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COUNTING PRIMES IN RESIDUE CLASSES

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ABSTRACT. We explain how the Meissel-Lehmer-Lagarias-Miller-Odlyzko method for computing $\pi(x)$ can be used for computing efficiently $\pi(x,k,l)$, the number of primes congruent to l modulo k up to x. As an application, we computed the number of prime numbers of the form $4n \pm 1$ less than x for several values of x up to 10^{20} and found a new region where $\pi(x,4,3)$ is less than $\pi(x,4,1)$ near $x = 10^{18}$.

1. INTRODUCTION

In the 1870's, the German astronomer Meissel designed a method for computing the value of $\pi(x)$, the number of prime numbers up to x. The method has been improved by many authors since then. The most important improvement is due to Lagarias-Miller-Odlyzko [LMO85] which obtained a method requiring $O(x^{2/3}/\log x)$ time and computed the value of $\pi(4 \cdot 10^{16})$. Further improvements were obtained by the first author and Rivat [DR96] with $O(x^{2/3}/\log^2 x)$ time and who computed $\pi(10^{18})$. Finally, Gourdon, using ideas originating from Lagarias-Miller-Odlyzko, implemented a parallel version of the algorithm and computed, to date, values of $\pi(x)$ up to $4 \cdot 10^{22}$.

For l and k two relatively prime positive integers, one defines $\pi(x,k,l)$ as the number of prime numbers up to x that are congruent to l modulo k. Asymptotically the numbers $\pi(x,k,l)$ are all of same size, $\varphi(k)^{-1}x/\log x$. However it has been known for quite some time that there are more primes in the congruence classes that are nonquadratic residues modulo k than in those that are. Heuristically, this bias can be explained from the fact that these classes contain more composite numbers than the latter since they contain all the squares (see also [RS94]).

For $k = 4$, there are two classes, the numbers congruent to 1 modulo 4, the quadratic residues, and the numbers congruent to 3 modulo 4, the nonquadratic residues. In this setting Littlewood proved that (see [Ing90] for the $\Omega_\pm$ notation)

$$\pi(x,4,3) - \pi(x,4,1) = \Omega_\pm\left(\frac{x^{1/2}}{\log x}\log\log\log x\right).$$

Therefore there are infinitely many sign changes for the function $\delta(x) = \pi(x,4,3) - \pi(x,4,1)$. Define two disjoint subsets of the set of integers:

$$\Delta^+ = \{x \geq 2 : \delta(x) > 0\},$$
$$\Delta^- = \{x \geq 2 : \delta(x) < 0\}.$$

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For A a subset of the positive integers, the logarithmic density $d(A)$ is defined as the following limit, if it exists:

$$d(A) = \lim_{x\to\infty} \frac{1}{\log x} \sum_{\substack{a\in A\\ a\le x}} \frac{1}{a}.$$

Note that any set A admitting a density in the usual sense admits also a logarithmic density, and the two densities are equal. However, there exist some sets (e.g., the set of numbers whose decimal expansion starts with 1) with a logarithmic density (in this example $\log 2/\log 10$) but not having a density in the usual sense.

In [RS94], Rubinstein and Sarnak proved that under suitable generalization of Riemann Hypothesis (RH) both sets admit a logarithmic density. More exactly, they proved, conditionally under these assumptions, that

$$d(\Delta^+) = 0.99592\ldots \quad \text{and} \quad d(\Delta^-) = 0.00407\ldots. \tag{1.1}$$

These results have been further generalized and improved in [FM00] and [BFHR01].

From the computational point of view, several people have been searching for a region containing elements of Δ^- (see [Lee57], [BH78], [BFHR01]). So far, eight regions have been found and we have discovered a new region using the method described in this paper. See the last section for more details.

In this article, we will prove the following theorem:

Theorem 1. *Let $x > 0$, and let k and l be two relatively prime positive integers. There exists an algorithm which computes $\pi(x,k,l)$ in time $O(x^{2/3}/\ln^2 x)$.*

Note that the computation time of this algorithm is exactly that of the algorithm for the computation of $\pi(x)$ given in [DR96]. Indeed, loops that ranged through the primes less than a given bound B in the computation of $\pi(x)$ are now replaced by $\varphi(k)$ loops, one for each invertible class modulo k ranging through the primes less than B in that class. Therefore, the total number of operations stays the same. In particular, the running time does not depend on the values of k or l. Of course, for fixed values of x and k, the computation of all $\pi(x,k,l)$ where l ranges through the $\varphi(k)$ invertible residue classes modulo k is done in $O(\varphi(k)x^{2/3}/\ln^2 x)$ time, and therefore the computation time of the two values $\pi(x,4,1)$ and $\pi(x,4,3)$ is twice that of $\pi(x)$.

