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Pascal's Triangle Modulo 3

AVERY WILSON

If you colour the odd entries of Pascal's triangle red and the even entries blue, a beautiful fractal pattern known as *Sierpinski's gasket* appears. A well-known problem is to determine how many odd entries appear in any given row of Pascal's triangle. A natural generalization of this problem is to ask, if we look at the nth row of Pascal's triangle modulo any positive integer m, how many occurrences of each residue class $0, 1, 2, \ldots, m-1$ will we find? Patterns are hard to come by for composite moduli, but nice formulae can be found for prime moduli. In this article, I derive the solution for the modulo 3 case of the problem using Lucas' theorem on binomial coefficients modulo a prime.

1. Introduction

Pascal's triangle is rife with patterns to explore. This famous triangle – which is generated by beginning with a 1 and letting each successive entry be the sum of the two entries directly above it – has been the subject of the curiosity of many mathematicians, from Chia Hsien in 11th century China to the triangle's namesake, Blaise Pascal, who published *Trait du Triangle Arithmétique* on the fundamental properties of the triangle in 1654. Today, Pascal's triangle is ubiquitous, and it is commonly known that the kth entry of the nth row is the binomial coefficient $\binom{n}{k}$.

A well-known problem is to determine the number of odd entries in the nth row of Pascal's triangle; that is, how many $\binom{n}{k} \equiv 1 \pmod 2$? In general, if we look at Pascal's triangle modulo a prime p by replacing each entry with its modulo p residue class, how many occurrences of each residue class will we find in the nth row? For example, if $p = 3$, how many zeroes, how many ones, and how many twos are there in the nth row? The solution quickly becomes difficult for $p > 3$, not to mention replacing p by a composite number. A general formula for any prime p is given in reference 1. In this article, I derive the formula for the case of $p = 3$.

2. Lucas' theorem

A useful result is *Lucas' theorem*, which gives a congruence for $\binom{n}{k}$ modulo a prime p in terms of the base p digits of n and k.

Theorem 1 (Lucas' theorem) *Let p be a prime number. If n and k are nonnegative integers with base p expansions $n = n_0+n_1p+n_2p^2+\cdots+n_dp^d$ and $k = k_0+k_1p+k_2p^2+\cdots+k_dp^d$, then*

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_d}{k_d} \pmod p.$$

Proof Expand $(1+x)^n$ two different ways: on one side using the binomial theorem as per usual, and, on the other, breaking n up into its base p expansion before applying the binomial theorem. We have

$$\sum_{i=0}^{n}\binom{n}{i}x^i \equiv (1+x)^n \equiv \prod_{i=0}^{d}(1+x)^{n_ip^i} \equiv \prod_{i=0}^{d}(1+x^{p^i})^{n_i} \equiv \prod_{i=0}^{d}\biggl(\sum_{j=0}^{n_i}\binom{n_i}{j}x^{jp^i}\biggr) \pmod p.$$

On the left-hand side of the congruence, the coefficient of x^k is $\binom{n}{k}$. Upon expanding the product on the right-hand side of the congruence, the x^k term has coefficient $\binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_d}{k_d}$, if k is written as $k = k_0 + k_1 p + \cdots + k_d p^d$ in base p. Two polynomials are congruent modulo p if and only if their coefficients are congruent modulo p, so this completes the proof.

For a fixed n and $k = 0, 1, 2, \ldots, n$, the number of $\binom{n}{k}$ divisible by p (i.e. the number of zeroes in the nth row of Pascal's triangle modulo p) comes quickly from Lucas' theorem. Using the convention that $\binom{n_i}{k_i} = 0$ for $k_i > n_i$, observe that $p \nmid \binom{n}{k}$ if and only if $0 \le k_i \le n_i$ for all i. Thus, there are $n_i + 1$ bad choices for each k_i, so the total number of $\binom{n}{k}$ *not* divisible by p is $(n_0 + 1)(n_1 + 1)\cdots(n_d + 1)$. Since there are $n + 1$ of the $\binom{n}{k}$ in total, we get the following corollary.

