Newton's Method

Solving Roots with Derivatives

Colby Cox
University of Central Arkansas

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Quadratic Equation

$$ax^2 + bx + c = 0$$

Quadratic Equation

$$ax^2 + bx + c = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubic Equation

$$ax^3 + bx^2 + cx + d = 0$$

Cubic Equation

$$ax^3 + bx^2 + cx + d = 0$$

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} + \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} + \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

Quartic Equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

Quartic Equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$x_1 = -\frac{b}{4a} - \frac{1}{2}\sqrt{\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}{3\sqrt[3]{2}a} + \frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{\sqrt[3]{2}\left(c^2 - 3bd + 12ae\right)}{3a\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}}$$

$$- \frac{1}{2}\sqrt{-\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}{3\sqrt[3]{2}a} + \frac{b^2}{2a^2} - \frac{4c}{3a} - \frac{\sqrt[3]{2}\left(c^2 - 3bd + 12ae\right)}{3a\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}} - \frac{-\frac{b^3}{a^3} + \frac{4cb}{a^2} - \frac{8d}{a}}{4\sqrt{\frac{\sqrt[3]{2c^3-9bdc-72aec+27ad^2+27b^2e+\sqrt{\left(2c^3-9bdc-72aec+27ad^2+27b^2e\right)^2-4\left(c^2-3bd+12ae\right)^3}}}{3\sqrt[3]{2}a} + \frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{\sqrt[3]{2}\left(c^2-3bd+12ae\right)}{3a\sqrt[3]{2c^3-9bdc-72aec+27ad^2+27b^2e+\sqrt{\left(2c^3-9bdc-72aec+27ad^2+27b^2e\right)^2-4\left(c^2-3bd+12ae\right)^3}}}}}}$$

Quartic Equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$
\begin{aligned}
x_2 = -\frac{b}{4a}
&- \frac{1}{2}\sqrt{\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}{3\sqrt[3]{2}a} + \frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{\sqrt[3]{2}\left(c^2 - 3bd + 12ae\right)}{3a\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}} \\
&+ \frac{1}{2}\sqrt{-\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}{3\sqrt[3]{2}a} + \frac{b^2}{2a^2} - \frac{4c}{3a} - \frac{\sqrt[3]{2}\left(c^2 - 3bd + 12ae\right)}{3a\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}} - \frac{-\frac{b^3}{a^3} + \frac{4cb}{a^2} - \frac{8d}{a}}{4\sqrt{\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}{3\sqrt[3]{2}a} + \frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{\sqrt[3]{2}\left(c^2 - 3bd + 12ae\right)}{3a\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}}}}
\end{aligned}
$$

Quartic Equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$x_3 = -\frac{b}{4a} + \frac{1}{2}\sqrt{\frac{\sqrt[3]{2c^3-9bdc-72aec+27ad^2+27b^2e+\sqrt{\left(2c^3-9bdc-72aec+27ad^2+27b^2e\right)^2-4\left(c^2-3bd+12ae\right)^3}}}{3\sqrt[3]{2}a}+\frac{b^2}{4a^2}-\frac{2c}{3a}+\frac{\sqrt[3]{2}\left(c^2-3bd+12ae\right)}{3a\sqrt[3]{2c^3-9bdc-72aec+27ad^2+27b^2e+\sqrt{\left(2c^3-9bdc-72aec+27ad^2+27b^2e\right)^2-4\left(c^2-3bd+12ae\right)^3}}}} - \frac{1}{2}\sqrt{-\frac{\sqrt[3]{2c^3-9bdc-72aec+27ad^2+27b^2e+\sqrt{\left(2c^3-9bdc-72aec+27ad^2+27b^2e\right)^2-4\left(c^2-3bd+12ae\right)^3}}}{3\sqrt[3]{2}a}+\frac{b^2}{2a^2}-\frac{4c}{3a}-\frac{\sqrt[3]{2}\left(c^2-3bd+12ae\right)}{3a\sqrt[3]{2c^3-9bdc-72aec+27ad^2+27b^2e+\sqrt{\left(2c^3-9bdc-72aec+27ad^2+27b^2e\right)^2-4\left(c^2-3bd+12ae\right)^3}}}+\frac{-\frac{b^3}{a^3}+\frac{4cb}{a^2}-\frac{8d}{a}}{4\sqrt{\frac{\sqrt[3]{2c^3-9bdc-72aec+27ad^2+27b^2e+\sqrt{\left(2c^3-9bdc-72aec+27ad^2+27b^2e\right)^2-4\left(c^2-3bd+12ae\right)^3}}}{3\sqrt[3]{2}a}+\frac{b^2}{4a^2}-\frac{2c}{3a}+\frac{\sqrt[3]{2}\left(c^2-3bd+12ae\right)}{3a\sqrt[3]{2c^3-9bdc-72aec+27ad^2+27b^2e+\sqrt{\left(2c^3-9bdc-72aec+27ad^2+27b^2e\right)^2-4\left(c^2-3bd+12ae\right)^3}}}}}}$$

