Lab 8 by Cole Bardin   Section 61

Water Clocks: The Polyvascular Clepsydra and the Laplace Transform

Questions 1-2

```
clc, clear, close all

% Question 1
syms s t Y
y0 = 1 % initial condition
```

y0 = 1

```
Y1 = s*Y - y0 % transform of the derivative
```

Y1 = $Y\,s - 1$

```


LTofDE = Y1 == -1*Y
```

LTofDE = $Y\,s - 1 = -Y$

```
Sol = solve(LTofDE, Y)
```

Sol =

$$\frac{1}{s+1}$$

```
Y = matlabFunction(Sol); Y(s);
Y = partfrac(Y(s)) % express solution in partial fraction form
```

Y =

$$\frac{1}{s+1}$$

```
sol = ilaplace(Sol,s,t);
y = matlabFunction(sol); y(t)
```

ans = e^{-t}

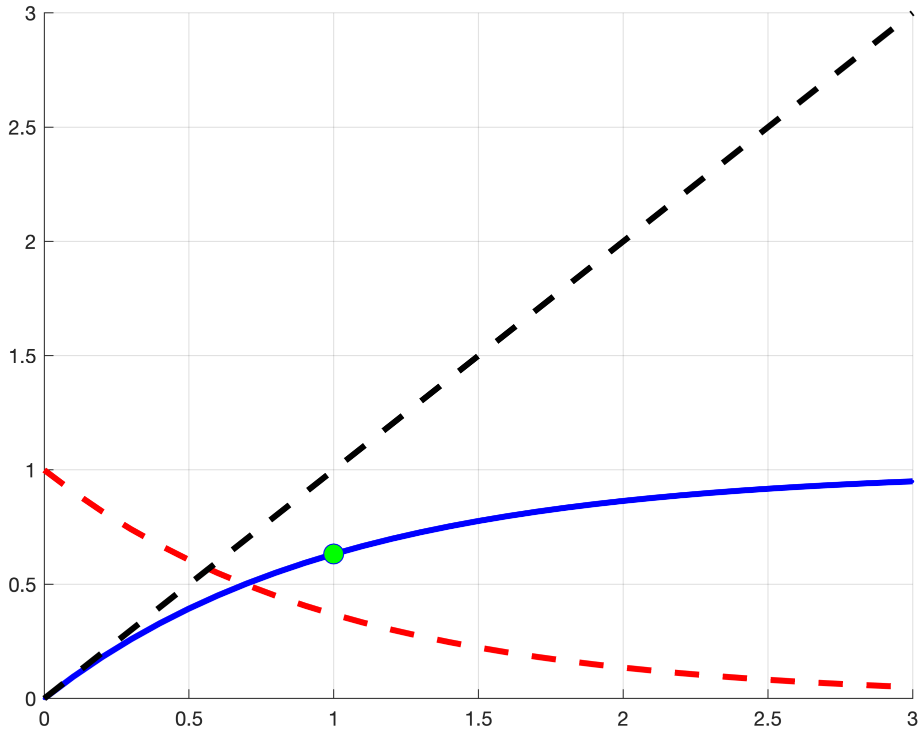
```
% Question 2
H = matlabFunction(1-y(t)); H(1)
```

ans = 0.6321

```


time=0:0.1:3;
hold on
grid on
plot(time, y(time), 'r--', "LineWidth",3);
```

```
plot(time, 1-y(time), 'b.-', "LineWidth",3)
plot(time, time, 'k--', "LineWidth",3)
plot(1, H(1), 'bo', "MarkerFaceColor", 'g', "MarkerSize",10)
```



Questions 5-8

```
clc, clear, close all

% Question 5
syms y1(t) y2(t) f1(t) f2(t) F(s);
x = [y1; y2];
A= [-1, 0; 1, -1]; x0 = [1;1];
DE = diff(x,t) == A*x
```

