In-Lab Group Activity for Week 7: Eigenvalues and Eigenvectors

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Problem 1: Diagonalization - The eigenvectors for the matrix $A = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$ can be chosen to be

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

a. Find the eigenvalue λ_1 that matches the eigenvector $\vec{\mathbf{v}}_1$. **Hint**: Apply $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$ to each eigenvector:

$$\lambda_1 = 0$$

b. Find the eigenvalue λ_2 that matches the eigenvector $\vec{\mathbf{v}}_2$.

$$\lambda_2 = 10$$

c. Give the **matrix** V **of eigenvectors** V:

$$V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and the matching **matrix of eigenvalues** D**:**

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}$$

d. Find the determinant of the matrix V of eigenvectors, and then give its inverse V^{-1}.

$|V| =$ **-2**

$$V^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

e. Diagonalization and Powers: Since $AV = A[\vec{\mathbf{v}}_1 \quad \vec{\mathbf{v}}_2] = [\lambda_1\vec{\mathbf{v}}_1 \quad \lambda_2\vec{\mathbf{v}}_2] = [\vec{\mathbf{v}}_1 \quad \vec{\mathbf{v}}_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = VD$, we can also write $A = VDV^{-1}$ and hence any power of A such as A^n can be found using $A^n = VD^nV^{-1}$. Clearly $10^6 = 1{,}000{,}000$ is one million. Find (by hand) an exact expression for $A^6 = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}^6$ using the result $A^6 = VD^6V^{-1}$. Show all details!

$$A^6 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} * \begin{bmatrix} 0 & 0 \\ 0 & 10^6 \end{bmatrix} * \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$A^6 = \begin{bmatrix} 0 & 10^6 \\ 0 & -10^6 \end{bmatrix} * \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$A^6 = \begin{bmatrix} 500000 & -500000 \\ -500000 & 500000 \end{bmatrix}$$

Problem 2: The Dog Ate My Homework!

The family dog chewed on an old homework assignment you are trying to review before an exam. The bottom row of the matrix A is missing, so let's represent it by symbols: $A = \begin{bmatrix} 5 & -5 \\ a & b \end{bmatrix}$

CHOMP!

The chewed-up homework still shows the eigenvectors for A are $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

a. Find the eigenvalues λ_1 and λ_2 for each eigenvector. Both are integers.

Hint: Apply the fundamental identity $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$ to each eigenvector:

$\lambda_1 = 0$

Similarly, for the second eigenvector we find:

$\lambda_2 = 10$

b. Combine your two equations into a linear system for the missing values a and b and then give the complete matrix.

System:

a+b=10

a-b=-10

Solution (for *a* and *b*):

a=-5

b=5

$A = \begin{bmatrix} 5 & -5 \\ -5 & -5 \end{bmatrix}$

Does this matrix look familiar?

c. Above you obtained a system with two linear equations for the two unknowns a and b. Let's collect them into a vector $\vec{\mathbf{x}} = \begin{bmatrix} a \\ b \end{bmatrix}$ and write your system in the matrix form $M\vec{\mathbf{x}} = \vec{\mathbf{d}}$. Give the matrix M and the vector $\vec{\mathbf{d}}$.

$M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\vec{\mathbf{d}} = \begin{bmatrix} 0 \\ -10 \end{bmatrix}$

Problem 3: Magic Matrices in MATLAB: Let $M(n)$ denote the magic matrix returned by MATLAB where n is a positive integer. Here are a few.

$$M(2) = \cancel{\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}}, \quad M(3) = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}, \quad M(4) = \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}$$

Exception for $n = 2$

Every number from 1 to n^2 appears exactly once in the magic matrix $M(n)$. Further, every column sum, row sum and diagonal sum is the same. These properties do **not** all hold for magic(2) which is an **exception**, so we will toss it out for the purposes of this exploration! Using MATLAB's `eig` command, explore the following properties for magic matrices `magic(n)` with n varying from 3 to 10. Remember we are excluding the case $n = 2$. Test the code below with different values of n in the range from 3 to 10.

```
clear, clc
n = 8 % size of magic matrix
M = magic(n)
[v, d] = eig(M) % eigenvectors and eigenvalues
```

a. The largest positive **eigenvalue** for magic(n) is:

i. n **ii.** n^2 **iii.** $\frac{n}{2}(n^2+1)$ **iv.** $\frac{n}{2}(n+1)$

b. The **eigenvector** corresponding to the largest eigenvalue of $M(n)$ can be chosen to be:

i. $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ **ii.** $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ **iii.** $\begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}$ **iv.** $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Each column vector above has n components.

c. Explore! Excluding the largest eigenvalue λ_{max}, if $\lambda < \lambda_{max}$ is an eigenvalue then so is $-\lambda$.

Check for all n in the range from 3 to 10. If false, give n for your exception.

i. True **ii.** False when n = _ _

d. Explore! What is the multiplicity of the eigenvalue $\lambda = 0$ for the case of $M(8)$?

i. 1 **ii.** 2 **iii.** 3 **iv.** 4 **v.** 5

e. What is the **nullity** (dimension of the null space) for the case of $M(8)$?

i. 1 **ii.** 2 **iii.** 3 **iv.** 4 **v.** 5