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p-Integral harmonic sums

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Abstract

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For $n > 0$, let $\sum_{k=1}^{n} 1/k = a(n)/b(n)$ where $a(n)$ and $b(n)$ are relatively prime positive integers. For a prime p, let $J(p) = \{n \geqslant 0 \mid p \text{ divides } a(n)\}$ where $a(0) = 0$ by definition. It is conjectured that $J(p)$ is finite for all p. Here, the conjecture is proved for $p \leqslant 7$, but remains unproved for $p = 11$. The largest integer belonging to $J(7)$ is $n = 102728$. It is shown that for $p > 2$, $J(p) \supset \{0, p-1, p(p-1), p^2-1\}$. When $J(p) = \{0, p-1, p(p-1), p^2-1\}$, the prime p is called *harmonic*. Many small primes are harmonic. It is conjectured that the set of harmonic primes is infinite.

1. Introduction

We investigate various arithmetic properties of the harmonic sums

$$H(n) = \sum_{k=1}^{n} \frac{1}{k}, \quad n \geqslant 0. \tag{1.1}$$

Let

$$H(n) = \frac{a(n)}{b(n)}, \tag{1.2}$$

where $a(n)$ and $b(n)$ are relatively prime nonnegative integers. Specifically, $a(0) = 0$, $b(0) = 1$ and it is well known that

(i) 2 divides $a(n)$ if and only if $n = 0$, and
(ii) 2 divides $b(n)$ if and only if $n > 1$.

From (ii) it follows that $H(n)$ is not an integer for $n > 1$. In attempting to generalize these facts, we consider any prime p and introduce the sets of integers

$$J = J(p) = \{n \geqslant 0 \mid a(n) \equiv 0 \pmod{p}\}, \tag{1.3}$$

$$I = I(p) = \{n \geqslant 0 \mid b(n) \not\equiv 0 \pmod{p}\}. \tag{1.4}$$

These definitions clearly imply

$$J(p) \subset I(p). \tag{1.5}$$

A natural objective is to determine these sets $J(p)$ and $I(p)$. In Section 4, we show that either one of these sets determines the other. From (i) above, $J(2) = \{0\}$ and in Section 4, we show that

$$J(3) = \{0, 2, 7, 22\}. \tag{1.6}$$

The primes 2 and 3 are exceptional in that for $p > 3$

$$J(p) \supset \{0, p-1, p(p-1), p^2-1\}. \tag{1.7}$$

Primes for which

$$J(p) = \{0, p-1, p(p-1), p^2-1\} \tag{1.8}$$

are singled out for special attention and are called *harmonic.* A prime $p > 3$ which is not harmonic will be called *anharmonic.* In Section 5, necessary and sufficient conditions are obtained for p to be harmonic, and these have been applied to determine all such $p < 200$. There are 16 of these, the first few of which are 5, 13, 17, 23, 41. The prime 7 is anharmonic and we have determined that $|J(7)| = 14$. The set $J(7)$ is given explicitly in Section 6. For a related discussion, see [3, problem 6.52 and solution].

Two main conjectures arise.

Conjecture A. For all primes p, $J(p)$ is finite.

In this regard, it even seems quite difficult to show that $J(11)$ is finite. In fact, the prime 7 is the only anharmonic prime for which we have been able to settle the conjecture.

Conjecture B. The set of harmonic primes is infinite.

Based upon numerical evidence and heuristic arguments it would appear that anharmonic primes occur more frequently than harmonic primes. However, we cannot show that there are infinitely many anharmonic primes.

2. The relation between $I(p)$ and $J(p)$

For a fixed prime p and integer $n \geqslant 0$, setting $\bar{n} = [n/p]$, we have $n = p\bar{n} + r$, where $r = r(n)$ with $0 \leqslant r < p$. Letting

$$H^*(n) = \sum_{k=1,(k,p)=1}^{n} \frac{1}{k}, \tag{2.1}$$

we have

$$H(n) = H^*(n) + \frac{1}{p} H(\bar{n}). \tag{2.2}$$

Let Q^p denote the set of rational numbers whose reduced forms have denominators *not* divisible by p. In [1], a rational $\theta \in Q^p$ is called *p-integral.* In this context, $I(p)$ consists precisely of those $n \geq 0$ for which $H(n)$ is p-integral.