DE(t) =

$$\begin{pmatrix} \frac{\partial}{\partial t}\, y_1(t) = -y_1(t) \\ \frac{\partial}{\partial t}\, y_2(t) = y_1(t) - y_2(t) \end{pmatrix}$$

```
sol = dsolve(DE, x(0)==x0)
```

```
sol = struct with fields:
    y2: exp(-t) + t*exp(-t)
    y1: exp(-t)
```

```
% Question 6
f1 = int(sol.y1, 0, t)
```

f1 = $1 - e^{-t}$

```
f2 = simplify(int(sol.y2, 0, t))
```

f2 = $2 - e^{-t}\,(t+1) - e^{-t}$

```
% Question 7
XL = inv((s*eye(2) - A))*x0
```

XL =

$$\begin{pmatrix} \frac{1}{s+1} \\ \frac{1}{s+1} + \frac{1}{(s+1)^2} \end{pmatrix}$$

```
xl = ilaplace(XL)
```

xl =

$$\begin{pmatrix} e^{-t} \\ e^{-t} + t\,e^{-t} \end{pmatrix}$$

```
% Question 8
syms s
A=[-1 0; 1 -1] % system matrix
```

```
A = 2×2
    -1     0
     1    -1
```

```
x0 = [1;1] % initial conditions
```

```
x0 = 2×1
     1
     1
```

```
X =  inv((s*eye(2) - A))*x0 % find X here using inv()
```

X =

$$\begin{pmatrix} \frac{1}{s+1} \\ \frac{1}{s+1} + \frac{1}{(s+1)^2} \end{pmatrix}$$

```
F = X / s % find F here. Integration is division by s.
```

F =

$$\begin{pmatrix} \frac{1}{s\,(s+1)} \\ \frac{\frac{1}{s+1}+\frac{1}{(s+1)^2}}{s} \end{pmatrix}$$

```
f = ilaplace(F) % find f here using ilaplace.
```

f =

$$\begin{pmatrix} 1-e^{-t} \\ 2-t\,e^{-t}-2\,e^{-t} \end{pmatrix}$$

Question 9

```
clc, clear, close all
syms s;

A = [-1, 0, 0, 0; 1, -1, 0, 0; 0, 1, -1, 0; 0, 0, 1, -1]
```

```
A = 4×4
    -1     0     0     0
     1    -1     0     0
     0     1    -1     0
     0     0     1    -1
```

```
x0= [1;1;1;1]
```

```
x0 = 4×1
     1
     1
     1
     1
```

```
X = inv(s*eye(4) - A) * x0
```

X =

$$\begin{pmatrix} \frac{1}{s+1} \\ \frac{1}{s+1}+\frac{1}{(s+1)^2} \\ \frac{1}{s+1}+\frac{1}{(s+1)^2}+\frac{1}{(s+1)^3} \\ \frac{1}{s+1}+\frac{1}{(s+1)^2}+\frac{1}{(s+1)^3}+\frac{1}{(s+1)^4} \end{pmatrix}$$

```
F = X/s
```

F =

$$\begin{pmatrix} \dfrac{1}{s\,(s+1)} \\ \dfrac{\dfrac{1}{s+1}+\dfrac{1}{(s+1)^2}}{s} \\ \dfrac{\dfrac{1}{s+1}+\dfrac{1}{(s+1)^2}+\dfrac{1}{(s+1)^3}}{s} \\ \dfrac{\dfrac{1}{s+1}+\dfrac{1}{(s+1)^2}+\dfrac{1}{(s+1)^3}+\dfrac{1}{(s+1)^4}}{s} \end{pmatrix}$$

```
f = ilaplace(F) % Cumulative outflows in the time domain
```

