

Newton's Method

Solving Roots with Derivatives

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Quadratic Equation

$$ax^2 + bx + c = 0$$

Quadratic Equation

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubic Equation

$$ax^3 + bx^2 + cx + d = 0$$

Cubic Equation

$$ax^3 + bx^2 + cx + d = 0$$

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} + \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} + \\ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

Quartic Equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

Quartic Equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$x_1 = -\frac{b}{4a}$$

$$\begin{aligned} & -\frac{1}{2} \sqrt{\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e} + \sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}{3\sqrt[3]{2a}}} + \frac{b^2}{4a^2} - \frac{2c}{3a} \\ & + \frac{\sqrt[3]{2}(c^2 - 3bd + 12ae)}{3a \sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e} + \sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}} \\ & -\frac{1}{2} \sqrt{\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e} + \sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}{3\sqrt[3]{2a}}} + \frac{b^2}{2a^2} - \frac{4c}{3a} \\ & - \frac{\sqrt[3]{2}(c^2 - 3bd + 12ae)}{3a \sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e} + \sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}} \\ & - \frac{-\frac{b^2}{a^3} + \frac{4cb}{a^2} - \frac{8d}{a}}{4 \sqrt{\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e} + \sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}{3\sqrt[3]{2a}}} + \frac{b^2}{4a^2} - \frac{2c}{3a}} \\ & + \frac{\sqrt[3]{2}(c^2 - 3bd + 12ae)}{3a \sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e} + \sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}} \end{aligned}$$

Quartic Equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$x_2 = -\frac{b}{4a}$$

$$\begin{aligned} & -\frac{1}{2} \sqrt{\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e} + \sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}{3\sqrt[3]{2}a} + \frac{b^2}{4a^2} - \frac{2c}{3a}} \\ & + \frac{\sqrt[3]{2}(c^2 - 3bd + 12ae)}{3a\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e} + \sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}} \\ & + \frac{1}{2} \sqrt{\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e} + \sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}{3\sqrt[3]{2}a} + \frac{b^2}{2a^2} - \frac{4c}{3a}} \\ & - \frac{\sqrt[3]{2}(c^2 - 3bd + 12ae)}{3a\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e} + \sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}} \\ & - \frac{\frac{b^2}{a^3} + \frac{4cb}{a^2} - \frac{8d}{a}}{4\sqrt{\frac{\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e} + \sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}{3\sqrt[3]{2}a} + \frac{b^2}{4a^2} - \frac{2c}{3a}} \\ & + \frac{\sqrt[3]{2}(c^2 - 3bd + 12ae)}{3a\sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e} + \sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}} \end{aligned}$$

Quartic Equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$x_3 = -\frac{b}{4a}$$

$$\begin{aligned}
 & + \frac{1}{2} \sqrt[3]{\frac{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}{\sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}} + \frac{b^2}{4a^2} - \frac{2c}{3a} \\
 & + \frac{\sqrt[3]{2}a}{3a} \sqrt[3]{\frac{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}{\sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}} \\
 & - \frac{1}{2} \sqrt[3]{\frac{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}{\sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}} + \frac{b^2}{2a^2} - \frac{4c}{3a} \\
 & + \frac{\sqrt[3]{2}a}{3a} \sqrt[3]{\frac{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}{\sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}} \\
 & + \frac{-\frac{b^3}{a^2} + \frac{4cb}{a^2} - \frac{8d}{a}}{4} \sqrt[3]{\frac{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}{\sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}} + \frac{b^2}{4a^2} - \frac{2c}{3a} \\
 & + \frac{\sqrt[3]{2}a}{3a} \sqrt[3]{\frac{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}{\sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}}
 \end{aligned}$$

Quartic Equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$x_4 = -\frac{b}{4a}$$

$$\begin{aligned}
 & + \frac{1}{2} \sqrt[3]{\frac{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}{\sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}} + \frac{b^2}{4a^2} - \frac{2c}{3a} \\
 & + \frac{\sqrt[3]{2}a}{3a \sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}} + \frac{\sqrt[3]{2}(c^2 - 3bd + 12ae)}{3a \sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}} \\
 & + \frac{1}{2} \sqrt[3]{\frac{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}{\sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}} + \frac{b^2}{2a^2} - \frac{4c}{3a} \\
 & - \frac{\sqrt[3]{2}a}{3a \sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}} + \frac{\sqrt[3]{2}(c^2 - 3bd + 12ae)}{3a \sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}} \\
 & + \frac{-\frac{b^2}{a^3} + \frac{4cb}{a^2} - \frac{8d}{a}}{4 \sqrt[3]{\frac{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}{\sqrt{(2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e)^2 - 4(c^2 - 3bd + 12ae)^3}}} + \frac{b^2}{4a^2} - \frac{2c}{3a}} \\
 & + \frac{\sqrt[3]{2}a}{3a \sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}} + \frac{\sqrt[3]{2}(c^2 - 3bd + 12ae)}{3a \sqrt[3]{2c^3 - 9bdc - 72aec + 27ad^2 + 27b^2e}}
 \end{aligned}$$

