1.3 2A are in homework 1

(b) $x_1 + x_2 - x_3 + 2x_4 + x_5 = 0$
$x_1 + 2x_2 - x_3 + x_4 + x_5 = 0$
$2x_1 + 3x_2 - x_3 + 2x_4 + x_5 = 0$
$4x_1 + 5x_2 - 2x_3 + 5x_4 + 2x_5 = 0$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 1 & 2 & -1 & 1 & 1 \\ 2 & 3 & -1 & 2 & 1 \\ 4 & 5 & -2 & 5 & 2 \end{bmatrix} \end{array} \xrightarrow[\substack{R_3 \to R_3 + (-2)R_1 \\ R_4 \to R_4 + (-4)R_1}]{R_2 \to R_2 + (-1)R_1} \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -2 & -1 \\ 0 & 1 & 2 & -3 & -2 \end{bmatrix} \xrightarrow[R_4 \to R_4 + (-1)R_2]{R_3 \to R_3 + (-1)R_2} \begin{bmatrix} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -2 & -2 \end{bmatrix} \xrightarrow[R_1 \to R_1 + R_3]{R_4 \to R_4 + (-2)R_3} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_2 + x_4 = 0 \Rightarrow x_1 = -x_2 - x_4$
$x_2 - x_4 = 0 \Rightarrow x_2 = x_4$
$x_3 - x_4 - x_5 = 0 \Rightarrow x_3 = x_4 + x_5$
$x_4 = s$
$x_5 = t$

$\Rightarrow$ $x_1 = -2s$, $x_2 = s$, $x_3 = s + t$, $x_4 = s$, $x_5 = t$ $\Rightarrow$ $\boxed{x = s \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$

Problem 3. The augmented matrix of a linear system is given below. Check whether the system is consistent or not.

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ -2 & 4 & 0 & 6 \\ 2 & 0 & 4 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 3 & 5 & 7 \\ -2 & 4 & 0 & 6 \\ 2 & 0 & 4 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc:c} 1 & 3 & 5 & 7 \\ 0 & 4 & 4 & 8 \\ 2 & 0 & 4 & 2 \end{array}\right] \xrightarrow[\substack{R_2 \to \frac{1}{4}R_2 \\ R_3 \to \frac{1}{6}R_3}]{R_3 \to R_3 + (-2)R_1} \left[\begin{array}{ccc:c} 1 & 3 & 5 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + R_2]{R_1 \to R_1 + (-3)R_2} \left[\begin{array}{ccc:c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -2x_3 + 1$
$x_2 = -x_3 + 2$
$x_3 = s$
$\Rightarrow$ $x_1 = -2s + 1$, $x_2 = -s + 2$, $x_3 = s$ $\Rightarrow$ $x = s \cdot \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

consistent

What if the last column of the above augmented matrix is replaced by the column vector $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$?

$$\left[\begin{array}{ccc:c} 1 & 3 & 5 & 4 \\ -2 & 4 & 0 & 2 \\ 2 & 0 & 4 & 3 \end{array}\right] \xrightarrow[R_3 \to R_3 + (-2)R_1]{R_2 \to R_2 + R_3} \left[\begin{array}{ccc:c} 1 & 3 & 5 & 4 \\ 0 & 4 & 4 & 5 \\ 0 & -6 & -6 & -5 \end{array}\right] \xrightarrow[\substack{R_2 \to \frac{1}{4}R_2 \\ R_3 \to \frac{1}{2}R_3}]{R_3 \to R_3 + R_2} \left[\begin{array}{ccc:c} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{2}R_2]{R_1 \to R_1 + R_3} \left[\begin{array}{ccc:c} 1 & 3 & 5 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

zero in the last column
$\therefore$ inconsistent

Problem 4. Let A be a matrix with 4 rows and 3 columns.

$m \times n$
4×3 $\rightarrow$ 3 columns, so 3 places for leading 1s

$\therefore \boxed{r = 0, 1, 2, 3}$
no leading 1/zero matrix

(a) What are the possible values for the rank of A ?

(b) For each such a value, write down all the potential reduced row-echelon (RRE) forms of A.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad r = 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad r = 1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad r = 2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad r = 3$$

Problem 5. The augmented matrix of a linear system is given below.Find the values of a such that the system has

find if consistent $\Rightarrow$ plug ?a=?=0

(1) no solutions
(2) a unique solution
(3) infinitely many solutions

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 1 \\ -1 & a-2 & 1 & -1 \\ 2 & 2 & a-2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & a & 1 & 1 \\ -1 & a-2 & 1 & -1 \\ 2 & 2 & a-2 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 + (-2)R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc:c} 1 & a & 1 & 1 \\ 0 & 2a-2 & 2 & 0 \\ 0 & 2-2a & a-4 & -1 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc:c} 1 & a & 1 & 1 \\ 0 & 2a-2 & 2 & 0 \\ 0 & 0 & a-2 & -1 \end{array}\right]$$

$a = 1$ $\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_2]{R_2 \to \frac{1}{2}R_2} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$ inconsistent, no solution

$a = 2$ $\left[\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$ inconsistent, no solution

if $a = 1, 2 \Rightarrow$ no solution
if $a \neq 1, 2 \Rightarrow$ unique solution
no value for a such that there are infinitely many solutions

$r = n$ unique solution
$r < n$ infinitely many solutions

Problem 6. The augmented matrix of a linear system is given below. Find the values of a and b such that the system has

(1) no solutions
(2) a unique solution
(3) infinitely many solutions

$$\left[\begin{array}{ccc|c} -1 & 3 & 2 & -8 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & a & b \end{array}\right]$$

$$\left[\begin{array}{ccc:c} -1 & 3 & 2 & -8 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & a & b \end{array}\right] \xrightarrow[R_3 \to R_3 + (-3)R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{ccc:c} 0 & 3 & 3 & -6 \\ 1 & 0 & 1 & 2 \\ 0 & 3 & a-3 & b-6 \end{array}\right] \xrightarrow[R_1 \to \frac{1}{3}R_1]{R_3 \to R_3 + (-1)R_1} \left[\begin{array}{ccc:c} 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & a-6 & b \end{array}\right]$$

if $a = 6$ & $b \neq 0$ it would put a one in the last column making it inconsistent $\Rightarrow$ no solution (1)

$a \neq 6$ $\left[\begin{array}{ccc:c} 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & a-6 & b \end{array}\right] \xrightarrow{R_3 \to \frac{1}{a-6}R_3} \left[\begin{array}{ccc:c} 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & b \end{array}\right]$ 3 leading 1s so $r = n$ meaning there is a unique solution if $a \neq 6$ (2)

Lastly if $a = 6$ & $b = 0$ then $r < n$ meaning there would be infinitely many solutions (3)

Exercise 1.1.1 In each case verify that the following are solutions for all values of s and t.

a. $x = 19t - 35$
$y = 25 - 13t$
$z = t$
is a solution of
$2x + 3y + z = 5$
$5x + 7y - 4z = 0$ ✓

b. $x_1 = 2s + 12t + 13$
$x_2 = s$
$x_3 = -s - 3t - 3$
$x_4 = t$
is a solution of
$2x_1 + 5x_2 + 9x_3 + 3x_4 = -1$
$x_1 + 2x_2 + 4x_3 = 1$ ✓

$2(19t - 35) + 3(25 - 13t) + t = 5$
$5(19t - 35) + 7(25 - 13t) - 4(t) = 0$

True for all t
$38t - 70 + 75 - 39t + t = 5 \Rightarrow 5 = 5$
True for all t
$95t - 175 + 175 - 91t - 4t = 0 \Rightarrow 0 = 0$

Since both are true for all t we know the given solution is valid

$2(2s + 12t + 13) + 5(s) + 9(-s - 3t - 3) + 3(t) = -1$

$2s + 12t + 13 + 2(s) + 4(-s - 3t - 3) = 1$

$4s + 24t + 26 + 5s - 9s - 27t - 27 + 3t = -1$
$\cancel{9s} - \cancel{9s} + \cancel{24t} - \cancel{24t} + 26 - 27 = -1 \Rightarrow -1 = -1$

Also valid

$2s + 12t + 13 + 2s - 4s - 12t - 12 = 1$
$4s - 4s + 12t - 12t + 13 - 12 = 1 \Rightarrow 1 = 1$

Solution valid for all t

So this is a valid solution for all t

Exercise 1.3.2 In each of the following, find all values of a for which the system has nontrivial solutions, and determine all solutions in each case.

