Lab 5: Method of Undetermined Coefficients **ENGR-232 Summer 2022**

In this lab, we study a differential equation with complex roots and force it with a sinusoidal function. This will help you master the **Method of Undetermined Coefficients**. We will first look at the homogeneous DE, and then add various multiples of the forcing term to illustrate linearity and superposition.

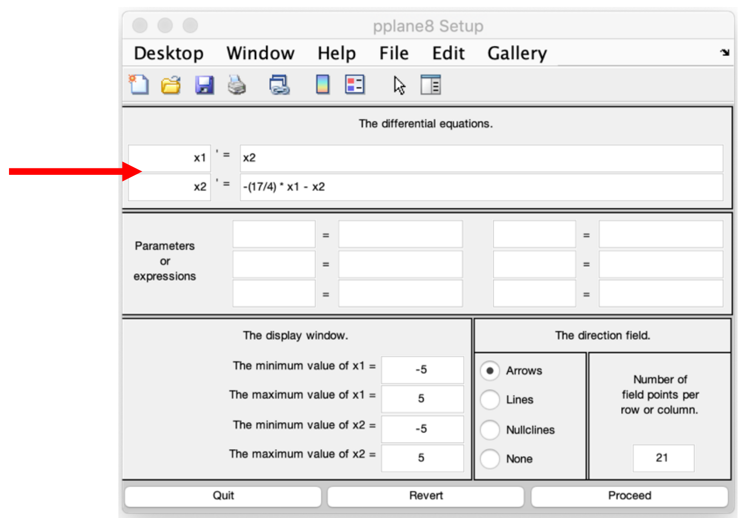
Part A: Homogeneous DE

Question 1: Consider the initial value problem:

DE: $4y'' + 4y' + 17y = 0$ **IC:** $y(0) = 4 \quad y'(0) = 0$

Note this 2nd-order equation is linear and homogeneous. We will add a forcing term in part B.
Using pplane8, create a phase diagram, for this DE.

We decompose this 2nd ODE into two 1st ODE to use with pplane8.
These two equations are: $x_1' = x_2 \qquad x_2' = -(17/4) * x_1 - x_2$



a. Proceed and show the **nullclines**. In MATLAB, nullclines are **Contours**.

Under **Solutions** on the pplane8 Display window, select **Show nullclines**. All solutions that cross a nullcline will do so at a local max or min. There are two nullclines, one for each of the variables x_1 and x_2.

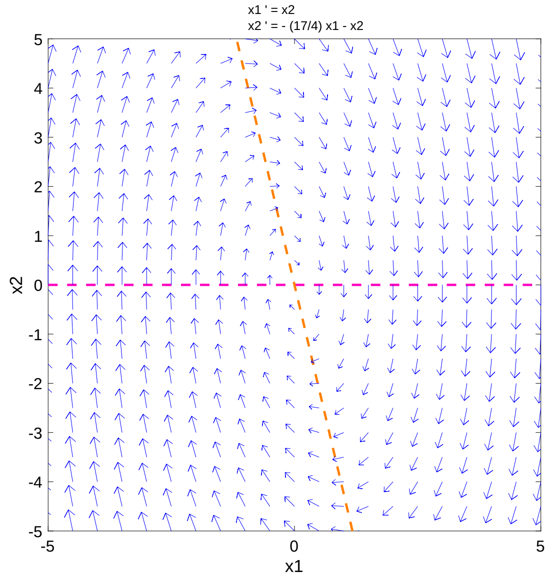
The respective nullclines are where the derivative of either variable is zero.

Nullcline for x_1: $\frac{d}{dt}x_1 = x_2 = 0$ So this nullcline is the horizontal axis $x_2 = 0$

Nullcline for x_2: $\frac{d}{dt}x_2 = -\left(\frac{17}{4}\right) * x1 - x2 = 0$ So this nullcline is the line $x_2 = -\left(\frac{17}{4}\right) * x1$

Make the axes equal so circles look like circles and not ellipses.

So far, your phase plane should look like this.