2. Proof of Theorem 1

We now explain the method we used to compute $\pi(x,k,l)$ for large values of x. It is the natural adaptation of the method used in [DR96]; in particular the total time complexity is the same (for a fixed k and l). From now on, we assume that k is fixed and we write $\pi(x,l)$ instead of $\pi(x,k,l)$.

Let y be a real positive number and let $T(x,y,l)$ be the set of positive integers n such that

$$\begin{cases} n \le x, \\ n \equiv l \pmod{k}, \\ p \mid n \Rightarrow p > y. \end{cases}$$

Assume that y is such that $x^{1/3} \le y \le x^{1/2}$. Then each element n of $T(x,y,l)$ has at most two (not necessarily distinct) prime factors. Thus we can split this set into three disjoint subsets $T_0(x,y,l)$, $T_1(x,y,l)$, and $T_2(x,y,l)$, according to the number of (not necessarily distinct) prime factors.

Let $F(x,y,l)$ be the cardinality of $T(x,y,l)$. The set $T_0(x,y,l)$ contains only 1 (resp. is empty) if $l=1$ (resp. $l\neq 1$). Its cardinality is thus $\delta_{l,1}$. The set $T_1(x,y,l)$ contains all the prime numbers p with $y<p\leq x$ and $p\equiv l \pmod k$. Therefore, its cardinality is $\pi(x,l)-\pi(y,l)$. Finally, let $P_2(x,y,l)$ denote the cardinality of $T_2(x,y,l)$. Putting everything together and rearranging terms, we get

$$\pi(x,l) = F(x,y,l) - \delta_{l,1} + \pi(y,l) - P_2(x,y,l). \tag{2.1}$$

2.1. **Computation of $P_2(x,y,l)$.** We have

$$\begin{aligned} P_2(x,y,l) &= \sum_{y<p\leq x^{1/2}} \sum_{\substack{p\leq q\leq x/p\\ pq\equiv l \ (\mathrm{mod}\ k)}} 1 \\ &= \sum_{y<p\leq x^{1/2}} \left[\pi(x/p,lp^{-1}) - \pi(p-1,lp^{-1})\right] \\ &= \sum_{y<p\leq x^{1/2}} \pi(x/p,lp^{-1}) - \sum_{y<p\leq x^{1/2}} \pi(p-1,lp^{-1}) \end{aligned} \tag{2.2}$$

with the implicit convention that $\pi(a,lp^{-1})=\pi(a,n)$ with $n\equiv lp^{-1} \pmod k$.

We use an auxiliary sieve to obtain all primes up to $x^{1/2}$ and a parallel sieve of all invertible classes modulo k up to x/y to get the value of $\pi(x/p,n)$. We thus compute the first sum of equation (2.2) in time $O((x/y)\log\log x)$.

The second sum in (2.2) is computed directly using the primes p coming from the auxiliary sieve. The computation time is $O(x^{\frac{1}{2}+\epsilon})$, which is negligible compared to $O(x^{\frac{2}{3}}/\ln^2 x)$.

2.2. **Computation of $\pi(y,l)$.** We compute a table of all the prime numbers up to y partitioned according to their class modulo k using a sieve. The value of $\pi(y,n)$ for all classes n invertible modulo k is deduced directly from this table. This table and the values $\pi(x,n)$ will prove useful later. This can be done in $O(y\ln y)$ time, which is again negligible compared to $O(x^{\frac{2}{3}}/\ln^2 x)$.

2.3. **Computation of $F(x,y,l)$.** Recall that $F(x,y,l)$ counts the number of elements in $T(x,y,l)$. Let us number the prime numbers $p_1=2,\ p_2=3,\ldots$. For a positive integer a, let $\tilde{T}(x,a,l)=T(x,p_a,l)$ and $\tilde{F}(x,a,l)=F(x,p_a,l)$. Thus, $F(x,y,l)=\tilde{F}(x,a,l)$ where a is the largest index such that $p_a\leq y$. We also set $\tilde{T}(x,0,l)=T(x,0,l)$ and $\tilde{F}(x,0,l)=F(x,0,l)$.