f =

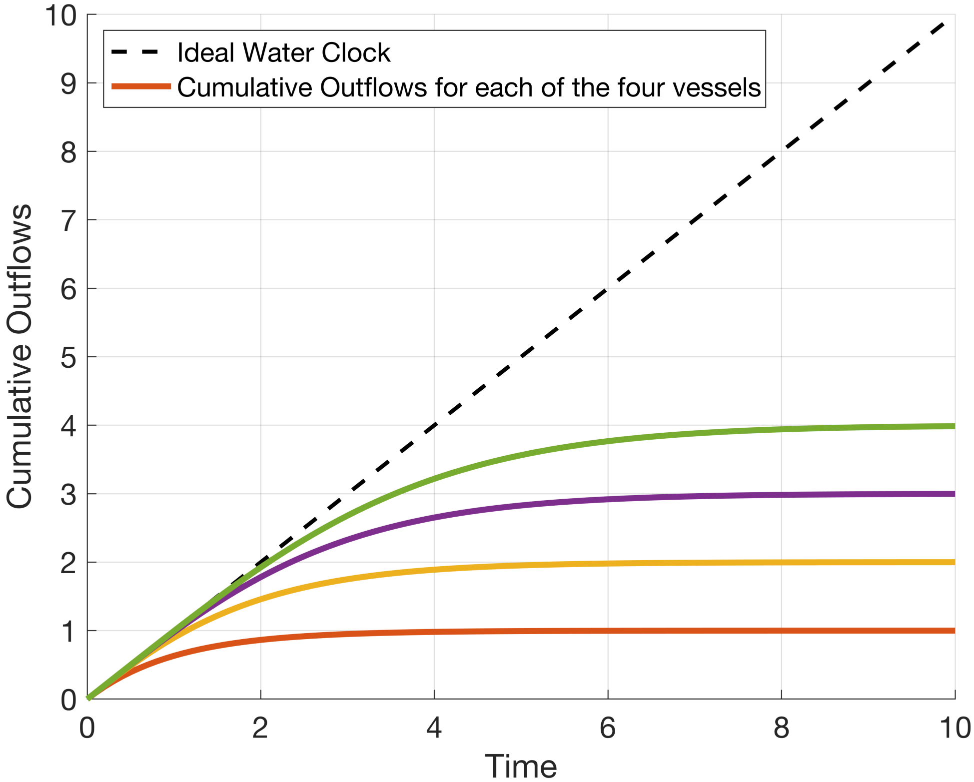
$$\begin{pmatrix} 1-e^{-t} \\ 2-t\,e^{-t}-2\,e^{-t} \\ 3-2\,t\,e^{-t}-\dfrac{t^2\,e^{-t}}{2}-3\,e^{-t} \\ 4-3\,t\,e^{-t}-t^2\,e^{-t}-\dfrac{t^3\,e^{-t}}{6}-4\,e^{-t} \end{pmatrix}$$

```
f = matlabFunction(f) % convert f to a function
```

```
f = function_handle with value:
    @(t)[-exp(-t)+1.0;exp(-t).*-2.0-t.*exp(-t)+2.0;exp(-t).*-3.0-t.*exp(-t).*2.0-(t.^2.*exp(-t))./2.0+3.0;
```

```
time = 0:0.1:10;
Y = f(time);

grid on
hold on
axis([0, 10, 0, 10])
set(gca, "FontSize", 15)
ideal = plot(time, time, 'k--', "LineWidth",2);
y1 = plot(time, Y(1,:), "LineWidth", 3);
y2 = plot(time, Y(2,:), "LineWidth", 3);
y3 = plot(time, Y(3,:), "LineWidth", 3);
y4 = plot(time, Y(4,:), "LineWidth", 3);
xlabel("Time")
ylabel("Cumulative Outflows")
legend([ideal, y1], ["Ideal Water Clock", "Cumulative Outflows for each of the four ve
```




```
f_of_1 = f(1)
```

```
f_of_1 = 4×1
    0.6321
    0.8964
    0.9767
    0.9957
```

