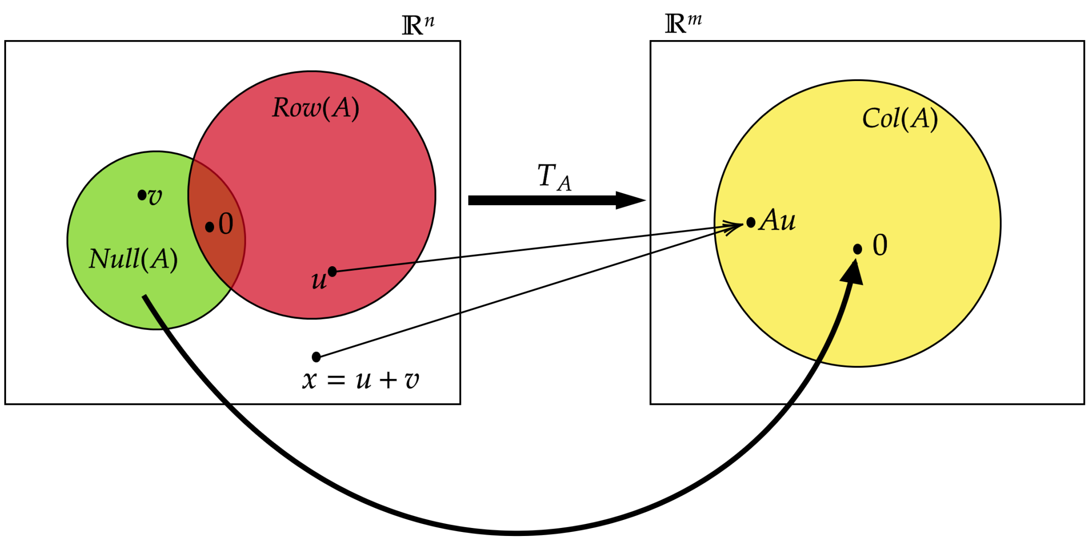
MATH 3333 PRACTICE PROBLEM SET 5

Problem 1. A is an $m \times n$ matrix. Fully explain the figure shown below.



Problem 2. Let $\mathcal{S}$ be a subspace of $\mathbb{R}^4$.

Suppose $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for $\mathcal{S}$ and

$\mathcal{B}' = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathcal{S}^\perp$.

(a) What is the dimension of $\mathcal{S}$?
Solution: 3

(b) What is the dimension of $\mathcal{S}^\perp$?
Solution: 1

(c) If $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ write down bases for $Row(A)$ and $Null(A)$ only using the information given above. (You do not have to do any new computations.)

Solution: $Row(A) = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

But $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathcal{S}$ which means it spans $\mathcal{S}$. Hence $Row(A) = \mathcal{S}$

and a basis for $Row(A)$ is $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

We know $Null(A) = (Row(A))^\perp$. So, $Null(A) = S^\perp$ and hence a basis for $Null(A)$ is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(d) Using your answer to (c), write down a basis for $\mathbb{R}^4$.
Solution: In the class, we saw $\mathbb{R}^n = Row(A) + Null(A)$ and that we can form a basis for $\mathbb{R}^n$ by taking the union of vectors in a basis of $Row(A)$ and a basis of $Null(A)$.

So $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^4$.

Problem 3. Let A be an $m \times n$ matrix which has the rank r. Prove the following statements. If needed, you can use theorems 1,2 or 3 from the class.

(1) We showed in the class any vector X in $\mathbb{R}^n$ can be written as $X = u_1 + v_1$ where u_1 is a vector in $Row(A)$ and v_1 is a vector in $Nulll(A)$. Show that this expression of X is unique. That is, if $X = u_2 + v_2$ for some other vector u_2 in in $Row(A)$ and a vector v_2 in $Null(A)$ then $u_1 = u_2$ and $v_1 = v_2$.
Solution: If $X = u_1 + v_1 = u_2 + v_2$ then $u_1 - u_2 = v_2 - v_1$.
Since $u_1, u_2 \in Row(A)$, so is $u_1 - u_2$ as $Row(A)$ is a subspace.