Now, we split the elements of $\tilde{T}(x,a,l)$ into two subsets: the first one containing those which are divisible by p_{a+1} and the second containing those which are not. Clearly, the cardinality of the first set is $\tilde{F}(x/p_{a+1},a,lp_{a+1}^{-1})$ and the cardinality of the second is $\tilde{F}(x,a+1,l)$. We have proved the induction formula

$$\tilde{F}(x,a+1,l) = \tilde{F}(x,a,l) - \tilde{F}(x/p_{a+1},a,lp_{a+1}^{-1}). \tag{2.3}$$

Together with the initial conditions

$$\tilde{F}(x,0,l) = \left\lceil \frac{x+1-l}{k} \right\rceil \quad \text{and} \quad \tilde{F}(x,a,l)=0 \text{ whenever } x<1,$$

we could use equation (2.3) to compute $F(x,y,l)$. However, such a method would require more than $x^{1-\varepsilon}$ time.

Another extreme method would be to sieve all the positive integers congruent to l modulo k up to x by all the prime numbers up to y and count what is left. But, this is even worse since that would take more than $x \log\log x$ time.

In fact, the best way to compute $F(x, y, l)$ is to use a mix between these two methods as was already done in [LMO85, p. 542]. Let $z \geq y$ be a real number. Using the induction formula (2.3) to unfold the terms $F(x/m, p, n)$ while $m \leq z$ and $p \geq 2$, we get an expression with terms of the form $F(u, 0, n)$ which are easily computed and terms of the form $F(u, p, n)$ with $u < x/z$ which can be computed using a sieve up to x/z (instead of x in a "sieve only" method). More precisely, we get the formula

$$F(x, y, l) = S_0 + S$$

with

$$S_0 = \sum_{\substack{m \leq z \\ \gamma(m) \leq y}} \mu(m) \tilde{F}\left(\frac{x}{m}, 0, lm^{-1}\right),$$

$$S = -\sum_{b<a} \sum_{\substack{m \leq z < mp_b \\ \delta(m) > p_b \\ \gamma(m) \leq y}} \mu(m) \tilde{F}\left(\frac{x}{mp_b}, b-1, l(mp_b)^{-1}\right),$$

where $\delta(m)$ (resp. $\gamma(m)$) denotes the smallest (resp. largest) prime number dividing m if $m > 1$, and $\delta(1) = \gamma(1) = 1$.

2.4. **Computation of S.** We split the sum (recall that a is the largest integer such that $p_a \leq y$)

$$S = -\sum_{p_b<y} \sum_{\substack{m \leq z < mp_b \\ \delta(m) > p_b \\ \gamma(m) \leq y}} \mu(m) F\left(\frac{x}{mp_b}, p_{b-1}, l(mp_b)^{-1}\right) = S_1 + S_2 + S_3$$

into three parts according to the size of p_b:

$$S_1 = -\sum_{x^{1/3}<p_b<y} \sum_{\substack{m \leq z < mp_b \\ \delta(m) > p_b \\ \gamma(m) \leq y}} \mu(m) F\left(\frac{x}{mp_b}, p_{b-1}, l(mp_b)^{-1}\right),$$

$$S_2 = -\sum_{x^{1/4}<p_b\leq x^{1/3}} \sum_{\substack{m \leq z < mp_b \\ \delta(m) > p_b \\ \gamma(m) \leq y}} \mu(m) F\left(\frac{x}{mp_b}, p_{b-1}, l(mp_b)^{-1}\right),$$

$$S_3 = -\sum_{p_b\leq x^{1/4}} \sum_{\substack{m \leq z < mp_b \\ \delta(m) > p_b \\ \gamma(m) \leq y}} \mu(m) F\left(\frac{x}{mp_b}, p_{b-1}, l(mp_b)^{-1}\right).$$

The sum S_1 is easy to deal with. For each p_b and each m, we have $mp_b > x^{2/3}$, so

$$\frac{x}{mp_b} < x^{1/3} < p_b$$

and therefore

$$F\left(\frac{x}{mp_b}, p_{b-1}, l(mp_b)^{-1}\right) = \begin{cases} 1 & \text{if } l(mp_b)^{-1} = 1, \\ 0 & \text{else,} \end{cases}$$

since $T(x/(mp_b), b-1, l(mp)^{-1})$ is respectively $\{1\}$ or $\emptyset$.

