

CS270: LAB #15

Mathematical Induction

You may work in teams of ideally one or two people (three is acceptable in the event of an unscheduled absence). Unless stated otherwise, the lab is due to be submitted into Gradescope at the end of the day (11:59pm).

In order to receive credit, follow these instructions:

[a] Every team member should be discussing simultaneously the same problem – do NOT try to divvy up the labor and assign different problems to different students since the material is cumulative.

[b] Directly edit this lab PDF with your answers (extra pages can be added in the rare event you need more than the allotted space)

[c] Each lab, rotate which member has the responsibility of being the Scribe. This is the person that is typing the answers and uploading the final PDF – note that only a single copy of the filled in PDF is turned into Gradescope. Only one lab needs to be submitted for the entire team, and all members receive the same score. Make sure to use a font that your PDF editor is compatible with (otherwise you might find your answers appear as weird shapes/sizes or simply disappear entirely!)

[d] The Gradescope submission must have each answer properly tagged with the appropriate question. Moreover, every member of the team must be listed as a submitter. Although it is the Scribe which executes these actions, it is still the responsibility of the entire team to make certain this is done properly (thus it is highly recommended that the Scribe share their screen so the entire team can witness it). Answers which are improperly tagged cannot be seen by the grader and thus cannot be scored.

REMOTE ONLY

[e] Each lab, rotate which member has the responsibility of being the Recorder. This is the person who hits the Zoom Record button (once the technical permission is granted by the TA/RCF/Professor) and ensures that everyone has their camera/microphone on. They are also the member that is responsible to make sure the DrexelStream video is marked as viewable and entered into the <https://tinyurl.com/VidLinkForm> webform before 11:59pm (they should also email the rest of their team as confirmation.) Note that the video file doesn't get created/processed until after the Recorder has quit Zoom.

[f] Each lab, rotate which member has the responsibility of being the Manager. This is the person that ensures that everyone is participating equally and honestly, keeps the group on task, ensures that all team members understand a solution before going on to the next question, and presses the “hand up” button in Zoom to summon a TA or the professor (but they only do so after surveying the group to make sure everyone has the same question).

Team Name (CS pioneer): Tim Berners-Lee

Scribe name: Jackson Masterson

Recorder name: Cole Bardin

Manager name: Jeremy Mathews

Other team member (if any): Brendan Hoag

Make sure your proofs include written sentences like the ones demonstrated in the video (that is, they should all start with a sentence reading "This proof proceeds by induction on ..." etc).

Your submissions should NOT just be a bunch of scribbled algebraic expressions!

This lab is worth a total of 100pts (each of the 5 proofs is worth 20 points)

#1. Prove by Induction that $\neg (A_1 \vee A_2 \vee \dots \vee A_n) \leftrightarrow (\neg A_1 \wedge \neg A_2 \wedge \dots \wedge \neg A_n)$

This proof will proceed by induction on n .

Base Case: anchored at $n=2$

Leap: We obtain the IH by replacing the n in our claim with k ; so in other words, we get to assume that

IH: $\neg (A_1 \vee A_2 \vee \dots \vee A_n) \leftrightarrow (\neg A_1 \wedge \neg A_2 \wedge \dots \wedge \neg A_n)$

Now we just start with the new LHS and by using the base case with $n=2$, we get

$\neg ((A_1 \vee A_2 \vee \dots \vee A_n) \vee A_{n+1}) \leftrightarrow ((\neg A_1 \wedge \neg A_2 \wedge \dots \wedge \neg A_n) \wedge \neg A_{n+1})$ by treating $(A_1 \vee A_2 \vee \dots \vee A_n)$ as one A .

$\leftrightarrow (\neg A_1 \wedge \neg A_2 \wedge \dots \wedge \neg A_n \wedge \neg A_{n+1})$ by using IH

Consequently, since both the base case and leap have been demonstrated, this proves by POMI that \neg distributes over \vee for any size clause.

#2. Prove that $2^0 + 2^1 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 1$

Prove: $2^0 + 2^1 + 2^2 + \dots + 2^n = 2^{n+1} - 1$

This proof will proceed by induction on n .

Base case: anchored at $n = 1$

LHS: $2^0 + 2^1 = 3$

RHS: $2^{1+1} - 1 = 2^2 - 1 = 4 - 1 = 3$

Since LHS = RHS, this establishes the base case.

