Inverse scattering method for a soliton cellular automaton

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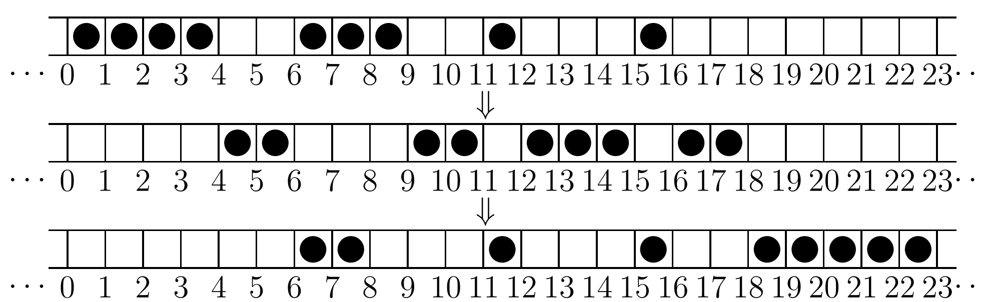
Abstract

A set of action-angle variables for a soliton cellular automaton is obtained. It is identified with the rigged configuration, a well-known object in Bethe ansatz. Regarding it as the set of scattering data an inverse scattering method to solve initial value problems of this automaton is presented. By considering partition functions for this system a new interpretation of a fermionic character formula is obtained.

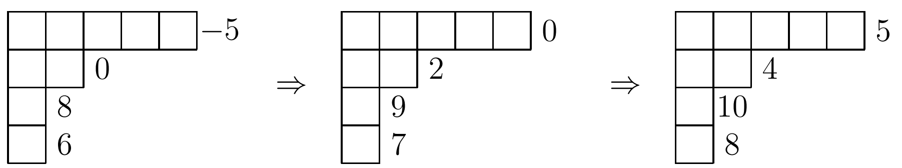
1 Introduction

1.1 Action-angle variables for the box-ball system

The box-ball system is a well-known example of soliton cellular automata [1, 2]. Here we show its simplest version.



The arrows show the time evolution process of the automaton whose precise rule will be given in the main text (See Section 2, also for the convention of the position coordinate). This system and its generalization to accommodate several kinds of balls are known as integrable systems by a certain limiting procedure on discrete KdV/Toda equations [3, 4] or by explicit constructions of their conserved quantities [5, 6]. Recently Kuniba, Okado, Yamada and the author presented a conjecture [7] that a set of action-angle variables for this system was given by the *rigged configuration*, an object known in Bethe ansatz [8]. In the simplest case a rigged configuration is a Young diagram with a set of integers *rigged* to its rows. For instance according to the above example the associated rigged configuration evolves as



The rows of the Young diagrams are regarded as the *action* variables and the numbers (riggings) associated with each row the *angle* variables. As one sees the former are invariant under the time evolution while the latter increase by the lengths of the associated rows. In other words the time evolution is linear in these variables.

In this paper we present a new method to construct a rigged configuration from a given state of the box-ball system. Here the procedure to construct a Young diagram is the same as that in [6, 9] but our method can extract additional data, the riggings. Having the additional data our method provides a bijection between the set of automaton states and the set of rigged configurations. To borrow a name from the theory of nonlinear evolution equations [10] we call this bijection the *inverse scattering transform* for the box-ball system, where we regard the rigged configuration as a set of scattering data. Using the technique of the inverse scattering transform we give a proof of the assertion in [7] that the rigged configuration is indeed playing the role of a set of action-angle variables.

1.2 The box-ball system and the fermionic formulas

We observe that a certain crystallization of solvable lattice models can produce soliton cellular automata [11, 12, 13], while there is a connection between solvable lattice models and Bethe ansatz in a character level [14, 15]. Now the inverse scattering transform for the box-ball system is in some sense providing a direct connection between soliton cellular automata and Bethe ansatz. The second purpose of this paper is to complete a study on this connection in the character level which was also proposed in [7].

Let us consider a particular case where the character formula comes from two different expressions for Kostka polynomials [16]. One is from the Bethe ansatz [17] and yields the fermionic character formula which is made of sums of products of q-binomial coefficients. The other is from the solvable lattice models in statistical mechanics [18] and yields the one-dimensional configuration sums over paths (tensor products of crystals). To be more specific we consider the Kostka polynomial $K_{\mu,\nu}(q)$ with $\mu = (L-s, s)$ and $\nu = (1^L)$ where L and s are two integers obeying $L/2 \geq s \geq 0$. Then the two expressions for the Kostka polynomial lead to the identity [14]

$$\begin{aligned} K_{\mu,\nu}(q) &= \sum_{\lambda \vdash s} q^{\phi(\lambda) - L\sum_{i\geq 1} m_i(\lambda) + \frac{L(L-1)}{2}} \prod_{i \geq 1} \begin{bmatrix} p_i(\lambda) + m_i(\lambda) \\ m_i(\lambda) \end{bmatrix} \\ &= \sum_{\mathbf{b} \in P_{L,s}} q^{\sum_{j=1}^{L-1}(L-j)(1-\theta(b_j < b_{j+1}))}, \end{aligned} \tag{1}$$

where $\lambda \vdash s$ means that λ is a partition of s. Here $p_i(\lambda), m_i(\lambda)$ and $\phi(\lambda)$ will be defined in the main text (equations (11),(12)), and $\theta(\text{true}) = 1, \theta(\text{false}) = 0$; the set of highest weight paths is defined as

$$P_{L,s} = \left\{ \mathbf{b} = (b_1, \ldots, b_L) \middle| b_i \in \{0,1\}, \sum_{j=1}^{i} b_j \leq \left\lfloor \frac{i}{2} \right\rfloor \text{ for any } i, \sum_{j=1}^{L} b_j = s \right\}.$$

There is a decomposition $P_{L,s} = \sqcup_{\lambda \vdash s} P_L(\lambda)$ such that if $\mathbf{b} \in P_L(\lambda)$ then the relation $\sum_{j=1}^{L-1} \theta(b_j < b_{j+1}) = \sum_{i \geq 1} m_i(\lambda)$ holds. Therefore equation (1) leads to

$$\sum_{\lambda \vdash s} \left(q^{\phi(\lambda)} \prod_{i \geq 1} \begin{bmatrix} p_i(\lambda) + m_i(\lambda) \\ m_i(\lambda) \end{bmatrix} - \sum_{\mathbf{b} \in P_L(\lambda)} q^{\sum_{j=1}^{L-1} j\theta(b_j < b_{j+1})} \right) = 0.$$

Furthermore the decomposition makes each term of the sum $\sum_{\lambda \vdash s}$ vanish separately. Thus there is actually a set of refined identities behind (1) with respect to λ. In the context of Bethe ansatz this λ has a clear meaning; it is a label for the associated eigenstate of the Heisenberg magnet. However in the context of solvable lattice models its meaning is so far unclear. By identifying the space of paths for a lattice model (the six-vertex model in ferro-magnetic regime) with the space of states of the box-ball system, we shall find that the path associated with $\lambda = (1^{m_1} 2^{m_2} \ldots)$ has m_i "solitons" of length i for any i. This interpretation is based on an expression for partition functions for the box-ball system with specified soliton contents, which turns out to be the above mentioned refined identities of the fermionic formula. This expression for partition functions and its variant for including non-highest weight paths were proposed in [7]. We derive these identities (Theorem 34) as a result of the inverse scattering transform for the box-ball system.

1.3 Outline of the paper

In Section 2 the box-ball system is introduced and its updating rule is explained in terms of arcs. In Section 3 we define a mapping Ξ which sends any state of the box-ball system to a rigged configuration (set of scattering data). The inverse map of Ξ (inverse scattering transform) is defined in Section 4. The notion of wave tails and wave fronts of automaton states is also introduced here. In Section 5 we give a proof that the time evolution is linearized in the scattering data , and present an inverse scattering method for this system.

In Section 6 we derive formulas for the right-most wave front and the left-most wave tail of the automaton state associated with a given rigged configuration. In Section 7 automaton states in a finite interval are studied and the notion of highest weight states is introduced. By taking summations over these states we define partition functions for the box-ball system in Section 8 and establish the conjectured fermionic formulas proposed in [7]. In Section 9 we prove that our inverse scattering transform is equivalent to the $sl(2)$ case of the bijection in [8].

In Appendix A we give a proof and an explanation of two formulas for one-dimensional configuration sums expressed by q-binomial coefficients.

2 The box-ball system

We consider a one-dimensional array of infinite number of boxes that extends towards both directions. As a position coordinate we put successive integers to the *walls* between the boxes rather than to the *boxes* themselves.

Any box is either an empty box or a filled box. The latter means that there is a ball within the box. We assume that there are at most finite number of balls in the system. The empty box is denoted by $\boxed{}$ and the filled box is by $\boxed{\bullet}$. Clearly there are four types of configurations of adjacent boxes, $\boxed{}\boxed{}$, $\boxed{}\boxed{\bullet}$, $\boxed{\bullet}\boxed{}$ and $\boxed{\bullet}\boxed{\bullet}$. We adopt the updating rule of the box-ball system in [9]:

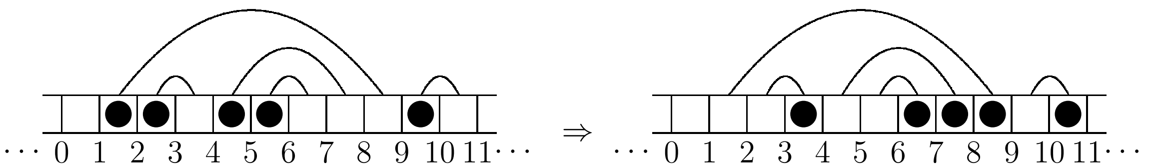
1. For every $\boxed{\bullet}\boxed{}$ connect its two boxes with an arc.