Furthermore, note that m is prime since all its prime factors are larger than $p_b > x^{1/3}$ and $m \leq z \leq x^{1/2}$. Thus, $\mu(m)$ is always equal to -1 and S_1 actually counts the primes congruent to lp_b^{-1} modulo k:

$$S_1 = \sum_{x^{1/3} < p_b < y} \sum_{\substack{p_b < q \leq y \\ q \equiv lp_b^{-1} \pmod k}} 1.$$

The sum S_1 is computed in negligible time $O(y)$.

Consider the sum S_2. Reasoning as above, it is clear that m is a prime number. Therefore, we will write q instead of m to emphasize this fact. We get

$$S_2 = \sum_{x^{1/4} < p_b \leq x^{1/3}} \sum_{p_b < q \leq y} F\left(\frac{x}{qp_b}, p_{b-1}, l(qp_b)^{-1}\right).$$

Let u be an element of $T(x/(qp_b), p_{b-1}, l(qp_b)^{-1})$. Then u has at most one prime factor since all its prime factors must be larger than or equal to $p_b > x^{1/4}$, and, on the other hand, u must be smaller than $x/(qp_b) \leq x^{1/2}$. Thus, u must be a prime unless $l \equiv qp_b \pmod k$ in which case $u = 1$ is also valid. So, we get the formula (writing simply p instead of p_b):

$$S_2 = \sum_{x^{1/4} < p \leq x^{1/3}} \sum_{p < q \leq y} \left[\max\left\{\pi\left(\frac{x}{qp}, l(qp)^{-1}\right) - \pi(p-1, l(qp)^{-1}), 0\right\} + \delta_{qp,l}\right]$$

where $\delta_{qp,l}$ equals 1 if $qp \equiv l \pmod k$ and 0 otherwise. The max in the sum is due to the fact that, whenever $\pi(x/(qp), l(qp)^{-1}) - \pi(p-1, l(qp)^{-1}) < 0$, the corresponding set $T(x/(qp), p-1, l(qp)^{-1})$ contains only 1 if $qp \equiv l \pmod k$ and is empty otherwise.

We split this sum again:

$$S_2 = U_1 + U_2 + U_3$$

with (note that the max condition translates to the fact that $q < x/p^2$)

$$\begin{aligned}
U_1 &= \sum_{x^{1/4} < p \leq x^{1/3}} \sum_{p < q \leq \min\{y, x/p^2\}} \pi\left(\frac{x}{qp}, l(qp)^{-1}\right), \\
U_2 &= \sum_{x^{1/4} < p \leq x^{1/3}} \sum_{p < q \leq y} \delta_{qp,l}, \\
U_3 &= -\sum_{x^{1/4} < p \leq x^{1/3}} \sum_{p < q \leq \min\{y, x/p^2\}} \pi(p-1, l(qp)^{-1}).
\end{aligned}$$

We rewrite the sums U_2 and U_3 in the following way:

$$\begin{aligned}
U_2 &= \sum_{\substack{1\le m<k\\(m,k)=1}} \sum_{\substack{x^{1/4}<p\le x^{1/3}\\ p\equiv m \pmod k}} \sum_{\substack{p<q\le y\\ q\equiv lm^{-1} \pmod k}} 1\\
&= \sum_{\substack{1\le m<k\\(m,k)=1}} \sum_{\substack{x^{1/4}<p\le x^{1/3}\\ p\equiv m \pmod k}} \left[\pi(y,lm^{-1}) - \pi(p,lm^{-1})\right]\\
&= \sum_{\substack{1\le m<k\\(m,k)=1}} \pi(y,lm^{-1}) \left[\pi(x^{1/3},m) - \pi(x^{1/4},m)\right]\\
&\qquad - \sum_{x^{1/4}<p\le x^{1/3}} \pi(p,lp^{-1})
\end{aligned}$$

and, letting $y(p)$ denote the minimum between y and x/p^2:

$$\begin{aligned}
U_3 &= -\sum_{\substack{1\le m<k\\(m,k)=1}} \sum_{x^{1/4}<p\le x^{1/3}} \sum_{\substack{p<q\le y(p)\\ q\equiv lm^{-1} \pmod k}} \pi(p-1,mp^{-1})\\
&= -\sum_{\substack{1\le m<k\\(m,k)=1}} \sum_{x^{1/4}<p\le x^{1/3}}\\
&\qquad \times \pi(p-1,mp^{-1})\Big[\pi(y(p),lm^{-1}) - \pi(p,lm^{-1})\Big].
\end{aligned}$$

Each sum is computed in a negligible time $O(x^{1/3})$ using the precomputed table of prime numbers sorted by congruences classes mentioned above.