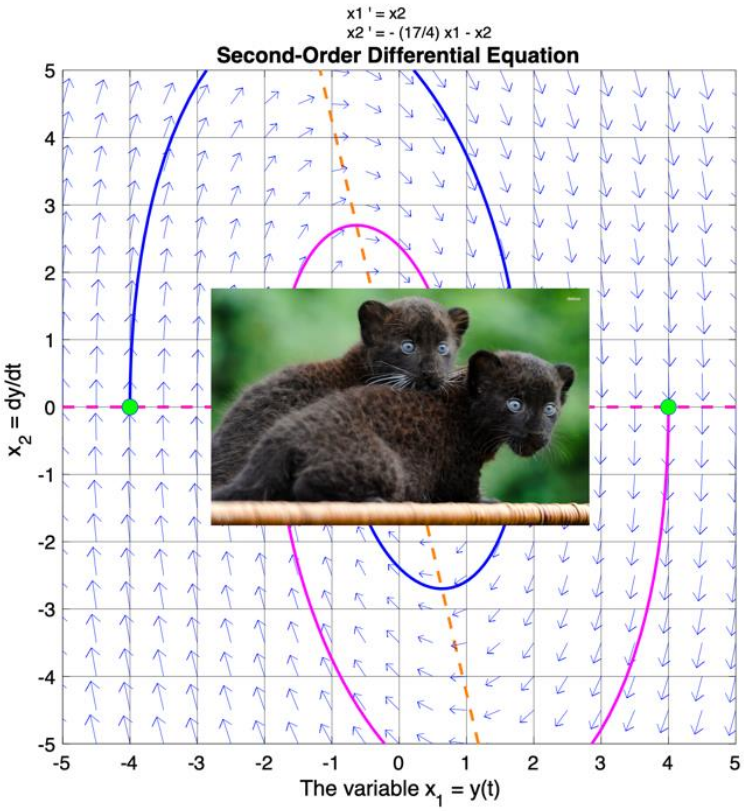
b. Next let's add some solution curves. First, use **Options** and set the **Solution direction** to **Forward** only. Next, using **Keyboard input**, find the solutions for each of the initial points below. Show the first in **magenta** and second in blue. Increase their **LineWidth** to 3.

IC: $(4, 0)$, $(-4, 0)$

Q1: Paste your completed phase plot into the answer template.

Be sure the solution for (4,0) is in **magenta** and that for (–4,0) is in **blue**. Be sure the nullclines are visible. Replace this sample graph, which has some of the graph hidden with an unrelated image of panther cubs.

Sample



Question 2: Find the exact solution to this same initial value problem using **dsolve**.

DE: $4y'' + 4y' + 17y = 0$ **IC:** $y(0) = 4 \quad y'(0) = 0$

Tips: Declare **y(t)** as symbolic first using **`syms`**. Then within **dsolve**, use **==** to denote equality in equations.

Q2: Copy the exact solution into the answer template including the code used to find it.

```
%% Q2 - Solve the Homogeneous IVP exactly using dsolve.
syms y(t)
Dy = diff(y,t);    D2y = diff(y,t, t);



% The exact solution is: y(t) = ⋯
```

Question 3: Use the help to review `matlabFunction()`.
Use it to define the functions Y(t) and DY(t), its derivative. These represent the exact solution and its derivative.
Hint: If you let sol denote the exact solution from dsolve,

```
sol = dsolve( ... )  % Arguments hidden.
```

then you can make that answer into a function using:
`Y = matlabFunction(sol)`

> **Alert:** High risk for the MATLAB Exam!

Now define DY using `Y`, the `diff()` command and another `matlabFunction()`.
Next evaluate $Y(0)$ and $DY(0)$ to verify both your functions are working.
Of course, we expect $Y(0) = 4$ and $DY(0) = 0$, since $Y(t)$ is the solution to the IVP.

Q3: Now evaluate both functions at time *t* =1. Paste in your values for Y(1) and DY(1) below.

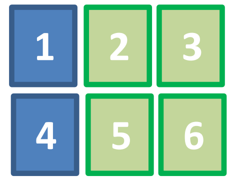
Q3: Answer
>> $Y(1)$ = _ _ _ _ _

>> $DY(1)$ = _ _ _ _ _

This demonstrates you can evaluate the solution and its derivative anywhere!