2. Ignore those boxes connected with the arcs and regard the other boxes as if they were successively adjoining.

3. Repeat steps 1 and 2 as many times as possible.

4. For every pair of connected boxes interchange $\boxed{\bullet}$ and $\boxed{}$.

Example 1. We have



Definition 2 (advanced/retarded arcs). The arcs that one will obtain by applying the items 1 (resp. 1 but $\boxed{\bullet}\boxed{}$ was replaced by $\boxed{}\boxed{\bullet}$), 2, and 3 in the above procedure to any automaton state are called the *advanced arcs* (resp. *retarded arcs*) of that state.

By definition we have that

Lemma 3. *Let p and p' be two automaton states. Suppose the set of advanced arcs of p is equal to the set of retarded arcs of p'. Then p' is the state that one will obtain from p by updating the system once.*

For later use we introduce the notion of

Definition 4 (depth). If an arc has no arc within it, its *depth* is set to be one. If an arc has arc(s) within it, its depth is given by $1 + \max\{\text{depths of inside arc(s)}\}$.

For instance there are three depth 1, one depth 2, and one depth 3 arcs in each figure of Example 1.

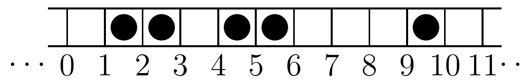
3 Scattering data and soliton contents

Given a set of integers $\{\alpha_i\}_{1 \le i \le 2n}$ we introduce a matrix of the form

$$M = \begin{pmatrix} \alpha_1 & \alpha_3 & \cdots & \alpha_{2n-1} \\ \alpha_2 & \alpha_4 & \cdots & \alpha_{2n} \end{pmatrix}. \tag{2}$$

Let $\mathcal{H}_n$ (resp. $\mathring{\mathcal{H}}_n$) be the set of all matrices of this form subject to the condition $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_{2n}$ (resp. $\alpha_1 < \alpha_2 < \cdots < \alpha_{2n}$). We define $\mathcal{H} := \sqcup_{n=0}^{\infty} \mathcal{H}_n$ (resp. $\mathring{\mathcal{H}} := \sqcup_{n=0}^{\infty} \mathring{\mathcal{H}}_n$). Here the set $\mathcal{H}_0 = \mathring{\mathcal{H}}_0$ consists of a formal two-row matrix with no element. We associate any matrix $M \in \mathring{\mathcal{H}}_n$ with an automaton state that has balls between the walls α_{2k-1} and α_{2k} for $1 \le k \le n$. We shall occasionally identity such a matrix M with the associated automaton state itself.

Example 5. We have the identification



$$\leftrightarrow \quad M = \begin{pmatrix} 1 & 4 & 9 \\ 3 & 6 & 10 \end{pmatrix}.$$

We show a procedure that associates a rigged configuration, a Young diagram with a set of integers, for any $M \in \mathring{\mathcal{H}}$. Suppose $M \in \mathring{\mathcal{H}}_n$ and let $n_1 = n$ and $M_1 = M$. Given any non-negative integer i and a matrix of the form

$$M_i = \begin{pmatrix} \alpha_1^i & \alpha_3^i & \cdots & \alpha_{2n_i-1}^i \\ \alpha_2^i & \alpha_4^i & \cdots & \alpha_{2n_i}^i \end{pmatrix} \in \mathring{\mathcal{H}}_{n_i} \tag{3}$$

we define a set of integers $h_i, v_i, a_1^i, \dots, a_{v_i}^i$ (the upper indices are not exponents) and a matrix $M_{i+1} \in \mathring{\mathcal{H}}_{n_{i+1}}$. Here $n_{i+1} = n_i - v_i$. We set $h_i = \min_{1 \le k \le 2n_i - 1}(\alpha_{k+1}^i - \alpha_k^i)$ and $\beta_k^i = \alpha_k^i - kh_i$. From the set of integers $\{\beta_k^i\}_{1 \le k \le 2n_i}$ we remove pairs of identical integers repeatedly until there is no such pair. Let $v_i = \#$(removed pairs). We arrange the removed integers (after their multiplicity was divided by two) in increasing order and call them $a_1^i, \dots, a_{v_i}^i$; we call the remaining integers $\alpha_1^{i+1}, \dots, \alpha_{2n_{i+1}}^{i+1}$ in increasing order. Then let M_{i+1} be a matrix of the form (3) but with all the i's were replaced by $i+1$'s. By repeating this we shall obtain a matrix M_{l+1} which has no element after a finite number (l) of steps. Then let λ be the Young diagram in Figure 1 and $a = \sqcup_{1 \le i \le l} \{a_j^i\}_{1 \le j \le v_i}$ the set of integers obtained through this procedure.

We denote $\mathcal{RC}$ (resp. $\mathcal{Y}$) the set of all rigged configurations (resp. Young diagrams). Let $\Xi : \mathring{\mathcal{H}} \to \mathcal{RC}$ (resp. $\xi : \mathring{\mathcal{H}} \to \mathcal{Y}$) be the mapping that sends any matrix $M \in \mathring{\mathcal{H}}$ to a rigged configuration $\Xi(M) = (\lambda, a) \in \mathcal{RC}$ (resp. a Young diagram $\xi(M) = \lambda \in \mathcal{Y}$) by the above procedure. For a reason that will be clear afterwards we call them

Definition 6 (scattering data, soliton content). For any $M \in \mathring{\mathcal{H}}$ (or the automaton state associated with M) the rigged configuration $\Xi(M)$ is called its *scattering data*, and the Young diagram $\xi(M)$ is called its *soliton content*.

The mapping Ξ becomes a bijection between $\mathring{\mathcal{H}}$ and $\mathcal{RC}$. Its inverse map will be discussed in the next section.

Example 7. We consider the matrix M in Example 5. Then we have $h_1 = \min\{3-1, 4-3, 6-4, 9-6, 10-9\} = 1$. Since

$$\begin{pmatrix} 1 & 4 & 9 \\ 3 & 6 & 10 \end{pmatrix} - \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \times 1 = \begin{pmatrix} 0 & \mathit{1} & \mathit{4} \\ \mathit{1} & 2 & \mathit{4} \end{pmatrix},$$

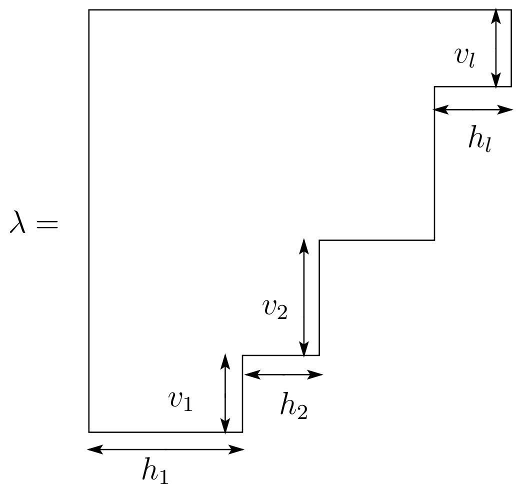


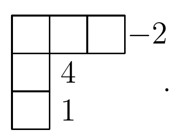
Figure 1: The Young diagram.

we obtain $v_1 = 2, a^1_1 = 1, a^1_2 = 4$, and $M_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Clearly $h_2 = 2$ and since

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \times 2 = \begin{pmatrix} -2 \\ -2 \end{pmatrix},$$

we obtain $v_2 = 1$ and $a^2_1 = -2$. Therefore we have

$$\begin{pmatrix} 1 & 4 & 9 \\ 3 & 6 & 10 \end{pmatrix} \quad \overset{\Xi}{\mapsto}$$



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We usually depict a rigged configuration by putting the riggings along the associated vertical edges of the Young diagram (See Example 7). This convention matches to the following step by step description of the mapping Ξ. Suppose we are on a square lattice made of vertices connected by unit length bonds, and have a matrix $M \in \mathring{\mathcal{H}}$. We start at an arbitrary vertex of the lattice. Every step we proceed rightward or upward by one unit length and replace our matrix M by the following rule. First suppose $M \notin \mathring{\mathcal{H}}_0$.

1. If $M \in \mathring{\mathcal{H}}$ we proceed rightward and replace M by $M - \left(\begin{smallmatrix} 1 3 \cdots \\ 2 4 \cdots \end{smallmatrix}\right)$ that is in $\mathcal{H}$. We place a horizontal segment of unit length on the way.

2. If $M \in \mathcal{H}$ but $M \notin \overset{\circ}{\mathcal{H}}$ there are pairs of same integers in M. Find a pair of smallest same integers, one of which is in the first row and the other in the second row. We proceed upward and remove the pair from M. We place a vertical segment of unit length on the way and put the removed integer on its right-hand side.

We repeat this procedure until our matrix M has no more entry. Then we stop there. By connecting the segments we obtain a path of lower-left to upper-right direction. By attaching an upper left corner to it (to complete a Young diagram) we obtain the rigged configuration.