a. $x - 2y + z = 0$
$x + ay - 3z = 0$
$-x + 6y - 5z = 0$

b. $x + 2y + z = 0$
$x + 3y + 6z = 0$
$2x + 3y + az = 0$

c. $x + y - z = 0$
$ay - z = 0$
$x + y + az = 0$

d. $ax + y + z = 0$
$x + y - z = 0$
$x + y + az = 0$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & a & -3 & 0 \\ -1 & 6 & -5 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & a & -3 & 0 \\ 0 & 4 & -4 & 0 \end{bmatrix} \xrightarrow[\substack{R_1 \to R_1 + R_3 \\ R_3 \to \frac{1}{4}R_3}]{R_1 \to R_1 + (-1)R_2} \begin{bmatrix} 0 & 2-a & 0 \\ 1 & a & -3 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + (-3)R_3} \begin{bmatrix} 0 & \frac{2-a}{a-3} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

if $a = 2$ we get a zero row which

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$x_1 + x_2 = 0$
$x_2 - x_3 = 0$

$x_1 = -x_2$
$x_2 = x_3$
$x_3 = s$
$\Rightarrow$ $\boxed{\begin{array}{l} x_1 = -s \\ x_2 = s \\ x_3 = s \end{array}}$ $a = 2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 6 \\ 2 & 3 & a \end{bmatrix} \xrightarrow[R_3 \to R_3 + (-2)R_1]{R_2 \to R_2 + (-1)R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 5 \\ 0 & -1 & a-2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & a+3 \end{bmatrix}$$

$a = -3$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ $x + 2y + z = 0$, $y + 5z = 0$

$x = -2y - z$
$y = -5z$
$z = s$
$\Rightarrow$ $\boxed{\begin{array}{l} x = 10s - s = 9s \\ y = -5s \\ z = s \end{array}}$ $a = -3$

$a = -1$ $\begin{bmatrix} a & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & a \end{bmatrix} \xrightarrow[R_1 \to R_1 + R_2]{R_3 \to R_3 + (-1)R_2} \begin{bmatrix} a+1 & 2 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & a+1 \end{bmatrix}$

$a = -1$ $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{R_1 \to \frac{1}{2}R_1, \; R_3 \to R_3 + (-1)R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x - z = 0$, $y = 0$

$x = z$
$y = 0$
$z = s$
$\Rightarrow$ $\boxed{\begin{array}{l} x = s \\ y = 0 \\ z = s \end{array}}$ $a = -1$

$a = 1$ $\begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $z = 0$

b. $\begin{bmatrix} 0 & -1 & 3 & 1 & 3 & 2 & 1 \\ 0 & -2 & 6 & 1 & -5 & 0 & -1 \\ 0 & 3 & -9 & 2 & 4 & 1 & -1 \\ 0 & 1 & -3 & -1 & 3 & 0 & 1 \end{bmatrix}$

Problem 1. Suppose A and B are two matrices of the same size and X is a column vector such that AX and BX are defined.
State whether following statements are true of false. If true, briefly justify the statement. If false, provide a counterexample.

(a) If $A + B = \mathbf{0}$ then at least one of A and B should be $\mathbf{0}$.

(b) If $A = B^T$ then A and B are both square matrices.

(c) If A is symmetric and $A = 2B^T$ then B is also symmetric.

(d) If A and B are symmetric then so is $2A - 5B$. $A^T = A$

(e) If A and B are skew-symmetric then so is $2A - 5B$. $A^T = -A$

(f) If A is a square matrix then $A + A^T$ is always symmetric.

(g) If A is a square matrix then $2A + 3A^T$ is always symmetric.

a) if $A+B=0$ then at least A or $B=0$
False, for $A+B=0$ $A=-B$ neither of which need to be 0 to be true, similarly $1+(-1)=0$, creating 0 without either element $=0$

b) if $A=B^T$ then A & B are both square
True, A & $B = m\times n$ so $B^T = n\times m$ meaning $A=B^T \to m\times n = n\times m \to m=n \to$ square matrix

c) if $A=2B^T$ & A is symmetric then so is B
True $(A=2B^T)^T \to A^T = 2B$
since A is symmetric $A = A^T$
$A = 2B$
$2B^T = 2B \to \underline{B^T = B}$

d) if A & B are sym. then so is $2A-5B$
True $(S=2A-5B)^T$ $S^T = 2A^T - 5B^T$
$S=S^T \Leftarrow S^T = 2A-5B = S \Leftarrow S^T = \underline{2A-5B}$

e) if A & B are skew sym then so is $2A-5B$
True $(S=2A-5B)^T \Rightarrow S^T = 2A^T - 5B^T$
$\to S^T = -2A+5B \Rightarrow S^T = -(2A-5B)$
$(S=2A-5B)\cdot(-1) = -S = -(2A+5B) \to S^T = -S$

f) if A is square then $A+A^T$ is always symmetric
True, sym. if $A^T = A \to (S=A+A^T)^T$ $S^T = (A+A^T)^T = A^T + (A^T)^T = A^T + A$
so $S = A+A^T$ & $S^T = A^T + A$ since $A+A^T = A^T+A$ $S = S^T$

g) if A is square $2A+3A^T$ is always sym
False, the scalars mess with symmetry
$\hookrightarrow 2A+3A^T \neq 2A^T+3A$

Problem 2. Suppose A is a matrix and X is a column vector such that AX is defined.
State following statements are true of false. If true, briefly justify the statement. If false, provide a counterexample.

(a) If AX has a zero entry then A has a row of zeros.

(b) If A has a row of zero then AX should have a zero entry.

(c) If $AX = 0$ and $X \neq 0$ then $A = 0$.

(d) If $V = AX$ then V is in the range of T_A where T_A is the transformation induced by A.

(e) If $V = AX$ then V is in $col(A)$.

a) If AX has a zero entry the A has a zero row
False, for an entry to be zero the entire row doesn't HAVE to be zero the dot product just have to be zero Ex. $AX = \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\end{bmatrix}$

b) if A has a zero row then AX has a zero entry
True, if an entire row of A is zero the dot product for that row will also be zero.
Ex. $A = \begin{bmatrix}0&0\\1&4\end{bmatrix}$ $X = \begin{bmatrix}3\\2\end{bmatrix}$ $AX = \begin{bmatrix}0&0\\4&9\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = 3\begin{bmatrix}0\\1\end{bmatrix} + 2\begin{bmatrix}0\\4\end{bmatrix} = \begin{bmatrix}0\\11\end{bmatrix}$

c) if $AX = 0$ & $X \neq 0$ then $A = 0$
False, some principal as part a $\Rightarrow a_{ij}x_j = 0$
Ex. $A = \begin{bmatrix}1&-1\\2&-2\end{bmatrix}$ $X = \begin{bmatrix}1\\1\end{bmatrix}$ $AX = 1\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\-2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

d) If $V = AX$ then V is in the range of T_A where T_A is the transformation induced by A
True, the range of T_A is the set of all outputs $T_A(x) = Ax$. Since $V = AX$, T_A is by definition in the range of T_A

e) If $V = AX$ then V is in $col(A)$
True, $col(A)$ is every possible linear combination of A, so the linear combination V which is made with A will be in $col(A)$

Problem 3. Let $A = \begin{bmatrix}4&5&1&-1\\2&0&3&6\\1&-1&0&4\end{bmatrix}$, $B = \begin{bmatrix}1&1&0\\-4&0&2\\1&5&0\\2&-1&5\end{bmatrix}$ and $V = \begin{bmatrix}1&-2&2\end{bmatrix}^T$

Compute the following whenever they are defined.

(a) AV $3\times\underline{4}$ $\underline{3}\times 1$ undefined

(b) A^TV 4×3 3×1 ✓

(c) $A(A^TV)$ 3×4 $(4\times 3$ $3\times 1) \Rightarrow 3\times 4$ 4×1 ✓

(d) $BV + 6A$ 4×3 $3\times 1 \Rightarrow \underline{4\times 1} + \underline{4\times 3}$ undefined

