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ON THE AVERAGE LENGTH OF A CLASS OF FINITE CONTINUED FRACTIONS

1. Introduction

Many years ago Dr. J. Gillis asked me the following question: Let N and a be coprime natural integers, $1 \leqq a < N$, so that a/N can be represented by a finite continued fraction

$$a/N = 1/c_1 + 1/c_2 + \cdots + 1/c_{n(a)},$$

where the c_i are natural integers depending on N and a, and where $c_{n(a)} > 1$ (to make the representation unique). What can be said about the sum

$$L(N) = \sum_{\substack{a=1 \\ (a,N)=1}}^{N} n(a)?$$

At the time I was able to make only the trivial statement that

$$L(N) = O(N \log N).$$

Recently I discovered a connection between $L(N)$ and the number $r(N)$ of representation of N by the bilinear form $N = xx' + yy'$, where the natural integers x, x', y, y' are subject to the restrictions $x > y$, $x' > y'$, $(x, y) = 1$, $(x', y') = 1$.

Theorem 1.

$$L(N) = \frac{3}{2}\varphi(N) + 2r(N) \quad \textit{for} \quad N > 2.$$

This raises the question, what can be said about the behaviour of $r(N)$. The answer is given by

Theorem 2.

$$r(N) = 6\pi^{-2} \log 2 \; \varphi(N) \log N + O(N\sigma_{-1}^{3}(N)),$$

where $\sigma_{-1}(N)$ denotes the sum of the reciprocals of the positive divisors of N.

It is clear that the main term dominates the error term by a factor at least of the order $\log N(\log\log N)^{-4}$. It appears very difficult to obtain a substantially better error term, though numerical evidence suggests that the error term is much too large.

Combining the two theorems leads to

Theorem 3.

$$L(N) = 12\pi^{-2} \log 2\varphi(N) \log N + O(N\sigma_{-1}^3(N)).$$

A slight extension of the method of the proof of theorem 2 leads to the following result. Let $L_c(N)$ be the number of times that the denominator c occurs in the continued fraction of the rationals a/N, $1 < a < N$, $(a, N) = 1$. Then we have

Theorem 4.

$$L_c(N) = 12\pi^{-2} \log (1 - (c + 1)^{-2})^{-1} \ \varphi(N) \log N + O(N\sigma_{-1}^3(N)).$$

This theorem suggests that in some sense the frequency of the denominator c equals

$$\log (1 - (c + 1)^{-2})^{-1}/\log 2,$$

if we consider the continued fraction of all real numbers. This is indeed the case, as Khintchine has shown [1].

2. Preliminaries

Small roman letters with or without indices are restricted to positive rational integers. The symbols $\varphi(n)$, $\mu(n)$, $\sigma_\tau(n)$, $d(n)$ have the meaning usual in elementary number theory, i.e. $\varphi(n)$ denotes the Euler function, $\mu(n)$ the Moebius function, $\sigma_\tau(n)$ the sum of the τ^{th} powers of the positive divisors of n, $d(n) = \sigma_0(n)$. The symbol O holds uniformly in all variables except possibly in $\varepsilon > 0$.

We shall make frequent use of the Moebius inversion formula, and such well known results as

$$d(n) = O(n^\varepsilon),$$

$$\sum_{n=1}^{z} \varphi(n)\, n^{-1} = 6\pi^{-2} z + O(\log z),$$

$$\sigma_{-1}(n) = O(\log\log N),$$

$$6\pi^{-2} < \varphi(n)\, \sigma_{-1}(n)\, n^{-1} \leqq 1.$$

We have already defined $r(N)$ as the number of solutions of $N = xx' + yy'$ subject to

$$x > y, \quad x' > y', \quad (x, y) = (x', y') = 1.$$

We further define $R(N)$ as the number of solutions subject to

$$x > y, \quad x' > y';$$

and for each $d \geqq 1$ we define $\varrho(N, d)$ as the number of solutions subject to

$$x > y, \quad x' > y', \quad (x, y) = 1, \quad x' > dx.$$

Then

$$R(N) = \sum_{bb'/N} r(N(bb')^{-1}),$$

and by a repeated application of the Moebius formula

$$r(N) = \sum_{bb'/N} \mu(b)\, \mu(b')\, R(N(bb')^{-1}).$$

Utilizing the symmetry in the definition of $R(N)$ between the primed and the 'unprimed' variables, we obtain

$$R(N) = 2 \sum_{d/N} \varrho(N d^{-1}, d) + O \sum_{x^2 < N} d(N - x^2),$$

$$r(N) = 2 \sum_{dbb'/N} \mu(b)\, \mu(b')\, \varrho(N(dbb')^{-1}, d) + O(N^{1/2+\varepsilon}). \tag{2}$$

3. Continued fractions

We define the function $[c_1, \ldots, c_n]$ in the usual way by

$$[\,] = 1, \quad [c_1] = c_1, \quad [c_1, c_2] = c_2 c_1 + 1,$$

$$[c_1, \ldots, c_n] = c_n[c_1, \ldots, c_{n-1}] + [c_1, \ldots, c_{n-2}] \quad \text{for} \quad n \geqq 3.$$

Our first object is to prove the formula

$$[c_1, \ldots, c_n] = [c_1, \ldots, c_m]\,[c_{m+1}, \ldots, c_n] + [c_1, \ldots, c_{m-1}]\,[c_{m+2}, \ldots, c_n]$$

for $1 \leqq m < n$. (I don't believe this formula is new, but I have not been able to find it in the literature.) The formula is obviously true for $1 \leqq m = n - 1$ and also, as a little calculation shows, for $1 \leqq m = n - 2$. Hence by induction with respect to m, it is true for $m \geqq 1$ and all $n > m$. Our formula makes it evident that

$$[c_1, \ldots, c_n] = [c_n, \ldots, c_1].$$

Now we introduce a pair N, a with $a < \frac{1}{2}N$, $(N, a) = 1$ and develop a/N as a continued fraction

$$a/N = 1/c_1 + 1/c_2 + \cdots + 1/c_n, \quad c_n \geqq 2, \tag{3}$$

and we have automatically

$$c_1 \geqq 2, \quad N = [c_1, \ldots, c_n].$$

If $a \neq 1$, then $n = n(a) > 1$ and we can choose m in the interval $1 \leqq m \leqq n - 1$ in $n - 1$ different ways. Put

$$x = [c_1, \ldots, c_m], \quad y = [c_1, \ldots, c_{m-1}], \tag{4}$$

$$x' = [c_n, \ldots, c_{m+1}], \quad y' = [c_n, \ldots, c_{m+2}]. \tag{5}$$

These integers, by virtue of our identity, satisfy the relation

$$N = xx' + yy',$$

and they also fulfil the conditions

$$x > y, \quad x' > y', \quad (x, y) = (x', y') = 1.$$

(If $m = 1$ or $m = n - 1$, remember $c_1 \geqq 2$ or $c_n \geqq 2$.)

Moreover we have for $m > 2$

$$\frac{y}{x} = \frac{[c_1, \ldots, c_{m-1}]}{[c_1, \ldots, c_m]} = \frac{[c_1, \ldots, c_{m-1}]}{c_m[c_1, \ldots, c_{m-1}] + [c_1, \ldots, c_{m-2}]}$$
$$= \left(c_m + \frac{[c_1, \ldots, c_{m-2}]}{[c_1, \ldots, c_{m-1}]}\right)^{-1}$$

and hence by induction

$$y/x = 1/c_m + \cdots + 1/c_1; \tag{6}$$

similarly

$$y'/x' = 1/c_{m+1} + \cdots + 1/c_n. \tag{7}$$

Conversely, given a representation of N by our bilinear form with our restrictions, we can find a unique sequence $c_1, \ldots, c_n$ not starting or finishing with 1 such that (6), (7), (4), (5) are satisfied and

$$N = [c_1, \ldots, c_n].$$

Putting $a = [c_2, \ldots, c_n]$, it is clear that (3) holds.

To sum up, we have a $(1-1)$ relation between all suitably restricted representations of N by the bilinear form, and all pairs of sequences

$$c_1, \ldots, c_m; \quad c_{m+1}, \ldots, c_n \quad \text{with} \quad c_1 \geqq 2, \quad c_n \geqq 2, \quad 1 \leqq m < n.$$

Thus, for $N > 2$

$$r(N) = \sum_{\substack{1<a<N/2\\(a,N)=1}} (n(a)-1) = \sum_{\substack{a<N/2\\(a,N)=1}} n(a) - \frac{1}{2}\varphi(N).$$

As for $0 < \alpha < \frac{1}{2}$

$$1 - \alpha = \cfrac{1}{1+\cfrac{1}{\cfrac{1}{\alpha}-1}},$$

it follows that, for $a < \frac{1}{2}N$, $(a, N) = 1$, (3) implies

$$(N-a)\,N = 1/1 + 1/(c_1 - 1) + 1/c_2 + \cdots + 1/c_n.$$

Hence

$$\sum_{\substack{N/2<a<N\\(a,N)=1}} n(a) = \frac{1}{2}\varphi(N) + \sum_{\substack{a<N/2\\(a,N)=1}} n(a),$$

and theorem 1 is proved.

4. Proof of theorem 2

We require the following

Lemma. $\varrho(N,d) = 3\pi^{-2}\log 2\; N\log(Nd^{-1}) + O(N)$ *for* $d \leqq N$.

Proof. Fix a pair x, y with

$$y < x < (Nd^{-1})^{1/2}, \quad (x,y) = 1. \tag{8}$$

We have to count the number of positive integers x' which satisfy

$$xx' \equiv N \pmod{y}, \quad x' > x, \quad x' > y' = y^{-1}(N - xx') > 0.$$

The last two inequalities can be written as

$$N(x+y)^{-1} < x' < Nx^{-1}.$$

This means we have to find the number, say $P(N, d, x, y)$ of solutions of the congruence in the interval

$$\operatorname{Max}(xd, N(x+y)^{-1}) < x' < Nx^{-1}.$$

Clearly

$$|P(N, d, x, y) - y^{-1}(Nx^{-1} - \operatorname{Max}(xd, N(x+y)^{-1}))| < 1$$

and

$$\varrho(N, d) = \sum_{x,y} P(N, d, x, y)$$

where the sum is extended over all x, y satisfying (8). We distinguish two cases.

Case 1. $xd \geqq N(x+y)^{-1}$.

Then

$$P(N, d, x, y) < 1 + y^{-1}(Nx^{-1} - xd)$$
$$\leqq 1 + y^{-1}(xd(x+y)x^{-1} - xd) = 1 + d,$$

Hence, in this case, summing over x, y satisfying (8)

$$\sum_{x,y} P(N, d, x, y) \leqq (1+d) \sum_{x<(Nd^{-1})^{1/2}} x = O(N).$$

Case 2. $xd < N(x+y)^{-1}$.

This implies $6d < N$. If $6d \geqq N$, the lemma is trivial. Keeping y fixed, x is now restricted by

$$(x, y) = 1, \quad x > y, \quad x(x+y)\, d < N. \tag{9}$$

As

$$P(N, d, x, y) = O(1) + y^{-1}(Nx^{-1} - N(x+y)^{-1}),$$

we obtain, summing over all relevant x,

$$\sum_{x} P(N, d, x, y) = O(N^{1/2}d^{-(1/2)}) + y^{-1}N \sum_{\substack{y<x<2y\\(x,y)=1}} x^{-1} - y^{-1}N \sum_{\substack{x_0 \leqq x < x_0+y\\(x,y)=1}} x^{-1}, \tag{10}$$

where x_0 is the smallest integer for which $x_0(x_0+y)\,d \geqq N$. As $x_0 > \left(\frac{1}{2}Nd^{-1}\right)^{1/2}$, the last sum including the factor $y^{-1}N$ is at most $O(N^{1/2}\,d^{1/2})$. Further

$$\sum_{\substack{y<x<2y\\(x,y)=1}} x^{-1} = \sum_{b/y} \mu(b)\, b^{-1} \sum_{r=yb^{-1}+1}^{2yb^{-1}-1} r^{-1} = \varphi(y)\, y^{-1} \log 2 + O(d(y)\, y^{-1}).$$

Hence

$$\sum_{x} P(N, d, x, y) = N\varphi(y)\, y^{-2} \log 2 + O(N^{1/2}d^{1/2}) + O(Nd(y)\, y^{-2}).$$

This expression has to be summed over all $y < \left(\frac{1}{2} Nd^{-1}\right)^{1/2}$. Thus

$$\sum_{x,y} P(N, d, x, y) = O(N) + N \log 2 \sum_{y < (Nd^{-1}/2)^{1/2}} \varphi(y)\, y^{-2},$$

and a simple summation by parts gives the lemma.

It is now easy to obtain theorem 2. As

$$\begin{aligned} r(N) &= 2 \sum_{dbb'/N} \mu(b)\, \mu(b')\, \varrho(N(dbb')^{-1} d) + O(N^{1/2+\varepsilon}) \\ &= 6\pi^{-2} \log 2\, N \sum_{dbb'/N} \mu(b)\, \mu(b')\, (dbb')^{-1} (\log N - \log (d^2 bb')) + O(1). \end{aligned}$$

From this theorem 2 follows as the error term has the required form, and as

$$\begin{aligned} \sum_{dbb'/N} \mu(b)\, \mu(b')\, (dbb')^{-1} &= \sum_{b/N} \mu(b)\, b^{-1} \sum_{n/Nb^{-1}} n^{-1} \sum_{b'/n} \mu(b') \\ &= \sum_{b/N} \mu(b)\, b^{-1} = \varphi(N)\, N^{-1}, \end{aligned}$$

whilst

$$\begin{aligned} &\sum_{dbb'/N} (dbb')^{-1} \mu(b)\, \mu(b') \log (d^2 bb') \\ &= 2 \sum_{dbb'/N} (dbb')^{-1} \mu(b)\, \mu(b') \log (db') \\ &= 2 \sum_{b/N} \mu(b)\, b^{-1} \sum_{n/Nb^{-1}} n^{-1} \log n \sum_{b/N} \varphi(b') = 0. \end{aligned}$$

5. Proof of theorem 4

Assume that a fixed number c is given. We denote by $r_c(N)$, $R_c(N)$, $\varrho_c(N, d)$ the number of representations of N by the bilinear form subject to the restriction mentioned before and the additional restriction

$$cy \leqq x < (c + 1)\, y. \tag{11}$$

Reverting to the pattern of the proof of theorem 1, we see that $r_c(N)$ equals the number of times that c is a denominator in the continued fraction of the rationals a/N, $1 < a < \frac{1}{2}N$, $(N, a) = 1$, not counting any possible $c_{n(a)}$, as $m < n$. These exceptional values are at most $\varphi(N)$. If we include the rationals a/N, $\frac{1}{2}N < a < N$, $(N, a) = 1$, every c_m will occur as often as before, except for changes in the first or second place. Hence we have

$$L_c(N) = 2r_c(N) + O(\varphi(N)).$$

Formula (1) works as before if we replace R and r by R_c and r_c respectively.

Further our argument continues to be true if we replace the new restriction (11) by

$$cy' \leqq x' < (c + 1)\,y',$$

because with the sequence $c_1, \ldots, c_n$ the sequence $c_n, \ldots, c_1$ also occurs. Hence $R_c(N)$ equals twice the number of solutions subject to the restrictions

$$cy \leqq x < (c + 1)\,y,\ x' > y',\quad x' > x,$$

with an error $O(N^{1/2+\varepsilon})$. This implies that (2) remains true if we replace r and ϱ by r_c and ϱ_c.

The proof of the lemma goes as before but (9) has to be replaced by (11) and $(x, y) = 1$, $x(x + y)\,d < N$. Then (10) has to be replaced by

$$\begin{aligned}\sum_x P(N, d, x, y) &= O(N^{1/2}d^{-(1/2)}) + y^{-1}N \sum_{\substack{cy<x<(c+1)y \\ (x,y)+1}} (x^{-1} - (x + y)^{-1}) \\ &= O(N^{1/2}d^{-(1/2)}) + y^{-1}N \sum_{b/y} \mu(b)\,b^{-1} \\ &\quad \times \sum_{r=cyb^{-1}+1}^{(c+1)yb^{-1}-1} (r^{-1} - (r + yb^{-1})^{-1}) \\ &= O(N^{1/2}d^{-(1/2)}) + \varphi(y)\,y^{-2}N\,(\log(1 + c^{-1})) \\ &\quad - \log(1 + (c + 1)^{-1})) + O(Nd(y)\,y^{-2}) \\ &= O(N^{1/2}d^{-(1/2)}) + O(Nd(y)y^{-2}) + N\varphi(y)y^{-2}\log(1-(c+1)^{-2})^{-1}.\end{aligned}$$

The rest of the calculation goes as before, with $\log(1 - (c + 1)^{-2})^{-1}$ in place of $\log 2$.

Reference

[1] A. Khintchine, Metrische Kettenbruchprobleme, Compositio Mathematica **1** (1935), 361—382.