4 Inverse scattering transform

In this section we study the inverse of the mapping Ξ. For any integer x we define a map $\Omega_x : \mathcal{H} \to \mathcal{H}$ by the following rule. Given $M \in \mathcal{H}_n$ we label its elements as (2). Let i be the largest integer obeying the condition $\alpha_i \leq x$; if $\alpha_1 > x$ we let $i = 0$. For an odd $i(= 2l-1)$ we set

$$\Omega_x(M) := \begin{pmatrix} \alpha_1 & \cdots & \alpha_{2l-1} & x & \cdots & \alpha_{2n-1} \\ \alpha_2 & \cdots & x & \alpha_{2l} & \cdots & \alpha_{2n} \end{pmatrix},$$

and for an even $i(= 2l)$

$$\Omega_x(M) := \begin{pmatrix} \alpha_1 & \cdots & \alpha_{2l-1} & x & \alpha_{2l+1} & \cdots & \alpha_{2n-1} \\ \alpha_2 & \cdots & \alpha_{2l} & x & \alpha_{2l+2} & \cdots & \alpha_{2n} \end{pmatrix}.$$

We extend our definition to any set of integers X. Given $M \in \mathcal{H}$ and $X = \{x_1, \ldots, x_p\}$ we set

$$\Omega_X(M) := \Omega_{x_1} \circ \cdots \circ \Omega_{x_p}(M).$$

In addition we define the following maps $\Phi_i : \mathcal{H} \to \overset{\circ}{\mathcal{H}} \quad (i = 0, 1)$. Given $M \in \mathcal{H}_n$ we set

$$\Phi_0(M) = M + \begin{pmatrix} 0 & 2 & \cdots & 2n-2 \\ 1 & 3 & \cdots & 2n-1 \end{pmatrix}, \Phi_1(M) = M + \begin{pmatrix} 1 & 3 & \cdots & 2n-1 \\ 2 & 4 & \cdots & 2n \end{pmatrix}.$$

For later analyses it is important to notice that the automaton state associated with $\Phi_1 \circ \Omega_X(M)$ (resp. $\Phi_0 \circ \Omega_X(M)$) is made by the following algorithm. We begin with the automaton state associated with M.

1. At the wall positions $x_1, \ldots, x_p$ and $\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}$ (resp. $\alpha_2, \alpha_4, \ldots, \alpha_{2n}$), put marks, say $\vee$. Here their multiplicity should be taken into account.

2. Split the array of boxes at the left-most wall with the mark(s). If it has $p(\geq 1)$ mark(s), translate every box and ball on the right-hand side of the wall rightwards by the width of $2p$ boxes. Delete the mark(s) at the wall.

3. Repeat item 2 as many times as possible.

4. Fill $\boxed{}\!\boxed{\bullet}$s (resp. $\boxed{\bullet}\!\boxed{}$s) into the gaps between the arrays of boxes made by the above procedures.

Remark 8. This algorithm also applies to Φ_0 and Φ_1 themselves, because we can regard them as $\Phi_0 \circ \Omega_X$ and $\Phi_1 \circ \Omega_X$ with $X = \emptyset$.

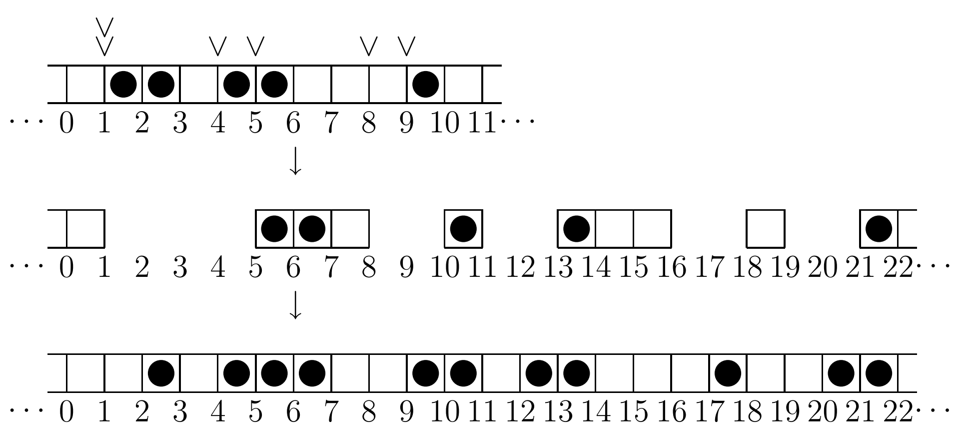
Example 9. Let M be the matrix in Example 5 and let $X = \{1, 5, 8\}$. Then we have

$$\begin{aligned}\Phi_1 \circ \Omega_X(M) &= \begin{pmatrix} \mathit{1} & 1 & 4 & \mathit{5} & \mathit{8} & 9 \\ \mathit{1} & 3 & \mathit{5} & 6 & \mathit{8} & 10 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 2 & 4 & 6 & 8 & 10 & 12 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 4 & 9 & 12 & 17 & 20 \\ 3 & 7 & 11 & 14 & 18 & 22 \end{pmatrix},\end{aligned}$$

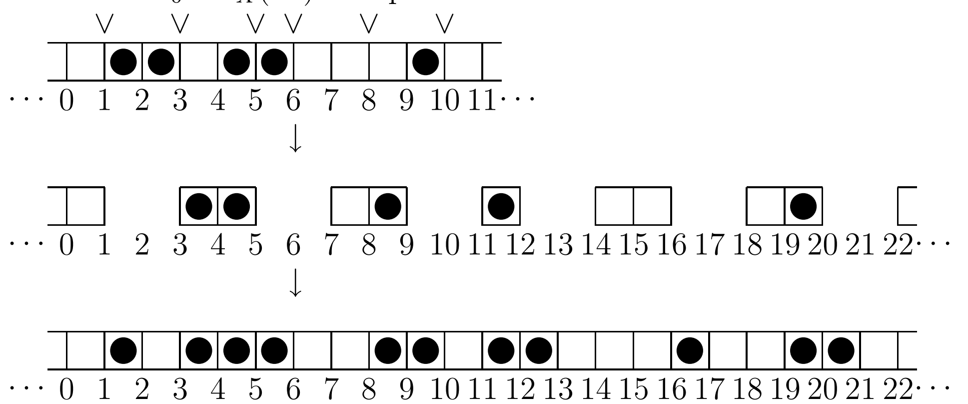
and

$$\Phi_0 \circ \Omega_X(M) = \begin{pmatrix} 1 & 3 & 8 & 11 & 16 & 19 \\ 2 & 6 & 10 & 13 & 17 & 21 \end{pmatrix}.$$

The algorithm to make the automaton state associated with $\Phi_1 \circ \Omega_X(M)$ is depicted as



In contrast that for $\Phi_0 \circ \Omega_X(M)$ is depicted as



Definition 10 (wave tails/fronts). In the state of the box-ball system associated with $M \in \mathring{\mathcal{H}}$ the wall positions specified by the first row of M are called *wave tails*. The wall positions specified by the second row of M are called *wave fronts*.

Remark 11. For any set of integers X the mapping $\Phi_1 \circ \Omega_X$ (resp. $\Phi_0 \circ \Omega_X$) embeds the box pair (resp.) at every wave tail (resp. wave front) of M, and at every wall position specified by X. This implies that the set of retarded arcs of $\Phi_1 \circ \Omega_X(M)$ (resp. advanced arcs of $\Phi_0 \circ \Omega_X(M)$) consists of — 1) depth ≥ 2 arcs made out of the retarded arcs (resp. advanced arcs) of M, being stretched but their topology unchanged, and — 2) depth 1 arcs introduced along with the newly embedded box pairs. See Figure 2 which corresponds to the latter case in Example 9.

Let (λ, a) be a rigged configuration where λ is the Young diagram in Figure 1 and $a = \sqcup_{1 \leq i \leq l} \{a^i_j\}_{1 \leq j \leq v_i}$ a set of integers. For any i we denote $\{a^i_j\}_{1 \leq j \leq v_i}$ by a^i (Again, the upper indices are not exponents). Let $L_{l+1}(\lambda, a) = M_{l+1}(\lambda, a) \in \mathring{\mathcal{H}}_0$ be the two-row matrix that has no element. For any $i \, (1 \leq i \leq l)$ we define the matrices $L_i(\lambda, a), M_i(\lambda, a) \in \mathring{\mathcal{H}}_{n_i}$ (where $n_i = \sum_{k=i}^{l} v_k$) recursively as

$$L_i(\lambda, a) := (\Phi_0)^{h_i} \circ \Omega_{a^i}(L_{i+1}(\lambda, a)),$$
$$M_i(\lambda, a) := (\Phi_1)^{h_i} \circ \Omega_{a^i}(M_{i+1}(\lambda, a)).$$

We write $L(\lambda, a) := L_1(\lambda, a)$ and $M(\lambda, a) := M_1(\lambda, a)$. Then we can regard L and M as mappings from $\mathcal{RC}$ to $\mathring{\mathcal{H}}$. It is easy to see that the map M is the inverse of the map Ξ in Section 3.

