Colby Frison | Math-3333 || Homework 3 | 7/23/2025

Problem 8. Compute the determinant of the following matrices in terms of x.

(2) $B = \begin{bmatrix} \cancel{0} & \cancel{1} & \cancel{1} & \cancel{1} \\ 1 & 0 & x & x \\ 1 & x & 0 & x \\ 1 & x & x & 0 \end{bmatrix}$

First row: $[\underset{\text{dont include}}{0} \ 1 \ 1 \ 1]$

$|B| = a_{11}C_{11}(B) + a_{12}C_{12}(B) + a_{13}C_{13}(B) + a_{14}C_{14}(B)$

$= 0 + 1\cdot(-x^2) + 1\cdot(-x^2) + 1\cdot(-x^2)$

$= -3x^2$

$\boxed{|B| = -3x^2}$

$(-1)^3(\quad)$

$C_{12}(B) = 1\cdot(-1)^4\cdot\begin{vmatrix} 0 & x \\ x & 0 \end{vmatrix} + x\cdot(-1)^5\cdot\begin{vmatrix} 1 & x \\ 1 & 0 \end{vmatrix} + x\cdot(-1)^6\cdot\begin{vmatrix} 1 & 0 \\ 1 & x \end{vmatrix}$

$= (0\cdot 0 - x\cdot x) - x(1\cdot 0 - x\cdot 1) + x(1\cdot x - 0\cdot 1)$

$= -x^2 + x^2 + x^2$

$C_{12}(B) = (-1)^3\cdot x^2$

$= -x^2$

$C_{13}(B) = 1\cdot\begin{vmatrix} x & x \\ x & 0 \end{vmatrix} - 0\cdot(\quad) + x\begin{vmatrix} 1 & x \\ 1 & x \end{vmatrix}$

$= (x\cdot 0 - x\cdot x) + (1\cdot x - x\cdot 1)$

$= -x^2 + 0$

$C_{13}(B) = (-1)^4\cdot(-x^2)$

$= -x^2$

$C_{14}(B) = 1\cdot\begin{vmatrix} x & 0 \\ x & x \end{vmatrix} - 0\cdot(\quad) + x\cdot\begin{vmatrix} 1 & x \\ 1 & x \end{vmatrix}$

$= 1\cdot(x\cdot x - x\cdot 0) + x\cdot(1\cdot x - x\cdot 1)$

$= x^2 - x(0) = x^2$

$C_{14}(B) = (-1)^5\cdot(x^2)$

$= -x^2$

Problem 11. The matrix A is given by

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ -8 & -4 & -4 \end{bmatrix}$$

Show that A is a) diagonalizable and compute b) A^{2024}.

$A = PDP^{-1}$

$A^{2024} = PD^{2024}P^{-1}$

$P = ?$

$D = ?$

$\Rightarrow (A - \lambda I) = \begin{bmatrix} 4-\lambda & 2 & 2 \\ 2 & 1-\lambda & 1 \\ -8 & -4 & -4-\lambda \end{bmatrix}$

$\Rightarrow |(A-\lambda I)| = (4-\lambda)\big((1-\lambda)(-4-\lambda) - (1)(-4)\big) - 2\big((2)(-4-\lambda) - (1)(-8)\big) + 2\big(2(-4) - (-8)(1-\lambda)\big)$

$= (4-\lambda)\big((-4-\lambda+4\lambda+\lambda^2)+4\big) - 2(\cancel{-8} - 2\lambda \cancel{+8}) + 2(\cancel{-8}\cancel{+8} - 8\lambda)$

$= (4-\lambda)(\lambda^2+3\lambda+0) + 4\lambda - 16\lambda$

$= 4\lambda^2 + 12\lambda - \lambda^3 - 3\lambda^2 + 4\lambda - 16\lambda$

$= -\lambda^3 + \lambda^2 = -\lambda^2(\lambda+1)$

$\lambda = 0$ (2 mult)

