W5 InLab Activity: **Name:** **Cole** **Bardin**

first *last*

Problem 1: Method of Undetermined Coefficients

Use Method of Undetermined Coefficients to find the IVP solution to the second-order linear non-homogeneous differential equation:

DE: $y'' - 2y' - 3y = 3e^{2t}$ **IC:** $y(0) = 1, \quad y'(0) = 0$

a. First find the general solution for the corresponding homogeneous equation: $y'' - 2y' - 3y = 0$

The characteristic equation factors nicely over the integers. Show it.

$$aux: r^2 - 2r - 3 = 0$$
$$(r-3)(r+1) = 0, r = 3, -1$$

The general solution for our homogeneous DE is:

$$y(t) = c_1 e^{3t} + c_2 e^{-t}$$

b. Next find the particular solution to the original DE using the method of undetermined coefficients.
We need to guess a form $y_p(t)$ which includes the forcing term $g(t) = 3e^{2t}$ and all its derivatives.
Since exponentials are proportional to their derivatives we only need that one term: Use *A* as the constant.

Form of Guess: $y_p(t) = \mathrm{A}e^{2t}$

To perform the substitution of the particular solution into the DE we also need the derivatives:

$y_p' = 2\mathrm{A}e^{2t}$ and $y_p'' = 4\mathrm{A}e^{2t}$

Plugging into the DE, find *A*.

$$\mathrm{A}4e^{2t} - 4\mathrm{A}e^{2t} - 3\mathrm{A}e^{2t} = 3e^{2t}$$
$$-3\mathrm{A}e^{2t} = 3e^{2t}$$

$$A = -1$$

Thus, the particular solution is: $y_p(t) = -e^{2t}$ . . . and the general solution is:

$$y(t) = y_p(t) + y_h(t) = -e^{2t} + c_1 e^{3t} + c_2 e^{-t}$$

c. Find the unique solution that matches the initial conditions.

First find the derivative:

$$y'(t) = -2e^{2t} + 3c_1 e^{3t} - c_2 e^{-t}$$

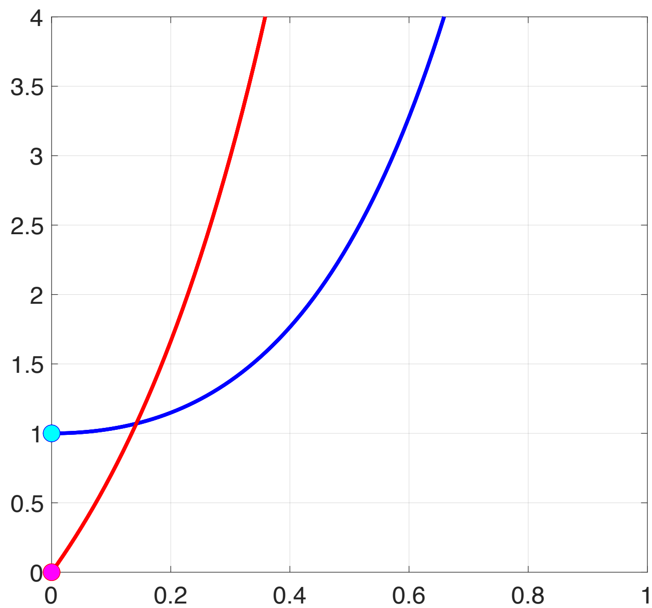
Now find c_1 and c_2.

$$y(0) = 1 = -1 + c_1 + c_2$$
$$c_1 + c_2 = 2$$
$$y'(0) = 0 = -2 + 3c_1 - c_2$$
$$3c_1 - c_2 = 2$$

$$c_1 = 2 - c_2$$
$$3(2 - c_2) - c_2 = 2$$
$$6 - 4c_2 = 2$$
$$4c_2 = 4, \qquad c_2 = 1$$
$$c_1 + 1 = 2$$
$$c_1 = 1$$

The unique solution matching the initial conditions is: $y(t) = -\mathrm{e}^{2t} + e^{3t} + e^{-t}$

Plot for Problem 1: $y(t)$ in blue, $y'(t)$ in red



Problem 2: Method of Undetermined Coefficients

Use the Method of Undetermined Coefficients for solving the second-order linear non-homogeneous differential equation. Find the general form of the particular solution, the particular solution, and the general solution to the differential equations

$$\textbf{DE: } y'' - 2y' - 3y = 3 + 4\sin(2t)$$

a. First find the general solution to the homogeneous equation: $y'' - 2y' - 3y = 0$

Since this is the same as the homogeneous **DE on Problem 1 and 2** we repeat:

$$y_h(t) = c_1 e^{3t} + c_2 e^{-t}$$

The general solution for our homogeneous DE is:

Done for you!

