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MATH 3333 PRACTICE PROBLEM SET 4

Problem 1. Determine whether each of the subset given below is a subspace of $\mathbb{R}^4$ and if so, find a spanning set.

(a) $\left\{ \begin{bmatrix} 0 \\ a \\ b \\ c \end{bmatrix} \middle| \quad a, b \text{ and } c \text{ are real numbers} \right\}$

Solution: Let $\mathcal{S} = \left\{ \begin{bmatrix} 0 \\ a \\ b \\ c \end{bmatrix} \middle| \quad a, b \text{ and } c \text{ are real numbers} \right\}$

In other words, $\mathcal{S}$ is the set of vectors in $\mathbb{R}^4$ which has 0 as the first coordinate.

If we take $a = b = c = 0$ then $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{S}$

Take $u = \begin{bmatrix} 0 \\ a_1 \\ b_1 \\ c_1 \end{bmatrix} \in \mathcal{S}$ and $v = \begin{bmatrix} 0 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix} \in \mathcal{S}$

Then $u + v = \begin{bmatrix} 0 \\ a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix} \in \mathcal{S}$ because the first coordinate of $u + v$ is 0.

Let r be any scalar. Then $r.u = \begin{bmatrix} 0 \\ ra_1 \\ rb_1 \\ rc_1 \end{bmatrix} \in \mathcal{S}$ because the first coordinate of $r.u$ is 0.

To find a spanning set, $u = \begin{bmatrix} 0 \\ a_1 \\ b_1 \\ c_1 \end{bmatrix} = a_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

So any vector in $\mathcal{S}$ can be written as a linear combination of $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

On the other hand, we can clearly see, any linear combination of these vectors have the first coordinate 0 and hence should belong to $\mathcal{S}$.

We can conclude $\mathcal{S} = span \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ a \\ b \\ c \end{bmatrix} \middle| \quad a, b \text{ and } c \text{ are real numbers} \right\}$

Solution: $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is not in this subset. So it cannot be a subspace of $\mathbb{R}^4$.

(c) $\left\{ \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} \middle| \quad a - 2b + 3c = 0 \right\}$

Solution: Let $\mathcal{S} = \left\{ \begin{bmatrix} a \\ b \\ c \\ 0 \end{bmatrix} \middle| \quad a - 2b + 3c = 0 \right\}$

Take $a = b = c = 0$. Then it satisfies the equation $a - 2b + 3c = 0$.

Then $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{S}$

Take $u = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ 0 \end{bmatrix} \in \mathcal{U}$ and $v = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ 0 \end{bmatrix} \in \mathcal{U}$

Then we have $a_1 - 2b_1 + 3c_1 = 0$ and $a_2 - 2b_2 + 3c_2 = 0$.

Then $u + v = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ 0 \end{bmatrix} \in \mathcal{S}$ because

$(a_1 + a_2) - 2(b_1 + b_2) + 3(c_1 + c_2) = (a_1 - 2b_1 + 3c_1) + (a_2 - 2b_2 + 3c_2) = 0 + 0 = 0$

Let r be any scalar. Then $r.u = \begin{bmatrix} ra_1 \\ rb_1 \\ rc_1 \\ 0 \end{bmatrix} \in \mathcal{S}$ because

the first coordinate of $ra_1 - 2rb_1 + 3rc_1 = r(a_1 - 2b_1 + 3c_1) = r \times 0 = 0$.

So $\mathcal{S}$ is a subspace of $\mathbb{R}^4$.

To find a spanning set, take any vector $u = \begin{bmatrix} 0 \\ a_1 \\ b_1 \\ c_1 \end{bmatrix} \in \mathcal{S}$. Then $a_1 - 2b_1 + 3c_1 = 0 \Rightarrow$ $a_1 = 2b_1 - 3c_1$.

So we can write u as $u = \begin{bmatrix} 0 \\ a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2b_1 - 3c_1 \\ b_1 \\ c_1 \end{bmatrix} = b_1 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

So any vector in $\mathcal{S}$ can be written as a linear combination of $\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

On the other hand, a linear combination of these vectors takes the form

$r_1 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2r_1 - 3r_2 \\ r_1 \\ r_2 \end{bmatrix} \in \mathcal{S}$ because $(2r_1 - 3r_2) - 2r_1 + 3r_2 = 0$.

