**W9 InLab Activity:** **Name: Cole Bardin**

*first* *last*

**Problem 1: Solve this DE using Eigenvalues and Eigenvectors**

**Summer 2022**

Solve the given initial value problem: **DE**: $\vec{\mathbf{x}}\,'(t) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{\mathbf{x}}(t)$, **IC:** $\vec{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

**a.** First find the eigenvalues of the matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

The characteristic equation is: $\det(A - \lambda I) =$ $\boxed{\lambda^2 + 4\lambda + 3}$

So, the eigenvalues are $\lambda_1 =$ $\boxed{-1}$ and $\lambda_2 =$ $\boxed{-3}$

**b.** Is the system stable or unstable? **Stable**

**c.** Find the corresponding eigenvectors.

**i.** Case $\lambda_1 = -1$: Show work then fill on the boxes.

$$\begin{bmatrix} -2+1 & 1 \\ 1 & -2+1 \end{bmatrix} * \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$v_1 - v_2 = 0$$
$$v_1 = v_2$$
$$v = v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

**ii.** Case $\lambda = -3$: Show work then fill on the boxes.

$$\begin{bmatrix} -2+3 & 1 \\ 1 & -2+3 \end{bmatrix} * \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$v_1 + v_2 = 0$$
$$v_1 = -v_2$$
$$v = v_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The matching eigenvectors are: $\vec{\mathbf{x}}_1 = \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ and $\vec{\mathbf{x}}_2 = \boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

**d.** Write out the general solution using constants $c_1$ and $c_2$. Recall our **EEE** mnemonic. Each **fundamental solution** is a scalar function of time involving the <u>E</u>xponential of an <u>E</u>igenvalue, multiplied by the matching <u>E</u>igenvector.

$$\vec{\mathbf{x}}(t) = \boxed{c_1 e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-3t}\begin{bmatrix}-1\\1\end{bmatrix}}$$

**e.** Evaluate at time 0 to match the initial conditions: $\vec{\mathbf{x}}(0) = \begin{bmatrix}0\\4\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix}$

$c_1 =$ **2** and $c_2 =$ **2**

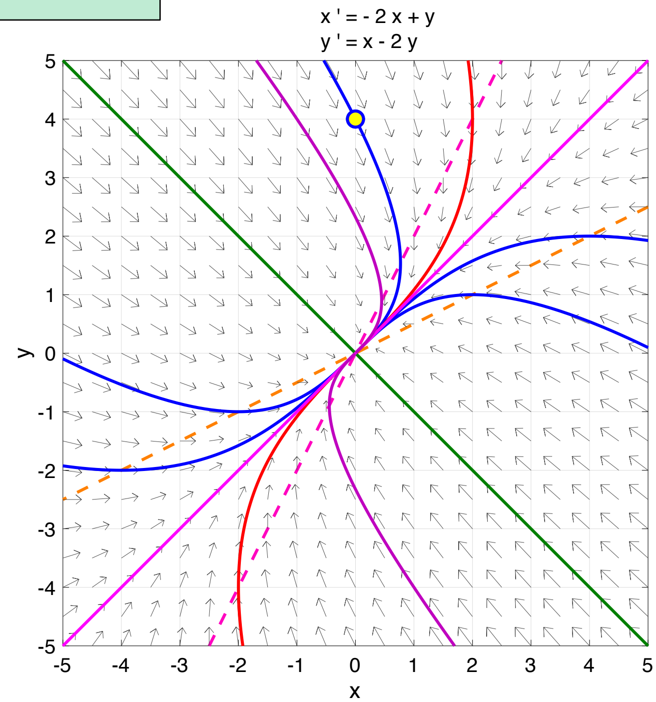
The solution matching the initial conditions is:

$$\vec{\mathbf{x}}(t) = \boxed{2e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + 2e^{-3t}\begin{bmatrix}-1\\1\end{bmatrix}}$$

