Chapter 5: Integer Compositions and Partitions and Set Partitions

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5.1. Compositions

- A *strict composition of n* is a tuple of *positive integers* that sum to n. The strict compositions of 4 are

$$(4) \quad (3,1) \quad (1,3) \quad (2,2) \quad (2,1,1) \quad (1,2,1) \quad (1,1,2) \quad (1,1,1,1)$$

- It's a tuple, so $(2,1,1)$, $(1,2,1)$, $(1,1,2)$ are all distinct. Later, we will consider *integer partitions*, in which we regard those as equivalent and only use the one where the entries are decreasing, $(2,1,1)$.
- A *weak composition of n* is a tuple of *nonnegative integers* that sum to n.

  $(1,0,0,3)$ is a weak composition of 4.
- If strict or weak is not specified, a *composition* means a *strict composition*.

Notation and drawings of compositions

- *Tuple notation:* To properly distinguish between $3+1+1$ and $1+3+1$, which both evaluate to the number 5, we represent them as tuples, $(3,1,1)$ and $(1,3,1)$, since tuples are distinguishable.
- Drawings:

Sum	**Tuple**	**Dots and bars**
$3+1+1$	$(3,1,1)$	$\cdot\cdot\cdot\mid\cdot\mid\cdot$
$1+3+1$	$(1,3,1)$	$\cdot\mid\cdot\cdot\cdot\mid\cdot$
$0+4+1$	$(0,4,1)$	$\mid\cdot\cdot\cdot\cdot\mid\cdot$
$4+1+0$	$(4,1,0)$	$\cdot\cdot\cdot\cdot\mid\cdot\mid$
$4+0+1$	$(4,0,1)$	$\cdot\cdot\cdot\cdot\mid\mid\cdot$
$4+0+0+1$	$(4,0,0,1)$	$\cdot\cdot\cdot\cdot\mid\mid\mid\cdot$

- If there is a bar at the beginning/end, the first/last part is 0.
  If there are any consecutive bars, some part(s) in the middle are 0.

How many strict compositions of n into k parts?

- A composition of n into k parts has n dots and $k-1$ bars.
- Draw n dots: $\bullet \bullet \bullet \bullet \bullet$
- There are $n-1$ spaces between the dots.
- Choose $k-1$ of the spaces and put a bar in each of them.
- For $n = 5$, $k = 3$: $\bullet | \bullet \bullet | \bullet \bullet$
- The bars split the dots into parts of sizes $\geqslant 1$, because there are no bars at the beginning or end, and no consecutive bars.
- Thus, there are $\binom{n-1}{k-1}$ strict compositions of n into k parts, for $n,k \geqslant 1$.
- For $n = 5$ and $k = 3$, we get $\binom{5-1}{3-1} = \binom{4}{2} = 6$.

Total # of strict compositions of n into any number of parts

- 2^{n-1} by placing bars in any subset (of any size) of the $n-1$ spaces.
- Or, $\displaystyle\sum_{k=1}^{n-1} \binom{n-1}{k-1}$, so the total is $\displaystyle 2^{n-1} = \sum_{k=1}^{n-1} \binom{n-1}{k-1}$.

How many weak compositions of n into k parts?

Review: We covered this when doing the Multinomial Theorem

- The diagram has n dots and $k-1$ bars in any order. No restriction on bars at the beginning/end/consecutively since parts=0 is OK.
- There are $n+k-1$ symbols.
  Choose n of them to be dots (or $k-1$ of them to be bars):
  $$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$
- For $n=5$ and $k=3$, we have
  $$\binom{5+3-1}{5} = \binom{7}{5} = 21 \qquad \text{or} \qquad \binom{5+3-1}{3-1} = \binom{7}{2} = 21.$$
- The total number of weak compositions of n of all sizes is infinite, since we can insert any number of 0's into a strict composition of n.

