ENGR 232 – Dynamic Engineering Systems

Lab 8: Water Clocks: The Polyvascular Clepsydra and the Laplace Transform **Summer 2022**

Introduction

In this lab, we'll study the differential equations associated with polyvascular clepsydrae. The word **clepsydra** is Greek for water thief and describes any of the historical instruments that use the flow of water under gravity to measure time. Let's start with the simple inflow clepsydra.

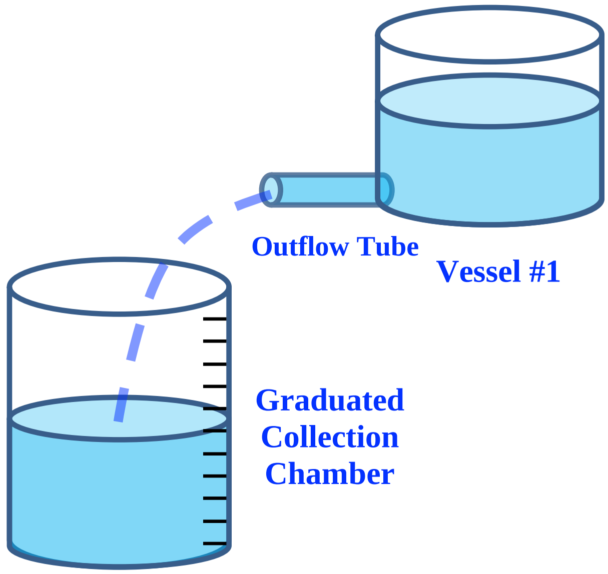


Figure 1: Clepsydra with One Vessel and Graduated Collection Chamber

This article discusses many of the early water clock designs: https://en.wikipedia.org/wiki/Water_clock

We assume the vessel is a cylinder with a cross-sectional area $A = 1$, maximum height of $h = 1$ and volume, when full, of $V = 1$. We also assume the outflow tube is sufficiently long and thin, so that viscosity dominates. The outflow rate Q is then governed by Poiseuille's Law:

$$Q = \frac{\pi r^4}{8\mu L} \cdot P$$

where L is the length of the outflow tube, r is its radius, $P = \rho g h$ is the hydrostatic pressure at the bottom of the vessel, and μ is the viscosity of the water. If $y_1(t)$ denotes the height of water in vessel #1, then since $h = y_1(t)$, the outflow rate is a <u>linear</u> function of the height:

$$Q = \frac{\pi r^4 \rho g}{8\mu L} \cdot y_1(t) \qquad \textbf{Linear!}$$

Jean Léonard Marie Poiseuille

A clever student adjusts the length L until the constant of proportionality is 1, and the outflow rate, with time measured in hours, is then:

$$Q = y_1(t)$$

Notice it is not constant! Instead the outflow rate starts off at 1, then gradually reduces towards zero. Not a very good water clock, since we would like the outflow rate to be as close as possible to 1 at all times.

Questions 1-2: With the length L adjusted so that $Q = 1 \cdot y_1(t)$, the differential equation describing the height of water in Vessel #1 is:

$$\textbf{DE:} \quad \frac{d}{dt} y_1(t) = -y_1(t) \qquad \textbf{IC:} \quad y_1(0) = 1$$

The height of water in the collection chamber, which also has cross-sectional area $A = 1$, is given by the integral:

$$\underbrace{H(t)}_{\text{Height in the collection chamber}} = -\int_0^t \frac{d}{dt}\left(y_1(t)\right) dt = -\, y_1(t)\Big|_0^t = 1 - y_1(t)$$

Since $A = 1$ for both vessel #1 and the collection chamber, the above equation just says the water is conserved. The water clock would be **ideal** if and only if the height of the accumulated water in the collection vessel satisfies:

$$H_{\text{ideal}}(t) = t$$

Using Laplace transforms, solve for $Y_1(s)$ in the transform domain, then use `ilapace` to find the solution $y_1(t)$ in the time domain. Here's a simultaneous plot showing $H_{\text{ideal}}(t) = t$, $H(t)$ and $y_1(t)$. Some starter code.

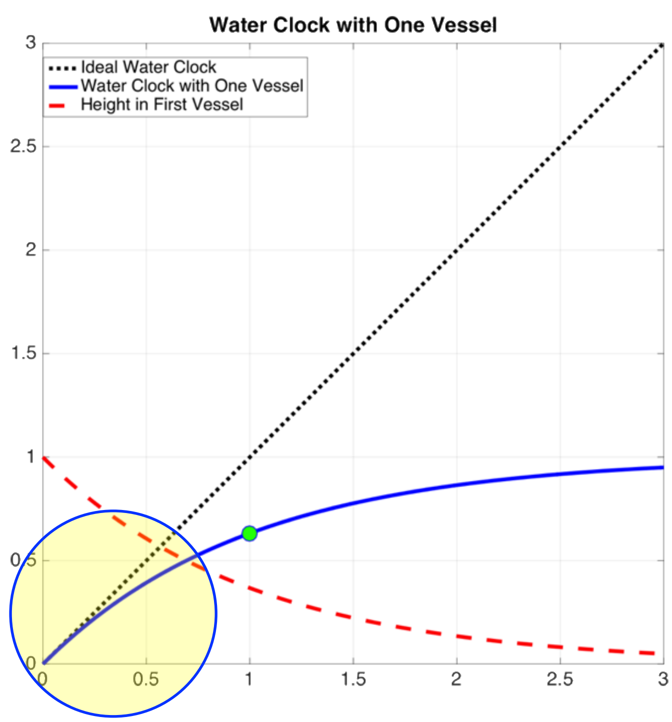
```
clear, clc, close all
syms s t Y
y0 = 1 % initial condition
Y1 = s*Y - y0  % transform of the derivative
```

An ideal water clock (which records perfect time) is shown by the black dotted line $H_{\text{ideal}}(t) = t$.

