In-Lab Group Activity for Week 8: Curve Fitting and Norms

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Problem 1: Norms: 1-norm, 2-norm and ∞-norm, orthogonality Consider these four vectors from $\mathbb{R}^5$.

$$\vec{\mathbf{a}} = \begin{bmatrix} -3 \\ -2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{\mathbf{b}} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ 2 \end{bmatrix} \qquad \vec{\mathbf{c}} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{\mathbf{d}} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

a. Which of these vectors has the largest ∞-**norm**? $\vec{\mathbf{a}}$ $\vec{\mathbf{b}}$ $\vec{\mathbf{c}}$ $\vec{\mathbf{d}}$

b. Which of these vectors in $\mathbb{R}^5$ has the smallest **1-norm**? $\vec{\mathbf{a}}$ $\vec{\mathbf{b}}$ $\vec{\mathbf{c}}$ $\vec{\mathbf{d}}$

c. Do all the vectors have the same **2-norm** as $\vec{\mathbf{a}}$? **Yes** **No**

d. Which of these vectors is orthogonal to $\vec{\mathbf{a}}$? $\vec{\mathbf{a}}$ $\vec{\mathbf{b}}$ $\vec{\mathbf{c}}$ $\vec{\mathbf{d}}$
Use the 2-norm.

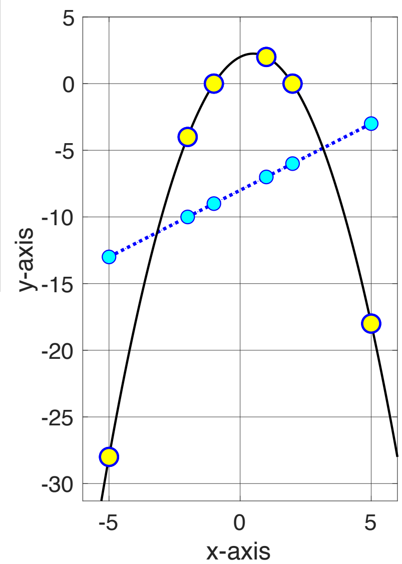
e. What is the angle θ (in degrees) between vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$? Use the 2-norm.
i. 0 **ii.** 45 **iii.** 60 **iv.** 90 **v.** 120

f. Find the **cosine** of the angle θ between vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{c}}$. **Hint**: $\cos\theta = \frac{\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}}{\|\vec{\mathbf{a}}\| \, \|\vec{\mathbf{c}}\|}$

Use the 2-norm. **i.** 0 **ii.** $\frac{1}{2}$ **iii.** $\frac{1}{3}$ **iv.** $\frac{1}{4}$ **v.** $\frac{1}{16}$

Problem 2: Curve Fitting: Six data points have these x-values and y-values: $\vec{\mathbf{x}} = \begin{bmatrix} -5 \\ -2 \\ -1 \\ 1 \\ 2 \\ 5 \end{bmatrix}$ and $\vec{\mathbf{y}} = \begin{bmatrix} -28 \\ -4 \\ 0 \\ 2 \\ 0 \\ -18 \end{bmatrix}$.

The plot shows these data points and both the best-fit **line** and the best-fit **parabola**.



a. Record the design matrix D you would use to find the best-fit **line**: $y = \beta_0 + \beta_1\, x \qquad D = [\vec{\mathbf{1}} \quad \vec{\mathbf{x}}] =$ $\begin{bmatrix} 1 & -5 \\ 1 & -2 \\ 1 & -1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$

b. The product $D^T D$ for your design matrix is: $D^T D = \begin{bmatrix} c & 0 \\ 0 & 60 \end{bmatrix}$.

What is the value of c? $c =$ 6

c. Find the inverse of $D^T D$ and place it in the adjacent box.

The inverse of $D^T D$ is:

$$(D^T D)^{-1} = \begin{bmatrix} 1/6 & 0 \\ 0 & 1/60 \end{bmatrix}$$

d. Solve the normal equation $(\, D^T D\,)\, \vec{\boldsymbol{\beta}} = D^T \vec{\mathbf{y}}$ to find the vector $\vec{\boldsymbol{\beta}} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$ with the best-fit parameters for the line.

Here's $D^T \vec{\mathbf{y}} = \begin{bmatrix} -48 \\ 60 \end{bmatrix}$ for free! Use it!

Parameter vector $\vec{\boldsymbol{\beta}} = (\, D^T D)^{-1}\, D^T \vec{\mathbf{y}}$.

$\vec{\beta} = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$

e. It can be shown that the best-fit **line** gives the estimates

$\vec{\mathbf{y}}_{est} = D\vec{\boldsymbol{\beta}} = \begin{bmatrix} -13 \\ -10 \\ -9 \\ -7 \\ -6 \\ -15 \end{bmatrix}$. **i.** Find the **error** vector: $\vec{\mathbf{y}} - \vec{\mathbf{y}}_{\text{est}}$. **ii.** Then give the missing value k for the $RMSE = \sqrt{\dfrac{\vec{\mathbf{e}}^{\,T}\, \vec{\mathbf{e}}}{N}}$

i. Error vector: $\begin{bmatrix} -15 \\ 6 \\ 9 \\ 9 \\ 6 \\ 21 \end{bmatrix}$

ii. $\text{RMSE} = \sqrt{k}$ where $k =$ 114

f. Let's start over and try a **parabolic** fit. Record the new design matrix D you would use to find the best-fit **parabola**: $y = \beta_0 + \beta_1\, x + \beta_2\, x^2$

Parabolic design matrix D:

$$\begin{bmatrix} 1 & -5 & 25 \\ 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 5 & 25 \end{bmatrix}$$

Best parabola is:

$\vec{\boldsymbol{\beta}} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Not required.