Since $v_1, v_2 \in Null(A)$, so is $v_2 - v_1$ as $Null(A)$ is a subspace.
Therefore, $u_1 - u_2 = v_2 - v_1 \in Row(A) \cap Null(A)$.
But $Row(A) \cap Null(A) = \{0\}$ as they are orthogonal complements of each other. Hence $u_1 - u_2 = v_2 - v_1 = 0 \Rightarrow u_1 = u_2$ and $v_1 = v_2$.

(2) If $X = u+v$ where u is a vector in $Row(A)$ and v is a vector in $Null(A)$ then $T_A(X) = T_A(u)$.

(This means when X is written as an addition of a vector in $Row(A)$ and a vector in $Null(A)$ then $T_A(X)$ depends only on the part of X that is in $Row(A)$. The other part in $Null(A)$ has no impact on $T_A(X)$.)
Solution: $T_A(X) = AX = A(u + v) = Au + Av$. But $Av = 0$ as $v \in Null(A)$.
Therefore $T_A(X) = Au = T_A(u)$

(3) If $u_1, u_2 \in Row(A)$ and $T_A(u_1) = T_A(u_2)$ then $u_1 = u_2$. (That is no two vectors in $Row(A)$ maps into the same vector under the transformation T_A.)
Solution: $T_A(u_1) = T_A(u_2) \Rightarrow Au_1 = Au_2 \Rightarrow A(u_1 - u_2) = 0 \Rightarrow u_1 - u_2 \in Null(A)$

But if $u_1, u_2 \in Row(A)$ then $u_1 - u_2 \in Row(A)$ as $Row(A)$ is a subspace.

Therefore, $u_1 - u_2 \in Row(A) \cap Null(A)$.
But $Row(A) \cap Null(A) = \{0\}$ as they are orthogonal complements of each other. Hence $u_1 - u_2 = 0 \Rightarrow u_1 = u_2$.

(4) If $\{u_1, u_2, \ldots u_r\}$ is a basis of $Row(A)$ then $\{T_A(u_1), T_A(u_2), \ldots, T_A(u_r)\}$ is basis of $Col(A)$.
Solution: $\{T_A(u_1), T_A(u_2), \ldots, T_A(u_r)\}$ is a set of vectors in the range of T_A which is same as $Col(A)$.We will first show this set is linearly independent.

Suppose $c_1T_A(u_1) + c_2T_A(u_2) + \ldots c_rT_A(u_r) = 0$. We need to show, only way this can happen is if $c + 1 = c_2 = \ldots c_r = 0$.

$c_1T_A(u_1) + c_2T_A(u_2) + \ldots c_rT_A(u_r) = 0$
$\Rightarrow c_1Au_1 + c_2Au_2 + \ldots c_rAu_r = 0$
$\Rightarrow A(c_1u_1 + c_2u_2 + \ldots c_ru_r) = 0$
$\Rightarrow c_1u_1 + c_2u_2 + \ldots c_ru_r \in Null(A)$

But $c_1u_1 + c_2u_2 + \ldots c_ru_r \in Row(A)$ as $\{u_1, u_2, \ldots u_r\}$ is a basis of $Row(A)$.

Hence $c_1u_1 + c_2u_2 + \ldots c_ru_r \in Row(A) \cap Null(A) = \{0\}$
$\Rightarrow c_1u_1 + c_2u_2 + \ldots c_ru_r = 0$
$\Rightarrow c_1 = c_2 = \ldots c_r = 0$ because the set $\{u_1, u_2, \ldots u_r\}$ is linearly independent because it is a basis of Row(A).

We just showed if $c_1T_A(u_1) + c_2T_A(u_2) + \dots c_rT_A(u_r) = 0$ then $c+1 = c_2 = \dots c_r = 0$
This means $\{T_A(u_1), T_A(u_2), \dots, T_A(u_r)\}$ is a set of linearly independent vectors in $Col(A)$.

Recall that the dimension of $Col(A)$ = dimension of $Row(A)$ which is given as r in this problem. (because Row(A) has a basis consists of r vectors.)

$\{T_A(u_1), T_A(u_2), \dots, T_A(u_r)\}$ is a set of r linearly independent vectors in $Col(A)$ and $dim(Col(A)) = r$. Then by theorem 3 from the class, this is a basis for $Col(A)$.