The hard part of the computation of $F(x,y,l)$ is the computation of the sum U_1. We write

$$\begin{aligned}
U_1 &= \sum_{x^{1/4}<p\le x^{1/3}} \sum_{p<q\le \min\{y,x/p^2\}} \pi\left(\frac{x}{qp}, l(qp)^{-1}\right)\\
&= \sum_{x^{1/4}<p\le (x/y)^{1/2}} \sum_{p<q\le y} \pi\left(\frac{x}{qp}, l(qp)^{-1}\right)\\
&\qquad + \sum_{(x/y)^{1/2}<p\le x^{1/3}} \sum_{p<q\le x/p^2} \pi\left(\frac{x}{qp}, l(qp)^{-1}\right)\\
&= \sum_{x^{1/4}<p\le x/y^2} \sum_{p<q\le y} \pi\left(\frac{x}{qp}, l(qp)^{-1}\right)\\
&\qquad + \sum_{x/y^2<p\le (x/y)^{1/2}} \sum_{p<q\le y} \pi\left(\frac{x}{qp}, l(qp)^{-1}\right)\\
&\qquad + \sum_{(x/y)^{1/2}<p\le x^{1/3}} \sum_{p<q\le x/p^2} \pi\left(\frac{x}{qp}, l(qp)^{-1}\right)\\
&= W_1 + (W_2+W_3) + (W_4+W_5)
\end{aligned}$$

with

$$W_1 = \sum_{x^{1/4}<p\le x/y^2} \sum_{p<q\le y} \pi\left(\frac{x}{qp}, l(qp)^{-1}\right),$$

$$W_2 = \sum_{x/y^2<p\le (x/y)^{1/2}} \sum_{p<q\le (x/p)^{1/2}} \pi\left(\frac{x}{qp}, l(qp)^{-1}\right),$$

$$W_3 = \sum_{x/y^2<p\le (x/y)^{1/2}} \sum_{(x/p)^{1/2}<q\le y} \pi\left(\frac{x}{qp}, l(qp)^{-1}\right),$$

$$W_4 = \sum_{(x/y)^{1/2}<p\le x^{1/3}} \sum_{p<q\le (x/p)^{1/2}} \pi\left(\frac{x}{qp}, l(qp)^{-1}\right),$$

$$W_5 = \sum_{(x/y)^{1/2}<p\le x^{1/3}} \sum_{\sqrt{x/p}<q\le x/p^2} \pi\left(\frac{x}{qp}, l(qp)^{-1}\right).$$

The sums W_1 and W_2 are computed directly. Since x/qp can be as large as $x^{1/2}$, we use a parallel sieve of all invertible classes modulo k up to $x^{1/2}$ to get the values of $\pi(x/(qp), l(qp)^{-1})$.

For W_3, since q is larger than $(x/p)^{1/2}$, a large number of consecutive values of q gives the same value of $\pi(x/(qp), l(qp)^{-1})$; henceforth this sum can be evaluated more efficiently by grouping these consecutive values of q. The same technique applies to W_5.

Finally, the sum W_4 is computed using, once again, the precomputed table.

The exact time complexity of the computation of these sums is given in [DR96]; in any case they are $O(x^{2/3}/\log^2 x)$.

3. Numerical results

We have implemented the method described above in C++ on a DEC Alpha EV6 500MHz and a Pentium III 1GHz. We have computed the values of $\pi(x,4,1)$ and $\pi(x,4,3)$ for $x = d\cdot 10^j$ with $1\le d\le 9$ and $10\le j\le 19$ and also for $x = 10^{20}$. These values are given in Table 2.