The blue line, shows $H(t)$, which is the time predicted by our one-vessel water clock.

You should produce a similar graph, but that will not be graded.

For small times, the water clock is quite accurate. Inside the yellow oval, the blue line and black dotted line are tangent at time $t = 0$.



Question 1: Record your answer for $H(t) = 1 - y_1(t)$ here. **Answer:** $H(t) =$ _________

Comment: You should check that the tangent line to your answer at the origin is indeed the line $H_{\text{ideal}}(t) = t$
The code below assumes you have already defined the solution $H(t)$ as a function.

```
% Equation for the tangent line at the origin is:
tangent_line = taylor(H(t), t, 0, 'Order', 2)
```

Question 2: How well does the single-vessel water clock perform at time $t = 1$ hour? See the **green** dot!
Hint: The error is quite large!

Question 2: Record your answer for $H(1)$ here. **Answer:** $H(1) =$ _________
Give answer to at least three decimals.

Comment: After one hour, our water clock is off by about one third. That's not so good!
However, it works quite well for smaller units of time, for which $y_1(t)$ is approximately still 1.
For example, $H(0.1) = 0.095$ is only off by about 5%.

Questions 3-4: Inflow Clepsydra with two Vessels and Graduated Collection Chamber

One early attempt to improve the performance of Water Clocks was to add additional vessels. Such an arrangement is called a **polyvascular clepsydra**. The figure below shows a clepsydra with two vessels and one collection chamber.

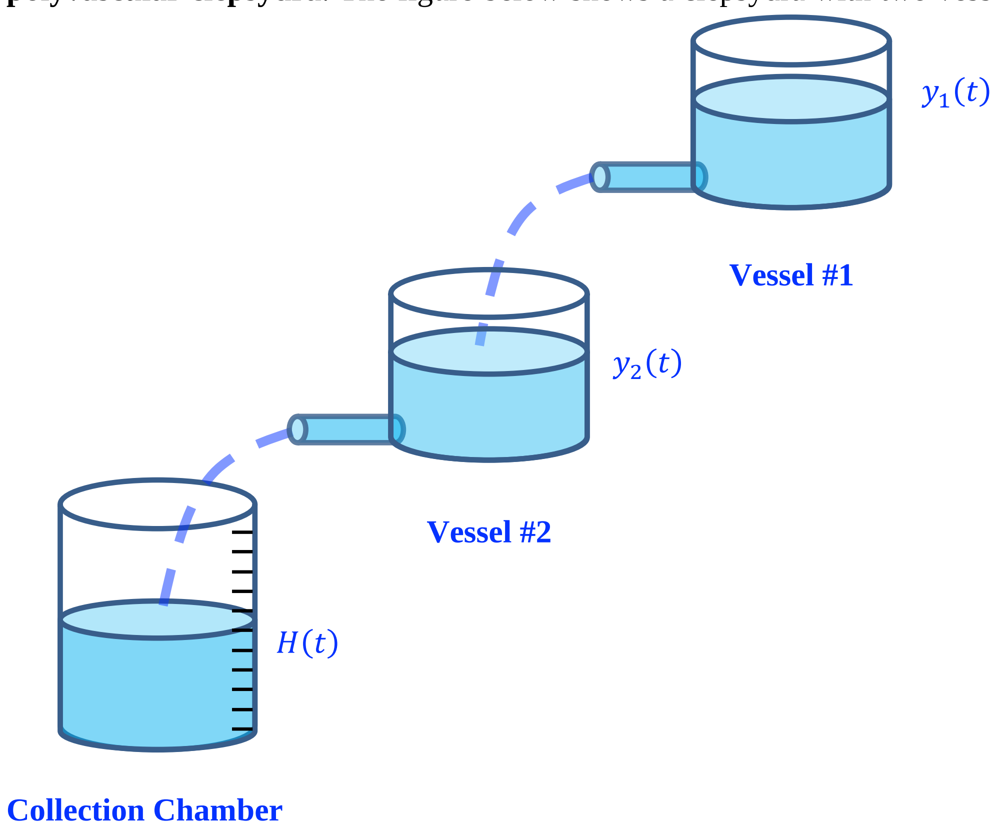


Figure 2: Clepsydra with two Vessels and Graduated Collection Chamber

We assume each vessel is a cylinder with a cross-sectional area $A = 1$, maximum height of $h = 1$ and volume when full of $V = 1$. However, the collection chamber has added height, so it can collect all the water. We also assume both outflow tubes are long and thin, so that viscosity dominates. Length of the tubes is adjusted so that for both vessels the outflow rate is equal to the height of water in that vessel:

$$Q_i = 1 \cdot y_i(t)$$

A perfect chronometer would give:

$$H_{\text{ideal}}(t) = t$$

The height $H(t)$ of water in the collection chamber gives the elapsed time according to the water clock.

The height $H(t)$ of water in the collection chamber gives the elapsed time according to the water clock.
Conservation of the water implies:

$$H(t) = 2 - y_1(t) - y_2(t)$$

Conservation of water gives the differential equation for both vessels and the height for the collection chamber. Remember, all cross-sectional areas are $A = 1$.