Corollary 1 *For a nonnegative integer n with base p digits $n_0, n_1, \ldots, n_d$ and $k = 0, 1, 2, \ldots, n$, the number of $\binom{n}{k}$ divisible by p is $n + 1 - (n_0 + 1)(n_1 + 1)\cdots(n_d + 1)$.*

As an example, take $p = 3$ and $n = 6$. Since $6 = 2 \cdot 3$ in base 3, the number of $\binom{n}{k}$ divisible by three is $6 + 1 - 3 = 4$. They are $\binom{6}{1} = \binom{6}{5} = 6$ and $\binom{6}{2} = \binom{6}{4} = 15$.

The number of odd entries in the nth row of Pascal's triangle is an immediate consequence of corollary 1. If $n_0, n_1, \ldots, n_d$ are the base 2 digits of n, then the number of odd entries in the nth row of Pascal's triangle is $(n_0 + 1)(n_1 + 1)\cdots(n_d + 1) = 2^m$, where m is the number of digits $n_i = 1$. As an example, since $10 = 2 + 2^3$, the number of odd entries in the tenth row of Pascal's triangle is $2^2 = 4$, namely 1 and $\binom{10}{2} = 45$, both twice.

3. Pascal's triangle modulo 3

Using Lucas' theorem and basic properties of the integers modulo 3, we can find the number of zeroes, ones, and twos in the nth row of Pascal's triangle modulo 3.

Fix a nonnegative integer n with base 3 digits $n_0, n_1, \ldots, n_d$, and let $k = 0, 1, 2, \ldots, n$. Denote by $M(r, n)$ the number of $\binom{n}{k}$ congruent to r modulo 3, and denote by N_s the number of digits n_i equal to s. Using corollary 1, we have that $M(0, n) = n + 1 - 2^{N_1} 3^{N_2}$. Since $M(0, n) + M(1, n) + M(2, n)$ must add up to the total of $n + 1$ binomial coefficients with which we are concerned, we need only find $M(1, n)$ and then can subtract from the total to get $M(2, n)$ with no further calculation.

Let us find $M(1, n)$. Recalling Lucas' theorem, suppose that

$$\binom{n}{k} \equiv \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_d}{k_d} \equiv 1 \pmod 3. \qquad (1)$$

Since $3 \nmid \binom{n}{k}$, each $\binom{n_i}{k_i}$ must be nonzero; this occurs if and only if $0 \le k_i \le n_i$ for all i. Now, each $\binom{n_i}{k_i}$ can be replaced by a remainder of either one or two. If we let T be the number of $\binom{n_i}{k_i}$ that are replaced by a two, then we have

$$\binom{n}{k} \equiv 2^T \equiv 1 \pmod 3.$$

This congruence holds if and only if T is even, since $2^2 \equiv 1 \pmod 3$. Thus, to satisfy (1), it is necessary and sufficient that $0 \le k_i \le n_i$ for all i and T is even. Now we count the number of $\binom{n}{k} \equiv 1 \pmod 3$ by considering disjoint cases. For a nonnegative integer j, define the jth case as follows.

Case *j* $\binom{n}{k}$ falls into case j if and only if $0 \le k_i \le n_i$ for all i and $T = 2j$.

To count the number of $\binom{n}{k}$ falling into case j, we suppose that $\binom{n}{k}$ falls into case j and count the number of possibilities for the digits of k. Pick $2j$ of the N_2 digits with $n_i = 2$ to have $k_i = 1$ (and thereby $\binom{n_i}{k_i} \equiv 2 \pmod 3$). Then for the remaining $N_1 + N_2 - 2j$ digits of n that are nonzero, there are two options for each digit: $k_i = 0$ or $k_i = n_i$. Hence, the total number of $\binom{n}{k}$ falling into case j is $\binom{N_2}{2j} 2^{N_1+N_2-2j}$.

Now, j is a nonnegative integer that cannot exceed half of N_2, so j ranges over $0 \le j \le \lfloor N_2/2 \rfloor$. Thus, adding up all possible cases gives us

$$M(1,n) = \sum_{j=0}^{\lfloor N_2/2 \rfloor} \binom{N_2}{2j} 2^{N_1+N_2-2j}.$$

This sum may seem somewhat ugly, but it is just the sum of the even-index terms of a certain binomial expansion. There is in fact a nice closed form for such a sum, given by the following proposition.