We can conclude $\mathcal{S} = span \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix} \middle| \quad a^2 = b^2 \right\}$

Solution: Let $\mathcal{S} = \left\{ \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix} \middle| \quad a^2 = b^2 \right\}$
This is not a subspace. To see this,

Take $a = 1$ and $b = 1$. Then $a^2 = b^2$. Hence the vector $u = \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in \mathcal{U}$

Take $a = 1$ and $b = -1$. Then $a^2 = b^2$. Hence the vector $u = \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \in \mathcal{U}$

But $u + v = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin \mathcal{U}$ because $2^2 \neq 0^2$

(e) $\left\{ \begin{bmatrix} 2a \\ 0 \\ a-2 \\ b \end{bmatrix} \middle| \quad a \text{ and } b \text{ are real numbers} \right\}$

Solution: This is not a subspace because we cannot find values for a and b such that $\begin{bmatrix} 2a \\ 0 \\ a-2 \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Hence $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin \mathcal{S}$

(f) $\left\{ \begin{bmatrix} 2a \\ 0 \\ b-2 \\ c \end{bmatrix} \middle| \quad a, b \text{ and } c \text{ are real numbers} \right\}$

Solution: Let $\mathcal{S} = \left\{ \begin{bmatrix} 2a \\ 0 \\ b-2 \\ c \end{bmatrix} \middle| \quad a, b \text{ and } c \text{ are real numbers} \right\}$

Take $a = 0, b = 2, c = 0$ then $\begin{bmatrix} 2a \\ 0 \\ b-2 \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathcal{S}$

Take $u = \begin{bmatrix} 2a_1 \\ 0 \\ b_1 - 2 \\ c_1 \end{bmatrix} \in \mathcal{S}$ and $v = \begin{bmatrix} 2a_2 \\ 0 \\ b_2 - 2 \\ c_2 \end{bmatrix} \in \mathcal{S}$

Then $u + v = \begin{bmatrix} 2a_1 + 2a_2 \\ 0 \\ b_1 + b_2 - 4 \\ c_1 + c_2 \end{bmatrix}$

If we take $a = a_1 + a_2, b = b_1 + b_2 - 2$ and $c = c_1 + c_2$

then $\begin{bmatrix} 2a \\ 0 \\ b-2 \\ c \end{bmatrix} = \begin{bmatrix} 2a_1 + 2a_2 \\ 0 \\ b_1 + b_2 - 4 \\ c_1 + c_2 \end{bmatrix} = u + v$

So, $u + v \in \mathcal{S}$

Let s be any scalar. Then $s.u = \begin{bmatrix} 2sa_1 \\ 0 \\ s(b_1 - 2) \\ sc_1 \end{bmatrix}$

If we take $a = sa_1$, $b = sb_1 - 2s + 2$ and $c = sc_1$

then $\begin{bmatrix} 2a \\ 0 \\ b-2 \\ c \end{bmatrix} = \begin{bmatrix} 2sa_1 \\ 0 \\ sb_1 - 2s + 2 - 2 \\ sc_1 \end{bmatrix} = \begin{bmatrix} 2sa_1 \\ 0 \\ sb_1 - 2s \\ sc_1 \end{bmatrix} = s.u$

Hence $s.u \in \mathcal{S}$.

So, $\mathcal{S}$ is a subspace of $\mathbb{R}^4$.

Now we want to find a spanning set for $\mathcal{S}$.
Take any vector in $\mathcal{S}$.

$$\begin{bmatrix} 2a \\ 0 \\ b-2 \\ c \end{bmatrix} = a \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (b-2) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So $\mathcal{S} \subseteq span \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

On the other hand, a linear combination of these vectors takes the form

$$r_1 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2r_1 \\ 0 \\ r_2 \\ r_3 \end{bmatrix}$$

If we take $a = r_1, b = r_2 + 2, c = r_3$ then $\begin{bmatrix} 2r_1 \\ 0 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2a \\ 0 \\ b-2 \\ c \end{bmatrix} \in \mathcal{S}$

We can conclude $\mathcal{S} = span \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(g) $\left\{ \begin{bmatrix} r \\ 0 \\ s \\ 0 \end{bmatrix} \middle| \quad r^2 + s^2 = 0 \quad \text{and} \quad r \text{ and } s \text{ are real numbers} \right\}$

Solution: As I'm typing solutions, I realized this is not a good problem. The only way $r^2 + s^2$ is equal to 0 is if $r = s = 0$. So the only vector in this subset is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. So it is a subspace and a spanning set would be $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Problem 2. Let $\mathcal{S}$ and $\mathcal{T}$ be two subspaces of $\mathbb{R}^n$. Show that $\mathcal{S} \cap \mathcal{T}$ is also a subspace of $\mathbb{R}^n$.
Solution: This was proved in the class. Please refer to your class notes

Problem 3. Let $u_1, u_2, \dots, u_k$ be vectors in $\mathbb{R}^n$ and $\mathcal{S}$ be a subspace of $\mathbb{R}^n$ that contains all those vectors. Show that $span\{u_1, u_2, \dots u_k\} \subseteq \mathcal{S}$.
Solution: Any vector in $span\{u_1, u_2, \dots u_k\}$ takes the form $p = r_1u_1 + r_2u_2 + \dots r_ku_k$.

