In-Lab Group Activity for Week 9: Orthogonal Projection and Curve Fitting II

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Problem 1: Orthogonal Projection onto a Line / Orthogonal Decomposition

a. i. Find the component of $\vec{\mathbf{y}} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ parallel to the vector $\vec{\mathbf{a}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Denote this component as $\hat{\mathbf{y}}$.

Hint: Use the formula: $\hat{\mathbf{y}} = \frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{a}}}{\vec{\mathbf{a}} \cdot \vec{\mathbf{a}}} \vec{\mathbf{a}}$ **PS:** You can see this answer in your head.

$$\hat{\mathbf{y}} = \frac{5}{1} * \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

ii. Now give $\vec{\mathbf{z}}$, the component of $\vec{\mathbf{y}}$ <u>perpendicular</u> to the line through $\vec{\mathbf{a}}$. As always, $\vec{\mathbf{y}} = \hat{\mathbf{y}} + \vec{\mathbf{z}}$.

$$\vec{\mathbf{z}} = \vec{\mathbf{y}} - \hat{\mathbf{y}} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

iii. Find the dot product of the two components. $\hat{\mathbf{y}} \cdot \vec{\mathbf{z}} =$ **0*3+0*4+5*0 = 0**

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b. i. Find the component of $\vec{\mathbf{y}} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ parallel to the vector $\vec{\mathbf{b}} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$. Denote this component as $\hat{\mathbf{y}}$.

Hint: Use the formula: $\hat{\mathbf{y}} = \frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{b}}}{\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}} \vec{\mathbf{b}}$

$$\hat{\mathbf{y}} = \frac{\mathbf{3 * 3 + 4 * 4}}{\mathbf{3 * 3 + 4 * 4}} * \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

ii. Now give $\vec{\mathbf{z}}$, the component of $\vec{\mathbf{y}}$ <u>perpendicular</u> to the line through $\vec{\mathbf{b}}$. As always, $\vec{\mathbf{y}} = \hat{\mathbf{y}} + \vec{\mathbf{z}}$.

$$\vec{\mathbf{z}} = \vec{\mathbf{y}} - \hat{\mathbf{y}} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

iii. Find the dot product of the two components. $\hat{\mathbf{y}} \cdot \vec{\mathbf{z}} = \mathbf{3 * 0 + 4 * 0 + 0 * 5 = 0}$

Problem 2: Orthogonal Projection onto a Plane / Orthogonal Decomposition

a. i. Find the component of $\vec{\mathbf{y}} = \begin{bmatrix}3\\4\\5\end{bmatrix}$ parallel to the **plane** spanned by the vectors $\vec{\mathbf{a}}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\vec{\mathbf{a}}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$.

Denote this component as $\hat{\mathbf{y}}$. As always, $\vec{\mathbf{y}} = \hat{\mathbf{y}} + \vec{\mathbf{z}}$.

Hint: Use the formula: $\hat{\mathbf{y}} = \frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{a}}_1}{\vec{\mathbf{a}}_1 \cdot \vec{\mathbf{a}}_1}\vec{\mathbf{a}}_1 + \frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{a}}_2}{\vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_2}\vec{\mathbf{a}}_2$ **PS**: You can also see this answer in your head.

$$\hat{\mathbf{y}} = \mathbf{3} * \begin{bmatrix}1\\0\\0\end{bmatrix} + \mathbf{4} * \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}3\\4\\0\end{bmatrix}$$

ii. Now give $\vec{\mathbf{z}}$, the component of $\vec{\mathbf{y}}$ perpendicular to the **plane** spanned by $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$.

$$\vec{\mathbf{z}} = \vec{\mathbf{y}} - \hat{\mathbf{y}} = \begin{bmatrix}3\\4\\5\end{bmatrix} - \begin{bmatrix}3\\4\\0\end{bmatrix} = \begin{bmatrix}0\\0\\5\end{bmatrix}$$

iii. Find the dot product of the two components $\hat{\mathbf{y}}$ and $\vec{\mathbf{z}}$. $\hat{\mathbf{y}} \cdot \vec{\mathbf{z}} = \mathbf{3} * \mathbf{0} + \mathbf{4} * \mathbf{0} + \mathbf{0} * \mathbf{5} = \mathbf{0}$

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b. i. Find the component of $\vec{\mathbf{y}} = \begin{bmatrix}3\\4\\5\end{bmatrix}$ parallel to the **plane** spanned by $\vec{\mathbf{b}}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\vec{\mathbf{b}}_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$.

Denote this component as $\hat{\mathbf{y}}$. As always, $\vec{\mathbf{y}} = \hat{\mathbf{y}} + \vec{\mathbf{z}}$.