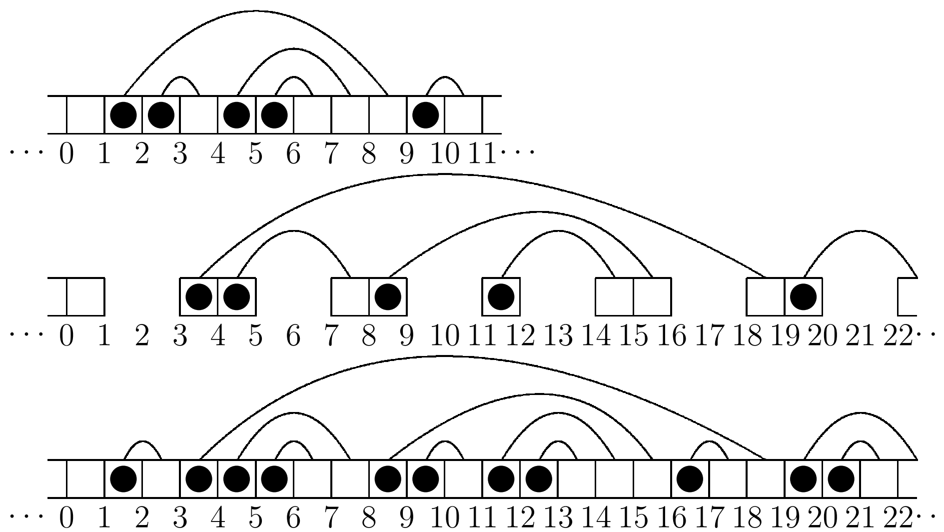


Figure 2: The embedding of box pairs and the advanced arcs.

Proposition 12. $\Xi(M(\lambda, a)) = (\lambda, a)$.

We call this map

Definition 13 (inverse scattering transform). For any $(\lambda, a) \in \mathcal{RC}$ the mapping (or its image) $(\lambda, a) \mapsto M(\lambda, a) \in \overset{\circ}{\mathcal{H}}$ is called its *inverse scattering transform*.

The inverse scattering transform and the mapping Ξ (which may be called direct scattering transform) form a bijection between $\mathcal{RC}$ and $\overset{\circ}{\mathcal{H}}$. We will also call this bijection itself the inverse scattering transform for the box-ball system.

5 A proof of the linearized time evolution of the scattering data

The generic appearance of the automaton states in the *real space* changes in not a simple way under the time evolution. According to the time evolution the set of associated scattering data also evolves. In this section we show that the time evolution of the scattering data is rather simple; it turns out to be linear.

Recall the two-row matrices $M_i(\lambda, a)$ and $L_i(\lambda, a)$ in Section 4. For any i we have

Lemma 14. *The first row of $M_i(\lambda, a)$ is equal to the second row of $L_i(\lambda, a)$.*

Proof. We give a proof by descending induction on i. For $i = l + 1$ the claim holds since the matrices have no entry. Let $(\beta_1, \dots, \beta_p)$ be the first row of $M_{i+1}(\lambda, a)$ and suppose it is equal to the second row of $L_{i+1}(\lambda, a)$. Then the first row of $\Omega_{a^i}(M_{i+1}(\lambda, a))$ is equal to the second row of $\Omega_{a^i}(L_{i+1}(\lambda, a))$ since both of them are made out of $\{a^i_1, \dots, a^i_{v_i}, \beta_1, \dots, \beta_p\}$ by rearranging them in increasing order. The claim of the lemma follows immediately because of the definitions of the mappings Φ_0 and Φ_1. $\square$

For any i we have

Lemma 15. *The set of advanced arcs of $L_i(\lambda, a)$ is equal to the set of retarded arcs of $M_i(\lambda, a)$.*

Proof. We give a proof by descending induction on i. For $i = l + 1$ the claim holds since the matrices have no entry. Suppose that the claim is true if i was replaced by $i + 1$. First we will show the following assertion:

- The set of advanced arcs of $\Phi_0 \circ \Omega_{a^i}(L_{i+1}(\lambda, a))$ is equal to the set of retarded arcs of $\Phi_1 \circ \Omega_{a^i}(M_{i+1}(\lambda, a))$.

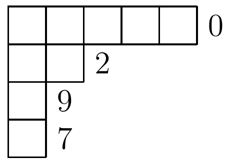
Here the $\Phi_0 \circ \Omega_{a^i}$ embeds $\boxed{\bullet}\boxed{}$s to the wave fronts of $L_{i+1}(\lambda, a)$ and the wall positions specified by the set a^i; the $\Phi_1 \circ \Omega_{a^i}$ embeds $\boxed{}\boxed{\bullet}$s to the wave tails of $M_{i+1}(\lambda, a)$ and the wall positions specified by the same a^i. But the wave fronts of $L_{i+1}(\lambda, a)$ are the wave tails of $M_{i+1}(\lambda, a)$ by Lemma 14. Therefore every embedding point coincides exactly between both cases. This together with the assumption of the induction concludes that the above assertion is true (See Remark 11). By repeating this argument we can obtain the claim of the lemma for i (See Remark 8). The proof is completed. $\square$

By Lemma 3 it leads to

Corollary 16. *If the state of the box-ball system associated with $L(\lambda, a)$ is updated once, it is the state of the system associated with $M(\lambda, a)$.*

We give an example for Lemmas 14 and 15.

Example 17. Recall the second rigged configuration of the example in Section 1.

$(\lambda, a) =$ 

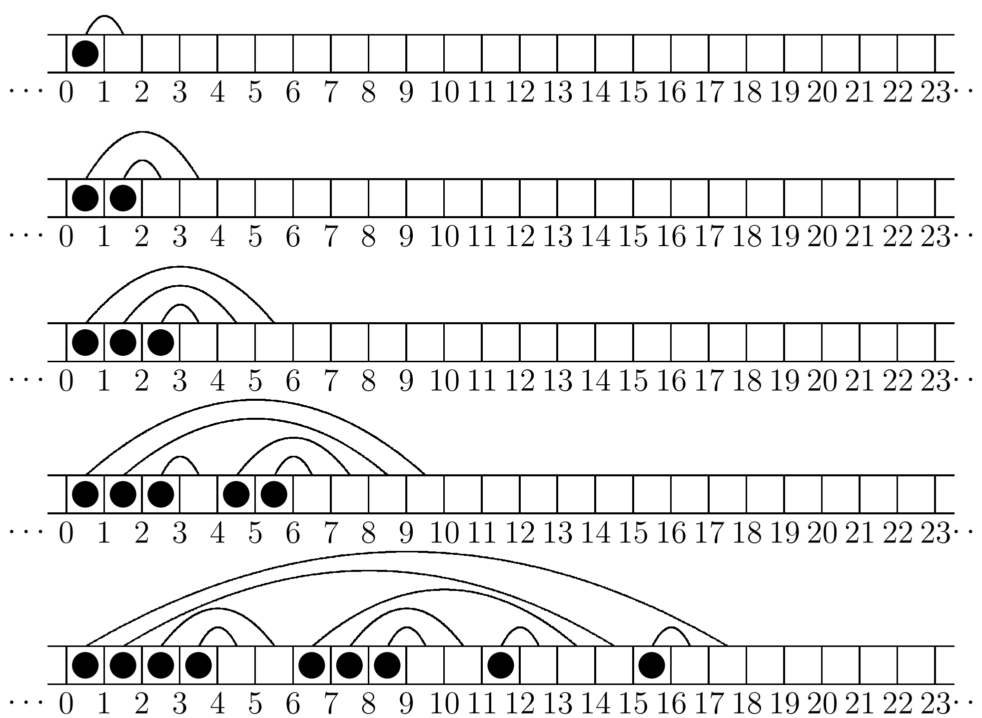
The λ has the shape of $h_1 = h_2 = 1, h_3 = 3$ and $v_1 = 2, v_2 = v_3 = 1$. The set of riggings is $a = a^1 \sqcup a^2 \sqcup a^3$ with $a^1 = \{7, 9\}, a^2 = \{2\}, a^3 = \{0\}$. Thus we have (by denoting $L_i(\lambda, a)$ by L_i)

$$\Phi_0 \circ \Omega_{a^3}(L_4) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \Phi_0^2 \circ \Omega_{a^3}(L_4) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad L_3 = \Phi_0^3 \circ \Omega_{a^3}(L_4) = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

$$L_2 = \Phi_0 \circ \Omega_{a^2}(L_3) = \begin{pmatrix} 0 & \mathit{2} \\ \mathit{2} & 3 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 3 & 6 \end{pmatrix},$$

$$L_1 = \Phi_0 \circ \Omega_{a^1}(L_2) = \begin{pmatrix} 0 & 4 & \mathit{7} & \mathit{9} \\ 3 & 6 & \mathit{7} & \mathit{9} \end{pmatrix} + \begin{pmatrix} 0 & 2 & 4 & 6 \\ 1 & 3 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 11 & 15 \\ 4 & 9 & 12 & 16 \end{pmatrix}.$$

Their advanced arcs are given as follows.