DE1: $\frac{d}{dt}y_1(t) = -y_1(t)$ **IC**: $y_1(0) = 1$

DE2: $\frac{d}{dt}y_2(t) = y_1(t) - y_2(t)$ **IC**: $y_2(0) = 1$

Let's collect together the two unknown heights $y_i(t)$ to form the **state vector**: $\vec{\mathbf{x}}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$

The system matrix for the water clock is: $A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$ and **IC:** $\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $H(0) = 0$

Thus, the matrix representation for our two-vessel water clock is:

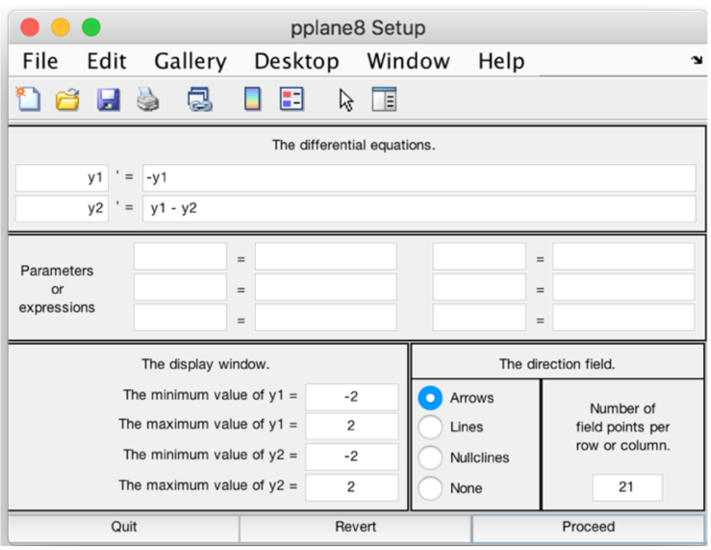
DE: $\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}$, where $A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$ **IC**: $\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Since the matrix A is invertible, the **DE** has a unique equilibrium point given by: $\vec{\mathbf{x}}_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (both vessels empty).

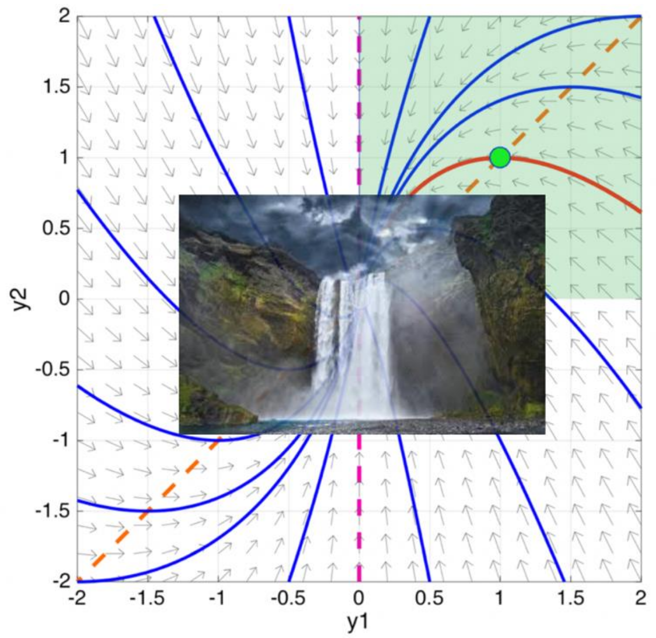
Questions 3-4: Using **pplane8**, create a phase plot for this two-vessel water clock. Select both ranges to be from –2 to 2, even though for this water clock, the meaningful range lies between 0 to 1. Show the nullclines and draw representative solution curves. Display the solution through the initial point $\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in **red**. Add at least ten representative solution curves.

Comment: Both eigenvalues are –1, and the equilibrium point at the origin is a **stable node**.

Paste your completed phase plot in the answer template.



Setup in pplane8



Sample plot with center occluded by a waterfall

Only the shaded green region where both vessels have positive height makes physical sense.

Question 5: Find the exact solutions in the time domain using **dsolve**.
Recall the matrix representation for our two-vessel water clock is:

$$\textbf{DE:}\quad \frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}, \quad \text{where } A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \qquad \textbf{IC:}\quad \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hint: Start with the code below. Then enter A, x0, and the DE. Use **dsolve**.

```
syms y1(t) y2(t)
x = [y1; y2];
```

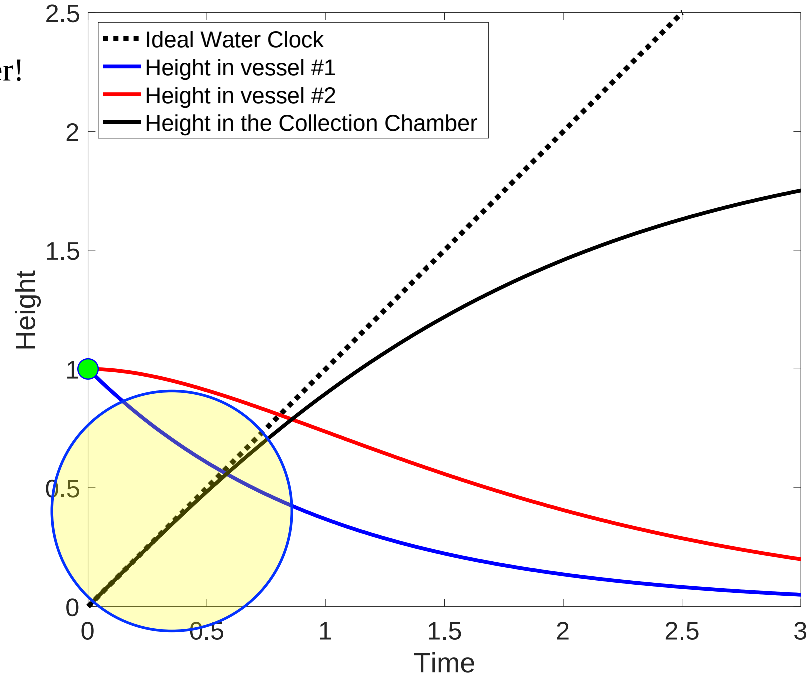
Use dot notation to access the components of the solution.

