Modified Monty Hall Problems

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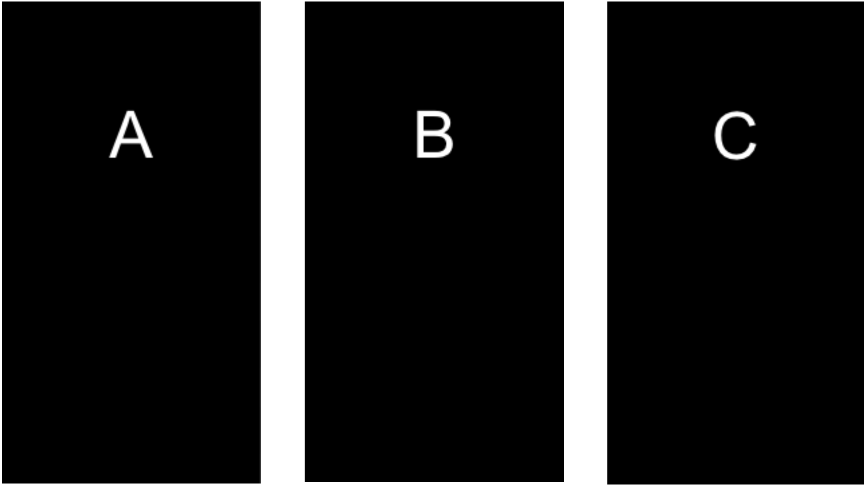
The Question

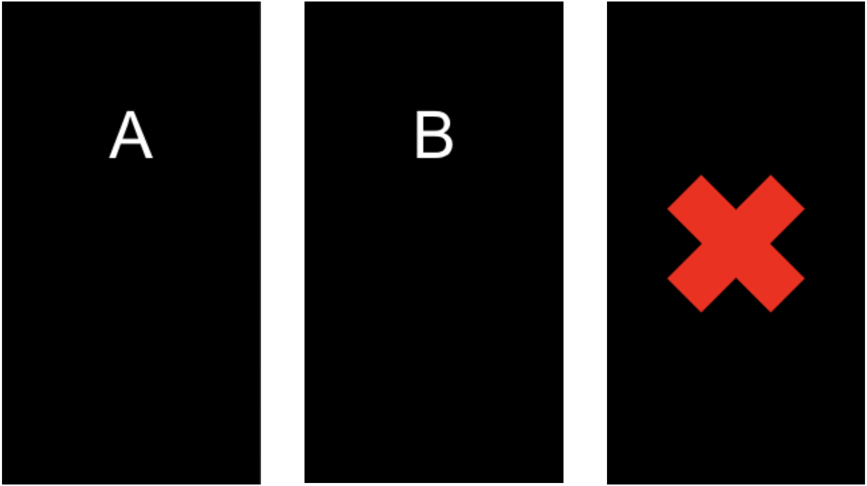
“Suppose you’re on a game show and given a choice of three doors. Behind one is a car; behind the others are goats. You pick door No. 1, and the host, who knows what’s behind them, opens No. 3, which has a goat. He then asks if you want to pick No. 2. Should you switch?” [Morgan et al., 1991, p. 284].

Example

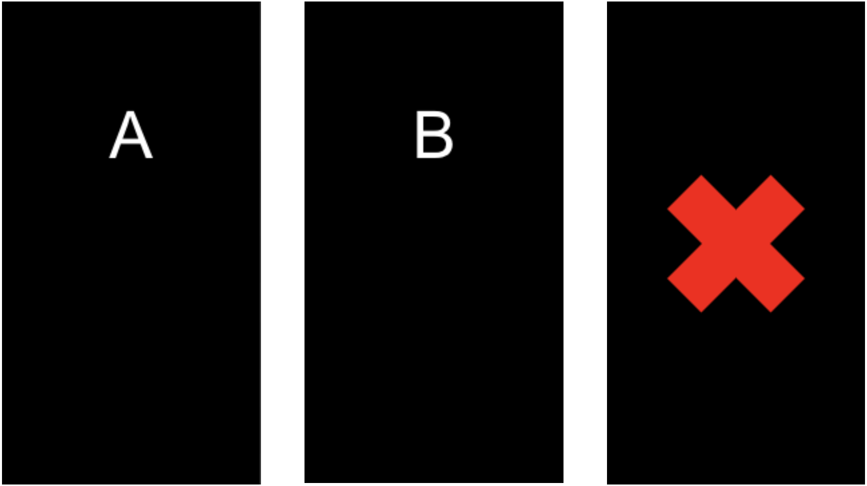
Here we will do the iconic example of the Monty Hall Problem:



