In-Lab Group Activity for Week 6: Linear Transformations, Block Matrices

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Question 1: Matrix of a Linear Transformation

Consider a linear transformation T that maps vectors from $\mathbb{R}^3$ to vectors in $\mathbb{R}^4$.
We are given the action of T on the standard basis for $\mathbb{R}^3$.

$$T(\vec{\mathbf{e}}_1) = \begin{bmatrix}1\\0\\0\\1\end{bmatrix},\quad T(\vec{\mathbf{e}}_2) = \begin{bmatrix}0\\1\\0\\1\end{bmatrix},\quad T(\vec{\mathbf{e}}_3) = \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \text{ where } \vec{\mathbf{e}}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\ \vec{\mathbf{e}}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \text{ and } \vec{\mathbf{e}}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

a. What are the dimensions of the matrix A for T in the standard basis?

i. 3×3 **ii.** 4×3 **iii.** 3×4 **iv.** 4×4

b. Find the matrix A for T in the standard basis such that for any vector $\vec{\mathbf{x}}$ in $\mathbb{R}^3$, $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$

$$A = [\,T(\vec{\mathbf{e}}_1) \quad T(\vec{\mathbf{e}}_2) \quad T(\vec{\mathbf{e}}_3)] = \begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{1} & \mathbf{1}\end{bmatrix}$$

c. Find the image of the vector $\vec{\mathbf{x}} = \begin{bmatrix}1\\2\\-3\end{bmatrix}$ under the linear transformation using your matrix A.

$$A\vec{\mathbf{x}} = \begin{bmatrix}\mathbf{1}\\ \mathbf{2}\\ \mathbf{-3}\\ \mathbf{0}\end{bmatrix}$$

d. Row reduce your matrix A to find the rank of T. (**Hint**: Same as the rank of its matrix A).

$$A = [\,T(\vec{\mathbf{e}}_1) \quad T(\vec{\mathbf{e}}_2) \quad T(\vec{\mathbf{e}}_3)] = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 1 & 1\end{bmatrix} \leftrightarrow \begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}$$

So, the rank of A (and T) is: 3

e. What is the **nullity** of T? **Hint**: Same as the number of free variables. 0

Question 2: Matrix of a new Linear Transformation S with Missing Information

Consider a new linear transformation S that also maps vectors from $\mathbb{R}^3$ to vectors in $\mathbb{R}^4$. Unfortunately, we do not (yet) know its action on a complete **basis**, but only for the first two standard basis vectors.

$$S(\vec{\mathbf{e}}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad S(\vec{\mathbf{e}}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{where } \vec{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

We are told nothing about its action on the third basis vector $\vec{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. It could be anything.

a. Select the form which includes every matrix A which might represent S, the unknown values of which will depend on $S(\vec{\mathbf{e}}_3)$. We can denote an arbitrary vector in $\mathbb{R}^4$ as $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

i. $A = \begin{bmatrix} a & 0 & a \\ b & 1 & b \\ d & 0 & c \\ d & 1 & d \end{bmatrix}$ **ii.** $A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \\ 1 & 1 & d \end{bmatrix}$ **iii.** $A = \begin{bmatrix} 1 & 1 & a \\ 1 & 0 & b \\ 1 & 0 & c \\ 1 & 0 & d \end{bmatrix}$ **iv.** $A = \begin{bmatrix} 1 & c & 1 \\ a & 0 & 1 \\ b & 0 & 1 \\ 1 & d & 1 \end{bmatrix}$

b. Find the image of the vector $\vec{\mathbf{a}} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$ under the linear transformation.

i. First write $\vec{\mathbf{a}}$ as a linear combination of the vectors $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$, so that $\vec{\mathbf{a}} =$ 5*e1+3*e2

ii. Now apply the linearity of S to find $S(\vec{\mathbf{a}})$:

$$S(\vec{\mathbf{a}}) = \mathbf{5} * \boldsymbol{S(e1)} + \mathbf{3} * \boldsymbol{S(e2)} = \mathbf{5} * \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} + \mathbf{3} * \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{5} \\ \mathbf{3} \\ \mathbf{0} \\ \mathbf{8} \end{bmatrix}$$

c. What is $S(\vec{\mathbf{0}})$?

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

d. For which of these can you **not** find its image under *S* without knowing the additional information for $S(\vec{\mathbf{e}}_3)$?

i. $\begin{bmatrix}1\\0\\1\end{bmatrix}$ **ii.** $\begin{bmatrix}5\\3\\2\end{bmatrix}$ **iii.** $\begin{bmatrix}1\\2\\3\end{bmatrix}$ **iv.** $\begin{bmatrix}1\\1\\1\end{bmatrix}$ **v.** We can't find any of these.