Relation between weak and strict compositions

- Let $(a_1, \ldots, a_k)$ be a weak composition of n (parts $\geqslant 0$).
- Add 1 to each part to get a strict composition of $n + k$:

  $$(a_1 + 1) + (a_2 + 1) + \cdots + (a_k + 1) = (a_1 + \cdots + a_k) + k = n + k$$

  The parts of $(a_1 + 1, \ldots, a_k + 1)$ are $\geqslant 1$ and sum to $n + k$.
- $(2, 0, 3)$ is a weak composition of 5.
  $(3, 1, 4)$ is a strict composition of $5 + 3 = 8$.
- This is reversible and leads to a bijection between
  Weak compositions of n into k parts
  $\longleftrightarrow$ Strict compositions of $n + k$ into k parts
  (Forwards: add 1 to each part; reverse: subtract 1 from each part.)
- Thus, the number of weak compositions of n into k parts
  = The number of strict compositions of $n + k$ into k parts
  = $\binom{n+k-1}{k-1}$.

5.2. Set partitions

- A *partition of a set A* is a set of nonempty subsets of A called *blocks*, such that every element of A is in exactly one block.
- A set partition of $\{1,2,3,4,5,6,7\}$ into three blocks is
$$\big\{\{1,3,6\},\{2,7\},\{4,5\}\big\}.$$
- This is a set of sets. Since sets aren't ordered, the blocks can be put in another order, and the elements within each block can be written in a different order:
$$\big\{\{1,3,6\},\{2,7\},\{4,5\}\big\} = \big\{\{5,4\},\{6,1,3\},\{7,2\}\big\}.$$
- Define $S(n,k)$ as the number of partitions of an n-set into k blocks. This is called the *Stirling Number of the Second Kind*.
We will find a recursion and other formulas for $S(n,k)$.

How do partitions of $[n]$ relate to partitions of $[n-1]$?

- Define $[0] = \emptyset$ and $[n] = \{1, 2, \ldots, n\}$ for integers $n > 0$.
  It is convenient to use $[n]$ as an example of an n-element set.
- Examine what happens when we cross out n in a set partition of $[n]$, to obtain a set partition of $[n-1]$ (here, $n = 5$):

$$\begin{array}{rcl} \{\{1,3\},\{2,4,\cancel{5}\}\} & \to & \{\{1,3\},\{2,4\}\} \\ \{\{1,3,\cancel{5}\},\{2,4\}\} & \to & \{\{1,3\},\{2,4\}\} \\ \{\{1,3\},\{2,4\},\cancel{\{5\}}\} & \to & \{\{1,3\},\{2,4\}\} \end{array}$$

- For all three of the set partitions on the left, removing 5 yields the set partition $\{\{1,3\},\{2,4\}\}$.
- In the first two, 5 was in a block with other elements, and removing it yielded the same number of blocks.
- In the third, 5 was in its own block, so we also had to remove the block $\{5\}$ since only nonempty blocks are allowed.

Recursion for $S(n, k)$

Insert n into a partition of $[n-1]$ to obtain a partition of $[n]$ into k blocks:

- **Case: partitions of $[n]$ in which n is not in a block alone:**
  Choose a partition of $[n-1]$ into k blocks ($S(n-1, k)$ choices)
  Insert n into any of these blocks (k choices).

  Subtotal: $k \cdot S(n-1, k)$

- **Case: partitions of $[n]$ in which n is in a block alone:**
  Choose a partition of $[n-1]$ into $k-1$ blocks ($S(n-1, k-1)$ ways)
  and add a new block $\{n\}$ (one way).

  Subtotal: $S(n-1, k-1)$

- **Total:** $S(n, k) = k \cdot S(n-1, k) + S(n-1, k-1)$

- This recursion requires using $n-1 \geqslant 0$ and $k-1 \geqslant 0$, so $n, k \geqslant 1$.

Initial conditions for $S(n, k)$

When $n = 0$ or $k = 0$

$n = 0$: Partitions of $\emptyset$

- It is not valid to partition the null set as $\{\emptyset\}$, since that has an empty block.
- However, it *is* valid to partition it as $\{\} = \emptyset$. It does not have any blocks (so no empty blocks), and the union of no blocks equals $\emptyset$.
- This is the only partition of $\emptyset$, so $S(0, 0)=1$ and $S(0, k)=0$ for $k>0$.

Other cases:

- $S(n, 0) = 0$ when $n > 0$
  since every partition of $[n]$ must have at least one block.

- This is not an initial condition, but note:
  $S(n, k) = 0$ for $k > n$
  since the partition of $[n]$ with the most blocks is $\{\{1\}, \ldots, \{n\}\}$.

