Lab 9: Matrix Laplace Method and Linear Systems **ENGR-232 Summer 2022**

Introduction: Resolvent and the State-Transition Matrix

Consider the homogeneous linear system of differential equations:

$$\textbf{DE: } \frac{d}{dt}\vec{\mathbf{x}} = A\vec{\mathbf{x}} \qquad \textbf{IC: } \vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_{\mathbf{0}}$$

Here, $\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ is the **state vector** and the constant square matrix A is the **system matrix**.

Denote the laplace transform of the state vector $\vec{\mathbf{x}}(t)$ by $\vec{\mathbf{X}}(s)$. Taking the laplace transform of the entire DE and applying the fundamental derivative identity to each component, we find:

$$s\vec{\mathbf{X}}(s) - \vec{\mathbf{x}}(0) = A\vec{\mathbf{X}}(\boldsymbol{s})$$

or after inserting the identity matrix *I*:

$$sI\,\vec{\mathbf{X}}(s) - \vec{\mathbf{x}}(0) = A\vec{\mathbf{X}}(\boldsymbol{s})$$

so that:

$$(sI - A)\,\vec{\mathbf{X}}(s) = \vec{\mathbf{x}}(0)$$

and the vector solution in transform space can now be written:

$$\vec{\mathbf{X}}(s) = (sI - A)^{-1}\,\vec{\mathbf{x}}(0)$$

The inverse matrix $R(s) = (sI - A)^{-1}$ is called the **resolvent** of the system matrix *A*. The resolvent $R(s)$ is defined for all $s \in \mathbb{C}$ except the eigenvalues of *A*, where it is undefined.

We can also obtain the solution in the original time domain by taking the inverse Laplace transform:

$$\vec{\mathbf{x}}(t) = \mathcal{L}^{-1}((sI - A)^{-1}) \cdot \vec{\mathbf{x}}(0)$$

The time-dependent matrix $\Phi(t) = \mathcal{L}^{-1}((sI - A)^{-1})$ is called the **state-transition matrix**. Note that $\Phi(t)$ maps the initial state to the state at time *t* so that is also sometimes called the evolution operator.

Part A: The Harmonic Oscillator **2/2**

Consider the linear differential system:

$$\textbf{DE: } \frac{d}{dt}\vec{\mathbf{x}} = A\vec{\mathbf{x}} \qquad \text{with } A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad \textbf{IC: } \vec{\mathbf{x}}(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

a. Find the resolvent $R(s) = (sI - A)^{-1}$ for the harmonic oscillator. A common factor has been pulled outside this matrix for you.

$$R(s) = \frac{1}{s^2 + 1} \cdot \begin{bmatrix} s & ? \\ ? & ? \end{bmatrix}$$

b. Find the state-transition matrix $\Phi(t) = \mathcal{L}^{-1}((sI - A)^{-1}) = \mathcal{L}^{-1}(R)$ using the inverse laplace transform in MATLAB which is **`ilaplace()`**. One component has been given for you.

$$\Phi(t) = \begin{bmatrix} \cos(t) & ? \\ ? & ? \end{bmatrix}$$

c. Find the solution $\vec{\mathbf{x}}(t)$ at any time *t* using the state-transition matrix $\Phi(t)$ and the initial condition $\vec{\mathbf{x}}(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. The top component has been given for you.

$$\vec{\mathbf{x}}(t) = \Phi(t)\,\vec{\mathbf{x}}(0) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3\cos(t) + 4\sin(t) \\ ? \end{bmatrix}$$

d. Plot $x_2(t)$ (vertical axis) versus $x_1(t)$ and verify you get a circle of radius $r = 5$. Add a small yellow circle at the initial point $\vec{\mathbf{x}}(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Drop your plot inside the answer template.

Tip: Once you have found the state vector $\vec{\mathbf{x}}(t)$ using `x = phi * x0`, you can extract and convert its components into functions using **`matlabFunction`** as follows:

```
% d. Plot the solution x2 (vertical axis) versus x1
x1 = matlabFunction(x(1))
x2 = matlabFunction(x(2))
time = 0:0.01: 2*pi;
```

Then apply both functions to the vector named **time** above to get the points for your graph, which should show a circle.