Problem 4. Let $\mathcal{V} = \mathbb{R}^2$.
Suppose a vector addition on $\mathcal{V}$ is defined as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$$

and a scalar multiplication is defined by

$$s. \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} sx_1 \\ sx_2 + s - 1 \end{bmatrix}$$

Determine if $\mathcal{V}$ with these operations a vector space or not.

Solution: Let $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $v = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $w = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

(A.1) $u + v = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$ is still a 2×1 column vector. Hence $u + v$ is in $\mathcal{V} = \mathbb{R}^2$.

(A.2) $u + v = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$ and $v + u = \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 + 1 \end{bmatrix}$
Hence $u + v = v + u$

(A.3) $u + (v + w) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 + z_1 \\ y_2 + z_2 + 1 \end{bmatrix} = \begin{bmatrix} x_1 + (y_1 + z_1) \\ x_2 + (y_2 + z_2 + 1) + 1 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + z_2 + 2 \end{bmatrix}$
On the other hand
$(u + v) + w = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} (x_1 + y_1) + z_1 \\ (x_2 + y_2 + 1)z_2 + 1 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + z_2 + 2 \end{bmatrix}$
Hence $u + (v + w) = (u + v) + w$

(A.4) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \\ x_2 - 1 + 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Hence $O = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is the 'zero vector'.

(A.5) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Hence $-u = \begin{bmatrix} -x_1 \\ -x_2 - 2 \end{bmatrix}$

Let s and r be scalars.

(S.1) $s.u = \begin{bmatrix} sx_1 \\ sx_2 + s - 1 \end{bmatrix}$ is still a 2×1 column vector. Hence $s.u$ is in $\mathcal{V} = \mathbb{R}^2$.

(1) $s.(u + v) = s.\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix} = \begin{bmatrix} s(x_1 + y_1) \\ s(x_2 + y_2 + 1) + s - 1 \end{bmatrix} = \begin{bmatrix} sx_1 + sy_1 \\ sx_2 + sy_2 + 2s - 1 \end{bmatrix}$

On the other hand

$s.u+s.v = \begin{bmatrix} sx_1 \\ sx_2 + s - 1 \end{bmatrix} + \begin{bmatrix} sy_1 \\ sy_2 + s - 1 \end{bmatrix} = \begin{bmatrix} sx_1 + sy_1 \\ (sx_2 + s - 1) + (sy_2 + s - 1) + 1 \end{bmatrix} = \begin{bmatrix} sx_1 + sy_1 \\ sx_2 + sy_2 + 2s - 1 \end{bmatrix}$

Hence $s.(u + v) = s.u + s.v$

(S.3) $(s + r).u = s.\begin{bmatrix} (s + r)x_1 \\ (s + r)x_2 + (s + r) - 1 \end{bmatrix} = \begin{bmatrix} sx_1 + rx_1 \\ sx_2 + rx_2 + s + r - 1 \end{bmatrix}$

On the other hand

$s.u+r.u = \begin{bmatrix} sx_1 \\ sx_2 + s - 1 \end{bmatrix} + \begin{bmatrix} rx_1 \\ rx_2 + r - 1 \end{bmatrix} = \begin{bmatrix} sx_1 + rx_1 \\ (sx_2 + s - 1) + (rx_2 + r - 1) + 1 \end{bmatrix} = \begin{bmatrix} sx_1 + rx_1 \\ sx_2 + rx_2 + s + r - 1 \end{bmatrix}$

Hence $(s + r).u = s.u + r.u$

(S.4) $s.(r.u) = s.\begin{bmatrix} rx_1 \\ rx_2 + r - 1 \end{bmatrix} = \begin{bmatrix} s(rx_1) \\ s(rx_2 + r - 1) + s - 1 \end{bmatrix} = \begin{bmatrix} srx_1 \\ srx_2 + sr - 1 \end{bmatrix}$

On the other hand, $(sr).u = \begin{bmatrix} srx_1 \\ srx_2 + sr - 1 \end{bmatrix}$

Hence $s.(r.u) = (sr).u$

(S.5) $1.u = \begin{bmatrix} 1x_1 \\ 1x_2 + 1 - 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = u$

Hence $\mathcal{V}$ with given operations is a vector space.