Leap: We may assume that the claim that the "shortcut" LHS=RHS is true when $n=k$. In other words, our Inductive Hypothesis is:

IH: $2^0 + 2^1 + 2^2 + \dots + 2^k = 2^{k+1} - 1$

Our goal is to show that LHS=RHS when $n=k+1$. To see this we do:

LHS: $2^0 + 2^1 + 2^2 + \dots + 2^{k+1} = (2^0 + 2^1 + 2^2 + \dots + 2^k) + 2^{k+1}$, by separating out last term,
 $= 2^{k+1} - 1 + 2^{k+1} = 2 * 2^{k+1} - 1 = 2^{k+2} - 1$, by algebra.

RHS: $2^{(k+1)+1} - 1 = 2^{k+2} - 1$, by algebra.

Since the LHS=RHS both simplified to be the same answer, this establishes the leap.

Consequently, since both the base case and leap have been demonstrated, this proves by POMI the equation holds for all n .

#3. Prove that $5 \cdot 8^n + 2$ is always a multiple of 7 (note: there is no need to use mods with this problem. use induction)

This proof proceeds by induction on n .

IH: $5 \times 8^n + 2 = 7m$

Base case: anchored at $n = 0$

LHS: $5 \times 8^0 + 2 = 5 \times 1 + 2 = 7$

RHS: choose $m=1$. so $7 \times 1 = 7$

Leap: We may assume that $5 \times 8^k + 2$ is a multiple of 7 with $5 \times 8^k + 2 = 7m$ (this is the IH). Our goal for the leap is to show that $5 \times 8^{k+1} + 2$ is a multiple of 7. (i.e. it is some integer multiplied by 7).

Observe that:

$5 \times 8^{k+1} + 2 = 5 \times 8 \times 8^k + 2 = 40 \times 8^k + 2 = (7 \times 5 + 5) \times 8^k + 2 = 7 \times (5 \times 8^k) + (5 \times 8^k + 2)$, by algebra

$= 7 \times (5 \times 8^k) + 7m$, by IH

$= 7 \times (5 \times 8^k + m)$, by algebra. Note this is $7 \times \text{integer}$ as desired, and thus the leap is established

Consequently, since both the base case and leap have been demonstrated, this proves by POMI that $5 \times 8^n + 2$ is a multiple of 7 for all $n \geq 0$

#4. Prove that factorial growth dominates exponential growth. In other words, show that $n! > 2^n$ provided $n \geq 4$

4)

Prove: $n! > 2^n$ for $n \geq 4$

This proof will proceed by induction on n

Base case: Anchored at $n = 4$

LHS: $4! = 24$

RHS: $2^4 = 16$

Since LHS = RHS, this establishes the base case.

Leap: We may assume that the claim that the "shortcut" LHS=RHS is true when $n = k$. In other words, our Inductive Hypothesis is:

IH: $k! > 2^k$

Our goal is to show that LHS=RHS when $n = k + 1$. To see this, we do:

RHS: 2^{k+1}

LHS: $(k + 1)! = (k + 1) * k!$, by algebra.

$(k + 1) * k! > (k + 1) * 2^k$, by inductive hypothesis.

$(k + 1) * 2^k > 2 * 2^k$, since $k + 1 > 2 \forall k \geq 4$.

$(k + 1)! > 2^{k+1} \forall k \geq 4$, by algebra

Since the LHS > RHS for all k greater than or equal to 4, this establishes the leap.

Consequently, since both the base case and the leap have been demonstrated, this proves by POMI the equation holds for all $n \geq 4$.

#5. If tofu comes in 5-packs and 7-packs, show that for a “sufficiently large” amount of tofu, you can buy exactly the amount that you need.

(note: if you find the minimum amount N that is guaranteed to be done, you must also show mathematically that $N-1$ **cant** be done!)

This proof will proceed by induction on BLOCKS of 5 integers at a time, starting with

Block 0: (24, 25, 26, 27, 28), Block 1: (29, 30, 31, 32, 33), etc

It is not possible to get 23 tofu packs because there are no combinations of 5 and 7 that add to 23.

Claim: Every number in block# n can be achieved using only combinations of 5 and 7 tofu packs.

Base Case: anchored at $n = 0$

We need to verify that we can get every number in Block#0. Which we can do as follows:

24 = two 5-packs, two 7-packs, 25 = five 5-packs, 26 = one 5-pack, three 7-packs, 27 = four 5-packs, one 7-pack

28 = four 7-packs

Thus everything in Block#0 is doable, and so this establishes the base case

Assuming the IH that everything in Block# k is doable, how can we show the next block is doable?

We simply include one more 5-pack to each member of Block# k .

This has the effect of adding +5 to the total number of tofu, which is exactly the numbers of the next block.

This completes the leap step.

Consequently, since both the base case and leap have been demonstrated, this proves by POMI that provided n is “sufficiently large”, it is possible to get exactly n tofu from packs of 5 and 7.