation also does not change the determinant. So the determinant of the above matrix is still 6.

Then $R_2 \to 4R_2$ yields

$$\begin{bmatrix} x & -2 & x^2 \\ 4y & -8 & 4y^2 \\ x-y+1 & 2 & x^2-y^2+3 \end{bmatrix}$$

This row operation multiply the determinant by 4. So the determinant of the above matrix is still 24.

Then $R_1 \to -R_1$ yields

$$B = \begin{bmatrix} -x & 2 & -x^2 \\ 4y & -8 & 4y^2 \\ x-y+1 & 2 & x^2-y^2+3 \end{bmatrix}$$

This row operation multiply the determinant by -1. So the determinant of the above matrix is still -24 and since this matrix is B whe have $det(B) = -24$.

Problem 7. Explain what can be said about the determinant of A in each of the following cases. A is an $n \times n$ matrix.

(1) $A^2 = A$
Solution: $det(A^2) = det(A) \Rightarrow det(A).det(A) = det(A) \Rightarrow det(A)(det(A)-1) = 0$. Hence $det(A) = 0$ or 1

(2) A is symmetric
Solution: $det(A^T) = det(A) \Rightarrow det(A) = det(A)$. This does not tell us anything about $det(A)$.

(3) $A = PA$ where P is an invertible matrix.
Solution: $det(P)det(A) = det(P) \Rightarrow det(A)(1 - det(P)) = 0$.
If $det(P) \neq 1$ then $det(A) = 0$. If $det(P) = 1$ we cannot say anything about $det(A)$.

(4) $A^2 = PA$ where P is an invertible matrix and $|P| = 4$
Solution: $(det(A))^2 = det(P)det(A) = 4det(A) \Rightarrow det(A)(det(A)-4) = 0 \Rightarrow det(A) = 0$ or 4.

(5) $A^2 + I = 0$ and n is even
Solution: $A^2 = -I \Rightarrow (det(A))^2 = (-1)^n det(I) = (-1)^n$. If n is even then $(-1)^n$ is 1. So. $(det(A))^2 = 1 \Rightarrow det(A) = 1$ or -1.

(6) $A^2 + I = 0$ and n is odd
Solution: If n is odd then $(-1)^n$ is -1. So. $(det(A))^2 = -1$ which is not possible. So if n is odd, there cannot be an $n \times n$ matrix A such that $A^2 + I = 0$.

Problem 8. Compute the determinant of the following matrices in terms of x.

(1) $$\begin{bmatrix} 1 & x & x^2 & x^3 \\ x & x^2 & x^3 & 1 \\ x^2 & x^3 & 1 & x \\ x^3 & 1 & x & x^2 \end{bmatrix}$$

Solution:
Let's call the matrix given in this problem as A. We perform the following operations on A $R_2 \to R_2 - xR_1, \quad R_3 \to R_2 - x^2R_1$ and $R_2 \to R_2 - x^3R_1$
and obtain the following matrix which we will call as B.

$$B = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 0 & 0 & 1-x^4 \\ 0 & 0 & 1-x^4 & x-x^5 \\ 0 & 1-x^4 & x-x^5 & x^2-x^6 \end{bmatrix}$$

The three row operations we applied do not change the determinant.
Hence $det(A) = det(B) = b_{11}c_{11}(B)$

$$c_{11}(B) = (-1)^{1+1} \begin{vmatrix} 0 & 0 & 1-x^4 \\ 0 & 1-x^4 & x-x^5 \\ 1-x^4 & x-x^5 & x^2-x^6 \end{vmatrix}$$

Note that each row of the 3×3 matrix above has a common factor $(1-x^4)$. By factoring it out from each row we have the following.

$$c_{11}(B) = (1-x^4)^3 \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & x \\ 1 & x & x^2 \end{vmatrix} = (1-x^4)^3\Big(1.(0\times x - 1\times 1)\Big) = (1-x^4)^3(-1) = -(1-x^4)^3$$

So, $det(A) = det(B) = b_{11}(-1)(1-x^4)^3 = -(1-x^4)^3$.

(2) $$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & x & x \\ 1 & x & 0 & x \\ 1 & x & x & 0 \end{bmatrix}$$

Solution:
Let's call the matrix given in this problem as A. We perform following operations on the matrix
$R_3 \to R_3 - R_2, \quad R_4 \to R_4 - R_2$ and obtain the following matrix which we will call as B.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & x & x \\ 0 & x & -x & 0 \\ 0 & x & 0 & -x \end{bmatrix}$$

The two row operations we applied do not change the determinant. Hence $det(A) = det(B) = b_{21}c_{21}(B)$

$c_{21}(B) = (-1)^{2+1} \begin{vmatrix} 1 & 1 & 1 \\ x & -x & 0 \\ x & 0 & -x \end{vmatrix}$

To find the determinant of the matrix $P = \begin{bmatrix} 1 & 1 & 1 \\ x & -x & 0 \\ x & 0 & -x \end{bmatrix}$ we perform $R_2 \to R_2 - xR_1$ and $R_3 \to R_3 - xR_3$ and obtain

$Q = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2x & -x \\ 0 & -x & -2x \end{bmatrix}$

The two row operations we applied do not change the determinant.
Hence $det(P) = det(Q) = 1((-2x)(-2x) - (-x)(-x)) = 3x^2$

So, $det(A) = det(B) = b_{21}c_{21}(B) = 1 \times (-1)^3 \times 3x^2 = 3x^2 = -3x^2$

Problem 9. For each of the following matrix, find its eigenvalues and eigenvectors associated with those values.