Table of values of $S(n,k)$

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1 \qquad S(n,k) = {\color{blue} k \cdot S(n-1,k)}$$
$$S(n,0) = 0 \text{ if } n > 0 \qquad + {\color{red} S(n-1,k-1)}$$
$$S(0,k) = 0 \text{ if } k > 0 \qquad \text{if } n \geqslant 1 \text{ and } k \geqslant 1$$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$n=0$	1	0	0	0	0
		$\downarrow \cdot 1$	$\downarrow \cdot 2$	$\downarrow \cdot 3$	$\downarrow \cdot 4$
$n=1$	0	1	0	0	0
		$\downarrow \cdot 1$	$\downarrow \cdot 2$	$\downarrow \cdot 3$	$\downarrow \cdot 4$
$n=2$	0	1	1	0	0
		$\downarrow \cdot 1$	$\downarrow \cdot 2$	$\downarrow \cdot 3$	$\downarrow \cdot 4$
$n=3$	0	1	3	1	0
		$\downarrow \cdot 1$	$\downarrow \cdot 2$	$\downarrow \cdot 3$	$\downarrow \cdot 4$
$n=4$	0	1	7	6	1

Example and Bell numbers

- $S(n,k)$ is the number of set partitions of $[n]$ into k blocks. For $n=4$:

$k=1$	$k=2$	$k=3$	$k=4$
	$\{\{1,2,3\},\{4\}\}$	$\{\{1,2\},\{3\},\{4\}\}$	
	$\{\{1,2,4\},\{3\}\}$	$\{\{1,3\},\{2\},\{4\}\}$	
	$\{\{1,3,4\},\{2\}\}$	$\{\{1,4\},\{2\},\{2\}\}$	
$\{\{1,2,3,4\}\}$	$\{\{2,3,4\},\{1\}\}$	$\{\{2,3\},\{1\},\{4\}\}$	$\{\{1\},\{2\},\{3\},\{4\}\}$
	$\{\{1,2\},\{3,4\}\}$	$\{\{2,4\},\{1\},\{3\}\}$	
	$\{\{1,3\},\{2,4\}\}$	$\{\{3,4\},\{1\},\{2\}\}$	
	$\{\{1,4\},\{2,3\}\}$		
$S(4,1)=1$	$S(4,2)=7$	$S(4,3)=6$	$S(4,4)=1$

- The *Bell number* B_n is the total number of set partitions of $[n]$ into any number of blocks:

$$B_n = S(n,0) + S(n,1) + \cdots + S(n,n)$$

- Total: $B_4 = 1+7+6+1 = 15$

Table of Stirling numbers and Bell numbers

Compute $S(n,k)$ from the recursion and initial conditions:

$$S(0,0) = 1 \qquad S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1) \text{ if } n \geqslant 1 \text{ and } k \geqslant 1$$

$$S(n,0) = 0 \text{ if } n > 0$$

$$S(0,k) = 0 \text{ if } k > 0$$

$S(n,k)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	Row total B_n
$n=0$	1	0	0	0	0	0	1
$n=1$	0	1	0	0	0	0	1
$n=2$	0	1	1	0	0	0	2
$n=3$	0	1	3	1	0	0	5
$n=4$	0	1	7	6	1	0	15
$n=5$	0	1	15	25	10	1	52

Simplex locks

- Simplex brand locks were a popular combination lock with 5 buttons.
- The combination 13-25-4 means:
  - Push buttons 1 and 3 together.
  - Push buttons 2 and 5 together.
  - Push 4 alone.
  - Turn the knob to open.
- Buttons cannot be reused.
- We first consider the case that all buttons are used, and separately consider the case that some buttons aren't used.

Represent the combination 13-25-4 as an ordered set partition

- We may represent 13-25-4 as an *ordered set partition*
$$(\{1,3\},\{2,5\},\{4\})$$
Block $\{1,3\}$ is first, block $\{2,5\}$ is second, and block $\{4\}$ is third.
Blocks are sets, so can replace $\{1,3\}$ by $\{3,1\}$, or $\{2,5\}$ by $\{5,2\}$.
- Note that if we don't say it's ordered, then a set partition is a set of blocks, not a tuple of blocks, and the blocks can be reordered:
$$\{\{1,3\},\{2,5\},\{4\}\} = \{\{5,2\},\{4\},\{1,3\}\}$$