Problem 5. Let $\mathcal{V} = \mathbb{R}^+ =$ The set of all positive real numbers. Suppose a vector addition on $\mathcal{V}$ is defined as

$$u \boldsymbol{+} v = uv$$

for any u and v in $\mathcal{V}$ and a scalar multiplication is defined by

$$s \boldsymbol{.} u = u^s$$

for any u in $\mathcal{V}$ and a scalar s.

(a) Determine if $\mathcal{V}$ with these operations is a vector space or not.

Solution: Let $u, v =$ and $w \in \mathbb{R}^+$

(A.1) $u + v = uv$
Since u and v are positive numbers, so is uv. Hence $u + v = uv \in \mathbb{R}^+$

(A.2) $u + v = uv$ and $v + u = vu$
Hence $u + v = v + u$

(A.3) $u + (v + w) = u + (vw) = u(vw) = uvw$

On the other hand
$(u + v) + w = (uv) + w = (uv)w = uvw$

Hence $u + (v + w) = (u + v) + w$

(A.4) $u + 1 = u.1 = u$
Hence $O = 1$ is the 'zero vector'.

(A.5) $u + \frac{1}{u} = u\frac{1}{u} = 1$
Hence $-u = \frac{1}{u}$

Let s and r be scalars.

(S.1) Since u is a positive number, u^s is also a positive number for any scalar s. Hence $s.u = u^s$ is in $\mathbb{R}^+$.

(a) $s.(u + v) = s.(uv) = (uv)^s = u^s v^s$

On the other hand
$s.u + s.v = u^s + v^s = u^s v^s$

Hence $s.(u + v) = s.u + s.v$

(S.3) $(s+r).u = u^{r+s}$

On the other hand
$s.u + r.u = u^s + u^r = u^s u^r = u^{s+r}$

Hence $(s+r).u = s.u + r.u$

(S.4) $s.(r.u) = s.(u^r) = (u^r)^s = u^{rs}$

On the other hand, $(sr).u = u^{rs}$

Hence $s.(r.u) = (sr).u$

(S.5) $1.u = u^1 = u$

Hence $\mathbb{R}^+$ with given operations is a vector space.

(b) If we keep operations the same, but change $\mathcal{V}$ to $\mathcal{V} = \mathbb{R} - \{0\}$ = The set of all the real numbers except 0, will $\mathcal{V}$ be a vector space ?
Solution: No. Then $\mathcal{V}$ will not satisfy $S.1$ as $s.u = u^s$ is not defined in some cases. For an example, take $u = -1$ and $s = \frac{1}{2}$. Then $s.u = u^s = (-1)^{\frac{1}{2}} = \sqrt{-1}$ is not defined.

(c) If we keep operations the same, but change $\mathcal{V}$ such that $\mathcal{V} = \mathbb{R}^+ \cup \{0\}$ = The set of all the non negative numbers, will $\mathcal{V}$ be a vector space ? **Solution:** No. The same issue as in part b arises. In addition, it is not possible to find a 'negative vector' of 0 as $0 + u = 0 \times u = 0$ can never be equal to the 'zero vector' 1 for any $u \in \mathbb{R}$.

Problem 6. Let $\mathcal{V}$ be a vector space with the vector addition $+$ and the scalar multiplication $\boldsymbol{.}$

(a) Using only the vector space axioms, show that $0.u = \mathbf{0}$ for any vector $u \in \mathcal{V}$. (Here $\mathbf{0}$ is the zero vector from the axiom $A.4$)
Solution: This was discussed in the class. Please refer to your class notes.

(b) Using only the vector space axioms, show that $-1.u = -u$ for any vector $u \in \mathcal{V}$. (Here $-u$ is the negative of u from the axiom $A.5$)
Solution: Start with $-1 + 1 = 0$. Then we have $(-1+1).u = 0.u$

By $S.3$ we have $(-1+1).u = -1.u + 1.u$.
In part (a) we proved $0.u = \mathbf{0}$.

Hence we have $-1.u + 1.u = \mathbf{0}$

By $S.5$ we have $1.u = u$ So the equation becomes $-1.u + u = \mathbf{0}$.

Since $u \in \mathcal{V}$, it has a negative vector $-u$ by $A.5$. Add this negative vector to both sides.