We have also made a thorough search of regions where $\delta(x)$ is negative for most of the values of x. Indeed, equation (1.1) shows that a search with a step of 0.004 on a logarithmic scale for a large interval of values of x would hit values x for which $\delta(x) < 0$ with a good chance. We performed a computation of the values of $\delta(x_0\times r^n)$ for $x_0 = 1,000$ and $r = 1.004$, up to $x = 1,088,537,721,123,564,252$ (as far as today). When the value of $\delta(x)$ obtained was positive but relatively small, we computed several values of δ near x to see whether or not there was a region in the area. This method led to the rediscovery of all the previous regions already known (see Table 1) and also to the discovery of a new region around $x = 10^{18}$. Note that

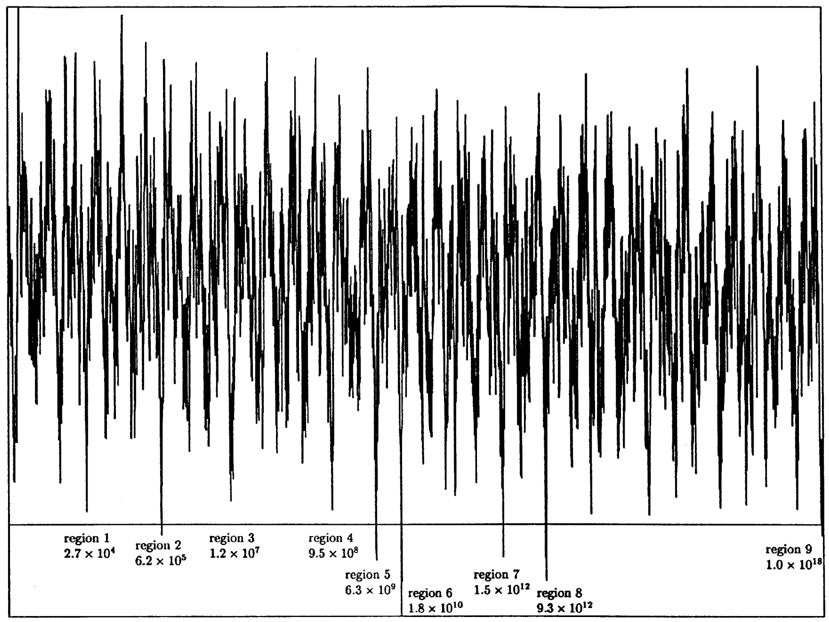

FIGURE 1. $(\log(x), \delta(x)\log(x)/\sqrt(x))$

we do not claim this search to be exhaustive since the method may miss narrow regions. Figure 1 gives a graph of these computations. On the horizontal axis are the values of x on a logarithmic scale, and on the vertical axis are the values of $\delta(x)\log(x)/\sqrt{x}$.

The new region extends as far as 1.005×10^{18}, so it surrounds 10^{18}. It should be noted that 10^{18} *does not* belong to Δ^-, but still the value of $\delta(10^{18})$ is relatively small.

TABLE 1. Known regions where $\delta(x) < 0$

Region	Starts at	
1	$26,861$	Leech [Lee57], 1957
2	6.16×10^5	Leech [Lee57], 1957
3	1.23×10^7	Lehmer, 1969
4	9.51×10^8	Lehmer, 1969
5	6.31×10^9	Bays and Hudson [BH78], 1979
6	1.85×10^{10}	Bays and Hudson [BH78], 1979
7	1.49×10^{12}	Bays and Hudson, 1996
8	9.32×10^{12}	Bays *et al.* [BFHR01], 2001
9	9.97×10^{17}	

TABLE 2.

