1. Systems of linear equations

1.1 Solutions & elementary operations

A linear equation takes the form $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

$x_1, x_2, \dots, x_n$ = variables

$a_1, a_2, \dots, a_n$ and b are real numbers
$\underbrace{a_1, a_2, \dots, a_n}_{\text{coefficient}}$, $\underbrace{b}_{\text{constant term}}$

A linear system is a collection of linear equations

↳ Eg. $2x_1 + x_2 + x_3 = 5$ $x_1 + x_2 + x_3 = 4$
$x_1 - x_2 + 2x_3 = 1$

↳ $x_1 = 1,\ x_2 = 2,\ x_3 = 3$ is a

A solution to a linear system is a sequence of numbers $x_1 = s_1,\ x_2 = s_2,\ x_3 = s_3, \dots, x_n = s_n$ that satisfies every equation in the system

A system may have:
1. A unique solution
2. Infinetely many solutions
3. No solutions

⇒ $x_1 + 3x_2 + 4x_4 = 2$ ①
$3x_2 - x_3 = 5$ ②

show that for any values of s & t
$x_1 = 2 - 3s - 4t$
$x_2 = s$
$x_3 = 3s - 5$
$x_4 = t$

a general solution
s & t are parameters

Plug into ①
$\overset{x_1}{(2-3s-4t)} + \overset{3x_2}{3(s)} + \overset{4x_4}{4t} = 2$
cancels ↳ $2 = 2$ ✓

same with ② too lazy

By plugging in different values for $s+t$, we can get different solutions.

Eg. $s = 0$ $t = 1$ $x_1 = 2$, $x_2 = 0$, $x_3 = -5$, $x_4 = 1$

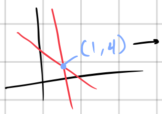
⇒ The general solution can be written in different ways:

Eg. $x_1 = s$ $x_2 = \dfrac{5+t}{5}$ $x_3 = t$ $x_4 = \dfrac{-3-5-t}{4}$

⇒ by plugging $s = \overset{2}{??}$ & $t = \overset{-5}{??}$ to get
$x_1 = 2$ $x_2 = 0$ $x_3 = 5$ $x_4 = 1$

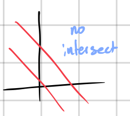
We can visualize these 3 different cases (posibilities) when there are only 2 variables

Eg 1
$x + y = 5$
$2x + y = 6$



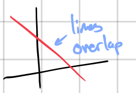
unique solution
$x = 1$, $y = 4$

Eg 2
$x + y = 5$
$2x + 2y = 6$



no solution

Eg 3
$x + y = 5$
$3y + 3y = 15$



infinetley many solutions

$\boxed{\text{A linear system}} \xrightarrow[\text{of operations}]{\text{Perform a sequence}} \boxed{\text{a new easier system to solve}}$

Solve the linear system:
$2x + 5y = 9$ ①
$x + 3y = 5$ ②

subtract 2 times second equation eq^n from eq^n one
gets rid of x

$-y = -1$ ⇒ $y = 1$
$x + 3y = 5$ $x + 3y = 5$

subtract 3 times eq^n one from eq^n two

$y = 1$
$x = 2$

Elementary operations on a system

① Interchanging eq^n
② Multiplying an eq^n by a non-zero number
③ Add a multiple of an eq^n to another

A matrix is a rectangular array of numbers

$$\begin{bmatrix} 2 & 1 & 7 & -7 \\ 1 & 6 & 5 & 0 \end{bmatrix}$$

we can use matricies to record linear systems

row 1 becomes row 1 subtracted by row 2 times 2

$$\Rightarrow \underbrace{\left[\begin{array}{cc|c} 2 & 5 & 9 \\ 1 & 3 & 5 \end{array}\right]}_{\text{Augmented matrix of the system}} \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cc|c} 0 & -1 & -1 \\ 1 & 3 & 5 \end{array}\right] \xrightarrow{R_1 \to -R_1} \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 3 & 5 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 2 \end{array}\right] \xrightarrow[R_2 \to R_1]{R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

Elementary operations on an augmented matrix

① Interchanging two rows
② Multiplying a row by a number
③ Add a multiple of a row to another row

Goal is to perform the elementary operations on the augmented matrix & obtain a matrix of the form: $\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$

Sol^n

Eg. $x + y + 2z = -1$
$2x + y + 3z = 0$
$-2y + z = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 2 & 1 & 3 & 0 \\ 0 & -2 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 + (-2)R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 2 \\ 0 & -2 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \to (-1)R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 + (-1)R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -2 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + (2)R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & -3 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Don't do subtraction instead add with coefficient -1

$x = 2$
$y = -1$
$z = -1$

Can we always get this "nice" form by performing elementary row operations?? NO → If we can't we try to get something close enough
↳ Row-echelon form ↳ reduced row echelon form

A matrix is row-echelon form if:
① The first non zero entry from left in each row is 1 ("leading 1's")
② Each "leading 1" is to the right of all leading 1s above it
③ If there are rows with all zeros, they are at the bottom

⇒ Example
$$\begin{bmatrix} 0 & 1 & 7 & 5 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

non example
violates rule 2
$$\begin{bmatrix} 1 & 0 & 7 & 2 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
can use interchange to make row echelon form

Reduced Row-echelon Form (RRE form)

↳ in addition to the 3 rules of the row-echelon form, if a matrix satisfies:

④ Each leading 1 is the only non-zero entry in the column that contains it

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{array}{l} R_2 \to R_2 + (-2/3)R_3 \\ R_1 \to R_1 + (-2)R_2 \\ R_1 \to R_1 + (-2)R_3 \end{array} \to \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solutions & RRE form

① If the RRE form of the augmented matrix has a row of the form: $\left[\begin{array}{cccc|c} 0 & 0 & 0 \dots & 0 & 1 \end{array}\right]$ ← inconsistent
↳ Then the system has no solutions $0x_1 + 0x_2 + \dots 0x_n \neq 1$

② If the RRE form has no such row (no leading 1s in the last column) then the system has sol^n(s) ← consistant → unique or infinitly many

From example to right
to solve for free variables we assign parameters
$x_3 = s$ → then solve in terms of free variables

$x_1 - \frac{9}{5}s = \frac{14}{5}$ → $x_1 = \frac{9}{5}s + \frac{14}{5}$
$x_2 + \frac{4}{5}s = \frac{1}{5}$ → $x_2 = -\frac{4}{5}s + \frac{1}{5}$
$x_3 = s$

1.2 Gaussian algorithm (a guide to compute the row-echelon form)

Eg. Compute the row-echelon form of

$$\begin{bmatrix} 0 & 0 & 3 & 2 \\ 0 & 2 & 4 & 4 \\ 0 & 7 & 5 & 9 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 2 & 4 & 4 \\ 0 & 0 & 3 & 2 \\ 0 & 7 & 5 & 9 \end{bmatrix}$$

Step 1: Starting from left, find the column with a non-zero entry (say 'a')
Move the row with this entry to the top

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 7 & 5 & 9 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -9 & -5 \end{bmatrix}$$

Step 2: Multiply the new top row by ½ (to make leading 1)

Step 3: Make each entry below the leading 1 zero by adding multiples of row with leading 1s to other rows

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & -9 & -5 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}$$

Step 4: Repeat above steps for the matrix consisting of remaining rows

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example/take home problem:

Check if the system has solutions or not by computing the RRE form of the augmented matrix

$x_1 + x_2 - x_3 = 3$
$-x_1 + 4x_2 + 5x_3 = -2$
$x_1 + 6x_2 + 3x_3 = 4$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 + (-1)R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 + (-1)R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{5}R_2} \underbrace{\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 1 & 4/5 & 1/5 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{row-echelon form}} \xrightarrow{R_1 \to R_1 + (-1)R_2} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -9/5 & 14/5 \\ 0 & 1 & 4/5 & 1/5 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array}$$

to make RRE make entries above leading 1s zero

$x_1 - \frac{9}{5}x_3 = \frac{14}{5}$
$x_2 + \frac{4}{5}x_3 = \frac{1}{5}$

leading variables = variables corresponding to leading 1s
other variables = free variables

If there is a leading 1 in the last column $\leftrightarrow$ No solution

↳ RRE has form: $[0\ 0\ 0\ \dots\ 0 \mid 1]$

If not:

- No parameters $\leftrightarrow$ Unique solution
- There are free variables $\leftrightarrow$ Infinitely many solutions

Eg. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ no solutions — leading 1

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \hline \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ Unique solution — no leading 1

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right]$$ unique solution — NOT a leading 1

$$\left[\begin{array}{cccc|c} 1 & 7 & 0 & 6 & 5 \\ 0 & 0 & 1 & 4 & 2 \end{array}\right]$$ Infinitely many solutions — Free variables

Eg. Augmented matrix of a system has the RRE form

$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & 3 & 0 & 0 & -1 \\ 0 & \boxed{1} & -1 & 0 & 1 & 1 \end{array}\right]$$

leading variables: x_1, x_2

free variables: x_3, x_4, x_5

$x_3 = s$, $x_4 = t$, $x_5 = u$

$x_1 + 3x_3 = -1 \Rightarrow x_1 = -3s - 1$

$x_2 - x_3 + x_5 = 1 \Rightarrow x_2 = s - u + 1$

x_4 doesn't appear here, but it is a free variable so it must be included

$x_1 = -3s - 1$
$x_2 = s - u + 1$
$x_3 = s$
$x_4 = t$
$x_5 = u$

Eg. $\left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 2 & 6 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{4}R_3]{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + (-1)R_2} \left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to (-1)R_3} \left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 4 & 8 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow$ A given matrix may have more than one row-echelon form

↳ But any two different row-echelon forms of the matrix has the same number of leading 1s

→ RRE form: $\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$ $\Rightarrow$ A given matrix has only one RRE form

It has the same number of leading 1s as any row echelon form.

Rank of a matrix = Number of leading 1s in its RRE form (or in any of its row-echelon forms)

For a system with n variables & the augmented matrix has rank r number of free variables $= n - r$

If $r = n$ unique solution, if $r < n$ infinitely many solutions

Find the values of 'h' such that the system
$$2x + 4y = 2$$
$$hx + 2y = -3$$
i) is inconsistent (no sol^n)
ii) has infinitely many solutions
iii) has a unique sol^n

iii) $\left[\begin{array}{cc|c} 2 & 4 & 2 \\ h & 2 & -3 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ h & 2 & -3 \end{array}\right] \xrightarrow{R_2 \to R_2 + (-h)R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 2-2h & -3-h \end{array}\right] \xrightarrow[\text{assuming } 2-2h \neq 0]{R_2 \to \frac{1}{2-2h}R_2} \left[\begin{array}{cc|c} \boxed{1} & 2 & 1 \\ 0 & \boxed{1} & \frac{-3-h}{2-2h} \end{array}\right]$ $r = 2$, $n = 2$

no leading 1s in the last column

(iii) unique sol^n as long as $2-2h \neq 0$ (if $h \neq 1$) $r = n$

plugging in $h=1$ works here as well, this was just the way it was done in class

$h = 1$:

$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -4 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{4}R_2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & \boxed{1} \end{array}\right]$ (i) no sol^n because leading 1 in last column

now with this system
$$-6x - 8y = h$$
$$-9x + 12y = -1$$

$\left[\begin{array}{cc|c} 6 & -8 & h \\ -9 & 12 & -1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{6}R_1} \left[\begin{array}{cc|c} 1 & -4/3 & h/6 \\ -9 & 12 & -1 \end{array}\right] \xrightarrow{R_2 \to R_2 + 9R_1} \left[\begin{array}{cc|c} 1 & -4/3 & h/6 \\ 0 & 0 & 9h/6 - 1 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{9h/6 - 1}R_2} \left[\begin{array}{cc|c} 1 & -4/3 & h/6 \\ 0 & 0 & \boxed{1} \end{array}\right]$ → no sol^n (i) assuming $9h/6 - 1 \neq 0$ ($h \neq 2/3$)

if $h = 2/3$: $\left[\begin{array}{cc|c} 1 & -4/3 & 2/18 \\ 0 & 0 & 0 \end{array}\right]$ $r = 1$, $n = 2$ $\Rightarrow$ infinitely many solutions

so there is no value for h to get a unique solution (iii)

1.3 Homogeneous Systems

a linear eq^n is homogeneous if the constant term (right side) is zero

↳ $a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$

If all the eq^{n}s in a system are homogeneous then the system is called homogeneous

→ The augmented matrix of a such system

$$\left[\begin{array}{cccc|c} * & * & * & \dots & 0 \\ & & \vdots & & \vdots \\ * & * & * & \dots & 0 \end{array}\right] \Rightarrow \text{No leading 1s in the last column}$$

For a homogeneous system with n variables and the augmented matrix has rank r

- $r = n \Leftrightarrow$ Unique sol^n (in this case unique sol^n)
- $r < n \Leftrightarrow$ Infinitely many sol^n

Also $x_1 = 0, x_2 = 0, \dots, x_n = 0$ is a sol^n for any homogeneous system (trivial solution)

A detour on column vectors

- a matrix with a single column

Eg. $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 7 \end{bmatrix}$

- If two column vectors have the same number of entries we can add them

$u = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$ $v = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$ $\Rightarrow u + v = \begin{bmatrix} p_1 + q_1 \\ p_2 + q_2 \\ \vdots \\ p_n + q_n \end{bmatrix}$

if s is a scalar (a real number)

$s \cdot u = \begin{bmatrix} sp_1 \\ sp_2 \\ \vdots \\ sp_n \end{bmatrix}$

Eg. $u = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ $v = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ $2u + 3v = \begin{bmatrix} 2 \\ 14 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 23 \end{bmatrix}$

Let $s_1, s_2, \dots, s_n$ be scalars. $u_1, u_2, \dots, u_n$ be column vectors

↳ $s_1u_1 + s_2u_2 + \dots + s_nu_n$ is a linear combination of vectors $u_1, u_2, \dots, u_n$

↳ span $\{u_1, \dots, u_n\}$ = All the linear combinations of the column vectors $u_1, u_2, \dots, u_n$

Solve the homogeneous system

with the RRE form:

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + 3s - t = 0 \\ x_3 + 3t = 0 \end{array} \Rightarrow \begin{array}{l} x_1 = -3s + t \\ x_2 = s \\ x_3 = -3t \\ x_4 = t \end{array}$$

leading: x_1, x_3 — $x_2 = s$

free: x_2, x_4 — $x_4 = t$

we can record this solution in a column vector:

$$X = \begin{bmatrix} -3s + t \\ s \\ -3t \\ t \end{bmatrix} = \begin{bmatrix} -3s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ -3t \\ t \end{bmatrix} = s\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

The column vectors $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$ are called basic sol^n to the homogeneous system

sol^{n}s for this system = All of the combinations of $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$ = span $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$

$$\boxed{\text{solutions to a homogeneous system} = \text{span of basic solutions}}$$

The RRE form of a system is

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 7 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{array}\right]$$

Find a set of basic sol^n to this system

leading: x_1, x_4

free: $x_2 = s$, $x_3 = t$, $x_5 = u$

$x_1 + 2s + 3t + 7u = 0$
$x_4 - 2u = 0$

$x_1 = -2s - 3t - 7u$
$x_2 = s$
$x_3 = t$
$x_4 = 2u$
$x_5 = u$

We can write sol^n for systems that are non-homogeneous using column vectors

But there is no such thing as 'basic sol^{n}s' for these systems

Eg. solve the system with the RRE form

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 7 & 10 \\ 0 & 0 & 0 & 1 & -2 & 12 \end{array}\right]$$

$x_1 + 2x_2 + 3x_3 + 7x_5 = 10 \Rightarrow x_1 = -2x_2 - 3x_3 - 7x_5 + 10$

$x_4 - 2x_5 = 12 \Rightarrow x_2 = s$, $x_3 = t$, $x_4 = 2x_5 + 12$, $x_5 = u$

$$X = s\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + t\begin{bmatrix}-3\\0\\1\\0\\0\end{bmatrix} + u\begin{bmatrix}-7\\0\\0\\2\\1\end{bmatrix} + 1\begin{bmatrix}10\\0\\0\\12\\0\end{bmatrix}$$

All sol^n to this system $\neq$ span$\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-7\\0\\0\\2\\1\end{bmatrix}, \begin{bmatrix}10\\0\\0\\12\\0\end{bmatrix}\right\}$

Since it's not homogeneous the final vector cannot be multiplied by anything besides 1, hence it is not ALL solutions

2 Matrix algebra

A matrix is a rectangular array of numbers

If a matrix has m rows & n columns, we say it is $m \times n$

$A = \begin{bmatrix}2 & 3\\7 & 1\\5 & -7\end{bmatrix}$ Size $= 3\times 2$

$(A)_{11} = a_{11} = 2$ $(A)_{22} = a_{22} = 1$

$(A)_{32} = a_{32} = -7$ $(A)_{31} = 5$

Let A be a matrix. The entry of A in row i & j is denoted b a_{ij} or $(A)_{ij}$

A zero matrix is a matrix with all the entries zero

↳ For each different size there is a zero matrix $\Rightarrow$ $O = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ $O = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

↳ All of the zero matricies are denoted by O

A column vector is a matrix with 1 column

A row vector is a matrix with 1 row

A square matrix is a matrix such that $m = n$

Some addition properties

- $A + B = B + A$
- $A + O = A$ zero matrix of same size

Addition of matricies:

If A & B are 2 maticies of the same size the $C = A + B$ is a matrix of the same size

↳ $C = A + B \Rightarrow (C)_{ij} = (A)_{ij} + (B)_{ij} \Rightarrow c_{ij} = a_{ij} + b_{ij}$

$\Rightarrow$ Eg. $A = \begin{bmatrix}2&3\\1&2\\5&1\end{bmatrix}$ $B = \begin{bmatrix}0&2\\7&1\\2&8\end{bmatrix}$ $C = \begin{bmatrix}2&2\\1&1\end{bmatrix}$

i) $A+B$ $\begin{bmatrix}2&3\\1&2\\5&1\end{bmatrix} + \begin{bmatrix}0&2\\7&1\\2&8\end{bmatrix} = \begin{bmatrix}2&5\\8&3\\7&9\end{bmatrix}$

ii) $B+A$ $\begin{bmatrix}0&2\\7&1\\2&8\end{bmatrix} + \begin{bmatrix}2&3\\1&2\\5&1\end{bmatrix} =$

Matrix addition is associative (Addition order does not matter)

iii) $A + C$ $\begin{bmatrix}2&3\\1&2\\5&1\end{bmatrix}$ $\begin{bmatrix}2&2\\1&1\end{bmatrix}$ = not defined

Matrix must be the same size

iv) $B+C$ $\begin{bmatrix}0&2\\7&1\\2&8\end{bmatrix}$ $\begin{bmatrix}2&2\\1&1\end{bmatrix}$ = not defined

vi) $B = -2\cdot A \begin{bmatrix}-4&-6\\-2&-4\\-10&-2\end{bmatrix}$

Scalar multiplication:

If A is a matrix & s is a scalar

↳ $B = s\cdot A$ is a matrix of the same size

If $B = s\cdot A$ then,

$(B)_{ij} = s\cdot(A)_{ij}$

$b_{ij} = s\cdot a_{ij}$

Also we denote $(-1)A = -A$

Some multiplication properties:

- $A + (-A) = O$ ← zero matrix
- $s\cdot(A+B) = s\cdot A + s\cdot B$ (distributive law)
- $0\cdot A = O$ ← zero matrix (scalar 0)

Transpose of a matrix

If A is a matrix, A^T is the matrix obtained by switching rows of A with columns of A

$\Rightarrow$ $A = \begin{bmatrix}2&7\\1&-2\\3&-1\end{bmatrix}$ $A^T = \begin{bmatrix}2&1&3\\7&-2&-1\end{bmatrix}$

$(A^T)_{ij} = (A)_{ji}$

If A is $m\times n$ then A^T is $n \times m$

(i,j) entry of $A^T = (j,i)$ entry of A

Transpose properties:

- $(A^T)^T = A$
- ★ $(A+B)^T = A^T + B^T$ Transpose of addition = addition of Transpose
- $(s\cdot A)^T = s\cdot A^T$

Proof of ★ $(A+B)^T = A^T + B^T$

We need to show (i,j) entry of $P = (i,j)$ entry of Q for any i & j or in other words $(P)_{ij} = (Q)_{ij}$

$(P)_{ij} = (i,j)$ entry of $(A+B)^T$
$= (j,i)$ entry of $A+B$
$= (A)_{ji} + (B)_{ji}$

$(Q)_{ij} = (i,j)$ entry of $A^T + B^T$
$= (i,j)$ entry of A^T + (i,j) entry of B^T
$= (j,i)$ entry of A + (j,i) entry of B
$= (A)_{ji} + (B)_{ji}$

so,

$(P)_{ij} = (Q)_{ij}$

$\Rightarrow P = Q$ ■

Solve for A if

$$\left(\left(A + 3\begin{bmatrix}1&-1&0\\1&2&4\end{bmatrix}\right)^T\right)^T = \left(\begin{bmatrix}2&1\\0&5\\3&8\end{bmatrix}\right)^T$$

$$A + 3\begin{bmatrix}1&-1&0\\1&2&4\end{bmatrix} = \begin{bmatrix}2&0&3\\1&5&8\end{bmatrix}$$

$$A + \underbrace{\begin{bmatrix}3&-3&0\\3&6&12\end{bmatrix}}_{B} = \begin{bmatrix}2&0&3\\1&5&8\end{bmatrix}$$

$-B$ $-B$

$$\boxed{A = \begin{bmatrix}-1&3&3\\-2&-1&-4\end{bmatrix}}$$

Solve for A

$$(2A - 3[1\ 2\ 0])^T = 3A^T + [2\ 1\ -1]^T$$

$$((\quad)^T)^T = ((\quad)^T)^T$$

$$2A - 3[1\ 2\ 0] = 3A + [2\ 1\ -1]$$

$-2A$ $-2A$

$[-3\ -6\ 0] = A + [2\ 1\ -1]$

$[-5\ -7\ 1] = A \Rightarrow A = [-5\ -7\ 1]$

Symmetric matrix

A matrix is symmetric if $A^T = A$

↳ Symmetric matricies are also always square matricies since $m = n$

Normally matrix A is $m\times n$ meaning A^T is $n \times m$

$\Rightarrow$ But with a symmetric matrix $m = n$ so $m\times n = m\times m = n\times n = n\times m$

Eg. $A = \begin{bmatrix}3&2\\2&1\end{bmatrix}$ $B = \begin{bmatrix}5&7&-2\\7&3&5\\-2&5&-6\end{bmatrix}$

so for a symmetric matrix $(A)_{ij} = (A)_{ji}$

Proving a property of symmetric matricies

If A and B are both symmetric matricies of the same size

Show that $A+B$ is also symmetric.