$(-1.u + u) + -u = \mathbf{0} + -u$

$(-1.u + u) + -u = -1.u + (u + -u)$ by $A.3$ and $\mathbf{0} + -u = -u$ by $A.4$

Hence we have $-1.u + (u + -u) = -u$.

Since $u + -u = \mathbf{0}$ by $A.5$ we have

$-1.u + \mathbf{0} = -u$

Since $-1.u + \mathbf{0} = -1.u$ by $A.4$ we finally have

$-1.u = -u$

(c) Using only the vector space axioms, show that if $u + u = 2.v$ then $u = v$
Solution: $u + u = 1.u + 1.u$ (by S.5)
$1.u + 1.u = (1+1).u = 2.u$ by(S.3)

Hence we have $2.u = 2.v$
Scalar multiply the both sides by 1/2.

$(1/2).(2.u) = (1/2).(2.v) \Rightarrow \left(\frac{1}{2}2\right).u = \left(\frac{1}{2}2\right).v$ (by S.4)
So we have $1.u = 1.v$.
But $1.u = u$ and $1.v = v$ by $S.5$. Hence $u = v$.

Problem 7. Let $\mathcal{V} = \mathcal{M}_{2,2}$ be the set of all the 2×2 matrices with usual operations. Determine whether the subsets given below are subspaces of $\mathcal{M}_{2,2}$.

(a) $\mathcal{U}$ = The set of symmetric matrices in $\mathcal{M}_{2,2} = \{A \in \mathcal{M}_{2,2} \mid A^T = A\}$
Solution:
(A.1) Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \in \mathcal{U}$.
Then $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{bmatrix}$ is a symmetric matrix and hence belongs to $\mathcal{U}$.

(S.1) $r.A = \begin{bmatrix} ra_{11} & ra_{12} \\ ra_{12} & ra_{22} \end{bmatrix}$ is a symmetric matrix and hence belongs to $\mathcal{U}$.

(A.4) The zero vector $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is symmetric matrix and hence belongs to $\mathcal{U}$.

So, $\mathcal{U}$ is a subspace of $\mathcal{M}_{2,2}$

(b) $\mathcal{U} = \{A \in \mathcal{M}_{2,2} \mid A^2 = A\}$
Solution: No. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{U}$ as $I^2 = I$.
But $2I = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \notin \mathcal{U}$ as $(2I)^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$.
Hence $\mathcal{U}$ does not satisfy $S.1$. So it is not a subspace of $\mathcal{M}_{2,2}$

(c) $\mathcal{U} =$ The set of non invertible matrices in $\mathcal{M}_{2,2}$
Solution: No. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{U}$ as they are non invertible.
But $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin \mathcal{U}$ as it is invertible. So $\mathcal{U}$ does not satisfy $A.1$. Hence it is not a subspace of $\mathcal{M}_{2,2}$

Problem 8. Let $\mathcal{V} = \mathcal{P}_3$ be the set of all the polynomials with a degree at most 3.
If p and q are in $\mathcal{P}_3$ their addition $p + q$ is defined as

$$(p + q)(x) = p(x) + q(x)$$

If s is a scalar, the scalar multiplication between s and a function p in $\mathcal{P}_3$; $s.p$ is defined as

$$(s.p)(x) = sp(x)$$

.
$\mathcal{P}_3$ with operations given as above is a vector space. Determine whether the subsets given below are subspaces of $\mathcal{P}_3$ or not.

(a) $\mathcal{U} = \{p \in \mathcal{P}_3 \mid p(2) = 0\}$
Solution: Let $p, q \in \mathcal{U}$. Then $p(2) = q(2) = 0$

Then $(p + q)(2) = p(2) + q(2) = 0 + 0 \Rightarrow p + q \in \mathcal{U}$

If r is a scalar, $(r.p)(2) = rp(2) = r \times 0 = 0 \Rightarrow r.p \in \mathcal{U}$

Finally, if $O(x) = 0 + 0x + 0x^2$ then $O(2) = 0 \Rightarrow O \in \mathcal{U}$

Hence $\mathcal{U}$ is a subspace of $\mathcal{P}_2$

(b) $\mathcal{U} = \{p \in \mathcal{P}_3 \mid p(x) = (1-x)g(x) \text{ for some } g \in \mathcal{P}_2\}$
Solution: First let's try to understand how should a polynomial look like, in order to be in $\mathcal{U}$. A polynomial is in $\mathcal{U}$ if it is the product between $(1-x)$ and another polynomial in $\mathcal{P}_2$.