(e) $-2A^TV + BV$ 4×3 3×1 + 4×3 3×1 $\Rightarrow 4\times 1 + 4\times 1$ ✓

b) $A^TV = \begin{bmatrix}2&-3&5&5\end{bmatrix}^T$

$$\begin{bmatrix}4&2&1\\5&0&-1\\1&3&0\\-1&6&4\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\2\end{bmatrix} = 1\cdot\begin{bmatrix}4\\5\\1\\-1\end{bmatrix} + (-2)\begin{bmatrix}2\\0\\3\\6\end{bmatrix} + 2\begin{bmatrix}1\\-1\\0\\4\end{bmatrix} = \begin{bmatrix}2\\3\\-5\\-5\end{bmatrix}$$

c) $A(A^TV) = \begin{bmatrix}23&-41&-21\end{bmatrix}^T$

$$\begin{bmatrix}4&5&1&-1\\2&0&3&6\\1&-1&0&4\end{bmatrix}\cdot\begin{bmatrix}2\\3\\-5\\-5\end{bmatrix} = 2\begin{bmatrix}4\\2\\1\end{bmatrix} + 3\begin{bmatrix}5\\0\\-1\end{bmatrix} + (-5)\begin{bmatrix}1\\3\\0\end{bmatrix} + (-5)\begin{bmatrix}-1\\6\\4\end{bmatrix} = \begin{bmatrix}8+15-5+5\\4+0-15-30\\2-3-0-20\end{bmatrix} = \begin{bmatrix}23\\-41\\-21\end{bmatrix}$$

e) $-2A^TV + BV = \begin{bmatrix}-1&-6&1&24\end{bmatrix}^T$

$$-2\cdot\begin{bmatrix}2\\3\\-5\\-5\end{bmatrix} + BV = \begin{bmatrix}-4\\-6\\10\\10\end{bmatrix} + \begin{bmatrix}1&1&0\\-4&0&2\\1&5&0\\2&-1&5\end{bmatrix}\cdot\begin{bmatrix}1\\-2\\2\end{bmatrix} = \begin{bmatrix}-4\\-6\\10\\10\end{bmatrix} + 1\cdot\begin{bmatrix}1\\-4\\1\\2\end{bmatrix} + (-2)\begin{bmatrix}1\\0\\5\\-1\end{bmatrix} + (2)\begin{bmatrix}0\\2\\0\\5\end{bmatrix} = \begin{bmatrix}-4\\-6\\10\\10\end{bmatrix} + \begin{bmatrix}1-2+0\\-4+0+4\\1-10+0\\2+2+10\end{bmatrix} = \begin{bmatrix}-4\\-6\\10\\10\end{bmatrix} + \begin{bmatrix}-1\\0\\-9\\14\end{bmatrix} = \begin{bmatrix}-5\\-6\\1\\24\end{bmatrix}$$

Problem 4. In each case, determine all values of s and t such that the given matrix is symmetric.

For it to be symmetric $A = A^T$

(a) $\begin{bmatrix}s&t\\st&1\end{bmatrix}$

(b) $\begin{bmatrix}s&2s&st\\t&-1&s\\t&s^2&s\end{bmatrix}$

a) $\begin{bmatrix}s&t\\st&1\end{bmatrix} = \begin{bmatrix}s&st\\t&1\end{bmatrix}$
$t \neq 0$ $s=1$
$t=0$ $s=\mathbb{R}$

b) $s=1$ $t=2$ or $s=0$ $t=0$

Problem 5.

(a) Consider the function $f : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$f(\mathbf{X}) = \begin{bmatrix}2x_1 - x_3\\x_2 + 3x_3 + x_4\end{bmatrix} \quad \text{where } \mathbf{X} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

Find a matrix A such that $f = T_A$

$$A\cdot\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}2x_1 - x_3\\x_2 + 3x_3 + x_4\end{bmatrix}$$
4×1 2×1

A is 2×4

$$Ax = x_1\begin{bmatrix}2\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} + x_3\begin{bmatrix}-1\\3\end{bmatrix} + x_4\begin{bmatrix}0\\1\end{bmatrix}$$

$$A = \begin{bmatrix}2&0&-1&0\\0&1&3&1\end{bmatrix}$$

(b) Consider the function $g : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$g(\mathbf{X}) = \begin{bmatrix}x_1\\x_2\\0\end{bmatrix} \quad \text{where } \mathbf{X} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Find a matrix B such that $g = T_B$

$$B\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\0\end{bmatrix}$$
2×1 3×1

so $B = 3\times 2$

$$Bx = x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\0\end{bmatrix}$$

$$B = \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

(c) Suppose $h : \mathbb{R}^4 \to \mathbb{R}^3$ is given by $h = g \circ f$. Compute $h(X)$ where $\mathbf{X} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ and then find a matrix C such that $h = T_C$.

$$g\left(\begin{bmatrix}2x_1 - x_3\\x_2 + 3x_3 + x_4\end{bmatrix}\right) \to \begin{bmatrix}2x_1 - x_3\\x_2 + 3x_3 + x_4\\0\end{bmatrix} \to \begin{bmatrix}2&0&-1&0\\0&1&3&1\\0&0&0&0\end{bmatrix} = C$$

(d) How is C related to A and B? $B\cdot A = C$

$C = 3\times\underline{4}$ $A = \underline{2}\times 4$ $B = 3\times\underline{2}$

$$\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\cdot\begin{bmatrix}2&0&-1&0\\0&1&3&1\end{bmatrix} = \begin{bmatrix}2&0&-1&0\\0&1&3&1\\0&0&0&0\end{bmatrix}$$

col 1 $2\begin{bmatrix}1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix}$
col 2 $0\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$
col 3 $-1\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\3\\0\end{bmatrix}$
col 4 $0\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$

✓

Problem 6. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^4$ is given by $f\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1 - x_2\\-x_1 + 7x_2\\0\\x_2\end{bmatrix}$.

Domain → inputs
Range → outputs

$n\times 1$

(a) Find two column vectors v_1 and v_2 in $\mathbb{R}^4$ such that range of f is the span $\{v_1, v_2\}$.

(b) If $V = 7v_1 + 11v_2$ then find a column vector $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ in $\mathbb{R}^2$ such that $f\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = V$

$$\begin{bmatrix}x_1 - x_2\\-x_1 + 7x_2\\0\\x_2\end{bmatrix} = f\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = x_1\underbrace{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}}_{v_1} + x_2\underbrace{\begin{bmatrix}-1\\7\\0\\1\end{bmatrix}}_{v_2}$$

so $v_1 = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$ & $v_2 = \begin{bmatrix}-1\\7\\0\\1\end{bmatrix}$

$f\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = x_1v_1 + x_2v_2$
$V = 7v_1 + 11v_2$ $\Rightarrow x_1 = 7$ & $x_2 = 11$

so the column vector would be $\begin{bmatrix}7\\11\end{bmatrix}$

Problem 7. Let A be an $m \times n$ matrix and T_A be the transform it induces. That is the function $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is given by $T_A(X) = AX$.
Prove the following statements.

(a) *T_A is an injective function if and only if the linear system $AX = b$ has a unique solution for any $m \times 1$ column vector b in $col(A)$.*

(b) *T_A is a surjective function if and only if any $m \times 1$ column vector b is in the column space of A.*

(c) If T_A is both injective and surjective, what can you say about the solutions to the linear system $AX = b$ for any $m \times 1$ column vector b ?

a) T_A is injective if $AX=b$ has a unique solution X for every $b \in col(A)$
$T_A(X)$ is the linear transformation of X imposed by matrix A. In other terms it is the linear combination of all of A represented by $AX=b$. So, since T_A operates essentially the same as we know the range of T_A can only be the column space of A

so if there is a unique solution for every b in $col(A)$ we know T_A is injective since its output range is $= col(A)$

b) Similar logic as 'a'; since range of $T_A = col(A)$ if every b is present then we know T_A will be surjective

c) Since range $T_A = col(A)$ we can relate their properties, $Ax=b$ has a solution for all X, & every one is unique

(2) $\begin{bmatrix}1&0&7&5\\0&1&3&6\\1&-1&5&2\\1&-1&5&1\end{bmatrix}$ → inverse

$R_4 - R_3$ $\begin{bmatrix}1&0&7&5\\0&1&3&6\\1&-1&5&2\\0&0&0&-1\end{bmatrix}$ $R_3 - R_1$ $\begin{bmatrix}1&0&7&5\\0&1&3&6\\0&-1&-2&-3\\0&0&0&-1\end{bmatrix}$ $R_4 + R_3$ $\begin{bmatrix}1&0&7&5\\0&0&1&3\\0&-1&-2&-3\\0&0&0&-1\end{bmatrix}$ $R_3 + R_2$ $\begin{bmatrix}1&0&7&5\\0&0&1&3\\0&-1&-1&0\\0&0&0&-1\end{bmatrix}$ $R_1 + 5R_4$, $R_2 + 3R_4$ $\begin{bmatrix}1&0&7&0\\0&0&1&0\\0&-1&-1&0\\0&0&0&-1\end{bmatrix}$ $R_1 - 7R_3$, $R_3 + R_2$ $\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&-1\end{bmatrix}$ $R_3 \to -R_3$, $R_4 \to -R_4$, $R_2 \leftrightarrow R_3$ $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$

$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ $R_4 - R_3$ $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-1&1\end{bmatrix}$ $R_3 - R_4$ $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-1\\0&0&-1&1\end{bmatrix}$ $R_3 + R_2$ $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}$ $R_2 + R_1$ $\begin{bmatrix}1&0&0&0\\-1&1&0&0\\-2&1&0&1\\0&0&-1&1\end{bmatrix}$

$\begin{bmatrix}1&0&-5&5\\-1&1&-3&3\\-2&1&0&1\\0&0&-1&1\end{bmatrix}$ $\begin{bmatrix}8&-7&16&-16\\-1&1&-3&3\\-3&2&-3&4\\0&0&-1&1\end{bmatrix}$ ← Gauss Jordan $\begin{bmatrix}8&-7&16&-16\\3&-2&3&-4\\-1&1&-3&3\\0&0&1&-1\end{bmatrix}$

$\begin{bmatrix}8&-7&4&-16\\3&-2&0&-3\\-1&1&-2&3\\0&0&1&-1\end{bmatrix}$

Right corner entries get horrendously hard! Gauss Jordan probably best

Problem 9. Solve the system of linear equations given below using your answers to the previous problem.