Quartic Equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$x_4 = -\frac{b}{4a} + \frac{1}{2}\sqrt{\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}{3\sqrt[3]{2}a} + \frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{\sqrt[3]{2}\left(c^2 - 3bd + 12ae\right)}{3a\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}}$$
$$+ \frac{1}{2}\sqrt{-\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}{3\sqrt[3]{2}a} + \frac{b^2}{2a^2} - \frac{4c}{3a} - \frac{\sqrt[3]{2}\left(c^2 - 3bd + 12ae\right)}{3a\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}} + \frac{-\frac{b^3}{a^3} + \frac{4cb}{a^2} - \frac{8d}{a}}{4\sqrt{\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}{3\sqrt[3]{2}a} + \frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{\sqrt[3]{2}\left(c^2 - 3bd + 12ae\right)}{3a\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e + \sqrt{\left(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e\right)^2 - 4\left(c^2 - 3bd + 12ae\right)^3}}}}}}$$

Quintic Equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

Quintic Equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

There is no quintic equation

Solving a Quintic Equation

$$f(x) = x^5 - 5x + 3 = 0$$

Solving a Quintic Equation

$$f(x) = x^5 - 5x + 3 = 0$$
$$f'(x) = 5x^4 - 5, x_0 = 2$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{25}{75} = 1.667$$

Solving a Quintic Equation

$$f(x) = x^5 - 5x + 3 = 0$$
$$f'(x) = 5x^4 - 5, x_0 = 2$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{25}{75} = 1.667$$

$$x_2 = 1.443, x_3 = 1.32, x_4 = 1.28$$
$$x_5 = 1.276, x_6 = 1.276, x_7 = 1.276$$

Newton's Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

- x_k: Our current guess for the root
- x_{k+1}: Our next guess for the root

Tolerance

Tolerance is extremely dependent on the application you are designing for. For example:

38 digits of π is sufficient to calculate the circumference of the universe.

We only need to calculate as many digits we can see on a calculator.

Square Root

Use Newton's Method to find any square root.

Let $\sqrt{a} = x$.

$$a = x^2, x^2 - a = 0$$

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$$f'(x) = 2x$$

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Let $f(x) = x^2 - a$.

$$f'(x) = 2x$$

Plug in to Newton's Method:

$$x_{k+1} = x_k - \frac{x_k^2 - a}{2x_k} = \frac{1}{2}\left(x_k - \frac{a}{x_k}\right)$$

Solving a Square Root

Using Newton's Method to solve $\sqrt{16}$.

Let's choose an initial guess of 16 and a tolerance of 0.001

$$f(x) = x^2 - 16, x_0 = 16$$

$$x_1 = \frac{1}{2}\left(16 - \frac{16}{16}\right) = 8.5$$

$$|8.5 - 16| = 7.5 \not< 0.001$$

Solving a Square Root

Using Newton's Method to solve $\sqrt{16}$.

Let's choose an initial guess of 16 and a tolerance of 0.001

$$f(x) = x^2 - 16, x_0 = 16$$
$$x_1 = \frac{1}{2}\left(16 - \frac{16}{16}\right) = 8.5$$
$$|8.5 - 16| = 7.5 \not< 0.001$$

Now let's use 8.5 as our x_k

$$x_2 = \frac{1}{2}\left(8.5 - \frac{16}{8.5}\right) = 5.191$$
$$|5.191 - 8.5| = 3.3088 \not< 0.001$$

