1 Rosenzweig-MacArthur consumer-resource model isoclines

Our Lotka-Volterra predator-prey model was one that tracked the rates of gains and losses of prey, V, and predator, P, populations:

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \text{prey gains} - \text{prey losses} \\
&= \text{density-independent growth} - \text{consumption of prey proportional to prey density} \\
&= \alpha V - \beta b V P \\
\frac{\mathrm{d}P}{\mathrm{d}t} &= \text{predator gains} - \text{predator losses} \\
&= \text{predator gain through prey conversion} - \text{density-independent mortality} \\
&= \gamma V P - \delta P.
\end{aligned}
\tag{1}
$$

Rosenzweig and MacArthur studied a model that modified two terms: (1) they replaced density-independent growth of prey with logistic/density-dependent growth and (2) they replaced consumption of prey proportional to prey density (type I functional response) with consumption of prey that includes the time it takes predators to handle the prey before returning to searching for prey (type II functional response). The predator-prey/consumer-resource model is now

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= rV\left(\frac{K-V}{K}\right) - \frac{aV}{1+ahV}P \\
\frac{\mathrm{d}P}{\mathrm{d}t} &= c\frac{aV}{1+ahV}P - mP.
\end{aligned}
\tag{2}
$$

To find the nullclines for each we set each to 0 and algebraically rearrange and plot. For the prey equation, we can first factor out V and we know that $V = 0$ is one nullcline. The other factor we can rearrange:

$$
\begin{aligned}
0 &= r\left(\frac{K-V}{K}\right) - \frac{a}{1+ahV}P \\
\frac{a}{1+ahV}P &= r\left(\frac{K-V}{K}\right) \\
\frac{1}{1+ahV}P &= \frac{r}{a}\left(\frac{K-V}{K}\right) \\
\frac{1}{1+ahV}P &= \frac{r}{a}\left(\frac{K-V}{K}\right) \\
P &= \frac{r}{a}\left(\frac{K-V}{K}\right)(1+ahV) \\
P &= \left(\frac{r}{a} - \frac{r}{Ka}V\right)(1+ahV) \\
P &= -\frac{rh}{K}V^2 + \left(rh - \frac{r}{Ka}\right)V + \frac{r}{a}.
\end{aligned}
\tag{3}
$$

This nullcline takes a quadratic form, $0 = ax^2 + bx + c$, so we can see that it is a convex parabola whose shape and location changes depending on parameters that affect a, b, or c.

For the predator equation, we can first factor out P and we know that $P = 0$ is one nullcline. The other factor we can rearrange:

$$
\begin{aligned}
0 &= c\frac{aV}{1+ahV} - m \\
c\frac{aV}{1+ahV} &= m \\
\frac{aV}{1+ahV} &= \frac{m}{c} \\
aV &= \frac{m}{c}(1+ahV) \\
V &= \frac{m}{ac}(1+ahV) \\
V &= \frac{m}{ac} + \frac{m}{ac}(ahV) \\
V - \frac{m}{ac}(ahV) &= \frac{m}{ac} \\
V(1 - \frac{mah}{ac}) &= \frac{m}{ac} \\
V\left(\frac{ac-mah}{ac}\right) &= \frac{m}{ac} \\
V &= \frac{m}{ac}\left(\frac{ac}{ac-mah}\right) \\
V &= \frac{m}{a(c-mh)}
\end{aligned}
\tag{4}
$$

Here, the predator rate of change is at 0 when $V =$ a combination of constants. This means $V =$ a constant whose location changes depending on parameters in the numerator or denominator (e.g., increasing a will increase the denominator thereby shifting the line toward 0).

Together, the prey nullcline will be a convex parabola and the predator nullcline with be a line.