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = c_1 e^{3t} + c_2 e^{-t}$$

b. Next find the particular solution to the original DE using the method of undetermined coefficients. We need to guess a form $y_p(t)$ which includes the forcing term $g(t) = 3 + 4\sin(2t)$ and all its derivatives. Use *A*, *B* and *C* as the unknown coefficients.

$$y_p(t) = A + B\sin(2t) + C\cos(2t)$$

To perform the substitution of the particular solution into the DE we also need the derivatives:

$$y_p' = 2B\cos(2t) - 2C\sin(2t) \quad \text{and} \quad y_p'' = -4Bin(2t) - 4C\cos(2t)$$

Plugging into the DE $y'' - 2y' - 3y = 3 + 4\sin(2t)$ we find:

$$[-4B\sin(2t) - 4C\cos(2t)] - 2[2B\cos(2t) - 2C\sin(2t)] - 3[A + B\sin(2t) + C\cos(2t)] = 3 + 4\sin(2t)$$

$$(-4B + 4C - 3B)\sin(2t) + (-4C - 4B - 3C)\cos(2t) = 3 + 4\sin(2t)$$

Scalar Coefficients: -3A

Coefficients of sin(2t): -7B + 4C

Coefficients of cos(2t): -7C - 4B

Solve for *A*, *B* and *C*:

$$\begin{aligned} &\boldsymbol{-3A = 3, A = -1} \\ &\boldsymbol{-7B + 4C = 4} \\ &\boldsymbol{-4B - 7C = 0} \\ &\boldsymbol{B = -\frac{28}{65}} \\ &\boldsymbol{C = \frac{16}{65}} \end{aligned}$$

Thus, the particular solution is: $y_p(t) = -1 - \frac{28}{65}\cos(2t) + \frac{16}{65}\sin(2t)$

and the general solution to the non-homogeneous DE is:

$$y(t) = y_p(t) + y_h(t) = \left(-1 - \frac{28}{65}\cos(2t) + \frac{16}{65}\sin(2t)\right) + \left[c_1 e^{3t} + c_2 e^{-t}\right]$$

Problem 3: Method of Undetermined Coefficients

Use the Method of Undetermined Coefficients for solving the second-order linear non-homogeneous differential equation. Find the general form of the particular solution, the particular solution, and the general solution to the differential equation.

DE: $y'' - 2y' - 3y = -3te^{-t}$ **Warning**: The factor e^{-t} is already a homogeneous solution!

a. First find the general solution to the homogeneous equation: $y'' - 2y' - 3y = 0$

Since this is the same as the homogeneous **DE on Problem 1 and 2** we repeat:

$$y_h(t) = c_1 e^{3t} + c_2 e^{-t}$$

Done for you!

b. Next find the **FORM** of the particular solution to the original DE using the method of undetermined coefficients. We need to guess a form $y_p(t)$ which includes the forcing term $g(t) = -3te^{-t}$ and all its derivatives:

Repeat Warning: The factor e^{-t} is already a homogeneous solution!

i. If this were not a homogeneous solution we would guess:

$$y_p(t) = A \cdot te^{-t} + Be^{-t}$$

Done for you!

ii. But since e^{-t} is already a homogeneous solution, we must bump up (the entire guess) by a factor of *t*. Write out the new guess using *A* and *B* as the constants:

$$y_p(t) = A \cdot t^2 e^{-t} + Bte^{-t}$$

c. Find the undetermined coefficients *A* and *B*.
To perform the substitution of the particular solution into the DE we also need the derivatives, which requires several applications of the product rule.

$$y_p' = A(2t - t^2)e^{-t} + B(1-t)e^{-t} \quad \text{and} \quad y_p'' = A(2 - 4t + t^2)e^{-t} + B(t-2)e^{-t}$$

Plugging into the DE $y'' - 2y' - 3y = -3te^{-t}$ gives:

$$[A(2 - 4t + t^2)e^{-t} + B(t-2)e^{-t}] - 2[A(2t - t^2)e^{-t} + B(1-t)e^{-t}] - 3[A \cdot t^2 e^{-t} + Bte^{-t}] = -3te^{-t}$$
$$-8Ae^{-t}t + (2A - 4B)e^{-t} = -3te^{-t}$$

Now match all the coefficients to find *A* and *B*.

Coefficient of t^2e^{-t}: 0
Coefficient of te^{-t}: -8A = -3
Coefficient of e^{-t}: 2A-4B = 0

So *A*= 3/8 and *B* = 3/16

Thus, the particular solution is: $y_p(t) = \frac{3}{8}t^2e^{-t} + \frac{3}{16}te^{-t}$

and the general solution to the non-homogeneous DE is:

$$y(t) = y_p(t) + y_h(t) = \left(\frac{3}{8}t^2e^{-t} + \frac{3}{16}te^{-t}\right) + [c_1e^{3t} + c_2e^{-t}]$$