Number of combinations

- Let there be n buttons (which must all be used) and k groups of button pushes.
- There are $S(n,k)$ ways to split the buttons into k blocks, and $k!$ ways to order the blocks, giving $k! \cdot S(n,k)$ combinations.
- The number of combinations on 5 buttons and 3 groups of pushes ($n = 5$ and $k = 3$) is
$$3! \cdot S(5,3) = 6 \cdot 25 = \boxed{\mathbf{150}}$$

Represent the combination 13-25-4 as a surjective (onto) function

- Define a function $f(i) = j$, where button i is in push number j:

$i =$ button number	$j =$ push number
1	1
2	2
3	1
4	3
5	2

- This gives a surjective (onto) function $f : [5] \to [3]$.
- The blocks of buttons pushed are
  1st: $f^{-1}(1) = \{1, 3\}$ 2nd: $f^{-1}(2) = \{2, 5\}$ 3rd: $f^{-1}(3) = \{4\}$

Theorem

The number of surjective (onto) functions $f : [n] \to [k]$ is $k! \cdot S(n, k)$.

Proof.

Split $[n]$ into k nonempty blocks in one of $S(n, k)$ ways.
Choose one of $k!$ orders for the blocks: $(f^{-1}(1), \ldots, f^{-1}(k))$. $\square$

How many combinations don't use all the buttons?

- The combination 3-25 does not use 1 and 4.
- Trick: write it as 3-25-(14)
- There are three groups of buttons and we don't use the 3rd group.
- The number of combinations with two pushes that do not use all the buttons is the same as the number of combinations with three pushes that do use all the buttons.

Lemma (General case)

For $n, k \geqslant 1$:

The number of combinations in an n button lock with $k-1$ pushes that do not use all of the buttons
= the number of combinations in k pushes that do use all of the buttons
= $k! \cdot S(n,k)$.

Counting the total number of functions $f : [n] \to [k]$

We will count the number of functions $f : [n] \to [k]$ in two ways.

First method

(k choices of $f(1)$) $\times$ (k choices of $f(2)$) $\times \cdots \times$ (k choices of $f(n)$) $= k^n$

Counting the total number of functions $f : [n] \to [k]$

Second method: Classify functions by their images and inverses

- Consider $f : [10] \to \{a, b, c, d, e\}$:

i =	1	2	3	4	5	6	7	8	9	10
f(i) =	a	c	c	a	c	d	c	a	c	d

- $\text{image}(f) = \{f(1), \ldots, f(10)\} = \{a, c, d\}$ and the inverse blocks are

$$f^{-1}(a) = \{1, 4, 8\} \qquad f^{-1}(c) = \{2, 3, 5, 7, 9\}$$
$$f^{-1}(d) = \{6, 10\} \qquad f^{-1}(b) = f^{-1}(e) = \emptyset$$

- $f : [10] \to \{a, b, c, d, e\}$ is not onto, but $f : [10] \to \{a, c, d\}$ is onto.
- There are $S(10, 3) \cdot 3!$ surjective functions $f : [10] \to \{a, c, d\}$.
- Classify all $f : [10] \to \{a, b, c, d, e\}$ according to $T = \text{image}(f)$.
  There are $\binom{5}{3}$ subsets $T \subset \{a, b, c, d, e\}$ of size $|T| = 3$.
  So $S(10, 3) \cdot 3! \cdot \binom{5}{3}$ functions $f : [10] \to \{a, \ldots, e\}$ have $|\text{image}(f)| = 3$.

Counting the total number of functions $f : [n] \to [k]$

Second method, continued

- Simplify: $3! \cdot \binom{5}{3} = 3! \cdot \frac{5!}{3!\,2!} = \frac{5!}{2!} = 5 \cdot 4 \cdot 3 = (5)_3$
  So $S(10,3) \cdot (5)_3$ functions $f : [10] \to [5]$ have $|\operatorname{image}(f)| = 3$.
- In general, $S(n,i) \cdot (k)_i$ functions $f : [n] \to [k]$ have $|\operatorname{image}(f)| = i$.
- Summing over all possible image sizes $i = 0, \ldots, n$ gives the total number of functions $f : [n] \to [k]$
  $$\sum_{i=0}^{n} S(n,i) \cdot (k)_i$$
- Putting this together with the first method gives
  $$k^n = \sum_{i=0}^{n} S(n,i) \cdot (k)_i \qquad \text{for all integers } n, k \geqslant 0$$
- *Note:* $i \leqslant n$ (it's n if $f(1), \ldots, f(n)$ are all distinct; $< n$ otherwise).
  Also, $i \leqslant k$ since $\operatorname{image}(f) \subset [k]$.
  $S(n,i)$ vanishes for $i > n$ and $(k)_i$ vanishes for $i > k$.
  So for the upper bound on the sum, we may use n, k, or $\min\{n,k\}$.

