CS270: LAB #14

FOL Natural Deduction

You may work in teams of one or two people (three is acceptable in the event of an unscheduled absence). Unless stated otherwise, the lab is due to be submitted into Gradescope at the end of the day.
In order to receive credit, follow these instructions:

[a] Every team member should be discussing simultaneously the same problem – do NOT try to divvy up the labor and assign different problems to different students since the material is cumulative.

[b] Directly edit this lab PDF with your answers (extra pages can be added in the rare event you need more than the allotted space)

[c] Each lab, rotate which member has the responsibility of being the Scribe. This is the person that is typing the answers and uploading the final PDF – note that only a single copy of the filled in PDF is turned into Gradescope. Only one lab needs to be submitted for the entire team, and all members receive the same score. Make sure to use a font that your PDF editor is compatible with (otherwise you might find your answers appear as weird shapes/sizes or simply disappear entirely!)

[d] The Gradescope submission must have each answer properly tagged with the appropriate question. Moreover, every member of the team must be listed as a submitter. Although it is the Scribe which executes these actions, it is still the responsibility of the entire team to make certain this is done properly (thus it is highly recommended that the Scribe share their screen so the entire team can witness it). Answers which are improperly tagged cannot be seen by the grader and thus cannot be scored.

REMOTE ONLY:
[e] Each lab, rotate which member has the responsibility of being the Recorder. This is the person who hits the Zoom Record button (once the technical permission is granted by the TA/RCF/Professor) and ensures that everyone has their camera/microphone on. They are also the member that is responsible to make sure the DrexelStream video is marked as viewable and entered into the https://tinyurl.com/VidLinkForm webform before 11:59pm (they should also email the rest of their team as confirmation.) Note that the video file doesn't get created/processed until after the Recorder has quit Zoom.

[f] Each lab, rotate which member has the responsibility of being the Manager. This is the person that ensures that everyone is participating equally and honestly, keeps the group on task, ensures that all team members understand a solution before going on to the next question, and presses the "hand up" button in Zoom to summon a TA or the professor (but they only do so after surveying the group to make sure everyone has the same question).

Team Name (CS Pioneer): Tim Berners Lee

Scribe name: Brendan Hoag

Recorder name: Jackson Masterson

Manager name: Cole Bardin

Other team member (if any): Jeremy Mathews

Question 1 : 26 points

In previous weeks, we have looked at **Predicate Logic**. We will now see how it works with Deduction.

The Greatest Common Denominator was defined as follows.

$$\text{divides}(g, a) \wedge \text{divides}(g, b) \wedge \forall e\, (\text{divides}(e, a) \wedge \text{divides}(e, b) \implies \text{divides}(e, g))$$

The number of parenthesis can quickly become confusing in these expressions. We use use a shorter notation in this lab. Functions will be capital letters and variables will be lowercase letters.

$$Dga \wedge Dgb \wedge \forall e\, (Dea \wedge Deb \implies Deg)$$

(a) (6 points) Convert the following expression to the new format.

$$\forall x(\text{even}(x)) \implies \exists y\, (ge(y, x) \wedge \text{even}(y))$$

∀x Ex ⇒ ∃y (GEyx ∧ Ey)

(b) (6 points) A **Bound Variable** is a variable that has a quantifier (either ∀ or ∃). What is the bound variables in the following expression?

$$Dga \wedge Dgb \wedge \forall e\, (Dea \wedge Deb \implies Deg)$$

The bound variable in the expression is e

(c) (6 points) A **Free Variable** is a variable that does not have a quantifier. What are the free variables in the following expression?

$$Dga \wedge Dgb \wedge \forall e\, (Dea \wedge Deb \implies Deg)$$

The free variables are a, g, b

(d) (8 points) For each variable in the below expression, state if it is **bound** or **free**.

$$Na \implies \forall x(Ma \implies Mx), Ma, \neg Mb \therefore \neg Na$$

X is a bounded variable, a & b are both free variables

Question 2 : 14 points

In an **Interpretation** we assign meanings to the functions and values in Predicate Logic expressions.