Part B: Falling Apple, Nonhomogeneous Equation (No friction)

An apple of mass *m* is tossed upwards from an initial height of $y_0 = h$ meters with initial speed v_0. If $y(t)$ denotes its height above the surface, then its motion is described by the equation:

$ma = -mg$ or **DE:** $\ddot{y} = -g$ and initial conditions: **IC:** $y(0) = h,\ y'(0) = v_0$

We can recast this as a first-order system by introducing the new variables $x_1 = y$ and $x_2 = y'$.
Collect the new variables together into the **state vector**: $\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$
Then the falling apple can be described by the **nonhomogeneous** system:

DE: $\frac{d}{dt}\vec{\mathbf{x}} = A\vec{\mathbf{x}} + \vec{\boldsymbol{f}}(\boldsymbol{t})$ with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\vec{\boldsymbol{f}}(t) = \begin{bmatrix} 0 \\ -g \end{bmatrix}$ and **IC:** $\vec{\mathbf{x}}(0) = \begin{bmatrix} h \\ v_0 \end{bmatrix}$

If we denote the transform of the forcing vector $\vec{\boldsymbol{f}}(t)$ by $\vec{\boldsymbol{F}}(s)$, and the transform of the state vector $\vec{\mathbf{x}}(t)$ by $\vec{\mathbf{X}}(s)$, then the laplace transform of the entire DE is:

$$s\vec{\mathbf{X}}(s) - \vec{\mathbf{x}}(0) = A\vec{\mathbf{X}}(\boldsymbol{s}) + \vec{\mathbf{F}}(s)$$

or after inserting the identity matrix I:

$$sI\,\vec{\mathbf{X}}(s) - \vec{\mathbf{x}}(0) = A\vec{\mathbf{X}}(\boldsymbol{s}) + \vec{\mathbf{F}}(s)$$

so that:

$$(sI - A)\,\vec{\mathbf{X}}(s) = \vec{\mathbf{x}}(0) + \vec{\mathbf{F}}(s)$$

and the vector solution in transform space for this **nonhomogeneous** equation can now be written:

$$\vec{\mathbf{X}}(s) = \underbrace{(sI - A)^{-1}\,\vec{\mathbf{x}}(0)}_{\textbf{Zero-input Solution}} \;+\; \underbrace{(sI - A)^{-1}\,\vec{\mathbf{F}}(s)}_{\textbf{Zero-state Solution}}$$

Notice that the resolvent $R(s) = (sI - A)^{-1}$ now multiplies two terms.
The term on the left is the **zero-input solution** and that on the right is the **zero-state solution**.

a. Find the resolvent $R(s) = (sI - A)^{-1}$ for the falling apple. One term has been given for you.

$$R(s) = \begin{bmatrix} 1/s & ? \\ ? & ? \end{bmatrix}$$

b. Find the state-transition matrix $\Phi(t) = \mathcal{L}^{-1}((sI - A)^{-1}) = \mathcal{L}^{-1}(R)$ for the falling apple. One component is free!

$$\Phi(t) = \begin{bmatrix} 1 & ? \\ ? & ? \end{bmatrix}$$

c. Find the transform $\vec{\mathbf{F}}(s)$ of the forcing vector $\vec{\boldsymbol{f}}(t) = \begin{bmatrix} 0 \\ -g \end{bmatrix}$. Leave g symbolic for now. Be aware that the `laplace()` command in MATLAB gets confused when you give it a constant. You can easily compute F by hand, but if you want to use laplace, after entering the forcing vector using:
`f = sym([0; -g])` then find the transform using the full form `F = laplace(f, t, s)`

$$\vec{\mathbf{F}}(s) = \begin{bmatrix} 0 \\ ? \end{bmatrix}$$

d. Find the **zero-input solution** $\vec{\mathbf{X}}_{zi}$ in the s-domain and then transform that back to the time domain.
Leave h, g and v_0 as symbolic quantities. One component given for free!

$$\vec{\mathbf{X}}_{zi} = (sI - A)^{-1}\,\vec{\mathbf{x}}(0) = \frac{1}{s^2}\cdot\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}\cdot\begin{bmatrix} h \\ v_0 \end{bmatrix} = \frac{1}{s^2}\cdot\begin{bmatrix} sh + v_0 \\ sv_0 \end{bmatrix}$$

Transforming back to the time domain, we find:

$$\vec{\mathbf{x}}_{zi}(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\cdot\begin{bmatrix} sh + v_0 \\ sv_0 \end{bmatrix}\right) = \begin{bmatrix} ? \\ v_0 \end{bmatrix}$$

Since we have "turned off" gravity to focus on the zero-input solution, the apple flies off at constant speed, travelling in a straight line forever rising higher and higher at a constant speed!