↳ We need to show $A+B$ is symmetric (ie. $(A+B)^T = A+B$)

$(A+B)^T = A + B$

$A^T + B^T =$

↳ with a symmetric matrix $A = A^T$

so we can convert $A^T + B^T$ into $A + B$

which is our right side showing equality ■

Skew-Symmetric matrix

A is skew symmetric if $A^T = -A$

$m = n$ so it is also a square matrix

→ $(A)_{ij} = -(A)_{ji}$

→ Diagonal has to be zero

$A = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$

$-A^T = \begin{bmatrix}0&-1\\1&0\end{bmatrix}$

what if $i = j$

$(A)_{ii} = -(A)_{ii}$

$\Rightarrow 2(A)_{ii} = 0$

$\Rightarrow (A)_{ii} = 0$

2.2 Multiplication between a matrix & a column vector

A is $m \times n$

X is a column vector with n entries $(n \times 1)$

$A \cdot X = x_1 \cdot \begin{pmatrix} \text{col } 1 \\ \text{of } A \end{pmatrix} + x_2 \cdot \begin{pmatrix} \text{col } 2 \\ \text{of } A \end{pmatrix} + \dots + x_n \begin{pmatrix} \text{col } n \\ \text{of } A \end{pmatrix}$ ← is a linear combination of the colums of A

→ results in a column vector of size $m \times 1$

$A = \begin{bmatrix} 3 & -2 & 0 \\ 5 & 0 & 1 \end{bmatrix}_{2\times 3}$ $\quad v = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}_{3\times 1}$

Compute $A \cdot V = 1 \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix} + 5 \begin{bmatrix} -2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 3 - 10 + 0 \\ 5 + 0 + 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \end{bmatrix}$

Eg. $A = \begin{bmatrix} 4 & 3 & 8 \\ 5 & 0 & -7 \\ 0 & 2 & 6 \\ -2 & 1 & 5 \end{bmatrix}$ $\quad V = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\quad \begin{bmatrix} 20 \\ -9 \\ 12 \\ 8 \end{bmatrix}$

$A \cdot V = 1 \cdot \begin{bmatrix} 4 \\ 5 \\ 0 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 8 \\ -7 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 16 \\ -14 \\ 12 \\ 10 \end{bmatrix} = \begin{bmatrix} 20 \\ -9 \\ 12 \\ 8 \end{bmatrix}$

$A \cdot x \in$ span of the columns of A

This multiplication can be used to describe linear systems

Eg. $2x_1 + 3x_2 + 7x_3 = 0$
$-x_1 + 2x_3 = 5$
$x_1 - 5x_2 + 4x_3 = 2$

$\Rightarrow A = \begin{bmatrix} 2 & 3 & 7 \\ -1 & 0 & 2 \\ 1 & -5 & 4 \end{bmatrix}$ also $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix}$

essentially we can summarize the linear system by say $A \cdot x = b$

A linear system has m eq^{n}s and n variables can be written as $Ax = b$

Where A = Matrix with coefficients of the system (has the size $m \times n$)

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ size $n \times 1$

b = Column vector with 'constant' terms (size $m \times 1$)

Eg. $2x_1 + 2x_4 + 5x_5 = -7$
$6x_2 + x_3 = 1$

$\underbrace{\begin{bmatrix} 2 & 0 & 0 & 1 & 5 \\ 0 & 6 & 1 & 0 & 0 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -7 \\ 1 \end{bmatrix}}_{b}$

Question: Can b be written as a linear combination of vectors $u, v, \& w$?

if $b = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ $\quad u = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $\quad v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $\quad w = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$

In other words, Can we find x_1, x_2, x_3 such that $x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

↳ Same as asking if the linear system given by $Ax = b$ has a solution

$\underbrace{\begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 1 \\ 0 & 2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}}_{b}$

$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 4 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 + (-2)R_1]{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 1 \end{array}\right] \xrightarrow[R_3 \to -R_3]{R_3 \to R_3 + (-2)R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$

leading 1 in last column so it is inconsistent

→ so no, b cannot be written in terms of u, v, w

The following statements all mean the same thing:
- b is in the column space of A
- The system $Ax = b$ is consistent
- b is in the range of T_A

Matrix-vector multiplication as a function.

$\mathbb{R}$ = The set of all the real numbers

$\mathbb{R}^2$ = set of all the points on a plain

$= \left\{ (x_1, x_2) \middle| \begin{matrix} x_1 \in \mathbb{R} \\ x_2 \in \mathbb{R} \end{matrix} \right\}$

= The set of all the ordered pairs of real numbers

$\mathbb{R}^3$ = Set of all real points on 3D space
= Set of all ordered triples of real numbers
$= \{ (x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R} \}$

$\Rightarrow$ We then have $\mathbb{R}^n$ which is: The set of all the ordered n-tuples of real numbers

↳ $\{ (x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in \mathbb{R} \}$

↳ In linear algebra we use the set of $n \times 1$ matricies to identify $\mathbb{R}^n$

$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| x_1, x_2, \dots, x_n \in \mathbb{R} \right\}$

Eg. Let A be a matrix

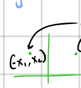
X (a column vector with n entries), $x \in \mathbb{R}^n$ → $\boxed{\text{Multiply by } A}$ → $Ax \in \mathbb{R}^m$

So, multiplying A can be thought as a function that takes an input from $\mathbb{R}^n$ and gives an output in $\mathbb{R}^m$

↳ We denote this function as T_A $\quad T_A : \mathbb{R}^n \to \mathbb{R}^m$

T_A = The transform endured by A $\quad T_A(x) = Ax$

Eg. 1 f = Reflection about y-axis



Reflection of (x_1, x_2) about the y-axis

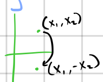
$f(x_1, x_2) = (-x_1, x_2)$

$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$

We can also find a matrix that achives the same results.

$\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}}_{b}$

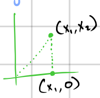
Eg. 2 like last one put across x-axis



$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$

Eg. 4 α = Projection on to x-axis

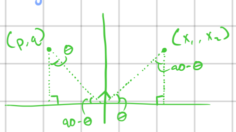


$\alpha\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{D} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

$\alpha = T_D$

Eg. 3 function h = Counter clockwise rotation by 90°



These triangles have the same angles & their diagonals have the same length } they are similar triangles

$OAB\Delta$ & $OCD\Delta$ are similar

$\Rightarrow \underset{x_2 = OD}{AB = OD}$ $\quad \Rightarrow \underset{x_1 = CD}{OB = CD}$ $\quad \Rightarrow \begin{matrix} p = -x_2 \\ q = x_1 \end{matrix}$

$\therefore h$ is the transformation induced by C or $h = T_C$

so, $h\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$

We can have transforms which has domains different from the target space

$\beta : \mathbb{R}^3 \to \mathbb{R}^2$

$\beta\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \underset{\substack{m \times n \\ 2 \times 3}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \underset{\substack{3 \times 1 \\ n \times 1}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\substack{2 \times 1 \\ m \times 1}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$

↳ Not all transforms have 'nice' geometric interpretations

$\beta : \mathbb{R}^3 \to \mathbb{R}^2$

$\beta\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 - x_3 \\ 2x_1 - 7x_2 + 5x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & -3 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_3 \\ 2x_1 - 7x_2 + 5x_3 \end{bmatrix}$

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1^2 - x_2/x_1 \\ x_2 \end{bmatrix} \Rightarrow$ Not induced by a matrix
↳ row 1 is not a linear equation

$\alpha : \mathbb{R}^n \to \mathbb{R}^n$

α = Does not change anything

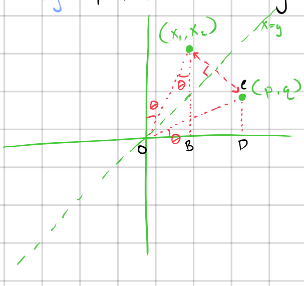
$\alpha\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

↳ Identity matrix $= I_n$ → size

identity matricies are square so $m = n$,

I_n → denotes size $(n \times n)$

Eg. ? ϕ = Reflection about $x = y$ line



$OCD\Delta$ & $OAB\Delta$ are similar

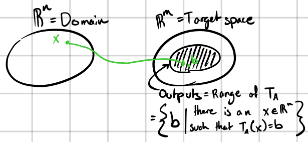
$OB = DC$
$x_1 = q$
$AB = OD$
$x_2 = p$

$\Rightarrow \phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$

$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{E} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$

$\phi = T_E$

$T_A : \mathbb{R}^n \to \mathbb{R}^m$



If $b \in$ Range of T_A, then $\exists$ should be an x such that

$\underbrace{T_A(x) = b}_{Ax = b}$ The linear system $Ax=b$ has a solⁿ

b is in the column space of A

2.3 Matrix multiplication

A is $m \times n$ B has p columns ← Any of these columns should have n entries

B is $n \times p$ say $b_1, b_2, \ldots, b_p$

we know how to compute

$\underset{m\times n}{A} \cdot \underset{n\times 1}{b_1}$

$A \cdot b_2$

and so on, but how do we do them all

$$\Rightarrow A \cdot B = \begin{array}{ccccc} \text{1st column of } A\cdot B & \text{2nd} & & & \text{col } p \\ \end{array}$$
$$A \cdot B = \begin{bmatrix} [A\cdot b_1] & [A\cdot b_2] & [A\cdot b_3] & \cdots & [A\cdot b_p] \end{bmatrix}$$

If A is $m \times n$, B is $n \times p$ } $A \cdot B$ has the size $m \times p$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & -1 & 2 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$A \cdot B =$

$$2\begin{bmatrix}1\\2\\1\end{bmatrix} + 1\begin{bmatrix}0\\-1\\-1\end{bmatrix} + 1\begin{bmatrix}1\\0\\2\end{bmatrix} + 0\begin{bmatrix}0\\0\\7\end{bmatrix} = \begin{bmatrix}3\\3\\3\end{bmatrix}$$

$$3\begin{bmatrix}1\\2\\1\end{bmatrix} + 0\begin{bmatrix}0\\-1\\-1\end{bmatrix} + 1\begin{bmatrix}1\\0\\2\end{bmatrix} + 1\begin{bmatrix}0\\0\\7\end{bmatrix} = \begin{bmatrix}4\\6\\12\end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} 3 & 4 \\ \boxed{3} & 6 \\ 3 & 12 \end{bmatrix}$$

Eg. $A = \begin{bmatrix} 2 & 3 \\ -1 & 7 \\ 0 & 1 \end{bmatrix}_{3\times 2}$ $B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}_{2\times 2}$

$$\underset{A\cdot b_1}{\begin{bmatrix}5\\6\\1\end{bmatrix}} \quad \underset{A\cdot b_2}{\begin{bmatrix}7\\5\\1\end{bmatrix}} \Rightarrow \begin{bmatrix} 5 & 7 \\ 6 & 5 \\ 1 & 1 \end{bmatrix}$$

Eg. now multiply B times A

$\underset{2\times 2}{B} \; \underset{3\times 2}{A}$

$2 \neq 3 \Rightarrow B \cdot A =$ not defined

$\hookrightarrow A \cdot B \neq B \cdot A$

Eg. $A = \begin{bmatrix} 5 & 1 \\ 7 & 6 \end{bmatrix}_{2\times 2}$ $B = \begin{bmatrix} -1 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}_{2\times 3}$

$A \cdot b_1 = -1\begin{bmatrix}5\\7\end{bmatrix} + 1\begin{bmatrix}1\\6\end{bmatrix} = \begin{bmatrix}-4\\-1\end{bmatrix}$

$A \cdot b_2 = 4\begin{bmatrix}5\\7\end{bmatrix} + 0\begin{bmatrix}1\\6\end{bmatrix} = \begin{bmatrix}20\\28\end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 20 & 2 \\ -1 & 28 & 12 \end{bmatrix}$

$A \cdot b_3 = 0\begin{bmatrix}5\\7\end{bmatrix} + 2\begin{bmatrix}1\\6\end{bmatrix} = \begin{bmatrix}2\\12\end{bmatrix}$

A is $m \times n$; entries of A are given by a_{ij}

B is $n \times p$; entries of B are given by b_{ij}

$AB = \begin{bmatrix} & \text{col } j & \\ \text{row } i & \square & \\ & & \end{bmatrix}$ position (i,j)

What is the (i,j) entry of $A \cdot B$?

Column j of AB $= A \cdot \begin{pmatrix} \text{Column } j \\ \text{of } B \end{pmatrix}$

$= b_{ij} \cdot \Big[$ picture

(i,j) entry of AB $= a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + a_{i3} \cdot b_{3j} + \ldots + a_{in} \cdot b_{nj}$

$$= \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \bullet \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

Dot product between (row i of A) and (column j of B)

Using Dot product method on previous example

Row 2 of A • Column 1 of B

$= 2 \cdot 2 + (-1) \cdot 1 + 0 \cdot 1 + 0 \cdot 0$

$= \boxed{3} = (AB)_{21}$

$\quad\quad\quad\quad\quad i\,j$

important properties of matrix multiplication:

- $AB \neq BA$ in general
- $A \cdot I_n = A = I_m \cdot A$ ← identity matrix
- $A(BC) = (AB)C$
- $A(B+C) = AB + BC$
- $(A+B)C = AC + BC$
- $A(s \cdot B) = (s \cdot A)B = s(AB)$ ← where s is scalar
- $(A \cdot B)^T = A^T \cdot B^T$ ☆

Proving $(AB)^T = A^T B^T$

We know (i,j) entry of AB = Dot product between Row i of A & Col j of B

(i,j) entry of $(AB)^T = (j,i)$ entry of AB

= Dot product betⁿ (row j of A) & (col i of B)

$$= \begin{bmatrix} a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix} \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix}$$

$= a_{j1} b_{1i} + a_{j2} b_{2i} + \ldots + a_{jn} b_{ni}$

(i,j) entry of $B^T A^T$ = Dot product between row i of B^T & colⁿ j of A^T

= colⁿ i of B & row j of A

same thing

so $(AB)^T = A^T B^T$ is a valid statement ∎

If A is a square matrix $(m \times m)$

$A \cdot A$ is defined (A^2)

and $\underbrace{A \cdot A \cdot A \cdot \ldots \cdot A}_{k\text{-times}} = A^k$

Eg. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ compute A^2

$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$= \begin{matrix} 0 \cdot \begin{bmatrix}0\\0\end{bmatrix} + 0 \cdot \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \\ 1 \cdot \begin{bmatrix}0\\0\end{bmatrix} + 0 \cdot \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \end{matrix}$$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

If a 2×2 matrix A commutes $(A \cdot B = B \cdot A)$

with $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

show that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some a & b

So, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

$B \cdot A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$

$(1,1)$ entry of $A \cdot B = [a\ b] \cdot \begin{bmatrix}0\\0\end{bmatrix} = 0$

$(1,2)$ entry of $A \cdot B = [a\ b] \cdot \begin{bmatrix}1\\0\end{bmatrix} = a$

$A \cdot B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$ $c = 0$, $d = a$

If ABC is defined & A is 3×3 and C is 5×5. Find the sizes of the matricies below

1) B → 3×5
2) ABC → 3×5

$$\underbrace{\underbrace{\underset{3\times 3}{A} \quad \underset{3\times q}{B}}_{\underset{3\times q}{AB}} \quad \underset{5\times 5}{C}}_{\underset{3\times 5}{ABC}} \qquad q = 5$$

Matrix multiplication & transformations induced by matricies

$f: \mathbb{R} \to \mathbb{R}$
$f(x) = 2x + 5$
$g: \mathbb{R} \to \mathbb{R}$
$g(x) = x^2 + 7$

$f \circ g: \mathbb{R} \to \mathbb{R}$
$f\circ g(x) = f(g(x))$
$= f(x^2+7)$
$= 2(x^2+7)+5$

A is $m\times n$
$\Rightarrow T_A: \mathbb{R}^n \to \mathbb{R}^m$
B is $n \times m$
$\Rightarrow T_B: \mathbb{R}^m \to \mathbb{R}^n$

Then the composition of T_A & T_B is
$T_A \circ T_B : \mathbb{R}^p \to \mathbb{R}^m$ & $T_A \circ T_B = T_{AB}$

In other terms
A is $m \times n$
B is $n \times p$

$\mathbb{R}^p \xrightarrow{T_B} \mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^m$
$x \longrightarrow T_B(x) = Bx \longrightarrow \underbrace{T_A(T_B(x)) = A(Bx)}_{T_A \circ T_B = T_{AB}}$

$A + O = A$
$A \cdot I_n = A$ } Operations to not change the matrix

Eg. $f\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1\\2x_1+x_2\\x_1-x_2\end{bmatrix} \Rightarrow f = T_A$ where $A = \begin{bmatrix}1&0\\2&1\\1&-1\end{bmatrix}$

$g\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_1+2x_2\\x_2-x_3\end{bmatrix} \Rightarrow g = T_B$ where $B = \begin{bmatrix}1&2&0\\0&1&-1\end{bmatrix}$

$g\circ f\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = g\left(f\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)\right) = g\left(\begin{bmatrix}x_1\\2x_1+x_2\\x_1-x_2\end{bmatrix}\right) = \begin{bmatrix}x_1+2(2x_1+x_2)\\(2x_1+x_2)-(x_1-x_2)\end{bmatrix}$

$\begin{bmatrix}5x_1+2x_2\\x_1+2x_2\end{bmatrix}$

$\Rightarrow$ This can also be done using matricies
$g\circ f = T_D$
where $D = \begin{bmatrix}5&2\\1&2\end{bmatrix}$

$g\left(A\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = B\left(A\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)$

$B \cdot A = \begin{bmatrix}5&2\\1&2\end{bmatrix}$

$\begin{bmatrix}1&2&0\\0&1&-1\end{bmatrix}\begin{bmatrix}1&0\\2&1\\1&-1\end{bmatrix} = 1\cdot\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}2\\1\end{bmatrix}+1\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix}$ 1st

$0\cdot\begin{bmatrix}1\\0\end{bmatrix}+1\begin{bmatrix}2\\1\end{bmatrix}+(-1)\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$ 2nd

$\Rightarrow \begin{bmatrix}5&2\\1&2\end{bmatrix}$ ✓

Same result as before so we know it is correct

2.4 Matrix inversions

A is a square matrix
B is another square matrix of the same size

If $AB = I = BA$ then we say B is the inverse of A $\quad B = A^{-1}$

If A has an inverse, then it is unique
↳ Proof:
Assume A has 2 different inverses say B & C
$AB = I = BA \qquad AC = I = CA$

cool exercise $\Rightarrow$
$C\cdot AB = I\cdot C$
$CAB = C \Rightarrow CA = I$
$\underline{B = C}$