Q4-5: Using your exact functions Y(t) and DY(t), and subplot commands, create **component plots** for Y and DY and their phase plot. Do **not** use ode45 or any other approximate solver. You already have the exact answers! Here is the start of the first of the three plots to get you started. We'll cut the plot into six tiles (instead of 4) to allow more room for the phase plot on the right. We'll draw Y(t) in **magenta** in **tile 1** and its derivative in **red** in **tile 4**. The phase plot on the right will take all four of the tiles 2, 3, 5 and 6.

```
figure
subplot(2,3,1) % tile 1
time = 0: 0.01 : 10;
plot(time, Y(time), 'm', 'LineWidth',3)
grid on
hold on
```

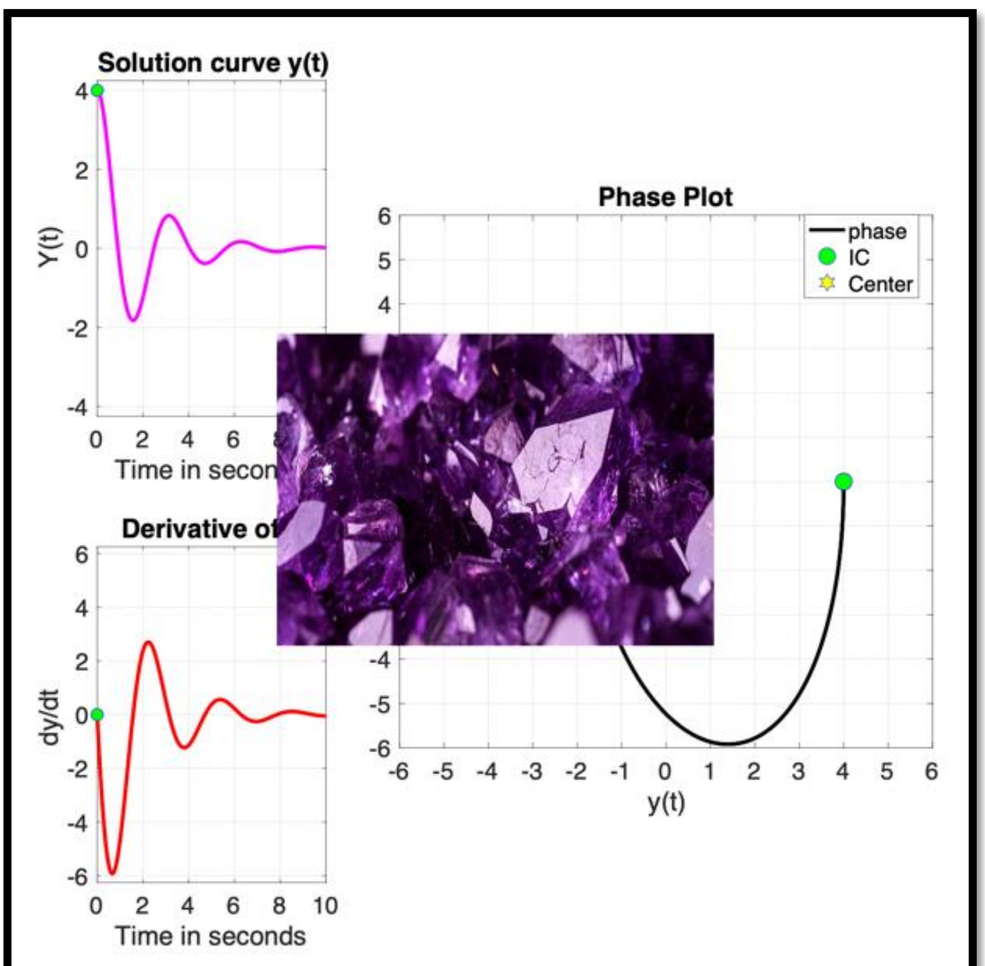


Now add a plot of DY(t) in tile 4 (bottom left) and add the phase plot on the right using both tiles 2, 3, 5 and 6, using [2, 3, 5, 6] in your code. Label the axes, give appropriate titles, set the 'FontSize' to 20 and draw DY in **red** and the phase curve in **black**. Mark the starting point (4,0) in the phase plot. Add a legend to the phase plot as shown. Mark the initial point with a blue circle filled in **green**. Mark the center of the circle point as a blue hexagram filled in **yellow**. Here's some sample code.

```
>> plot(0, 0, 'bh', 'MarkerSize', 14, 'MarkerFaceColor', 'yellow')  % Center Point
```

Questions 4-5: Paste your completed component and phase plots into the answer template for credit. Replace this sample graph, which has some of the image hidden by amethyst crystals.