(1) $-x_1 + x_2 + 2x_3 = 1$
$2x_2 - x_3 = 0$
$x_2 - x_3 = -1$

$A^{-1} = \begin{bmatrix} -1 & 3 & -5 \\ 0 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix}$

(2) $x_1 + 7x_3 + 5x_4 = 0$
$x_2 + 3x_3 + 6x_4 = 1$
$x_1 - x_2 + 5x_3 + 2x_4 = 2$
$x_1 - x_2 + 5x_3 + x_4 = 1$

$B^{-1} = \begin{bmatrix} 8 & -7 & 9 & -16 \\ 3 & -2 & 0 & -3 \\ -1 & 1 & -2 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

Problem set 3

Problem 1. Compute the cofactor matrix and the determinant of the matrix given below. If it is invertible, write down its inverse.

$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$C_{ij}(A) = (-1)^{i+j} \cdot M_{ij}$ ← Determinate of sub matrix made by deleting the i-th row & j-th column

$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$C_{11}(A) = 1 \quad C_{12}(A) = -3 \quad C_{13}(A) = -3$
$C_{21}(A) = -1 \quad C_{22}(A) = 1 \quad C_{23}(A) = 1$
$C_{31}(A) = -2 \quad C_{32}(A) = 6 \quad C_{33}(A) = 4$

$C = \begin{bmatrix} 1 & -3 & -3 \\ -1 & 1 & 1 \\ -2 & 6 & 4 \end{bmatrix}$

row 2 of A × row 2 of C

$\det(A) = 3\cdot(-1) + 1\cdot(1) + 0\,(1) = -2$

$\det \neq 0$ so A is invertible

$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A) \quad \text{adj}(A) = C^T$

$A^{-1} = \frac{1}{-2} \cdot \begin{bmatrix} 1 & -1 & -2 \\ -3 & 1 & 6 \\ -3 & 1 & 4 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} -1/2 & 1/2 & 1 \\ 3/2 & -1/2 & -3 \\ 3/2 & -1/2 & -2 \end{bmatrix}$

Problem 2. Determine whether following statements are true or false. If true, write down a short proof. If false, provide a counter example.

(a) $det(A+B) = det(A) + det(B)$

(b) If $det(A) = 0$ then A has two identical rows.

(c) If R is the RRE form of A, then $det(a) = det(R)$

(d) If A is 2×2 then $det(7A) = 49det(A)$

(e) If A has a row of all zeroes then so does $cof(A)$.

(f) Recall that the transpose of $cof(A)$ is called as the adjoint of A and denoted by $adj(A)$. If $adj(A) = A^{-1}$ then $det(A) = 1$

a) false, the determinant is not additive. For example,
$A = [1] \quad |A| = 1$
$B = [2] \quad |B| = 2$
$|A+B| = 3$

b) false, $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$
$|A| = (1)(4) - (2)(2) = 4 - 4 = 0$

c) false, row operations applied to matrices change the values, causing differing determinants.
$A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad |A| = (4)(4) - (0)(0) = 4$
$R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad |R| = (1)(1) - (0)(0) = 1$
$4 \neq 1$

d) True, for a 2x2 (square) matrix where k is a scalar, $|k \cdot A| = k^n \cdot |A|$
$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \Rightarrow 2 \cdot |A| = 4 \cdot ((1)(3) - (2)(2)) = 4 \cdot (-1) = -4$
$2 \cdot A = \begin{bmatrix} 2 & 4 \\ 4 & 6 \end{bmatrix} \Rightarrow |2 \cdot A| = (2)(6) - (4)(4) = 12 - 16 = -4$
$-4 = -4$ ■

e) True, due to the nature of cofactor at some point an entire row of the cofactor matrix will rely on the zero row of the matrix & since the formula of cofactor $((a)(d) - (b)(c))$ has the elements of each row go into both halves of the statement it will cause a zero entry as $0 - 0 = 0$, the pattern will continue down the row creating a zero row in cof(A) due to the zero row in A

f) True, $A^{-1} = \frac{1}{|A|} \text{adj}(A)$, so the only way for $A^{-1} = \text{adj}(A)$ is if $\frac{1}{|A|} = 1$ which is only possible $\Leftrightarrow |A| = 1$

Problem 3. Suppose that the first row of a matrix A is $\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$ and the first row of the cofactor matrix of a A is $\begin{bmatrix} 3 & 1 & 2 \end{bmatrix}$

(a) Determine whether A is invertible.

(b) Why cannot the second row the cofactor matrix be $\begin{bmatrix} 0 & -1 & 2 \end{bmatrix}$?

a) $|A| = 1\cdot(3) + (-1)(1) + 2\cdot(2)$
$= 6$
$6 \neq 0$ so A is invertable

b) Taking entries of 1 row and cofactors of another for a cofactor expansion should result in 0, but in this case it does not:
$(1)(0) + (-1)(-1) + (2)(2) = 5 \neq 0$
→ Since this expansion resulted in something other than 0 the proposed row of the cofactor expansion is not true.
This type of expansion is sometimes referred to as "Expansion of alien cofactors"

Problem 4. Let S and T be two linear transformations as described below.

$T : \mathbb{R}^3 \to \mathbb{R}^3$ is the projection onto $yz-$plane. That is for any vector U in $\mathbb{R}^3$, $T(U)$ is the projection of U onto $yz-$plane.

$S : \mathbb{R}^3 \to \mathbb{R}^3$ is the function that scales vectors by 5. That is for any vector U in $\mathbb{R}^3$, $S(U) = 5U$.

(1) Write down a matrix A such that T is the transformation induced by A.

(2) Find eigenvalues and associated eigenvectors of A. Can you geometrically interpret your answer ?

(3) Write down a matrix B such that S is the transformation induced by B.

(4) Find eigenvalues and associated eigenvectors of B. Can you geometrically interpret your answer ?

(5) Find a matrix C such that $T \circ S$ is the transformation induced by C.

(6) Find the eigenvalues and associated eigenvectors of C. Can you geometrically interpret your answer ?

T: projection onto yz plane, so removing x component.
if $U = (x,y,z) \quad T(U) = (0,y,z)$

1) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2) $(A - \lambda I) = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \quad \det(A - \lambda I) = (-\lambda)(1-\lambda)(1-\lambda) - 0$
$= (-\lambda)(1-\lambda)^2$

$\lambda = 0$
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad y = 0, z = 0 \quad v = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$ Free variable
vanishes under projection

$\lambda = 0, 1$
$\lambda = 1$
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad -x + 0 \quad v = \begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$
unchanged by projection

3) scale by 5 ⇒ $5 \cdot (U) = (x, y, z)$
$\Rightarrow (5x, 5y, 5z)$
$B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

4) $(B - \lambda I) = \begin{bmatrix} 5-\lambda & 0 & 0 \\ 0 & 5-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} \quad |B - \lambda I| = (5-\lambda)^3$
$\lambda = 5$

$\lambda = 5$
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
so any non zero vector $n \in \mathbb{R}^3$ is an eigen vector of B
all vector persevered on scaling

5) $T \circ S = T(S(U)) = T(5x, 5y, 5z) = (0, 5y, 5z)$
$C = AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

6) $(C - \lambda I) = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 5-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} \quad \lambda = 0, 5$

$\lambda = 0$
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad y = 0, z = 0 \quad \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$
vanishes under projection

$\lambda = 5$
$\begin{bmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x = 0 \quad v = \begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$
scaled by 5

Problem 5. Let A, B and C be 4×4 matrices such that $det(A) = -2, det(B) = 7$ and $det(C) = 5$. Compute the determinants of following matrices.

a) $BC^2(A^{-1})^2(C^2)^T$

b) $E_1B^2(A^T)^3E_2$ where E_1 and E_2 are the elementary matrices obtained by performing the row operations $R_2 \to R_2 + 2R_1$ and $R_3 \to -3R_3$ on I respectively.

a) $|BC^2(A^{-1})^2(C^2)^T| = |B| \cdot |C^2| \cdot |(A^{-1})^2| \cdot |(C^2)^T|$
$= 7 \cdot (5)^2 \cdot (\frac{1}{-2})^2 \cdot (5)^2$
$= 7 \cdot 25 \cdot 1/4 \cdot 25 = \frac{4375}{4}$

b) $E_1B^2(A^T)^3E_2 = 1 \cdot (49) \cdot (-8)(-3) = 1176$
$E_1: R_2 \to R_2 + 2R_1 \quad E_2: R_3 \to -3R_3$
$\left|\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 1$
$\left|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}\right| = 1 \cdot (1)(-3) - 0$
$= -3$

Problem 6. Let $A = \begin{bmatrix} x & -2 & x^2 \\ y & -2 & y^2 \\ 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -x & 2 & -x^2 \\ 4y & -8 & 4y^2 \\ x - y + 1 & 2 & x^2 - y^2 + 3 \end{bmatrix}$

If $|A| = 6$ compute $|B|$.