**f.** Classify the system using the table below.

The system is a: **Asymptotically stable improper node**

| Roots (eigenvalues) | Type of Critical Point | Stability |
|---|---|---|
| distinct, positive | improper node | unstable |
| distinct, negative | improper node | asymptotically stable |
| opposite signs | saddle point | unstable |
| equal, positive | proper node or improper node | unstable |
| equal, negative | proper node or improper node | asymptotically stable |
| complex-valued: | | |
| positive real part | spiral point | unstable |
| negative real part | spiral point | asymptotically stable |
| pure imaginary | center | stable |




## Problem 2: Solve the same Problem using the Laplace Transform (in Matrix Form)!

Solve the given initial value problem: **DE:** $\vec{\mathbf{x}}'(t) = A\,\vec{\mathbf{x}}(t)$ where $A = \begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}$ **IC:** $\vec{\mathbf{x}}(0) = \begin{bmatrix}0\\4\end{bmatrix}$

**Tip:** The solution in the time-domain is $\vec{\mathbf{x}}(t)$. Denote the solution is the s-domain as $\vec{\mathbf{X}}(s)$ or just $\vec{\mathbf{X}}$ for short.

**a.** Recalling that the Laplace Transform is linear, what is the Laplace transform of the **RHS**?
**Hint:** the matrix $A$ can be treated just like a constant.

$$\mathcal{L}\{A\,\vec{\mathbf{x}}(t)\} = \boxed{\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix} * X(s)}$$

**b.** What is the transform of the **LHS**? Use the fundamental derivative identity!

$$\mathcal{L}\{\vec{\mathbf{x}}'(t)\} = \boxed{sX(s) - \begin{bmatrix}0\\4\end{bmatrix}}$$

Combining the two sides and solving for $\vec{\mathbf{X}}$ you can show the solution is: $\vec{\mathbf{X}}(\boldsymbol{s}) = (sI - A)^{-1}\,\vec{\mathbf{x}}(0)$

**c.** Write out the matrix $(sI - A)$ where $I$ denotes the $2 \times 2$ identity matrix.

$$(sI - A) = \begin{bmatrix} 2+s & -1 \\ -1 & 2+s \end{bmatrix}$$

**d.** Give the **determinant** of this matrix. It will be a quadratic polynomial in the s variable. Factor it!

$$|sI - A| = s^2 + 4s + 3$$

**Tip:** Compare your answer to the characteristic polynomial in Problem 1.

**e.** Find the **inverse** of the matrix $(sI - A)$. Don't forget to divide by the determinant!

$$(sI - A)^{-1} = \frac{1}{(s+3)(s+1)} * \begin{bmatrix} 2+s & 1 \\ 1 & 2+s \end{bmatrix}$$

**f.** Now solve for $\vec{\mathbf{X}}$ using the formula given previously: $\vec{\mathbf{X}} = (sI - A)^{-1}\,\vec{\mathbf{x}}(0)$

$$\vec{\mathbf{X}}(\boldsymbol{s}) = (sI - A)^{-1}\,\vec{\mathbf{x}}(0) = \frac{1}{(s+3)(s+1)} *$$

**g.** Solve in the time domain.

**i.** The partial fraction for the top component of $\vec{\mathbf{X}}(\boldsymbol{s}) = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$ is: $X_1 = \frac{4}{s^2+4s+3} = \frac{2}{s+1} - \frac{2}{s+3}$

Find the top component $x_1(t)$ in the **time domain**.

$$x_1(t) = 2e^{-t} - 2e^{-3t}$$

**ii.** The partial fraction for the bottom component of $\vec{\mathbf{X}}(\boldsymbol{s}) = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$ is: $X_2 = \frac{4s+8}{s^2+4s+3} = \frac{2}{s+1} + \frac{2}{s+3}$

Find the bottom component $x_2(t)$ in the **time domain**.

$$x_2(t) = 2e^{-t} + 2e^{-3t}$$

**h.** Are these components the same as you found in **Problem 1**? Yes or No? **Yes**