The following predicate logic expression can be interpreted in many ways.

$$Pxy \implies \exists z(Pxz \land Pzy)$$

If we interpret $P(x, y) = x < y$ then we are saying

$$(x < y) \implies \exists z((x < z) \land (z < y))$$

This is true if x and y are real numbers. It is false if x and y are integers and $y = x + 1$.

Come up with two interpretations of the following expression. Hint: try $<, >, \neq, =, \leq, \geq$.

$$(Pxy \land Pyz) \implies Pxz$$

(a) (7 points) Come up with a function for predicate P that will make the statement True for integers

Which of the following operators $(<, >, \neq, =, \leq, \geq.)$ can P be defined as: $P(a, b) = (a?b)$ so that the statement $\forall(x, y, z) \in \mathbb{Z}\,((Pxy \land Pyz) \implies Pxz)$ is True.

All operators except for not equals work.

(b) (7 points) Come up with a function for predicate P that will make the statement False for integers

Which of the following operators $(<, >, \neq, =, \leq, \geq.)$ can P be defined as: $P(a, b) = (a?b)$ so that the statement $\forall(x, y, z) \in \mathbb{Z}\,((Pxy \land Pyz) \implies Pxz)$ is False.

Not equals is the only one that is false.

Question 3 : 1 point

Quantifiers can be used in our proof checker. https://proof-tool.herokuapp.com/

There are two rules for working with **For All** statements.

We will prove $\forall x \in V\ (Px \rightarrow Qx)\ ;\ \forall x \in V\ Px \therefore \forall x \in V\ Qx$

Line #	Expression	Rule
1	∀x∈V (Px→Qx)	Premise
2	∀x∈V Px	Premise
3	Pa→Qa	∀E 1
4	Pa	∀E 2
5	Qa	→E 3, 4
6	∀x∈V Qx	∀I 5

On Line 3, we state that since the statement is true for all variables, we can create a variable c that the statement holds for. This is called a **For All Elimination**. We complete the same process on Line 4.

Line 5 uses **Conditional Elimination** from deduction.

Line 6 says that since we have shown Qc to be true for an arbitrary c under our premise, we can conclude that it is true for all variables. This uses **For All Introduction**.

(a) Copy the above proof into the checker.

Note: To enter these into the Proof Checker, select FOL. or choose the "Lab 14" assignment in the proof tool. Put a screenshot of your solution below.

Line #	Expression	Rule
1	∀x∈V (Px→Qx)	Premise
2	∀x∈V Px	Premise
3	Pa→Qa	∀E 1
4	Pa	∀E 2
5	Qa	→E 3 4
6	∀x∈V Qx	∀I 5

Check Proof Save

Great job! Don't forget to save your work by clicking "Save"! This will allow you to view this correct proof later!

Question 4 : 12 points

Using the same proof (duplicated below), answer the following questions. Use the proof tool to test each question.

Line #	Expression	Rule
1	∀x∈V (Px→Qx)	Premise
2	∀x∈V Px	Premise
3	Pa→Qa	∀E 1
4	Pa	∀E 2
5	Qa	→E 3, 4
6	∀x∈V Qx	∀I 5

(a) (4 points) Change the a to an x in lines 3-5. Does the proof still pass the checker? Explain why or why not.

It does not because x is already defined in the premise

(b) (4 points) In lines 3 and 4, change Pa to Pc. In lines 3 and 5, change Qa to Qb.
Does the proof still pass the checker? Why or why not?

Note that as of May 2023, Proof Buddy has a bug and thinks the proof is valid with this change, even though it is not (that bug is currently being fixed). Can you explain why in the real world it should NOT logically work.

The proof does not pass because we cannot assume that the two variables are the same

(c) (4 points) In lines 3 and 4, change Pa to Pb. In lines 3 and 5, change Qa to Qb.
Does the proof still pass the checker? Why or Why not?

It does still pass because all the variables are consistent regardless of what they are called.

Question 5 : 1 point

There are also two rules related to the **Exists** law.