Quintic Equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

Quintic Equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

There is no quintic equation

Solving a Quintic Equation

$$f(x) = x^5 - 5x + 3 = 0$$

Solving a Quintic Equation

$$f(x) = x^5 - 5x + 3 = 0$$

$$f'(x) = 5x^4 - 5, x_0 = 2$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{25}{75} = 1.667$$

Solving a Quintic Equation

$$f(x) = x^5 - 5x + 3 = 0$$

$$f'(x) = 5x^4 - 5, x_0 = 2$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{25}{75} = 1.667$$

$$x_2 = 1.443, x_3 = 1.32, x_4 = 1.28$$

$$x_5 = 1.276, x_6 = 1.276, x_7 = 1.276$$

Newton's Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

- x_k : Our current guess for the root
- x_{k+1} : Our next guess for the root

Tolerance

Tolerance is extremely dependent on the application you are designing for. For example:



38 digits of π is sufficient to calculate the circumference of the universe.



We only need to calculate as many digits we can see on a calculator.

Square Root

Use Newton's Method to find any square root.

Let $\sqrt{a} = x$.

$$a = x^2, x^2 - a = 0$$

Square Root

Use Newton's Method to find any square root.

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Let $f(x) = x^2 - a$.

$$f'(x) = 2x$$

Square Root

Use Newton's Method to find any square root.

Let $\sqrt{a} = x$.

$$a = x^2, x^2 - a = 0$$

Let $f(x) = x^2 - a$.

$$f'(x) = 2x$$

Plug in to Newton's Method:

$$x_{k+1} = x_k - \frac{x_k^2 - a}{2x_k} = \frac{1}{2} \left(x_k + \frac{a}{x_k} \right)$$

Solving a Square Root

Using Newton's Method to solve $\sqrt{16}$.

Let's choose an initial guess of 16 and a tolerance of 0.001

$$f(x) = x^2 - 16, x_0 = 16$$

$$x_1 = \frac{1}{2} \left(16 - \frac{16}{16} \right) = 8.5$$

$$|8.5 - 16| = 7.5 \not< 0.001$$

Solving a Square Root

Using Newton's Method to solve $\sqrt{16}$.

Let's choose an initial guess of 16 and a tolerance of 0.001

$$f(x) = x^2 - 16, x_0 = 16$$

$$x_1 = \frac{1}{2} \left(16 - \frac{16}{16} \right) = 8.5$$

$$|8.5 - 16| = 7.5 \not< 0.001$$

Now let's use 8.5 as our x_k

$$x_2 = \frac{1}{2} \left(8.5 - \frac{16}{8.5} \right) = 5.191$$

$$|5.191 - 8.5| = 3.3088 \not< 0.001$$

Let's continue iterating until the tolerance is satisfied

Solving a Square Root

Using Newton's Method to solve $\sqrt{16}$

$$x_3 = \frac{1}{2} \left(5.191 - \frac{16}{5.191} \right) = 4.136664$$

$$|4.136664 - 5.191| = 1.0545 \not\leq 0.001$$

$$x_4 = \frac{1}{2} \left(4.136664 - \frac{16}{4.136664} \right) = 4.0022\dots$$

$$|4.136664 - 4.0022| = 0.1344 \not\leq 0.001$$

$$x_5 = \frac{1}{2} \left(4.022 - \frac{16}{4.022} \right) = 4.0000006367$$

$$|4.022 - 4.0000006367| = 0.002 \not\leq 0.001$$

Solving a Square Root

Using Newton's Method to solve $\sqrt{16}$

$$x_6 = \frac{1}{2} \left(4.0000006367 - \frac{16}{4.0000006367} \right) = 4$$

$$|4 - 4.0000006367| = 6.367 \times 10^{-7} < 0.001$$

$$\text{So } \sqrt{16} = 4$$

nth Root

Use Newton's Method to find the nth root.

Let $\sqrt[n]{a} = x$.

$$a = x^n, x^n - a = 0$$

Let $f(x) = x^n - a$.

$$f'(x) = nx^{n-1}$$

Plug in to Newton's Method:

$$x_{k+1} = x_k - \frac{x_k^n - a}{nx_k^{n-1}} = \frac{1}{n} \left(x_k(n-1) + \frac{a}{x_k^{n-1}} \right)$$

Leetcode Problem

The screenshot displays the LeetCode web application. On the left, the problem page for '367. Valid Perfect Square' is shown, including its description, difficulty level (Easy), and two examples. On the right, the 'Code' tab is active, showing a Python solution that implements a binary search algorithm to check if a number is a perfect square.

367. Valid Perfect Square Solved

Easy Topics Companies

Given a positive integer `num`, return `true` if `num` is a perfect square or `false` otherwise.