e. Find the **zero-state solution** $\vec{\mathbf{X}}_{zs}$ in the s-domain and then transform that back to the time domain. This will contain all of the effects due to gravity!

$$\vec{\mathbf{X}}_{zs} = (sI - A)^{-1}\,\vec{\mathbf{F}}(s) = -\frac{g}{s^3}\cdot\begin{bmatrix} 1 \\ ? \end{bmatrix}$$

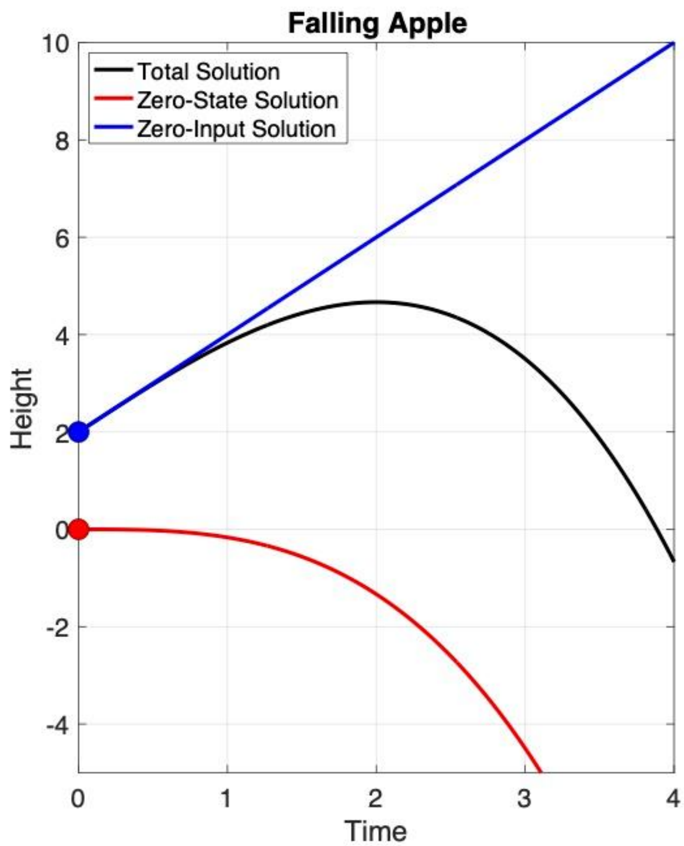
Transforming back to the time domain, we find:

$$\vec{\mathbf{x}}_{zs}(t) = \mathcal{L}^{-1}\left(-\frac{g}{s^3}\cdot\begin{bmatrix}1\\ s\end{bmatrix}\right) = -g\begin{bmatrix}t^2/2\\ ?\end{bmatrix}$$

f. Combine the zero-input and zero-state solution to obtain the **total solution** in the time domain.

$$\vec{\mathbf{x}}_{\mathbf{total}}(t) = \vec{\mathbf{x}}_{zi}(t) + \vec{\mathbf{x}}_{zs}(t) \quad = \quad \begin{bmatrix} ? \\ v_0 - gt\end{bmatrix}$$

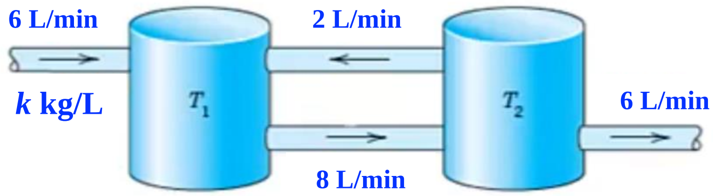
You do not need to produce this plot for the lab – but you may be asked a similar skill on the MATLAB EXAM. Setting $g = 9.81\ \frac{m}{sec^2}$, $h = 2$ meters and $v_0 = 2$ meters/sec we obtain the following component plot for the first component $x(t)$ (the height) of the state vector.



Notice that the zero-input solution (shown in **blue**) does not feel the effects of gravity while the zero-state solution (in **red**) does - but must start at the origin (in a relaxed state). The sum of these is the **total** solution (in **black**).

Part C: Two Tanks – Laplace Matrix Method **2/2**

Two tanks, each hold $V = 24$ liters of a well-stirred brine mixture. Let $Q_1(t)$ and $Q_2(t)$ denote the amount of salt in kilograms in each tank at time t (in minutes). The initial salt quantities are $Q_1(0) = 2$ and $Q_2(0) = 4$ pounds. The flow rates for the four interconnecting pipes are shown in the diagram. The inflow pipe at the top left of Tank 1 delivers brine from an infinite reservoir of concentration k kilograms per liter, so needs to be treated symbolically.