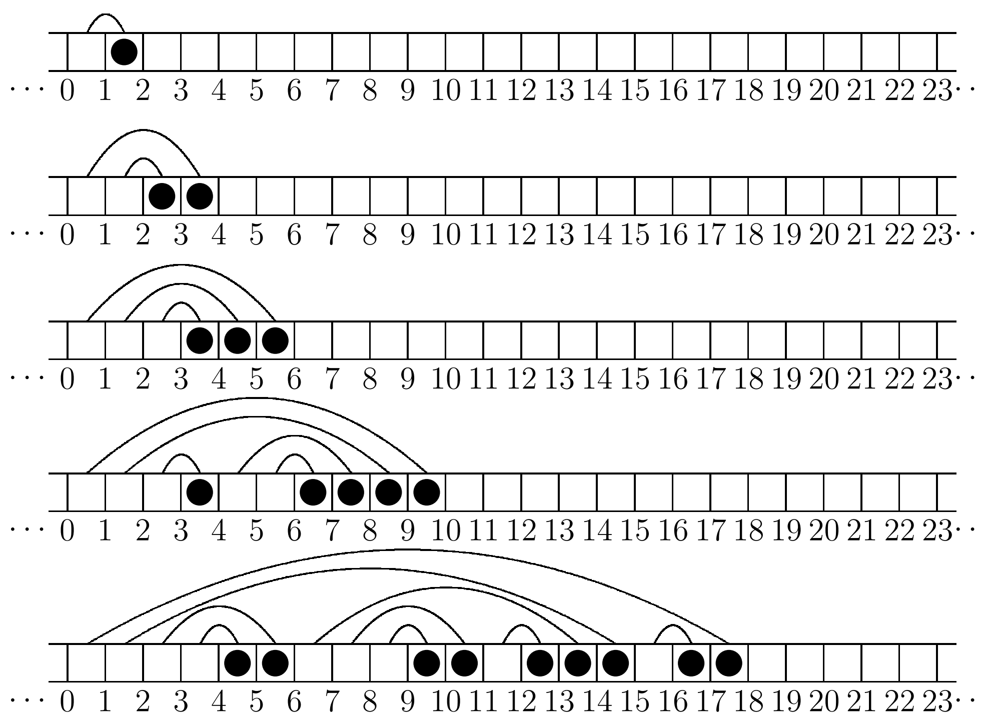
On the other hand we have (by denoting $M_i(\lambda, a)$ by M_i)

$$\Phi_1 \circ \Omega_{a^3}(M_4) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \Phi_1^2 \circ \Omega_{a^3}(M_4) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad M_3 = \Phi_1^3 \circ \Omega_{a^3}(M_4) = \begin{pmatrix} 3 \\ 6 \end{pmatrix},$$

$$M_2 = \Phi_1 \circ \Omega_{a^2}(M_3) = \begin{pmatrix} 2 & 3 \\ 2 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 4 & 10 \end{pmatrix},$$

$$M_1 = \Phi_1 \circ \Omega_{a^1}(M_2) = \begin{pmatrix} 3 & 6 & 7 & 9 \\ 4 & 7 & 9 & 10 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix} = \begin{pmatrix} 4 & 9 & 12 & 16 \\ 6 & 11 & 15 & 18 \end{pmatrix}.$$

Their retarded arcs are given as follows.



Let $\underline{a}$ be the rigging made out of a by replacing a^i_j by $\underline{a}^i_j := a^i_j - \sum_{k=1}^{i} h_k$. Let $\underline{a}^i$ denote the set $\{\underline{a}^i_j\}_{1 \le j \le v_i}$. We set $T_1 := \Phi_0^{-1} \circ \Phi_1$. By the same symbol we write as $T_1(X) = \{x_1 + 1, \ldots, x_p + 1\}$ for any set of integers $X = \{x_1, \ldots, x_p\}$. For instance we have $a^i = T_1^{h_1 + \cdots + h_i}(\underline{a}^i)$. It is easy to see that

Lemma 18. *For any $M \in \mathcal{H}$ and any set of integers X the relation $\Omega_{T_1(X)}(T_1(M)) = T_1 \circ \Omega_X(M)$ holds.*

Then for any i we have

Lemma 19. *The following relation holds:* $L_i(\lambda, a) = T_1^{h_1+\cdots+h_{i-1}}(M_i(\lambda, \underline{a}))$.

Proof. We give a proof by descending induction on i. For $i = l+1$ the claim holds because both sides are equal to a matrix with no entry. Suppose $L_{i+1}(\lambda, a) = T_1^{h_1+\cdots+h_i}(M_{i+1}(\lambda, \underline{a}))$. By repeated use of Lemma 18 we have $\Omega_{a^i}(L_{i+1}(\lambda, a)) = T_1^{h_1+\cdots+h_i} \circ \Omega_{\underline{a}^i}(M_{i+1}(\lambda, \underline{a}))$. Then

$$\begin{aligned} L_i(\lambda, a) &= \Phi_0^{h_i} \circ \Omega_{a^i}(L_{i+1}(\lambda, a)) \\ &= \Phi_0^{h_i} \circ (\Phi_0^{-1} \circ \Phi_1)^{h_1+\cdots+h_i} \circ \Omega_{\underline{a}^i}(M_{i+1}(\lambda, \underline{a})) \\ &= (\Phi_0^{-1} \circ \Phi_1)^{h_1+\cdots+h_{i-1}} \circ \Phi_1^{h_i} \circ \Omega_{\underline{a}^i}(M_{i+1}(\lambda, \underline{a})) \\ &= T_1^{h_1+\cdots+h_{i-1}}(M_i(\lambda, \underline{a})). \end{aligned}$$

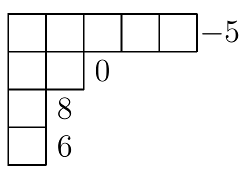
$\square$

By setting $i = 1$ in this lemma we have

Corollary 20. *The following relation holds:* $L(\lambda, a) = M(\lambda, \underline{a})$.

We give an example for Lemma 19.

Example 21. Recall the first rigged configuration of the example in Section 1.

$(\lambda, \underline{a}) =$ 

The set of riggings is $\underline{a} = \underline{a}^1 \sqcup \underline{a}^2 \sqcup \underline{a}^3$ with $\underline{a}^1 = \{6, 8\}, \underline{a}^2 = \{0\}, \underline{a}^3 = \{-5\}$. We have (by denoting $M_i(\lambda, \underline{a})$ by M_i)

$$\Phi_1 \circ \Omega_{\underline{a}^3}(M_4) = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad \Phi_1^2 \circ \Omega_{\underline{a}^3}(M_4) = \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \quad M_3 = \Phi_1^3 \circ \Omega_{\underline{a}^3}(M_4) = \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

$$M_2 = \Phi_1 \circ \Omega_{\underline{a}^2}(M_3) = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix},$$

$$M_1 = \Phi_1 \circ \Omega_{\underline{a}^1}(M_2) = \begin{pmatrix} -1 & 3 & 6 & 8 \\ 2 & 5 & 6 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 11 & 15 \\ 4 & 9 & 12 & 16 \end{pmatrix}.$$

Compare the M_3, M_2, M_1 in this example with the L_3, L_2, L_1 in Example 17.

Now we present the main result of this paper. By combining Corollaries 16 with 20 we have

Theorem 22. *If the state of the box-ball system associated with $M(\lambda, \underline{a})$ is updated once, it is the state of the system associated with $M(\lambda, a)$.*

By this theorem we can regard the rows of λ as the action variables for the box-ball system, and the riggings as the associated angle variables. Initial value problems of the box-ball system can now be solved by an *inverse scattering method*. Let $T(a)$ be the rigging made out of a by replacing a^i_j by $a^i_j + \sum_{k=1}^{i} h_k$. Let $T(M) \in \overset{\circ}{\mathcal{H}}$ be the matrix for the automaton state which is made out of the state for $M \in \overset{\circ}{\mathcal{H}}$ by updating once. Then for any integer N we have

$$\begin{tikzcd}
\begin{bmatrix} \text{Automaton State} \\ M \in \overset{\circ}{\mathcal{H}} \end{bmatrix} \arrow[r, "\text{Direct Scattering}"] \arrow[d, "\text{Time Evolution}"'] & \begin{bmatrix} \text{Scattering Data} \\ (\lambda, a) \in \mathcal{RC} \end{bmatrix} \arrow[d, "\text{Linearized Time Evolution}"] \\
\begin{bmatrix} \text{Automaton State} \\ T^N(M) \in \overset{\circ}{\mathcal{H}} \end{bmatrix} & \begin{bmatrix} \text{Scattering Data} \\ (\lambda, T^N(a)) \in \mathcal{RC} \end{bmatrix} \arrow[l, "\text{Inverse Scattering}"']
\end{tikzcd}.$$

This diagram shows that the time evolution of any automaton state is given by a composition of the direct scattering, the linearized time evolution of the scattering data, and the inverse scattering.

To close the section we recall the notion of soliton content in Definition 6. We can write the soliton content λ as a partition

$$\lambda = (1^{m_1} 2^{m_2} \dots h^{m_h}). \tag{4}$$

Here $m_i \in \mathbb{Z}_{\geq 0}$ is the multiplicity of i in λ. We assume $m_h \neq 0$; if we take λ as in Figure 1 then $h = \sum_{k=1}^{l} h_k$, namely the width of the Young diagram. We can say that there are m_h solitons of length h, m_{h-1} solitons of length $h-1$, ... , and m_1 solitons of length 1 in the automaton state associated with λ. The numbers of successive balls in the real space can change. However the solitons (in the space of scattering data) keep their independence. To see the soliton content in the real space one generally has to apply the updating process many times.

6 Upper and lower bounds for the inverse scattering transform

Before describing the aim of this section, here we give a representation for rigged configurations that is slightly different from the previous one. Let λ be the Young diagram of the form (4). Denote the set of riggings for i by $J^i = \{J^i_j\}_{1\le j\le m_i}$, $J^i_j \in \mathbb{Z}$. In other words J^i is associated with the set of λ's rows of length i. Then if $m_i = 0$ we have $J^i = \emptyset$. Let $J = \sqcup_{1\le i\le h} J^i$. Now the pair (λ, J) specifies a rigged configuration.

Remark 23. For reasons of simpler descriptions of some statements we shall assume $J^i_1 \le J^i_2 \le \cdots \le J^i_{m_i}$ for any i.

In this section we study the upper and lower bounds for the inverse scattering transform. The result is used in the following sections to study the relation between partition functions for the box-ball system and fermionic formulas. It is also used to prove that our inverse scattering transform is equivalent to a bijection in [8] (Section 9).