Question 5: Record your solution for the two unknowns $y_1(t)$ and $y_2(t)$. The first is given for you.

Answer: $y_1(t) = e^{-t}, \quad y_2(t) =$ ________

Here's a plot of both solutions $y_1(t)$ and $y_2(t)$. Notice the red curve (vessel #2) remains higher longer than the blue curve. That is why it is more accurate.

Even more exciting is the **black** curve. This shows the height in the collection chamber! Notice how it starts off as almost perfectly equal to $H_{\text{ideal}}(t) = t$.



Optional Review: At home, solve the first DE using **separation of variables**. Then solve the second using the **method of undetermined coefficients**. Great review before the final!

Optional: Solution using Generalized Eigenvectors - Not graded

Find the solution using an eigenvector $\vec{\mathbf{v}}$ and a generalized eigenvector $\vec{\mathbf{w}}$.

i. The `eig` command shows that there is only one eigenvalue, $\lambda = -1$, which is repeated.

```
A = [-1 0; 1 -1];   [V,D] = eig(A)
```

And there is only one linearly independent eigenvector, we'll choose $\vec{\mathbf{v}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We will need to find a GEV.

ii. Solve the GEV equation to find a generalized eigenvector $\vec{\mathbf{w}}$. Use MATLAB's **solve** command.

$$\textbf{GEV Eq:}\ \ (A - \lambda I)\vec{\mathbf{w}} = \vec{\mathbf{v}}$$

You should find one choice for the GEV is: $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

iii. The general solution is thus:

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \cdot e^{\lambda t}\vec{\mathbf{v}} + c_2 \cdot e^{\lambda t}(t\vec{\mathbf{v}} + \vec{\mathbf{w}}) = c_1 \cdot e^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \cdot e^{-t}\begin{bmatrix} 1 \\ t \end{bmatrix}$$

iv. Match the initial conditions by evaluating at time 0.

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \overset{[\vec{\mathbf{v}}\ \ \vec{\mathbf{w}}]}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}$$

Thus, $c_1 = c_2 = 1$.

v. Substitute $c_1 = c_2 = 1$ into the general solution to find the specific solution to our IVP:

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = 1 \cdot e^{\lambda t}\vec{\mathbf{v}} + 1 \cdot e^{\lambda t}(t\vec{\mathbf{v}} + \vec{\mathbf{w}}) \ = \ e^{-t}\begin{bmatrix} 1 \\ 1 + t \end{bmatrix}$$

Question 6: Cumulative Outflows

Let's now focus on the collection chamber and the total outflows from each vessel.
Let $f_1(t)$ and $f_2(t)$ be the cumulative outflows from time 0 to time t, for each vessel. Because of the identity:

$$Q_i = 1 \cdot y_i(t)$$

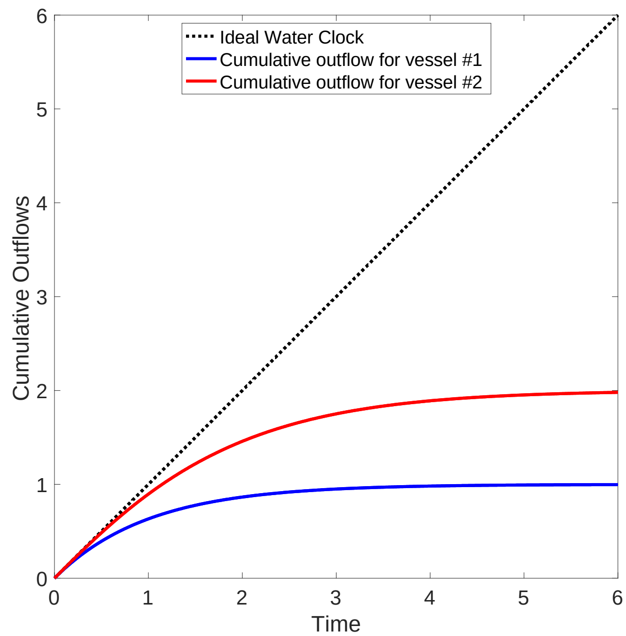
these integrated outflows are just:

$$f_1(t) = \int_0^t y_1(t)\,dt \quad \text{and} \quad f_2(t) = \int_0^t y_2(t)\,dt$$

Evaluate each integral using MATLAB's **int** command, then create a simultaneous plot of the three quantities: $H_{ideal}(t) = t$, $f_1(t)$ and $f_2(t) = H(t)$. The plot is given for free.

Question 6: Record your answers for the cumulative outflows. The first is given for free.

Answer: $f_1(t) = 1 - e^{-t}$, $f_2(t) =$ ____________



Cumulative outflows from each vessel and the ideal curve.

Comment: The Taylor series for $f_2(t)$ about the origin is: $f_2(t) \sim t - \frac{t^3}{6} + \cdots$
Notice the error is now in the third term! That's better (for small t)!
The Taylor series for $f_1(t)$ about the origin displays a quadratic error: $f_1(t) \sim t - \frac{t^2}{2} + \cdots$

Comment: You can use the **sympref** command so that polynomials appear in ascending order. Examples:

```
sympref('PolynomialDisplayStyle','ascend'); % display polynomials in ascending order

sympref('default') % return symbolic preferences to the default
```

Question 7: Solve the entire problem using Laplace Transforms. Recall the DE for our two-vessel water clock is:

$$\textbf{DE}: \quad \frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}, \quad \text{where } A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \qquad \textbf{IC}: \quad \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $\vec{\mathbf{X}}(s)$ denote the Laplace transform of $\vec{\mathbf{x}}(t)$. Then:

$$\vec{\mathbf{X}}(s) = (sI - A)^{-1}\, \vec{\mathbf{x}}(0)$$

There is no forcing term, so this is just the **zero-input** or homogeneous solution.
Solve for $\vec{\mathbf{X}}(s)$ and record your answer in the answer template. The first component has been given for you.

