CS270: LAB #16

Racket Induction

You may work in teams of ideally one or two people (three is acceptable in the event of an unscheduled absence). Unless stated otherwise, the lab is due to be submitted into Gradescope at the end of the class day (11:59pm).
In order to receive credit, follow these instructions:

[a] Every team member should be discussing simultaneously the same problem – do NOT try to divvy up the labor and assign different problems to different students since the material is cumulative.

[b] Directly edit this lab PDF with your answers (extra pages can be added in the rare event you need more than the allotted space)

[c] Each lab, rotate which member has the responsibility of being the Scribe. This is the person that is typing the answers and uploading the final PDF – note that only a single copy of the filled in PDF is turned into Gradescope. Only one lab needs to be submitted for the entire team, and all members receive the same score. Make sure to use a font that your PDF editor is compatible with (otherwise you might find your answers appear as weird shapes/sizes or simply disappear entirely!)

[d] The Gradescope submission must have each answer properly tagged with the appropriate question. Moreover, every member of the team must be listed as a submitter. Although it is the Scribe which executes these actions, it is still the responsibility of the entire team to make certain this is done properly (thus it is highly recommended that the Scribe share their screen so the entire team can witness it). Answers which are improperly tagged cannot be seen by the grader and thus cannot be scored.

REMOTE ONLY

[e] Each lab, rotate which member has the responsibility of being the Recorder. This is the person who hits the Zoom Record button (once the technical permission is granted by the TA/RCF/Professor) and ensures that everyone has their camera/microphone on. They are also the member that is responsible to make sure the DrexelStream video is marked as viewable and entered into the https://tinyurl.com/VidLinkForm webform before 11:59pm (they should also email the rest of their team as confirmation.) Note that the video file doesn't get created/processed until after the Recorder has quit Zoom.

[f] Each lab, rotate which member has the responsibility of being the Manager. This is the person that ensures that everyone is participating equally and honestly, keeps the group on task, ensures that all team members understand a solution before going on to the next question, and presses the "hand up" button in Zoom to summon a TA or the professor (but they only do so after surveying the group to make sure everyone has the same question).

Team Name (CS pioneer): Tim Berners-Lee

Scribe name: Brendan Hoag

Recorder name: Cole Bardin

Manager name: Jeremy Mathews

Other team member (if any): Jackson Masterson

Question1: 0 points [this question is merely reading the example over carefully. There is nothing to write here]

2 Proofs by Induction

```
(define (f n) (if (zero? n) 0 (+ (- (* 2 n) 1) (f (- n 1)))))
```

We shall prove the the principle of Racket induction that $\forall n \in \mathbb{N}$ (f n) = n^2

Base Case:

The smallest value of n is n = 0. Equational reasoning can is used to verify this.

1.	(f 0)	LHS of base case, anchor at n=0
2.	(if (zero? 0) 0 (+ (- (* 2 0) 1) (f (- 0 1))))	Apply definition of f
3.	(if #t 0 (+ (- (* 2 0) 1) (f (- 0 1))))	Evaluate zero?
4.	0	Evaluate If

For the RHS base case at n=0: $0^2 = 0$

since LHS came out to be the same as the RHS (both 0), we know that the theorem holds for the Base Case.

Inductive Hypothesis

This is our Inductive Hypothesis (IH): Assume that for some integer k, (f k) = k^2

Note that this is essentially the same as the theorem, but with n replaced by k

Leap Step

We knows show that as long as the IH is true, that the theorem holds when the n is replaced by k+1.

In other words, we wish to use the IH to show our new Goal: that (f (+ k 1)) = $(k+1)^2$

1.	(f (+ k 1))	premise of LHS with n replaced by k+1
2.	(if (zero? (+ k 1)) 0 (+ (- (* 2 (+ k 1)) 1) (f (- (+ k 1) 1)))	Apply definition of f
3.	(if #f 1 (+ (- (* 2 (+ k 1)) 1) (f (- (+ k 1) 1))))	Evaluate zero? (#f since k≠-1 by contract)
4.	(+(-(*2(+k 1))1)(f(-(+k 1)1)))	Evaluate If
5.	(+ (- (* 2 (+ k 1)) 1) (f k))	Evaluate -+

Note the conversion to Algebra, from this point on once we invoke the IH we have transitioned out of the Racket world.