Identity for real numbers

This identity generalizes to real numbers! (Replacing integer k by real x.)

Theorem

For all real numbers x and all integers $n \geqslant 0$, $\quad x^n = \sum_{i=0}^{n} S(n,i) \cdot (x)_i$

Examples

For $n = 2$:

$$\begin{aligned} S(2,0)(x)_0 + S(2,1)(x)_1 + S(2,2)(x)_2 &= 0 \cdot 1 + 1 \cdot x + 1 \cdot x(x-1) \\ &= 0 + x + (x^2 - x) = x^2 \end{aligned}$$

For $n = 3$:

$$\begin{aligned} &S(3,0)(x)_0 + S(3,1)(x)_1 + S(3,2)(x)_2 + S(3,3)(x)_3 \\ &\quad = 0 \cdot 1 + 1 \cdot x + 3 \cdot x(x-1) + 1 \cdot x(x-1)(x-2) \\ &\quad = 0 + x + 3(x^2 - x) + (x^3 - 3x^2 + 2x) \\ &\quad = x^3 + (3-3)x^2 + (1-3+2)x \qquad = x^3 \end{aligned}$$

Lemma from Abstract Algebra

Lemma

If $f(x)$ and $g(x)$ are polynomials of degree $\leqslant n$ that agree on more than n distinct values of x, then $f(x) = g(x)$ as polynomials.

Proof.

- Suppose that $x_1, \ldots, x_m$ are distinct (with $m > n$) and that $f(x_i) = g(x_i)$ for $i = 1, \ldots, m$.
- If the polynomial $h(x) = f(x) - g(x)$ is not identically 0, it factors as
$$h(x) = p(x)(x - x_1)^{r_1}(x - x_2)^{r_2} \cdots (x - x_m)^{r_m} \cdots$$
for some polynomial $p(x) \neq 0$ and some integers $r_1, \ldots, r_m \geqslant 1$.

  This has degree $\geqslant m > n$.
- But $h(x)$ has degree $\leqslant n$, a contradiction. $\square$

Identity for real numbers

Theorem

For all real numbers x and all integers $n \geqslant 0$, $\quad x^n = \sum_{i=0}^{n} S(n,i) \cdot (x)_i$

Proof.

- Both sides of the equation are polynomials in x of degree n.
- They agree at an infinite number of values $x = 0, 1, \ldots$
- Since $\infty > n$, they're identical polynomials. $\square$

5.3. Integer partitions

- The compositions $(2,1,1)$, $(1,2,1)$, $(1,1,2)$ are different. Sometimes the number of 1's, 2's, 3's, … matters but not the order.
- An *integer partition* of n is a tuple $(a_1,\ldots,a_k)$ of positive integers that sum to n, with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant 1$.
  The partitions of 4 are:
  $$(4) \qquad (3,1) \qquad (2,2) \qquad (2,1,1) \qquad (1,1,1,1)$$
- Define
  $$p(n) = \text{\# integer partitions of } n$$
  $$p_k(n) = \text{\# integer partitions of } n \text{ into exactly } k \text{ parts}$$
  $$p(4) = 5$$
  $$p_1(4) = 1 \quad p_2(4) = 2 \quad p_3(4) = 1 \quad p_4(4) = 1$$
- We will learn a method to compute these in Chapter 8.

Type of a set partition

- Consider this set partition of $[10]$:

$$\Big\{\{1,4\},\{7,6\},\{5\},\{8,2,3\},\{9\},\{10\}\Big\}$$

- The block lengths in the order it was written are $2, 2, 1, 3, 1, 1$.
- But the blocks of a set partition could be written in other orders. To make this unique, the *type* of a set partition is a tuple of the block lengths listed in decreasing order: $(3, 2, 2, 1, 1, 1)$.
- For a set of size n partitioned into k blocks, the type is an integer partition of n in k parts.

How many set partitions of $[10]$ have type $(3,2,2,1,1,1)$?