Since $u_i \in \mathcal{S}$ and $\mathcal{S}$ is a subspace $\Rightarrow r_iu_i \in \mathcal{S}$.

So, for $i = 1, 2, \dots, k$ we have $r_iu_i \in \mathcal{S}$.

Since $\mathcal{S}$ is a subspace, addition of any two vectors in $\mathcal{S}$ should also be in $\mathcal{S}$.

This means $r_1u_1 + r_2u_2 \in \mathcal{S}$. Then $(r_1u_1 + r_2u_2) + r_3u_3$ should also be in $\mathcal{S}$ for the same reason. Continuing this way, we see that $p = r_1u_1 + r_2u_2 + \dots r_ku_k \in \mathcal{S}$. Hence for any $p \in span\{u_1, u_2, \dots u_k\}$ belongs to $\mathcal{S}$.

Therefore $span\{u_1, u_2, \dots u_k\} \subseteq \mathcal{S}$.

Problem 4. Let u, v and w be three vectors in $\mathbb{R}^n$. Show the following.

(a) $span\{u, v\} = span\{u, v, u - 2v\}$
Solution: Let $p = r_1u + r_2v \in span\{u, v\}$.
We can write $p = r_1u + r_2v + 0(u - 2v) \in span\{u, v, u - 2v\}$
Hence $span\{u, v\} \subset span\{u, v, u - 2v\}$

On the other hand, let $q = s_1u + s_2v + s_3(u - 2v) \in span\{u, v, u - 2v\}$. We can write $q = (s_1 - s_3)u + (s_2 - 2s_3)v \in span\{u, v\}$
Hence $span\{u, v, u - 2v\} \subset span\{u, v\}$

We conclude $span\{u, v\} = span\{u, v, u - 2v\}$

(b) $span\{u, v, w\} = span\{u - 2v, v, w\} = span\{5u, 2v, v - w\}$
Solution: Let $p = r_1u + r_2v + r_3w \in span\{u, v, w\}$.
We can write $p = r_1(u - 2v) + 2r_1v + r_2v + r_3w = r_1(u - 2v) + (2r_1 + r_2)v + r_3w \in span\{u - 2v, v, w\}$
Hence $span\{u, v, w\} \subset span\{u - 2v, v, w\}$

On the other hand, let $q = s_1(u - 2v) + s_2v + s_3w \in span\{u - 2v, v, w\}$.
We can write $q = s_1u + (s_2 - 2s_1)v + s_3w \in span\{u, v, w\}$
Hence $span\{u - 2v, v, w\} \subset span\{u, v, w\}$

We conclude $span\{u - 2v, v, w\} = span\{u, v, w\}$

You can prove the remaining equation similarly.

Problem 5. Show that any set of vectors with the zero vector is linearly dependent.
Solution: This was discussed in the class. Please refer to your class notes.

Problem 6. Determine whether the sets given below are linearly independent or not.

(a) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -6 \\ 6 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -6 \end{bmatrix} \right\}$

Solution: To answer problem 6 part a and b, write the vectors in the given set as columns of a matrix and then compute the RRE form of the matrix.
If the rank = number of vectors then the set is linearly independent.
If the rank < number of vectors then the set is linearly dependent.

Problem 7. Find a basis and write down the dimension of the following subspaces of $\mathbb{R}$.

(a) $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \middle| a + b = c + d \right\}$

Solution: Let $\mathcal{S}$ be this subspace.

Any vector in S can be written as $\begin{bmatrix} a \\ b \\ c \\ a+b-c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$

$\Rightarrow \mathcal{S} \subset span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

On the other hand, any linear combination of vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ takes the form

$$r_1\begin{bmatrix}1\\0\\0\\1\end{bmatrix}+r_2\begin{bmatrix}0\\1\\0\\1\end{bmatrix}+r_3\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}=\begin{bmatrix}r_1\\r_2\\r_3\\r_1+r_2-r_3\end{bmatrix}\in\mathcal{S} \text{ because } r_1+r_2=r_3+(r_1+r_2-r_3)$$

Therefore, we have $\mathcal{S} = span\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}\right\}$

If we can show, this set is linearly independent then it will be a basis for $\mathcal{S}$.