6.	$2(k+1) - 1 + (k)^2$	invoke IH
7.	$2k+2 - 1 + k^2$	Algebra(technicallyDistributiveproperty)
8.	$k^2 + 2k + 1$	Algebra(technicallyCommutativeproperty)

Now we must look at the RHS and show it comes out the same as the LHS we just computed.

RHS with n replaced by k+1: $(k+1)^2 = (k+1)(k+1) = k^2 + 2k + 1$

Since the LHS came out to the same as the RHS, this established the Leap step.

Consequently, since we demonstrated both the base case and the leap step, then by the principle of racket induction the theorem holds for all inputs. In other words, $\forall n \in \mathbb{N}$ (f n) = n^2

Question 2 : 18 points

(a) (3 points) What does ∀ n∈ℕ mean in plain English?

For all numbers n in the set of Natural Numbers

(b) (3 points) What value of n is being used in the base case? why?

0 is used for the base case because it is easy to compute (f 0) by hand.
0 is arguably a part of the set of Natural Numbers so this question implies that 0∈ℕ.

(c) (3 points) Assume the function was

```
; input contract: n is at least a double digit number
(define (f n) (if (equal? n 10) 0 (+ (- (* 2 n) 1) (f (- n 1)))))
```

What value of n would you use in the base case? why?

10 would be used for the base case because it's the lowest double digit number

(d) (3 points) Many students confuse the IH with the theorem to be proven. Explain the difference between n and k.

n is used to represent all natural numbers but k (and k + 1) is a natural number that is not the base case.

(e)(3points)WhyareweallowedtousetheInductiveHypothesistogetfromLine5toLine6?

Since we assumed the IH that (f k) = k^2, then we are allowed to substitute all instances of (f k) with k^2.

(f) (3 points) This proof uses a combination of Equational Reasoning and Algebraic Reasoning. For example, on Line 3 of the inductive proof we know that $k+1$ is not equal to zero, because of the input contract. In this line, k a variable, so Racket could NOT actually execute this code (it would crash saying k undefined). By using variables like this, we can describe how the code executes in a general sense. We are not looking at a specific input, we are looking at a general execution path.

One line 6 of the leap step, we switch to algebraic notation. At this point, the programming logic has executed. We now need to work with the return value algebraically to confirm it is the formula we want.

Would it be possible to write Line 3 of the Inductive Proof Algebraically? Why or why not?

Yes it can be rewritten algebraically because the condition of the if statement is already evaluated to be false.
Therefore the false case of the if-statement body can be written in purely algebraic terms.

Question 3 : 24 points

Before starting a proof, it is useful to have some intuition that the proof will hold. We can get this by experimentation.

```
(define (f n) (if (zero? n) 0 (+ (- (* 2 n) 1) (f (- n 1)))))
```

Conjecture 1: (f n) = n ^{2}

We can test this for arbitrary values, 1 through 5. (note that we missed 0, as well as 6 and higher, so this is NOT a proof!)

$$\begin{aligned}
(\mathrm{f}\ 1) &= 1 = 1^2 \\
(\mathrm{f}\ 2) &= (2*2-1)+1 = 4 = 2^2 \\
(\mathrm{f}\ 3) &= (2*3-1)+4 = 9 = 3^2 \\
(\mathrm{f}\ 4) &= (2*4-1)+9 = 16 = 4^2 \\
(\mathrm{f}\ 5) &= (2*5-1)+16 = 25 = 5^2
\end{aligned}$$

(a) (12 points) Conjecture 2:

```
(define (g n) (if (zero? n) 0 (+ (* 2 n) (g (- n 1)))))
```

test the conjecture by experimentation that (g n) = n ^{2} +n for n=1...4.

