W8 InLab Activity: **Name: Cole Bardin**

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Problem 1: Review: Solve this DE using the Method of Undetermined Coefficients

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Consider the following linear, non-autonomous differential equation which describes a **cycloid**:

DE: $y'' + 0 \cdot y' + 1 \cdot y = 25t$ **IC:** $y(0) = 50, \quad y'(0) = 0$

Above, the forcing term $f(t) = 25t$ grows linearly in time.

a. Find the particular solution $y_p = At + b$

i. Give the derivative of your guess: $y' =$ A

ii. Give the double derivative of your guess: $y'' =$ 0

iii. Solve for the unknown coefficients *A* and *B*.

Plugging into the full DE we find: (show work here.)

$$0 + 0 * (A) + 1(At + b) = 25t$$
$$At + b = 25t$$
$$A = 25, b = 0$$

so, the particular solution is: $y_p =$ 25t

b. Find the general solution to the **homogeneous DE**: $y'' + 0 \cdot y' + 1 \cdot y = 0$

i. Give the characteristic equation.

$$r^2 + 1 = 0$$

ii. The roots are the complex conjugates:

$r_1 =$ -i and $r_2 =$ +i

iii. So, the <u>general</u> **homogeneous** solution y_h is: (Use c_1 for the cosine term and c_2 for the sine.)

$y_h =$ $c_1\sin(t) + c_2\cos(t)$

c. The complete solution to the nonhomogeneous DE is:

$y(t) = y_p(t) + y_h(t) =$ $c_1 \sin(t) + c_2 \cos(t) + 25t$

d. Find the coefficients c_1 and c_2 that match the initial conditions.

$c_1 =$ -25 and $c_2 =$ 50

$$y(0) = 50 = c_1 sin(0) + c_2 cos(0) + 25 * 0$$
$$50 = c_2$$
$$y'(0) = c_1 cos(t) - c_2 sin(t) + 25$$
$$y'(0) = 0 = c_1 cos(0) - c_2 sin(0) + 25$$
$$-25 = c_1$$

Type equation here.

Problem 2: The Cycloid Revisited! Equations in Normal Form

Consider the same differential equation for a **cycloid** seen in the previous problem.

DE: $y'' + 0 \cdot y' + 1 \cdot y = 25t$ **IC:** $y(0) = 50, \quad y'(0) = 0$

a. Represent the system in **normal form** after defining $x_1 = y$ and $x_2 = y'$ so that the **state vector** is:

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} \text{ and its derivative is } \vec{\mathbf{x}}\,' = \begin{bmatrix} y' \\ y'' \end{bmatrix}$$

Give the matrix A and the vector $\vec{\mathbf{b}}(t)$ so that our DE is equivalent to: $\frac{d}{dt}\vec{\mathbf{x}} = A\,\vec{\mathbf{x}} + \vec{\mathbf{b}}(t)$

$A = \boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$ $\vec{\mathbf{b}}(t) = \boxed{\begin{bmatrix} 0 \\ 25t \end{bmatrix}}$

b. Give the initial value of the state vector: $\vec{\mathbf{x}}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \boxed{\begin{bmatrix} 50 \\ 0 \end{bmatrix}}$

c. Verify that $\vec{\mathbf{x}}_p(t) = \begin{bmatrix} 25t \\ 25 \end{bmatrix}$ is a particular solution to the DE: $\frac{d}{dt}\vec{\mathbf{x}} = A\,\vec{\mathbf{x}} + \vec{\mathbf{b}}(t)$

LHS: $\frac{d}{dt}\vec{\mathbf{x}} = \boxed{\begin{bmatrix} 25 \\ 0 \end{bmatrix}}$ **RHS:** $A\,\vec{\mathbf{x}} + \vec{\mathbf{b}}(t) = \boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 25t \\ 25 \end{bmatrix} + \begin{bmatrix} 0 \\ 25t \end{bmatrix}}$

d. Verify that $\vec{\mathbf{x}}_h(t) = \begin{bmatrix} c_1\cos(t) + c_2\sin(t) \\ -c_1\sin(t) + c_2\cos(t) \end{bmatrix}$ is a solution to the <u>homogeneous</u> DE: $\frac{d}{dt}\vec{\mathbf{x}} = A\,\vec{\mathbf{x}}$

LHS: $\frac{d}{dt}\vec{\mathbf{x}} = \boxed{\begin{bmatrix} -c_1\sin(t) + c_2\cos(t) \\ -c_1\cos(t) - c_2\sin(t) \end{bmatrix}}$ **RHS:** $A\,\vec{\mathbf{x}} = \boxed{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} c_1\cos(t) + c_2\sin(t) \\ -c_1\sin(t) + c_2\cos(t) \end{bmatrix}}$

e. The general solution to the full DE is thus:

$$\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_p(t) + \vec{\mathbf{x}}_h(t) = \boxed{\begin{bmatrix} 25t \\ 25 \end{bmatrix} + \begin{bmatrix} c_1\cos(t) + c_2\sin(t) \\ -c_1\sin(t) + c_2\cos(t) \end{bmatrix}}$$

f. Find the specific solution matching the initial condition that: $\vec{\mathbf{x}}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \end{bmatrix}$. That is, find c_1 and c_2.

$c_1 = \boxed{50}$ and $c_2 = \boxed{-25}$

Note: The top component $x_1(t)$ gives the same solution as in the previous problem:

$$x_1(t) = y(t) = y_p(t) + y_h(t) \ = 25t \ + \ 50\cos(t) - \ 25\sin(t)$$

Problem 3: The Cycloid Revisited - Solve the cycloid DE using the Laplace Transform

Use Laplace Transforms to solve this differential equation for the cycloid.

DE: $y'' + 0 \cdot y' + 1 \cdot y = 25t$ **IC:** $y(0) = 50, \quad y'(0) = 0$

a. Find the transform of the RHS forcing function $f(t) = 25t$.

$\mathcal{L}\{25t\} =$ $\boxed{\frac{25}{s^2}}$

b. Find the transform of the double derivative term on the LHS. Denote the transform of the unknown $y(t)$ as $Y(s)$.

$\mathcal{L}\{y''\} = s\mathcal{L}\{y'\} - y'(0) = s^2Y - sy(0) - y'(0) =$ $\boxed{Y(s) * (s^2+1) - 50s}$

c. Solve for the solution $Y = \frac{N(s)}{D(S)}$ (in transform space) as a ratio of two polynomials in s.

$Y =$ $\boxed{\frac{50s^3 + 25}{(s^2+1)s^2}}$

d. Find the partial fraction expansion for $Y = \frac{50s^3+25}{s^2 \cdot (s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$

i. First find B using the Heaviside Cover-up Method. For free, you would find $A = 0$.

$B =$ $\boxed{25}$

e. From the partial fraction expansion for $Y_1 = Y - \frac{A}{s} - \frac{B}{s^2} = 25\ \frac{2s-1}{s^2+1}$ we see that $C = 50$ and $D = -25$.

Thus, the full partial fraction expansion is: $Y = \frac{50s^3+25}{s^2 \cdot (s^2+1)} = \frac{25}{s^2} + \frac{50s-25}{s^2+1}$

Give the solution in the time domain.

$y(t) =$ $\boxed{25t + 50\cos(t) - 25\sin(t)}$

Here's a plot of the solution in phase space with y' on the vertical axis and y on the horizontal axis.