For any set of integers X we denote $\Phi_1 \circ \Omega_X$ by Π_X, where Φ_1 and Ω_X are the maps introduced in Section 4. We set

$$\overline{M}_a(\lambda, J) = \Pi_{J^a} \circ \Pi_{J^{a+1}} \circ \cdots \circ \Pi_{J^h}(\overline{M}_{h+1}), \tag{5}$$

$$\overline{M}(\lambda, J) = \overline{M}_1(\lambda, J), \tag{6}$$

where $\overline{M}_{h+1} \in \mathring{\mathcal{H}}_0$ is the matrix with no entry. We can regard $\overline{M}$ as a mapping from $\mathcal{RC}$ to $\mathring{\mathcal{H}}$. It should be observed that this $\overline{M}$ is the same as the M in Section 5 although the descriptions of rigged configurations are different. By abuse of notation we shall denote Ξ^{-1} either the M or the $\overline{M}$.

We try to find expressions for the lower right and the upper left elements of the matrices $\overline{M}_a(\lambda, J)$ in (5) because they determine the upper and lower bounds for the inverse scattering transform. Given any i with $m_i > 0$ let $c^{i,a} = \sum_{k=a}^{h} (\min(k,i) - a + 1)\, m_k$ and $J^{i,a}_j = J^i_j + 2c^{i,a}$ for $1 \le a \le i+1$. It is easy to see that

$$J^{a-1,a}_j = J^{a-1}_j \tag{7}$$

and

$$c^{i,a+1} = c^{i,a} - (m_a + \cdots + m_h). \tag{8}$$

For $1 \le a \le h$ we have

Lemma 24.

1. *The lower right element of* $\Omega_{J^a}(\overline{M}_{a+1}(\lambda, J))$ *is* $\max_{a\le i\le h, m_i>0}\{J^{i,a+1}_{m_i}\}$.

2. *The lower right element of* $\overline{M}_a(\lambda, J)$ *is* $\max_{a\le i\le h, m_i>0}\{J^{i,a}_{m_i}\}$.

Proof. We give a proof by descending induction on a. For $a = h$ the item 1 holds because both sides are equal to $J^h_{m_h}$. Assume that the item 1 holds for a. Since $\overline{M}_a(\lambda, J) = \Phi_1 \circ \Omega_{J^a}(\overline{M}_{a+1}(\lambda, J))$ we have

$$\begin{aligned}\text{L. R. E. of}\,\overline{M}_a(\lambda, J) &= \text{L. R. E. of}\,\Omega_{J^a}(\overline{M}_{a+1}(\lambda, J)) + 2(m_a + \cdots + m_h)\\ &= \max_{a\le i\le h, m_i>0}\{J^{i,a+1}_{m_i}\} + 2(m_a + \cdots + m_h)\\ &= \max_{a\le i\le h, m_i>0}\{J^{i,a}_{m_i}\}.\end{aligned}$$

(L. R. E. is for *lower right element.*) Here the first equality is by the definition of Φ_1, the second is by the assumption of induction, and the third is from (8). Therefore the item 2 holds for a. Then we have

$$\begin{aligned}\text{L. R. E. of}\,\Omega_{J^{a-1}}(\overline{M}_a(\lambda, J)) &= \max\{J^{a-1}_{m_{a-1}}, \max_{a\le i\le h, m_i>0}\{J^{i,a}_{m_i}\}\}\\ &= \max_{a-1\le i\le h, m_i>0}\{J^{i,a}_{m_i}\}.\end{aligned}$$

Here we used (7). Therefore the item 1 holds for $a - 1$. The proof is completed. ☐

For $1 \le a \le h$ we have

Lemma 25.

1. *The upper left element of* $\Omega_{J^a}(\overline{M}_{a+1}(\lambda, J))$ *is* $\min_{a\le i\le h, m_i>0}\{J^i_1 + i - a\}$.

2. *The upper left element of* $\overline{M}_a(\lambda, J)$ *is* $\min_{a\le i\le h, m_i>0}\{J^i_1 + i - a + 1\}$.

Proof. We give a proof by descending induction on a. For $a = h$ the item 1 holds because both sides are equal to $J^h_{m_h}$. Assume that the item 1 holds for a. By the definition of Φ_1 the item 2 for a follows. Then we have

$$\begin{aligned}\text{U. L. E. of}\,\Omega_{J^{a-1}}(\overline{M}_a(\lambda, J)) &= \min\{J^{a-1}_1, \min_{a\le i\le h, m_i>0}\{J^i_1 + i - a + 1\}\}\\ &= \min_{a-1\le i\le h, m_i>0}\{J^i_1 + i - (a - 1)\}.\end{aligned}$$

(U. L. E. is for *upper left element.*) Therefore the item 1 holds for $a - 1$. The proof is completed. ☐

Then we have that

Theorem 26. *Let (λ, J) be a rigged configuration of the form given at the beginning of this section. For the automaton state associated with $\Xi^{-1}(\lambda, J)$ the following statements hold.*

1. *The number of balls is $|\lambda| := \sum_{h \geq i \geq 1} i m_i$.*

2. *The left-most wave tail of the state is $\min_{1 \leq i \leq h, m_i > 0}\{J^i_1 + i\}$, and the right-most wave front is $\max_{1 \leq i \leq h, m_i > 0}\{J^i_{m_i} + 2\sum_{j=1}^{h} \min(i, j) m_j\}$.*

Proof. Item 1 follows from the fact that each of the Π_{J^a} in (5) is embedding as many $\boxed{}\boxed{\bullet}$s as the height of the a-th column of the Young diagram λ, which is equal to $m_a + \cdots + m_h$. Item 2 follows from Lemmas 24 and 25. $\square$

7 Automaton states in a finite interval

In the set of all automaton states $\overset{\circ}{\mathcal{H}}$ we define a set of all the states, and a set of all the *highest weight* states in a finite interval with a specified soliton content. Then we determine the associated subsets in the set of all rigged configurations $\mathcal{RC}$. The results are used in the next section.

To begin with we introduce the notion of

Definition 27 (highest weight state). A state of the box-ball system is called a *highest weight state* if all the balls are on the right-hand side of the wall 0, and for any ℓ the number of empty boxes between the walls 0 and ℓ is not less than that of filled boxes there.

Consider an array of L boxes. We put the numbers $0, 1, \ldots, L$ to its walls. Recall the matrix notation in Section 3. Let $\mathcal{H}_n(L, \lambda)$ be the set of all matrices of the form (2) obeying the conditions $0 \leq \alpha_1, \alpha_{2n} \leq L$ and $\xi(M) = \lambda$, and $\mathcal{H}^+_n(L, \lambda)$ be its subset with additional conditions $2\sum_{k=1}^{j-1} \alpha_{2k} + \alpha_{2j} \leq 2\sum_{k=1}^{j} \alpha_{2k-1}$ for $1 \leq j \leq n$. In the latter case the set of additional conditions is equivalent to that for the highest weight states in Definition 27. We set

$$\mathcal{H}(L, \lambda) = \sqcup_{n \geq 0} \mathcal{H}_n(L, \lambda), \tag{9}$$

$$\mathcal{H}^+(L, \lambda) = \sqcup_{n \geq 0} \mathcal{H}^+_n(L, \lambda). \tag{10}$$

They are sets of automaton states in the finite interval between walls 0 and L, and with the specified soliton content λ.

Recall the rigged configuration (λ, J) of the form given at the beginning of Section 6. Fix a positive integer L. As functions of the variable λ we write

$$p_i(\lambda) := L - 2\sum_{j\geq 1}\min(i,j)m_j, \tag{11}$$

$$m_i(\lambda) := m_i, \quad \phi(\lambda) := \sum_{i,j\geq 1}\min(i,j)m_i m_j. \tag{12}$$

The latter will be used in the next section (Lemma 33). We set

$$\mathcal{RC}(L,\lambda) = \{(\mu, J) \in \mathcal{RC} | \mu = \lambda, -i \leq J^i_1 \leq \cdots \leq J^i_{m_i} \leq p_i(\lambda)\,\text{for any } i\}, \tag{13}$$

$$\mathcal{RC}^+(L,\lambda) = \{(\mu, J) \in \mathcal{RC} | \mu = \lambda, 0 \leq J^i_1 \leq \cdots \leq J^i_{m_i} \leq p_i(\lambda)\,\text{for any } i\}. \tag{14}$$

By definition we have $\mathcal{H}^+(L,\lambda) \subset \mathcal{H}(L,\lambda)$ and $\mathcal{RC}^+(L,\lambda) \subset \mathcal{RC}(L,\lambda)$. From Theorem 26 it is clear that

Lemma 28. *The inverse scattering transform is a bijection between $\mathcal{H}(L,\lambda)$ and $\mathcal{RC}(L,\lambda)$.*

Moreover we have that

Lemma 29. *The inverse scattering transform is a bijection between $\mathcal{H}^+(L,\lambda)$ and $\mathcal{RC}^+(L,\lambda)$.*

Proof. Recall that the map Ξ^{-1} is so defined as to make automaton states by embedding $\boxed{}\boxed{\bullet}$s recursively at the wave tails, and at the wall positions specified by the riggings. We assume $J^i_{m_i} \leq p_i(\lambda)$ for any i; if otherwise the right-most wave front of $\Xi^{-1}(\lambda, J)$ would exceed L. Suppose $(\lambda, J) \in \mathcal{RC}^+(L,\lambda)$. By induction we prove the automaton state associated with $\overline{M}_a(\lambda, J)$ in (5) is a highest weight state. If $a = h + 1$ the claim holds by definition. Suppose the automaton state associated with $\overline{M}_{a+1}(\lambda, J)$ is a highest weight state. By the embeddings we are shifting the divided arrays of boxes to the right (See Example 9). Therefore if a state is a highest weight state, then the state obtained from it by embedding a $\boxed{}\boxed{\bullet}$ at any wave tail, or at any non-negative wall position is a highest weight state. Thus by the assumption of induction the automaton state associated with $\overline{M}_a(\lambda, J)$ is also a highest weight state. By taking $a = 1$ we have $\Xi^{-1}(\lambda, J) \in \mathcal{H}^+(L,\lambda)$. Thus $\Xi^{-1}(\mathcal{RC}^+(L,\lambda)) \subset \mathcal{H}^+(L,\lambda)$.