$R_1 \to -R_1$ second ①
$R_2 \to 4R_2$ third ③
$R_3 \to R_1 - R_2 + R_3$ first ②

$|A| = 6$
① → 6
② → -6
③ → -24
→ $|B| = -24$

Problem 7. Explain what can be said about the determinant of A in each of the following cases. A is an $n \times n$ matrix.

(1) $A^2 = A$

(2) A is symmetric

(3) $A = PA$ where P is an invertible matrix.

(4) $A^2 = PA$ where P is an invertible matrix and $|P| = 4$

(5) $A^2 + I = 0$ and n is even

(6) $A^2 + I = 0$ and n is odd

1) $A^2 = A \to |A^2| = |A| \to |A|^2 = |A|$
only possible if $|A| = 1, 0$

2) Transposing has no effect on det. Since you can take the det. from any row or column

3) if $A = PA$, P is either I or A is the zero matrix. Either way $|A| = 0$

4) $A^2 = PA \to |A^2| = |P| \cdot |A|$
$|A|^2 = 4 \cdot |A| \to$ only true if $|A| = 0$ or 4

5) $A^2 + I = 0$ A is $n \times n$, n is odd
$A^2 = -I \quad |-I| = (-1)^n$ so for odd n
$|A|^2 = 1 \Rightarrow |A| = \sqrt{1} \Rightarrow |A| = \pm 1$

6) $A^2 + I = 0$ n odd
$A^2 = -I \quad |-I| = (-1)^n \Rightarrow |-I| = -1$
$|A|^2 = -1 \Rightarrow |A| = \sqrt{-1} = |A| = \pm i$
$|A|$ has no real solution when n is odd

from this we can see $|-I| = (-1)^n$ because when the n is even it results in an even amount of -1 which cancels, odd n causes opposite resulting in -1

4x4
$|-I| = (-1) \cdot \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$
$= (-1)((-1)(-1) - 0)$
$= -1$
$= (-1)(-1)$
$= 1$

3x3
$|-I| = (-1) \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}$
$= (-1)((-1)(-1) - 0)$
$= -1 \cdot 1$
$= -1$

Problem 8. Compute the determinant of the following matrices in terms of x.

(1) $\begin{bmatrix} 1 & x & x^2 & x^3 \\ x & x^2 & x^3 & 1 \\ x^2 & x^3 & 1 & x \\ x^3 & 1 & x & x^2 \end{bmatrix}$ $(x^4 - 1)^3$

(2) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & x & x \\ 1 & x & 0 & x \\ 1 & x & x & 0 \end{bmatrix}$ $-3x^2$

Problem 9. For each of the following matrix, find its eigenvalues and eigenvectors associated with those values.

(1) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(2) $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

1) $(A - \lambda I) = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \quad |(A - \lambda I)| = (-\lambda)((-\lambda)(-\lambda) - 1) - 1((-\lambda) - 1) + 1((1) + (-\lambda))$
$= -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda)$
$= -\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda$
$= -\lambda^3 + 3\lambda + 2$
$= (\lambda + 1)^2(\lambda - 2)$

$\lambda = -1, 2$ (twice)
$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = -1$
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x = -y - z \\ y = s \\ z = t \end{matrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + t\underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$

$\lambda = 2$
$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x = z \\ y = z \\ z = s \end{matrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2) $(A - \lambda I) = \begin{bmatrix} 6-\lambda & 3 & -8 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & -3-\lambda \end{bmatrix} \quad |(A - \lambda I)| = 0 + (-2-\lambda)((6-\lambda)(-3-\lambda) - (-8)(1)) + 0$
$= (-2-\lambda)(-18 - 6\lambda + 3\lambda + \lambda^2 + 8)$
$= (-2-\lambda)(\lambda^2 - 3\lambda - 10)$

$\lambda = -2, 5$ (twice)
$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix} \quad v_3 =$ defective matrix

$\lambda = -2$
$\begin{bmatrix} 8 & 3 & -8 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x = z \\ y = 0 \\ z = s \end{matrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{v_1}$
Only one eigen vector for eigen vector of multiplicity 2 indicating defective matrix of λ=-2

$\lambda = 5$
$\begin{bmatrix} 1 & 3 & -8 \\ 0 & -7 & 0 \\ 1 & 0 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x = 8z \\ y = 0 \\ z = s \end{matrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\underbrace{\begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$

$A = PDP^{-1}$
$PD = \begin{bmatrix} 0 & 1 & 3 \\ -2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 12 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$
$(PD)P^{-1} = \begin{bmatrix} 0 & 0 & 12 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} \cdot 1/4 \begin{bmatrix} 0 & -1 & 2 \\ 4 & -3 & -6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 6 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} = A$
∴ A is diagonalizable

Problem 10. The matrix A is given by

$\begin{bmatrix} 0 & 3 & 6 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

(a) Find all eigenvalues of A.

(b) For each eigenvalue λ of A, find a basis for $Null(A - \lambda I)$.

(c) Determine if A is diagonalizable. If it is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

1) $(A - \lambda I) = \begin{bmatrix} -\lambda & 3 & 6 \\ 0 & 2-\lambda & 4 \\ 0 & 1 & 2-\lambda \end{bmatrix} \quad |(A - \lambda I)| = (-\lambda)((2-\lambda)(2-\lambda) - 4) + 0 + 0$
$= (-\lambda)(4 - 2\lambda - 2\lambda + \lambda^2 - 4)$
$= (-\lambda)(\lambda^2 - 4\lambda)$
$= (-\lambda)(\lambda)(\lambda - 4)$
$= (-\lambda^2)(\lambda - 4)$

$\lambda = 0, 4$ (twice)
$\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}_{v_3} \right\}$

$\lambda = 0$
$\begin{bmatrix} 0 & 3 & 6 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} y = -2z \\ x = s \\ z = t \end{matrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + t\underbrace{\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}}_{v_2}$

$\lambda = 4$
$\begin{bmatrix} -4 & 3 & 6 \\ 0 & -2 & 4 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x = 3z \\ y = 2z \\ z = s \end{matrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\underbrace{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}_{v_3}$

A is diagonalizable if it can be represented by $A = PDP^{-1}$

$P = \begin{bmatrix} 0 & 1 & 3 \\ -2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$C = \begin{bmatrix} 0 & 4 & 0 \\ -1 & -3 & 1 \\ 2 & -6 & 2 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 0 & -1 & 2 \\ 4 & -3 & -6 \\ 0 & 1 & 2 \end{bmatrix}$
$|P| = 1 \cdot ((-2)(1) - (2)(0)) = -4$
$P^{-1} = 1/4 \cdot (\text{adj}(P))$

Problem 11. The matrix A is given by

$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ -8 & -4 & -4 \end{bmatrix}$

Show that A is diagonalizable and compute A^{2024}.

In home work 3, essentially
$A = PDP^{-1}$
↳ $A^k = PD^kP^{-1}$
↳ so we use this to solve for A^k
⇒ Simply apply the exponent to the entries of D then compute PDP^{-1} to get A^k

Problem set 4

A subspace S of $\mathbb{R}^n$ is a subset that is closed under vector addition & scalar multiplication, and contains the zero vector

Problem 1. Determine whether each of the subset given below is a subspace of $\mathbb{R}^4$ and if so, find a spanning set.