Question 10

```
clc, clear, close all

syms s;

N = 12 % Number of vessels, excluding the collection chamber
```

```
N = 12
```

```
subdiagonal = 1; diagonal = -1; superdiagonal = 0
```

```
superdiagonal = 0
```

```
A = full(gallery('tridiag', N, subdiagonal, diagonal, superdiagonal))
```

```
A = 12×12
    -1     0     0     0     0     0     0     0     0     0     0     0
     1    -1     0     0     0     0     0     0     0     0     0     0
```

```
     0     1    -1     0     0     0     0     0     0     0     0     0
     0     0     1    -1     0     0     0     0     0     0     0     0
     0     0     0     1    -1     0     0     0     0     0     0     0
     0     0     0     0     1    -1     0     0     0     0     0     0
     0     0     0     0     0     1    -1     0     0     0     0     0
     0     0     0     0     0     0     1    -1     0     0     0     0
     0     0     0     0     0     0     0     1    -1     0     0     0
     0     0     0     0     0     0     0     0     1    -1     0     0
     ⋮
```

```
x0 = ones(N,1) % Initial conditions
```

```
x0 = 12×1
     1
     1
     1
     1
     1
     1
     1
     1
     1
     1
     ⋮
```

```
X = inv(s*eye(N) - A) * x0
```

```
X =
```

$$
\begin{pmatrix}
\frac{1}{s+1} \\
\frac{1}{s+1} + \sigma_1 \\
\frac{1}{s+1} + \sigma_1 + \sigma_2 \\
\frac{1}{s+1} + \sigma_1 + \sigma_2 + \sigma_3 \\
\frac{1}{s+1} + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 \\
\frac{1}{s+1} + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 \\
\frac{1}{s+1} + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 \\
\frac{1}{s+1} + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 + \sigma_7 \\
\frac{1}{s+1} + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 + \sigma_7 + \sigma_8 \\
\frac{1}{s+1} + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 + \sigma_7 + \sigma_8 + \sigma_9 \\
\frac{1}{s+1} + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 + \sigma_7 + \sigma_8 + \sigma_9 + \sigma_{10} \\
\frac{1}{s+1} + \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 + \sigma_7 + \sigma_8 + \sigma_9 + \sigma_{10} + \frac{1}{(s+1)^{12}}
\end{pmatrix}
$$

where

$$\sigma_1 = \frac{1}{(s+1)^2}$$

$$\sigma_2 = \frac{1}{(s+1)^3}$$

$$\sigma_3 = \frac{1}{(s+1)^4}$$

$$\sigma_4 = \frac{1}{(s+1)^5}$$

$$\sigma_5 = \frac{1}{(s+1)^6}$$

$$\sigma_6 = \frac{1}{(s+1)^7}$$

$$\sigma_7 = \frac{1}{(s+1)^8}$$

$$\sigma_8 = \frac{1}{\ }$$

```
F = X/s
```

```
F =
```

$$
\begin{pmatrix}
\frac{1}{s\,(s+1)} \\[2ex]
\frac{\frac{1}{s+1}+\sigma_1}{s} \\[2ex]
\frac{\frac{1}{s+1}+\sigma_1+\sigma_2}{s} \\[2ex]
\frac{\frac{1}{s+1}+\sigma_1+\sigma_2+\sigma_3}{s} \\[2ex]
\frac{\frac{1}{s+1}+\sigma_1+\sigma_2+\sigma_3+\sigma_4}{s} \\[2ex]
\frac{\frac{1}{s+1}+\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5}{s} \\[2ex]
\frac{\frac{1}{s+1}+\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6}{s} \\[2ex]
\frac{\frac{1}{s+1}+\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_7}{s} \\[2ex]
\frac{\frac{1}{s+1}+\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_7+\sigma_8}{s} \\[2ex]
\frac{\frac{1}{s+1}+\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_7+\sigma_8+\sigma_9}{s} \\[2ex]
\frac{\frac{1}{s+1}+\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_7+\sigma_8+\sigma_9+\sigma_{10}}{s} \\[2ex]
\frac{\frac{1}{s+1}+\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_7+\sigma_8+\sigma_9+\sigma_{10}+\frac{1}{(s+1)^{12}}}{s}
\end{pmatrix}
$$

where

$$\sigma_1 = \frac{1}{(s+1)^2}$$

$$\sigma_2 = \frac{1}{(s+1)^3}$$

$$\sigma_3 = \frac{1}{(s+1)^4}$$

$$\sigma_4 = \frac{1}{(s+1)^5}$$

```
f = ilaplace(F) % Cumulative outflows in the time domain
```

```
f =
```