Conversely suppose $(\lambda, J) \notin \mathcal{RC}^+(L,\lambda)$. Then there is a negative rigging $J^i_1 (< 0)$ for some i. By embedding a $\boxed{}\boxed{\bullet}$ at J^i_1 we obtain a state that is not a highest weight state. One cannot obtain a highest weight state from a non-highest weight state by embedding a $\boxed{\ \ }\!\boxed{\bullet}$ at any position. Therefore we have $\Xi^{-1}(\lambda, J) \notin \mathcal{H}^+(L, \lambda)$. Thus $\Xi(\mathcal{H}^+(L, \lambda)) \subset \mathcal{RC}^+(L, \lambda)$. $\square$

8 Partition functions with a specified soliton content

In this section we study partition functions for the box-ball system. Recall the quantity m_i in (4). In the context of the box-ball system the m_i was the number of *solitons* of length i. By considering the partition function taken over the highest weight states, we shall find that the m_i coincides with a well-known quantity in the context of Bethe ansatz, the number of *strings* of length i. We also consider a partition function taken over not necessarily highest weight states.

First we introduce the notion of energy for states of the box-ball system. Among the four types of configurations of adjacent boxes, let $\boxed{\ \ }\!\boxed{\ \ }$, $\boxed{\bullet}\!\boxed{\bullet}$, $\boxed{\bullet}\!\boxed{\ \ }$ do not cost the energy, and let $\boxed{\ \ }\!\boxed{\bullet}$ cost the energy of its center wall position.

Definition 30 (energy of an automaton state). For any matrix $M \in \mathring{\mathcal{H}}$ of the form (2) (or for the automaton state associated with M), the quantity $E_{\mathrm{CTM}}(M) := \sum_{k \geq 1} \alpha_{2k-1}$ is called its *energy.*

This energy is not a conserved quantity of the box-ball system. It has its origin in the corner transfer matrix (CTM) analyses of the solvable lattice models [19]. In particular the energy in Definition 30 is for the six-vertex model [20] but the sign was reversed according to the change of the model to its ferromagnetic regime.

Next we introduce the notion of

Definition 31 (energy of a rigged configuration). For any rigged configuration (λ, J) of the form at the beginning of Section 6 the quantity $E_{\mathrm{RC}}(\lambda, J) := \sum_{h \geq i,j \geq 1} \min(i,j) m_i m_j + \sum_{i=1}^{h} \sum_{j=1}^{m_i} J^i_j$ is called its *energy.*

We observe that the inverse scattering transform is an energy-preserving bijection in the sense of these energies.

Lemma 32. *If $\Xi^{-1}(\lambda, J) = M$ then the identity $E_{\mathrm{RC}}(\lambda, J) = E_{\mathrm{CTM}}(M)$ holds.*

Proof. Fill 1's, 3's, 5's, ... into the boxes of the first, second, third, ... rows of the Young diagram λ. The sum of these numbers in the a-th column is $(\sum_{a=k}^{h} m_a)^2$

because the length of the column is $(\sum_{a=k}^{h} m_a)$. By summing them for $1 \le a \le h$ we obtain $\sum_{h \ge i,j \ge 1} \min(i,j) m_i m_j$. Then by adding the sum of all the riggings we have $E_{\mathrm{RC}}(\lambda, J)$. Let $M = \Xi^{-1}(\lambda, J)$. Recall the map $\Pi_{J^a} = \Phi_1 \circ \Omega_{J^a}$ in (5) and consider what it does on the first row of $\overline{M}_{a+1}(\lambda, J)$. The Ω_{J^a} embeds $J^a_1, \dots, J^a_{m_a}$ somewhere and the Φ_1 adds $1, 3, 5, \dots$ to the first, second, third, $\dots$ columns. By summing them for $1 \le a \le h$ we obtain $E_{\mathrm{CTM}}(M)$ that is certainly equal to the above $E_{\mathrm{RC}}(\lambda, J)$. $\square$

Let us introduce the notion of partition functions for the box-ball system with a specified soliton content [7]. By means of the sets of automaton states (9), (10) and the energy in Definition 30 we define

$$Z(L, \lambda) := \sum_{M \in \mathcal{H}(L,\lambda)} q^{E_{\mathrm{CTM}}(M)}, \tag{15}$$

$$Z^+(L, \lambda) := \sum_{M \in \mathcal{H}^+(L,\lambda)} q^{E_{\mathrm{CTM}}(M)}. \tag{16}$$

On the other hand we define the partition functions for the rigged configurations by using (13), (14) and the energy in Definition 31 as

$$Y(L, \lambda) := \sum_{(\lambda, J) \in \mathcal{RC}(L,\lambda)} q^{E_{\mathrm{RC}}(\lambda, J)}, \tag{17}$$

$$Y^+(L, \lambda) := \sum_{(\lambda, J) \in \mathcal{RC}^+(L,\lambda)} q^{E_{\mathrm{RC}}(\lambda, J)}. \tag{18}$$

Recall the definition of the q-binomial coefficient [21]

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \frac{\prod_{k=1}^{n}(1-q^k)}{\prod_{k=1}^{m}(1-q^k)\prod_{k=1}^{n-m}(1-q^k)} & \text{if } 0 \le m \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have that

Lemma 33. *The following relations hold,*

$$Y(L, \lambda) = q^{-|\lambda| + \phi(\lambda)} \prod_{i \ge 1} \begin{bmatrix} p_i(\lambda) + m_i(\lambda) + i \\ m_i(\lambda) \end{bmatrix},$$

$$Y^+(L, \lambda) = q^{\phi(\lambda)} \prod_{i \ge 1} \begin{bmatrix} p_i(\lambda) + m_i(\lambda) \\ m_i(\lambda) \end{bmatrix}.$$

Here $p_i(\lambda), m_i(\lambda)$ and $\phi(\lambda)$ are given by (11),(12).

Proof. The latter relation follows from the identity of the q-binomial coefficient [21]

$$\sum_{0\le J_1\le\cdots\le J_m\le p} q^{J_1+\cdots+J_m} = \begin{bmatrix} p+m \\ m \end{bmatrix}.$$

The former relation also follows from this identity by replacing p by $p+i$ and J_j's by J_j+i's. $\square$

By Lemmas 28, 29 and 32 the identities $Z(L,\lambda) = Y(L,\lambda)$ and $Z^+(L,\lambda) = Y^+(L,\lambda)$ hold. Thus we have that

Theorem 34. *The following relations hold,*

$$\sum_{M\in\mathcal{H}(L,\lambda)} q^{E_{\mathrm{CTM}}(M)} = q^{-|\lambda|+\phi(\lambda)} \prod_{i\ge1} \begin{bmatrix} p_i(\lambda)+m_i(\lambda)+i \\ m_i(\lambda) \end{bmatrix},$$

$$\sum_{M\in\mathcal{H}^+(L,\lambda)} q^{E_{\mathrm{CTM}}(M)} = q^{\phi(\lambda)} \prod_{i\ge1} \begin{bmatrix} p_i(\lambda)+m_i(\lambda) \\ m_i(\lambda) \end{bmatrix}.$$

These are the identities proposed in [7].

By taking a sum over all the soliton contents with a fixed number ($s = |\lambda|$) of balls their left-hand sides yield simple expressions (See Appendix A). Then we have that

Proposition 35. *The following relations hold. For* $0\le s\le L$

$$\begin{bmatrix} L \\ s \end{bmatrix} = q^{-s} \sum_{\lambda\vdash s} q^{\phi(\lambda)} \prod_{i\ge1} \begin{bmatrix} p_i(\lambda)+m_i(\lambda)+i \\ m_i(\lambda) \end{bmatrix},$$

and for $0\le s\le L/2$

$$\begin{bmatrix} L \\ s \end{bmatrix} - \begin{bmatrix} L \\ s-1 \end{bmatrix} = \sum_{\lambda\vdash s} q^{\phi(\lambda)} \prod_{i\ge1} \begin{bmatrix} p_i(\lambda)+m_i(\lambda) \\ m_i(\lambda) \end{bmatrix}.$$

The latter is a q-analogue of the Bethe's formula [22]. In the context of Bethe ansatz the summation variable $\lambda = (1^{m_1}2^{m_2}\dots)$ has meant that the associated eigenstate of the Heisenberg magnet is given by a set of variables which satisfy an algebraic equation (the Bethe equation), where the multiplicity of the strings (particular configurations of its roots in the complex plane) of length i was given by m_i. Thus we obtained a new interpretation of a well-known fermionic character formula; *the summation variable λ is representing the soliton content of a cellular automaton.*

On the other hand the former formula may be new and the author expects that it can also be used in a completeness problem of Bethe ansatz analysis for some quantum mechanical system.

9 An alternative description of the inverse scattering transform

In this section we describe $sl(2)$ case of the bijection in [8] in our setting and prove that it is equivalent to our inverse scattering transform.