Eg. $A = \begin{bmatrix}1&1\\0&2\end{bmatrix}$ $B = \begin{bmatrix}1&-1/2\\0&1/2\end{bmatrix}$
Show that $B = A^{-1}$

col 1: $A\cdot B = 1\cdot\begin{bmatrix}1\\0\end{bmatrix}+0\cdot\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad -\frac{1}{2}\cdot\begin{bmatrix}1\\0\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ $\Rightarrow A\cdot B = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I_2$

$B\cdot A = 1\cdot\begin{bmatrix}1\\0\end{bmatrix}+0\begin{bmatrix}-1/2\\1/2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad 1\cdot\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}-1/2\\1/2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ $\Rightarrow B\cdot A = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I_2$

Eg 2 $A = \begin{bmatrix}3&0\\4&0\end{bmatrix}$ show that A does not have an inverse

↳ Solution: Assume A has an inverse
$A^{-1} = \begin{bmatrix}a&b\\c&d\end{bmatrix}$
Then $\begin{bmatrix}3&0\\4&0\end{bmatrix}\begin{bmatrix}a&c\\b&d\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

col 1: $A\cdot B = a\cdot\begin{bmatrix}3\\4\end{bmatrix}+c\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}3a\\4a\end{bmatrix}$ col 2: $b\begin{bmatrix}3\\4\end{bmatrix}+d\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}3b\\4b\end{bmatrix}$ $\quad \begin{bmatrix}3a&3b\\4a&4b\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$3a = 1$
$4a = 0$ $\Rightarrow a = 1/3$, $a = 0$ $\Rightarrow 1/3 \neq 0$

There is no value for a, b, c, d where $A^{-1} = \begin{bmatrix}a&b\\c&d\end{bmatrix}$ so our assumption that A has an inverse is incorrect proving A does not have an inverse

Eg 3 If $A^{15} = I$ ← 15th power
find A^{-1} in terms of A $\Rightarrow$ $\left.\begin{matrix}A^{14}\cdot A = I\\ A\cdot A^{14} = I\end{matrix}\right\} A^{-1} = A^{14}$

$\begin{matrix}x_1+x_2=5\\2x_2=7\end{matrix} \Rightarrow x_1\begin{bmatrix}1\\0\end{bmatrix}+x_2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}5\\7\end{bmatrix} \Rightarrow \underset{A}{\begin{bmatrix}1&1\\0&2\end{bmatrix}}\underset{x}{\begin{bmatrix}x_1\\x_2\end{bmatrix}} = \underset{b}{\begin{bmatrix}5\\7\end{bmatrix}} \Rightarrow A\cdot x = b$

In fact any linear system can be written as $Ax = b$
If A has an inverse:
$Ax = b \Rightarrow A^{-1}\cdot A\cdot x = A^{-1}\cdot b \Rightarrow x = A^{-1}b$

If A has an inverse, we saw:
$Ax = b$ has a unique solution for any $n\times 1$ column vector b
In other words, if columns of A are $A_1, A_2, \ldots, A_n$
$\underbrace{x_1A_1 + x_2A_2 + \ldots + x_nA_n = b}_{Ax=b}$

So, any $n\times 1$ column vector b can be written uniquely as a linear combination of columns of A

We know: $T_A \circ T_B = T_{AB}$
If A has an inverse:
$T_A \circ T_{A^{-1}} = T_{AA^{-1}} = T_I$ = Identity transform
$T_A \circ T_{A^{-1}}\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\right) = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$
Similarly $T_{A^{-1}} \circ T_A\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\right) = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$
} The function T_A has an inverse function

A is an $n\times n$ matrix, the following statements are equivalent:
1. A has an inverse
2. The system $Ax = b$ has a unique solution for any $n\times 1$ vector b
3. Any $n\times 1$ vector b can be uniquely written as a linear combination of columns of A.
4. T_A has an inverse function (given by $T_{A^{-1}}$)
5. RREF of $A = I$
6. Rank of $A = n$
7. A = the product of elementary matricies

$f: \mathbb{R} \to \mathbb{R}$
$g: \mathbb{R} \to \mathbb{R}$
if $\left.\begin{matrix}f\circ g(x) = x\\ g\circ f(x) = x\end{matrix}\right\}$ we call g as the inverse function of f

more generally
$f: \mathbb{R}^n \to \mathbb{R}^n$
$g: \mathbb{R}^n \to \mathbb{R}^n$
$f\circ g\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\right) = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$
$g\circ f\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}\right) = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$
} just saying only way for input to be = output is if $g = f^{-1}$

Inverse of a 2×2 matrix.
$A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$
If $ad - cb \neq 0$, then A has an inverse given by $A^{-1} = \frac{1}{ad-bc} = \begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

Eg. $A = \begin{bmatrix}7&5\\1&1\end{bmatrix}$ find A^{-1}
$7\cdot 1 - 5\cdot 1 = 2 \neq 0$
$A^{-1} = \frac{1}{2}\cdot\begin{bmatrix}1&-5\\-1&7\end{bmatrix} = \begin{bmatrix}1/2&-5/2\\-1/2&7/2\end{bmatrix}$

Inverse of a square matrix of any size
- Check if A can be carried to I through a sequence of row operations
↳ (checking if RREF of A is I)
- If so, apply the same sequence to I
↳ The matrix you end up with is A^{-1}

Eg. $A = \begin{bmatrix}1&0&1\\0&2&0\\3&1&1\end{bmatrix} \xrightarrow[R_3\to R_3+(-3)R_1]{R_2\to\frac{1}{2}R_2} \begin{bmatrix}1&0&1\\0&1&0\\0&1&-2\end{bmatrix} \xrightarrow[R_1\to R_1+(-1)R_3]{R_3\to R_3+(-1)R_2,\ R_3\to -\frac{1}{2}R_3} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = I_3$

$I_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&0\\0&1/2&0\\-3&0&1\end{bmatrix} \longrightarrow \begin{bmatrix}-1/2&-1/4&1/2\\0&1/2&0\\3/2&1/4&-1/2\end{bmatrix} = A^{-1}$

① Cancellation
However if A has an inverse
$AB = AC \Rightarrow B = C$
$A^{-1}(AB) = A^{-1}(AC)$
$(A^{-1}A)B = (A^{-1}A)C$
$IB = IC$
$B = C$

② $(AB)^{-1} = B^{-1}A^{-1}$
$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = A\cdot A^{-1} = I$
$(B^{-1}A^{-1})(AB) = B^{-1}(AA^{-1})B = B^{-1}(I)B = B^{-1}B = I$
$\Rightarrow$ Inverse of $AB = A^{-1}B^{-1}$

③ $I^{-1} = I$
④ $(A^{-1})^{-1} = A$
$A^{-1}\,\boxed{A} = I$

use the result from Friday $(AB)^T = B^TA^T$

⑤ $(A^T)^{-1} = (A^{-1})^T$
$A^T\cdot(A^{-1})^T = (A^{-1}\cdot A)^T = I^T = I$
$(A^{-1})^T A^T = (A\cdot A^{-1})^T = I^T = I$

$A^{-1} = \begin{bmatrix}-1/2&-1/4&1/2\\0&1/2&0\\3/2&1/4&-1/2\end{bmatrix}$ find $(A^T)^{-1} = (A^{-1})^T$

$(A^T)^{-1} = \begin{bmatrix}-1/2&0&3/2\\1/4&1/2&-1/4\\1/2&0&-1/2\end{bmatrix}$

In general if $AB = AC$ we CANNOT say $B = C$ $\Rightarrow$
For example, $A = \begin{bmatrix}0&1\\0&0\end{bmatrix}$ $B = \begin{bmatrix}7&3\\1&6\end{bmatrix}$ $C = \begin{bmatrix}2&6\\1&6\end{bmatrix}$
$AB = \begin{bmatrix}1&6\\0&0\end{bmatrix}$
$AC = \begin{bmatrix}1&6\\0&0\end{bmatrix}$
} These are the same, but we can see $B \neq C$

2.5 Elementary matricies

- An elementary matrix is a matrix obtained by performing a single row operation on I

$A \xrightarrow{\text{elementary row operation}} E\cdot A$ where E = elementary matrix obtained by performing the same row operation on I

$A = \begin{bmatrix}2&1\\7&-5\\6&1\end{bmatrix} \xrightarrow{R_1\to R_1+3R_2} \begin{bmatrix}23&-14\\7&-5\\6&1\end{bmatrix} = E\cdot A \Rightarrow E = \begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}$

- Any elementary matrix has an inverse

Inverse of an elementary matrix = Elementary matrix corresponding to "inverse" of that row operation

$\begin{bmatrix}1&0\\0&1\end{bmatrix} \xrightarrow{R_1\to R_1+R_2} \begin{bmatrix}1&1\\0&1\end{bmatrix} = E_1$
$\xrightarrow{R_1\to R_1-R_2} \begin{bmatrix}1&-1\\0&1\end{bmatrix} = E_1^{-1}$
$+R_2$ vs. $-R_2$ inverse operations

To the right we showed $A^{-1} = E_3E_2E_1 \Rightarrow (A^{-1})^{-1} = (E_3E_2E_1)^{-1}$
→ So, $A = E_1^{-1}E_2^{-1}E_3^{-1}$ $\Rightarrow$ This is important if we need to represent a matrix as a product of it's elementary matricies

Eg. $A = \begin{bmatrix}1&0&0&0\\-4&1&0&0\\0&3&1&0\\2&0&0&1\end{bmatrix} \xrightarrow{R_2\to R_2+4R_1} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&3&1&0\\2&0&0&1\end{bmatrix} \xrightarrow{R_4\to R_4+(-2)R_1} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&3&1&0\\0&0&0&1\end{bmatrix} \xrightarrow{R_3\to R_3+(-3)R_2} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

$E_1\cdot A =$ ⤴ $\quad E_2\cdot E_1\cdot A =$ ⤴ $\quad E_3\cdot E_2\cdot E_1\cdot A =$ ⤴ $\longrightarrow$ $E_3\cdot E_2\cdot E_1\cdot A = I$, $A^{-1}\cdot A = I$ $\Rightarrow A^{-1} = E_3\cdot E_2\cdot E_1$

$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&0&0\\4&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&0&0\\4&1&0&0\\0&0&1&0\\-2&0&0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&0&0\\4&1&0&0\\-12&-3&1&0\\-2&0&0&1\end{bmatrix}$

step 1: $E_1 = \begin{bmatrix}1&0&0&0\\4&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ $\quad$ step 2: $E_2 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-2&0&0&1\end{bmatrix}$ $\quad$ step 3: $E_3 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-3&1&0\\0&0&0&1\end{bmatrix}$

Row operation	inverse row operation
$R_i \to k\cdot R_i$	$R_i \to 1/k\, R_i$
$R_i \leftrightarrow R_j$	$R_j \leftrightarrow R_i$
$R_i \to R_i + kR_j$	$R_i \to R_i + (-k)R_j$

Elementary matricies cont...

Problem 10. Let A be a 4×4 matrix. The following elementary row operations are applied on A to obtain the matrix B.

First row operation : $R_2 \to R_2 - 3R_1$

Second row operation : $R_3 \leftrightarrow R_2$

(1) Find two elementary matrices E_1 and E_2 such that $E_2E_1A = B$

(2) Find two elementary matrices E_3 and E_4 such that $E_4E_3B = A$

(3) If $B = I$ then compute A.

1) $E_1 = \begin{bmatrix} 1&0&0&0\\ -3&1&0&0\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix}$ $E_2 = \begin{bmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1\end{bmatrix}$

$E_2^{-1}\cdot(E_2)\cdot E_1\cdot A = (B)\cdot E_2^{-1}$

$E_1^{-1}\cdot(E_1)\cdot A = (E_2^{-1}B)\cdot E_1^{-1}$

$A = E_1^{-1}E_2^{-1}B$

2) $E_4E_3B = A$

$E_4 = E_1^{-1} = \begin{bmatrix} 1&0&0&0\\ 3&1&0&0\\ 0&0&1&0\\ 0&0&0&1\end{bmatrix}$

$E_3 = E_2^{-1} = \begin{bmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1\end{bmatrix}$

3) $B = I$ $\quad A = ?$

$A = E_4\cdot E_3\cdot B \Rightarrow A = E_4\cdot E_3\cdot I$

$\Rightarrow A = E_4\cdot E_3$

$E_4\cdot E_3 = \begin{bmatrix} 1&0&0&0\\ 3&0&1&0\\ 0&1&0&0\\ 0&0&0&1\end{bmatrix}$

Exam 1 1.1, 1.2, 1.3 ← Set 1; 2.1 through 2.5 ← Set 2

3.1 Cofactor expansion

$\begin{bmatrix} a&b\\ c&d\end{bmatrix}$ $\begin{bmatrix} 1&0\\ 0&1\end{bmatrix}$ some row operations $\begin{bmatrix} 1&0\\ 0&1\end{bmatrix}$ $\frac{1}{ad-bc}\cdot\begin{bmatrix} d&-c\\ -b&a\end{bmatrix}$

$\begin{bmatrix} a&b&c\\ d&e&f\\ g&h&i\end{bmatrix}$ $\begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&0&1\end{bmatrix}$ $\begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&0&1\end{bmatrix}$ $\frac{1}{a(ei-fh)-b(di-fg)+c(dh-eg)}\cdot\begin{bmatrix} *&*&*\\ *&*&*\\ *&*&*\end{bmatrix}$ $\neq 0$

Determinant of matrix obtained by deleting...

$ei-fh$ = row 1 col^n 1

$di-fg$ = row 1 col^n 2

$dh-eg$ = row 1 col^n 3

$A = \begin{bmatrix} 1&2&3\\ 4&5&6\\ 7&8&9\end{bmatrix}$

$C_{11}(A) = (-1)^{1+1}\begin{vmatrix} 5&6\\ 8&9\end{vmatrix} = 1\cdot(45-48) = -3$ $\quad$ (5·9 − 6·8)

$C_{12}(A) = (-1)^{1+2}\begin{vmatrix} 4&6\\ 7&9\end{vmatrix} = -1\cdot(-6) = 6$

$C_{13}(A) = (-1)^{1+3}\begin{vmatrix} 4&5\\ 7&8\end{vmatrix} = -3$

maybe do the rest

cofactor expansion of row 1: $|A| = 1\cdot(-3) + 2(6) + 3(-3) = 0$ → Since its zero there is no A^{-1}

A is $n\times n$

The (i,j) cofactor of A $= (-1)^{i+j}\cdot$ (Determinant of the matrix obtained by deleting row i & col^n j)

This is written as: $C_{ij}(A)$

For a $n\times n$ matrix A we define the determinant of A as

$|A| = a_{11}C_{11}(A) + a_{12}C_{12}(A) + \dots + a_{1n}C_{1n}(A)$ → Doesn't have to be the first row; usually best to just pick the easiest row

cofactor expansion along row 1

In fact $|A|$ can be computed using the cofactor expansion along any row or column, yielding the same result regardless

$A = \begin{bmatrix} 1&2&3&\boxed{-1}\\ 0&3&0&\boxed{0}\\ 2&-1&1&0\\ 1&3&1&\boxed{0}\end{bmatrix}$ Expand along column 4

$|A| = a_{14}C_{14}(A) + a_{24}C_{24}(A) + a_{34}C_{34}(A) + a_{44}C_{44}(A)$

$= (-1)(3) + 0 + 0 + 0$

$= -3$

$C_{14}(A) = (-1)^{1+4}\begin{vmatrix} 0&3&0\\ 2&-1&1\\ 1&3&1\end{vmatrix}$ → B

$C_{14}(A) = 3$

find det B = −3

$|B| = 0 + a_{12}C_{12}(A) + 0 = -3$

$3\cdot(-1)^3\begin{vmatrix} 2&1\\ 1&1\end{vmatrix} = -3\cdot 1$

How do row operations change the determinate:

$A \xrightarrow{R_i \to R_j} B \Rightarrow |A| \to -|B|$

$A \xrightarrow{R_i \to kR_i} B \Rightarrow |A| \to k|B|$

$A \xrightarrow{R_i \to R_i + kR_j} B \Rightarrow |A| \to |B|$

All these operations can also be done to columns ☆

the operation creates the same determinant

$A = \begin{bmatrix} 1&0&3&1\\ 2&2&6&0\\ -1&0&-3&1\\ 4&1&12&0\end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1&0&3&1\\ 2&2&6&0\\ 0&0&0&2\\ 4&1&12&0\end{bmatrix}$

$|A| = a_{34}C_{34}(A)$

$= 2\cdot 0$

$= 0$

$C_{34}(A) = (-1)^{3+4}\begin{vmatrix} 1&0&3\\ 2&2&6\\ 4&1&12\end{vmatrix}$

$= -1\cdot 0 = 0$

$B = \begin{bmatrix} 1&0&3\\ 2&2&6\\ 4&1&12\end{bmatrix} \xrightarrow{R_3 \to R_3 + (-4)R_1} \begin{bmatrix} 1&0&3\\ 2&2&6\\ 0&1&0\end{bmatrix}$

$|B| = a_{32}C_{32}(B)$

$= 1\cdot 0 = 0$

$C_{32}(B) = (-1)^5\begin{vmatrix} 1&3\\ 2&6\end{vmatrix}$

$= 0$

The same result can be achieved by modifying the columns

$\begin{bmatrix} 1&0&3&1\\ 2&2&6&0\\ -1&0&-3&1\\ 4&1&12&0\end{bmatrix} \xrightarrow{C_3 \to C_3 + (-3)C_1} \begin{bmatrix} 1&0&0&1\\ 2&2&0&0\\ -1&0&0&1\\ 4&1&0&0\end{bmatrix} = B$

$|A| = |B| = 0$

$|A| = |B| = b_{13}C_{13}(B) + b_{23}C_{23}(B) + b_{33}C_{33}(B) + b_{43}C_{43}(B)$

$0 + 0 + 0 + 0 = 0$

as seen to the right this is much easier than row operators

NOTE: Column manipulation can only be done while getting the determinant

① If A has a row (or column) of all zeros $\Rightarrow |A| = 0$

↳ Proof: Use the cofactor expansion along that row

② If A has two identical rows (or columns)

↳ Proof: rows/col^{n}s can be zeroed & use co. exp.

③ if A is $n\times n$ $\quad |k\cdot A| = k^n|A|$

Eg. $A = \begin{bmatrix} a&b&c\\ d&e&f\\ g&h&i\end{bmatrix}$ $B = \begin{bmatrix} 3d&3e&3f\\ a+g&b+h&c+i\\ 7g&7h&7i\end{bmatrix}$ if $|A| = 2$, $|B| = ?$

det=2 $\quad A \xrightarrow{R_1 \leftrightarrow R_2}$ −2 $\begin{bmatrix} d&e&f\\ a&b&c\\ g&h&i\end{bmatrix} \xrightarrow{R_2 \to R_2 + R_3}$ −2 $\begin{bmatrix} d&e&f\\ a+g&b+h&c+i\\ g&h&i\end{bmatrix} \xrightarrow{R_1 \to 3R_1}$ −6 $\begin{bmatrix} 3d&3e&3f\\ a+g&b+h&c+i\\ g&h&i\end{bmatrix} \xrightarrow{R_3 \to 7R_3}$ −42 $\begin{bmatrix} 3d&3e&3f\\ a+g&b+h&c+i\\ 7g&7h&7i\end{bmatrix}$

Upper triangular matrix = A square matrix with all the entries below the main diagonal are 0.