Question 7: The solution in the transform domain is:

$$\vec{\mathbf{X}}(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} \quad \text{where } X_1(s) = \frac{1}{s+1} \text{ and } X_2(s) = \_\_\_\_\_\_\_\_\_\_$$

Question 8: Integrating the Flows
We saw earlier, that the situation is clearer if we plot the <u>cumulative</u> outflows $f_1(t)$ and $f_2(t)$, where:

$$f_1(t) = \int_0^t y_1(t)\, dt \quad \text{and} \quad f_2(t) = \int_0^t y_2(t)\, dt$$

Now recall **identity #32** in our Table of Laplace transforms for the <u>transform of an integral</u>.
Integration in the time domain is simply division by s in the transform domain.

$$\textbf{32:} \quad \mathcal{L}\left\{\int_0^t f(v)\, dv\right\} = \frac{F(s)}{s}$$

If the transform of $f(t)$ is $F(s)$, then the transform of its integral (from 0 to t) is $\frac{F(s)}{s}$. Just divide by s!

Find the cumulative outflow vector $\vec{\mathbf{f}}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$, in two steps.

First define $\vec{\mathbf{F}}(s) = \frac{\vec{\mathbf{X}}(s)}{s}$, then apply MATLAB's command **ilaplace** to find $\vec{\mathbf{f}}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$.

<u>**Complete**</u> this code to find the $\vec{\mathbf{f}}(t)$ vector in the time domain. The answer should be the same as above.

Question 8: Complete this code to find the cumulative flow vector $\vec{\mathbf{f}}(t)$ using Laplace transforms:

```
syms s
A=[-1 0; 1 -1]   % system matrix
x0 = [1;1]       % initial conditions
X = …            % find X here using inv()
F = …            % find F here. Integration is division by s.
f = …            % find f here using ilaplace.
```

Ungraded Challenge: Apply the FVT to show that: $\lim_{t\to\infty} \vec{\mathbf{f}}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. This just says that all the water drains out.

Question 9: Water Clock with four vessels plus the collection chamber.
The figure below shows a clepsydra with four vessels and graduated collection chamber.

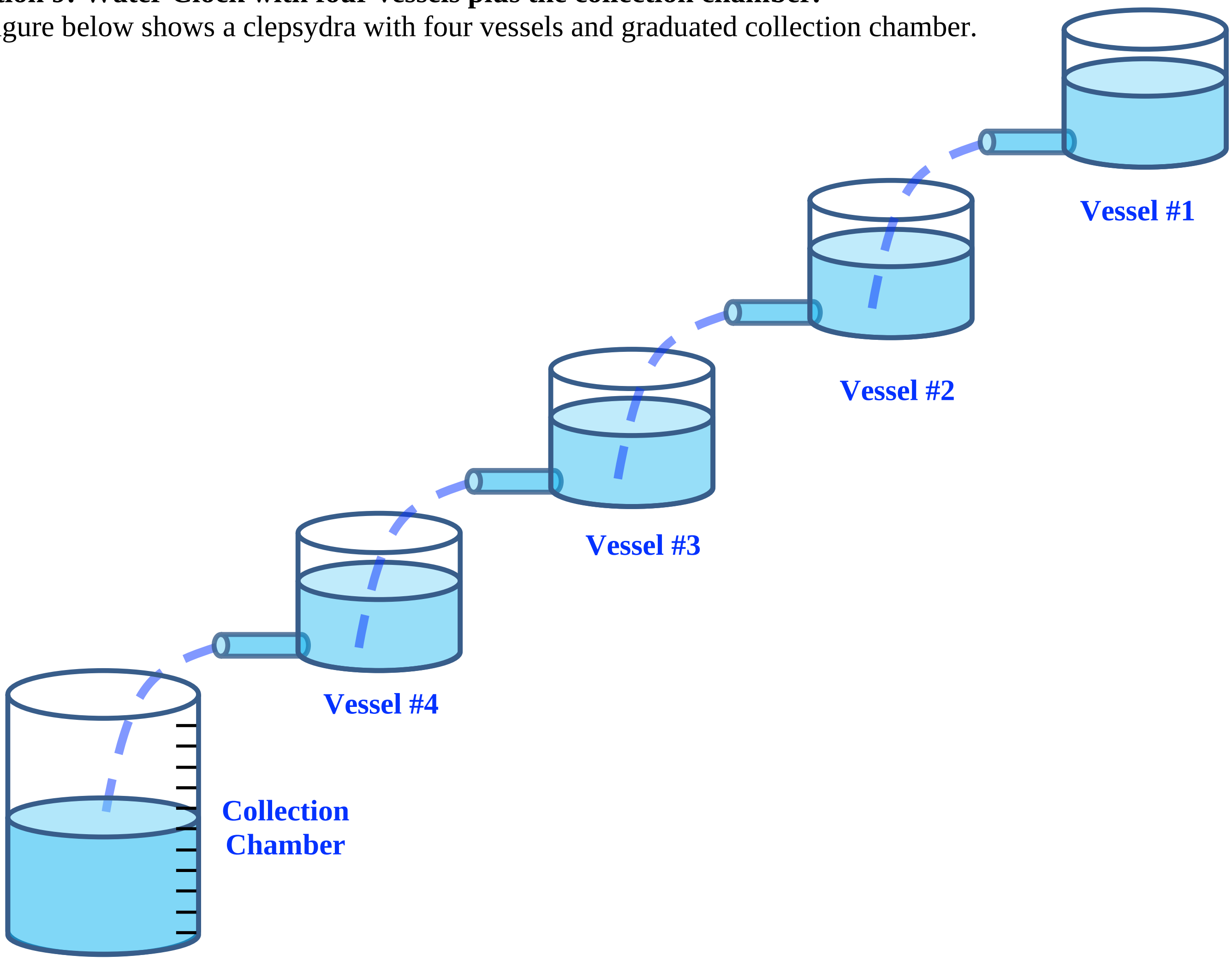


Figure 3: Clepsydra with four Vessels and Collection Chamber

All four vessels have cross-sectional area $A = 1$, and start off full, with initial heights of 1. In vector form, the DE is:

