1 On the Dynamics of Crab Submarines

Let us begin by considering a group of crab submarines with some arbitrary position, which we denote x_i for the ith crab submarine. A set of n crab submarines may then be denoted as the n-tuple $(x_1, \ldots, x_n)$. We must first consider the fuel expended by this set of crab submarines in moving to another point k. To do so, we consider an arbitrary crab submarine, x_i. It is well-known that the fuel expenditure of a crab submarine moving from x_i to k, which we denote $f_i(k)$, may be written as

$$\begin{aligned} f_i(k) &= 1 + 2 + 3 + \cdots + |k - x_i| \\ &= \sum_{n=0}^{|k-x_i|} n \\ &= \frac{|k - x_i| \left(|k - x_i| + 1\right)}{2} \\ &= \frac{(k - x_i)^2 + |k - x_i|}{2} \end{aligned}$$

where the closed form of the series is due to a result by Gauss. It must then be the case that the fuel expenditure of all n crab submarines may be obtained by summing the expenditures of each individual submarine, so the total fuel expenditure of the crab submarines, $f(k)$, is given by

$$\begin{aligned} f(k) &= \sum_{i=1}^{n} f_i(k) \\ &= \sum_{i=1}^{n} \frac{(k - x_i)^2 + |k - x_i|}{2} \end{aligned}$$

which is the amount of fuel expended by a set of n crab submarines with initial positions x_i in moving to a point k.

2 On the Minimization of Fuel Expenditure

We are now at a point where we have an expression for the fuel expenditure of a set of crab submarines, and we wish to find some point k to which the crab submarines may be moves which minimizes the total fuel expenditure. In other words, given $f(k)$ we wish to find some k which minimizes f. This is a very common problem in calculus, and it is known that the minimum of f, if one exists, must occur at a point where

$$\frac{\mathrm{d}f}{\mathrm{d}k} = 0$$

Given the definition of f found in part 1 of this paper, the derivative may be readily calculated, and we find

$$\frac{\mathrm{d}f}{\mathrm{d}k} = \sum_{i=1}^{n} \frac{2(k - x_i) + \operatorname{sgn}(x_i - k)}{2}$$

where $\operatorname{sgn}(x)$ is a function called the *sign function*, and is equal to 1 if $x > 0$, 0 if $x = 0$, and -1 if $x < 0$. We are looking for the value of k which minimizes f, which occurs when

$$\sum_{i=1}^{n} \frac{2(k - x_i) + \operatorname{sgn}(x_i - k)}{2} = 0$$

The usefulness of the above equation may not be immediately obvious, but by rearranging terms we may put it in the form

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \left(k + \frac{\operatorname{sgn}(x_i - k)}{2} \right)$$

at which point k may be drawn out of the sum to yield

$$\sum_{i=1}^{n} x_i = nk + \sum_{i=1}^{n} \frac{\operatorname{sgn}(x_i - k)}{2}$$

which gives us an expression for k that is very nearly in a closed form, namely that

$$k = \frac{\sum_{i=1}^{n} x_i}{n} - \frac{1}{2}\frac{\sum_{i=1}^{n} \operatorname{sgn}(x_i - k)}{n}$$

It is now valuable to note that $\frac{\sum_{i=1}^{n} x_i}{n}$ is precisely the mean of the x_i, which we denote $\overline{x}$. This gives us the simplified form

$$k = \overline{x} - \frac{1}{2}\frac{\sum_{i=1}^{n} \operatorname{sgn}(x_i - k)}{n}$$

but unfortunately another term persists, and remains dependent on k. Eliminating this term is intractable, but it may be bounded. Namely, we may observe that $\sum_{i=1}^{n} \operatorname{sgn}(x_i - k)$ is maximally n, when $x_i > k$ for all i, and minimally $-n$, when $x_i < k$ for all i. As such, the second term is maximally equal to $\frac{1}{2}$ and minimally equal to $-\frac{1}{2}$. This means that while k may not be exactly calculated, it is bounded by

$$\overline{x} + \frac{1}{2} \geq k \geq \overline{x} - \frac{1}{2}$$

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