Lower triangular matrix = same, but 0 above

$A = \begin{bmatrix} 2&0&-5\\ 0&7&6\\ 0&0&6\end{bmatrix}$ $|A| = a_{11}C_{11}(A) + \cancel{a_{21}C_{21}(A)} + \cancel{a_{31}C_{31}(A)}$ $\quad |A| = 48$

$2\cdot(-1)^2\, 42 = 48$

$B = \begin{bmatrix} 5&0&0\\ 1&6&0\\ 7&6&-2\end{bmatrix}$ $|B| = \cancel{b_{13}C_{13}(B)} + \cancel{b_{23}C_{23}(B)} + b_{33}C_{33}(B)$ $\quad |B| = -60$

$-2\cdot(-1)^6(30) = -60$

$\Rightarrow$ For an upper/lower triangular matrix A, $|A|$ = Product of the entries in the main diagonal

Diagonal matrix = A matrix with all entries above & below main diagonal are zero $\Rightarrow D = \begin{bmatrix} 1&0&0\\ 0&6&0\\ 0&0&5\end{bmatrix} \Rightarrow |D| = 1\cdot 6\cdot 5 = 30$

For the identity matrix $I_n \to |I_n| = 1$

Important properties

① $|AB| = |A|\cdot|B|$

↳ $\det(AB) = \det(A)\cdot\det(B)$

② $|A^T| = |A|$

$A^2 = A\cdot A$

$|A^2| = |A|\cdot|A| = (|A|)^2$

↳ $\det(A^2) = (\det(A))^2$

$\Rightarrow$ In general $|A^k| = (|A|)^k$ for any positive k integer

Suppose k has an inverse

$A\cdot A^{-1} = I$

$|A\cdot A^{-1}| = |I|$

$|A|\cdot|A^{-1}| = |I|$

$\to |A|\cdot|A^{-1}| = 1$

$|A^{-1}| = 1/|A|$ $\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)} \Rightarrow |(A^{-1})^k| = \frac{1}{|A|^k}$

Eg. $|A| = -1 \quad |B| = 2 \quad |C| = 3$

$A^3B(A^{-1})^2C^3B^2 = |A^3|\cdot|B|\cdot|(A^{-1})^2|\cdot|C^3|\cdot|B^2| = -216$

$(-1)^3 \quad 2 \quad \frac{1}{(-1)^2} \quad 3^3 \quad 2^2$

$|(A^T)^3BC^TB^{-1}| = -1\cdot 2\cdot 3\cdot \frac{1}{2} = -3$

$(|A|)^3\cdot|B|\cdot|C|\cdot\frac{1}{|B|}$

3.2 Determinants & Matrix inverses

A is $n\times n$

Definition

Cofactor matrix of A is denoted by $\mathrm{cof}(A)$ & is given by (i,j) entry of $\mathrm{cof}(A) = C_{ij}(A)$

$A\cdot(\mathrm{cof}(A))^T = |A|\cdot I_n$

If $|A| \neq 0 \Rightarrow \left(\frac{1}{|A|}\right)\cdot A\cdot(\mathrm{cof}(A))^T = I_n$

$\boxed{A^{-1} = \frac{1}{|A|}\cdot(\mathrm{cof}(A))^T}$

We also call $(\mathrm{cof}(A))^T$ = adjoint of A = adj(A)

Lets compute $A\cdot(\mathrm{cof}(A))^T$

$= \begin{bmatrix} a_{11}&a_{12}&\cdots&a_{1n}\\ \vdots&&&\vdots\\ a_{i1}&a_{i2}&\cdots&a_{in}\\ \vdots&&&\vdots\\ a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}\cdot\begin{bmatrix} C_{11}(A)&C_{1n}(A)&\cdots&C_{1n}(A)\\ C_{21}(A)&C_{22}(A)&\cdots&C_{2n}(A)\\ \vdots&&&\vdots\\ C_{n1}(A)&C_{n2}(A)&\cdots&C_{nn}(A)\end{bmatrix}^T \Rightarrow \begin{bmatrix} a_{11}&a_{12}&\cdots&a_{1n}\\ \vdots&&&\vdots\\ a_{i1}&a_{i2}&\cdots&a_{in}\\ \vdots&&&\vdots\\ a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}\cdot\begin{bmatrix} C_{11}(A)&C_{21}(A)&\cdots&C_{n1}(A)\\ C_{12}(A)&C_{22}(A)&\cdots&C_{n2}(A)\\ \vdots&&&\vdots\\ C_{1n}(A)&C_{2n}(A)&\cdots&C_{nn}(A)\end{bmatrix} = \begin{bmatrix} |A|&0&\cdots&0\\ 0&|A|&&0\\ \vdots&&\ddots&\vdots\\ 0&0&\cdots&|A|\end{bmatrix} = |A|\cdot I_n$

(i,j) entry of the product between A & $(\mathrm{cof}(A))^T$ $= a_{i1}C_{j1}(A) + a_{i2}C_{j2}(A) + \dots + a_{in}C_{jn}(A)$

if $i = j \Rightarrow a_{i1}C_{i1}(A) + a_{i2}C_{i2}(A) + \dots + a_{in}C_{in}(A) = |A|$

if $i \neq j \Rightarrow 0$

3.2 Cont...

Eg. Compute A^{-1} for

$$A=\begin{bmatrix}3&2&1\\1&1&2\\-1&2&1\end{bmatrix}$$

$C_{11}(A)=-3$ $C_{12}(A)=-3$ $C_{13}(A)=3$
$C_{21}(A)=0$ $C_{22}(A)=4$ $C_{23}(A)=-8$
$C_{31}(A)=3$ $C_{32}(A)=-5$ $C_{33}(A)=1$

$$\text{cof}(A)=\begin{bmatrix}-3&-3&3\\0&4&-8\\3&-5&1\end{bmatrix}$$

$|A|=0+4-16=-12$

$$A^{-1}=\frac{1}{|A|}\cdot(\text{cof}(A))^T$$

$$=\frac{1}{-12}\cdot\begin{bmatrix}-3&0&3\\-3&4&-5\\3&-8&1\end{bmatrix}=\begin{bmatrix}1/4&0&-1/4\\1/4&-1/3&5/12\\-1/4&2/3&-1/12\end{bmatrix}$$

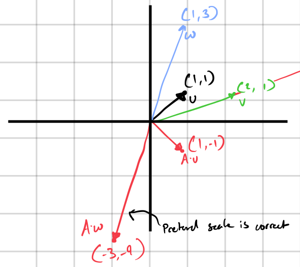
3.3 Diagnolization & eigen values

$A=\begin{bmatrix}3&-2\\3&-4\end{bmatrix}$ $u=\begin{bmatrix}1\\1\end{bmatrix}$ $v=\begin{bmatrix}2\\1\end{bmatrix}$ $w=\begin{bmatrix}1\\3\end{bmatrix}$

$A\cdot u=1\begin{bmatrix}3\\3\end{bmatrix}+1\begin{bmatrix}-2\\-4\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$

$A\cdot v=2\begin{bmatrix}3\\3\end{bmatrix}+1\begin{bmatrix}-2\\-4\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$

$A\cdot w=1\begin{bmatrix}3\\3\end{bmatrix}+3\begin{bmatrix}-2\\-4\end{bmatrix}=\begin{bmatrix}-3\\-9\end{bmatrix}$



We are interested in vectors that does not change the direction OR vectors that change the direction by 180° when multiplied by A

In the example to the right
$\lambda=2$ is an eigen value of A and $\begin{bmatrix}2\\1\end{bmatrix}$ is an eigen vector associated with $\lambda=2$
$\lambda=-3$ is an eigen value of A & $\begin{bmatrix}1\\3\end{bmatrix}$ is an eigen vector associated with $\lambda=-3$

There is a non-zero vector X such that

$A\cdot X=\lambda\cdot X$ for some scalar λ $\Rightarrow$ λ = an eigen value of A; X = an eigen vector associated with λ

matrix, number

$AX-\lambda X=0$ ← zero column vector

$(A-\lambda I)x=0$ ← normally we cannot subtract a matrix by a matrix so we multiply λ by I to make it a matrix

Does $(A-\lambda I)$ have an inverse?
↳ it does not since after assuming yes we got $x=0$

→ Assume yes

$(A-\lambda I)^{-1}(A-\lambda I)x=(A-\lambda I)^{-1}0$

$Ix=0 \Rightarrow x=0$

Step 1: Find λ values such that $A-\lambda I=0$ Step 2: For each λ solve the homogeneous linear system $(A-\lambda I)x=0$

Eg. Find eigen values & corresponding eigen vectors

$A=\begin{bmatrix}1&2\\3&2\end{bmatrix}$ $A-\lambda I=\begin{bmatrix}1&2\\3&2\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}1-\lambda&2\\3&2-\lambda\end{bmatrix}$

$|A-\lambda I|=(1-\lambda)(2-\lambda)-(2\times3)=0$

$2-3\lambda+\lambda^2-6=\lambda^2-3\lambda-4=0$

$(\lambda-4)(\lambda+1)=0$ $\lambda=4,-1$

$\lambda=4$

$\begin{bmatrix}-3&2\\3&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\left[\begin{array}{cc|c}-3&2&0\\3&-2&0\end{array}\right]\xrightarrow{R_2\to R_2+R_1}\left[\begin{array}{cc|c}-3&2&0\\0&0&0\end{array}\right]\xrightarrow{R_1\to -1/3R_1}\left[\begin{array}{cc|c}1&-2/3&0\\0&0&0\end{array}\right]$

$x_1-2/3x_2=0$
$x_1=2/3x_2$
$x_2=s$

$x_1=2/3s$
$x_2=s$

$\Rightarrow\begin{bmatrix}x_1\\x_2\end{bmatrix}=s\begin{bmatrix}2/3\\1\end{bmatrix}\cdot3$

$\begin{bmatrix}x_1\\x_2\end{bmatrix}=s\begin{bmatrix}2\\3\end{bmatrix}$

$\lambda=-1$

$\left[\begin{array}{cc|c}2&2&0\\3&3&0\end{array}\right]=\left[\begin{array}{cc|c}1&1&0\\0&0&0\end{array}\right]$

$x_1=-x_2$
$x_2=s$

$\begin{bmatrix}x_1\\x_2\end{bmatrix}=s\begin{bmatrix}-1\\1\end{bmatrix}$

$\begin{bmatrix}x_1\\x_2\end{bmatrix}=s\begin{bmatrix}-1\\1\end{bmatrix}$

any scalar multiple of this is also an eigen vector

Eg. 2 $A=\begin{bmatrix}7&0&-4\\0&5&0\\5&0&-2\end{bmatrix}$ $A-\lambda I=\begin{bmatrix}7-\lambda&0&-4\\0&5-\lambda&0\\5&0&-2-\lambda\end{bmatrix}$

$|A-\lambda I|=(5-\lambda)\cdot(\lambda^2-5\lambda+6)$

$(5-\lambda)(\lambda-3)(\lambda-2)$

$\lambda=5,3,2$

$\begin{vmatrix}7-\lambda&-4\\5&-2-\lambda\end{vmatrix}$

$(7-\lambda)(-2-\lambda)-(-20)$

$-14-7\lambda+2\lambda+\lambda^2$

$\lambda^2-5\lambda+6$

$\lambda=5$

$\begin{bmatrix}2&0&-4\\0&0&0\\5&0&-7\end{bmatrix}\begin{bmatrix}2&0&-4\\0&0&0\\0&0&1\end{bmatrix}\left[\begin{array}{ccc|c}1&0&0&0\\0&0&1&0\\0&0&0&0\end{array}\right]$

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=s\begin{bmatrix}0\\1\\0\end{bmatrix}$ $x_1=0$, $x_2=s$, $x_3=0$

$\lambda=2$

$\begin{bmatrix}5&0&-4\\0&3&0\\5&0&-4\end{bmatrix}\left[\begin{array}{ccc|c}1&0&-4/5&0\\0&1&0&0\\0&0&0&0\end{array}\right]$

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=s\begin{bmatrix}4/5\\0\\1\end{bmatrix}$ $x_1=4/5x_3$, $x_2=0$, $x_3=s$

$\lambda=3$

$\begin{bmatrix}4&0&-4\\0&2&0\\5&0&-5\end{bmatrix}\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&0&0\\0&0&0&0\end{array}\right]$

$x_1=x_3$
$x_2=0$
$x_3=s$

for this example I wrote no row operations, but they are proper, a lot of shortcuts were taken though

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=s\begin{bmatrix}1\\0\\1\end{bmatrix}$

vector form

general solution

$\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$

basic solution

Definition

A $n\times n$ matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that...

$$A=PDP^{-1}$$

↳ $AP=PDP^{-1}P\to AP=PD$

$\Rightarrow$ How to find P & D

$P=\begin{bmatrix}x_1&x_2&\cdots&x_n\end{bmatrix}$

&

$D=\begin{bmatrix}\lambda_1&0&\cdots&0\\0&\lambda_2&&\vdots\\\vdots&&\ddots&\\0&0&\cdots&\lambda_n\end{bmatrix}$

$P\cdot D=\begin{bmatrix}\lambda_1x_1&\lambda_2x_2&\cdots&\lambda_nx_n\end{bmatrix}$

$\parallel\quad\parallel\quad\parallel$

$A\cdot P=\begin{bmatrix}Ax_1&Ax_2&\cdots&Ax_n\end{bmatrix}$

To construct P & D we need the column vectors $x_1,x_2,\dots,x_n$ & real numbers $\lambda_1,\lambda_2,\dots,\lambda_n$

should be eigen vectors
should be eigen values

$Ax_1=\lambda_1x_1$
$Ax_2=\lambda_2x_2$
$\vdots$
$Ax_n=\lambda_nx_n$

So for the previous example

eigen values

$D=\begin{bmatrix}5&0&0\\0&3&0\\0&0&2\end{bmatrix}$

$P=\begin{bmatrix}0&1&4/5\\1&0&0\\0&1&1\end{bmatrix}$ ← columns correlate to eigen values in matrix D (matching eigen vector)

Eg. Compute Eigen values/vectors of:

$\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$ ① Find λ such that $|A-\lambda I|=0$
② for each λ, solve the system $(A-\lambda I)\cdot X=0$

$\left|\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}\right|=\begin{vmatrix}-\lambda&1&1\\1&-\lambda&1\\1&1&-\lambda\end{vmatrix}$

$|A-\lambda I|=a_{11}C_{11}(A-\lambda I)+a_{12}C_{12}(A-\lambda I)+a_{13}C_{13}(A-\lambda I)$

$=-\lambda\begin{vmatrix}-\lambda&1\\1&-\lambda\end{vmatrix}+1\begin{vmatrix}1&1\\1&-\lambda\end{vmatrix}+1\begin{vmatrix}1&-\lambda\\1&1\end{vmatrix}$

$=-\lambda(\lambda^2-1)-(-\lambda-1)+(1+\lambda)=0$

$=-\lambda(\lambda-1)(\lambda+1)+(\lambda+1)+(\lambda+1)=0$

$=(\lambda+1)(-\lambda(\lambda-1)+1+1)=0$

$=-(\lambda+1)(\lambda^2-\lambda-2)=0$

$=-(\lambda+1)^2(\lambda-2)=0$ $\Rightarrow\lambda=-1$ & $\lambda=2$

Eigen values:
$\lambda=-1$
$\lambda=2$

$\lambda=-1$
$X=s\begin{bmatrix}-1\\0\\1\end{bmatrix}+t\begin{bmatrix}-1\\1\\0\end{bmatrix}$

$\lambda=2$
$X=s\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\lambda=2$

$\left[\begin{array}{ccc|c}-2&1&1&0\\1&-2&1&0\\1&1&-2&0\end{array}\right]\Rightarrow\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{array}\right]$ $x_1=x_2$, $x_2=x_3$, $x_3=s$ $X=s\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\lambda=-1$

$\left[\begin{array}{ccc|c}1&1&1&0\\1&1&1&0\\1&1&1&0\end{array}\right]\to\left[\begin{array}{ccc|c}1&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right]$ $x_1=-x_2-x_3$, $x_2=s$, $x_3=t$ $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=s\begin{bmatrix}-1\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}$

Is A diagonalizable? (Can we find an invertible matrix P & a diagonal matrix D such that $A=PDP^{-1}$?)

$D=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&2\end{bmatrix}$ $P=\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}$

We write $\lambda=-1$ two times in the diagonal of D because it is repeated 2 times in the equation

$|A-\lambda I|=0$ ← characteristic equation
$-(\lambda+1)^2(\lambda-2)=0$

What can we do if A is diagnolizable?

$A=PDP^{-1}$

① Compute $|A|$

$|A|=|PDP^{-1}|$
$=|P||D||P^{-1}|$

$|A|=|D|$
$=$ Product of diagonal entries of D
$=$ Products of eigen values of A (including multiplicity)

for A in the above example
$|A|=-1\cdot-1\cdot2=2$

Eg. $A^{2025}=PD^{2025}P^{-1}$

$=\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&2\end{bmatrix}^{2025}\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}^{-1}$

☆ be able to compute this

② If $A=PDP^{-1}$, compute A^2 in terms of P & D?

$A^2=(PDP^{-1})(PDP^{-1})$
$=P\cdot D\cdot\cancel{P^{-1}}\cancel{P}\cdot D\cdot P^{-1}$
$A^2=P\cdot D^2\cdot P^{-1}$

D^2

$=\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&4\end{bmatrix}\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}^{-1}$

So when apply k power to A the eigen values are also take to the k power, but the eigen values stay the same.

If D is diagonal, then D^2 is also diagonal ← same for A
If the diagonal entries of D are $\lambda_1,\lambda_2,\lambda_3,\dots,\lambda_n$
then the diagonal entries of D^2 is $\lambda_1^2,\lambda_2^2,\lambda_3^2,\dots,\lambda_n^2$

If A is Invertable & diag.?

$A=PDP^{-1}$
$A^{-1}=(P^{-1})^{-1}D^{-1}P^{-1}$
$=PD^{-1}P^{-1}$

$\begin{bmatrix}1/2&0&\cdots&0\\0&1/2&&\vdots\\\vdots&&\ddots&\\0&\cdots&\cdots&0\end{bmatrix}$

Eigen values & Diagonalization Cont...

Another example

$A=\begin{bmatrix}6&3&-8\\0&-2&0\\1&0&-3\end{bmatrix}$ $A-\lambda I=\begin{bmatrix}6-\lambda&3&-8\\0&-2-\lambda&0\\1&0&-3-\lambda\end{bmatrix}$

$|A-\lambda I| = \overset{0}{a_{21}}C_{21}(A-\lambda I)+a_{22}C_{22}(A-\lambda I)+\overset{0}{a_{23}}C_{23}(A-\lambda I)$

$= (-2-\lambda)(-1)^{2+2}\begin{vmatrix}6-\lambda & -8\\ 1 & -3-\lambda\end{vmatrix}$

$= (-2-\lambda)((6-\lambda)(-3-\lambda)+8)$

$= (-2-\lambda)(-18-6\lambda+3\lambda+\lambda^2+8)$

$= (-2-\lambda)(\lambda^2-3\lambda-10)$

$= -(2+\lambda)(\lambda-5)(\lambda+2)$

eigen values: $\lambda = -2, -2, 5$

$\lambda = 5 \Rightarrow x = s\begin{bmatrix}8\\0\\1\end{bmatrix}$

$\lambda = -2 \Rightarrow x = s\begin{bmatrix}1\\0\\1\end{bmatrix}$

Only 2 solutions
So not enough eigen vectors to construct matrix P, so A is not diagonalizable

↳ If A is mxn there must be m eigen values (not unique) & there must be m UNIQUE eigen vectors

An nxn matrix A is diagonalizable if and only if it satisfies the conditions below:

For each eigen value λ of A if λ is repeated k-times in the characteristic equation then $(A-\lambda I)x=0$ should have k basic solutions.

Showing satisfied => have to show ALL are true not just examples
Showing not satisfied => Just need one example that doesn't satisfy

Ex. Show that

$\text{span}\overset{S}{\{u,v\}} = \text{span}\overset{T}{\{u,v,5u+2v\}}$ linear combination of first 2 vectors

take any $p\in S$ — take any $q\in T$

$p = r_1\cdot u + r_2\cdot v$
$= r_1u + r_2u + (0)(5u+2v)$
$p\in \text{span}\{u,v,5u+2v\}$
$S\subseteq T$

$q = t_1u+t_2v+t_3(5u+2v)$
$= t_1u+t_2v+t_3 5u+t_3 2v$
$= t_1t_3 5u + t_2t_3 3v$
$= (t_1t_3 5)u + (t_2t_3 3)v$
$\therefore q\in \text{span}\{u,v\}$
$T\subseteq S$

$S\subseteq T\subseteq S \Rightarrow S=T$

← should be at bottom

5.1 Subspaces of $\mathbb{R}^n$

$\mathbb{R}^n = \left\{\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} \middle| x_1,x_2,\dots,x_n\in\mathbb{R}\right\} \Rightarrow \mathbb{R}^n$ = column vector with n entries

A subset S of $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if:
1. $0 \in S$ zero vector of $\mathbb{R}^n$
2. If u & v are any 2 vectors in S then u+v is also in S
3. For any scaler 'r' & any vector u in S then $r\cdot u\in S$

Eg. S = all points on $y=x^2$
1. $0=\begin{bmatrix}0\\0\end{bmatrix}$ $0=0^2$ ✓
2. $u=\begin{bmatrix}x_1\\y_1\end{bmatrix}$ $v=\begin{bmatrix}x_2\\y_2\end{bmatrix}$

$u+v=\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix}\Rightarrow y_1+y_2=(x_1+x_2)^2$

$y_1+y_2=x_1^2+2x_1x_2+x_2^2$

$y_1+y_2\neq y_1+2x_1x_2+y_2$

$u=\begin{bmatrix}1\\1\end{bmatrix}$ $v=\begin{bmatrix}2\\4\end{bmatrix}$

$1=1^2$ $4=2^2$ $5\neq 3^2$

S is not a ⊆ of $\mathbb{R}^n$ as it fails rule number 2 & has the example $u=\begin{bmatrix}1\\1\end{bmatrix}$ $v=\begin{bmatrix}2\\4\end{bmatrix}$ failing case 2

S = Set of all points on the line $y=2x+5$

Eg. Is S a subset of $\mathbb{R}^n$?

