RECURSIVE FORMULAE FOR THE MULTIPLICATIVE PARTITION FUNCTION

JUN KYO KIM and SANG GUEN HAHN

ABSTRACT. For a positive integer n, let $f(n)$ be the number of essentially different ways of writing n as a product of factors greater than 1, where two factorizations of a positive integer are said to be essentially the same if they differ only in the order of the factors. This paper gives a recursive formula for the multiplicative partition function $f(n)$.

Keywords and phrases. Partitions, multiplicative partitions.

1991 Mathematics Subject Classification. 11P82.

A multi-partite number of order j is a j-dimensional vector, the components of which are nonnegative integers. A partition of $(n_1, n_2, \ldots, n_j)$ is a solution of the vector equation

$$\sum_k (n_{1k}, n_{2k}, \ldots, n_{jk}) = (n_1, n_2, \ldots, n_j) \tag{1}$$

in multi-partition numbers other than $(0, 0, \ldots, 0)$. Two partitions which differ only in the order of the multi-partite numbers are regarded as identical. We denote by $p(n_1, n_2, \ldots, n_j)$ the number of different partitions of $(n_1, n_2, \ldots, n_j)$. For example, $p(3) = 3$ since $3 = 2 + 1 = 1 + 1 + 1$ and $p(2, 1) = 4$ since $(2, 1) = (2, 0) + (0, 1) = (1, 0) + (1, 0) + (1, 0) = (1, 0) + (1, 1)$. Let $f(1) = 1$ and for any integer $n > 1$, let $f(n)$ be the number of essentially different ways of writing n as a product of factors greater than 1, where two factorizations of a positive integer are said to be essentially the same if they differ only in the order of the factors. For example, $f(12)p(2, 1) = 4$ since $12 = 2 \cdot 6 = 3 \cdot 4 = 2 \cdot 2 \cdot 3$. In general, if $n = p_1^{n_1} p_2^{n_2} \cdots p_j^{n_j}$, then $f(n) = p(n_1, n_2, \ldots, n_j)$. We find recursive formulas for the multi-partite partition function $p(n_1, n_2, \ldots, n_j)$. The most useful formula known to this day for actual evaluation of the multi-partite partition function is presented in Theorem 4.

For convenience, we define some sets used in this paper. For a positive integer r, let M_r^0 be the set of r-dimensional vectors with nonnegative integer components and M_r be the set of r-dimensional vectors with nonnegative integer components not all of which are zero. The following three theorems are well known.

THEOREM 1 (Euler [3]; see also [1, p. 2]). *If $n \geq 0$, then*

$$p(n) = \sum_{m=1}^{\infty} (-1)^{m+1} \left(p\left(n - \tfrac{1}{2}m(3m - 1)\right) + p\left(n - \tfrac{1}{2}(3m + 1)\right) \right), \tag{2}$$

where we recall that $p(k) = 0$ for all negative integers k.

THEOREM 2. *If $n \geq 0$, then $p(0) = 1$ and*

$$n \cdot p(n) = \sum_{k=1}^{n} \sigma(k) \cdot p(n-k), \tag{3}$$

where $\sigma(m) = \sum_{d|m} d$.

THEOREM 3 ([1, Ch. 12]). *If $g(x_1, x_2, \ldots, x_r)$ is the generating function for $p(\vec{n})$ and $|x_i| < 1$ for $i \leq r$, then*

$$\begin{aligned}
g(x_1, x_2, \ldots, x_r) &= \prod_{\vec{n} \in M_r} \frac{1}{1 - x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}} \\
&= 1 + \sum_{\vec{m} \in M_r} p(\vec{m}) x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}.
\end{aligned} \tag{4}$$

Similarly, we can extend the equation of Theorem 2 to multi-partite numbers as follows.

THEOREM 4. *For $\vec{n} \in M_r$, we have*

$$n_i \cdot p(\vec{n}) = \sum_{\substack{l_j \leq n_j \text{ for } j \leq r \\ \vec{l} \in M_r}} \frac{\sigma(\gcd[\vec{l}])}{\gcd[\vec{l}]} \cdot l_i \cdot p(\vec{n} - \vec{l}). \tag{5}$$

PROOF. Let $g(x_1, x_2, \ldots, x_r)$ be the function defined in Theorem 3. Taking the ith partial logarithmic derivative of the product formula for $g(x_1, x_2, \ldots, x_r)$ in (4), we get

$$\begin{aligned}
\frac{\partial g(x_1, x_2, \ldots, x_r)}{\partial x_i} \cdot \frac{x_i}{g(x_1, x_2, \ldots, x_r)} &= \sum_{\vec{l} \in M_r} \frac{l_i \cdot \prod_{j=1}^{r} x_j^{l_j}}{1 - \prod_{j=1}^{r} x_j^{l_j}} \\
&= \sum_{\vec{l} \in M_r} \sum_{k=1}^{\infty} l_i \cdot \left(\prod_{j=1}^{r} x_j^{l_j} \right)^k.
\end{aligned} \tag{6}$$