Using the basic conservation identity, the two-tank system can be shown to be governed by the differential equations:

$$\frac{d}{dt}Q_1(t) = -8\cdot\frac{Q_1}{V_1} + 2\cdot\frac{Q_2}{V_2} + 6k = -\frac{Q_1}{3} + \frac{Q_2}{12} + 6k$$

$$\frac{d}{dt}Q_2(t) = +8\cdot\frac{Q_1}{V_1} - 8\cdot\frac{Q_2}{V_2} = +\frac{Q_1}{3} - \frac{Q_2}{3}$$

Define the state vector $\vec{\mathbf{q}}(t) = \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix}$, the initial conditions vector $\vec{\mathbf{q}}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, the system matrix $A = \frac{1}{12}\cdot\begin{bmatrix} -4 & 1 \\ 4 & -4 \end{bmatrix}$ and the forcing vector $\vec{\mathbf{b}} = \begin{bmatrix} 6k \\ 0 \end{bmatrix}$. Then our two-tank system is described by the matrix equation:

DE: $\frac{d}{dt}\vec{\mathbf{q}} = A\vec{\mathbf{q}} + \vec{\mathbf{b}}$ with $A = \frac{1}{12}\cdot\begin{bmatrix} -4 & 1 \\ 4 & -4 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 6k \\ 0 \end{bmatrix}$ **IC:** $\vec{\mathbf{q}}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Taking the laplace transform of the entire DE and using capital letters for the transforms we find:

$$s\vec{\mathbf{Q}}(s) - \vec{\mathbf{q}}(0) = A\vec{\mathbf{Q}}(s) + \vec{\mathbf{B}}(s)$$

so that:

$$(sI - A)\,\vec{\mathbf{Q}}(s) = \vec{\mathbf{q}}(0) + \vec{\mathbf{B}}(s)$$

and finally, we obtain the familiar **resolvent** equation:

$$\vec{\mathbf{Q}}(s) = \underbrace{(sI - A)^{-1}\cdot\vec{\mathbf{q}}(0)}_{\textbf{Zero-input Solution}} + \underbrace{(sI - A)^{-1}\cdot\vec{\mathbf{B}}(s)}_{\textbf{Zero-state Solution}}$$

a. Enter the system matrix $A = \frac{1}{12} \cdot \begin{bmatrix} -4 & 1 \\ 4 & -4 \end{bmatrix}$, the forcing vector $\vec{\mathbf{b}} = \begin{bmatrix} 6k \\ 0 \end{bmatrix}$ and the initial conditions vector $\vec{\mathbf{q}}0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ into MATLAB. Declare *s*, *t* and *k* to be symbols first, of course.

Now find the resolvent $R(s) = (sI - A)^{-1}$ for the two-tank system. A common denominator has been given for you.

$$R(s) = \frac{1}{12s^2 + 8s + 1} \cdot \begin{bmatrix} 4 + 12s & ? \\ ? & ? \end{bmatrix}$$

b. Using **`ilaplace()`**, record the **state-transition matrix** $\Phi(t) = \mathcal{L}^{-1}((sI - A)^{-1})$ for the two-tank system. The first column has been given for you.

$$\Phi(t) = \begin{bmatrix} \frac{1}{2} \cdot e^{-\frac{t}{2}} + \frac{1}{2} \cdot e^{-t/6} & ? \\ -e^{-\frac{t}{2}} + e^{-t/6} & ? \end{bmatrix}$$

c. Find the **zero-input solution**. This is the same as setting $k = 0$, so that the infinite reservoir is imagined to deliver pure, fresh water though the volume flow rate remains unchanged.

$$\vec{\mathbf{Q}}_{\mathbf{zi}}(s) = (sI - A)^{-1} \cdot \vec{\mathbf{q}}(0) = R(s)\,\vec{\mathbf{q}}(0) = \frac{1}{12s^2 + 8s + 1} \cdot \begin{bmatrix} 4 + 12s & 1 \\ 4 & 4 + 12s \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$$= \frac{12}{(6s + 1)(2s + 1)} \cdot \begin{bmatrix} 2s + 1 \\ 4s + 2 \end{bmatrix} = \frac{2}{(s + 1/6)} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

You can do the inverse transform in your head.