We can write S as

$S=\left\{\begin{bmatrix}x\\y\end{bmatrix}\middle| y=2x+5\right\}$

$S\not\subseteq\mathbb{R}^n$ since it violates the first rule

1. is $0=\begin{bmatrix}0\\0\end{bmatrix}$ in S?
↳ NO $0\neq 2(0)+5$

Eg. S = all points on $y=3x$ $S\subseteq\mathbb{R}^n$

1. $0=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow 0=3(0) = 0=0$ ✓
↳ does not violate rule 1
2. $u=\begin{bmatrix}x_1\\y_1\end{bmatrix}\Rightarrow y_1=3x_1$
$v=\begin{bmatrix}x_2\\y_2\end{bmatrix}\Rightarrow y_2=3x_2$
$u+v=\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix}\Rightarrow y_1+y_2=3x_1+3x_2 = 3(x_1+x_2)$
↳ $u+v\in S$ ✓
we are able to express in for $y=3x$: $y=3(x)$, $x=x_1+x_2$
3. $u=\begin{bmatrix}x_1\\y_1\end{bmatrix}$ r = scaler
$r\cdot u=\begin{bmatrix}r\cdot x_1\\r\cdot y_1\end{bmatrix} = \underbrace{r\cdot y_1 = 3(r\cdot x_1)}_{\text{proper form}}$
↳ $r\cdot u\in S$ ✓

$\mathbb{R}^2$ = Straight lines through Origin (0,0)
$\mathbb{R}^3$ = Planes through (0,0,0), or lines through origin (doesn't have to be straight)

$S=\left\{\begin{bmatrix}x+y\\x-2y\\2y\end{bmatrix}\middle| x+y\in\mathbb{R}\right\}$

is $S\subseteq\mathbb{R}^3$?

Two trivial examples:
1. $S=\{0\}$ → smallest subset of $\mathbb{R}^n$
2. $S=\mathbb{R}^n$ → biggest subset of $\mathbb{R}^n$

Eg. S & T are subspaces of $\mathbb{R}^n$
1. Show that $S\cap T$ is also a subspace of $\mathbb{R}^n$ Intersection
2. Show $S\cup T$ is NOT a subspace of $\mathbb{R}^n$

=> 1. Is $0\in S\cap T$? ✓
we know S & T is a subspace
↳ $0\in S$ & $T \rightarrow 0\in S\cap T$

2. Take $u,v\in S\cap T$ ✓
is u+v in $S\cap T$
$u\in S$ & T, $v\in S$ & T => $u+v\in S$ & T
↳ $u+v\in S\cap T$

3. take r scaler & $u\in S\cap T$
is $r(u)\in S\cap T$
$r\cdot u\in S$ & $T \rightarrow r\cdot u\in S\cap T$

=> 1. ✓ 2. ✓ 3. example

$S=\left\{\begin{bmatrix}x\\y\end{bmatrix}\middle| y=2x\right\}$ $T=\left\{\begin{bmatrix}x\\y\end{bmatrix}\middle| y=3x\right\}$

$u=\begin{bmatrix}1\\2\end{bmatrix}\in S$
$v=\begin{bmatrix}1\\3\end{bmatrix}\in T$
=> u & $v\in S\cup T$

but, $u+v=\begin{bmatrix}2\\5\end{bmatrix}$
$u+v\notin S$ or T
$5\neq 4$ $5\neq 6$

S & $J\subseteq\mathbb{R}$

Also the complement of any subspace of $\mathbb{R}^n$ is not a subspace

Defn: A $\text{span}\{u_1,u_2,\dots,u_k\}$ = set of all the linear combinations of $u_1,u_2,u_3,\dots,u_k$ wich are $\in\mathbb{R}^n$

$=\left\{r_1\cdot u_1+r_2\cdot u_2+\dots+r_ku_k \middle| r_1,r_2,\dots,r_k \text{ are real numbers}\right\}$

Eg. is $\text{span}\{u_1,\dots,u_k\}\subseteq\mathbb{R}$?
1. Is $0\in\text{span}\{\}$? ✓
are there values for r where the linear combination is 0: $r_1\cdot u_1+\dots+r_ku_k=0$
if $r_1=r_2=\dots=r_k=0$ then yes

2. is $u+v\in\text{span}\{..\}$?
$u,v\in\text{span}\{..\}$
$u=r_1u_1+r_2u_2+\dots+r_ku_k$
$v=t_1v_1+t_2u_2+\dots+t_kv_k$
$u+v=(r_1+t_1)u_1+(r_2+t_2)u_2+\dots(r_kt_k)u_k$

So u+v = a linear combination of $u_1,u_2,\dots,u_k$
$u+v\in\text{span}\{u_1,\dots,u_k\}$

3. is $r\cdot u\in\text{span}\{\dots\}$
t is a scaler
$u=r_1u_1+r_2u_2+\dots+r_ku_k$
$t\cdot u=(tr_1)u_1+(tr_2)u_2+\dots+(tr_k)u_k$
so, $t\cdot u$ = a linear combination on $u_1,u_2,u_3,\dots,u_k \Rightarrow t\cdot u\in\text{span}\{\dots\}$

Theorem 1:

Consider the set of vectors $u_1,u_2,\dots,u_k$

If we change one vector by
$u_i\rightarrow u_i+ku_j$
or
$u_i\rightarrow ku_j$
or
$u_i\leftrightarrow u_j$

The span does not change

$\text{span}\{u_1,\dots,u_i,\dots,u_k\}=\text{span}\{u_1,\dots,u_i+ku_j,\dots,u_k\}=\text{span}\{u_1,\dots,ku_i,\dots,u_k\}$

Ex. u, v, w are vectors of $\mathbb{R}^n$

Show that $\text{span}\overset{S}{\{u,v,w\}}=\text{span}\overset{T}{\{u,v,w+5u\}}$

Take an vector $p\in S$

$p=r_1u+r_2v+r_3w$
$=r_1u+r_2v+r_3(w+5u)-5r_3u$
$=(r_1-5r_3)u+r_2v+r_3(w+5u)$
$p\in T\rightarrow S\subseteq T$

now take any vector $q\in T$

$q=t_1\cdot u+t_2v+t_3(w+5u)$
$=t_1u+t_2v+t_3w+5t_3u$
$=(t_1+5t_3)\cdot u+t_2v+t_3w$
$q\in S\rightarrow T\subseteq S$

$S\subseteq T\subseteq S \Rightarrow S=T$

Since they are subsets of each other they are equal

now with $\text{span}\overset{T}{\{u,7v,w\}}$

take any vector p in S — any vector q in T

$p=r_1u+r_2v+r_3w$
$=r_1u+\frac{r_2}{7}(7v)+r_3w$ matches form of linear combination of T
$p\in T\rightarrow S\subseteq T$

$q=t_1u+t_27v+t_3w$
$=t_1u+7t_2v+t_3w$ matches form of linear combination of S
$q\in S\rightarrow T\subseteq S$

$S\subseteq T\subseteq S\rightarrow S=T$

Definition: Let S be a subspace of $\mathbb{R}^n$

If $S=\text{span}\{u_1,u_2,\dots,u_k\}$
we say, the set $\{u_1,\dots,u_k\}$ is a spanning set of S

Eg. $S=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\middle| y=0\right\}$ $S\subseteq\mathbb{R}^n$

$\begin{bmatrix}x\\y\\z\end{bmatrix}=u\in S$ & $y=0$

$u=\begin{bmatrix}x\\0\\z\end{bmatrix}=x\begin{bmatrix}1\\0\\0\end{bmatrix}+z\begin{bmatrix}0\\0\\1\end{bmatrix}$

$u\in\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

↳ $S\subseteq\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

=> $v\in\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

$v=r_1\begin{bmatrix}1\\0\\0\end{bmatrix}+r_2\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}r_1\\0\\r_2\end{bmatrix}$

↳ $\begin{bmatrix}r_1\\0\\r_2\end{bmatrix}\in S$ because $y=0$

=> $\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\subseteq S$

so

$S=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\middle| y=0\right\}=\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ not unique

Eg. S = all points on the plane $3x-2y+z=0$. Show $S\subseteq\mathbb{R}^3$

$S=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\middle| 3x-2y+z=0\right\}$

$\begin{bmatrix}x\\y\\z\end{bmatrix}=u\in S$ where $3x-2y+z=0$

$3x-2y+z=0 \Rightarrow z=-3x+2y$

$u=\begin{bmatrix}x\\y\\-3x+2y\end{bmatrix}=x\begin{bmatrix}1\\0\\-3\end{bmatrix}+y\begin{bmatrix}0\\1\\2\end{bmatrix}$

$u\in\text{span}\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$

$S\subseteq\text{span}\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$

$v\in\text{span}\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$

$v=r_1\begin{bmatrix}1\\0\\-3\end{bmatrix}+r_2\begin{bmatrix}0\\1\\2\end{bmatrix}$

$=\begin{bmatrix}r_1\\r_2\\-3r_1+2r_2\end{bmatrix}$

$=3x-2y+z$

$v\in S\Rightarrow\text{span}\left\{\begin{bmatrix}1\\0\\-3\end{bmatrix},\begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$

Subspace cont...

Linear independence/dependence

Consider a set of vectors in $\mathbb{R}^n$

say $\{u_1, u_2, \dots, u_k\}$

1. If one vector in this set can be written as a linear combination of others, we say the set is linearly dependent

An equivalent definition:

If we can find scalars $r_1, r_2, \dots, r_k$ (not all of which are zero) such that,...

$r_1u_1 + r_2u_2 + \dots + r_ku_k = 0$ then we say the set of vectors $u_1, \dots, u_k$ is linearly dependent

If NOT, the only way for $r_1u_1 + r_2u_2 + \dots + r_ku_k = 0$ is if $r_1 = r_2 = \dots = r_k = 0$. If this is the case then the set is linearly independent

$\Rightarrow$ Since not all scalars are 0) say $r_1 \neq 0$

$$u_1 + \left(\frac{r_2}{r_1}\right)u_2 + \left(\frac{r_3}{r_1}\right)u_3 + \dots + \left(\frac{r_k}{r_1}\right)u_k = 0$$

$$u_1 = \left(\frac{-r_2}{r_1}\right)u_2 + \left(\frac{-r_3}{r_1}\right)u_3 + \dots \left(\frac{-r_k}{r_1}\right)u_k$$

↳ u_1 is a linear combination of other vectors

Special case: when A is a square matrix if

$\det(A) \neq 0 \Rightarrow$ linearly independent

$\det(A) = 0 \Rightarrow$ linearly dependent

Eg. $\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\0\\6\end{bmatrix}, \begin{bmatrix}4\\5\\1\end{bmatrix}\right\} \Rightarrow \begin{bmatrix}1&2&4\\0&0&5\\1&6&1\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ ⇓ linearly independent

$|A| = a_{32}C_{32}(A) = 5\cdot(-1)^5\begin{vmatrix}1&2\\1&6\end{vmatrix} = -5(6-2) = -20 \neq 0 \Rightarrow$ linearly independent

Eg. Determine if this set is linearly independent

$\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}4\\4\end{bmatrix}\right\} \Rightarrow \left[\begin{array}{ccc|c}1&1&4&0\\2&0&4&0\end{array}\right]$

↳ $\left[\begin{array}{ccc|c}1&0&2&0\\0&1&2&0\end{array}\right]$ $r < n$ so infinitely many sol^n ↳ linearly dependent

↳ Since its 2×3 its not possible for $r = n$ so matrices of size $m\times n$ if $m < n$ it will always either be infiniteley many solutions or no solution, since it a homogenous system it will always have infinite solutions → always linearly dependent

Eg. Show that if a set has the zero vector then it is linearly dependent.

$\{0, u_2, \dots, u_k\} \Rightarrow A = \begin{bmatrix}0&\uparrow&\uparrow&&\uparrow\\ \vdots&u_2&u_3&\dots&u_n\\0&\downarrow&\downarrow&&\downarrow\end{bmatrix}$

Since a row will always be 0 rank will always be less than the number of columns

∴ if the zero matrix is present it will always be linearly dependent

↳ Because of this $S = \{0\}$ does NOT have a basis

S is a subset of $\mathbb{R}^n$. the set of vectors $\{u_1, u_2, \dots, u_k\}$ in S is called a basis of S if:

1. If $S = \text{span}\{u_1, u_2, \dots, u_k\}$
2. If $\{u_1, u_2, \dots, u_k\}$ is linearly independent

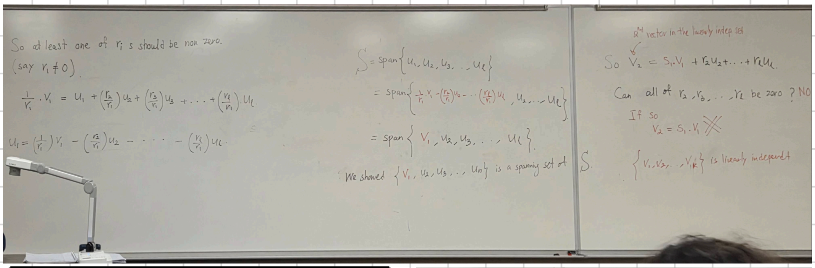
Theorem 2:

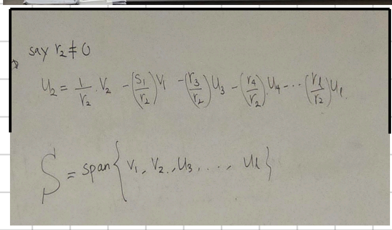
S is a subspace of $\mathbb{R}^n$ if...

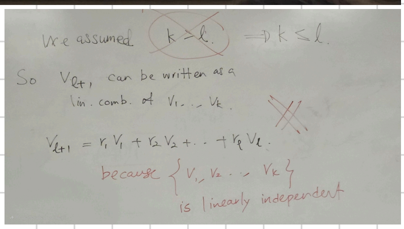
1. $S = \text{span}\{u_1, u_2, \dots, u_n\}$

and

2. $\{v_1, v_2, \dots, v_k\}$ is a linearly independent set of vectors in S







Eg. Determine if the set of vectors $\left\{\begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\3\\0\end{bmatrix}\right\}$ is linearly independent or dependent

$r_1\begin{bmatrix}1\\2\\1\\1\end{bmatrix} + r_2\begin{bmatrix}0\\1\\1\\1\end{bmatrix} + r_3\begin{bmatrix}1\\3\\3\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix}1&0&1\\2&1&3\\1&1&3\\1&1&0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix}}_{x} = \underbrace{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}}_{b}$

Does the linear system $Ax = b$ only have the trivial solution or does it have non-trivial solutions?

trivial $\Rightarrow$ Set is linearly independent

non-trivial $\Rightarrow$ Set is linearly dependent

$\left[\begin{array}{ccc|c}1&0&1&0\\2&1&3&0\\1&1&3&0\\1&1&0&0\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}\boxed{1}&0&0&0\\0&\boxed{1}&0&0\\0&0&\boxed{1}&0\\0&0&0&0\end{array}\right]$

rank = number of variables

$r = 3 = n \Rightarrow$ unique solution $\Rightarrow$ linearly independent

Eg. $\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\3\\1\\0\end{bmatrix}\right\} \Rightarrow \left[\begin{array}{ccc|c}1&0&2&0\\1&1&3&0\\0&1&1&0\\0&0&0&0\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c}1&0&\boxed{2}&0\\0&1&\boxed{1}&0\\0&0&0&0\\0&0&0&0\end{array}\right]$ $\Rightarrow$ Free variable $r < n$ linearly dependent

$x_1 = -2x_3$
$x_2 = -x_3$
$x_3 = s$
$\Rightarrow \begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix} = s\begin{bmatrix}-2\\-1\\1\end{bmatrix}$

The following statements are all equivalent:

1. A has an inverse
2. Columns of A are linearly independent
3. Rows of A are linearly independent

Eg. Biggest subset of $\mathbb{R}^n$ is $\mathbb{R}^n$, prove it

$e_1 = \begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix} e_2 = \begin{bmatrix}0\\1\\ \vdots\\0\end{bmatrix} \dots e_n = \begin{bmatrix}0\\0\\ \vdots\\1\end{bmatrix}$ show $\{e_1, e_2, e_3, \dots, e_n\}$ is a basis for $\mathbb{R}^n$

1) Show $\mathbb{R}^n = \text{span}\{e_1, e_2, \dots, e_n\}$

Take any vector in $\mathbb{R}^n$

$\begin{bmatrix}x_1\\x_2\\x_3\\ \vdots\\x_n\end{bmatrix} = (x_1)\cdot e_1 + (x_2)\cdot e_2 + (x_3)e_3 + \dots + (x_n)e_n$

∴ $\begin{bmatrix}x_1\\ \vdots\\x_n\end{bmatrix} \in \text{span}\{e_1, e_2, \dots, e_n\} \Rightarrow \mathbb{R}^n \subseteq \text{span}\{e_1, \dots, e_n\}$

$\text{span}\{e_1, \dots, e_n\} \subseteq \mathbb{R}^n \Rightarrow \mathbb{R}^n = \text{span}\{e_1, \dots, e_n\}$

2) Show $\{e_1, \dots, e_n\}$ is linearly independent

$A = \begin{bmatrix}1&0&0&\dots&0\\0&1&0&&0\\0&0&1&&\vdots\\ \vdots&&&\ddots&\\0&0&0&\dots&1\end{bmatrix} = I_n \Rightarrow \det(I_n) = 1 \neq 0$ ∴ linearly dependent

So, we call the set $\{e_1, e_2, \dots, e_n\}$ as the standard basis of $\mathbb{R}^n$

Eg. $S = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \middle| y = 0\right\} \Rightarrow S = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

Show $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is linearly independent

$A = \begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix} \Rightarrow \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$ $r = n$ so $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is a basis for S

S is a subspace of $\mathbb{R}^n$

Any 2 basis of S should have the same number of vectors.

↳ This number is called the dimension of S

↳ denoted by: $\dim(S)$

Proof: lets say $B_1 = \{u_1, u_2, \dots, u_l\}$

$B_2 = \{v_1, v_2, \dots, v_k\}$ are 2 basis of S

Need to show $l = k$

B_1 spans S ; B_1 is linearly independent

B_2 spans S ; B_2 is lin indp

Theorem 2 says $k \leq l$

Theorem 2 says $l \leq k$

$k \leq l \leq k \Rightarrow k = l$

For any positive 'n' $\dim(\mathbb{R}^n) = n$

$S = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \middle| y = 0\right\} \Rightarrow \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is the basis of S

$\dim(S) = 2$

Eg. $S = \left\{\begin{bmatrix}x\\y\\z\\w\end{bmatrix} \middle| \begin{matrix}x = z + w\\y = z - w\end{matrix}\right\}$ find $\dim(S)$

Take any vector in S

$u = \begin{bmatrix}x\\y\\z\\w\end{bmatrix}$ $\begin{matrix}x = z + w\\y = z - w\end{matrix} \Rightarrow u = \begin{bmatrix}z+w\\z-w\\z\\w\end{bmatrix} = z\begin{bmatrix}1\\1\\1\\0\end{bmatrix} + w\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}$

$u \in \text{span}\left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\right\}$

↳ $S \subseteq \text{span}\left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\right\}$

On the other hand, take any vector in $\text{span}\left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\right\}$

$u = r_1\begin{bmatrix}1\\1\\1\\0\end{bmatrix} + r_2\begin{bmatrix}1\\-1\\0\\1\end{bmatrix} = \begin{bmatrix}r_1+r_2\\r_1-r_2\\r_1\\r_2\end{bmatrix}$

$\Rightarrow \text{span}\left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\right\} \subseteq S$

∴ $S = \text{span}\left\{\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\right\}$

$\begin{bmatrix}1&1\\1&-1\\1&0\\0&1\end{bmatrix} \rightarrow \begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix} \rightarrow r = n \Rightarrow$ linearly independent

$\dim(S) = 2$

Eg. Show that $B = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^n$

1. Show B is a spanning set

Take any vector in $\mathbb{R}^2$

$\begin{bmatrix}a\\b\end{bmatrix} = (r_1)\begin{bmatrix}1\\2\end{bmatrix} + (r_2)\begin{bmatrix}2\\3\end{bmatrix} \Rightarrow \begin{matrix}a = r_1 + 2r_2\\b = 2r_1 + 3r_2\end{matrix}$ $\begin{matrix}r_1 = 2b - 3a\\r_2 = 2a - b\end{matrix}$

2. $A = \begin{bmatrix}1&2\\2&3\end{bmatrix} \Rightarrow |A| = -1 \neq 0 \Rightarrow$ lin. indp.

