## MATH 3333 PRACTICE PROBLEM SET 1

**Problem 1.** Solve the following linear system by using elementary row operations. If the system is consistent, write down the general solution.

$$\begin{aligned} 2x_1 + 2x_2 + x_3 - x_4 &= -1 \\ x_1 + x_2 + x_3 + x_4 &= 2 \\ 3x_1 + 3x_2 + 2x_3 &= 1 \\ x_1 + x_2 - 2x_4 &= -3 \end{aligned}$$

**Solution:** 22 The augmented matrix is

$$\left[\begin{array}{cccc|c} 2 & 2 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2 \\ 3 & 3 & 2 & 0 & 1 \\ 1 & 1 & 0 & -2 & -3 \end{array}\right]$$

By interchanging the first and second rows, we obtain

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & -1 & -1 \\ 3 & 3 & 2 & 0 & 1 \\ 1 & 1 & 0 & -2 & -3 \end{array}\right]$$

Then we perform the following row operations
$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$ and
$R_4 \to R_4 - R_1$ to obtain

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & -1 & -3 & -5 \end{array}\right]$$

The next two operations are
$R_3 \to R_3 - 2R_2$ and
$R_4 \to R_4 - 3R_2$. We obtain the following.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then $R_1 \to R_1 + R_2$ yields

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & -3 \\ 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and finally by $R_2 \to -R_2$ we obtain the RRE form

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The leading variables are $x_1$ and $x_3$.
Free variables are $x_2$ and $x_4$. By taking $x_2 = s$ and $x_4 = t$ we can write the general solution as

$$x_1 = -3 - s + 2t$$

$$x_2 = s$$

$$x_3 = 5 - 3t$$

$$x_4 = t$$

**Problem 2.** Find a set of basic solutions for each of the following homogeneous linear systems. Write the solutions as linear combinations of those basic solutions.

(a) $x_1 + 2x_2 - x_3 + x_4 + x_5 = 0$
$- x_1 - 2x_2 + 2x_3 + x_5 = 0$
$- x_1 - 2x_2 + 3x_3 + x_4 + 3x_5 = 0$

**Solution:** The augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 1 & 0 \\ -1 & -2 & 2 & 0 & 1 & 0 \\ -1 & -2 & 3 & 1 & 3 & 0 \end{array}\right]$$

First we perform the following row operations
$R_2 \to R_2 + R_1$ and
$R_3 \to R_3 + R_1$ to obtain

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 4 & 0 \end{array}\right]$$

Then $R_3 \to R_3 - 2R_2$ yields

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Finally $R_1 \to R_1 + R_2$ yields the RRE form

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The leading variables are $x_1$ and $x_3$.
Free variables are $x_2, x_4$ and $x_5$. By taking $x_2 = s, x_4 = t$ and $x_5 = u$ we can write the general solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s-2t-3u \\ s \\ -t-2u \\ t \\ u \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Hence the basic solutions are $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

(b) $x_1 + x_2 - x_3 + 2x_4 + x_5 = 0$
$x_1 + 2x_2 - x_3 + x_4 + x_5 = 0$
$2x_1 + 3x_2 - x_3 + 2x_4 + x_5 = 0$
$4x_1 + 5x_2 - 2x_3 + 5x_4 + 2x_5 = 0$

**Solution:** The augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & 0 \\ 1 & 2 & -1 & 1 & 1 & 0 \\ 2 & 3 & -1 & 2 & 1 & 0 \\ 4 & 5 & -2 & 5 & 2 & 0 \end{array}\right]$$

First we perform the following row operations
$R_2 \to R_2 - R_1$
$R_{33} - 2R_1$ and
$R_4 \to R_4 - 4R_1$ to obtain

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & -3 & -2 & 0 \end{array}\right]$$

Then $R_3 \to R_3 - R_2$ and
$R_4 \to R_4 - R_2$
yield

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & -2 & -2 & 0 \end{array}\right]$$

Then $R_4 \to R_4 - 2R_3$ yields

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Clear the entries above the second leading 1 by $R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Finally clear the entries above the third leading 1 by $R_1 \to R_1 + R_3$ and obtain the RRE form

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The leading variables are $x_1, x_2$ and $x_3$.
Free variables are $x_4$ and $x_5$. By taking $x_4 = s$ and $x_5 = t$ we can write the general solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ s+t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the basic solutions are $\begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

**Problem 3.** The augmented matrix of a linear system is given below. Check whether the system is consistent or not.

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ -2 & 4 & 0 & 6 \\ 2 & 0 & 4 & 2 \end{array}\right]$$

What if the last column of the above augmented matrix is replaced by the column vector $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ ?

**Solution:** The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ -2 & 4 & 0 & 6 \\ 2 & 0 & 4 & 2 \end{array}\right]$$

First we perform the following row operations.
$R_2 \to R_2 + 2R_1$ and
$R_3 \to R_3 - 2R_1$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 10 & 10 & 20 \\ 0 & -6 & -6 & -12 \end{array}\right]$$

Then $R_2 \to \frac{1}{10}R_2$ and $R_3 \to \frac{-1}{6}R_3$ provide

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

Finally we obtain a row-echelon form by $R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There are no leading 1s in the last column of this row-echelon form. Hence the system is consistent. (In fact, we can say more; the system has infinitely many solutions because $r = 2 < n = 3$.)

Now, if we replace the last column by the vector given in the second part, we have the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 4 \\ -2 & 4 & 0 & 2 \\ 2 & 0 & 4 & 3 \end{array}\right]$$

is consistent or not.