(a) $\left\{ \begin{bmatrix} 0 \\ a \\ b \\ c \end{bmatrix} \middle| \ a, b \text{ and } c \text{ are real numbers} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ a \\ b \\ c \end{bmatrix} \middle| \ a, b \text{ and } c \text{ are real numbers} \right\}$

(c) $\left\{ \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} \middle| \ a - 2b + 3c = 0 \right\}$

(d) $\left\{ \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix} \middle| \ a^2 = b^2 \right\}$

(e) $\left\{ \begin{bmatrix} 2a \\ 0 \\ a-2 \\ b \end{bmatrix} \middle| \ a \text{ and } b \text{ are real numbers} \right\}$

(f) $\left\{ \begin{bmatrix} 2a \\ 0 \\ b-2 \\ c \end{bmatrix} \middle| \ a, b \text{ and } c \text{ are real numbers} \right\}$

(g) $\left\{ \begin{bmatrix} r \\ 0 \\ s \\ 0 \end{bmatrix} \middle| \ r^2 + s^2 = 0 \quad \text{and} \quad r \text{ and } s \text{ are real numbers} \right\}$

a) ✓ 1) ✓ $\{0\} \in \begin{bmatrix} 0 \\ 0 \\ 0 \\ c \end{bmatrix}$ if $0=a=b=c$

2) ✓ if abc are real numbers, any real number + any real number is still a real number

3) ✓ $\mathbb{R} \cdot \mathbb{R} = \mathbb{R}$

b) ✗ 1) ✗ $\{0\} \notin \begin{bmatrix} 0 \\ 0 \\ 0 \\ c \end{bmatrix}$ as $1 \neq 0$

c) ✓ 1) ✓ if $0=a=b=c$ then $\{0\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

2) ✓ if $u = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ 0 \end{bmatrix}$

$u+v = (a_1-2b_1+3c_1)+(a_2-2b_2+3c_2) = 0$

$= (a_1+a_2) - 2(b_1+b_2) + 3(c_1+c_2) = 0$

$u+v = \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \\ 0 \end{bmatrix}$

3) ✓ $ku = \begin{bmatrix} ka \\ kb \\ kc \\ 0 \end{bmatrix} = ka - 2kb + 3kc = k(a-2b+3c) = 0$

$= k(0) = 0$

$= 0 = 0$ ✓

Do parts d - g

Problem 2. Let $\mathcal{S}$ and $\mathcal{T}$ be two subspaces of $\mathbb{R}^n$. Show that $\mathcal{S} \cap \mathcal{T}$ is also a subspace of $\mathbb{R}^n$.

Since both S & T are both a subspace of $\mathbb{R}^n$ The intersection of the two would also be a subspace of $\mathbb{R}^n$.

1) ✓ $\{0\} \in S$ & T so $\{0\} \in S \cap T$ 2) ✓ $u, v \in S$ & T so $u+v \in S \cap T$ 3) ✓ $ku \in S$ & T so $ku \in S \cap T$

Problem 3. Let $u_1, u_2, \ldots, u_k$ be vectors in $\mathbb{R}^n$ and $\mathcal{S}$ be a subspace of $\mathbb{R}^n$ that contains all those vectors. Show that $span\{u_1, u_2, \ldots u_k\} \subseteq \mathcal{S}$.

We can say span $\{u_1, \ldots, u_k\} \subseteq S$ because S is closed under scalar multiplication & vector addition. This means any linear combination of $u_1, \ldots, u_k$ (i.e., any vector in their span) must also lie in S

Problem 4. Let u, v and w be three vectors in $\mathbb{R}^n$. Show the following.

(a) $span\{u, v\} = span\{u, u - 2v\}$

(b) $span\{u, v, w\} = span\{u - 2v, v, w\} = span\{5u, 2v, v - w\}$

a) span $\{u, v\}$ = span $\{u, v, u-2v\}$

$u-2v$ is a linear combination of the other 2 vectors making it redundant meaning it can be removed leaving us with two identical spans

b) span $\{u, v, w\}$ = span $\{u-2v, v, w\}$ = span $\{5u, 2v, v-w\}$

All of these set span the same space as each set can be represented as a linear combination of the others

← Should do the forward/backward proof

Problem 5. Show that any set of vectors with the zero vector is linearly dependent.

A set is linearly independent if the linear combination of all vectors creates the zero vector only when ALL coefficients are zero. ⇒ So a set is linearly dependent when a zero vector is created from the linear combination, but not all coefficients are zero. ⇒ If a set contains a zero vector the coefficient can be non-zero & the linear combination still results in the zero vector.

$c_1 \cdot 0 + c_2 v_2 + \ldots + c_k v_k = 0 \Rightarrow \underbrace{1}_{\text{non-zero coefficient}} \cdot 0 + c_2 v_2 + \ldots + c_k v_k = \underbrace{0}_{\text{result is 0 vector}} \Rightarrow 0 + c_2 v_2 + \ldots c_k v_k = 0$

Problem 6. Determine whether the sets given below are linearly independent or not.

(a) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -6 \\ 6 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -6 \end{bmatrix} \right\}$

a) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 9 \\ 2 & 0 & -6 \\ 0 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x = 3z$, $y = -2z$, $z = z$

$r < n \rightarrow$ infinitely many solutions

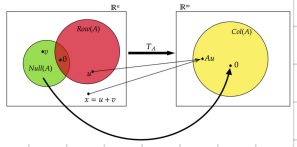
↳ linearly dependent

b) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 9 \\ 2 & 0 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $r = n \rightarrow$ unique solution

↳ linearly independent

Problem set 5

Problem 1. A is an $m \times n$ matrix. Fully explain the figure shown below.



Domain ($\mathbb{R}^n$) contains the Row(A) & Null(A)

Every vector $x \in \mathbb{R}^n$ can be written as $u+v$ where:

$u \in Row(A)$ & $v \in Null(A)$

$0_v \in Row(A) \cap Null(A)$

Applying A (T_A) to x gives:

$A \cdot x = A(u+v) = Au + Av = Au + 0 = Au$

↳ The codomain ($\mathbb{R}^m$) shows the result $A \cdot u \in Col(A)$

Because $v \in Null(A)$ the component $A \cdot v = 0_v$

The transformation A flattens the component v & maps all vectors with the same row-space components to the same point in $\mathbb{R}^m$

Problem 2. Let $\mathcal{S}$ be a subspace of $\mathbb{R}^4$.

Suppose $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for $\mathcal{S}$ and

$v_1 \quad v_2 \quad v_3$

$\mathcal{B}' = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathcal{S}^\perp$.

u_1

(a) What is the dimension of $\mathcal{S}$? $\dim(S) = 3$

(b) What is the dimension of $\mathcal{S}^\perp$? $\dim(S^\perp) = 1$

(c) If $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ write down bases for $Row(A)$ and $Null(A)$ only using the information given above. (You do not have to do any new computations.)

(d) Using your answer to (c), write down a basis for $\mathbb{R}^4$.

c) $Row(A) = span\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$= S$

$Null(A) = (Row(A))^\perp$

$= (S)^\perp$

$= S^\perp$ $\quad \therefore$ $Row(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ & $Null(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

d) Basis for $\mathbb{R}^n$

$\mathbb{R}^n = Null(A) + Row(A)$

$\mathbb{R}^n$ = basis of Null(A) ∪ basis of Row(A)

$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cup \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

basis of $\mathbb{R}^n = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Problem 3. Let A be an $m \times n$ matrix which has the rank r. Prove the following statements. If needed, you can use theorems 1,2 or 3 from the class.

(1) We showed in the class any vector X in $\mathbb{R}^n$ can be written as $X = u_1 + v_1$ where u_1 is a vector in $Row(A)$ and v_1 is a vector in $Nullll(A)$. Show that this expression of X is unique. That is if $X = u_2 + v_2$ for some other vector u_2 in in $Row(A)$ and a vector v_2 in $Null(A)$ then $u_1 = u_2$ and $v_1 = v_2$.

(2) If $X = u+v$ where u is a vector in $Row(A)$ and v is a vector in $Null(A)$ then $T_A(X) = T_A(u)$.

(This means when X is written as an addition of a vector in $Row(A)$ and a vector in $Null(A)$ then $T_A(X)$ depends only on the part of X that is in $Row(A)$. The other part in $Null(A)$ has no impact on $T_A(X)$.)

(3) If $u_1, u_2 \in Row(A)$ and $T_A(u_1) = T_A(u_2)$ then $u_1 = u_2$. (That is no two vectors in $Row(A)$ maps into the same vector under the transformation T_A.)

(4) If $\{u_1, u_2, \ldots u_r\}$ is a basis of $Row(A)$ then $\{T_A(u_1), T_A(u_2), \ldots, T_A(u_r)\}$ is basis of $Col(A)$.

1) $X = U_1 + V_1$ & $X = U_2 + V_2$ where $U_1, U_2 \in Row(A)$, $V_1, V_2 \in Null(A)$

$U_1 + V_1 = U_2 + V_2 \Rightarrow U_1 - U_2 = V_2 - V_1 = \omega$

$U_1 - U_2 \in Row(A)$ & $V_2 - V_1 \in Null(A)$

$\omega \in Row(A)$ $\quad$ $\omega \in Null(A)$

$\omega \in Row(A) \cap Null(A)$

$\Rightarrow Row(A) \cap Null(A) = \{0\} = \omega$

so $U_1 - U_2 = V_2 - V_1 = 0$ so $U_1 = U_2$ & $V_1 = V_2$

Meaning $X = U + V$ is unique.

