Colby Frison Math-3333 || Homework 1 7/9/2025

Problem 1. Solve the following linear system by using elementary row operations. If the system is consistent, write down the general solution.

$$
\begin{aligned}
2x_1 + 2x_2 + x_3 - x_4 &= -1 \\
x_1 + x_2 + x_3 + x_4 &= 2 \\
3x_1 + 3x_2 + 2x_3 &= 1 \\
x_1 + x_2 - 2x_4 &= -3
\end{aligned}
$$

$$
\left[\begin{array}{cccc:c} 2 & 2 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2 \\ 3 & 3 & 2 & 0 & 1 \\ 1 & 1 & 0 & -2 & -3 \end{array}\right]
\xrightarrow{R_1 \to R_1 + (-1)R_4}
\left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 \\ 3 & 3 & 2 & 0 & 1 \\ 1 & 1 & 0 & -2 & -3 \end{array}\right]
\xrightarrow{R_2 \to R_2 + (-1)R_4}
\left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 & 5 \\ 3 & 3 & 2 & 0 & 1 \\ 1 & 1 & 0 & -2 & -3 \end{array}\right]
\xrightarrow{R_3 \to R_3 + (-3)R_4}
\left[\begin{array}{cccc:c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 6 & 10 \\ 1 & 1 & 0 & -2 & -3 \end{array}\right]
\xrightarrow{R_1 \to R_1 + (-1)R_4}
\left[\begin{array}{cccc:c} 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 6 & 10 \\ 1 & 1 & 0 & -2 & -3 \end{array}\right]
$$

$$
\left[\begin{array}{cccc:c} 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 6 & 10 \\ 1 & 1 & 0 & -2 & -3 \end{array}\right]
\xrightarrow{R_3 \to R_3 + (-2)R_1}
\left[\begin{array}{cccc:c} 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -2 & -3 \end{array}\right]
\xrightarrow{R_1 \to R_1 + (-1)R_2}
\left[\begin{array}{cccc:c} 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -2 & -3 \end{array}\right]
\xrightarrow{R_1 \to \frac{1}{3}R_1}
\left[\begin{array}{cccc:c} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -2 & -3 \end{array}\right]
\xrightarrow[R_3 \leftrightarrow R_4]{R_1 \leftrightarrow R_4}
\begin{array}{c}
\begin{array}{ccccc} x_1 & & x_3 & & \end{array} \\
\left[\begin{array}{cccc:c} \underline{1} & 1 & 0 & -2 & -3 \\ 0 & 0 & \underline{1} & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\
\begin{array}{ccccc} & x_2 & & x_4 & \end{array}
\end{array}
$$

so,

$\underline{x_1} + x_2 - 2x_4 = -3 \Rightarrow x_1 = -x_2 + 2x_4 - 3$

$\underline{x_3} + 3x_4 = 5 \Rightarrow x_3 = -3x_4 + 5$

$x_2 = s$, $x_4 = t$ where $s, t \in \mathbb{R}$

$x_1 = -s + 2t - 3$
$x_2 = s$
$x_3 = -3t + 5$
$x_4 = t$

$$
\xrightarrow{\text{vector form}}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 5 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix}
$$

Problem 2. Find a set of basic solutions for each of the following homogeneous linear systems. Write the solutions as linear combinations of those basic solutions.

(a)
$$
\begin{aligned}
x_1 + 2x_2 - x_3 + x_4 + x_5 &= 0 \\
-x_1 - 2x_2 + 2x_3 + x_5 &= 0 \\
-x_1 - 2x_2 + 3x_3 + x_4 + 3x_5 &= 0
\end{aligned}
$$

$$
\left[\begin{array}{ccccc:c} 1 & 2 & -1 & 1 & 1 & 0 \\ -1 & -2 & 2 & 0 & 1 & 0 \\ -1 & -2 & 3 & 1 & 3 & 0 \end{array}\right]
\xrightarrow{R_2 \to R_2 + R_1}
\left[\begin{array}{ccccc:c} 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ -1 & -2 & 3 & 1 & 3 & 0 \end{array}\right]
\xrightarrow{R_3 \to R_3 + R_1}
\left[\begin{array}{ccccc:c} 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 4 & 0 \end{array}\right]
\xrightarrow{R_3 \to R_3 + (-2)R_2}
\left[\begin{array}{ccccc:c} 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

$$
\left[\begin{array}{ccccc:c} 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
\xrightarrow{R_1 \to R_1 + R_2}
\left[\begin{array}{ccccc:c} \underline{1} & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & \underline{1} & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

$x_1 + 2x_2 + 2x_4 + 3x_5 = 0 \Rightarrow x_1 = -2x_2 - 2x_4 - 3x_5$

$x_3 + x_4 + 2x_5 = 0 \Rightarrow x_3 = -x_4 - 2x_5$

$x_2 = s$, $x_4 = t$, $x_5 = u$ $\Rightarrow$

$x_1 = -2s - 2t - 3u$
$x_2 = s$
$x_3 = -t - 2u$
$x_4 = t$
$x_5 = u$

$$
\longrightarrow
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + u \cdot \begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}
$$