MATH 3333 PRACTICE PROBLEM SET 6

Problem 1. For each of the linear transformations T given below, compute a basis for $Null(T)$. Using the basis you computed, determine if T is onto and/or one-to-one.

- $T : \mathcal{M}_{2,2} \to \mathbb{R}$
  $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + c - b - d$

  Solution: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Null(T)$ then $a + c - b - d = 0 \Rightarrow a = b + d - c$

  $\Rightarrow A = \begin{bmatrix} b+d-c & b \\ c & d \end{bmatrix} = b\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

  Then $Null(T) = span\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

  To check this is a linearly independent set, we write
  $r_1\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + r_2\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + r_3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

  It is clear then $r_1 = r_2 = r_3 = 0$ and hence the set is linearly independent, therefore forms a basis for $Null(T)$

  Hence $dim(Null(T)) = 3$. So T is not injective (because dimension of the kernal is not 0.)

  Using rank-nullity theorem,
  $3 + dim(Range(T)) = 4 \Rightarrow dim(Im(T)) = 1$. Hence T is surjective (because the dimension of the range $=1 =$ dimension of the co-domain $\mathbb{R}$.)

- $T : \mathcal{M}_{2,2} \to \mathcal{M}_{2,2}$
  $T(A) = A^T$

  Solution: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Null(T)$

  then $T(A) = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a = b = c = d = 0$

  $\Rightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

  So $Null(T)$ contains only the zero matrix. Hence $dim(Null(T)) = 0$ and T is injective

Using rank-nullity theorem,
$0 + dim(Range(T)) = 4 \Rightarrow dim(Range(T)) = 4$. Hence T is surjective. (because the dimension of the range = 4= dimension of the co-domain $\mathcal{M}_{2,2}$.)

- $T : \mathcal{M}_{2,2} \to \mathcal{M}_{2,2}$
  $T(A) = A + A^T$
  Solution: If $A \in ker(T)$ then $T(A) = A + A^T = 0 \Rightarrow A = -A^T$.
  Hence the kernel of T is the set of 2×2 skew-symmetric matrices.

  You can show that the set $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ is a basis for the set of 2×2 skew-symmetric matrices. Hence the dim(Null(T)) is 1 and T is not injective.

  Using rank-nullity theorem,
  $1 + dim(Range(T)) = 4 \Rightarrow dim(Range(T)) = 3$. Hence T is not surjective. (because the dimension of the image is $3 \neq 4$ = dimension of the co-domain $\mathcal{M}_{2,2}$.)

- $T : \mathcal{P}_2 \to \mathbb{R}$
  $T\left(a_0 + a_1 x + a_2 x^2\right) = a_0 + a_1 + a_2$

  Solution: We did a similar example in the class today. The answers are $dim(Null(T)) = 2$ and $dim(Range(T)) = 1$. So T is surjective but not injective.

- $T : \mathcal{M}_{3,3} \to \mathbb{R}$
  $T(A) = trace(A)$
  Solution: If $A \in ker(T)$ then trace(A) $= a_{11} + a_{22} + a_{33} = 0 \Rightarrow a_{11} = -a_{22} - a_{33}$.
  Hence $A = \begin{bmatrix} -a_{22} - a_{33} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
  Arguing as in the previous problems, you can show the following set is a basis for $Null(T)$

  $$\left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right.$$
  The you can compute the dimensions of $Null(T)$ to be 8 and the range to be 1 and conclude T is surjective but not injective.

- $T : \mathbb{R}^3 \to \mathcal{P}_2$
  $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a + (a+b)x + (b-c)x^2$

Solution: If $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in Null(T)$ then $a = 0, a+b = 0$ and $b-c = 0$ which means $a = b = c = 0$.

Hence $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

So $Null(T)$ only has the zero vector. You can conclude that T is injective and compute the dimension of the range to be 3 and hence T is surjective.

- $T : \mathbb{R}^5 \to \mathcal{M}_{2,2}$

  $$T\left(\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}\right) = \begin{bmatrix} a & b-c \\ d & c+e \end{bmatrix}$$

  Solution: If $\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \in Null(T)$ then $a = 0, b - c = 0, d = 0$ and $c + e = 0$ which means

  $a = d = 0$ and $b = c = -e$. Hence $\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ b \\ 0 \\ -b \end{bmatrix} = b \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

  So the set $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a spanning set for $Null(T)$ and you can easily show it is linearly independent (it only has 1 vector!) and hence it is basis for $Null(T)$.

  Then T is not injective as $dim(Null(T)) \neq 0$. By rank-nullity theorem, the dimension of the range can be computed to be 4 and hence T is surjective.