2) $T_A(X) = AX = A(U+V)$

$= A(U) + A(V)$ ($A(V) \in Null(A)$ so $A \cdot V = 0$)

$= AU + 0$

$T_A(X) = AU = T_A(U)$

3) $U_1, U_2 \in Row(A)$ $\quad U_1 = U_2$

$T_A(U_1) = T_A(U_2) \Rightarrow AU_1 = AU_2$

$AU_1 - AU_2 = 0 \Rightarrow A(U_1 - U_2) = 0$

$U_1 - U_2 \in Null(A)$ but Row(A) is also closed under addition so $U_1 - U_2 \in Row(A)$

↳ $U_1 - U_2 \in Null(A) \cap Row(A) = 0$

$U_1 - U_2 = 0$

4) both Row(A) & Col(A) have dim r

$T_A(U_i) = AU_i$ which is a linear combination of the columns of A (with the coefficients being U_i), so $T_A(U_i) \in Col(A)$, but we dont know if basis lin. indep.

$c_1 T_A(U_1) + \ldots + c_r T_A(U_r) = 0$, only way for

$c_1 AU_1 + c_2 AU_2 + \ldots + c_r AU_r = A(c_1 U_1 + \ldots + c_r U_r) = 0$

$A \cdot V = 0 \Rightarrow V = \Rightarrow c_1 U_1 + \ldots + c_r U_r \in Null(A)$

but we know $c_1 U_1 + \ldots + c_r U_r \in Row(A)$ as $\{U_1, \ldots, U_r\}$ is basis of $c = Row(A) \cap Null(A) = 0$

Hence $c_1 U_1 + \ldots + c_r U_r = 0$, we already know it is linearly independent since $\{U_1, \ldots, U_r\}$ is a basis for Row(A)

So, $\{T_A(U_1), \ldots, T_A(U_r)\}$ is a set of r lin. indep. vectors $\in Col(A)$, and the $\dim(Col(A)) = r$ so $\{T_A(U_1), \ldots, T_A(U_r)\}$ is a basis of Col(A)

Problem 4. Let $\mathcal{V} = \mathbb{R}^2$.

Suppose a vector addition on $\mathcal{V}$ is defined as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 + 1 \end{bmatrix}$$

and a scalar multiplication is defined by

$$s \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} sx_1 \\ sx_2 + s - 1 \end{bmatrix}$$

Determine if $\mathcal{V}$ with these operations a vector space or not.

just going thru the Axioms proving them correct

⇒ look through Axioms to make sure we familiar

Problem 5. Let $\mathcal{V} = \mathbb{R}^+ =$ The set of all positive real numbers.

Suppose a vector addition on $\mathcal{V}$ is defined as

$$u + v = uv$$

for any u and v in $\mathcal{V}$ and a scalar multiplication is defined by

$$s.u = u^s$$

for any u in $\mathcal{V}$ and a scalar s.

(a) Determine if $\mathcal{V}$ with these operations is a vector space or not.

(b) If we keep operations the same, but change $\mathcal{V}$ to $\mathcal{V} = \mathbb{R} - \{0\} =$ The set of all the real numbers except 0, will $\mathcal{V}$ be a vector space ?

(c) If we keep operations the same, but change $\mathcal{V}$ such that $\mathcal{V} = \mathbb{R}^+ \cup \{0\} =$ The set of all the non negative numbers, will $\mathcal{V}$ be a vector space ?

a) all axioms on HW

b) No, (S.1) false as $(.5) \cdot (-1) = -1^{.5} = \sqrt{-1} = i \notin V$

c) No, (S.1) false $(-1) \cdot (0) = 0^{-1} = 1/0 \notin V$

Problem 6. Let $\mathcal{V}$ be a vector space with the vector addition $+$ and the scalar multiplication $\cdot$

(a) Using only the vector space axioms, show that $0.u = \mathbf{0}$ for any vector $u \in \mathcal{V}$. (Here $\mathbf{0}$ is the zero vector from the axiom $A.4$)

(b) Using only the vector space axioms, show that $-1.u = -u$ for any vector $u \in \mathcal{V}$. (Here $-u$ is the negative of u from the axiom $A.5$)

(c) Using only the vector space axioms, show that if $u+u = 2.v$ then $u = v$

a) $0 \cdot u = 0 \Rightarrow 0 \cdot u = 0 \cdot u$ $\quad 0 + 0 = 0$

$0 \cdot u = (0+0)u \Rightarrow 0 \cdot u = 0 \cdot u + 0 \cdot u$ (distribute, $0 = 0+0$)

$(0 \cdot u) + (-(0 \cdot u)) = 0 \cdot u + 0 \cdot u + (-(0 \cdot u))$ (subtract from both sides)

simplify

$0 = 0 \cdot u$

b) Prove $-1 \cdot u = -u$ by S.3 $\quad -u + u = 0$

$|-1| = 0$ $\quad (1-1) \cdot u = 0 \cdot u$ $\quad 1u - 1u = 0$

$1 \cdot u = u$ (S.S) $\quad u + (-1 \cdot u) = 0$ $\quad (-1) \cdot u = -u$

c) Prove if $u + u = 2v$ $\quad u = v$

$u + u = 2v \Rightarrow u + u = 1 \cdot u + 1 \cdot u \Rightarrow (1+1)u = 2u$

$2u = 2v \Rightarrow u = v$

Problem 7. Let $\mathcal{V} = \mathcal{M}_{2,2}$ be the set of all the 2×2 matrices with usual operations. Determine whether the subsets given below are subspaces of $\mathcal{M}_{2,2}$.

(a) $\mathcal{U} =$ The set of symmetric matrices in $\mathcal{M}_{2,2} = \{A \in \mathcal{M}_{2,2} \mid A^T = A\}$ (symmetric)

To be a subspace it needs to pass S.1 A.4 & A.1

(b) $\mathcal{U} = \{A \in \mathcal{M}_{2,2} \mid A^2 = A\}$

(c) $\mathcal{U} =$ The set of non invertible matrices in $\mathcal{M}_{2,2}$

a) (A.1) $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ $B = \begin{bmatrix} d & e \\ e & f \end{bmatrix}$

$A + B = \begin{bmatrix} a+d & b+e \\ b+e & c+f \end{bmatrix}$ as same so true, $\in U$

(A.4) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A = A^T$ so $\in U$

(S.1) $r \cdot A = \begin{bmatrix} r \cdot a & r \cdot b \\ r \cdot b & r \cdot c \end{bmatrix} \Rightarrow A = A^T$ so $\in U$

b) (S.1) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad r A \in V$

$2 \cdot A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow (2 \cdot A)^2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$\neq$

$\therefore 2 \cdot A \neq (2 \cdot A)^2$ so the condition $A = A^2$ does not hold under scalar multiplication

c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

No n invertible $\quad$ invertable

So, U is not closed under addition (A.1), hence it is not a subspace of $M_{2,2}$

Problem 8. Let $\mathcal{V} = \mathcal{P}_3$ be the set of all the polynomials with a degree at most 3.

If p and q are in $\mathcal{P}_3$ their addition $p + q$ is defined as

$$(p+q)(x) = p(x) + q(x)$$

If s is a scalar, the scalar multiplication between s and a function p in $\mathcal{P}_3$; $s.p$ is defined as

$$(s.p)(x) = sp(x)$$

$\mathcal{P}_3$ with operations given as above is a vector space. Determine whether the subsets given below are subspaces of $\mathcal{P}_3$ or not.