Theorem 3:

S is a subspace of $\mathbb{R}^n$, lets say $\dim(S) = k$

Lets say $\{u_1, u_2, \dots, u_k\}$ is a set of vectors S

1. If this set is lin. indp. then it is a basis for S
2. If this set is a spanning set of S then it is a basis for S

Eg. $S = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \middle| y = 0\right\}$, show $\left\{\begin{bmatrix}7\\0\\1\end{bmatrix}, \begin{bmatrix}6\\0\\5\end{bmatrix}\right\}$ is a basis for S

$u_1 = \begin{bmatrix}7\\0\\1\end{bmatrix} \in S$

$u_2 = \begin{bmatrix}6\\0\\5\end{bmatrix} \in S$

linear independent?

$\begin{bmatrix}7&6\\0&0\\1&5\end{bmatrix} \rightarrow \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} \Rightarrow r = n$ ∴ lin. indp.

due to theorem 3 it is a basis

Eg. S & T are subspaces of $\mathbb{R}^n$ s.t. $S \subseteq T$

1. Show that $\dim(S) \leq \dim(T)$

if $B_1 = \{u_1, \dots, u_l\}$ is a basis of T

$B_2 = \{v_1, \dots, v_k\}$ " " " " S

Need to show $k \leq l$

B_2 is linearly independent set of vectors in $S \subseteq T$

$B_2 = \{v_1, \dots, v_k\}$ is a lin. indep. set of vectors in C

$B_1 = \{u_1, \dots, u_l\}$ is a spanning set of C

Theorem 2 says $k \leq l \Rightarrow \dim(S) \leq \dim(T)$

2. If $\dim(S) = \dim(T)$ then $S = T$

↳ Say $\dim(S) = \dim(T) = k$ & $B = \{u_1, \dots, u_k\}$ is a basis of $S = \text{span}\{u_1, \dots, u_k\}$

and $\{u_1, \dots, u_k\}$ is a set of linearly independent vectors in $S \subseteq T$

so $\{u_1, \dots, u_k\}$ is a set of linearly independent vectors in T

$\Rightarrow$ By Theorem 3 $\{u_1, u_2, \dots, u_k\}$ is a basis of T

as $T = \text{span}\{u_1, \dots, u_k\} = S \Rightarrow T = S$

We don't need to have the theorems memorized, but we do need to know how to use them in order to complete a proof.

5.4 Rank of a matrix

A is an $m\times n$ matrix

Row space of A = Row(A) = span{rows of A}

Row(A) is a subspace of $\mathbb{R}^n$

Eg. Find a basis for Row(A)

$$A=\begin{bmatrix}2&-4&6&8\\2&-1&3&2\\4&-5&9&10\\0&-1&1&2\end{bmatrix}\begin{array}{l}R_1\\R_2\\R_3\\R_4\end{array}\xrightarrow[\text{operations}]{\text{row}}\begin{bmatrix}1&0&1&0\\0&1&-1&-2\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Don't add anything, so can be removed

$\text{Row}(A)=\text{span}\{R_1,R_2,R_3,R_4\}=\text{span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\\-2\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right\}$

A basis for Row(A) is: $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\\-2\end{bmatrix}\right\}$ so $\dim(A)=2$

$\longrightarrow$ A basis for Row(A) = Non-zero rows of the RREF of A

$\dim(\text{Row}(A))=\text{Rank}(A)$

Non-zero rows in a matrix which is in the RRE form are linearly independent

A is $m\times n$

Column space of A = col(A)

= span{columns of A} is a subspace of $\mathbb{R}^m$

Eg.
$$\begin{array}{c}\begin{array}{cccc}C_1&C_2&C_3&C_4\end{array}\\ A=\begin{bmatrix}2&-4&6&8\\2&-1&3&2\\4&-5&9&10\\0&-1&1&2\end{bmatrix}\end{array}\xrightarrow{R_2\to R_2+(-1)R_1}\begin{array}{c}\begin{array}{cccc}D_1&D_2&D_3&D_4\end{array}\\ \begin{bmatrix}2&-4&6&8\\0&3&-3&-6\\4&-5&9&10\\0&-1&1&2\end{bmatrix}\end{array}$$

$\text{col}(A)=\text{span}\{C_1,\dots,C_4\}\neq\text{span}\{D_1,\dots,D_4\}$

Column space changes under row operations

↳ A basis for col(A) = $\left\{\begin{bmatrix}2\\2\\4\\0\end{bmatrix},\begin{bmatrix}-4\\-1\\-5\\-1\end{bmatrix}\right\}\Rightarrow\dim(\text{col}(A))=\text{Rank}(A)=2$

→ If the RREF has leading 1's in column $j_1,j_2,\dots,j_r$ then corresponding columns of A forms a basis for col(A)

basis = bare minimum

spanning set = set of vector, not the bare minimum to represent a set

Eg. Find a basis for col(A) & row(A)

$$A=\begin{bmatrix}2&-1&1\\-2&1&1\\4&-2&3\\-6&3&0\end{bmatrix}\to\begin{bmatrix}1&-1/2&0\\0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

$\Rightarrow$ A basis for Row(A) = $\left\{\begin{bmatrix}1\\-1/2\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

$\Rightarrow$ A basis for Col(A) = $\left\{\begin{bmatrix}2\\-2\\4\\-6\end{bmatrix},\begin{bmatrix}1\\1\\3\\0\end{bmatrix}\right\}$

$\dim(A)=2$

The column space of a matrix can be used to truncate a spanning set of a subspace to a basis.

Eg. S is a subspace of $\mathbb{R}^4$ & $\left\{\begin{bmatrix}1\\-1\\0\\3\end{bmatrix},\begin{bmatrix}2\\1\\5\\1\end{bmatrix},\begin{bmatrix}4\\-1\\5\\7\end{bmatrix}\right\}$ (u_1, u_2, u_3) is a spanning set of S. Truncate this set to a basis of S. $S=\text{span}\{u_1,u_2,u_3\}$

take
$$A=\begin{bmatrix}1&2&4\\-1&1&-1\\0&5&5\\3&1&7\end{bmatrix}\to\begin{bmatrix}1&0&2\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}$$
(u_1, u_2, u_3) $\Rightarrow$ A basis for S = span col(A) = $\left\{\begin{bmatrix}1\\-1\\0\\3\end{bmatrix},\begin{bmatrix}2\\1\\5\\1\end{bmatrix}\right\}$

A is $m\times n$

Null space of A = All the solutions to the linear system $Ax=0$ (zero vector in $\mathbb{R}^m$)

$=\{x\in\mathbb{R}^n \mid A\cdot x=0\}\subseteq\mathbb{R}^n$

Show that Null(A) is a subspace of $\mathbb{R}^n$

① Is O_n (zero vector in $\mathbb{R}^n$) in null(A)

↳ $A\cdot O_n=O_m$

↳ ∴ $O_n\in\text{Null}(A)$

② Take $u,v\in\text{Null}(A)$, is $u+v\in\text{Null}(A)$?

↳ we know $u\in\text{Null}(A)\Rightarrow A\cdot u=O_m$, $v\in\text{Null}(A)\Rightarrow A\cdot v=O_m$

↳ So, $A\cdot(u+v)=\underbrace{A\cdot u}_{O_m}+\underbrace{A\cdot v}_{O_m}=O_m$

↳ ∴ $u+v\in\text{Null}(A)$

③ Take scalar s & $u\in\text{Null}(A)$, is $s\cdot u\in\text{Null}(A)$?

↳ We know $A\cdot u=O_m$

↳ So $A\cdot(s\cdot u)=s(A\cdot u)=s\cdot O_m=O_m$

↳ ∴ $s\cdot u\in\text{Null}(A)$

A is $m\times n$

Find a basis for Null(A)

$$Ax=O=\left[\begin{array}{cccc|c}2&-4&6&8&0\\2&-1&3&2&0\\4&-5&9&10&0\\0&-1&1&2&0\end{array}\right]=\left[\begin{array}{cccc|c}1&0&1&0&0\\0&1&-1&-2&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right]\quad\begin{array}{l}x_1=-x_3\\x_2=x_3+2x_4\\x_3=s\\x_4=t\end{array}$$

$x=s\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}+t\begin{bmatrix}0\\2\\0\\1\end{bmatrix}\Rightarrow\left\{\begin{bmatrix}-1\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\1\end{bmatrix}\right\}$

$\text{Null}(A)=\text{span}\left\{\begin{bmatrix}-1\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\1\end{bmatrix}\right\}$

we can show is linearly independent by:

$s\begin{bmatrix}-1\\1\\1\\0\end{bmatrix}+t\begin{bmatrix}0\\2\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\Rightarrow\begin{bmatrix}-s\\s+2t\\s\\t\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

Since this is only true if $s,t=0$ we know this is linearly independent

$\Longrightarrow$ ∴ A basis for Null(A) = $\left\{\begin{bmatrix}-1\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\1\end{bmatrix}\right\}$

A basis for Null(A) = The set of basic solutions to $Ax=0$

↳ So the dim(Null(A)) = number of free variables

$= \underset{\text{number of col's}}{n} - \underset{\text{num of leading 1s}}{\text{Rank}(A)}$

A is $m\times n$

The null space is just the solution to a homogeneous system $Ax=0$

Summary of things above

A is $m\times n$, Rank(A) = r

	dimension	a basis
Row(A) $\subseteq\mathbb{R}^n$	r	Non zero rows in RRE
Col(A) $\subseteq\mathbb{R}^m$	r	Columns of A corresponding to leading 1s
Null(A) $\subseteq\mathbb{R}^n$	$n-r$	Basic solutions to $Ax=0$

let $u,v\in\mathbb{R}^n$

$u=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$ $v=\begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix}$

Dot Product between u & v = $x_1y_1+x_2y_2+\dots+x_ny_n=\langle u,v\rangle$

notation for dot product

For any $u\in\mathbb{R}^n$

$\langle u,u\rangle=x_1^2+x_2^2+\dots+x_n^2\geq 0$

if $\langle u,u\rangle=0$, then $x_1^2+\dots+x_n^2=0$

↳ $x_1=x_2=\dots=x_n=0$

↳ $u=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$

If $\langle u,v\rangle=0$ we say u is orthogonal to v

Properties: ① $\langle r\cdot u,v\rangle=r\cdot\langle u,v\rangle$ for any scalar r

② $\langle u+v,w\rangle=\langle u,w\rangle+\langle v,w\rangle$

③ $\langle u,v\rangle=\langle v,u\rangle$

④ $\langle u,0\rangle=0$

S is a subspace of $\mathbb{R}^n$

$S^\perp$ = Set of all the vectors in $\mathbb{R}^n$, orthogonal to any vector in S.

Orthogonal complement of S

↳ $=\{x\in\mathbb{R}^n \mid \langle x,u\rangle=0$ for any u in $S\}$

$S^\perp$ is a subspace of $\mathbb{R}^n$

① Is O_n in $S^\perp$?

↳ Take any vector in S say u:

↳ $\langle O_n,u\rangle=0$

② Take any two vectors in $S^\perp$ say X_1 & X_2, is X_1+X_2 is $S^\perp$?

↳ $X_1\in S^\perp\to\langle x_1,u\rangle=0$, $X_2\in S^\perp\to\langle x_2,u\rangle=0$ for any u in S

↳ $\langle x_1+x_2,u\rangle=\underbrace{\langle x_1,u\rangle}_{O_n}+\underbrace{\langle x_2,u\rangle}_{O_n}$

↳ ∴ $X_1+X_2\in S^\perp$

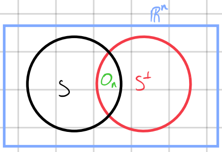
③ Take scalar r & $u,v\in S^\perp$, is $r\cdot u\in S^\perp$

↳ $u\in S^\perp\to\langle u,v\rangle=0$

↳ $\langle r\cdot u,v\rangle=r\cdot\langle u,v\rangle=r\cdot 0=0$

↳ ∴ $r\cdot u\in S^\perp$

$S\cap S^\perp=\{0\}$



$S\cap S^\perp=\{O_n\}$

lets say $x\in S\cap S^\perp$

↳ $x\in S$, $x\in S^\perp\Rightarrow\langle x,x\rangle=0$

↳ $X=O_n$

Eg. S & T are two subspaces of $\mathbb{R}^n$

$S+T=\{u+v \mid u\in S, v\in T\}$

Show that $S+T$ is a subspace of $\mathbb{R}^n$

Any vector in $S+T$ can be written as an addition of a vector in S and a vector in T

① Is O_n is $S+T$

$O_n=O_n+O_n$

$O_n\in S+T$

② Take p & $q\in S+T$, is $u+v\in S+T$?

$p=u_1+v_1$, $u_1\in S$, $v_1\in T$

$q=u_2+v_2$, $u_2\in S$, $v_2\in T$

$p+q=u_1+v_1+u_2+v_2$

↳ $p+q=(u_1+u_2)+(v_1+v_2)$

③ Take scalar $r\in\mathbb{R}$, & $w\in S+T$, is $rw\in S+T$

$w=u+v$ $rw=ru+rv$ $ru\in S$, $rv\in T$

so $rw\in S+T$

$$A\cdot x=\begin{bmatrix}R_1\\R_2\\\vdots\\R_m\end{bmatrix}\cdot\begin{bmatrix}X\end{bmatrix}=\begin{bmatrix}\langle R_1,x\rangle\\\langle R_2,x\rangle\\\vdots\end{bmatrix}$$

Dot product between rows of A & column vector X

so $A\cdot x=\begin{bmatrix}\langle R_1,x\rangle\\\langle R_2,x\rangle\\\vdots\\\langle R_m,x\rangle\end{bmatrix}$

If $x\in\text{Null}(A)$

$Ax=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$

$\begin{bmatrix}\langle R_1,x\rangle\\\vdots\\\langle R_m,x\rangle\end{bmatrix}=\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix}$

Dot product between a vector in Null(A) and a row of A is always zero

$\to$ Null(A) = (Row(A))$^\perp$

Their Dot product = 0

A is $m\times n$ & rank(A) = r

& let $\{u_1,u_2,\dots,u_r\}$ be a basis of Row(A)

let $\{v_1,v_2,\dots,v_{n-r}\}$ be a basis of Null(A)

① Show that the set is linearly independent

$\{u_1,u_2,\dots,u_r,v_1,v_2,\dots,v_{n-r}\}$

has $r+n-r$ vectors $\Rightarrow n$ vectors in this set

$s_1u_1+s_2u_2+\dots+s_ru_r+t_1v_1+t_2v_2+\dots+t_{n-r}v_{n-r}$

$P=\underbrace{s_1u_1+\dots+s_ru_r}_{\text{a vector in row}(A)}=\underbrace{-t_1v_1-\dots-t_{n-r}v_{n-r}}_{\text{a vector in null}}$

so $P\in\text{Row}(A)\cap\text{Null}(A)\Rightarrow$ so $P=0$

Since the set $\{u_1,u_2,\dots,u_r,v_1,v_2,\dots,v_{n-r}\}$ is the same dimension as $\mathbb{R}^n$ (n) & it is linearly independent we know it is a basis of $\mathbb{R}^n$ by Theorem 3

$$A=\begin{bmatrix}2&-4&6&8\\2&-1&3&2\\4&-5&9&10\\0&-1&1&2\end{bmatrix}$$

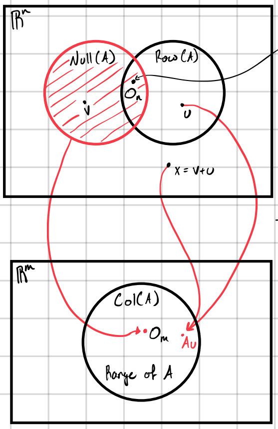
A basis for row(A) = $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\\-2\end{bmatrix}\right\}$

A basis for Null(A) = $\left\{\begin{bmatrix}-1\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\1\end{bmatrix}\right\}$

A basis for $\mathbb{R}^4$ is: $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\\-2\end{bmatrix},\begin{bmatrix}-1\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\2\\0\\1\end{bmatrix}\right\}$

$\Rightarrow\mathbb{R}^n=\text{Row}(A)+\text{Null}(A)$

$\Rightarrow$ In other words, for any X in $\mathbb{R}^n$, $X=U+V$ where $U\in\text{Row}(A)$, $V\in\text{Null}(A)$



only point in the intersection

$V=O_n+V$

$U=O_n+U$

V is in Null(A)

$T(v)=A\cdot v=O_m$

if $X=U+V$ $\quad U\in\text{Row}(A)$, $V\in\text{Null}(A)$

$T_A(x)=A(x)$

$=A(u+v)$

$=Au+\underbrace{Av}_{\to O_m}$

$=Au$

6.1 Vector Spaces

A vector space is a set that is "similar" to $\mathbb{R}^n$

- A set with two operations
  - + addition, $\cdot$ scalar multiplication
  - These 2 operations behave similar to addition & scalar multiplication in $\mathbb{R}^n$

A vector space is a set V with 2 operations that satisfies the following axioms:

Let u, v, w be any three objects in V

(A.1) $u+v \in V$
(A.2) $u+v = v+u$
(A.3) $(u+v)+w = u+(v+w)$
(A.4) There is an object in V called the "zero vector" and denoted by 0_V s.t.: $u + 0_V = u$ for any $u \in V$
(A.5) For any $u \in V$ there is a vector called "negative of u" such that $u + (-u) = 0_V$

A non-example:

① V = Set of all matricies $\rightarrow u = [1]$ $v = \begin{bmatrix}1&2\\2&4\end{bmatrix}$ $u+v$ is not possible so $u+v \notin V$

So "set of all matricies" does not fit as some matricies cannot be added

"V = Set of all matricies of a specific size" is true

more rules: r & s are scalars

$(S.1)$ $s \cdot u \in V$ for any scalar s
$(S.2)$ $s \cdot (u+v) = s \cdot u + s v$
$(S.3)$ $(s+r) \cdot u = s \cdot u + r \cdot u$
$(S.4)$ $(s \cdot r) \cdot u = s \cdot (r \cdot u)$
$(S.5)$ $1 \cdot u = u$

Examples:

① $V = \mathbb{R}^n$
② V = Set of $m \times n$ matricies $= M_{m,n}$
③ P_n = Set of polynomials with degree at most n
④ $V = \mathbb{R}^+$ = Set of all positive numbers

	Zero vectors in A.4
$M_{2,2}$	$0 = \begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\Rightarrow$ Show $V = M_{2,2}$ is a vector space

Take $u = \begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$ $v = \begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}$ $w = \begin{bmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{bmatrix}$

A.1 $\rightarrow$ $u+v = \begin{bmatrix}a_{11}+b_{11}&a_{12}+b_{12}\\a_{21}+b_{21}&a_{22}+b_{22}\end{bmatrix}$ is a 2×2 matrix so $u+v \in M_{2,2}$

A.2 $u+v = v+u$ } yea
A.3 } too lazy to write rn

A.4 Take $0_V = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ $u + 0_V = u$

A.5 $-u = \begin{bmatrix}-a_{11}&-a_{12}\\-a_{21}&-a_{22}\end{bmatrix} \Rightarrow u + (-u) = 0$

addition given by $x+y = xy$
Scalar mult given by $r \cdot x = x^r$ where r is a scalar

More examples of vector spaces:

① P_n = Polynomials with degree at most 'n'

if $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$
$q(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n$
$(p+q)(x) = (a_0+b_0) + (a_1+b_1)x + \ldots + (a_n+b_n)x^n$
if r is a scalar
$(r \cdot p)(x) = (r \cdot a_0) + (r \cdot a_1)x + \ldots + (r \cdot a_n)x^n$

Check if P_n with addition and scalar mult as described to the left is a vector space.