$$\textbf{DE}: \quad \frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} \quad \text{with} \quad A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & -1 \end{bmatrix} \qquad \textbf{IC}: \quad \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

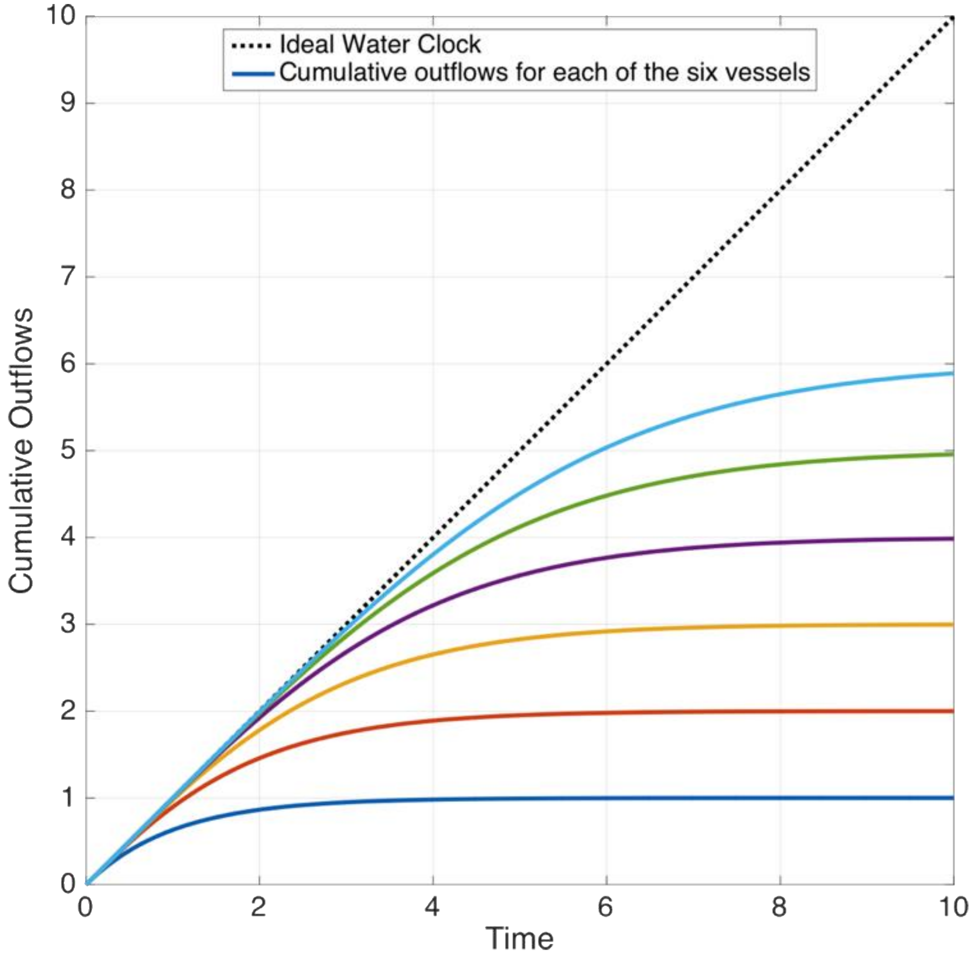
Enter A and $\vec{\mathbf{x}}(0)$**,** then solve for the <u>**cumulative outflow vector**</u> $\vec{\mathbf{F}}(s) = \frac{\vec{\mathbf{X}}(s)}{s}$ in the transform domain using:

```
X = inv(s*eye(4) - A) * x0
F = X/s
f = ilaplace(F) % Cumulative outflows in the time domain
f = matlabFunction(f) % convert f to a function
```

This sample plot is for a different clepsydra with **six** vessels (instead of four) plus the collection chamber. You may color the curves anyway you wish. But be sure the ideal curve is a **dotted black** line.

Question 9: Replace the sample plot with the correct graph for a clepsydra with **four** vessels. (not 6!)

Sample is for Water Clock with Six Vessels



Check 1: Evaluating the cumulative flow vector $\vec{\mathbf{f}}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{bmatrix}$ at time one hour, we find: $\vec{\mathbf{f}}(1) = \begin{bmatrix} 0.6321 \\ 0.8964 \\ 0.9767 \\ 0.9957 \end{bmatrix}$

Since the fourth component gives the height $H(t)$ in the collection vessel, we see our error at time 1 hour is now less than half a percent! An exact chronometer would register $H_{\text{ideal}}(1) = 1$.

Check 2: Apply the Final Value Theorem to the outflow vector $\vec{\mathbf{F}}(s) = \frac{\vec{\mathbf{X}}(s)}{s}$ in the transform domain to show that the total outflow through the n^{th} vessel approaches n. That is show: $\lim_{t\to\infty} \vec{\mathbf{f}}(t) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

Comment: The Taylor series for $f_4(t)$ about the origin is: $f_4(t) \sim t - \frac{t^5}{120} + \cdots$

Notice the error is now in the fifth power! That's much better (for small t)!