(1) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **Solution:** The characteristic equation $|A - \lambda I| = 0$ can be simplified into

$-(\lambda + 1)^2(\lambda - 2) = 0$. So, the eigenvalues are -1 and 2.

When $\lambda = -1$, corresponding eigenvectors are given by $s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

When $\lambda = 2$, corresponding eigenvectors are given by $s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(2) $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

Solution: The characteristic equation $|A - \lambda I| = 0$ can be simplified into $-(\lambda + 2)^2(\lambda - 5) = 0$. So, the eigenvalues are -1 and 2.

When $\lambda = -2$, corresponding eigenvectors are given by $s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

When $\lambda = 5$, corresponding eigenvectors are given by $s \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$

Problem 10. The matrix A is given by

$$\begin{bmatrix} 0 & 3 & 6 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

(a) Find all eigenvalues of A.

Solution: $|A - \lambda I| = -\lambda\Big((2 - \lambda)(2 - \lambda) - 4\Big) = -\lambda\big(4 - 4\lambda + \lambda^2 - 4)$
$= -\lambda(-4\lambda + \lambda^2) = -\lambda^2(\lambda - 4) = 0 \Rightarrow \lambda = 0, 4$

(b) For each eigenvalue λ of A, find a basis for $Null(A - \lambda I)$. **Solution:** Take $\lambda = 0$. We need to find the solutions to the homogeneous system , $(A - 0I)X = 0 \Rightarrow AX = 0$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 3 & 6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

and its RRE form is

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is given by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ -2t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

Therefore, a basis for $Null(A - \lambda I)$ when $\lambda = 0$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

Take $\lambda = 4$. We need to find the solutions to the homogeneous system , $(A - 4I)X = 0$

The augmented matrix is

$$\left[\begin{array}{ccc|c} -4 & 3 & 6 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

and its RRE form is

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is given by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3s \\ 2s \\ s \end{bmatrix} = s\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Therefore, a basis for $Null(A - \lambda I)$ when $\lambda = 4$ is $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$

(c) Determine if A is diagonalizable. If it is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. **Solution:** A is diagonalizable because for each eigenvalue λ of A, we have the following : If λ is repeated k times in the characteristic equation then the linear system $(A - \lambda I)X = 0$ has k basic solutions.

We can take $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

Problem 11. The matrix A is given by

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ -8 & -4 & -4 \end{bmatrix}$$

Show that A is diagonalizable and compute A^{2024}.

Solution: The characteristic polynomial can be simplified in to $-\lambda^2(\lambda-1)=0 \Rightarrow \lambda = 0, 1$

Take $\lambda = 0$. We need to find the solutions to the homogeneous system , $(A-0I)X = 0 \Rightarrow AX = 0$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ -8 & -4 & -4 & 0 \end{array}\right]$$

and its RRE form is

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is given by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2s - 1/2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$

$= s\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Take $\lambda = 1$. We need to find the solutions to the homogeneous system , $(A - I)X = 0$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 2 & 2 & 0 \\ 2 & 0 & 1 & 0 \\ -8 & -4 & -5 & 0 \end{array}\right]$$

and its RRE form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution is given by $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2s \\ -1/4s \\ s \end{bmatrix} = s\begin{bmatrix} -1/2 \\ -1/4 \\ 1 \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$

So, A is diagonalizable (because it satisfies the same condition mentioned in the problem 10(c).)

We can take $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & 1 \\ 0 & -2 & -4 \end{bmatrix}$

For P and D as above, we have $A = PDP^{-1}$.
Multiplying this equation with itself, we have $A.A = PDP^{-1}PDP^{-1} \Rightarrow A^2 = PD^2P^{-1}$.
By doing this again, we can obtain $A^3 = PD^3P^{-1}$

By continuing this process we can obtain $A^{2024} = PD^{2024}P^{-1}$

D is a diagonal matrix. So, $D^{2024} = \begin{bmatrix} 0^{2024} & 0 & 0 \\ 0 & 0^{2024} & 0 \\ 0 & 0 & 1^{2024} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$

Hence $A^{2024} = PD^{2024}P^{-1} = PDP^{-1} = A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ -8 & -4 & -4 \end{bmatrix}$