(A.1) $p+q$ is defined as a polynomial with degree at most n, so $p+q \in P_n$

(A.2)
(A.3) } Order of addition is irrelevant

(A.4) Take the zero vector as:
$O(x) = 0 + 0x + \ldots + 0x^n$
$= 0$ for all x

(A.5) $-P(x) = -a_0 - a_1 x - \ldots - a_n x^n$
$P(x) + (-P(x)) = O(x)$

(S.1) $(r \cdot p)(x) = (r \cdot a_0) + (r \cdot a_1)x + \ldots + (r \cdot a_n)x^n$
$r \cdot p$ is a polynomial with degree at most 'n'
So $r \cdot p \in P_n$

② $V = \mathbb{R}^n$ $\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} + \begin{bmatrix}y_1\\\vdots\\y_n\end{bmatrix} = \begin{bmatrix}x_1+y_1\\\vdots\\x_n+y_n\end{bmatrix}$

$r \cdot \begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} = \begin{bmatrix}r \cdot x_1\\\vdots\\r \cdot x_n\end{bmatrix}$ zero vector in (A.4) $= \begin{bmatrix}0\\\vdots\\0\end{bmatrix}$

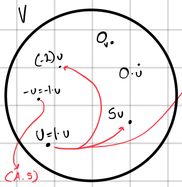
③ $M_{m,n}$ = Set of all matricies of size $m \times n$ has usual addition & scalar multiplication

zero vector = zero matrix of size $m \times n$ $\begin{bmatrix}0&0&\cdots&0\\0&0&&\\\vdots&&\ddots&\vdots\\0&\cdots&&0\end{bmatrix}$

④ $V = \mathbb{R}^+$ $x+y = xy$ $r \cdot x = x^r$
Homework question zero vector = 1

⑤ $V = \mathbb{C}$ = Set of all the complex numbers $z_1 = x_1 + i y_1$, $z_2 = x_2 + i y_2$
addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
multiplication: $r \cdot z_1 = (r x_1) + i(r \cdot y_1)$
zero matrix: $O = 0 + i0$

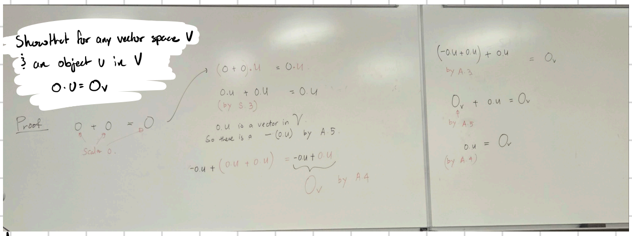
V = a vector space u = an object in V



Since u is in V the axioms say $r \cdot u$ is also in V (S.1)

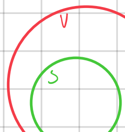
① $0 \cdot u = 0_V$
scalar; zero vector in (A.4)

② $(-1) \cdot u = -u$
Scalar (-1); negative vector from (A.5)



$\Rightarrow$ $-1 + 1 = 0 \rightarrow (-1+1)u = 0 \cdot u$
Scalars
$-1u + 1u = 0_V$ by S.3
$-1u + u = 0_V$ by S.5
proved in image to left

u is an object in V, so there is $-u$
$(-1 \cdot u + u) + (-u) = 0_V + (-u)$
$-1u + (u + (-u)) = -u$ (A.3, A.4)
$-1u + 0_V = -u$ (A.5) $\Rightarrow -1 \cdot u = -u$ (A.4)



V = a vector space

A subset S of V is a subspace of V if S itself is a vector space

Any subset S automatically satisfies the following axioms:

A.2 S.2 S.4
A.3 S.3 S.5

So if we want to check if $\Rightarrow$ a subset is a subspace we must check the other 4

A.1) If u & $v \in S$, then $u+v \in S$
A.4) We know $0_V \in V$, but we need to see if $0_V \in S$
$\Rightarrow$ A.5) If $u \in S$ we need to check if $-u \in S$
& we know the equation is true & $0_V/-u$ exists in V, so we are only checking if it exists in S
S.1) If $u \in S$ & $r \in \mathbb{R}$, then $r \cdot u \in S$
$(-1) \cdot u = -1u = -u \in S$ $\Rightarrow$ A.5 is true if S.1 is true so we only need to check S.1
&scalar

Eg ① $V = M_{2,2}$ $S = \left\{ \begin{matrix}\text{Set of all } 2\times2\\ \text{diagonal matrices}\end{matrix} \right\}$

Check if S is a subspace of V

Take $u = \begin{bmatrix}a&0\\0&b\end{bmatrix}$ $v = \begin{bmatrix}c&0\\0&d\end{bmatrix}$

$u + v = \begin{bmatrix}a+c&0\\0&b+d\end{bmatrix} \in S$ because $u+v$ is a 2×2 diagonal matrix (A.1)

$r \cdot u = \begin{bmatrix}r \cdot a&0\\0&r \cdot b\end{bmatrix} \in S$ because $r \cdot u$ is still a 2×2 diagonal matrix (S.1)

$0_V = \begin{bmatrix}0&0\\0&0\end{bmatrix} \in S$ because 0_V is a 2×2 diagonal matrix (A.4)

Simply formatting of the equation
We are proving the subset with this condition is a sub space, so we are focusing on that during our solution

② $V = P_2$ $S = \left\{ \begin{matrix}\text{Set of all polynomials}\\ \text{with degree exactly 2}\end{matrix} \right\}$ $\rightarrow$ Also could have checked (A.4) which would have shown the 0_V is not of degree two showing again S is not a subspace of V

$p(x) = a_0 + a_1 x + a_2 x^2$
$q(x) = b_0 + b_1 x + b_2 x^2$ & $a_2 \cdot b_2 \neq 0$ (if 0 then $\neq P_2$)

$a+b = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 \notin S$ as if $a_2 + b_2 = 0$ then it is not degree 2 (A.1)

counter $p(x) = 1 + x^2 \in S$, $q(x) = 1 - x^2 \in S$ $\rightarrow (1+x^2) + (1-x^2) = 2 + x^2 - x^2 = 2$ not degree 2

$\therefore$ S is not a subspace of V

③ $S = \left\{ \begin{matrix}\text{Set of all } 2\times2\\ \text{invertible matrices}\end{matrix} \right\}$ $\Rightarrow$ not a subspace of V as the 0_V is not invertable so $0_V \notin S$ (A.4)

④ $V = P_2$ $S = \left\{ a_0 + a_1 x + a_2 x^2 \mid a_1 = 2a_2 \right\}$

$p(x) = a_0 + a_1 x + a_2 x^2 \in S$ $a_1 = 2a_2$
$q(x) = b_0 + b_1 x + b_2 x^2 \in S$ $b_1 = 2b_2$

$p(x) + q(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2$ (A.1)
$= (a_0+b_0) + (2a_2 + 2b_2)x + (a_2+b_2)x^2$
$= (a_0+b_0) + 2(a_2+b_2)x + (a_2+b_2)x^2$

$r \cdot p(x) = r a_0 + r a_1 x + r a_2 x^2$
(S.1) $r a_1 = 2 r a_2 = 2(r a_2)$

$0_V = O(x) = 0 + 0x + 0x^2$
$a_1 = 0$, $a_2 = 0$ } $0 = 2 \cdot 0$

V = a vector space

$u_1, u_2, \ldots, u_k$ is a set of "vectors" (objects) of V

$r_1, r_2, \ldots, r_k$ are scalars

$r_1 \cdot u_1$, $r_2 \cdot u_2$, $\vdots$, $r_k \cdot u_k$ } These are vectors in V (S.1)

so $r_1 u_1 + r_2 u_2 + \ldots + r_k u_k$ is a vector in V (S.1)
linear combination of $u_1, u_2, \ldots, u_k$

Members/objects of a vector space are called "vector"

$\text{span}\{u_1, \ldots, u_k\}$ = Set of all linear combinations of $u_1, u_2, \ldots, u_k$

For example
$3 \cdot \begin{bmatrix}0&1\\2&4\end{bmatrix} + 2\begin{bmatrix}1&0\\7&3\end{bmatrix} + 5\begin{bmatrix}1&-1\\3&4\end{bmatrix}$
is a linear comb. of
$u_1 = \begin{bmatrix}0&1\\2&4\end{bmatrix}$ $u_2 = \begin{bmatrix}1&0\\7&3\end{bmatrix}$ $u_3 = \begin{bmatrix}1&-1\\3&4\end{bmatrix}$

S is a subspace of a vector space V. If $S = \text{span}\{u_1, \ldots, u_k\}$ then we call $\{u_1, \ldots, u_k\}$ a spanning set of S

Eg. $V = M_{2,3}$, find a spanning set of $M_{2,3}$

Take $u = \begin{bmatrix}u_{11}&u_{12}&u_{13}\\u_{21}&u_{22}&u_{23}\end{bmatrix} = a_{11}\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix} + a_{12}\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix} + \ldots$

$\left\{ \underbrace{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}}_{E_{11}}, \underbrace{\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}}_{E_{12}}, \underbrace{\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}}_{E_{13}}, \underbrace{\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}}_{E_{21}}, \underbrace{\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}}_{E_{22}}, \underbrace{\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}}_{E_{23}} \right\}$ is a spanning set for $M_{2,3}$

There are infinitely many spanning sets for any given matrix

For $V = M_{m \times n}$ $V = \text{span}\left\{ E_{ij} \;\middle|\; \begin{matrix}1 \le i \le m\\ 1 \le j \le n\end{matrix} \right\}$ where E_{ij} = The matrix with 1 at position (i,j) & 0 for every other entry

$V = M_{2\times2}$ $S = \left\{ \begin{matrix}2\times2 \text{ symmetric}\\ \text{matrices}\end{matrix} \right\}$

Find a spanning set for S

$\begin{bmatrix}a_{11}&a_{12}\\a_{12}&a_{22}\end{bmatrix} = a_{11}\begin{bmatrix}1&0\\0&0\end{bmatrix} + a_{12}\begin{bmatrix}0&1\\1&0\end{bmatrix} + a_{22}\begin{bmatrix}0&0\\0&1\end{bmatrix}$

$\left\{ \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix} \right\}$ is a spanning set for S

$S = \left\{ \begin{matrix}2\times2 \text{ skew}\\ \text{sym. matrix}\end{matrix} \right\}$

$\begin{bmatrix}0&a_{12}\\-a_{12}&0\end{bmatrix} = a_{12}\begin{bmatrix}0&1\\-1&0\end{bmatrix} = (-a_{12})\begin{bmatrix}0&-1\\1&0\end{bmatrix}$

$\left\{ \begin{bmatrix}0&1\\-1&0\end{bmatrix} \right\}$ is a spanning set for S $\rightarrow \left\{ \begin{bmatrix}0&-1\\1&0\end{bmatrix} \right\}$

$V = P_n$ $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \in V$

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = x^2$
$\vdots$

$= a_0 \cdot P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \ldots + a_n P_n(x)$

$P_n = \text{span}\{P_0, P_1, P_2, \ldots, P_n\}$

$V = P_2$ $S = \left\{ a_0 + a_1 x + a_2 x^2 \mid a_1 = 2a_2 \right\}$

$p(x) = a_0 + a_1 x + a_2 x^2 \Rightarrow a_1 = 2a_2$ So any polynomial in S can be written as a linear comb. of a_1 & a_2

$= a_0 + 2a_2 x + a_2 x^2$
$= a_0 + a_2(2x + x^2)$
$= a_0 \cdot 1 + a_2 \cdot (2x + x^2)$ $\hookrightarrow \text{span}\{a_1, a_2\}$

$V = P_2$ $S = \left\{ \begin{matrix}\text{Polynomial in } P_2\\ \text{which has } x=2 \text{ as a root}\end{matrix} \right\}$

Take $P \in S$ $p(x) = (x-2)(ax+b)$
$= ax(x-2) + b(x-2)$

$S \subseteq \text{span}\{x(x-2), x-2\}$

Any polynomial $p \in S$ can be written as a linear combination of

On the other hand, any polynomial in takes the form $ax(x-2) + b(x-2) = (x-2)(ax+b) \in S$

$\text{Span}\{x(x-2), x-2\} \subseteq S$ $\therefore$ $S = \text{span}\{\ldots\}$

Vector space cont...

V = a vector space, $\{u_1, \dots, u_k\}$ is a set of vectors in V

① This set is linearly dependent if one vector in this set can be written as a linear combination of other vectors in the set

② If not, the set is linearly independent

Equivalent definition:

$r_1 \cdot u_1 + r_2 \cdot u_2 + \dots + r_k \cdot u_k = 0$ ⓐ

① If there are scalars $r_1, r_2, \dots, r_k$ not all of which are zero such that ⓐ is true then the set is linearly dependent

② If ⓐ is only true when $r_1 = r_2 = \dots = r_k = 0$, then the set is linearly independent

Eg. Determine if $\left\{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right\}$ is a linearly independent set of vectors in $M_{2,3}$ basis

$r_1 \cdot \begin{bmatrix}1&0&0\\0&0&0\end{bmatrix} + r_2 \cdot \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix} + \dots + r_5 \cdot \begin{bmatrix}0&0&0\\0&1&0\end{bmatrix} + r_6 \cdot \begin{bmatrix}0&0&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$

Only true if all r is 0 => the set is linearly independent

↳ In general, the set of matricies $\left\{E_{ij} \,\middle|\, \begin{matrix}1 \le i \le m\\ 1 \le j \le n\end{matrix}\right\}$ is a linearly independent set in $M_{m,n}$ $\dim(M_{m,n}) = m \cdot n$

Where E_{ij} = The $m \times n$ matrix which has entry i in position (i,j) position & 0 else where

$V = P_3$, $\{1, 1+x, x^3 - x^2\}$ determine if this is a linearly independent set in P_3

$r_1 \cdot 1 + r_2 \cdot (1+x) + r_3 \cdot (x^3 - x^2) = 0_{(x)}$ $= 0 + 0x + 0x^2 + 0x^3$

Equate coefficients:

const $= r_1 + r_2 = 0$, $r_1 = 0$
$x = r_2 = 0$, $r_2 = 0$
$x^2 = -r_3 = 0$
$x^3 = r_3 = 0$ } $r_3 = 0$

$r_1 = r_2 = r_3 = 0$ ↳ linearly independent

V = a vector space S = a subspace of V. The set of vectors $\{u_1, u_2, \dots, u_k\}$ in S is a basis of S if:

① $S = \text{span}\{u_1, u_2, \dots, u_k\}$ and ← to show a set is a basis these two must be true

② $\{u_1, u_2, \dots, u_k\}$ is linearly independent

Eg. $P_n = \{1, x, x^2, \dots, x^n\}$ & is linearly independant.

So, $\{1, x, x^2, \dots, x^n\}$ is a basis for P_n

$V = M_{3,3}$, S = The set of 3×3 diagonal matricies

Find a basis for S. Take $v \in S$ $v = \begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix}$

$\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$

These are obviosly linearly independent, but you need to show this, which is done by show the linear combination of the spanning set is $=0$ only when all coefficients are 0.

$V = M_{2,2}$ S = The set of 2×2 upper triangular matricies

Find a basis for S $v = \begin{bmatrix}a&b\\0&c\end{bmatrix}$

$S = \text{span}\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$

$r_1\begin{bmatrix}1&0\\0&0\end{bmatrix} + r_2\begin{bmatrix}0&1\\0&0\end{bmatrix} + r_3\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ → only true if $r_1 = r_2 = r_3 = 0$

Span ✓ lin. indep ✓ ∴ $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ is a basis of S

$V = P_n = \{1, x, x^2, \dots, x^n\}$ determine if this is a linearly independent set in P_n

$r_1 \cdot 1 + r_2 \cdot x + \dots + r_n \cdot x^n = 0$ is only true when $r_1 = r_2 = r_3 = \dots = r_n = 0$, ∴ the set is linearly indep.

$\dim(P_n) = n+1$

Eg. $V = P_2 \,|\, S = \{a_0 + a_1x + a_2x^2 \,|\, a_1 = 2a_2\}$

$p(x) = a_0 + a_1x + a_2x^2 \in S$ $a_1 = 2a_2$
$= a_0 + 2a_2x + a_2x^2$
$= a_0 + a_2(2x + x^2)$ $S = \text{span}\{1, 2x + x^2\}$
$= a_0(1) + a_2(2x + x^2)$

$r_1 \cdot 1 + r_2 \cdot (2x + x^2) = 0$
$r_1 + r_2(2x) + r_2(x^2) = 0$

const $= r_1 = 0$
$x = 2r_2 = 0$
$x_2 = r_2 = 0$
$r_1 = r_2 = 0$

$\dim(S) = 2$

span ✓ linearly independent ✓

∴ $\{1, 2x + x^2\}$ is a basis of S

V = a vector space, S = a subspace of V.

Two different bases of S must be the same number of vectors

dimension of $S = \dim(S)$ = Number of vectors in a basis of S

$\{u_1, \dots, u_k, 0_v\}$

Show that any set of vectors with 0_v is linearly dependent

↳ $r_1u_1 + \dots + r_ku_k + (r_{k+1} \cdot 0_v) = 0$ Since this term is always 0 regardless of the coefficient it means there is a solution where not all of the coefficients are 0, meaning it is not lin. indep.

↳ ex. $0 \cdot u_1 + 0 \cdot u_2 + \dots + 0 \cdot u_k + \boxed{1} 0_v = 0$ ∴ if a set has 0_v it will always be linearly dependent

7.1 Linear Transformation

$T: V \longrightarrow W$ where V & W are vector spaces

T is a linear transformation if

(T.1) $T(v_1 + v_2) = T(v_1) + T(v_2)$ for any two vectors $v_1, v_2 \in V$

(T.2) $T(r \cdot v) = r \cdot T(v)$ for any $v \in V$ & any scalar r

Eg. $T: M_{2,2} \to P_2$

$T\left(\begin{bmatrix}a_{11}&a_{11}\\a_{12}&a_{22}\end{bmatrix}\right) = a_{11} + (2a_{22})x + a_{12}x^2$

(T.1) Take $A = \begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$ $B = \begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}$

$T(A+B) = T\left(\begin{bmatrix}a_{11}+b_{11}&a_{12}+b_{12}\\a_{21}+b_{21}&a_{22}+b_{22}\end{bmatrix}\right)$
$= (a_{11}+b_{11}) + 2(a_{22}+b_{22}) + (a_{12}+b_{12})x^2$
$= \underbrace{(a_{11} + 2a_{22}x + a_{12}x^2)}_{T(A)} + \underbrace{(b_{11} + 2b_{22} + b_{12}x^2)}_{T(B)}$ ✓

(T.2) $T(r \cdot A) = T\left(\begin{bmatrix}ra_{11}&ra_{12}\\ra_{21}&ra_{22}\end{bmatrix}\right)$
$= ra_{11} + r2a_{22} + ra_{12}x^2$
$= r(a_{11} + 2a_{22} + a_{12}x^2)$

$T(r \cdot A) = r \cdot T(A)$ ✓

"(T.1) & (T.2) are both valid, therefore, $T(A)$ is a linear transformation."

A is an $n \times n$ matrix

Trace of (A) = Sum of all diagonal entries of A
$= a_{11} + a_{22} + a_{33} + \dots + a_{nn}$

$T: M_{n,n} \to \mathbb{R}$ $T(A)$ = trace of A

$A, B \in M_{n,n}$ $A + B = \begin{bmatrix}a_{11}+b_{11}&a_{12}+b_{12}&\cdots&a_{1n}+b_{1n}\\a_{21}+b_{21}&a_{22}+b_{22}&&\\\vdots&&\ddots&\\a_{n1}+b_{n1}&\cdots&&a_{nn}+b_{nn}\end{bmatrix}$

(T.1) $T(A+B) = (a_{11}+b_{11}) + (a_{22}+b_{22}) + \dots + (a_{nn}+b_{nn})$
$= (a_{11} + a_{22} + \dots + a_{nn}) + (b_{11} + \dots + b_{nn})$

$T(A+B) = T(A) + T(B)$ ✓

$r \cdot A = \begin{bmatrix}r \cdot a_{11}&r \cdot a_{12}&\cdots\\r \cdot a_{21}&r \cdot a_{22}&\\\vdots&&\ddots\\&&&r \cdot a_{nn}\end{bmatrix}$

(T.2) $T(r \cdot A) = (r \cdot a_{11} + r \cdot a_{22} + \dots + r \cdot a_{nn})$
$= r(a_{11} + a_{22} + \dots + a_{nn})$

$T(r \cdot A) = r \cdot T(A)$ ✓

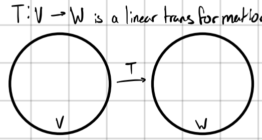
$T: P_n \to P_{n-1} \Rightarrow T(p(x)) = \dfrac{d(p(x))}{dx}$

(T.1) $T(p(x) + q(x)) = \dfrac{d}{dx}(p(x) + q(x)) = \dfrac{d(p(x))}{dx} + \dfrac{d(q(x))}{dx} = T(p(x)) + T(q(x))$

(T.2) $T(r \cdot p(x)) = \dfrac{d(r \cdot p(x))}{dx} = r \cdot \dfrac{d(p(x))}{dx} = r \cdot T(p(x))$

∴ $T(p(x))$ is a linear transformation

$T: V \to W$ is a linear transformation



let $\{v_1, v_2, \dots, v_n\}$ be a basis of V

If we know $T(v_1), T(v_2), \dots, T(v_n)$ then we can compute $T(u)$ for any vector in V, ... why?