Here, for free, are the total flows in the time domain so you can check your work.

$$\vec{\mathbf{f}}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - e^{-t} \cdot \begin{bmatrix} 1 \\ 2+t \\ 3+2t+\dfrac{t^2}{2} \\ 4+3t+t^2+\dfrac{t^3}{6} \end{bmatrix}$$

Recall the state vector $\vec{\mathbf{x}}(t)$ gives the height of water in each of the vessels: $\vec{\mathbf{x}}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$

You should have found the solution in the transform domain is:

$$\vec{\mathbf{X}}(s) = \begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{s+1} + \dfrac{1}{(s+1)^2} \\ \dfrac{1}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{1}{(s+1)^3} \\ \dfrac{1}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{1}{(s+1)^3} + \dfrac{1}{(s+1)^4} \end{bmatrix}$$

Find the solution in the time domain, (in your head) using the expression for $\vec{\mathbf{X}}(s)$ above.
Can you predict the form for a clepsydra with N vessels? Make a prediction now for $N = 12$ before proceeding.

Question 10: Polyvascular Clepsydra with TWELVE Vessels and Graduated Collection Chamber

Here again is the water clock system matrix A in the case of four vessels: $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & -1 \end{bmatrix}$

What would it look like if there were <u>twelve</u> vessels?? The pattern is already clear! The new A will be a 12×12 matrix with –1 on each diagonal spot and +1 all along the subdiagonal. All the other entrees are zero. Fortunately, MATLAB has a way to enter such a matrix for us! We will use the `'tridiag'` option available under the `pattern` command. Here's some starter code.

```
N = 12 % Number of  vessels, excluding the collection chamber
subdiagonal = 1;  diagonal = -1; superdiagonal = 0
A = full(gallery('tridiag', N, subdiagonal, diagonal, superdiagonal))
x0 = ones(N,1)  % Initial conditions
```

Check your matrix A is as expected. Solve the system using laplace transforms, then plot all twelve cumulative outflows plus the ideal solution $H_{\text{ideal}}(t) = t$.

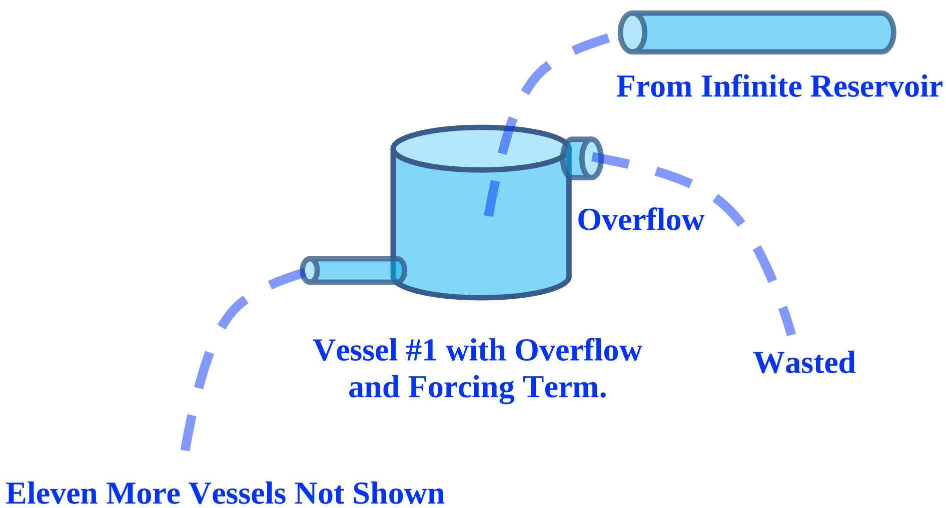
Question 10: Paste your completed cumulative flow graph for a clepsydra with **<u>twelve</u>** vessels in the answer template.

Check 1: Evaluating the cumulative flow vector $\vec{\mathbf{f}}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \\ f_5(t) \\ f_6(t) \\ f_7(t) \\ f_8(t) \\ f_9(t) \\ f_{10}(t) \\ f_{11}(t) \\ f_{12}(t) \end{bmatrix}$ at time one hour, we find: $\vec{\mathbf{f}}(1) = \begin{bmatrix} 0.63212056 \\ 0.89636168 \\ 0.97666307 \\ 0.99565123 \\ 0.99931108 \\ 0.99990526 \\ 0.99998850 \\ 0.99999875 \\ 0.99999988 \\ 0.99999999 \\ 1.00000000 \\ 1.00000000 \end{bmatrix}$

Since the twelfth (and last) component gives the height $H(t)$ in the collection vessel, we see our error at time 1 hour is now extremely tiny! An exact chronometer would register $H_{\text{ideal}}(1) = 1.000$

Check 2: Apply the Final Value Theorem to the outflow vector $\vec{\mathbf{F}}(s) = \frac{\vec{\mathbf{X}}(s)}{s}$ in the transform domain to show that the total outflow through the n^{th} vessel approaches n. All the water drains out. That is show: $\lim_{t\to\infty} \vec{\mathbf{f}}(t) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{bmatrix}$

Ungraded Challenge!
Suppose we add a forcing term, by pumping water into the top tank at a rate slightly in excess of one volume unit per hour and equipping vessel #1 with an overflow outlet. All the other vessels are unchanged.



As a result of this **control** mechanism, the height of water in vessel #1 will stay constant at $y_1(t) = 1$

The system matrix and DE is the same as before, except we need to add a forcing term for vessel 1.
The matrix A is 12×12, while the state vector $\vec{\mathbf{x}}$ and initial vector have 12 components, one for each vessel.

$$\textbf{DE}: \quad \frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} + \vec{\mathbf{b}}\, f(t) \;\text{ where }\; \vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad \textbf{IC}: \;\; \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$

The vector $\vec{\mathbf{b}}$ has 12 components, all of which are zero except the first which is 1. The pump from the infinite reservoir starts up at time zero, so we can take $f(t)$ to be the unit step function. It's just 1 for all times $t \geq 0$.