Let's continue iterating until the tolerance is satisfied

Solving a Square Root

Using Newton's Method to solve $\sqrt{16}$

$$x_3 = \frac{1}{2}\left(5.191 - \frac{16}{5.191}\right) = 4.136664$$

$$|4.136664 - 5.191| = 1.0545 \nless 0.001$$

$$x_4 = \frac{1}{2}\left(4.136664 - \frac{16}{4.136664}\right) = 4.0022...$$

$$|4.136664 - 4.0022| = 0.1344 \nless 0.001$$

$$x_5 = \frac{1}{2}\left(4.022 - \frac{16}{4.022}\right) = 4.0000006367$$

$$|4.022 - 4.0000006367| = 0.002 \nless 0.001$$

Solving a Square Root

Using Newton's Method to solve $\sqrt{16}$

$$x_6 = \frac{1}{2}\left(4.0000006367 - \frac{16}{4.0000006367}\right) = 4$$

$$|4 - 4.0000006367| = 6.367 \times 10^{-7} < 0.001$$

So $\sqrt{16} = 4$

nth Root

Use Newton's Method to find the nth root.

Let $\sqrt[n]{a} = x$.

$$a = x^n, x^n - a = 0$$

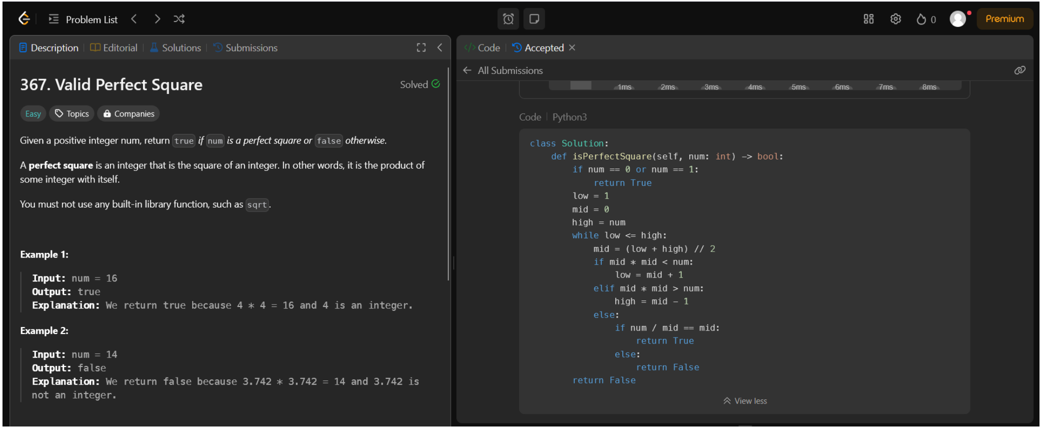
Let $f(x) = x^n - a$.

$$f'(x) = nx^{n-1}$$

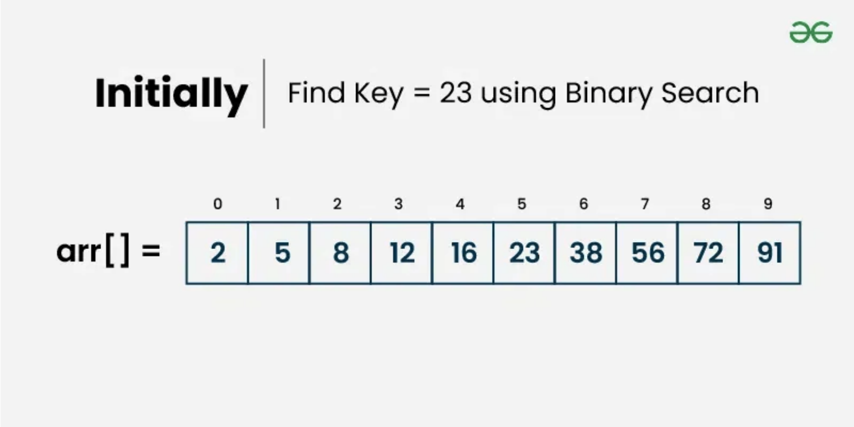
Plug in to Newton's Method:

$$x_{k+1} = x_k - \frac{x_k^n - a}{nx_k^{n-1}} = \frac{1}{n}\left(x_k(n-1) + \frac{a}{x_k^{n-1}}\right)$$