$\lambda = 1$

$\lambda = 0$

$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ -8 & -4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \xrightarrow[R_3 \to R_3 + (2)R_1]{R_2 \to R_2 - (1/2)R_1} \begin{bmatrix} 4 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x = -\frac{1}{2}y - \frac{1}{2}z \\ y = s \\ z = t \end{matrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\cdot\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + t\cdot\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} = \left(\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}\right)\cdot 2 \Rightarrow v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

a) $\boxed{\text{Matrix } A \text{ IS diagonalizable since we could find an eigen vector for every eigen value present}}$

$\lambda = 1$

$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 0 & 1 \\ -8 & -4 & -5 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 3 & 2 & 2 \\ 8 & 4 & 5 \\ -8 & -4 & -5 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 3R_1]{R_3 \to R_3 + R_2} \begin{bmatrix} 3 & 2 & 2 \\ -1 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{matrix} x = -\frac{1}{2}z \\ y = -\frac{1}{4}z \\ z = s \end{matrix} \to \begin{matrix} x = -\frac{2}{4}s \\ y = -\frac{1}{4}s \\ z = s \end{matrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} -2/4 \\ -1/4 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2/4 \\ -1/4 \\ 1 \end{bmatrix}\cdot(-4) \Rightarrow v_3 = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$

$\to \xrightarrow{R_1 \to R_1 + 3R_2} \begin{bmatrix} 0 & -4 & -1 \\ -1 & -2 & -1 \\ & & \end{bmatrix} \xrightarrow[\to R_1 \leftrightarrow R_2]{R_1 \,\&\, R_2 \to -(R_1 \,\&\, R_2)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{bmatrix}$

$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $P = \begin{bmatrix} -1 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 2 & -4 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} -2 & 0 & -1 \\ 8 & 4 & 5 \\ 4 & 2 & 2 \end{bmatrix}$

P^{-1} = transpose of cofactor matrix

$\begin{bmatrix} -1 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 2 & -4 \end{bmatrix}$

$C_{11}(P) = -2$ $C_{12}(P) = 8$ $C_{13}(P) = 4$

$C_{21}(P) = 0$ $C_{22}(P) = 4$ $C_{23}(P) = 2$

$C_{31}(P) = -1$ $C_{32}(P) = 5$ $C_{33}(P) = 2$

$C\begin{bmatrix} -2 & 8 & 4 \\ 0 & 4 & 2 \\ -1 & 5 & 2 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} -2 & 0 & -1 \\ 8 & 4 & 5 \\ 4 & 2 & 2 \end{bmatrix} = P^{-1}$

$A^{2024} = P\cdot D^{2024}\cdot P^{-1} \Rightarrow D = \begin{bmatrix} 0^{2024} & 0 & 0 \\ 0 & 0^{2024} & 0 \\ 0 & 0 & 1^{2024} \end{bmatrix} \Rightarrow D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^{2024} = \begin{bmatrix} -1 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 2 & -4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 & -1 \\ 8 & 4 & 5 \\ 4 & 2 & 2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} -2 & 0 & -1 \\ 8 & 4 & 5 \\ 4 & 2 & 2 \end{bmatrix}$

$\text{col}^n_1 \to -2\cdot[\,] + 8\cdot[\,] + 4\cdot\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -16 \end{bmatrix}$

$\text{col}^n_2 \to 0\cdot[\,] + 4\cdot[\,] + 2\cdot\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -8 \end{bmatrix}$

$\text{col}^n_3 \to -1\cdot[\,] + 5[\,] + 2\cdot\begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -8 \end{bmatrix}$

$\to A^{2024} = \begin{bmatrix} 8 & 4 & 4 \\ 4 & 2 & 2 \\ -16 & -8 & -8 \end{bmatrix}\cdot\frac{1}{2} \Rightarrow$ b) $\boxed{A^{2024} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ -8 & -4 & -4 \end{bmatrix}}$