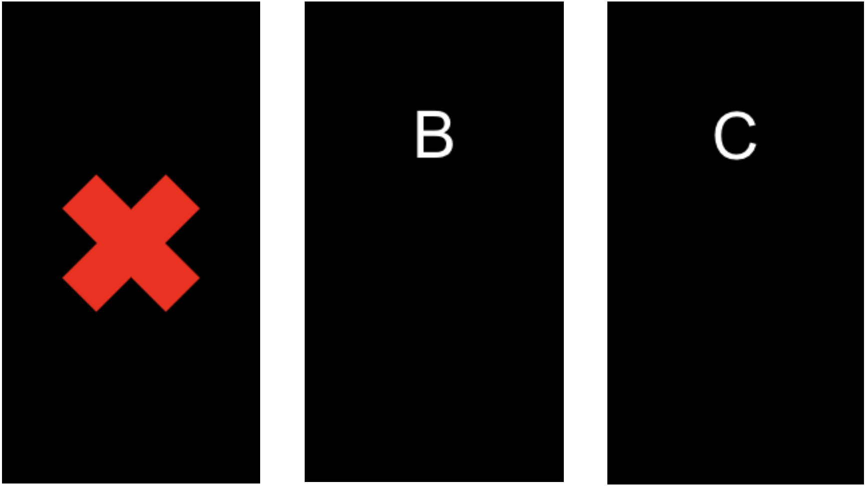
Door A



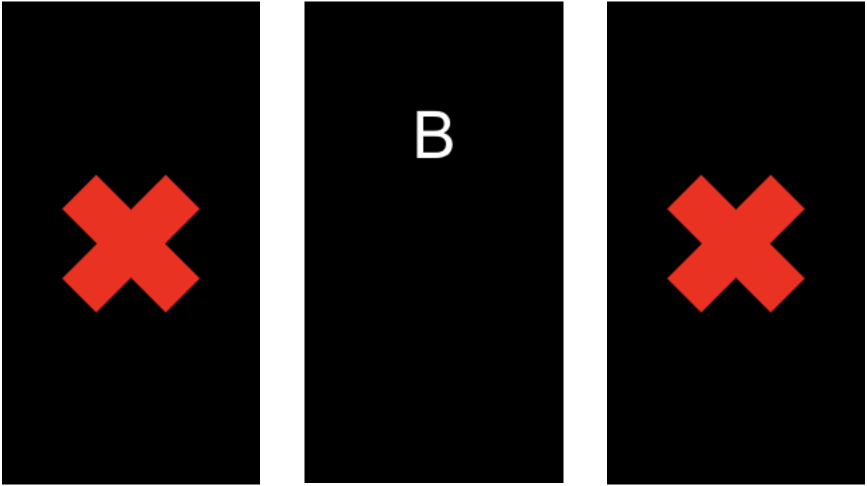
Door B



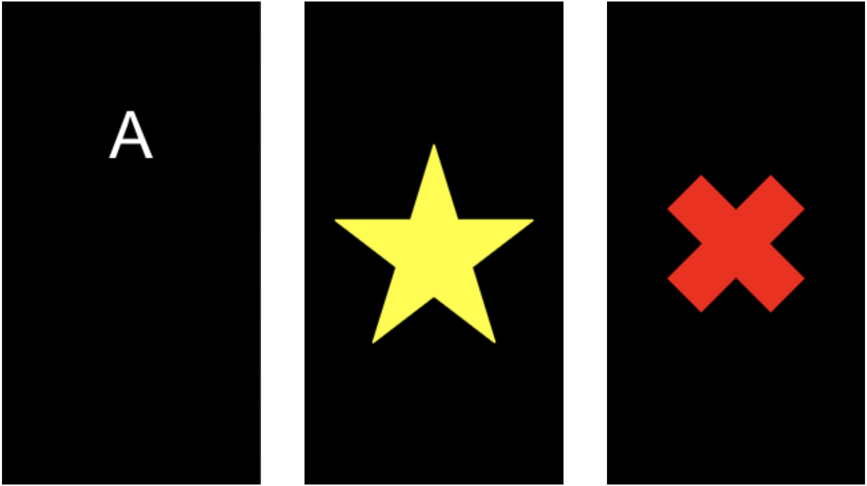
Door C



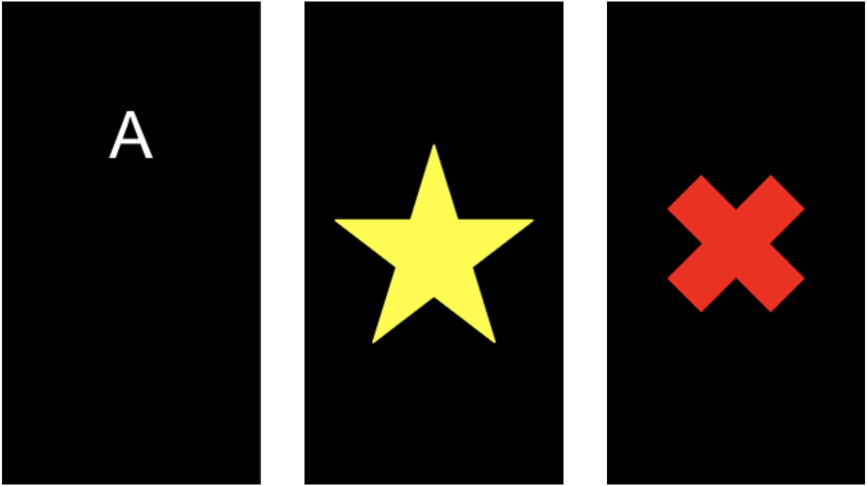
Door A Stay



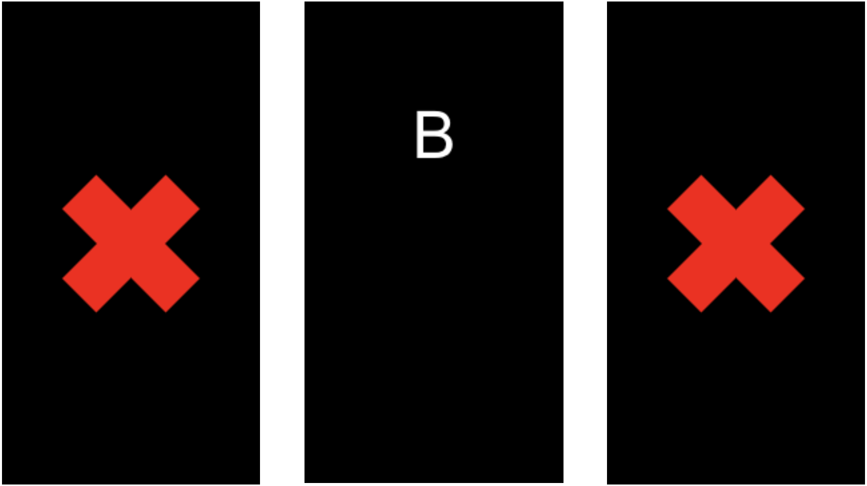
Door A Switch



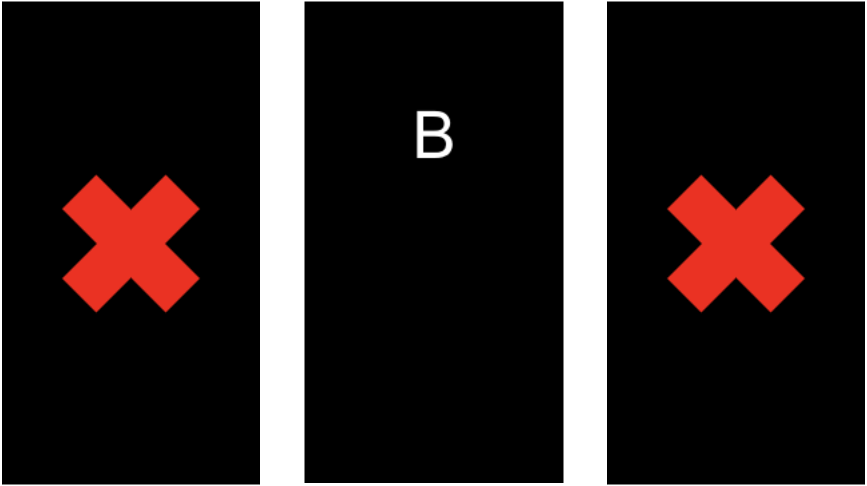
Door B Stay



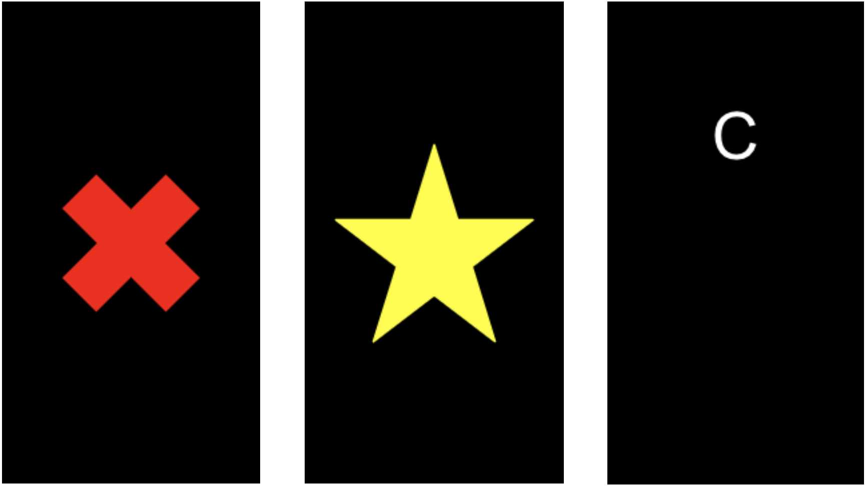
Door B Switch



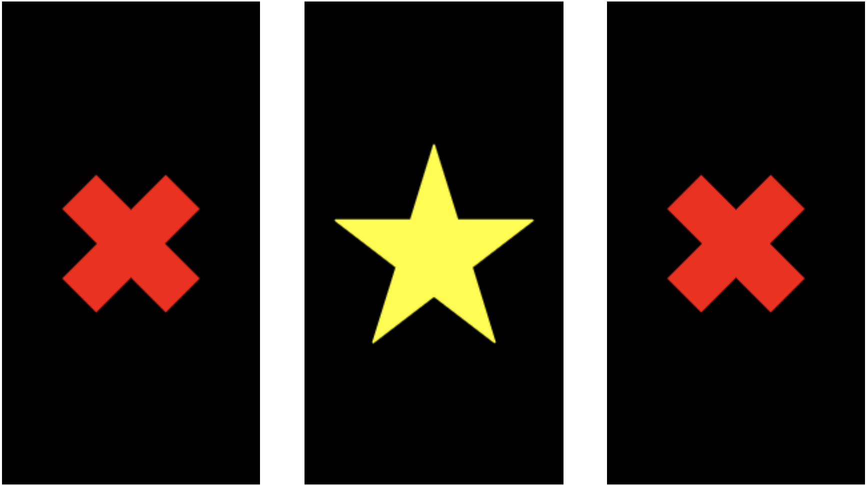
Door C Stay



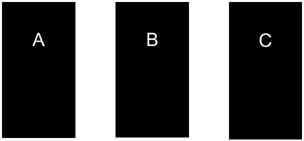
Door C Switch



End Of Show



Analysis



$$\Pr(A) = \Pr(B) = \Pr(C) = \frac{1}{3}.$$

Pick door A, then the probability of the non-picked doors winning is

$$\Pr(B) + \Pr(C) = \frac{2}{3}.$$

However, once a losing door is removed

$$\Pr(C) = 0.$$

Thus, the probability of the non-picked doors winning is

$$\Pr(B) + 0 = \Pr(B) = \frac{2}{3}.$$

Analysis (Cases)

Another way of looking at it is by cases, either you pick the winning door or you don't.

- Case 1: You pick the winning door.
  - Thus, when you switch you will lose.
  - The probability of this case is $\frac{1}{3}$.
- Case 2: You don't pick the winning door.
  - Thus, when you switch you are guaranteed to win because there is only one door left, the winning door.
  - The probability of this case is the complement of case 1. That is, the probability is $\frac{2}{3}$.