They are both shown in the example below. **∀x∈V (Qx → Rx) ; ∃x∈V (Px ∧ Qx) ∴ ∃x∈V (Px ∧ Rx)**

Line #	Expression	Rule
1	∀x∈V (Qx → Rx)	Premise
2	∃x∈V (Px ∧ Qx)	Premise
3.1	Pc ∧ Qc	Assumption
3.2	Qc → Rc	∀E 1
3.3	Qc	∧E 3.1
3.4	Rc	→E 3.2, 3.3
3.5	Pc	∧E 3.1
3.6	Pc ∧ Rc	∧I 3.5, 3.4
3.7	∃x∈V (Px ∧ Rx)	∃I 3.6
4	∃x∈V (Px ∧ Rx)	∃E 2, 3

In this problem, we need to make an assumption. On line 3, we assume that c is the value that exists to make the statement true.

On line 4, we state that since this premise is true for all, it must also be true for our c.

The basic deduction rules are now used. On line 9, we says that since $Pc \wedge Rc$ is true there must exists a value.

We can now leave the subproof. On line 10 we draw a conclusion. We state that a value exists (line 2). That value causes another statement to be true. We conclude this means $\exists x(Px \wedge Rx)$true.

(a) Copy the above proof into the checker.

Put a screenshot of your solution below.

Line #	Expression	Rule
1	∀x∈V (Qx → Rx)	Premise
2	∃x∈V (Px ∧ Qx)	Premise
3.1	Pc ∧ Qc	Assumption
3.2	Qc→Rc	∀E 1
3.3	Qc	∧E 3.1
3.4	Rc	→E 3.2, 3.3
3.5	Pc	∧E 3.1
3.6	Pc∧Rc	∧I 3.5, 3.4
3.7	∃x∈V (Px∧Rx)	∃I 3.6
4	∃x∈V (Px∧Rx)	∃E 2, 3

Check Proof Save

Great job! Don't forget to save your work by clicking "Save"! This will allow you to view this correct proof later!

Question 6 : 6 points

Using the same proof as the previous page answer the following questions.

Line #	Expression	Rule
1	∀x∈V (Qx → Rx)	Premise
2	∃x∈V (Px ∧ Qx)	Premise
3.1	Pc ∧ Qc	Assumption
3.2	Qc → Rc	∀E 1
3.3	Qc	∧E 3.1
3.4	Rc	→E 3.2, 3.3
3.5	Pc	∧E 3.1
3.6	Pc ∧ Rc	∧I 3.5, 3.4
3.7	∃x∈V (Px ∧ Rx)	∃I 3.6
4	∃x∈V (Px ∧ Rx)	∃E 2, 3

(a) (3 points) In plain English, explain why line 3.7 is true.

Since there is some value x in V that Px and Qx is true, Qx must be true on its own.
Due to the premise on line 1, for all values of x in V Qx implies Rx.
So, for some value in V, lets call it c, such that Pc and Qc is true, then Qc implies Rc.
Thus there must be some value c in V such that Pc and Rc are true

(b) (3 points) In plain English, explain why line 4 is true. What is the key difference between it and line 3.7?

Line 4 is true for the same reasons as line 3.7. The difference is that 3.7's logic follows from the assumptions on line 3.1, whereas line 4 is taking that subproof and applying the reality of the premises we're given.

Question 7 : 10 points

Prove the following. Put a screenshot of your proof below. $\forall x \in V\ \forall y \in V\ Gxy \therefore \exists x \in V\ Gxx$

Line #	Expression	Rule
1	$\forall x \in V\ \forall y \in V\ Gxy$	Premise
2	$\forall y \in V\ Gay$	$\forall$E 1
3	Gaa	$\forall$E 2
4	$\exists x \in V\ Gxx$	$\exists$I 3

Check Proof Save

Great job! Don't forget to save your work by clicking "Save"! This will allow you to view this correct proof later!

Question 8 : 10 points

Prove the following. Put a screenshot of your proof below. $\forall x \in V\ (Px \rightarrow Bx)\ ;\ \exists x \in V\ Px \therefore \exists x \in V\ Bx$

Consider what this is saying in English. For instance, suppose V is the set of all arctic animals, P is the predicate for Puffins, and B is the predicate for birds. As example P(Bob the polar bear) is false. and B(Tom the tern) is true. Then our first premise technically states "for all arctic animals, if it's a puffin, then it's a bird", however in plain English this would be stated simply as "all puffins are birds". The second premise is "there is a puffin in the arctic". How would you state the conclusion in plain English?