Wait! We were just handed some new info about *S*! Only use this new info in the parts below!

We are now given $S(\vec{\mathbf{b}}) = \begin{bmatrix}1\\1\\1\\3\end{bmatrix}$ where $\vec{\mathbf{b}} = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Note that $\vec{\mathbf{b}}$ is not one of the standard basis vectors!

e. Find the image of the third standard basis vector $\vec{\mathbf{e}}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ under *S*.

i. First you must write $\vec{\mathbf{e}}_3$ in terms of the given vectors $\{\vec{\mathbf{e}}_1, \ \vec{\mathbf{e}}_2, \ \vec{\mathbf{b}}\}$. You can do this in your head.

$$\vec{\mathbf{e}}_3 = x_1\vec{\mathbf{e}}_1 + x_2\vec{\mathbf{e}}_2 + x_3\vec{\mathbf{b}} = \boxed{x1 * \begin{bmatrix}1\\0\\0\end{bmatrix} + x2 * \begin{bmatrix}0\\1\\0\end{bmatrix} + x3 * \begin{bmatrix}1\\1\\1\end{bmatrix}}$$

ii. Now use your expansion (and linearity) to find:

$$S(\vec{\mathbf{e}}_3) = \boxed{\begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{1}\\\mathbf{1}\end{bmatrix}}$$

f. Give the matrix *A* for *S* in the standard basis using the new info. Now you know all three columns!

$$A = [\,S(\vec{\mathbf{e}}_1) \quad S(\vec{\mathbf{e}}_2) \quad S(\vec{\mathbf{e}}_3)] = \begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{1} & \mathbf{1}\end{bmatrix}$$

g. Using your matrix *A*, confirm $S(\vec{\mathbf{b}}) = \begin{bmatrix}1\\1\\1\\3\end{bmatrix}$ where we recall $\vec{\mathbf{b}} = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Just multiply *A* times $\vec{\mathbf{b}}$.

$$S(\vec{\mathbf{b}}) = A\vec{\mathbf{b}} = \begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{1} & \mathbf{1}\end{bmatrix} \begin{bmatrix}\mathbf{1}\\\mathbf{1}\\\mathbf{1}\end{bmatrix} = \begin{bmatrix}\mathbf{1}\\\mathbf{1}\\\mathbf{1}\\\mathbf{3}\end{bmatrix}$$

Question 3: Block Matrices: Find the inverse of the 2×2 block matrix A where a, b, c and d are symbolic variables. Note block forms are treated in Lecture 7, but you can do it now.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \left[\begin{array}{cccc:cc} 0 & 1 & 0 & 0 & a & b \\ 1 & 0 & 0 & 0 & b & a \\ 0 & 0 & 0 & 1 & c & d \\ 0 & 0 & 1 & 0 & d & c \\ \hdashline 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

This is a 2×2 block decomposition of the 6×6 matrix A.

a. Write out each of the block matrices A_{ij} below. Some blocks are partially filled for free.

$$A_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} a & b \\ b & a \\ c & d \\ d & c \end{bmatrix}, \qquad A_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_{22} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

b. Which of these block matrices are their own inverse? That's unusual but was designed to make your work easier. **Hint**: Just check if $M^2 = I$.

i. Only A_{11} **ii.** Only A_{22} **iii.** Both A_{11} and A_{22} **iii.** Neither A_{11} nor A_{22}

c. In lectures, we will derive a formula for the inverse of a block upper triangular matrix.

Recall if $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ is invertible with A_{11} and A_{22} both square, then the diagonal blocks A_{11} and A_{22} are also invertible; and the inverse of the entire matrix A is:

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

where we are <u>**also**</u> using B to denote the inverse. Compute B_{12} by explicitly performing the matrix products:

$$B_{12} = -A_{11}^{-1} A_{12} A_{22}^{-1} = -\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & a \\ c & d \\ d & c \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{b} & \boldsymbol{a} \\ \boldsymbol{c} & \boldsymbol{d} \\ \boldsymbol{d} & \boldsymbol{c} \end{bmatrix}$$

d. Notice that **magically,** for this special matrix, $B_{12} = A_{12}$. Thus, the inverse of the entire matrix A is:

i. $-A$ **ii.** A **iii.** the identity matrix I_6 **iv.** None of these