Modified Monty Hall Problems

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Abstract

Here we will write an abstract/summary of our project.

1. Introduction

In 1991, a question appeared in *Parade Magazine*, asking Marilyn vos Savant, the reportedly holder of the world's highest I.Q., the following: "Suppose you're on a game show and given a choice of three doors. Behind one is a car; behind the others are goats. You pick door No. 1, and the host, who knows what's behind them, opens No. 3, which has a goat. He then asks if you want to pick No. 2. Should you switch?" [Morgan et al., 1991, p. 284]. Here, vos Savant responded that you should switch because the first door had a 1/3 chance of winning while the second door had a 2/3 chance [Morgan et al., 1991]. Surprisingly, vos Savant's answer was met with backlash as 92% of the letters from the general public along with 65% of the letters from universities were against her answer [Morgan et al., 1991]. Here, many people believed that the probability of each remaining choice was 1/2, claiming that the revealing of the losing door did not matter [Morgan et al., 1991]. Hence, the objective of this paper is to, first, explore the original problem, and then to, second, simulate and derive different variants of the problem.

2. Explanation of the Problem

In this paper, we will be generalizing the above problem, hence the title of the paper. The modifications will include variation in total number of doors, number of winning doors, number of doors a contestant can pick, and the number of losing doors the host can open. For these variants, simulations and formulations will be included, as well as a focus on two different scoring methods: a comprehensive probability (i.e. the number of winning doors the contestant picked divided by the total number of doors the contestant picked) and an "at least one" probability (i.e. the probability of winning at least once per round).

3. Mathematical Theory

When dealing with method one, or the comprehensive probability, it is easy to deduce the probability of not switching doors. Let t represent the total number of doors, w the number of winning doors, p the number of picked doors, and r the number of losing doors revealed. Then the probability of picking the winning door without switching is

$$\Pr(\text{Not Switching}) = \frac{w}{t}. \tag{1}$$

This is obvious as for each door selected, the probability is uniform across each door. That is, if there are 10 doors and 3 winning doors, then the probability of winning without switching is 3/10 for each pick. On the other hand, the probability of switching is more difficult to notice. Let t, w, p, and r be defined as before and s be the number of doors that are available to be switched to, i.e. $s = t - p - r$. Then, the probability of winning is

$$\Pr(\text{Switching}) = \frac{t-p}{t} \cdot \frac{w}{s}. \tag{2}$$

Here, the first fraction denotes the fraction of remaining non-picked doors. We do this because we want to observe what doors are not picked after the initial phase. Afterwards, we divide that fraction by s, or the remaining doors that can be switched to, revealing the probability of each s door being a winning door. Then we multiply that probability by the number of winning doors, resulting in (2).

We can then observe method one, the "at least one" probability. First, let's look at the probability of not switching doors, which is

$$\Pr(\text{Not Switching}) = 1 - \frac{(t-p)!}{(t-w-p)!} \cdot \frac{(t-w)!}{t!} = 1 - \prod_{i=0}^{w-1} \frac{t-p-i}{t-i}. \tag{3}$$

This probability is discovered by taking the complement of the losing cases. Here, the first fraction denotes the values for the numerator and the second fraction denotes the values for the denominator. For example, let $t = 9, p = 3$, and $w = 2$, then the probability of winning when not switching is $1 - (6/9) \cdot (5/8)$, since there is a $(6/9)$ chance of the winning door being behind a non-picked door and a $(5/8)$ chance of the other winning door being behind another non-picked door. Remember that we those this is the losing case, so we take the complement and thus we get our original answer. But, note that this is equivalent to

$$1-(9-3)!/9! \cdot (9-2)!/(9-2-3)! = 1-6!/9! \cdot 7!/4! = 1-6!/4! \cdot 7!/9! = 1-(6 \cdot 5)/(9 \cdot 8).$$

This probability can also be expressed in the form of a finite product as seen above.

Finally, the other case is the probability of switching doors. This probability is

$$\Pr(\text{Switching}) = 1 - \sum_{i=0}^{p} \left(\binom{p}{i} \cdot \frac{(t-p)!}{t!} \cdot \prod_{k=0}^{p-i-1} (t-w-k) \cdot \prod_{\ell=0}^{i-1} (w-\ell) \cdot \prod_{j=0}^{p-1} \frac{s-w+i-j}{s-j} \right). \tag{4}$$

This probability can be derived by looking at different cases within the problem. These cases are classified by the number of winning doors selected by the contestant in the initial phase. The variable that represents this number of winning doors selected is i. The choose function at the beginning represents the number of different ways of choosing these winning doors, thus it is here to account for these combinations. The fraction afterward is the "denominator" fraction, where it is used to divide the following cases to get our probability. This fraction can also be written as

$$\frac{(t-p)!}{t!} = \prod_{n=0}^{p-1} \frac{1}{t-n}.$$

The first finite product represents the number of losing doors picked, while the second finite product represents the number of winning doors picked for each case. Then, the third finite product looks at the remaining doors and determines the probability of losing. All cases are then summed, indicating the probability of not picking a single door, so the complement must indicate the probability of winning at least once.

4. Simulations

5. Discussion

6. Future Work

$$\Pr(\text{Switching}) = 1 - \sum_{m=0}^{n-1} \left(\sum_{i=0}^{p} \left(\binom{p}{i} \cdot \prod_{k=0}^{p-1} \frac{1}{t-k} \cdot \prod_{k=0}^{p-i-1} (t-w-k) \cdot \prod_{\ell=0}^{i-1} (w-\ell) \cdot \binom{p}{m} \cdot \prod_{k=0}^{p-1} \frac{1}{s-k} \cdot \prod_{j=0}^{p-m-1} (s-w+i-j) \cdot \prod_{k=0}^{m-1} (w-i-k) \right) \right)$$

$$\Pr(\text{Not Switching}) = 1 - \sum_{m=0}^{n-1} \left(\binom{p}{m} \prod_{k=0}^{p-1} \frac{1}{t-k} \cdot \prod_{i=0}^{p-m-1} (t-w-i) \cdot \prod_{\ell=0}^{m-1} (w-\ell) \right)$$

References

[Morgan et al., 1991] Morgan, J. P., Chaganty, N. R., Dahiya, R. C., and Doviak, M. J. (1991). Let's make a deal: The player's dilemma. *The American Statistician*, 45(4):284–287.