To show this, we create a matrix A which has these vectors as its columns.

$A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&-1\end{bmatrix}$ has the RRE form $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$

So rank of $A = 3 =$ number of vectors.

Hence the set $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}\right\}$ is linearly independent.

It follows $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}\right\}$ is a basis of $\mathcal{S}$.

Hence $dim(\mathcal{S}) = 3$

(b) $\left\{\begin{bmatrix}a+b\\a-b\\b\\a\end{bmatrix}\middle| a,b\in\mathbb{R}\right\}$

Solution: By following the same steps as in part (a), you can show that the set $\left\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}\right\}$ is a basis for this subspace and its dimension is 2.

(c) $\left\{\begin{bmatrix}a+2b\\2a+4b\\0\\-a-2b\end{bmatrix}\middle| a,b\in\mathbb{R}\right\}$

Solution: Let $\mathcal{S}$ be this subspace.

By following the same steps as in part (a), you can first show that the set $\mathcal{S} = span\left\{\begin{bmatrix}1\\2\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\4\\0\\-2\end{bmatrix}\right\}$

The next part is to check if this set is linearly independent. To check this, we create a matrix A which has these vectors as its columns.

$A = \begin{bmatrix}1 & 2\\2 & 4\\0 & 0\\-1 & -2\end{bmatrix}$ has the RRE form $\begin{bmatrix}1 & 0\\0 & 0\\0 & 0\\0 & 0\end{bmatrix}$

So rank of $A = 1 < 2 =$ number of vectors. So this set of vectors is linearly dependent. Then how do we find a basis for $\mathcal{S}$?

Note that $\mathcal{S} = span\left\{\begin{bmatrix}1\\2\\0\\-1\end{bmatrix}, \begin{bmatrix}2\\4\\0\\-2\end{bmatrix}\right\} = col(A)$. This means all we need is a basis for the column space of A. We can find one by looking at the RRE form of A. The RRE form has a leading 1 in the first column. So the first column of A must form a basis for $col(A) = \mathcal{S}$.

So, a basis for $\mathcal{S}$ would be $\left\{\begin{bmatrix}1\\2\\0\\-1\end{bmatrix}\right\}$

$dim(\mathcal{S}) = 1$

Problem 8. Solve the following linear system by using elementary row operations.

$$x_2 - 6x_3 + 5x_4 = -7$$

$$x_1 + 2x_3 - x_4 = 5$$

$$3x_1 + x_2 + 2x_4 = 8$$

Solution: The augmented matrix is

$$\left[\begin{array}{cccc|c}0 & 1 & -6 & 5 & -7\\1 & 0 & 2 & -1 & 5\\3 & 1 & 0 & 2 & 8\end{array}\right]$$

By $R_3 \to R_3 - 3R_2$ have

$$\left[\begin{array}{cccc|c}0 & 1 & -6 & 5 & -7\\1 & 0 & 2 & -1 & 5\\0 & 1 & -6 & 5 & -7\end{array}\right]$$

Then $R_3 \to R_3 - R_1$ followed by $R_1 \leftrightarrow R_2$ yields

$$\left[\begin{array}{cccc|c}1 & 0 & 2 & -1 & 5\\0 & 1 & -6 & 5 & -7\\0 & 0 & 0 & 0 & 0\end{array}\right]$$

Solutions are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5-2s+t \\ -7+6s-5t \\ s \\ t \end{bmatrix}$$

Answer the following.

Let $A = \begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & 0 & 2 & -1 \\ 3 & 1 & 0 & 2 \end{bmatrix}$

(a) Find bases for the column space, row space and the null space of A.

Solution: The RRE form of A is $\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -6 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

A basis for $Row(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -6 \\ 5 \end{bmatrix} \right\}$

The RRE form of A has leading 1s in the first and second columns. So the first and second columns of A form a basis for $Col(A)$; a basis for $Col(A)$ is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s+t \\ 6s-5t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

The general solution to $AX = 0$ is given by Hence a basis for $Null(A)$ is $\left\{ \begin{bmatrix} -2 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) Determine whether the vector $v = \begin{bmatrix} -7 \\ 5 \\ 8 \end{bmatrix}$ is in the column space of A. If so, write v as a linear combination of the columns of A.