The solution in the transform domain is:

$$\vec{\mathbf{X}}(s) = (sI - A)^{-1}\,\vec{\mathbf{x}}(0) \; + \; (sI - A)^{-1}\,\vec{\mathbf{b}}\,F(s)$$

Zero Input Solution (homogeneous) **Zero State Soln. (forcing function)**

In the equation above, $F(s) = \frac{1}{s}$

With the overflow control added to vessel #1, the flow vector in the time domain reduces to:

$$\vec{\mathbf{f}}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \\ f_5(t) \\ f_6(t) \\ f_7(t) \\ f_8(t) \\ f_9(t) \\ f_{10}(t) \\ f_{11}(t) \\ f_{12}(t) \end{bmatrix} \text{ where each term is the same, and is given by } f_i(t) = \quad \_\,\_\,\_\,\_ \quad \text{ for } i = 1, 2, 3, \ldots, 12$$

Appendix on Generalized Eigenvectors

The only eigenvalue for the matrix A that describes a water clock with N vessels is $\lambda = -1$

In the case $N = 4$, you can readily check that the <u>only</u> independent eigenvector is $\vec{\mathbf{v}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, all zeros except for a one in the last position. We need <u>three</u> generalized eigenvectors which we shall label $\vec{\mathbf{w}}_1$, $\vec{\mathbf{w}}_2$ and $\vec{\mathbf{w}}_3$.

If there are N vessels, we will need $(N-1)$ GEVs. The GEV's can be found from the chain of linear equations:

GEV Chain: $(A - \lambda I)\vec{\mathbf{w}}_1 = \vec{\mathbf{v}}$, $\quad (A - \lambda I)\vec{\mathbf{w}}_2 = \vec{\mathbf{w}}_1$, $\quad (A - \lambda I)\vec{\mathbf{w}}_3 = \vec{\mathbf{w}}_2$

Solving one-by-one in MATLAB we find the eigenvector and the three GEVs are:

$$\vec{\mathbf{v}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{\mathbf{w}}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{\mathbf{w}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{\mathbf{w}}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice this is just the standard basis vectors in reverse order.

Returning to the differential equation, the fundamental solutions corresponding to the eigenvector and each GEV fit the pattern:

$$\vec{\mathbf{v}} \to e^{\lambda t}\vec{\mathbf{v}} \qquad \vec{\mathbf{w}}_1 \to e^{\lambda t}(t\vec{\mathbf{v}} + \vec{\mathbf{w}}_1) \qquad \vec{\mathbf{w}}_2 \to e^{\lambda t}\left(\frac{t^2}{2}\vec{\mathbf{v}} + t\vec{\mathbf{w}}_1 + \vec{\mathbf{w}}_2\right) \qquad \vec{\mathbf{w}}_3 \to e^{\lambda t}\left(\frac{t^3}{6}\vec{\mathbf{v}} + \frac{t^2}{2}\vec{\mathbf{w}}_1 + t\vec{\mathbf{w}}_2 + \vec{\mathbf{w}}_3\right)$$

Note the coefficients have the form: $\frac{t^k}{k!}$

Thus, the general solution for the water clock with four vessels can be written:

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} = c_1 \cdot e^{\lambda t}\vec{\mathbf{v}} + c_2 \cdot e^{\lambda t}(t\vec{\mathbf{v}} + \vec{\mathbf{w}}_1) + c_3 \cdot e^{\lambda t}\left(\frac{t^2}{2}\vec{\mathbf{v}} + t\vec{\mathbf{w}}_1 + \vec{\mathbf{w}}_2\right) + c_4 \cdot e^{\lambda t}\left(\frac{t^3}{6}\vec{\mathbf{v}} + \frac{t^2}{2}\vec{\mathbf{w}}_1 + t\vec{\mathbf{w}}_2 + \vec{\mathbf{w}}_3\right)$$

Plugging in the known eigenvalue, eigenvector and the three GEVS gives:

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} = c_1 \cdot e^{-t}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \cdot e^{-t}\begin{bmatrix} 0 \\ 0 \\ 1 \\ t \end{bmatrix} + c_3 \cdot e^{-t}\begin{bmatrix} 0 \\ 1 \\ t \\ t^2/2 \end{bmatrix} + c_4 \cdot e^{-t}\begin{bmatrix} 1 \\ t \\ t^2/2 \\ t^3/6 \end{bmatrix}$$

The solution satisfying the initial condition has each c_i equal to one.

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_4 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_4 \\ c_3 \\ c_2 \\ c_1 \end{bmatrix}$$

The solution matching the initial conditions is:

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix} = e^{-t} \begin{bmatrix} 1 \\ 1+t \\ 1+t+\dfrac{t^2}{2} \\ 1+t+\dfrac{t^2}{2}+\dfrac{t^3}{6} \end{bmatrix}$$

You should be able to guess the pattern for a water clock with $N = 12$ vessels.

Ancient Chinese Copper Kettle Clepsydra

Ready to Submit?

Be sure all ten questions are answered. When your lab is complete, be sure to submit three files:

1. Your **completed Answer Template** as a PDF file
2. A copy of your **MATLAB Live Script**
3. A **PDF** copy of your **MATLAB Live Script** (Save-Export to PDF…)

The due date is the day after your lab section by **11:59pm** to receive full credit. You have one more day, to submit the lab (but with a small penalty), and then the window closes for good and your grade will be zero.