NAME ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓

MATH 3333
Quiz 1
Form A

INSTRUCTIONS: Please show your work. This quiz is closed-book and notes are not allowed.

$\boxed{\textbf{1.}}$ **[10 points]** Consider the homogeneous linear system given below.

$$x_1 + 2x_2 - x_3 + 2x_4 + x_5 = 0$$
$$x_1 + 2x_2 + 2x_3 - x_4 + x_5 = 0$$

(a) Write down the augmented matrix of the system and compute its reduced row-echelon form. Clearly state the elementary row operations you are performing.

(b) Find a set of basic solutions for the system.

The augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 1 & 0 \\ 1 & 2 & 2 & -1 & 1 & 0 \end{array}\right]$$

First we perform the row operation
$R_2 \to R_2 - R_1$ to obtain

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 1 & 0 \\ 0 & 0 & 3 & -3 & 0 & 0 \end{array}\right]$$

Then $R_2 \to \frac{-1}{3} R_2$ yields

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

Finally $R_1 \to R_1 + R_2$ yields the RRE form

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

The leading variables are x_1 and x_3.
Free variables are x_2, x_4 and x_5. By taking $x_2 = s, x_4 = t$ and $x_5 = u$ we can write the general solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - t - u \\ s \\ t \\ t \\ u \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the basic solutions are $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{2.}$ **[10 points]** An augmented matrix of a linear system is given below.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & -2 & 3 & b \\ 3 & -1 & 2 & 3 \end{array}\right]$$

Find all values of b for which the system has

(a) a unique solution.
(b) infinitely many solutions.
(c) no solutions.

By performing the elementary row operations $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ we can obtain the following

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -4 & 5 & b-4 \\ 0 & -4 & 5 & -3 \end{array}\right]$$

The Gaussian algorithm tells us the next step is to divide the second row by -4. But we see R_2 and R_3 have same entries in several places. So we can pause dividing the second row by -4 and instead perform the following row operation ; $R_3 \to R_3 - R_2$. This yields the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -4 & 5 & b-4 \\ 0 & 0 & 0 & 1-b \end{array}\right]$$

Finally, $R_2 \to \frac{-1}{4}R_2$ yields

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -5/4 & \frac{b-4}{-4} \\ 0 & 0 & 0 & 1-b \end{array}\right]$$

We have leading $1s$ in R_1 and R_2. Whether we can have a leading 1 in R_3 depends on whether $1-b$ is zero or not.
The two cases are

(1) $b = 1$
(2) $b \neq 1$

If $b = 1$ then we have the row-echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -5/4 & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This has no leading 1s in the last column and $r = 2 < n = 3$. So the system has infinitely many solutions when $b = 1$.

If $b \neq 1$ then $R_3 \to \frac{1}{1-b}R_3$ yields the row-echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -5/4 & \frac{b-4}{-4} \\ 0 & 0 & 0 & 1 \end{array}\right]$$

So the system has no solutions if $b \neq 1$

For no value of b, the system can have a unique solution.