With that in mind, take two polynomials p and q in $\mathcal{U}$.
Then $p(x) = (1-x)g(x)$ for some $g \in \mathcal{P}_2$ and $q(x) = (1-x)h(x)$ for some $h \in \mathcal{P}_2$.
Then $p(x) + q(x) = (1-x)(g(x) + h(x))$. We know $\mathcal{P}_2$ is a vector space. So, g and $h \in \mathcal{P}_2$ means $g + h \in \mathcal{P}_2$

So we have $p(x) + q(x) = (1-x)(g(x) + h(x))$ with $g + h \in \mathcal{P}_2$; This shows $p + q$ is the product between $(1-x)$ and a polynomial in $\mathcal{P}_2$. Hence $p + q \in \mathcal{U}$

If r is a scalar, $(r.p)(x) = r(1-x)g(x) = (1-x)(r.g(x))$. We know $\mathcal{P}_2$ is a vector space. So, if $g \in \mathcal{P}_2$ and r is a scalar, $rg \in \mathcal{P}_2$

So we have $(rp)(x) = (1-x)(r.g(x)$ with $r.g \in \mathcal{P}_2$; This shows $r.p$ is the product between $(1-x)$ and a polynomial in $\mathcal{P}_2$. Hence $r.p \in \mathcal{U}$

The zero vector in $\mathcal{P}_3$ is the constant polynomial which is zero everywhere ; $O(x) = 0$ for all x. We can write $O(x)$ as $O(x) = (1-x)(0 + 0x + 0x^2)$. Hence $O \in \mathcal{U}$.

So, $\mathcal{U}$ is a subspace of $\mathcal{P}_2$.

(c) $\mathcal{U} = \{p \in \mathcal{P}_3 \mid p \text{ has degree } 3\}$
Solution: This is not subspace. If $p(x) = x^3 + x^2$ and $q(x) = -x^3$ then p and q are in $\mathcal{U}$. But $p(x) + q(x) = x^2$. Hence $p + q \notin \mathcal{U}$. So, $\mathcal{U}$ does not satisfy $A.1$.

Problem 9. Compute a basis for each of the following subspaces $\mathcal{U}$ of the given vector spaces $\mathcal{V}$. In each case, write down the dimension of $\mathcal{U}$.

- $\mathcal{V} = \mathcal{P}_2$
  $\mathcal{U} = \{p \mid p \text{ has no constant term}\}$
  Solution: Any $p \in \mathcal{U}$ can be written as $p(x) = a_1x + a_2x^2$ because p cannot have a constant term. It is clear now $\mathcal{U} = span\{x, x^2\}$.

  You can check that the set $\{x, x^2\}$ is a linearly independent set.

  Hence $\{x, x^2\}$ is a basis of $\mathcal{U}$ and the dimension of $\mathcal{U}$ is 2.
- $\mathcal{V} = \mathcal{P}_4$
  $\mathcal{U} = \{p \mid p \text{ has no odd power terms}\}$
  Solution: Any $p \in \mathcal{U}$ can be written as $p(x) = a_0 + a_2x^2 + a_4x^4$ because p has no odd

power terms. It is clear now $\mathcal{U} = span\{1, x^2, x^4\}$.

You can check that the set $\{1, x^2, x^4\}$ is a linearly independent set.

Hence $\{1, x^2, x^4\}$ is a basis of $\mathcal{U}$ and the dimension of $\mathcal{U}$ is 3.

- $\mathcal{V} = \mathcal{P}_2$
  $\mathcal{U} = \{p \mid p(x) = p(-x) \text{ for all } x\}$
  Solution: Take any $p \in \mathcal{U}$. Then $p(x) = a_0 + a_1x + a_2x^2 = p(-x) = a_0 - a_1x + a_2x^2$.

  So we have $a_0 + a_1x + a_2x^2 = a_0 - a_1x + a_2x^2$

  By equating coefficients we get $a_0 = a_0, a_1 = -a_1, a_2 = a_2$.