Line #	Expression	Rule
1	$\forall x \in V\ (Px \rightarrow Bx)$	Premise
2	$\exists x \in V\ Px$	Premise
3.1	Pa	Assumption
3.2	$Pa \rightarrow Ba$	$\forall$E 1
3.3	Ba	$\rightarrow$E 3.2, 3.1
3.4	$\exists x \in V\ Bx$	$\exists$I 3.3
4	$\exists x \in V\ Bx$	$\exists$E 2, 3

Check Proof Save

Great job! Don't forget to save your work by clicking "Save"! This will allow you to view this correct proof later!

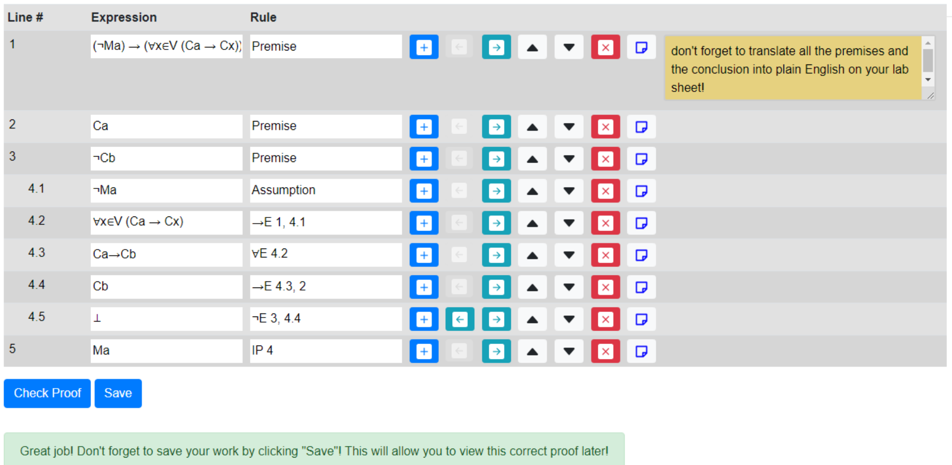
The conclusion is "there exists an arctic animal that is a bird".

Question 9 : 10 points

Prove the following. Put a screenshot of your proof below. **(¬Ma) → (∀x∈V (Ca → Cx)) ; Ca ; ¬Cb ∴ Ma**

Suppose V is the set of my relatives. M is the predicate for wearing a mask, and C is the predicate for having COVID.

a is the constant for my aunt alice, and b is the constant for my brother barry. Translate the above argument into plain English.



Line #	Expression	Rule
1	(¬Ma) → (∀x∈V (Ca → Cx))	Premise
2	Ca	Premise
3	¬Cb	Premise
4.1	¬Ma	Assumption
4.2	∀x∈V (Ca → Cx)	→E 1, 4.1
4.3	Ca→Cb	∀E 4.2
4.4	Cb	→E 4.3, 2
4.5	⊥	¬E 3, 4.4
5	Ma	IP 4

If aunt alice doesn't wear a mask, then for all relatives, if aunt alice has covid then the relatives have covid. Aunt alice has covid, but brother barry doesn't. Therefore aunt alice is wearing a mask.

Question 10: 10pts. Prove or disprove following sequent: **∃x∈V Px ∴ ∀x∈V Px**

If you believe it to be true, then screenshot a proof from the proof checker. If you don't think the conclusion logically follows from the premises, then you need to provide a counter-model. This means coming up with your own set V and predicate P where all the premises are true in your model, but the conclusion is false.

With the premise ∃x∈V Px, the conclusion ∀x∈V Px cannot be made.
This premise says that there is one element x in V such that Px is true therefore
ALL elements in V would make the predicate true. This is clearly not true.
A simple example of this would be V={0, 10} and P(x)=x<5. P(0) is true, P(10) is false.
It would be naive to assume that P is true for all values in V just because P(0) is true.