Sample



Part B: Non-Homogeneous DE: Method of Undetermined Coefficients

Consider the non-homogeneous initial value problem:

DE: $4y'' + 4y' + 17y = f(t)$ **IC:** $y(0) = 4 \quad y'(0) = 0$

Questions 6-7: Complete the table below, to find the unique solution matching the initial conditions and the given forcing function $f(t)$ which varies in each row. Simply use `dsolve()`.
The homogeneous part is given, so you just need to add on the particular terms right after the trailing + sign.
If you do not see the blue terms in the solution returned by **dsolve**, try **simplify()**.

Forcing function $f(t)$	Guess for Particular Solution	Unique Solution y(t)
a. $f(t) = 17 + 289t$	$At + B$	$y(t) = e^{-\frac{t}{2}} \cdot \left[7\cos 2t - \frac{27}{4}\sin 2t\right] + \ldots$
b. $f(t) = 100\, e^{-2t}$	$A\, e^{-2t}$	$y(t) = e^{-\frac{t}{2}} \cdot [4\sin 2t] + \ldots$
c. $f(t) = 260\cos 2t$	$A\cos 2t + B\sin 2t$	$y(t) = -e^{-\frac{t}{2}} \cdot [32\sin 2t] + \ldots$
d. $f(t) = 16e^{-\frac{t}{2}}\cos 2t + 32\, e^{-\frac{t}{2}}\sin 2t$	**Bump up!** $t\, e^{-t/2}[A\cos 2t + B\sin 2t]$	$y(t) = e^{-\frac{t}{2}} \cdot [4\cos 2t + 2\sin 2t] + \ldots$

*** Grader will randomly pick two to check for correctness.**

Question 8: Let's focus on part **(c)** above. This time, we will not rely on **dsolve**, but will implement the **Method of Undetermined Coefficients** from scratch! Based on Lecture 4-5, you should expect a particular solution of the form below, as this would be closed under differentiation.

$$y_p(t) = A \cdot \cos(2t) + B \cdot \sin(2t)$$

Since the functions $\cos(2t)$ and $\sin(2t)$ are **not** solutions of the homogeneous equation so we will **NOT** need to **bump it up**! Let's verify our solution has the above form:

Enter the above DE and find the exact solution by substituting our guess for $y_p(t)$ into the original DE, which now includes the forcing term $f(t) = 260\cos 2t$.

Let's solve for A and B as follows. See next page.

```
%% QUESTION 8: The Method of Undetermined Coefficients from SCRATCH
clc
syms A B t
syms y(t) a b
Dy = diff(y,t);    D2y = diff(y,t, t);
f = 260 * cos(2*t)
DE = 4* D2y + 4*Dy + 17*y - f == 0       % non-homogeneous differential equation

Y = A * cos(2*t)    +    B * sin(2*t)      % our guess
plug_it_in = subs(DE, y, Y)               % Plug our guess into the DE
eqn = collect(plug_it_in, [cos(2*t), sin(2*t)])  % arrange by similar terms
```

This gives:

```
(8*B + A)*cos(2*t) + (B - 8*A)*sin(2*t) == 260*cos(2*t)
```

The coefficients of both terms must match on each side:

```
cos(2t):    A + 8B = 260

sin(2t):   -8A + B = 0
```

so $B = 8A$

```
% MATLAB can find the undetermined coefficients for us
equations = coeffs( lhs(eqn), [cos(t), sin(t)] )
variables = [A, B]

% find and display the undetermined coefficients
[A, B] = solve(equations, variables)
```

Plugging $B = 8A$ into the first equation we find: $A = \frac{260}{65} = 4, \quad B = 32$

Thus, the method of undetermined coefficients leads to the particular solution:

$$y_p(t) = 4 \cdot \cos(2t) \ + \ 32 \cdot \sin(2t)$$

Question 8: Now that you have found the particular solution, record its <u>derivative</u> for one point.