- Split $[10]$ into sets A,B,C,D,E,F of sizes $3,2,2,1,1,1$, respectively, in one of $\binom{10}{3,2,2,1,1,1} = \frac{10!}{3!\,2!^2\,1!^3} = 151200$ ways.
- But $\{A, \quad B,C, \quad D,E,F\} = \{A, \quad C,B, \quad F,E,D\}$, so we overcounted:
  - B,C could be reordered C,B: $2!$ ways.
  - D,E,F could be permuted in $3!$ ways.
  - If there are m_i blocks of size i, we overcounted by a factor of $m_i!$.
- Dividing by the overcounts gives

$$\frac{\binom{10}{3,2,2,1,1,1}}{1!\,2!\,3!} = \frac{151200}{1 \cdot 2 \cdot 6} = \boxed{\mathbf{12600}}$$

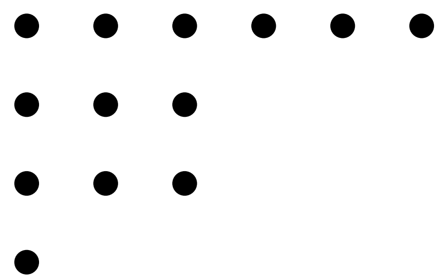
General formula

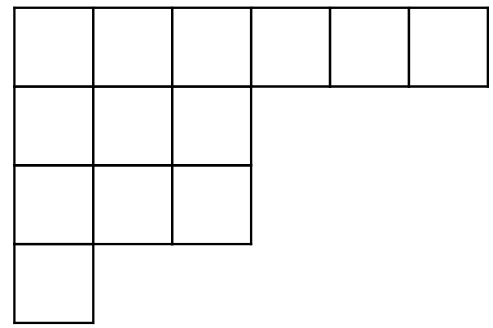
For an n element set, the number of set partitions of type $(a_1, a_2, \ldots, a_k)$ where $n = a_1 + a_2 + \cdots + a_k$ and m_i of the a's equal i, is

$$\frac{\binom{n}{a_1,a_2,\ldots,a_k}}{m_1!\,m_2!\,\cdots} = \frac{n!}{(1!^{m_1}\,m_1!)(2!^{m_2}\,m_2!)\cdots}$$

Ferrers diagrams and Young diagrams

Ferrers diagram of $(6,3,3,1)$ **Young diagram**



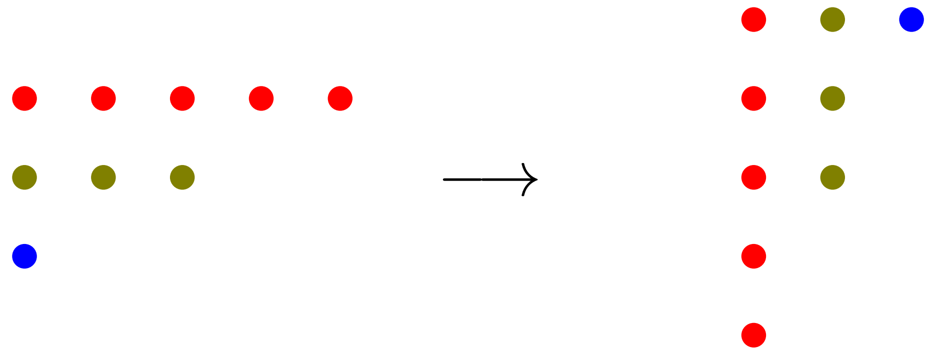


- Consider a partition $(a_1, \ldots, a_k)$ of n.
- *Ferrers diagram:* a_i dots in the ith row.
- *Young diagram:* squares instead of dots.
- The total number of dots or squares is n.

- Our book calls both of these *Ferrers diagrams*, but often they are given separate names.