Hint: Use the formula: $\hat{\mathbf{y}} = \frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{b}}_1}{\vec{\mathbf{b}}_1 \cdot \vec{\mathbf{b}}_1}\vec{\mathbf{b}}_1 + \frac{\vec{\mathbf{y}} \cdot \vec{\mathbf{b}}_2}{\mathbf{b}_2 \cdot \mathbf{b}_2}\vec{\mathbf{b}}_2$

$$\hat{\mathbf{y}} = \frac{\mathbf{7}}{\mathbf{2}} * \begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{\mathbf{1}}{\mathbf{2}} * \begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}3\\4\\0\end{bmatrix}$$

ii. Now give $\vec{\mathbf{z}}$, the component of $\vec{\mathbf{y}}$ perpendicular to the **plane** spanned by $\vec{\mathbf{b}}_1$ and $\vec{\mathbf{b}}_2$.

$$\vec{\mathbf{z}} = \vec{\mathbf{y}} - \hat{\mathbf{y}} = \begin{bmatrix}3\\4\\5\end{bmatrix} - \begin{bmatrix}3\\4\\0\end{bmatrix} = \begin{bmatrix}0\\0\\5\end{bmatrix}$$

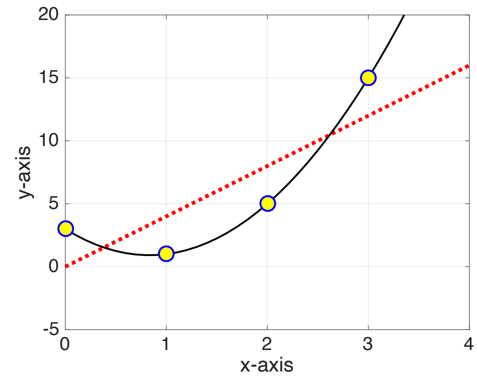
iii. Find the dot product of the two components. $\hat{\mathbf{y}} \cdot \vec{\mathbf{z}} = \mathbf{3} * \mathbf{0} + \mathbf{4} * \mathbf{0} + \mathbf{5} * \mathbf{0} = \mathbf{0}$

iv. Why are the two components $\vec{\mathbf{y}}$ and $\vec{\mathbf{z}}$ the same in both parts **a** and **b**?

Because the planes spanned by $\{\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2\}$ and $\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2\}$ are: _ _ _

i. orthogonal **ii.** the same **iii.** do not intersect

Problem 3: Curve Fitting: Four data points were observed (0,3), (1,1) (2, 5), (3, 15). Thus, the *x*-values and *y*-values are: $\vec{\mathbf{x}} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{\mathbf{y}} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 15 \end{bmatrix}$. The plot shows these data points and both the best-fit **line** and the best-fit **parabola**.



a. Record the design matrix D you would use to find the best-fit **line**:

$y = \beta_0 + \beta_1 \, x$

Design matrix D for best-fit line:

$$\begin{bmatrix} 1 & \boxed{0} \\ 1 & \boxed{1} \\ 1 & \boxed{2} \\ 1 & \boxed{3} \end{bmatrix}$$

b. The product $D^T D$ for your design matrix should be: $D^T D = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}$.

Find the inverse of $D^T D$ and place it in the adjacent box.

The inverse of $D^T D$.

$$\begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix}$$

c. Solve the normal equation $(D^T D)\, \vec{\boldsymbol{\beta}} = D^T \vec{\mathbf{y}}$ to find the vector $\vec{\boldsymbol{\beta}} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$ with the best-fit parameters for the line. **Hint**: $D^T \vec{\mathbf{y}} = \begin{bmatrix} 24 \\ 56 \end{bmatrix}$

Parameter vector $\vec{\boldsymbol{\beta}} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

d. It can be shown that the best-fit **line** gives the estimates $\vec{\mathbf{y}}_{est} = \begin{bmatrix} 0 \\ 4 \\ 8 \\ 12 \end{bmatrix}$. Find the **error** vector: $\vec{\mathbf{e}} = \vec{\mathbf{y}} - \vec{\mathbf{y}}_{est}$, then give the **root mean square error**. Below, $N = 4$ since there are four data points.

$$\vec{\mathbf{e}} = \begin{bmatrix} \boxed{3} \\ \boxed{-3} \\ \boxed{-3} \\ \boxed{3} \end{bmatrix} \qquad RMSE = \sqrt{\frac{\vec{\mathbf{e}} \cdot \vec{\mathbf{e}}}{N}} = 3$$

e. Let's start over and try a **parabolic** fit. Record the new design matrix D you would use to find the best-fit **parabola**: $y = \beta_0 + \beta_1 \, x + \beta_2 \, x^2$

$$D = \begin{bmatrix} 1 & 0 & \boxed{0} \\ 1 & 1 & \boxed{1} \\ 1 & 2 & \boxed{4} \\ 1 & 3 & \boxed{9} \end{bmatrix}$$

f. Nice! The parabola fits perfectly. The RMSE for the parabolic fit is:

i. 0   **ii.** 1   **iii.** 4   **iii.** $\sqrt{3}$

For free, the best parabola is:

$\vec{\boldsymbol{\beta}} = \begin{bmatrix} 3 \\ -5 \\ 3 \end{bmatrix}$. Not required.