Taking the ith partial derivative of the right-hand side of (4), we get

$$\begin{aligned}
\sum_{\vec{n} \in M_r} n_i \cdot p(\vec{n}) x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r} &= \frac{\partial g(x_1, x_2, \ldots, x_r)}{\partial x_i} \cdot x_i \\
&= g(x_1, x_2, \ldots, x_r) \sum_{\vec{t} \in M_r} \sum_{k=1}^{\infty} t_i \cdot \left(\prod_{j=1}^{r} x_j^{t_j} \right)^k \\
&= \left(\sum_{\vec{m} \in M_r^0} p(\vec{m}) x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} \right) \sum_{\vec{t} \in M_r} \sum_{k=1}^{\infty} t_i \cdot \left(\prod_{j=1}^{r} x_j^{t_j} \right)^k.
\end{aligned} \tag{7}$$

Comparing the coefficients of both sides of (7), we get

$$n_i \cdot p(\vec{n}) = \sum_{\substack{\vec{m}, \vec{t} \in M_r^0, k \in M_1 \\ \vec{m} + k\vec{t} = \vec{n}}} t_i \cdot p(\vec{m})$$

$$= \sum_{\vec{l} \in M_r} p(\vec{n} - \vec{l}) \sum_{k | \gcd(\vec{l})} \frac{l_i}{k} \tag{8}$$

$$= \sum_{\substack{l_j \le n_j \text{ for } j \le r \\ \vec{l} \in M_r}} \frac{\sigma(\gcd[\vec{l}])}{\gcd[\vec{l}]} \cdot l_i \cdot p(\vec{n} - \vec{l}).$$

The theorem is proved. $\qquad\qquad\square$

COROLLARY 5. *For $\vec{n} \in M_r$, we have*

$$\left(\sum_{i=1}^{r} n_i \right) \cdot p(\vec{n}) = \sum_{\substack{l_j \le n_j \text{ for } j \le r \\ \vec{l} \in M_r}} \frac{\sigma(\gcd[\vec{l}])}{\gcd[\vec{l}]} \left(\sum_{i=1}^{r} l_i \right) \cdot p(\vec{n} - \vec{l}). \tag{9}$$

For positive integers m and n, let

$$(m, n)_{\vDash} = \max_{\substack{k | m \\ n^{1/k} \text{ is an integer}}} k. \tag{10}$$

The following properties of $(m, n)_{\vDash}$ are easy to obtain:
  (1) $(m, p_1^{n_1} p_2^{n_2} \cdot p_k^{n_k}) = \gcd(m, n_1, n_2, \ldots, n_k)$
  (2) $(m, nk)_{\vDash} = \gcd[(m, n)_{\vDash}, (mk)_{\vDash}]$ *for $\gcd(n, k) = 1$*
  (3) $(mk, n)_{\vDash} = (m, n)_{\vDash} \cdot (k, n)_{\vDash}$ *for $\gcd(m, k) = 1$.*

From the point of view of the multiplicative partition function, Theorem 4 can be restated as the following theorem.

THEOREM 6. *let n, t be positive integers and let p be a prime number such that $p \nmid m$. Then we get*

$$t \cdot f(mp^t) = \sum_{i=1}^{t} \sum_{l | m} \frac{\sigma((i, l)_{\vDash})}{(i, l)_{\vDash}} i \cdot f\left(\frac{m}{l} p^{t-i} \right). \tag{11}$$

In [4], MacMahon presents a table of values of $f(n)$ for those n which divide one of $2^{10} \cdot 3^8$, $2^{10} \cdot 3 \cdot 5$, $2^9 \cdot 3^2 \cdot 5^1$, $2^8 \cdot 3^3 \cdot 5^1$, $2^6 \cdot 3^2 \cdot 5^2$, $2^5 \cdot 3^3 \cdot 5^2$. In [2], Canfield, Erdös, and Pomerance commented that they doubted the correctness of MacMahon's figures. Specifically,

$$p(10, 5) = 3804, \qquad \text{not } 3737, \tag{12}$$

$$p(9, 8) = 13715, \qquad \text{not } 13748, \tag{13}$$

$$p(10, 8) = 21893, \qquad \text{not } 21938, \tag{14}$$

$$p(4, 1, 1) = 38, \qquad \text{not } 28. \tag{15}$$

From Theorem 4 we can easily be sure that Canfield, Erdös and Pomerance comment is true.

References

[1]   G. E. Andrews, *The theory of partitions*, Encyclopedia of Mathematics and its Applications, vol. 2, Addison-Wesley Publishing Co., London,Amsterdam, 1976. MR 58 27738. Zbl 371.10001.

[2]   E. R. Canfield, Paul Erdos, and Carl Pomerance, *On a problem of Oppenheim concerning "factorisatio numerorum"*, J. Number Theory **17** (1983), no. 1, 1–28. MR 85j:11012. Zbl 513.10043.

[3]   L. Euler, *Introductio in Analysin Infinitorum*, ch. 16, Marcum-Michaelem Bousquet Lausannae, 1748 (Latin).

[4]   P. A. Macmahon, *The enumeration of the partitions of multipartite numbers*, Math. Proc. Cambridge Philos. Soc. **22** (1927), 951–963.

KIM AND HAHN: KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY 373-1, KUSONG DON, YUSUNG GU, TAEJON 305-701, SOUTH KOREA

Special Issue on
Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning

- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management

- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk