CLASSROOM CAPSULES

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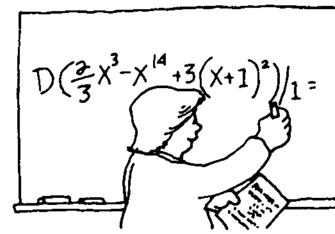

Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Warren Page, 30 Amberson Ave., Yonkers, NY 10705-3613.

Maximizing the Area of a Quadrilateral

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In this note, we use calculus to characterize the quadrilateral Q with side lengths a, b, c and d that has the largest possible area. Think of Q as having hinges at its vertices, A, B, C and D. We can assume that Q is convex, because every non-convex quadrilateral is contained in a convex quadrilateral having the same edge lengths and larger area.

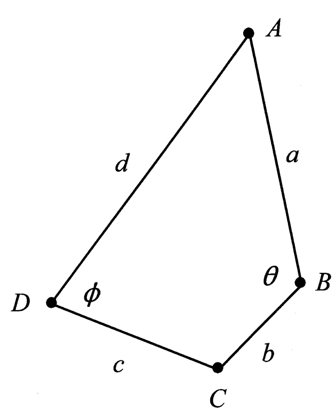

Figure 1.

Assume, by relabelling the sides if necessary, that $a + b \le c + d$. Then by pushing on the hinge at B, we can deform the quadrilateral into a triangle having A, B, C collinear. In this triangle, $\theta + \phi > \pi$. Assume without loss of generality that $b + c \le a + d$. Then by pulling on the hinge at B, we can deform the quadrilateral into a triangle having B, C, D collinear. In this triangle $\theta + \phi < \pi$. Therefore, we can find a position for B where $\theta + \phi = \pi$. A quadrilateral Q is said to be *cyclic* if its vertices lie on a circle. Using the fact that Q is cyclic if and only if opposite angles are supplementary, we have proven the following theorem.

Theorem 1. *For any quadrilateral with given edge lengths, there is a cyclic quadrilateral with the same edge lengths.*

Theorem 2. *The cyclic quadrilateral Q has the largest area of all quadrilaterals with sides of the same length as those of Q.*

Proof. Let $x = \cos\theta$ and $y = \cos\phi$, where θ and ϕ are in $(0, \pi)$. By applying the law of cosines to triangles ABC and ACD, we have $a^2 + b^2 - 2abx = c^2 + d^2 - 2cdy$, from which it follows that

$$\frac{dy}{dx} = \frac{ab}{cd}.$$

Adding the areas of triangles ABC and ACD, the area of the quadrilateral is

$$K = \frac{1}{2}ab\sin\theta + \frac{1}{2}cd\sin\phi = \frac{1}{2}\left(ab\sqrt{1-x^2} + cd\sqrt{1-y^2}\right).$$

Using

$$\frac{dy}{dx} = \frac{ab}{cd},$$

it follows that

$$\frac{dK}{dx} = \frac{-ab}{2}\left[\frac{x}{\sqrt{1-x^2}} + \frac{y}{\sqrt{1-y^2}}\right]. \tag{1}$$

Therefore, $\frac{dK}{dx} = 0$ if and only if $x = \pm y$. If $x = y$, then (1) and $\frac{dK}{dx} = 0$ imply that $x = 0 = y$ and $\theta = \frac{\pi}{2} = \phi$. If $x = -y$, then $\cos\theta = -\cos\phi = \cos(\pi - \phi)$ and, since both θ and ϕ are in $[0, \pi]$, we again have $\theta + \phi = \pi$.

Since

$$\frac{d^2K}{dx^2} = \frac{-ab}{2}\left[\frac{1}{(\sqrt{1-x^2})^3} + \frac{1}{(\sqrt{1-y^2})^3}\frac{dy}{dx}\right],$$

and $\frac{dy}{dx} = \frac{ab}{cd} > 0$, we have that $\frac{d^2K}{dx^2} < 0$ for all values of $x \in (0, 1)$. Therefore, the area K of quadrilateral Q is maximized when $\theta + \phi = \pi$. Hence, the area is maximized when Q is cyclic.