$$
\left(\begin{array}{c}
1-\mathrm{e}^{-t}\\
2-t\,\mathrm{e}^{-t}-2\,\mathrm{e}^{-t}\\
3-2\,t\,\mathrm{e}^{-t}-\frac{\sigma_{11}}{2}-3\,\mathrm{e}^{-t}\\
4-3\,t\,\mathrm{e}^{-t}-\sigma_{11}-\frac{\sigma_6}{6}-4\,\mathrm{e}^{-t}\\
5-4\,t\,\mathrm{e}^{-t}-\frac{\sigma_{10}}{2}-\frac{\sigma_6}{3}-\frac{\sigma_3}{24}-5\,\mathrm{e}^{-t}\\
6-5\,t\,\mathrm{e}^{-t}-2\,t^2\,\mathrm{e}^{-t}-\frac{\sigma_6}{2}-\frac{\sigma_3}{12}-\frac{\sigma_2}{120}-6\,\mathrm{e}^{-t}\\
7-6\,t\,\mathrm{e}^{-t}-\frac{\sigma_9}{2}-\frac{2\,t^3\,\mathrm{e}^{-t}}{3}-\frac{\sigma_3}{8}-\frac{\sigma_2}{60}-\frac{\sigma_1}{720}-7\,\mathrm{e}^{-t}\\
8-7\,t\,\mathrm{e}^{-t}-\sigma_{10}-\frac{5\,t^3\,\mathrm{e}^{-t}}{6}-\frac{\sigma_3}{6}-\frac{\sigma_2}{40}-\frac{\sigma_1}{360}-\frac{\sigma_4}{5040}-8\,\mathrm{e}^{-t}\\
9-8\,t\,\mathrm{e}^{-t}-\frac{7\,t^2\,\mathrm{e}^{-t}}{2}-\sigma_6-\frac{5\,t^4\,\mathrm{e}^{-t}}{24}-\frac{\sigma_2}{30}-\frac{\sigma_1}{240}-\frac{\sigma_4}{2520}-\frac{\sigma_5}{40320}-9\,\mathrm{e}^{-t}\\
10-9\,t\,\mathrm{e}^{-t}-4\,t^2\,\mathrm{e}^{-t}-\frac{7\,t^3\,\mathrm{e}^{-t}}{6}-\frac{\sigma_3}{4}-\frac{\sigma_2}{24}-\frac{\sigma_1}{180}-\frac{\sigma_4}{1680}-\frac{\sigma_5}{20160}-\frac{\sigma_7}{362880}-10\,\mathrm{e}^{-t}\\
11-10\,t\,\mathrm{e}^{-t}-\frac{9\,t^2\,\mathrm{e}^{-t}}{2}-\frac{4\,t^3\,\mathrm{e}^{-t}}{3}-\frac{7\,t^4\,\mathrm{e}^{-t}}{24}-\frac{\sigma_2}{20}-\frac{\sigma_1}{144}-\frac{\sigma_4}{1260}-\frac{\sigma_5}{13440}-\frac{\sigma_7}{181440}-\frac{\sigma_8}{3628800}-11\,\mathrm{e}^{-t}\\
12-11\,t\,\mathrm{e}^{-t}-\sigma_9-\frac{3\,t^3\,\mathrm{e}^{-t}}{2}-\frac{\sigma_3}{3}-\frac{7\,t^5\,\mathrm{e}^{-t}}{120}-\frac{\sigma_1}{120}-\frac{\sigma_4}{1008}-\frac{\sigma_5}{10080}-\frac{\sigma_7}{120960}-\frac{\sigma_8}{1814400}-\frac{t^{11}\,\mathrm{e}^{-t}}{39916800}-12
\end{array}\right.
$$