From (2.1) we observe that

$$H^*(n) \in Q^p. \tag{2.3}$$

Then (2.2) and (2.3) show that $H(n) \in Q^p$ is equivalent to $(1/p)H(\bar{n}) \in Q^p$. In other words

$$n \in I(p) \text{ if and only if } \bar{n} \in J(p). \tag{2.4}$$

The latter, together with (1.5) provides

$$n \in J(p) \text{ implies } \bar{n} \in J(p). \tag{2.5}$$

Finally, since $n = p\bar{n} + r$, $0 \leq r < p$, (2.4) yields the set theoretic identity

$$I(p) = pJ(p) + R(p) \tag{2.6}$$

where

$$R = R(p) = \{0, 1, \ldots, p-1\}. \tag{2.7}$$

Hence, determining either $I(p)$ or $J(p)$ gives a determination of the other.

3. *J* to *J* transitions

In the previous section, (2.4) provided a means of making a 'transition' from $I(p)$ to $J(p)$ and vice versa. By comparison, (2.5) shows $n \in J(p)$ only implies $\bar{n} \in J(p)$. Here, our concern is to find some additional condition which enables us to replace (2.5) by an equivalence.

Throughout, we will observe the usual convention that for a prime power p^t, $t > 0$, and any natural number $\theta \in Q^p$, the congruence

$$\theta \equiv 0 \pmod{p^t} \tag{3.1}$$

means that in the reduced form for θ, the numerator is divisible by p^t. When (3.1) holds, we say that p^t divides θ. Also, $\theta_1 \equiv \theta_2 \pmod{p^t}$ denotes $\theta_1 - \theta_2 \equiv 0 \pmod{p^t}$.

Lemma 3.1. *For integers $k \geq 0$, $t > 0$, and prime $p > 2$,*

$$H^*(p^t k) \equiv 0 \pmod{p^t}. \tag{3.2}$$

Specifically,

$$H^*(pk) \equiv 0 \pmod{p}. \tag{3.3}$$

Proof. Let $n = p^t k$. Then

$$H^*(n) = \sum_{j=1,(j,p)=1}^{n} \frac{1}{j} = \sum_{j=1,(j,p)=1}^{n} \frac{1}{(p^t k - j)} \equiv -H^*(n) \pmod{p^t},$$

and the result follows. $\square$

Definition 3.1. For integers $0 \leqslant k \leqslant n$, let $H^*(k, n) = H^*(n) - H^*(k)$.

Definition 3.2. For $n \in J$ we have $H(n) \equiv 0 \pmod p$, so that

$$\psi = \psi(n) \equiv \frac{1}{p} H(n) \pmod p, \tag{3.4}$$

$0 \leqslant \psi < p$, is a uniquely defined integer.

Lemma 3.2. *For $n = p\bar{n} + r$, $0 \leqslant r < p$, we have*

$$H^*(p\bar{n}, n) \equiv H(r) \pmod p. \tag{3.5}$$

Further, if $\bar{n} \in J$, then

$$H(n) \equiv H(r) + \psi(\bar{n}) \pmod p. \tag{3.6}$$

Proof. (3.5) is immediate since

$$H^*(p\bar{n}, n) = \sum_{k=1}^{r} \frac{1}{(p\bar{n} + k)} \equiv H(r) \pmod p.$$

Now, for $\bar{n} \in J$ we have using (2.2), (3.3) and (3.5),

$$\begin{aligned} H(n) &\equiv H^*(n) + \psi(\bar{n}) = H^*(p\bar{n}) + H^*(p\bar{n}, n) + \psi(\bar{n}) \\ &\equiv H(r) + \psi(\bar{n}) \pmod p, \end{aligned}$$

which is (3.6). $\square$

Note that our poof of (3.6) depended upon (3.3) which is not generally true for $p = 2$. Nevertheless, (3.6) is still true, but of little interest, when $p = 2$. For in this case, by our introductory remarks $\bar{n} \in J$ means $\bar{n} = 0$ and consequently $n = 0$ or 1.