x	$\pi(x,4,1)$	$\pi(x,4,3)$	$\delta(x)$
1×10^{10}	227523275	227529235	5960
2×10^{10}	441101890	441104825	2935
3×10^{10}	649997354	650008571	11217
4×10^{10}	855972440	855982992	10552
5×10^{10}	1059822165	1059832412	10247
6×10^{10}	1262014995	1262023159	8164
7×10^{10}	1462847357	1462852181	4824
8×10^{10}	1662521926	1662537319	15393
9×10^{10}	1861205914	1861223076	17162
1×10^{11}	2059020280	2059034532	14252
2×10^{11}	4003548492	4003556566	8074
3×10^{11}	5909207980	5909231154	23174
4×10^{11}	7790493403	7790512253	18850
5×10^{11}	9654058131	9654078010	19879
6×10^{11}	11503736012	11503765773	29761
7×10^{11}	13342013346	13342060963	47617
8×10^{11}	15170671955	15170711571	39616
9×10^{11}	16990975120	16991012465	37345
1×10^{12}	18803924340	18803987677	63337
2×10^{12}	36650920051	36650976087	56036
3×10^{12}	54170123581	54170175121	51540
4×10^{12}	71483076254	71483131871	55617
5×10^{12}	88645790439	88645871209	80770
6×10^{12}	105690668569	105690758469	89900
7×10^{12}	122638762289	122638926514	164225
8×10^{12}	139504962196	139505108614	146418
9×10^{12}	156300160163	156300193944	33781
1×10^{13}	173032709183	173032827655	118472
2×10^{13}	337947869842	337948039428	169586
3×10^{13}	500060778623	500060890229	111606
4×10^{13}	660405866854	660406104847	237993
5×10^{13}	819461739349	819462025217	285868
6×10^{13}	977505071501	977505356756	285255
7×10^{13}	1134716310961	1134716560342	249381
8×10^{13}	1291221836521	1291222276965	440444
9×10^{13}	1447116002078	1447116248704	246626
1×10^{14}	1602470783672	1602470967129	183457
2×10^{14}	3135212239502	3135212411812	172310
3×10^{14}	4643720595358	4643721004921	409563
4×10^{14}	6136911872530	6136912282960	410430
5×10^{14}	7618916303080	7618917351539	1048459
6×10^{14}	9092127220696	9092128070873	850177
7×10^{14}	10558104318534	10558104592488	273954
8×10^{14}	12017944798977	12017945569183	770206
9×10^{14}	13472462653549	13472463812671	1159122

Table 2 (continued)

x	$\pi(x,4,1)$	$\pi(x,4,3)$	$\delta(x)$
1×10^{15}	14922284735484	14922285687184	951700
2×10^{15}	29239107639569	29239108042321	402752
3×10^{15}	43344300693083	43344302117035	1423952
4×10^{15}	57315493601108	57315495302891	1701783
5×10^{15}	71188707903700	71188709292663	1388963
6×10^{15}	84984830526287	84984832028263	1501976
7×10^{15}	98717495795309	98717498283021	2487712
8×10^{15}	112396302108982	112396304209617	2100635
9×10^{15}	126028365161887	126028368292040	3130153
1×10^{16}	139619168787795	139619172246129	3458334
2×10^{16}	273931712869820	273931719080187	6210367
3×10^{16}	406380135935561	406380140853941	4918380
4×10^{16}	537646385801772	537646392951377	7149605
5×10^{16}	668047381490698	668047386273272	4782574
6×10^{16}	797767045802885	797767053786388	7983503
7×10^{16}	926925544457111	926925555169508	10712397
8×10^{16}	1055607507851023	1055607518369420	10518397
9×10^{16}	1183875715467888	1183875722942661	7474773
2×10^{17}	2576664675966205	2576664686679702	10713497
3×10^{17}	3825005948840463	3825005962380339	13539876
4×10^{17}	5062840596161843	5062840612149478	15987635
5×10^{17}	6292978272706233	6292978293865386	21159153
6×10^{17}	7517051002806033	7517051018457786	15651753
7×10^{17}	8736125733010690	8736125766616565	33605875
8×10^{17}	9950954267090255	9950954300876809	33786554
9×10^{17}	11162094600919585	11162094630455263	29535678
1×10^{18}	12369977142579584	12369977145161275	2581691
2×10^{18}	24322580623880090	24322580657858444	33978354
3×10^{18}	36127352391026284	36127352406660798	15634514
4×10^{18}	47838130416736104	47838130487151502	70415398
5×10^{18}	59479994798617422	59479994889656049	91038627
6×10^{18}	71067524678491295	71067524734130848	55639553
7×10^{18}	82610256979417864	82610257001551559	22133695
8×10^{18}	94114914605549098	94114914641880405	36331307
9×10^{18}	105586489592518919	105586489650739358	58220439
1×10^{19}	117028833597800689	117028833678543917	80743228
2×10^{19}	230318827545992966	230318827580012523	34019557
3×10^{19}	342279960248880580	342279960334204109	85323529
4×10^{19}	453395257443424108	453395257662152462	218728354
5×10^{19}	563889961853817581	563889961936366961	82549380
6×10^{19}	673895097943622446	673895098116473000	172850554
7×10^{19}	783496420076932640	783496420248547163	171614523
8×10^{19}	892754404995121348	892754405128443233	133321885
9×10^{19}	1001713975101251869	1001713975318165460	216913591
1×10^{20}	1110409801150582707	1110409801410336132	259753425

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