g(1) = 1^2 + 1 = 2
g(2) = 2^2 + 2 = 6
g(3) = 3^2 + 3 = 12
g(4) = 4^2 + 4 = 20

(b) (12 points) Conjecture 3:

```
(define (h n) (if (zero? n) 0 (+ (- (* 4 n) 5) (h (- n 1)))))
```

Test the conjecture by experimentation that (h n) = 2n^2 - 3n for n = 1 ... 4

h(1) = 2(1^2) - 3(1) = -1
h(2) = 2(2^2) - 3(2) = 2
h(3) = 2(3^2) - 3(3) = 9
h(4) = 2(4^2) - 3(4) = 20

Question 4 : 18 points

(define (g n) (if (zero? n) 0 (+ (* 2 n) (g (- n 1)))))

Prove using the principle of Racket induction that ∀n ∈ℕ (g n) = n^2 + n

(a) (6 points) Complete the Base Case Proof

Base Case: [hint: put the anchor at n=0]

Be sure to include the LHS (racket with equational reasoning), the RHS (algebra) and a concluding sentence.

Base Case: anchored at n = 0
LHS:
1. (g 0) LHS of base case, anchor at n=0
2. (if (zero? 0) 0 (+ (* 2 0) (g (- 0 1))))) apply def of g
3. (if #t 0 (+ (* 2 0) (g (- 0 1)))) evaluate zero?
4. 0 evaluate if

RHS:
0^2 + 0 = 0
LHS is the same as RHS, so theorem holds for base case

(b) (6 points) State the IH ("Inductive Hypothesis") which will be used in the leap step. [hint: replace n with k]

∀n ∈N (g k) = k^2 + k

(c) (6 points) Complete the Leap Step by filling in the missing expressions or justifications for each line of the equational reasoning proof below

LHS:		
1.		Premise of LHS with n=k+1
2.		Apply definition of g
3.	(if #f 0 (+ (* 2 (+ k 1)) (g (- (+ k 1) 1))))	
4.	(+(*2(+k 1))(g(-(+k 1)1)))	
5.		Evaluate -+

For step6 and onward, switch to Algebra by using the IH on step5

LHS:
1. (g (+ k 1)) Premise
2. (if (zero? (+ k 1)) 0 (+ (* 2 (+ k 1)) (g (- (+ k 1) 1))))) apply def of g
3. (if #f 0 (+ (* 2 (+ k 1)) (g (- (+ k 1) 1))))) evaluate zero?
4. (+ (* 2 (+ k 1)) (g (- (+ k 1) 1)))) evaluate if
5. (+ (* 2 (+ k 1)) (g k)) evaluate +-
6. (2(k + 1)) + k^2 + k invoke IH
7. 2k + 2 + k^2 + k Algebra
8. k^2 + 3k + 2 Algebra

Then show that the RHS (with n= k+1) also reduces to the same thing, hence LHS=RHS

RHS:
1. (k + 1)^2 + k + 1 Premise
2. k^2 + 2k + 1 + k + 1 Algebra
3. k^2 + 3k + 2

Finish this concluding sentence:

Consequently, since we demonstrated both the base case and the leap step, then by the principle of racket induction, the theorem holds for all inputs. In other words,....

Since LHS = RHS, the leap has been established.
The base and leap have been proven, (g n) = n^2 + n holds true for all non-negative numbers

Question 5 : 20 points

(define (h n) (if (zero? n) 0 (+ (- (* 4 n) 5) (h (- n 1)))))

Prove using the principle of Racket induction that ∀n ∈ℕ (h n) = $2n^2$ - 3n

(a) (6 points) Complete the Base Case Proof

The smallest value of n is 0. Equational reasoning can is used to verify this:

1) (h 0)	LHS of base case, anchor at n=0
2) (if (zero? 0) 0 (+ (- (* 4 0) 5) (h (- 0 1))))	Apply definition of h
3) (if #t 0 (+ (- (* 4 0) 5) (h (- 0 1))))	Evaluate zero?
4) 0	Evaluate If

For the RHS base case at n=0;2(0)^2 - 3(0) = 0
Since LHS came out to be the same as the RHS (both 0), we know that the theorem holds for the Base Case

(b) (4 points) State the IH

This is our Inductive Hypothesis (IH): Assume that for some integer k, (h k) = 2k^2 - 3k.
Note that this is essentially the same as the theorem, but with n replaced by k.