The desired strengthening of (2.5) may now be given by

Theorem 3.1. *Let $n = p\bar{n} + r$. Then $n \in J$ if and only if*

$$\bar{n} \in J \quad \text{and} \quad H(r) + \psi(\bar{n}) \equiv 0 \pmod p. \tag{3.7}$$

Proof. From (3.6) it is immediate that (3.7) implies $n \in J$. Conversely, if $n \in J$, (2.5) gives $\bar{n} \in J$, and consequently (3.6) yields

$$H(r) + \psi(n) \equiv H(n) \equiv 0 \pmod p. \quad \square$$

An obvious corollary to Theorem 3.1 is

$$n \notin J \quad \text{and} \quad \bar{n} \in J \quad \text{imply} \quad H(r) + \psi(\bar{n}) \not\equiv 0 \pmod{p}. \tag{3.8}$$

In applying the above transitions from n to $\bar{n}$, it is natural to consider the successive intervals of integers:

$$G_0 = \{0\}, \qquad G_t = \{n \mid p^{t-1} \leq n < p^t\}, \quad t = 1, 2, \ldots . \tag{3.9}$$

Clearly $n \in G_{t+1}$ if and only if $\bar{n} \in G_t$ for $t \geq 0$. Letting

$$J_t = J \cap G_t, \quad t = 0, 1, 2, \ldots \tag{3.10}$$

it is clear that $J = \bigcup_{t=0}^{\infty} J_t$. Using (3.7), we can then give J_t recursively by $J_0 = \{0\}$ and

$$J_{t+1} = \{n \mid \bar{n} \in J_t,\ r \in R, \text{ and } H(r) + \psi(\bar{n}) \equiv 0 \pmod{p}\}, \tag{3.11}$$

$t = 0, 1, 2, \ldots$ Here, R is as given by (2.7). From (3.11) it follows that if $J_t = \emptyset$ (the empty set), then $J_k = \emptyset$ for $k = t, t+1, \ldots$. Thus

$$J_t = \emptyset \text{ implies } J = \bigcup_{k=0}^{t-1} J_k \text{ (and that } J \text{ is finite).} \tag{3.12}$$

4. The case $p = 3$

We now apply the above to derive (1.6). To determine J_0, J_1 and J_2, we simply rely upon the following table.

n	0	1	2	3	4	5	6	7	8
$H(n)$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$

Clearly $J_0 = \{0\}$, and $J_1 = \{2\}$, and $J_2 = \{7\}$. Consider next $n \in J_3$, i.e., $n \in J$ and $9 \leq n < 27$. From (2.5) we have $\bar{n} \in J_2$ and hence $\bar{n} = 7$. Since

$$\tfrac{1}{3}H(7) = \tfrac{121}{140} \equiv \tfrac{1}{2} \pmod{3},$$

we have $\psi(7) = 2$. So, from (3.11)

$$\begin{aligned} J_3 &= \{n \mid \bar{n} = 7,\ r \in R \text{ and } H(r) + 2 \equiv 0 \pmod{3}\} \\ &= \{n \mid \bar{n} = 7,\ r = 1\} = \{22\}, \end{aligned}$$

i.e., $H(0)$, $H(1)$ and $H(2)$ are congruent to 0, 1, 0, respectively (mod 3). So the only solution to $H(r) + 2 \equiv 0 \pmod{3}$ with $r \in R$ occurs with $r = 1$. So, $n = 3\bar{n} + r = 22$.

Next, consider $n \in J_4$. Again an application of (2.5) shows $\bar{n} \in J_3$, hence $\bar{n} = 22$. Now, from (2.2) and (3.2)

$$\begin{aligned} H(22) &= H^*(22) + \tfrac{1}{3}H(7) = H^*(18) + H^*(18, 22) + \tfrac{1}{3}H(7) \\ &\equiv 0 + \left(\tfrac{1}{19} + \tfrac{1}{20} + \tfrac{1}{22}\right) + \tfrac{121}{140} \\ &\equiv 0 + \tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{4}{5} \equiv 3 \pmod{9}. \end{aligned}$$

Hence $\psi(22) = 1$ and we note that $H(r) + \psi(22) \equiv 0 \pmod 3$ has no solution with $r \in R$. Thus by (3.11)

$$J_4 = \{n \mid \bar{n} = 22,\ r \in R \text{ and } H(r) + \psi(22) \equiv 0 \pmod 3\} = \emptyset.$$

Since $J_4 = \emptyset$, we have from (3.12) that

$$J = \bigcup_{k=0}^{3} J_k = \{0, 2, 7, 22\},$$

which is (1.6). From (2.6) we also have

$$\begin{aligned} I(3) &= \{0, 6, 21, 66\} + \{0, 1, 2\} \\ &= \{0, 1, 2, 6, 7, 8, 21, 22, 23, 66, 67, 68\}. \end{aligned}$$

In particular, for all $n > 68$ the denominator $b(n)$ of $H(n)$ is divisible by 3.