Let (λ, J) be the rigged configuration given at the beginning of Section 6. We show a procedure to make another rigged configuration (λ', J') from it. Let s be the smallest integer with $m_s > 0$ that maximizes $J^{s,1}_{m_s} = J^s_{m_s} + 2\sum_{j=1}^h \min(s,j)m_j$. Call the row of the diagram λ of length s and with rigging $J^s_{m_s}$ the *shortest singular row* [8]. We first delete the right-most box of the shortest singular row, and then rearrange the order of the rows of the diagram so that the result is again a Young diagram. We replace the rigging of the shortest singular row by a specific manner; explicitly

1. If $s = 1$ let $\lambda' = (1^{m_1 - 1}2^{m_2}\dots h^{m_h})$, $J'^1 = J^1 \backslash \{J^1_{m_1}\}$, and $J'^i = J^i$ for $2 \leq i \leq h$.
2. If $s \geq 2$ let $\lambda' = (1^{m_1}\dots(s-1)^{m_{s-1}+1}s^{m_s - 1}\dots h^{m_h})$, $J'^{s-1} = J^{s-1} \sqcup \{J^{s,s}_{m_s} - 1\}$, $J'^s = J^s \setminus \{J^s_{m_s}\}$, and $J'^i = J^i$ for $i \neq s, s-1$.

Here $J^{s,s}_{m_s} = J^s_{m_s} + 2(m_s + \dots + m_h)$.

Theorem 36. *If $s = 1$ the matrix $\overline{M}(\lambda', J')$ is made out of $\overline{M}(\lambda, J)$ by removing its right-most column. If $s \geq 2$ it is made out of $\overline{M}(\lambda, J)$ by subtracting 1 from its lower right element.*

Proof. For both cases we have $\overline{M}_a(\lambda', J') = \overline{M}_a(\lambda, J)$ for $s+1 \leq a \leq h$. Since $J'^s = J^s \setminus \{J^s_{m_s}\}$ we see that the matrix $\Omega_{J'^s}(\overline{M}_{s+1}(\lambda', J'))$ is made out of $\Omega_{J^s}(\overline{M}_{s+1}(\lambda, J))$ by removing its right-most column ${}^t(J^s_{m_s}, J^s_{m_s})$. By applying Φ_1 we can deduce that $\overline{M}_s(\lambda', J')$ is made out of $\overline{M}_s(\lambda, J)$ by removing its right-most column ${}^t(J^{s,s}_{m_s} - 1, J^{s,s}_{m_s})$. Hence the $s = 1$ case follows. Suppose $s \geq 2$. Then since $J'^{s-1} = J^{s-1} \sqcup \{J^{s,s}_{m_s} - 1\}$ we see that $\Omega_{J'^{s-1}}(\overline{M}_s(\lambda', J'))$ and $\Omega_{J^{s-1}}(\overline{M}_s(\lambda, J))$ have the same number of columns; we also see that the right-most column of the former is ${}^t(J^{s,s}_{m_s} - 1, J^{s,s}_{m_s} - 1)$, that of the latter is ${}^t(J^{s,s}_{m_s} - 1, J^{s,s}_{m_s})$, and the other columns are equal to each other. By applying Φ_1 again we obtain $\overline{M}_{s-1}(\lambda', J')$ with its lower right element $J^{s,s-1}_{m_s} - 1$, and $\overline{M}_{s-1}(\lambda, J)$ with its lower right element $J^{s,s-1}_{m_s}$. By the proof of Lemma 24 one can deduce that their difference is kept untouched until we obtain $\overline{M}(\lambda', J')$ with its lower right element $J^{s,1}_{m_s} - 1$, and $\overline{M}(\lambda, J)$ with its lower right element $J^{s,1}_{m_s}$. $\square$

From this theorem and its proof we have that

Corollary 37. *The automaton state $\Xi^{-1}(\lambda', J')$ is made out of $\Xi^{-1}(\lambda, J)$ by removing its right-most ball.*

This implies that our inverse scattering transform for the box-ball system coincides with the bijection in [8]. In the $sl(2)$ case the latter method (in our terminology) determines the positions of the balls by repeating the procedure for obtaining (λ', J') from (λ, J). We note that although their method was defined only for the highest weight states, it can be also defined for the non-highest weight states in the $sl(2)$ case.

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Appendix A Partition functions without a specified soliton content

The following items are included here to make this paper to be self-contained, although they can be found elsewhere.

Recall the sets of the automaton states (9), (10). We define

$$\mathcal{H}(L, s) = \sqcup_{\lambda \vdash s} \mathcal{H}(L, \lambda),$$
$$\mathcal{H}^{+}(L, s) = \sqcup_{\lambda \vdash s} \mathcal{H}^{+}(L, \lambda).$$

(The notation is a bit ambiguous but no confusion should occur.) They are sets of automaton states in the finite interval between walls 0 and L, without a specified soliton content but with the number of balls set to be s. With the energy in Definition 30 we define

$$Z(L, s) := \sum_{M \in \mathcal{H}(L,s)} q^{E_{\mathrm{CTM}}(M)},$$
$$Z^{+}(L, s) := \sum_{M \in \mathcal{H}^{+}(L,s)} q^{E_{\mathrm{CTM}}(M)}.$$

Then we have that

Lemma 38. *The partition function $Z(L, s)$ satisfies the recursion relation*

$$Z(L, s) = Z(L-1, s) + \sum_{k=1}^{s} q^{L-k} Z(L-k-1, s-k). \qquad \text{(A 1)}$$

The $Z^+(L,s)$ also satisfies the recursion relation of the same form.

Proof. Consider the array of L boxes between walls 0 and L. The first term in the right-hand side of (A 1) is for those automaton states without ball in the right-most box. In the summation, the k-th term is for those states which have their right-most empty box on the left of the wall $L-k$. $\square$

Each of the partition functions is uniquely determined by the recursion relation of the form (A 1) under the boundary conditions

$$Z(L,0) = Z(L,L) = 1, \tag{A 2}$$

$$Z^+(L,0) = 1, Z^+(2s-1,s) = 0. \tag{A 3}$$

Proposition 39. *The following identities hold.*

$$Z(L,s) = \begin{bmatrix} L \\ s \end{bmatrix}, \tag{A 4}$$

$$Z^+(L,s) = \begin{bmatrix} L \\ s \end{bmatrix} - \begin{bmatrix} L \\ s-1 \end{bmatrix}. \tag{A 5}$$

Proof. Using the identities of the q-binomial coefficients [21]

$$\begin{bmatrix} L \\ s \end{bmatrix} = \begin{bmatrix} L-1 \\ s \end{bmatrix} + q^{L-s} \begin{bmatrix} L-1 \\ s-1 \end{bmatrix} = q^s \begin{bmatrix} L-1 \\ s \end{bmatrix} + \begin{bmatrix} L-1 \\ s-1 \end{bmatrix},$$

one can deduce that the expressions in the right hand sides of (A 4) and (A 5) satisfy the recursion relation of the form (A 1). They also satisfy the boundary conditions (A 2) and (A 3) respectively. $\square$

The right hand side of (A 4) is the generating function for the number of partitions of an integer into at most s part, and each part is less than or equal to $L-s$ [21]. An energy preserving bijection between $\mathcal{H}(L,s)$ and the set of all such partitions is given as follows. Given $M \in \mathcal{H}(L,s)$ consider the automaton state associated to M. If its left-most s boxes are filled, then it is mapped to $\emptyset$. Suppose the state is not the case. Then the state admits the following description: there are α_1 filled boxes from the left, then β_1 empty boxes, α_2 filled boxes, β_2 empty boxes, ..., α_p filled boxes, β_p empty boxes. Here p is an integer ≥ 2, $\sum_{k=1}^p \alpha_k = s$, and $\sum_{k=1}^p \beta_k = L-s$. We assume $\alpha_k, \beta_k \geq 1$ except $\alpha_1, \beta_p \geq 0$. Then the state is mapped to the largest Young diagram in Figure 3.

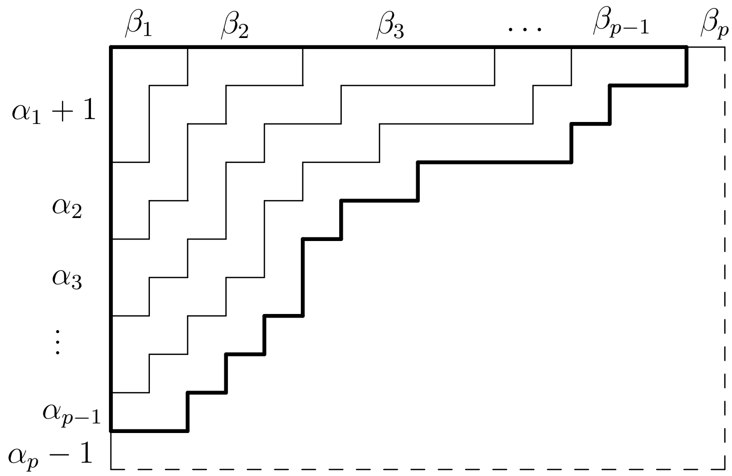

Figure 3: A graphical representation of the energy preserving bijection between automaton states to restricted partitions. The largest Young diagram (thick line) is made of $p-1$ border strips (skew Young diagrams without 2 by 2 square blocks).

Conversely given an arbitrary Young diagram we can read off these numbers α_k, β_k in the following way. We peel off a border strip from the south-east direction of the Young diagram. The result is also a Young diagram. Therefore we can repeat this procedure as many times as possible. In this way any Young diagram can be divided into such border strips uniquely. Then we can read off the numbers α_k, β_k from the lengths of the west and north edges of the border strips.

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