(c) (10 points) Complete the Leap step.

Be sure to note which line uses the Inductive Hypothesis (-5pt if missed).

Remember to do the the LHS (racket with n=k+1), the RHS (algebra with n=k+1), and a concluding sentence.

We know show that as long as the IH is true, that the theorem holds when the n is replaced by k+1.
In other words, we wish to use the IH to show our new Goal: that (h (+ k 1)) = 2(k+1)^2 - 3(k + 1)

1) (h (+ k 1))	LHS of base case, anchor at n=0
2) (if (zero? (+ k 1)) 0 (+ (- (* 4 (+ k 1)) 5) (h (- (+ k 1) 1))))	Apply definition of h
3) (if #f 1 (+ (- (* 4 (+ k 1)) 5) (h (- (+ k 1) 1))))	Evaluate zero?
4) (+ (- (* 4 (+ k 1)) 5) (h (- (+ k 1) 1)))	Evaluate If
5) (+ (- (* 4 (+ k 1)) 5) (h k))	Evaluate -+
6) 4(k+1) - 5 + (2k^2 - 3k)	Invoke IH
7) 4k + 4 - 5 + (2k^2 - 3k)	Algebra
8) 2k^2 + k - 1	Algebra

Now we must look at the RHS and show it comes out the same as the LHS we just computed.
RHS with n replaced by k+1: 2(k+1)^2 - 3(k+1) = 2k^2 + 4k + 2 - 3k -3 = 2k^2 + k - 1
Since the LHS came out to the same as the RHS, this established the Leap step.

Question 6 : 20 points

```
(define (Q n) (if (zero? n) 0 (+ 2 (Q (- n 1)))))
```

Use the principle of Racket induction to prove that $\forall n \in \mathbb{N}$ (Q n) = 2n

Remember you need to establish the base case (both sides!), state the IH, and then establish the leap step (both sides) and use concluding statements after each part.

(define (Q n) (if (zero? n) 0 (+ 2 (Q (- n 1)))))
prove: $\forall n \in \mathbb{N}$ (Q n) = 2n
base case: n=0
LHS:
(Q 0) Premise
(if (zero? 0) 0 (+ 2 (Q (- 0 1)))) Apply def of Q
(if #t 0 (+ 2 (Q (- 0 1)))) Evaluate zero?
0 Evaluate if statement
Therefore (Q 0) = 0
RHS:
2*0 = 0
Since LHS=RHS, the theorem holds for the base case.

IH:
For the Inductive Hypothesis, assume that for some positive integer not equal to 0:
(Q k) = 2*k

Leap:
If the IH is true, then the theorem should hold when n is replaced by k+1.
Goal: prove (Q (+ k 1)) = 2*(k + 1)
(Q (+ k 1)) Premise
(if (zero? (+ k 1)) 0 (+ 2 (Q (- (+ k 1) 1)))) Apply def of Q
(if #f 0 (+ 2 (Q (- (+ k 1) 1)))) Evaluate zero?
(+ 2 (Q (- (+ k 1) 1))) Evaluate if-statement
(+ 2 (Q k)) Evaluate -+
2 + 2*k Invoke IH, and convert to algebraic form
2*(k + 1) Factor out 2, algebraically

LHS when n=k+1: 2*(k+1)
RHS when n=k+1: 2*(k+1)
Since LHS=RHS, this establishes the leap.

Consequently, since we demonstrated both the base case and the leap step,
then by the principle of racket induction the theorem holds for all inputs.
In other words, $\forall n \in \mathbb{N}$ (Q n) = 2*n.