5. Harmonic primes

In this section our main objective is to provide a characterization of harmonic primes. Throughout this section, p denotes a prime >3. For a prime $p > 3$, Wolstenholme's theorem [2] asserts

$$H(p-1) \equiv 0 \pmod{p^2}. \tag{5.1}$$

A standard consequence of (5.1) which is not difficult to establish is that for any $k \geqslant 0$,

$$H^*(pk) \equiv 0 \pmod{p^2}. \tag{5.2}$$

(For $p > 3$, (5.2) is a stronger version of (3.3)). Two immediate consequences of (5.1) are

$$p - 1 \in J \quad (\text{in fact } p - 1 \in J_1), \tag{5.3}$$

and from Definition 3.2, that

$$\psi(p-1) = 0. \tag{5.4}$$

Definition 5.1. Using (5.1), we uniquely define

$$W = W(p) \equiv \frac{1}{p^2} H(p-1) \pmod p \tag{5.5}$$

where W is an integer $0 \leqslant W < p$. We refer to W as the 'W-index of p'.

Thusfar we have not yet established (1.7) and here is the appropriate place to do so. We have

Lemma 5.1. *For* $p>3$,

$$p(p-1)\in J, \qquad p^2-1\in J, \tag{5.6}$$

and

$$\psi(p(p-1))=\psi(p^2-1)=W. \tag{5.7}$$

Proof. From (2.2), (5.2) and (5.5)

$$\begin{aligned}H(p(p-1))&=H^*(p(p-1))+\frac{1}{p}H(p-1)\\&\equiv 0+pW \pmod{p^2}.\end{aligned}$$

Thus $p(p-1)\in J$ and $\psi(p(p-1))\equiv(1/p)H(p(p-1))\equiv W \pmod p$. Since $\psi(p(p-1))$ and W both belong to $R(p)$, it follows that $\psi(p(p-1)=W$.

In a similar way,

$$H(p^2-1)=H^*(p^2)+\frac{1}{p}H(p-1)\equiv pW \pmod{p^2}$$

yields the remaining portions of (5.6) and (5.7).

From (5.3), (5.6), and the fact that $0\in J$, we have established (1.7).

We now turn to our proposed characterization of harmonic primes. For this purpose it is convenient to introduce

$$H(r)\equiv H(p-1-r) \quad \text{for} \quad 0\leqslant r<p. \tag{5.8}$$

To prove (5.8), we can suppose by symmetry that $0\leqslant r<p/2$. Then

$$\begin{aligned}H(p-1-r)&=H(r)+\sum_{k=r+1}^{p-1-r}\frac{1}{k}=H(r)+\sum_{k=r+1}^{(p-1)/2}\left(\frac{1}{k}+\frac{1}{p-k}\right)\\&\equiv H(r) \pmod p.\end{aligned}$$

Now, let $\Delta_1=\Delta_1(p)$ be the set of $(p-1)/2$ fractions given by

$$\Delta_1(p)=\{H(1), H(2), \ldots, H((p-1)/2)\}, \tag{5.9}$$

and let $\Delta_2=\Delta_2(p)$ be the set of $(p+1)/2$ fractions given by

$$\Delta_2(p)=\{W+H(0), W+H(1), \ldots, W+H((p-1)/2)\}. \tag{5.10}$$

It is evident that Δ_1 and Δ_2 are both subsets of Q^p. For any set $S\subset Q^p$, we write $S\not\equiv 0 \pmod p$ to denote that $\theta\not\equiv 0 \pmod p$ for every $\theta\in S$.

Lemma 5.2. $\Delta_1(p)\not\equiv 0 \pmod p$ *if and only if*

$$J_1=\{p-1\} \quad \textit{and} \quad J_2=\{p(p-1), p^2-1\}.$$

Proof. Suppose $\Delta_1(p) \not\equiv 0 \pmod p$. Then, from (5.8) we have $H(r) \not\equiv 0 \pmod p$ for $1 \leq r \leq p-2$. This and (5.3) give $J_1 = \{p-1\}$.

If $n \in J_2$, then (2.5) yields $\bar{n} \in J_1$, hence $\bar{n} = p-1$. Then $n = p(p-1)+r$, $r \in R$ and by (3.7) and (5.4)

$$0 \equiv H(r) + \psi(p-1) \equiv H(r) \pmod p.$$

Thus, since $0 \leq r < p$, $r \in J_0 \cup J_1$. Hence $r = 0$ or $p-1$. Thus $n = p(p-1)$ or p^2-1. This together with (5.6) shows that $J_2 = \{p(p-1), p^2-1\}$.