A **perfect square** is an integer that is the square of an integer. In other words, it is the product of some integer with itself.

You must not use any built-in library function, such as `sqrt`.

Example 1:

Input: `num = 16`
Output: `true`
Explanation: We return `true` because $4 * 4 = 16$ and 4 is an integer.

Example 2:

Input: `num = 14`
Output: `false`
Explanation: We return `false` because $3.742 * 3.742 = 14$ and 3.742 is not an integer.

Code | Accepted X

All Submissions

Code | Python3

```
class Solution:
    def isPerfectSquare(self, num: int) -> bool:
        if num == 0 or num == 1:
            return True
        low = 1
        mid = 0
        high = num
        while low <= high:
            mid = (low + high) // 2
            if mid * mid < num:
                low = mid + 1
            elif mid * mid > num:
                high = mid - 1
            else:
                if num / mid == mid:
                    return True
                else:
                    return False
        return False
```

View less

Binary Search



Initially

Find Key = 23 using Binary Search

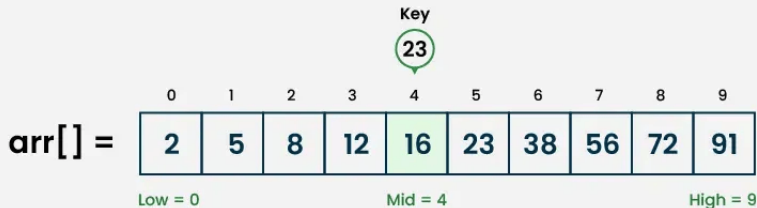
	0	1	2	3	4	5	6	7	8	9
arr[] =	2	5	8	12	16	23	38	56	72	91

Binary Search



Step 1

Key (i.e., 23) is greater than current mid element (i.e., 16). The search space moves to the right.

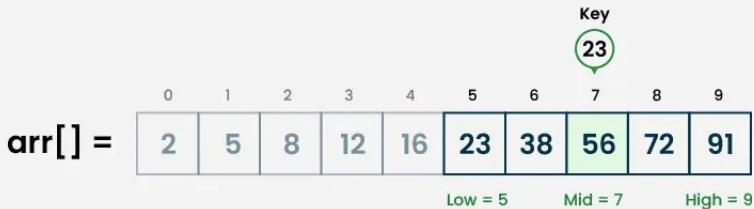


Binary Search



Step 2

Key is less than the current mid 56.
The search space moves to the left.

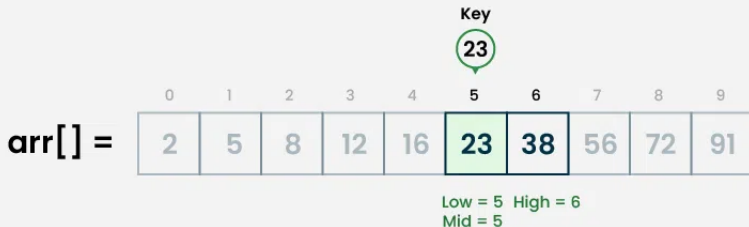


Binary Search



Step 3

If the key matches the value of the mid element, the element is found and stop search.



Edge Case for Newton's Method

Use Newton's Method to solve $f(x) = \sqrt[3]{x}$

Let's choose an initial guess of 1 and a tolerance of 0.001

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}, x_0 = 1$$

Edge Case for Newton's Method

Use Newton's Method to solve $f(x) = \sqrt[3]{x}$

Let's choose an initial guess of 1 and a tolerance of 0.001

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}, x_0 = 1$$

$$x_1 = 1 - \frac{1^{\frac{1}{3}}}{\frac{1}{3}1^{-\frac{2}{3}}} = -2$$

Edge Case for Newton's Method

Use Newton's Method to solve $f(x) = \sqrt[3]{x}$

Let's choose an initial guess of 1 and a tolerance of 0.001

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}, x_0 = 1$$

$$x_1 = 1 - \frac{1^{\frac{1}{3}}}{\frac{1}{3}1^{-\frac{2}{3}}} = -2$$

$$x_2 = -2 - \frac{-2^{\frac{1}{3}}}{\frac{1}{3} \cdot -2^{-\frac{2}{3}}} = 4$$

Edge Case for Newton's Method

Use Newton's Method to solve $f(x) = \sqrt[3]{x}$

Let's choose an initial guess of 1 and a tolerance of 0.001

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}, x_0 = 1$$

$$x_1 = 1 - \frac{1^{\frac{1}{3}}}{\frac{1}{3}1^{-\frac{2}{3}}} = -2$$

$$x_2 = -2 - \frac{-2^{\frac{1}{3}}}{\frac{1}{3} \cdot -2^{-\frac{2}{3}}} = 4$$

$$x_3 = -8, x_4 = 16, x_5 = -32, x_k = (-2)^k$$

When $f(x) = \sqrt[3]{x}$, Newton's method fails to find the solution of $x = 0$ when $x_0 \neq 0$.