where

$\sigma_1 = t^6\,\mathrm{e}^{-t}$

$\sigma_2 = t^5\,\mathrm{e}^{-t}$

$\sigma_3 = t^4\,\mathrm{e}^{-t}$

$\sigma_4 = t^7\,\mathrm{e}^{-t}$

$\sigma_5 = t^8\,\mathrm{e}^{-t}$

$\sigma_6 = t^3\,\mathrm{e}^{-t}$

$\sigma_7 = t^9\,\mathrm{e}^{-t}$

$\sigma_8 = t^{10}\,\mathrm{e}^{-t}$

$\sigma_9 = 5\,t^2\,\mathrm{e}^{-t}$

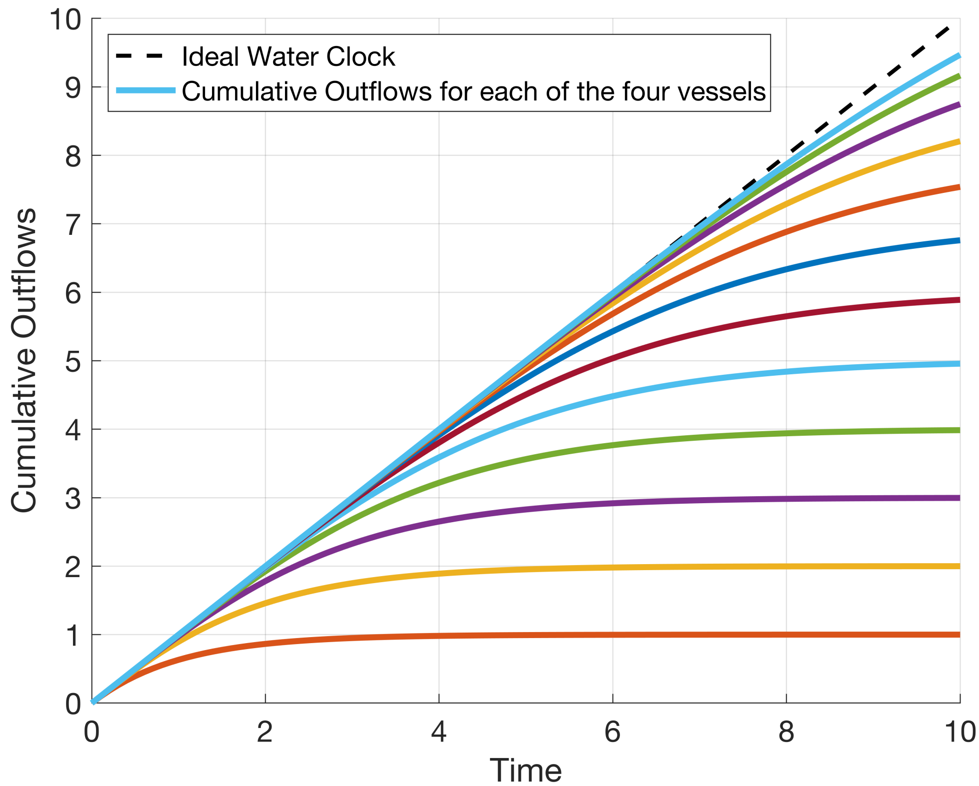
$\sigma_{10} = 3\,t^2\,\mathrm{e}^{-t}$

```
f = matlabFunction(f) % convert f to a function
```

```
f = function_handle with value:
    @(t)[-exp(-t)+1.0;exp(-t).*-2.0-t.*exp(-t)+2.0;exp(-t).*-3.0-t.*exp(-t).*2.0-(t.^2.*exp(-t))./2.0+3.0;
```

```
time = 0:0.1:10;
Y = f(time);

grid on
hold on
axis([0, 10, 0, 10])
set(gca, "FontSize", 15)
ideal = plot(time, time, 'k--', "LineWidth",2);
for i=1:12
    yh = plot(time, Y(i,:), "LineWidth", 3);
end
xlabel("Time")
ylabel("Cumulative Outflows")
legend([ideal, yh], ["Ideal Water Clock", "Cumulative Outflows for each of the four ves
```




```
f_of_1 = f(1)
```

```
f_of_1 = 12×1
    0.6321
    0.8964
```

```
0.9767
0.9957
0.9993
0.9999
1.0000
1.0000
1.0000
1.0000
 ⋮
```