Conjugate Partition

- Reflect a Ferrers diagram across its main diagonal:



$$\pi = (5, 3, 1) \qquad\qquad \pi' = (3, 2, 2, 1, 1)$$

- This transforms a partition π to its *conjugate partition*, denoted π'.
- The ith row of π turns into the ith column of π':
  the red, green, and blue rows of π turn into columns of π'.
  Also, the ith column of π turns into the ith row of π'.
- **Theorem:** $(\pi')' = \pi$
- **Theorem:** If π has k parts, then the largest part of π' is k.
  Here: π has 3 parts $\leftrightarrow$ the first column of π has length 3
  $\leftrightarrow$ the first row π' is 3
  $\leftrightarrow$ the largest part of π' is 3

Theorem

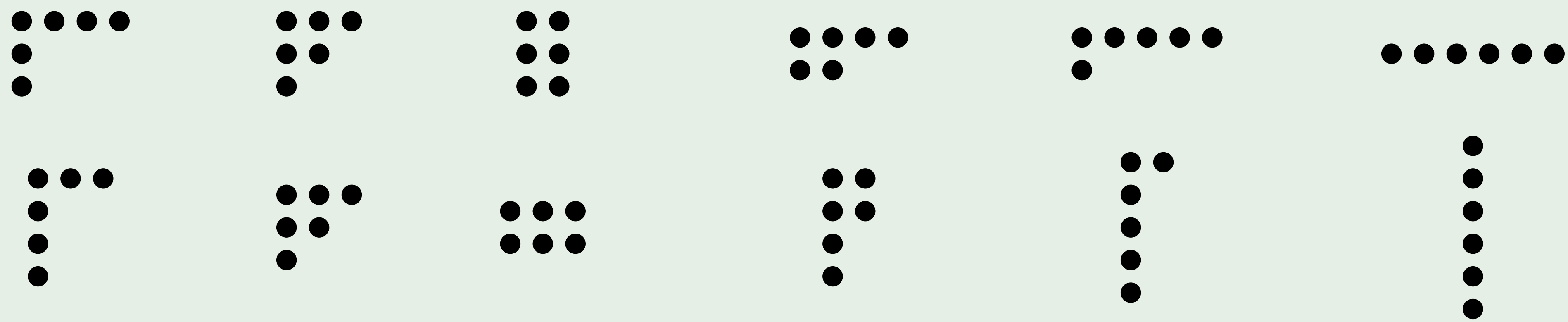
1. *The number of partitions of n into exactly k parts $(p_k(n))$ $=$ the number of partitions of n where the largest part $= k$.*
2. *The number of partitions of n into $\leqslant k$ parts $=$ the number of partitions of n into parts that are each $\leqslant k$.*

Proof: Conjugation is a bijection between the two types of partitions.

Example: Partitions of 6 into 3 or $\leqslant 3$ parts

π with exactly 3 parts — π with < 3 parts

$(4,1,1)$ $(3,2,1)$ $(2,2,2)$ $(4,2)$ $(5,1)$ (6)



$(3,1,1,1)$ $(3,2,1)$ $(3,3)$ $(2,2,1,1)$ $(2,1,1,1,1)$ $(1,1,1,1,1,1)$

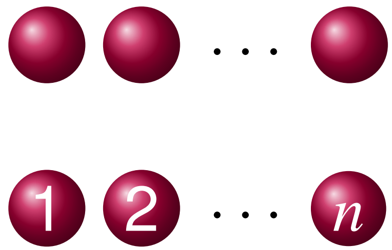
π' has largest part $= 3$ — π' has largest part < 3

Balls and boxes

Many combinatorial problems can be modeled as placing *balls* into *boxes*:

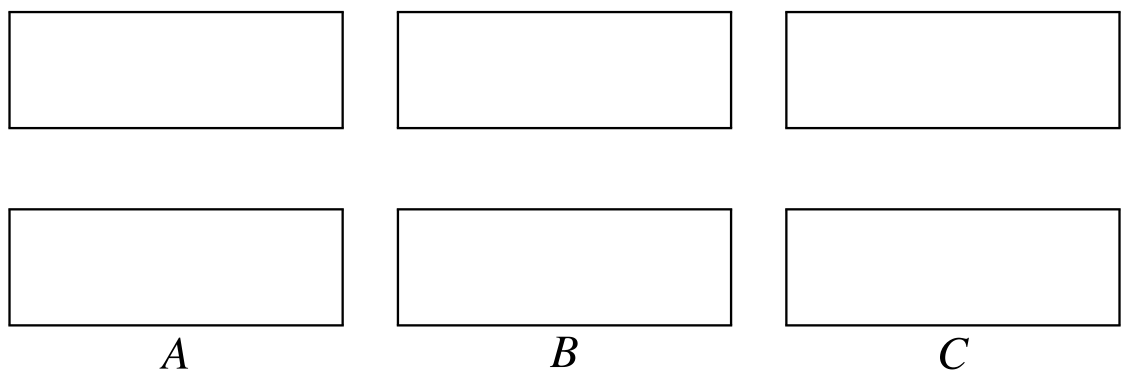
Indistinguishable balls:

Distinguishable balls:

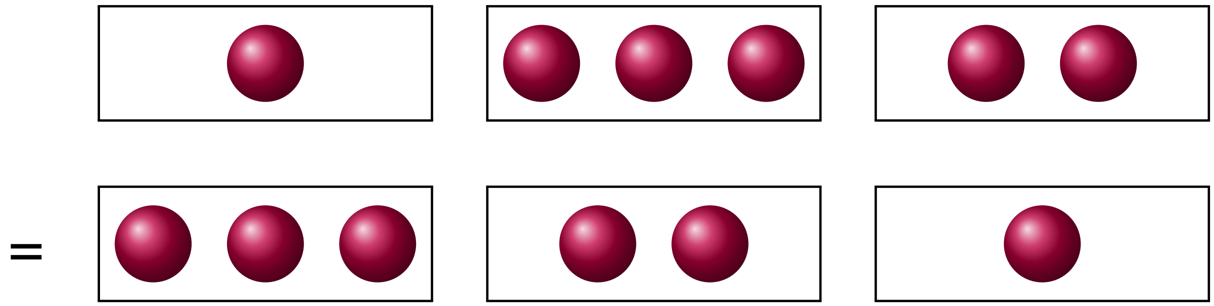


Indistinguishable boxes:

Distinguishable boxes:

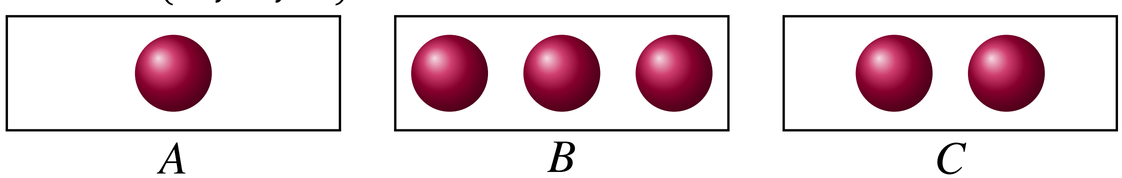


- **Integer partitions:** $(3, 2, 1)$



Indistinguishable balls.
Indistinguishable boxes.

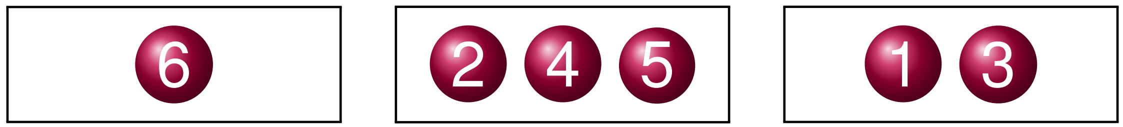
- **Compositions:** $(1, 3, 2)$



Indistinguishable balls.
Distinguishable boxes (which give the order).

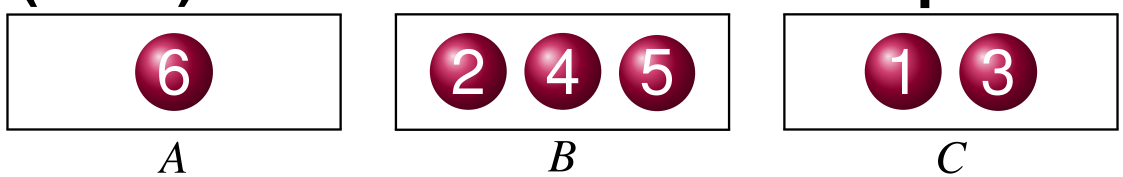
Balls and boxes
Distinguishable balls

- **Set partitions:** $\{\{6\},\{2,4,5\},\{1,3\}\}$



Distinguishable balls.
Indistinguishable boxes (so the blocks are not in any order).

- **Surjective (onto) functions / ordered set partitions:**



Distinguishable balls and distinguishable boxes.

Gives surjective function $f : [6] \to \{A, B, C\}$

$$f(6) = A \qquad f(2) = f(4) = f(5) = B \qquad f(1) = f(3) = C$$

or an ordered set partition $(\{6\},\{2,4,5\},\{1,3\})$