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MATH 3333
Quiz 1
Form A

INSTRUCTIONS: Please show your work. This quiz is closed-book and notes are not allowed.

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1. **[10 points]** Consider the matrix A given below.

$$A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}$$

The characteristic equation of this matrix can be simplified into $|A - \lambda I| = (\lambda - 5)(\lambda - 3)(\lambda - 2) = 0$

(a) Write down the eigenvalues of A.
$\lambda = 5, 3, 2$

(b) If $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ then find an invertible matrix P such that $A = PDP^{-1}$

When $\lambda = 5$

$A - 5I = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 0 & 0 \\ 5 & 0 & -7 \end{bmatrix}$

By applying the row operations $R_1 \to 1/2R_1$ followed by $R_3 \to R_3 - 5R_1$ we obtain

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Then we can obtain the RRE form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The general solution to this system can be written as

$$X = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

When $\lambda = 3$
$A - 3I = \begin{bmatrix} 4 & 0 & -4 \\ 0 & 2 & 0 \\ 5 & 0 & -5 \end{bmatrix}$
The RRE form of this matrix is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
The general solution to this system can be written as

$$X = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

When $\lambda = 2$

$A - 2I = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 3 & 0 \\ 5 & 0 & -4 \end{bmatrix}$
The RRE form of this matrix is $\begin{bmatrix} 1 & 0 & -4/5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
The general solution to this system can be written as

$$X = s \begin{bmatrix} 4/5 \\ 0 \\ 1 \end{bmatrix}$$

For D as given, we can take $P = \begin{bmatrix} 4/5 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ so that $A = PDP^{-1}$.

$\boxed{\textbf{2.}}$ **[5 points]** Let u, v and w be three vectors in $\mathbb{R}^n$.
Show that $span\{u, v, w\} = span\{u, v + 2w, w\}$
Take a vector $p \in span\{u, v, w\}$.

Then $p = r_1 u + r_2 v + r_3 w$ for some scalars r_1, r_2 and r_3.

$p = r_1 u + r_2 v + r_3 w = r_1 u + r_2(v + 2w) - 2r_2 w + r_3 w$
$= r_1 u + r_2(v + 2w) + (r_3 - 2r_2)v \in span\{u, v + 2w, w\}$

This shows $span\{u, v, w\} \subset span\{u, v + 2w, w\}$.

Now take a vector $q \in span\{u, v + 2w, w\}$.

Then $q = s_1 u + s_2(v + 2w) + s_3 w$ for some scalars s_1, s_2 and s_3.

$q = s_1 u + s_2(v + 2w) + s_3 w = s_1 u + s_2 v + (2s_2 + s_3)w \in span\{u, v, w\}$

This shows $span\{u, v + 2w, w\} \subset span\{u, v, w\}$.

Hence $span\{u, v + 2w, w\} = span\{u, v, w\}$

$\boxed{\textbf{3.}}$ **[5 points]** Consider the subset of $\mathbb{R}^3$ given below.

$$\mathcal{S} = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \middle| a^2 = b^2 \right\}$$

Determine whether $\mathcal{S}$ is a subspace of $\mathbb{R}^3$ or not.

It's obvious the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{S}$.

Now take two vectors $u, v \in \mathcal{S}$.

Say $u = \begin{bmatrix} a_1 \\ 0 \\ b_1 \end{bmatrix}$ and $v = \begin{bmatrix} a_2 \\ 0 \\ b_2 \end{bmatrix}$. Then $a_1^2 = b_1^2$ and $a_2^2 = b_2^2$

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$$u + v = \begin{bmatrix} a_1 + b_1 \\ 0 \\ a_2 + b_2 \end{bmatrix}.$$

Now the question you have to answer is whether $u + v \in \mathcal{S}$ or not. In other words, you need to check if $(a_1 + b_1)^2 = (a_2 + b_2)^2$ or not.

We know $(a_1 + b_1)^2 = a_1^2 + 2a_1b_1 + b_1^2$ and $(a_2 + b_2)^2 = a_2^2 + 2a_2b_2 + b_2^2$.
If these two things are equal, then we must have $a_1^2 + 2a_1b_1 + b_1^2 = a_2^2 + 2a_2b_2 + b_2^2 \Rightarrow 2a_1b_1 = 2a_2b_2$ (because $a_1^2 = b_1^2$ and $a_2^2 = b_2^2$.)

But it is not necessary for two vectors u and v as above to satisfy the condition $2a_1b_1 = 2a_2b_2$.
So it looks like $\mathcal{S}$ does not satisfy this condition.

To see this, we can take $u = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$

Then $u + v = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \notin \mathcal{S}$ because $3^2 \neq (-1)^2$

Hence $\mathcal{S}$ is not a subspace of $\mathbb{R}^3$.