Proposition 1 *For a nonnegative integer m,*

$$\sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m}{2j} x^{2j} = \frac{1}{2}((1+x)^m + (1-x)^m).$$

Proof Apply the binomial theorem to $(1+x)^m$ and $(1-x)^m$:

$$\frac{1}{2}((1+x)^m + (1-x)^m) = \frac{1}{2}\sum_{j=0}^{m} \binom{m}{j} x^j + \frac{1}{2}\sum_{j=0}^{m} \binom{m}{j} (-x)^j = \frac{1}{2}\sum_{j=0}^{m} \binom{m}{j} (x^j + (-x)^j).$$

Notice that the odd-index terms vanish, and we are left with only the even-index terms, i.e.

$$\frac{1}{2}\sum_{j \text{ even}} \binom{m}{j} (x^j + (-x)^j) = \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m}{2j} x^{2j}.$$

Taking $x = \frac{1}{2}$ in proposition 1 gives us a greatly simplified expression for $M(1,n)$:

$$M(1,n) = \sum_{j=0}^{\lfloor N_2/2 \rfloor} \binom{N_2}{2j} 2^{N_1+N_2-2j} = \frac{2^{N_1+N_2}}{2}\left(\left(\frac{3}{2}\right)^{N_2} + \left(\frac{1}{2}\right)^{N_2}\right) = 2^{N_1-1}(3^{N_2}+1).$$

Now, since $M(0,n) + M(1,n) + M(2,n)$ must add up to $n+1$, we can just subtract $M(0,n)$ and $M(1,n)$ from the total to find that

$$\begin{aligned} M(2,n) &= n + 1 - M(0,n) - M(1,n) \\ &= n + 1 - (n + 1 - 2^{N_1}3^{N_2}) - 2^{N_1-1}(3^{N_2}+1) \\ &= 2^{N_1-1}(3^{N_2}-1). \end{aligned}$$

The following theorem compiles these results.

Theorem 2 *If $M(r,n)$ denotes the number of entries congruent to r modulo 3 in the nth row of Pascal's triangle, and N_s denotes the number of base 3 digits of n that are equal to s, then*

(i) $M(0,n) = n + 1 - 2^{N_1}3^{N_2}$,

(ii) $M(1,n) = 2^{N_1-1}(3^{N_2} + 1)$,

(iii) $M(2,n) = 2^{N_1-1}(3^{N_2} - 1)$.

The formulae given in theorem 2 allow us to figure out the composition of the nth row of Pascal's triangle modulo 3, given that we can find the base 3 digits of n. As an example, take $n = 11 = 3^2 + 2$. The eleventh row of Pascal's triangle modulo 3 is

$$1 \quad 2 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 1,$$

so theorem 2 should give us $M(0,11) = 6$, $M(1,11) = 4$, and $M(2,11) = 2$. The number of digits equal to one in the base 3 expansion of 11 is $N_1 = 1$, and the number of digits equal to two is $N_2 = 1$. Reassuringly, when we substitute these numbers into the formulae we get

$$\begin{aligned} M(0,11) &= n + 1 - 2^{N_1}3^{N_2} = 12 - 2 \cdot 3 = 6, \\ M(1,11) &= 2^{N_1-1}(3^{N_2} + 1) = 2^0(3+1) = 4, \\ M(2,11) &= 2^{N_1-1}(3^{N_2} - 1) = 2^0(3-1) = 2. \end{aligned}$$

4. Further questions

(i) Using corollary 1, show that if p is a prime and the binomial coefficient $\binom{n}{k}$ is picked at random from the first m rows of Pascal's triangle, then the probability that p divides $\binom{n}{k}$ tends to one as m tends to infinity.

(ii) How many occurrences of each residue class are there in the nth row of Pascal's triangle modulo 5?

(iii) Patterns in Pascal's triangle become much more obscure when looked at modulo a composite number. Do the results on Pascal's triangle modulo 2 and 3 tell anything about Pascal's triangle modulo a power of 2 or 3? What about modulo $2 \cdot 3 = 6$?

References

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Avery Wilson *is an undergraduate mathematics student at the University of Colorado, Boulder, USA. In his spare time he enjoys cooking and exploring the Colorado mountains by bike.*