Leetcode Problem



Binary Search



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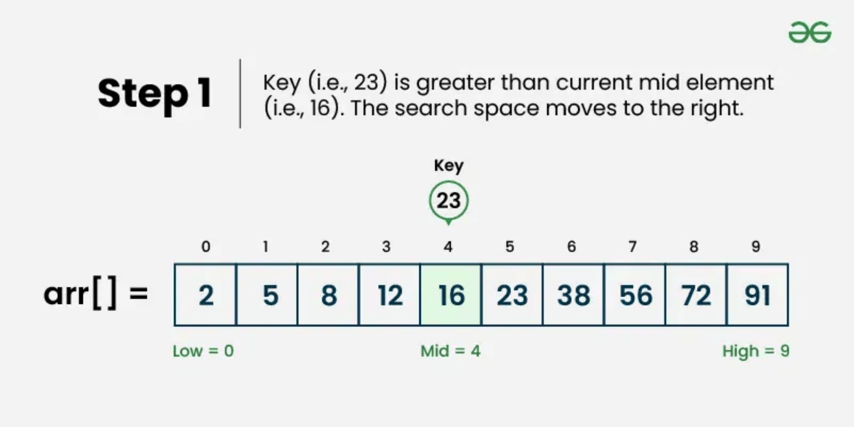

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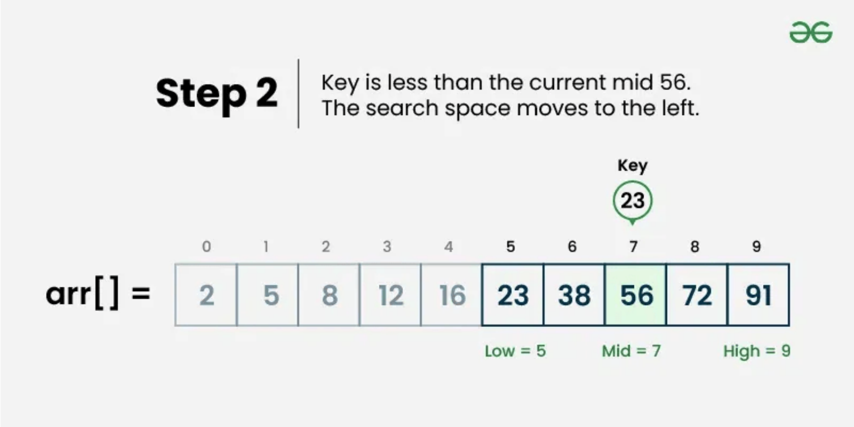

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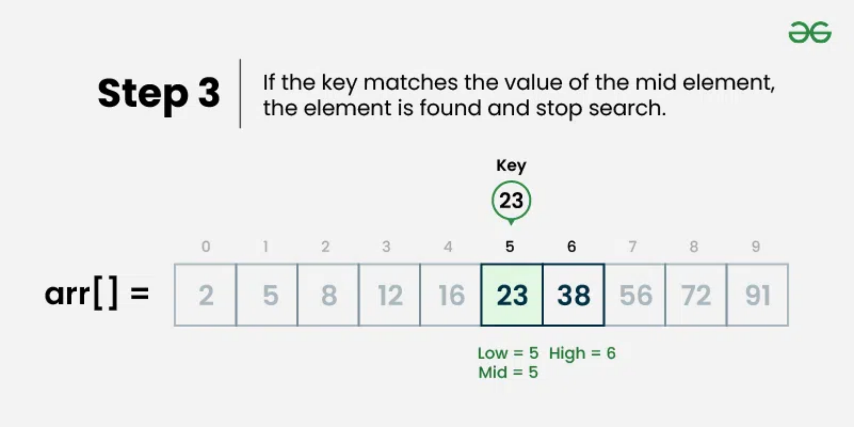

Edge Case for Newton's Method

Use Newton's Method to solve $f(x) = \sqrt[3]{x}$

Let's choose an initial guess of 1 and a tolerance of 0.001

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}, x_0 = 1$$

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$$x_1 = 1 - \frac{1^{\frac{1}{3}}}{\frac{1}{3}1^{-\frac{2}{3}}} = -2$$

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$$x_2 = -2 - \frac{-2^{\frac{1}{3}}}{\frac{1}{3}\cdot -2^{-\frac{2}{3}}} = 4$$

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$$x_3 = -8, x_4 = 16, x_5 = -32, x_k = (-2)^k$$

When $f(x) = \sqrt[3]{x}$, Newton's method fails to find the solution of $x = 0$ when $x_0 \neq 0$.