$y_p'(t) =$ _____

Questions 9-10: Graph the Non-Homogeneous Solution

Tip: You can clone most of your plotting code from Questions 4-5 to create the plot for this part. Of course, both the functions $y(t)$ and $y'(t)$ have changed. You will earn the remaining two points by pasting in your completed graph for these questions.

$$\textbf{DE: } 4y'' + 4y' + 17y = \mathbf{260\cos(2t)} \qquad \textbf{IC: } \ y(0) = 4 \quad y'(0) = 0$$

Create a new figure using:

```
fig3 = figure(3); % Start with a new figure using ***SIX TILES*** in a 2x3 grid
subplot(2,3,1)    % We'll plot the solution Y(t) in tile #1 in magenta
```

We will use the same 2x3 grid we used earlier in Question 4-5.

a. In the first tile, plot the solution $y(t)$ using your solution from **dsolve** found earlier. Use **magenta** and a linewidth of 3. Be sure your forcing function is $f(t) = 260\cos 2t$.

b. In the fourth tile, plot the solution for its derivative $y'(t)$ using your solution from **dsolve** found earlier and the **diff** command. Use **red** and a linewidth of 3.

c. In the combined tiles [2, 3, 5, 6], plot the phase plot with $y(t)$ along the horizontal axis and its derivative $y'(t)$ along the vertical axis. Use **black** and a linewidth of 3.

d. You can see the phase plot approaches a **limiting ellipse** as the transitory parts of the solution die out for large values of t. Add hat limiting ellipse now. It is drawn using just the particular solution:

$$y_p(t) = 4 \cdot \cos(2t) \ + \ 32 \cdot \sin(2t)$$

and its derivative:

$$y_p'(t) = -8sin \ + \ 64 \cdot \cos(2t)$$

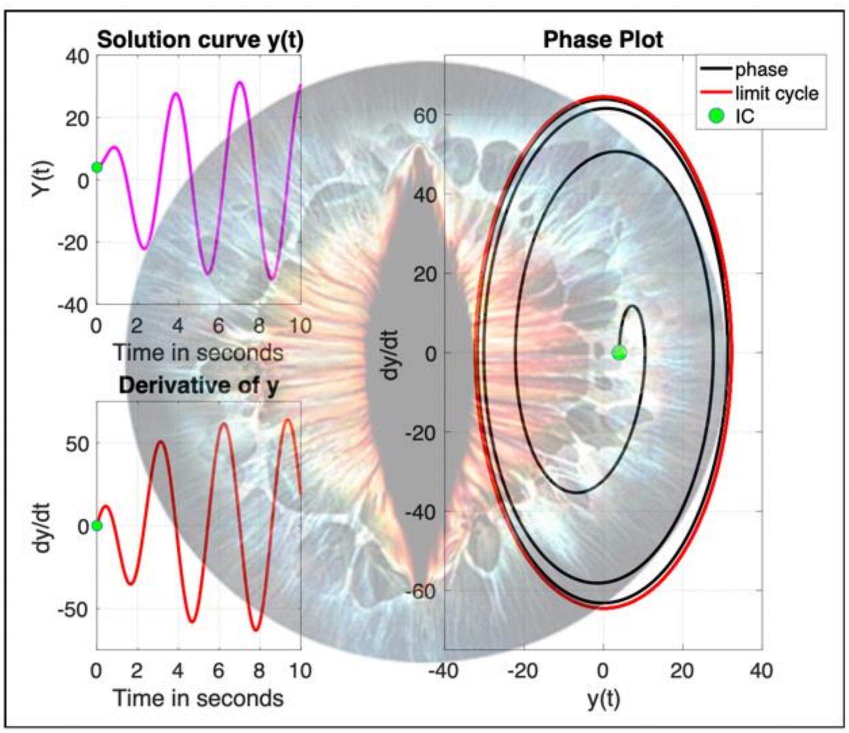
e. Add **legend** as shown in the sample.

Adjust all the axes to get a good view of your plots using the axis command.

Questions: 9-10: Paste your completed multiplot in the answer template.

Replace the image below with your completed multiplot. Some info has been obscured in the sample with a dragon's eye! Your submission must not include the dragon eye.

Sample



```
Grader will award 2 points as follows.
i. All three graphs are correct
ii. The red limit cycle and the legend are included in the phase plot.
```

Ready to Submit?

Be sure all ten questions are answered. When your lab is complete, be sure to submit three files:

1. Your **completed Answer Template** as a PDF file
2. A copy of your **MATLAB Live Script**
3. A **PDF** copy of your **MATLAB Live Script** (Save-Export to PDF…)

The due date is the day after your lab section by **11:59pm** to receive full credit. You have one more day, to submit the lab (but with a small penalty), and then the window closes for good and your grade will be zero.