What is Going to be Different?

In this presentation, we will be demonstrating different variants of the problem. The following will be subject to variation

- the total number of doors, t
- the number of winning doors, w
- the number of doors a contestant is allowed to pick, p
- the number of losing doors the host can open, r

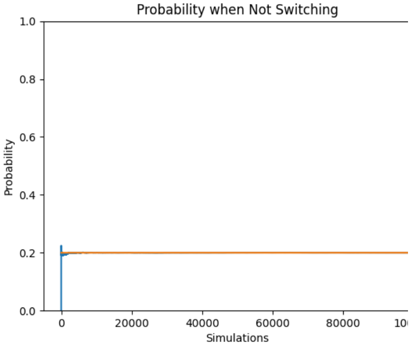
Restrictions so our problem makes sense:

- $t \geq r + w + p$
- $p \leq t - p - r = s$

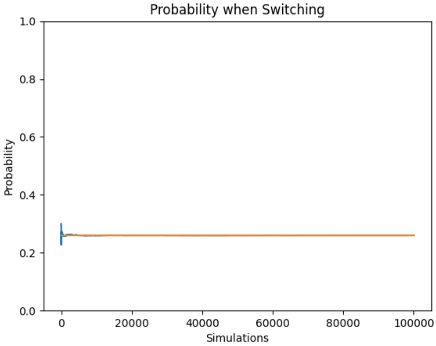
Pseudo-Code for Method 1

- Create 3 vectors that dictate the positions of each picked, winner, and opened door.
- Fill the picked and winner vector with random unique positions.
- Fill the opened vector with positions that are random, unique, and not in picked or winner.
- If the player will switch, replace all the positions in picked with positions that are both not in picked or opened.
- Cast picked and winners into sets and see if there is an intersection between picked and winners. If there is, return the number of intersecting elements.

Example for Method 1



Final Probability: 0.20018.



Final Probability: 0.26086.

Formulas for Method 1

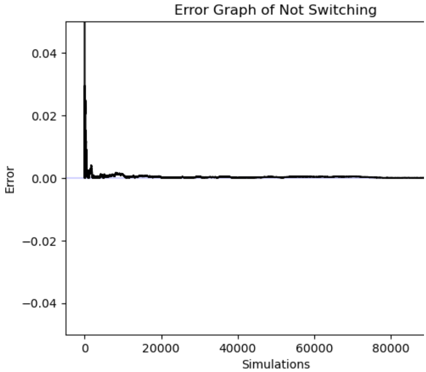
$$\Pr(\text{Not Switching}) = \frac{w}{t}$$

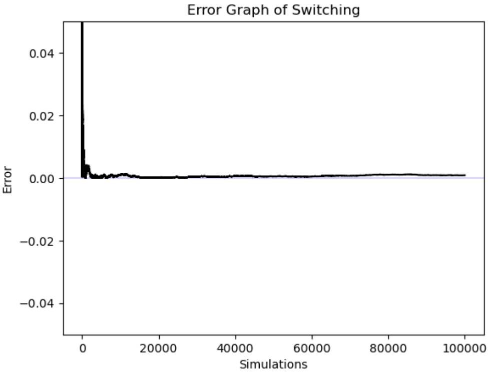
- This is obvious from our assumption of the winning door distribution being uniform

$$\Pr(\text{Switching}) = \frac{t-p}{t} \cdot \frac{w}{s}$$

- Where $s = t - p - r$
- The first fraction denotes the fraction of remaining non-picked doors
- The second fraction represents the probability of each remaining door, s, being a winning door

Comparing the Simulation and Formulas for Method 1





Simulation for not Switching = 0.20018 and Formula for not Switching = 0.2.