(a) $\mathcal{U} = \{p \in \mathcal{P}_3 \mid p(2) = 0\}$

(b) $\mathcal{U} = \{p \in \mathcal{P}_3 \mid p(x) = (1-x)g(x)$ for some $g \in \mathcal{P}_2\}$

(c) $\mathcal{U} = \{p \in \mathcal{P}_3 \mid p$ has degree $3\}$

a) $p(2) = 0$ $\quad q(2) = 0$

$(p+q)(2) = p(2) + q(2) = 0 + 0 = 0$

$(p+q)(2) = 0 \Rightarrow$ (A.1) ✓

$r \cdot (p)(2) = r \cdot p(2) = r \cdot 0 = 0 \Rightarrow$ (S.1) ✓

$0(x) = 0 + 0x + 0x^2 \Rightarrow 0(2) = 0 + 0(2) + 0(2)^2 = 0$

$0(2) = 0 \Rightarrow 0 \in U \Rightarrow$ (A.4) ✓

b) $U = \{p \in P_3 \mid p(x) = (1-x)g(x)$ for some $g \in P_2$

(A.1) $(p+q)(x) = (1-x)g(x) + (1-x)g_2(x)$ $\quad g_1, g_2 \in P_2$

$= (1-x)(g(x) + g_2(x))$ $\quad g_1 + g_2 \in P_2$

follows $f(x) = (1-x)(h(x))$ $\quad f \in P_3$, $h \in P_2$

So (A.1) holds

(S.1) $r \cdot p(x) = r \cdot (1-x)g(x)$

$= (1-x)(r \cdot g(x))$ $\quad r \cdot g \in P_2$

(S.1) holds as $r \cdot p \in U$

(A.4) $p(x) = (1-x)g(x) = (1-x)(a_0 + a_1 x + a_2 x^2)$

$0(x) = (1-x)(0 + 0x + 0x^2)$

$= (1-x)(0) = 0$

$\therefore 0(x) = 0$ exists & $0(x) \in U$

Since U holds under S.1 A.1 & A.5 we know U is a subspace of P_3

Set 5 continued

Problem 9. Compute a basis for each of the following subspaces U of the given vector spaces V. In each case, write down the dimension of U.

- $V = \mathcal{P}_2$
  $U = \{p \mid p \text{ has no constant term}\}$
- $V = \mathcal{P}_4$
  $U = \{p \mid p \text{ has no odd power terms}\}$
- $V = \mathcal{P}_2$
  $U = \{p \mid p(x) = p(-x) \text{ for all } x\}$
- $V = M_{3,3}$
  $U = \{A \mid A \text{ is a symmetric matrix}\}$
- $V = M_{3,3}$
  $U = \{A \mid A \text{ is a skew-symmetric matrix}\}$
- $V = M_{3,3}$
  $U = \{A \mid A \text{ is an upper triangular matrix}\}$
- $V = M_{2,2}$
  $U = \{A \mid \text{diagonal entries of } A \text{ are all zero}\}$
- $V = M_{3,2}$
  $U = \{A \mid a_{32} = a_{21} = 0\}$
- $V = M_{2,2}$
  $U = \left\{A \,\middle|\, A\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix} = \begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}A\right\}$
  ; or in other words, U is the set of 2×2 matrices commutes with $\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}$

$V = P_2$ $U = \{p \mid p \text{ has no constant term}\}$ $\dim(U) = 2$

$p(x) = a_0 + a_1x + a_2x^2$

$= a_1x + a_2x^2$ $\Rightarrow U = \text{span}\{x, x^2\}$

$r_1 \cdot x_1 + r_2 x_2^2 = 0 \Rightarrow$ only true if $r_1 = r_2 = 0$

$\therefore \{x, x^2\}$ is lin. indep., making it a basis of U

$V = P_4$ $U = \{p \mid p \text{ has no odd power terms}\}$

$p(x) = a_0 + a_2x^2 + a_4x^4 \Rightarrow \text{span}\{1, x^2, x^4\}$

$r_1 \cdot 1 + r_2 \cdot x + r_3 \cdot x^4 = 0$, only true when

$r_1 = r_2 = r_3 = 0$ so lin. indep.

$\therefore \{1, x, x^2\}$ is a basis of U with $\dim(U) = 3$

$V = P_2$ $U = \{p \mid p(x) = p(-x) \text{ for all } x\}$ $\dim(U) = 2$

$p(x) = a_0 + a_1x + a_2x^2 \Rightarrow a_0 + a_1(-x) + a_2(-x)^2 = a_0 - a_1x + a_2x^2$

$\hookrightarrow a_0 + a_1x + a_2x^2 = a_0 - a_1x + a_2x^2 \Rightarrow a_0 = a_0$ $a_2 = a_2$, but $a_1 = -a_1$, which can only be true if $a_1 = 0$ so $\Rightarrow p(x) = a_0 + a_2x^2 \Rightarrow \text{span}\{1, x^2\} \rightarrow$ lin. indep. $\rightarrow \{1, x^2\}$ is a basis of U

$V = M_{3,3}$ $U = \{A \mid A \text{ is sym. matrix}\}$ $\dim(U) = 6$

$$A = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \Rightarrow A = a\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + b\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} + \ldots$$

$$A = \text{span}\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\} \rightarrow \text{lin. indep.} \rightarrow \text{is a basis of } U$$

$V = M_{3,3}$ $U = \{A \mid A = -A^T\}$ $\dim(U) = 3$

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = a\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix} + b\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix} + c\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}$$

$$A = \text{span}\left\{\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}\right\} \rightarrow \text{lin. indep.} \rightarrow \text{basis of } U$$

$V = M_{3,3}$ $U = \{A \mid A \text{ is an upper triangular matrix}\}$

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = a\cdot\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + b\cdot\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} + c\cdot\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} + d\cdot\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} + e\cdot\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix} + f\cdot\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$A = \text{Span}\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\} \rightarrow \text{lin. indep.} \rightarrow \text{basis of } U \quad \dim(U) = 6$$

$V = M_{2,2}$ $U = \{A \mid \text{Diagonal entries of } A \text{ are all zero}\}$

$$A = \begin{bmatrix}0&a\\b&0\end{bmatrix} = a\cdot\begin{bmatrix}0&1\\0&0\end{bmatrix} + b\cdot\begin{bmatrix}0&0\\1&0\end{bmatrix} \rightarrow U = \text{span}\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}\right\}$$

$\rightarrow$ lin. indep. $\rightarrow \left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}\right\}$ is a basis of U ; $\dim(U) = 2$

$V = M_{3,2}$ $U = \{A \mid a_{32} = a_{21} = 0\}$

$$A = \begin{bmatrix}a&b\\0&c\\d&0\end{bmatrix} = a\cdot\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix} + b\cdot\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix} + c\cdot\begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix} + d\cdot\begin{bmatrix}0&0\\0&0\\1&0\end{bmatrix} \quad \dim(U) = 4$$

$$U = \text{span}\left\{\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\\1&0\end{bmatrix}\right\} \rightarrow \text{lin. indep.} \rightarrow \text{is a basis of } U$$

$V = M_{2,2}$ $U = \left\{A \,\middle|\, A\begin{bmatrix}1&1\\-1&0\end{bmatrix} = \begin{bmatrix}1&1\\-1&0\end{bmatrix}A\right\}$ $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$

$$A\cdot\begin{bmatrix}1&1\\-1&0\end{bmatrix} = \begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&1\\-1&0\end{bmatrix} = 1\cdot\begin{bmatrix}a\\c\end{bmatrix} + (-1)\begin{bmatrix}b\\d\end{bmatrix}^{1\cdot A} \quad 1\begin{bmatrix}a\\c\end{bmatrix} + 0\cdot\begin{bmatrix}b\\d\end{bmatrix}^{1\cdot A} = \begin{bmatrix}a+b & a\\ c-d & c\end{bmatrix}$$

$$\begin{bmatrix}1&1\\-1&0\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix} = a\cdot\begin{bmatrix}1\\-1\end{bmatrix} + c\cdot\begin{bmatrix}1\\0\end{bmatrix}^{1\cdot A} \quad b\begin{bmatrix}1\\-1\end{bmatrix} + d\begin{bmatrix}1\\0\end{bmatrix}^{1\cdot A} = \begin{bmatrix}a+c & b+d\\ -a & -b\end{bmatrix}$$

$$\begin{bmatrix}a+b & a\\ c-d & c\end{bmatrix} = \begin{bmatrix}a+c & b+d\\ -a & -b\end{bmatrix}$$

$a - b = a + c$ $\quad b = -c$

$a = b + d$ $\quad a = b + d$

$c - d = -a$ $\quad d = a + c$

$c = -b$ $\quad c = -b$

$$A = \begin{bmatrix}a&b\\c&d\end{bmatrix} \rightarrow \begin{bmatrix}a&-c\\c&a+c\end{bmatrix} = a\begin{bmatrix}1&0\\0&1\end{bmatrix} + c\begin{bmatrix}0&-1\\1&1\end{bmatrix}$$

$$\text{span}\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&-1\\1&1\end{bmatrix}\right\} \quad r_1\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix} + r_2\begin{bmatrix}0&-1\\1&1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&-1\\1&1\end{bmatrix}\right\} \text{ is a basis of } U \quad \begin{bmatrix}\boxed{r_1}\\ \boxed{r_2}\end{bmatrix} \begin{matrix}-r_2\\ r_1+r_2\end{matrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix} \quad \begin{matrix}r_1 = 0\\ r_2 = 0\end{matrix} \checkmark$$

$\dim(U) = 2$