  The first and last equations are trivial but the second one tells us $a_2 = 0$.
  Then any polynomial in $\mathcal{U}$ must be of the form $a_0 + a_2x^2$

  It is clear now $\mathcal{U} = span\{1, x^2\}$ and you can show that the set $\{1, x^2\}$ is linearly independent.

  Hence it is a basis for $\mathcal{U}$. The dimension of $\mathcal{U}$ is 2.

- $\mathcal{V} = \mathcal{M}_{3,3}$
  $\mathcal{U} = \{A \mid A \text{ is a symmetric matrix}\}$
  Solution: We can show that the set
  $$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$
  is a basis for $\mathcal{U}$. Hence the dimension of $\mathcal{U}$ is 6.

- $\mathcal{V} = \mathcal{M}_{3,3}$
  $\mathcal{U} = \{A \mid A \text{ is a skew-symmetric matrix}\}$
  Solution: We can show that the set
  $$\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}$$
  is a basis for $\mathcal{U}$. Hence the dimension of $\mathcal{U}$ is 3.

- $\mathcal{V} = \mathcal{M}_{3,3}$
  $\mathcal{U} = \{A \mid A \text{ is an upper triangular matrix}\}$
  Solution: Any 3×3 upper triangular matrix can be written as
  $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} =$$
  $$a_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{23} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} +$$

$a_{33}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$

Now it is clear that $\mathcal{U} = span\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$

You can check that the set

$$\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$$

is linearly independent. Hence it is a basis for $\mathcal{U}$ and the dimension of $\mathcal{U}$ is 6.

- $\mathcal{V} = \mathcal{M}_{2,2}$
  $\mathcal{U} = \{A \mid \text{ diagonal entries of A are all zero}\}$
  Solution: We can show that the set

  $\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}\right\}$ is a basis for $\mathcal{U}$ and the dimension of $\mathcal{U}$ is 2.
- $\mathcal{V} = \mathcal{M}_{3,2}$
  $\mathcal{U} = \{A \mid a_{32} = a_{21} = 0\}$
  Solution: We can show that the set

  $$\left\{\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\\1&0\end{bmatrix}\right\}$$

  is a basis for $\mathcal{U}$ and the dimension of $\mathcal{U}$ is 4.
- $\mathcal{V} = \mathcal{M}_{2,2}$
  $\mathcal{U} = \left\{A \mid A\begin{bmatrix}1&1\\-1&0\end{bmatrix} = \begin{bmatrix}1&1\\-1&0\end{bmatrix}A\right\}$

  ; or in other words, $\mathcal{U}$ is the set of 2×2 matrices commutes with $\begin{bmatrix}1&1\\-1&0\end{bmatrix}$

  Solution: Let $A = \begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix} \in \mathcal{U}$.

  $A\begin{bmatrix}1&1\\-1&0\end{bmatrix} = \begin{bmatrix}a_{11}-a_{12}&a_{11}\\a_{21}-a_{22}&a_{21}\end{bmatrix}$ and

  $\begin{bmatrix}1&1\\-1&0\end{bmatrix}A = \begin{bmatrix}a_{11}+a_{21}&a_{12}+a_{22}\\-a_{11}&-a_{12}\end{bmatrix}$

  Then we have $\begin{bmatrix}a_{11}-a_{12}&a_{11}\\a_{21}-a_{22}&a_{21}\end{bmatrix} = \begin{bmatrix}a_{11}+a_{21}&a_{12}+a_{22}\\-a_{11}&-a_{12}\end{bmatrix}$

By equating the corresponding entries, we have the following.

$a_{11} - a_{12} = a_{11} + a_{21} \Rightarrow -a_{12} = a_{21}$
$a_{11} = a_{12} + a_{22}$
$a_{21} - a_{22} = -a_{11}$
$a_{21} = -a_{12}$

Both the first and last equations yield $a_{21} = -a_{12}$ and both the second and third equations yield $a_{11} = a_{12} + a_{22}$.

Hence $A = \begin{bmatrix} a_{12} + a_{22} & a_{12} \\ -a_{12} & a_{22} \end{bmatrix} = a_{12} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

At this point, you can argue as in the previous parts to show that the set
$\left\{ \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
is a basis for $\mathcal{U}$ and the dimension of $\mathcal{U}$ is 2.