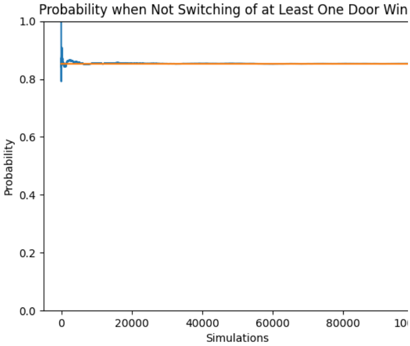
Simulation for Switching 0.26086 and Formula for Switching 0.26

What Changes from Method 1 to Method 2

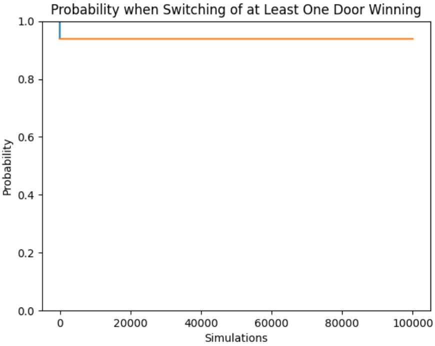
Instead of determining the probability of each pick, we want to observe the probability of each trial having at least one winning door picked

- So, in this method, if the intersection is not empty, the value is 1, otherwise it is 0
- We can then add all of our trials together and divide by the number of trials to determine the probability of winning at least once

Example for Method 2



Final Probability: 0.8513.



Final Probability: 0.93779.

Formula 1 for Method 2

$$\Pr(\text{Not Switching}) = 1 - \frac{(t-p)!}{(t-w-p)!} \cdot \frac{(t-w)!}{t!} = 1 - \prod_{i=0}^{p-1} \frac{t-w-i}{t-i}$$

- Probability discovered by taking the complement of the losing cases
- First fraction denotes the values for the numerator
- Second fraction denotes values for denominator
- Look at the non-winning doors and calculate the probability that they are all not picked

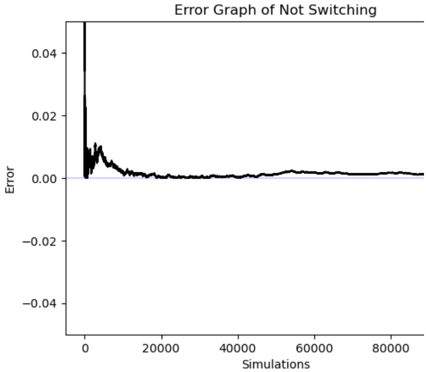
Formula 2 for Method 2

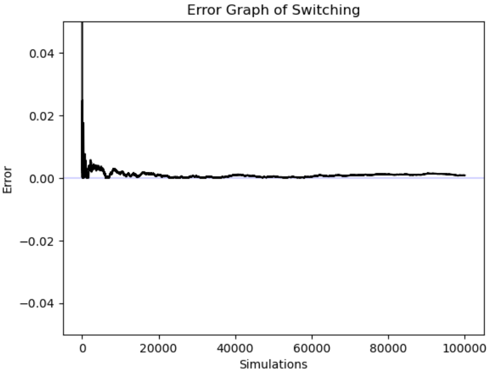
Pr(Switching) =

$$1-\sum_{i=0}^{p}\left(\binom{p}{i}\cdot\prod_{n=0}^{p-1}\frac{1}{t-n}\cdot\prod_{k=0}^{p-i-1}(t-w-k)\cdot\prod_{\ell=0}^{i-1}(w-\ell)\cdot\prod_{j=0}^{p-1}\frac{s-w+i-j}{s-j}\right)$$

- i denotes the number of winning doors picked in the initial phase
- First finite product is the denominator for the probability
- The second and third finite products represent the numerators for the probability of picking a losing door and winning door, respectively
- The last finite product is the probability of not picking a winning door after switching

Comparing the Simulation and Formulas for Method 2





Simulation for not Switching 0.8513 and Formula for not Switching 0.852425181

Simulation for Switching 0.93779 and Formula for Switching 0.938620571

Discussion

- It is almost always better to switch rather than stay
- The run time exponentially grows based on how many picked, winner, and opened doors there are, which makes it inefficient
- Highlights a fun everyday usage of probability theory

Future Work

Potential changes to our program:

- Change method 2 to check for at least a variable number of doors
- Making the switch function more specific (i.e. not binary)
- Could extend the problem to the continuous case, having the winning door placed on a point and the contestant chooses an interval of measure n bound above at point t and below at point 0

References I

Morgan, J. P., Chaganty, N. R., Dahiya, R. C., and Doviak, M. J. (1991).
Let's make a deal: The player's dilemma.
The American Statistician, 45(4):284–287.