First we perform the following row operations.
$R_2 \to R_2 + 2R_1$ and
$R_3 \to R_3 - 2R_1$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 4 \\ 0 & 10 & 10 & 10 \\ 0 & -6 & -6 & -5 \end{array}\right]$$

Then $R_2 \to \frac{1}{10}R_2$ and $R_3 \to \frac{-1}{6}R_3$ provide

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 5/6 \end{array}\right]$$

Then $R_3 \to R_3 - R_2$ yields

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1/6 \end{array}\right]$$

Finally we obtain a row-echelon form by $R_3 \to -6R_3$.

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

There is a leading 1 in the last column of this row-echelon form. Hence the system is inconsistent.

**Problem 4.** Let $A$ be a matrix with 4 rows and 3 columns.

(a) What are the possible values for the rank of $A$ ? **Solution:** Each row can have at most one leading 1. Also, no column can contain more than a single leading 1.

Therefore, the rank of $A$ cannot exceed the number of rows or the number of columns of A ; Rank of A $\leqslant 4$ and Rank of A $\leqslant 3$.

So the possible values for the Rank of A are $0, 1, 2$ and $3$.

(b) For each such a value, write down all the potential reduced row-echelon (RRE) forms of $A$.
**Solution:**

- Rank of A $= 0$

  No leading 1s in any of the rows. So the only possible RRE form is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- Rank of A $= 1$
  Only the first row can have a leading 1 and other three rows are all zero. There are three such possibilities.
  $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ where entries with $*$ can be any real number.

- Rank of A $= 2$
  First two rows have leading 1s and the last two rows are all zero. The possible RRE forms in this case are
  $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- Rank of A $= 3$
  In this case, each of the first three rows should have a leading 1 and there is only one such possibility. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

**Problem 5.** The augmented matrix of a linear system is given below.Find the values of $a$ such that the system has

(1) no solutions
(2) a unique solution

(3) infinitely many solutions

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ -1 & a-2 & 1 & -1 \\ 2 & 2 & a-2 & 1 \end{array}\right]$$

**Solution:** The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ -1 & a-2 & 1 & -1 \\ 2 & 2 & a-2 & 1 \end{array}\right]$$

Then apply $R_2 \to R_2 + R_1$ and
$R_3 \to R_3 - 2R_1$ to obtain

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 2a-2 & 2 & 0 \\ 0 & 2-2a & a-4 & -1 \end{array}\right]$$

We can see that the second and third entries in the second column are negatives of each other. We perform $R_3 \to R_3 + R_2$ and obtain

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 2a-2 & 2 & 0 \\ 0 & 0 & a-2 & -1 \end{array}\right]$$

Then multiply $R_2$ by $1/2$

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & a-1 & 1 & 0 \\ 0 & 0 & a-2 & -1 \end{array}\right]$$

**If a=1** this becomes

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

By $R_3 \to R_3 + R_2$ we obtain a row-echelon form given by

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

So the system is inconsistent.

Now suppose $a$ is not equal to 1. Then we can divide $R_2$ by $a-1$ to obtain

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1 & \frac{1}{a-1} & 0 \\ 0 & 0 & a-2 & -1 \end{array}\right]$$

**If a =2** then this becomes

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

So the system is inconsistent.

If not (if $a \neq 1, 2$) then we can divide $R_3$ by $a-2$ and obtain

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 1 & \frac{1}{a-1} & 0 \\ 0 & 0 & 1 & \frac{-1}{a-2} \end{array}\right]$$

Now there are no leading 1s in the last column and the rank of the augmented matrix $r = 3 = n$. So the system has a unique solution.
In summary ,

- If $a = 1$ or $a = 2$ then the system is inconsistent.
- For any other value of $a$, the system has a unique solution
- There is no value of $a$ for which the system has infinitely many solutions.

**Problem 6.** The augmented matrix of a linear system is given below. Find the values of $a$ and $b$ such that the system has

(1) no solutions
(2) a unique solution
(3) infinitely many solutions

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & -8 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & a & b \end{array}\right]$$

**Solution:** The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & -8 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & a & b \end{array}\right]$$

First interchange $R_2$ and $R_3$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ -1 & 3 & 2 & -8 \\ 3 & 3 & a & b \end{array}\right]$$

Then apply $R_2 \to R_2 + R_1$ and
$R_3 \to R_3 - 3R_1$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 3 & 3 & -6 \\ 0 & 3 & a-3 & b-6 \end{array}\right]$$

Then we can perform $R_3 \to R_3 - R_2$ and obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & a-6 & b \end{array}\right]$$

Then multiply $R_2$ by $1/3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & a-6 & b \end{array}\right]$$

**Case 1 : a =6 and b=0** Then we have

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

No leading 1s in the last column and $r = 2 < n$. So the system will have infinitely many solutions.

**Case 2 : a = 6 and b is not 0**. Then we have

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & * \end{array}\right]$$

where $*$ is a **non zero** number. So the system is inconsistent.

**Case 3 : a is not 6**. Then we can divide $R_3$ by $a - 6$ and obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & \frac{b}{a-6} \end{array}\right]$$

In this case there will be no leading 1s in the last column and $r = 3 = n$. So, the system will have a unique solution.

In summary ,

- If $a = 6$ **and** $b = 0$ then the system infinitely many solutions.is inconsistent.
- If $a = 6$ **and** $b \neq 0$ then the system is inconsistent.
- If $a \neq 6$ then the system has a unique solution. ($b$ can have any value in this case.)