Take any u in V

Since $\{v_1, v_2, \dots, v_n\}$ is a basis, $u = r_1v_1 + r_2v_2 + \dots + r_nv_n$

$T(u) = T(r_1v_1 + r_2v_2 + \dots + r_nv_n)$
T.1 $T(u) = T(r_1v_1) + T(r_2v_2) + \dots + T(r_nv_n)$
T.2 $T(u) = r_1T(v_1) + r_2T(v_2) + \dots + r_nT(v_n)$

Eg. $T: P_2 \to M_{2,2}$ is a linear transformation

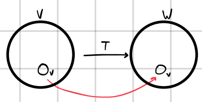
If $T(1) = \begin{bmatrix}-1&0\\0&0\end{bmatrix}$ $T(x) = \begin{bmatrix}1&0\\0&0\end{bmatrix}$ & $T(x^2) = \begin{bmatrix}0&1\\0&0\end{bmatrix}$

Determine the map T (i.e. compute $T(p(x))$ for any $p \in P_2$

↳ Take $p \in P_2 \Rightarrow p(x) = a_0 + a_1x + a_2x^2$

$T(p(x)) = T(a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2)$
$\overset{(T.1)}{=} T(a_0 \cdot 1) + T(a_1 \cdot x) + T(a_2 \cdot x^2)$
$= a_0T(1) + a_1T(x) + a_2 \cdot T(x^2)$
$= a_0 \cdot \begin{bmatrix}-1&0\\0&0\end{bmatrix} + a_1\begin{bmatrix}1&0\\0&0\end{bmatrix} + a_2\begin{bmatrix}0&1\\0&0\end{bmatrix} = \boxed{\begin{bmatrix}a_1 - a_0&a_2\\0&0\end{bmatrix}}$

$T: V \to W$ is a linear transformation



① $T(0_v) = 0_w$

Proof: $0 \cdot 0_v = 0_v$
$\Rightarrow T(0 \cdot 0_v) = T(0_v)$
$0 \cdot T(0_v) = T(0_v)$
$0_w = T(0_v)$

② $T(-v) = -T(v)$
$T(-v) = T((-1) \cdot v)$
$T(-v) = (-1) \cdot T(v)$
$T(-v) = -T(v)$

Best is to state what you are solving for before solving for it

Eg. A is an $m \times n$ matrix

$T_A: \mathbb{R}^n \to \mathbb{R}^m$ given by $T_A(x) = A \cdot x$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$

$T: V \to W$ $\overset{\text{Kernel of } T}{Null(T)} = \{v \in V \,|\, T(v) = 0_w\}$

$Null(T)$ is a subspace of V ⟶ (A.4) ✓

(A.1) Take $v_1, v_2 \in Null(T) \Rightarrow T(v_1) = 0_w, T(v_2) = 0_w$

$T(v_1 + v_2) = T(v_1) + T(v_2) = 0_w + 0_w = 0_w$

(S.1) r is a scalar & $v_1 \in Null(T) \to T(v_1) = 0_w$

$T(r \cdot v_1) = r \cdot T(v_1) = r \cdot 0_w = 0_w$

$T: P_2 \to \mathbb{R}^4 \Rightarrow T(a_0 + a_1x + a_2x^2) = \begin{bmatrix}a_0\\a_1\\0\\a_2\end{bmatrix}$

Find a basis for $Null(T)$

If $p(x) = a_0 + a_1x + a_2x^2 \in Null(T)$ then $T(p(x)) = \begin{bmatrix}a_0\\a_1\\0\\a_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$\left.\begin{matrix}a_0\\a_1\\a_2\end{matrix}\right\}$ $p(x) = 0 + 0x + 0x^2 = 0_{(x)}$ $= Null(T) = \{0_{(x)}\}$ → no basis

Eg. $T: P_2 \to \mathbb{R}^4 \Rightarrow T(a_0 + a_1x + a_2x^2) = \begin{bmatrix}a_0\\a_1 - 2a_2\\0\\0\end{bmatrix}$

Find a basis for $Null(T)$

If $p(x) = a_0 + a_1x + a_2x^2 \in Null(T)$ then

$T(p(x)) = \begin{bmatrix}a_0\\a_1 - 2a_2\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ $a_0 = 0$, $a_1 = 2a_2$

$p(x) = 0 + 2a_2x + a_2x^2$
$= a_2(2x + x^2)$

∴ $Null(T) = \text{span}\{2x + x^2\}$ => linearly indep, as

$r_1 \cdot (2x + x^2) = 0_{(x)} \to 2r_1x + r_1x^2 = 0 + 0x + 0x^2$

$r_1 = 0$ ∴ $\{2x + x^2\}$ is a basis for $Null(T)$

$\dim(Null(T)) = 1$

Finding basis of

Eg. $T: P_2 \to M_{2,3} \to T(p(x)) = \begin{bmatrix}a_1 - a_0&a_2\\0&0\end{bmatrix}$

If $p(x) = a_0 + a_1x + a_2x^2 \in Null(A)$ then

$\begin{bmatrix}a_1 - a_0&a_2\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix} \Rightarrow$ $a_1 - a_0 = 0$, $a_1 = a_0$, $a_2 = 0$

↳ $p(x) = a_0 + a_0x = a_0(1 + x)$

$Null(T) = \text{span}\{1 + x\}$

Eg. $T: M_{2,2} \to \mathbb{R}$ $T\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right) = a + b + c + d$

If $\begin{bmatrix}a&b\\c&d\end{bmatrix} \in Null(T)$, then

$T\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right) = a + b + c + d = 0 \Rightarrow a = -b - c - d$

$\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}-b-c-d&b\\c&d\end{bmatrix} = b\begin{bmatrix}-1&1\\0&0\end{bmatrix} + c\begin{bmatrix}-1&0\\1&0\end{bmatrix} + d\begin{bmatrix}-1&0\\0&1\end{bmatrix}$

$Null(T) = \text{span}\left\{\begin{bmatrix}-1&1\\0&0\end{bmatrix}, \begin{bmatrix}-1&0\\1&0\end{bmatrix}, \begin{bmatrix}-1&0\\0&1\end{bmatrix}\right\}$

determine if linearly independent

$r_1\begin{bmatrix}-1&1\\0&0\end{bmatrix} + r_2\begin{bmatrix}-1&0\\1&0\end{bmatrix} + r_3\begin{bmatrix}-1&0\\0&1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\begin{bmatrix}-r_1 - r_2 - r_3&r_1\\r_2&r_3\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ $r_1 = 0$, $r_2 = 0$, $r_3 = 0$ ✓

Eg. $T: M_{2,2} \to M_{2,2}$ $T(A) = A + A^T$

$T\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right) = \begin{bmatrix}a&b\\c&d\end{bmatrix} + \begin{bmatrix}a&c\\b&d\end{bmatrix} = \begin{bmatrix}2a&b+c\\c+b&2d\end{bmatrix}$

$\begin{bmatrix}2a&b+c\\c+b&2d\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ $2a = 0$ $2d = 0$ $c + b = 0 \to c = -b$; $a = 0$ $d = 0$

$A + A^T = 0$ $A = -A^T$ => skew symmetry => $Null(T)$ = {set of skew symmetric matricies}
$= \{A \in M_{2,2} \,|\, A = -A^T\}$

$\begin{bmatrix}a&b\\-b&d\end{bmatrix}$ $a = 0$, $d = 0$ $\Rightarrow b\begin{bmatrix}0&1\\-1&0\end{bmatrix}$

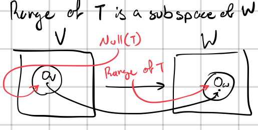
$Null(T) = \text{span}\left\{\begin{bmatrix}0&1\\-1&0\end{bmatrix}\right\}$

A linear transformation $T: V \to W$ injective (one to one) if $T(v_1) = T(v_2)$ then $v_1 = v_2$ for any $v_1, v_2 \in V$

T is one to one $\Leftrightarrow$ (if and only if) $Null(T) = \{0_v\}$

Range of $(T) = \{T(v) \,|\, v \in V\}$
$= \{w \in W \,|\,$ there is at least one $v \in V$ s.t. $T(v) = w\}$

Range of T is a subspace of W



$T: V \to W$

T is surjective (onto) if $Range(T) = W$

T is onto $\Longleftrightarrow \dim(Range(T)) = \dim(W)$

	$\dim(Null(T))$	$\dim(Range(T))$
$T: P_2 \to \mathbb{R}^4$, $T(a_0 + a_1x + a_2x^2) = \begin{bmatrix}a_0\\a_1\\0\\a_2\end{bmatrix}$	0 injective	$3 \neq 4$ not surj.
$T: P_2 \to \mathbb{R}^4$, $T(a_0 + a_1x + a_2x^2) = \begin{bmatrix}a_0\\a_1 - 2a_2\\0\\0\end{bmatrix}$	$1 \neq 0$ not inj.	$2 \neq 4$ not surj.
$T: M_{2,2} \to \mathbb{R}$, $T\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right) = a + b + c + d$	$3 \neq 0$ not inj.	$1 = 1$ surjective

Rank Nullity Theorem: $T: V \to W$

$\dim(Null(T)) + \dim(Range(T)) = \dim(V)$

$\dim(P_n) = n + 1$
$\dim(M_{m \times n}) = m \cdot n$
$\dim(\mathbb{R}^n) = n$
$\mathbb{R} = \mathbb{R}^1$

Review

Topics:

1. Cofactors/determinants
2. Eigen values/Eigen vectors/diagonalization
3. Given a matrix A, find bases for $Row(A)$, $Col(A)$, $Null(A)$
4. Vector space axioms
   ↳ Find 0_V, negative vector, etc.
5. $S \subseteq V$, show S is a subspace
6. S is a subspace => find a basis for S
7. Linear transformations
8. Prove something using a theorem from class — This is true due to theorem... or following theorem...

Theorems & Axioms will be given

55 points available, graded out of 50

---

(4) $V = \mathbb{R}^2$

Symbol denoting unique addition/mult.

with addition: $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 + 1 \end{bmatrix}$

with scalar mult: $s \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} s \cdot x_1 \\ s \cdot y_1 + s - 1 \end{bmatrix}$

__A.4__: What should be 0_V?

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus 0_V = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $\qquad 0_V = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ ✓

$\begin{bmatrix} x_1 + a \\ y_1 + b + 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \Rightarrow \begin{matrix} a = 0 \\ b = -1 \end{matrix}$

__A.5__: $u + (-u) = 0_V$, $-u = ?$

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\qquad -u = \begin{bmatrix} -x_1 \\ -y_1 - 2 \end{bmatrix}$ ✓

$\begin{bmatrix} x_1 + c \\ y_1 + d + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \begin{matrix} c = -x_1 \\ d = -y_1 - 2 \end{matrix}$

__S.3__: Show $(s+r) \odot u = s \odot u \oplus r \odot u$ (scalars; $u \in V$)

$(s+r) \odot u = \begin{bmatrix} (s+r)x_1 \\ (s+r)y_1 + (s+r) - 1 \end{bmatrix}$

$s \odot u \oplus r \odot u = \begin{bmatrix} sx_1 \\ sy_1 + s - 1 \end{bmatrix} \oplus \begin{bmatrix} rx_1 \\ ry_1 + r - 1 \end{bmatrix} = \begin{bmatrix} sx_1 + rx_1 \\ sy_1 + ry_1 + s + r - 2 + 1 \end{bmatrix}$

$\begin{bmatrix} (s+r)x_1 \\ (s+r)y_1 + (s+r) - 1 \end{bmatrix} = \begin{bmatrix} sx_1 + rx_1 \\ sy_1 + ry_1 + s + r - 2 + 1 \end{bmatrix}$ by (S.2)

✓ $= \begin{bmatrix} (s+r)x_1 \\ (s+r)y_1 + (s+r) - 1 \end{bmatrix}$

(4) $V = 2 \times 2$ invertable matricies

$A \oplus B = A \cdot B \qquad r \odot A = A^r$

__A.4__ $A \oplus 0_V = A$

$A \oplus I = A \cdot I = A \Rightarrow 0_V = I$ ✓

__A.5__ $A \oplus -A = 0_V$

$A \oplus A^{-1} = I = 0_V \Rightarrow -A = A^{-1}$ ✓

__A.2__: $A \oplus B = B \oplus A$

$A \oplus B = AB$ ✗

$B \oplus A = BA$ $\Rightarrow AB \neq BA$

(4) So for this topic we will be asked what is 0_V, what is $-u$ or be asked to prove certain axioms.

(5/6) To see if a vectorspace is a subspace you must check...

(A.1) If $u, v \in V$, then $u + v \in S$

(S.1) If $u \in V$, then $s \cdot u \in S$

(A.4) $0_V \in V$

Know these Axioms, but for proving Axioms it will be given

---

(1) $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 & 2 \\ 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 1 & 1 & 3 & 1 & 2 \\ 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{bmatrix}$

$|A| = -2 \cdot (-1)^{5+3} \begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 0 \end{vmatrix}$ (B)

$= -2 \cdot (-30)$

$|A| = 60$

$|B| = 2 \cdot (-1)^{4+3} \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \end{vmatrix}$ (C)

$= -2 \cdot 15$

$= -30$

$|C| = 3(-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$

$= 3 \cdot (6 - 1)$

$= 15$

Row operations

- $R_i \to R_i + kR_j$ $|A| \to$ no effect
- $R_i \leftrightarrow R_j$ $|A| \to -|A|$
- $R_i \to kR_i$ $|A| \to k \cdot |A|$

$\rightarrow \begin{bmatrix} 5/3 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \Rightarrow |A| = 5/3 \cdot 3 \cdot 2 \cdot 3 \cdot 2$

$5/3 \cdot 36$

$180/3 = \underline{60}$ ✓

---

(3) $A = \begin{bmatrix} 3 & -2 & 1 & 2 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ find basis for $Col(A)$, $Row(A)$ & $Null(A)$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$Col(A) = \left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ $dim(\ldots) = 3$

$Row(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $dim(\ldots) = 3$

$\begin{matrix} x_1 = -x_3 \\ x_2 = -x_3 \\ x_3 = s \\ x_4 = 0 \end{matrix} \Rightarrow \begin{matrix} x_1 = -s \\ x_2 = -s \\ x_3 = s \\ x_4 = 0 \end{matrix} = x = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \rightarrow Null(A) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ $dim(\ldots) = 1$

A basis for $\mathbb{R}^4 = Row(A) \cup Null(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(5) $\left\{ \begin{bmatrix} 2a \\ 0 \\ a-2 \\ b \end{bmatrix} \middle| \text{a \& b are real numbers} \right\}$ Is S a subspace of $\mathbb{R}^4$?

(A.4) There is no value for a & b s.t. $\begin{bmatrix} 2a \\ 0 \\ a-2 \\ b \end{bmatrix} = 0_V$

$\therefore 0_V \notin S$, so S is not a subspace of $\mathbb{R}^4$

(5) $S = \left\{ p \in P_5 \middle| \text{p has no } x^4 \text{ term} \right\}$

(A.4) $0(x) = 0 + 0x + 0x^2 + 0x^3 + 0x^4 + 0x^5$

$0(x) \in S$ as it has no x^4 term ($=0$)

$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_5x^5$

(A.1) $p, q \in S$ $\quad q(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_5x^5$

$(p+q)(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + (a_3+b_3)x^3 + (a_5+b_5)x^5$

$\therefore p + q \in S$ as it has no x^4

(S.1) $r \cdot p(x) = ra_0 + ra_1x + ra_2x^2 + ra_3x^3 + ra_5x^5$

$\therefore r \cdot p \in S$ as it has no x^4

There may be a question about showing spans are equal like last quiz

$span\{u, v, w\} = span\{u, v + 2w, w\}$

(6) $S = \left\{ \begin{matrix} \text{Set of all 3×3} \\ \text{symmetric} \\ \text{matricies} \end{matrix} \right\} \to A = A^T$

$A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

is linearly indep? $\Rightarrow S = span\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$

$c_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c_5 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + c_6 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} c_1 & c_2 & c_4 \\ c_2 & c_3 & c_6 \\ c_4 & c_6 & c_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \begin{matrix} c_1 = 0 & c_4 = 0 \\ c_2 = 0 & c_5 = 0 \\ c_3 = 0 & c_6 = 0 \end{matrix}$ $\Rightarrow$ since $c_1 = c_2 = \ldots = 0 \Rightarrow$ lin. indep. $\Rightarrow$ is a basis of S

(6) $S = \left\{ \begin{matrix} \text{Set of all } M_{33} \\ \text{with all 0 } 2^{nd} \text{ col}^n \end{matrix} \right\}$

$A = \begin{bmatrix} a & 0 & b \\ c & 0 & d \\ e & 0 & f \end{bmatrix} = a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$S = span\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \Rightarrow$ is lin. indep. $\Rightarrow$ basis of S

---

(4) $\{u_1, u_2, u_3\}$ is a set of linearly independent vectors in a vector space V. u_4 is NOT in $span\{u_1, u_2, u_3\}$

Show that $\{u_1, u_2, u_3, u_4\}$ is lin. indep.

$r_1 \cdot u_1 + r_2 \cdot u_2 + r_3 \cdot u_3 + r_4 u_4 = 0$

Proof by contradiction: (create assumption & prove wrong) $r_4 \neq 0$

$r_1 \cdot u_1 + r_2 \cdot u_2 + r_3 \cdot u_3 + r_4 \cdot u_4 = 0$

$u_4 = \left(\frac{-r_1}{r_4}\right)u_1 + \left(\frac{-r_2}{r_4}\right)u_2 + \left(\frac{-r_3}{r_4}\right)u_3$

$\therefore u_4$ is a linear combination of so $u_4 \in span\{u_1, u_2, u_3\}$

↳ This is stated false in the question, meaning the assumption is wrong & $r_4 = 0 \Rightarrow \{u_1, u_2, u_3, u_4\}$ is lin. indep.

(4/8) $\{u_1, u_2, u_3\}$ is a set of linearly independent vectors in a vector space $\mathbb{R}^4$. u_4 is NOT in $span\{u_1, u_2, u_3\}$

Show that $\{u_1, u_2, u_3, u_4\}$ is a basis for $\mathbb{R}^4$

↳ we have already proved that is lin. indep.

and $dim(\mathbb{R}^4) = 4 = dim(\quad)$ so by theorem 3

$\{u_1, u_2, u_3, u_4\}$ is a basis of $\mathbb{R}^4$

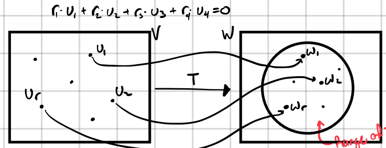
(7) $T: V \to W$ is a linear transformation

$\{w_1, w_2, \ldots, w_r\}$ is a basis of $Range(T)$

Let $u_1, u_2, \ldots, u_r$ be set of vectors in V

s.t. $T(u_1) = w_1, T(u_2) = w_2, \ldots$

Show that $\{u_1, u_2, \ldots, u_r\}$ is a set of linearly independent vectors in V



$\{w_1, w_2, \ldots, w_r\}$ is the basis of $Range(T)$

trying to show lin. indep. $s_1 = s_2 = \ldots = 0$

$T(s_1u_1 + s_2u_2 + \ldots + s_ru_r) = T(0_V)$

$s_1 \cdot T(u_1) + s_2 T(u_2) + \ldots + s_r T(u_r) = 0_W$

$s_1 w_1 + s_2 w_2 + \ldots + s_r w_r = 0_W$

We know $\{w_1, w_2, \ldots, w_r\}$ is lin. indep. (since its a basis of Range(T)) so we know $s_1 = s_2 = \ldots = 0$

So, $\{u_1, u_2, \ldots, u_r\}$ is lin. indep.

(7) $T: P_3 \to \mathbb{R}$ $\quad T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0 + a_1 + a_2 + a_3$

Find a basis for $Null(T)$. Then determine if T is injective and/or surjective.

↳ If $p \in Null(T)$, then $T(p) = T(a_0 + a_1x + a_2x^2 + a_3x^3) = 0$

$= a_0 + a_1 + a_2 + a_3 = 0$

$a_0 = -a_1 - a_2 - a_3$

$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$= (-a_1 - a_2 - a_3) + a_1x + a_2x^2 + a_3x^3$

$= a_1x - a_1 + a_2x^2 - a_2 + a_3x^3 - a_3$

$= a_1(x-1) + a_2(x^2-1) + a_3(x^3-1)$

$\Rightarrow Null(T) = span\{x-1, x^2-1, x^3-1\}$

$r_1(x-1) + r_2(x^2-1) + r_3(x^3-1) = 0 + 0x + 0x^2 + 0x^3$

$r_1x - r_1 + r_2x^2 - r_2 + r_3x^3 - r_3 =$ $\qquad x: r_1 = 0$

$(-r_1 - r_2 - r_3) + r_1x + r_2x^2 + r_3x^3 =$ $\qquad x^2: r_2 = 0$

$\qquad x^3: r_3 = 0$ → all zero lin. indep.

$\Rightarrow \{x-1, x^2-1, x^3-1\}$ is a basis of $Null(T)$

$dim(Null(T)) + dim(Range(T)) = dim(V)$

$3 + 1 = 4$

$3 \Downarrow$ not injective, $dim(Null(T))$ needs to be 0 to be injective

$1 \Downarrow$ $1 = dim(\mathbb{R})$ $\Downarrow$ it is surjective as $dim(Range(T)) = dim(codomain)$ (↓ output ↓ (W))