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MATH 3333
Exam 1

INSTRUCTIONS:

- This exam is closed-book, and notes are not allowed.
- You may use basic scientific calculators for the exam. However, calculators with graphing capabilities are not allowed.
- Show all of your work and use words to explain your reasoning. An answer lacking justification will earn you little or no credit. Points may be taken off for lack of explanation, even for correct final answers. Incorrect final answers with some correct work may earn some credit.
- Please make sure to organize your work and write neatly. It's important to write darkly to make your writing clear and easy to read. Avoid writing too small so that it can be easily understood.
- Phones and other electronic devices must be put away in your bag for the exam duration. Phones must be turned off or put in "do not disturb" mode". Having your phone or other electronics out during the exam is considered a violation of the University's Academic Integrity Code.
- It is against the University's Academic Integrity Code to look at another student's exam, and doing so carries a severe penalty.
- Keep your work out of view of other students.

$\boxed{1.}$ [**8 points**] An augmented matrix of a linear system is given below.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ -1 & a & -3 & -4 \\ 2 & 4 & a & 9 \end{array}\right]$$

Find all values of a for which the system has

(a) a unique solution.
(b) infinitely many solutions.
(c) no solutions.

We apply the row operations $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - 2R_1$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & a+2 & 0 & 0 \\ 0 & 0 & a-6 & 1 \end{array}\right]$$

Now we try to create a leading 1 in R_2 and that depends on whether $a+2=0$ or not.
Case 1: $a=-2$
In this case we have

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 1 \end{array}\right]$$

Now it is easy to see the rank of the augmented matrix is 2 which is less than the number of variables. $(r = 2 < n = 3)$. This means the system has infinitely many solutions when $a=-2$.

Case 2 : $a \neq -2$
In this case we can do $R_2 \to \frac{1}{a+2} R_2$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a-6 & 1 \end{array}\right]$$

Now we try to create a leading 1 in R_3 and that depends on whether $a-6=0$ or not.

Case 2.1 : $a=6$
Then we have

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

which means the system has no solutions.

Case 2.2 : $a \neq -2, 6$
In this case we can do $R_3 \to \frac{1}{a-6} R_3$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{a-6} \end{array}\right]$$

Now we see $r = 3 = n$. Hence the system has a unique solution when $a \neq -2, 6$.

$\boxed{\textbf{2.}}$ **[8 points]** Consider the linear system given below.

$$x_2 - 6x_3 + 5x_4 = -7$$
$$x_1 + 2x_3 - x_4 = 5$$
$$3x_1 + x_2 + 2x_4 = 8$$

(a) Write down the augmented matrix of the system and compute its reduced row-echelon form. At each step, clearly state the elementary row operations you are performing.

(b) Determine if the system is consistent or not. If consistent, write down its solution.

The augmented matrix is

$$\left[\begin{array}{cccc|c} 0 & 1 & -6 & 5 & -7 \\ 1 & 0 & 2 & -1 & 5 \\ 3 & 1 & 0 & 2 & 8 \end{array}\right]$$

First we perform the row operation
$R_1 \leftrightarrow R_2$ to obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 5 \\ 0 & 1 & -6 & 5 & -7 \\ 3 & 1 & 0 & 2 & 8 \end{array}\right]$$

Then $R_3 \to R_3 - 3R_1$ yields

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 5 \\ 0 & 1 & -6 & 5 & -7 \\ 0 & 1 & -6 & 5 & -7 \end{array}\right]$$

Finally $R_3 \to R_3 + R_2$ yields the RRE form

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 5 \\ 0 & 1 & -6 & 5 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Since there are no leading 1s in the last column, the system is consistent.
The leading variables are x_1 and x_2.
Free variables are x_3 and x_4. By taking $x_3 = s$ and $x_4 = t$ we can write the general solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s + t + 5 \\ 6s - 5t - 7 \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -5 \\ 0 \\ 1 \end{bmatrix} + u \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{\textbf{3.}}$ **[7 points]** Suppose $A = \begin{bmatrix} 6 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $Y = \begin{bmatrix} 4 & 1 \end{bmatrix}$

Compute $AB^T + XY$

$$AB^T = \begin{bmatrix} 7 & 13 \\ 1 & 4 \end{bmatrix}$$

$$XY = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}.$$

So $AB^T + XY = \begin{bmatrix} 11 & 14 \\ 9 & 6 \end{bmatrix}$

$\boxed{4.}$ **[8 points]** Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 4 & 6 \\ 0 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 20 \\ 8 \end{bmatrix}$.

B can be expressed as a linear combination of the columns of A in many different ways. Two of those ways are shown below.

- $B = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$

- $B = 4 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$

(a) Is B in the column space of A ? If so, why ?

Yes, because B can be written as a linear combination of the columns of A as shown above.

(b) Write down two different column vectors U and V in $\mathbb{R}^3$ such that $T_A(U) = B$ and $T_A(V) = B$.

$$U = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad V = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$$

(c) Using only the information given in the problem, find two different solutions to the linear system given by

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 4 & 6 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \\ 8 \end{bmatrix}$$

No points will be given, if you solve this system by computing the RRE form of the augmented matrix.

$$U = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad V = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$$

(d) Does A have an inverse ? Explain your answer briefly using part (c).

No. If A has an inverse, then the linear system $AX = B$ has a unique solution which is $X = A^{-1}B$. But as we saw in part (c), this system has more than one solution.

$\boxed{5.}$ **[8 points]** Consider the 4×4 matrix given below.

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(a) Compute the inverse of A using **elementary row operations.**

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \to R_1 - 2R_4 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \to R_2/3 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3/4 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{4} & 0 \end{bmatrix}$$

$$\text{Hence } A^{-1} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{4} & 0 \end{bmatrix}$$

(b) Solve the linear system given below **using the inverse you computed.**

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

No points will be given, if you solve the system using a different method.

The system can be written as $AX = B$ and multiplying both sides from left by A^{-1}, we obtain $X = A^{-1}B$.

Hence, the solution to this linear system is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1/4 \end{bmatrix}$$

$\boxed{\textbf{6.}}$ [**4 points**] If A is a square matrix, show that $A - A^T$ is a skew-symmetric matrix.
Take $B = A - A^T$.

Then $B^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -B$.

Hence $B = A - A^T$ is skew-symmetric.

$\boxed{7.}$ **[8 points]** E_1 and E_2 are two 4×4 elementary matrices as described below.

$E_1 =$ The elementary matrix obtained by performing $R_2 \to R_2 + 2R_1$ on I
$E_2 =$ The elementary matrix obtained by performing $R_3 \leftrightarrow R_4$ on I

where I is the 4×4 identity matrix.

(a) Write down the inverses of E_1 and E_2.

(b) Let A be a 4×4 matrix. If $E_2 E_1 A = I$, find A.

If E_1 is as above then E_1^{-1} is obtained by performing the inverse row operation of $R_2 \to R_2 + 2R_1$ which is $R_2 \to R_2 - 2R_1$ on I. Hence we have $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

If E_2 is as above then E_2^{-1} is obtained by performing the inverse row operation of $R_3 \leftrightarrow R_4$ which is the same operation on I. Hence we have $E_2^{-1} = E_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

If $E_2 E_1 A = I$ then $E_1 A = E_2^{-1} I$

$$\Rightarrow A = E_1^{-1} E_2^{-1} I = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$