The converse is evident. □

Lemma 5.3. *Suppose $J_2 = \{p(p-1), p^2-1\}$. Then*

$$\Delta_2(P) \not\equiv 0 \pmod p \text{ if and only if } J_3 = \emptyset.$$

Proof. Suppose $\Delta_2(p) \not\equiv 0 \pmod p$. Then in view of (5.8), we have

$$W + H(r) \not\equiv 0 \pmod p, \quad \text{for } 0 \leq r < p. \tag{5.11}$$

Now, say, $n \in J_3$, then by (3.7) $\bar{n} \in J_2$ and $H(r) + \psi(\bar{n}) \equiv 0 \pmod p$. But $\bar{n} \in J_2$ means that $\bar{n} = p(p-1)$ or p^2-1 and in either case by (5.7), $\psi(\bar{n}) = W$. Thus $H(r) + W \equiv 0 \pmod p$ in violation of (5.11). Consequently, $J_3 = \emptyset$.

In the converse direction, suppose $J_3 = \emptyset$. For any r with $0 \leq r < p$, set $n = p^3 - p + r$. Since $n \in G_3$, we have $n \notin J$ (otherwise $n \in J_3$). But $\bar{n} \in J$ since $\bar{n} = p^2 - 1$. Hence by (3.8) and (5.7)

$$0 \not\equiv H(r) + \psi(\bar{n}) = H(r) + W \pmod p$$

for any r with $0 \leq r < p$. This certainly implies $\Delta_2(p) \not\equiv 0 \pmod p$. □

Now, let

$$\Delta(p) = \Delta_1(p) \cup \Delta_2(p). \tag{5.12}$$

Then, recalling that a harmonic prime is a prime $p > 3$ satisfying (1.8), we have

Theorem 5.1. *Let $p > 3$. Then p is harmonic if and only if $\Delta(p) \not\equiv 0 \pmod p$.*

Proof. Suppose p is harmonic. Then $J_1 = \{p-1\}$, $J_2 = \{p(p-1), p^2-1\}$, and $J_3 = \emptyset$. Then by Lemma 1, $\Delta_1(p) \not\equiv 0 \pmod p$ and from Lemma 2, $\Delta_2(p) \not\equiv 0 \pmod p$. Thus $\Delta(p) \not\equiv 0 \pmod p$.

In the converse direction, we first note that $J_0 = \{0\}$ (which is the case for all primes). Now suppose $\Delta(p) \not\equiv 0 \pmod p$. This means $\Delta_1(p) \not\equiv 0 \pmod p$ and $\Delta_2(p) \not\equiv 0 \pmod p$. Then Lemma 1 yields that $J_1 = \{p-1\}$ and $J_2 = \{p(p-1), p^2-1\}$. From Lemma 2 we then obtain $J_3 = \emptyset$. Hence, using (3.12) we have

$$J = J_0 \cup J_1 \cup J_2 = \{0, p-1, p(p-1), p^2-1\},$$

i.e., p is harmonic. □

Example. Suppose $p=5$. Here $H(1)=1 \not\equiv 0 \pmod 5$ and $H(2)=\frac{3}{2} \not\equiv 0 \pmod 5$. Thus $\Delta_1 \not\equiv 0 \pmod 5$.

Also $W \equiv \frac{1}{25}H(4) = \frac{1}{12} \equiv 3 \pmod 5$, so $W=3$. Then $W+H(0)=3$, $W+H(1)=4$ and $W+H(2)=\frac{11}{2}$ and each is $\not\equiv 0 \pmod 5$. Thus $\Delta_2 \not\equiv 0 \pmod 5$. It follows that 5 is a harmonic prime.

For primes p in the range $3<p<200$, the harmonic primes are

$$5, 13, 17, 23, 41, 67, 73, 79, 107, 113, 139, 149, 157, 179, 191, 193.$$

6. The case $P=7$

To determine $J(7)$ one can compute the sets $J_0, j_1, \ldots$ recursively as given by (3.11). In doing so, we find that $J_7=\emptyset$, and consequently by (3.12) $J=\bigcup_{t=0}^{6} J_t$. From this, we obtain $|J(7)|=14$, where

$$J(7)=\{0, 6, 42, 48, 295, 299, 337, 341, 2096, 2390,$$
$$14675, 16731, 16735, 102728\}.$$

From the latter, we can easily determine $I(7)$ by use of (2.6) finding the largest integer in $I(7)$ is 719102. Thus $H(719102)$ is the last harmonic sum whose denominator is not divisible by 7.

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