Solution: If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix}$ then $AX = x_1 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$

We have already seen $AX = v$ has infinitely many solutions. Hence there are x_1, x_2, x_3 and x_4 such that $AX = x_1 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -6 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = v = \begin{bmatrix} -7 \\ 5 \\ 8 \end{bmatrix}$

This means the vector v can be written as a linear combination of columns of A

Hence $v \in Col(A)$.

Problem 9. Find a basis for $span\{u_1, u_2, u_3, u_4\}$

where $u_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}$ and $u_4 = \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}$

Solution: Create a matrix A such that columns of A are x_1, x_2, x_3 and x_4.

$A = \begin{bmatrix} 1 & 3 & 11 & 7 \\ 2 & 2 & 10 & 6 \\ 2 & 1 & 7 & 4 \end{bmatrix}$

Then $span\{x_1, x_2, x_3, x_4\} = Col(A)$

By $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 2R_1$ we obtain

$\begin{bmatrix} 1 & 3 & 11 & 7 \\ 0 & -4 & -12 & -8 \\ 0 & -5 & -15 & -10 \end{bmatrix}$

By $R_2 \to R_2/-4$ and $R_3 \to R_3/-5$, we have

$\begin{bmatrix} 1 & 3 & 11 & 7 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

By $R_3 \to R_3 - R_2$, we have $\begin{bmatrix} 1 & 3 & 11 & 7 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is a row-echelon form. It has leading 1s in column 1 and 2. Hence the column 1 and 2 of A forms a basis for $Col(A) = span\{x_1, x_2, x_3, x_4\}$.

Hence $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $span\{x_1, x_2, x_3, x_4\}$

Problem 10. Let $u_1, u_2, \dots u_{15}$ be vectors in $\mathbb{R}^{15}$. Construct a matrix A such that columns of A are $u_1, u_2, \dots u_{14}$ and u_{15}.

(a) What is the size of A ? **Solution:** 15×15

For each of the following cases, determine whether the set of vectors $\{u_1, u_2, \dots u_{15}\}$ is linearly independent or not.

(b) If the rank of A is 15. **Solution:** Linearly independent because rank(A) = No. of vectors
(c) If the rank of A is 14. **Solution:** Linearly dependent because rank(A) < No. of vectors
(d) If the rank of A^T is 12. **Solution:** Linearly dependent because $rank(A^T) = rank(A) <$ No. of vectors
(e) If the determinant of A is 12. **Solution:** Linearly independent because $|A| \neq 0 \Rightarrow$ rank(A)=15 = No. of vectors
(f) If the determinant of A^T is 0.01 **Solution:** Linearly independent because $|A^T| = |A| \neq 0 \Rightarrow$ rank(A)=15 = No. of vectors

Problem 11. Let u_1, u_2 and u_3 be three vectors in $\mathbb{R}^n$ such that the set $\{u_1, u_2, u_3\}$ is linearly independent.
If u_4 is another vector in $\mathbb{R}^n$ such that x_4 is NOT in $span\{u_1, u_2, u_3\}$, show that the set $\{u_1, u_2, u_3, u_4\}$ is linearly independent.

Hint : We do not know the value of n or the coordinates of the vectors. So we cannot create a matrix A with these vectors as its columns.
So, the only way to show this is to use the actual definition of linear independence. Set

$$r_1u_1 + r_2u_2 + r_3u_3 + r_4u_4 = 0$$

and try to show the only way this can happen is when $r_1 = r_2 = r_3 = r_4 = 0$.

First figure out what happens if $r_4 = 0$. Then think about the case when $r_4 \neq 0$.

Solution: Set

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$$

. If $c_4 \neq 0$ then we have $c_4x_4 = -c_1x_1 - c_2x_2 - c_3x_3$. By dividing by c_4 we have

$$x_4 = \left(\frac{-c_1}{c_4}\right)x_1 + \left(\frac{-c_2}{c_4}\right)x_2 + \left(\frac{-c_3}{c_4}\right)x_3$$

This means x_4 can be written as a linear combination of x_1, x_2 and x_3. Hence $x_4 \in span\{x_1, x_2, x_3\}$.

But this is not true as it is given $x_4 \notin span\{x_1, x_2, x_3\}$. Hence it is not possible for c_4 to be not zero.

So $c_4 = 0$. Then we have

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = c_1x_1 + c_2x_2 + c_3x_3 = 0$$

. Since $\{x_1, x_2, x_3\}$ is linearly independent, all of c_1, c_2 and c_3 should also be equal to 0.

This means $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0$.

Hence the set $\{x_1, x_2, x_3, x_4\}$ is linearly independent.