$$\vec{\mathbf{q}}_{\mathbf{zi}}(t) = \begin{bmatrix} 2e^{-t/6} \\ ? \end{bmatrix} = 2e^{-t/6} \cdot \begin{bmatrix} 1 \\ ? \end{bmatrix}$$

d. Now find the **zero-state solution** in terms of the concentration *k*.

$$\vec{\mathbf{Q}}_{\mathbf{zs}}(s) = (sI - A)^{-1} \cdot \vec{\mathbf{B}}(s) = \frac{1}{12s^2 + 8s + 1} \cdot \begin{bmatrix} 4 + 12s & 1 \\ 4 & 4 + 12s \end{bmatrix} \begin{bmatrix} \frac{6k}{s} \\ 0 \end{bmatrix}$$
$$= \frac{24k}{s \cdot (6s + 1)(2s + 1)} \cdot \begin{bmatrix} 1 + 3s \\ 1 \end{bmatrix}$$

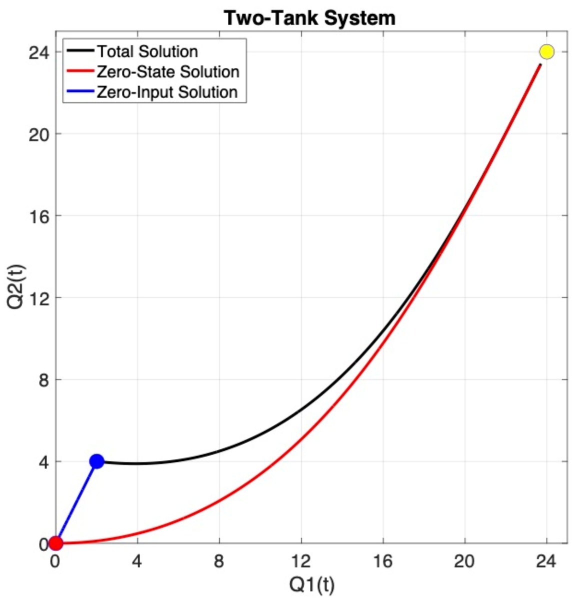
Perform the inverse transform using either **`ilaplace()`** or **`partfrac()`**.

$$\vec{\mathbf{q}}_{\mathbf{zs}}(t) = k \cdot \begin{bmatrix} 24 - 6e^{-t/2} - 18e^{-t/6} \\ ? \end{bmatrix}$$

e. Thus, the total solution in the time domain is:

$$\vec{\mathbf{q}}_{\text{total}}(t) = \vec{\mathbf{q}}_{\mathbf{zi}}(t) + \vec{\mathbf{q}}_{\mathbf{zs}}(t) = 2e^{-\frac{t}{6}} \cdot \begin{bmatrix} 1 \\ ? \end{bmatrix} + k \cdot \begin{bmatrix} 24 - 6e^{-t/2} - 18e^{-t/6} \\ ? \end{bmatrix}$$

Here for free, is the plot of the solution components after setting $k = 1$ so we can graph. Notice again how the total solution breaks cleanly into two pieces, one determined by the initial conditions and the other by the forcing term. Here are pictures of the zero-input solution (in **blue**), the zero-state solution (in **red**) and the total solution in **black**.



Part D: The Rose of Venus **3/3**

Consider the following coupled-system of first-order differential equations with unknowns $x(t)$ and $y(t)$ which describes the Earth-Venus Orbital Resonance. Here, $R_E = 1.00$ and $R_V = 0.72$ are the radius of the orbits for Earth and Venus respectively in astronomical units. The periods of the orbits are such that Venus completes 13 orbits in the time it takes Earth to complete 8 orbits. For Earth, the angular frequency in radians per year is $\omega_E = 2\pi$ while for Venus the angular frequency is: $\omega_V = 2\pi \cdot \frac{13}{8}$ All these values are given inside the function below. We defined $c = (\omega_V - \omega_E) \cdot R_V = 2.827$

DE: $\frac{dx}{dt} = -2\pi \cdot y - c \cdot \sin \omega_V t \quad \frac{dy}{dt} = +2\pi \cdot x + c \cdot \cos \omega_V t$ **IC:** $x(0) = -0.28, \ y(0) = 0$

Defining the state vector $\vec{\mathbf{x}} = \begin{bmatrix} x \\ y \end{bmatrix}$ and the initial condition $\vec{\mathbf{x}}(0) = \begin{bmatrix} -0.28 \\ 0.00 \end{bmatrix}$ this system can be represented in the matrix form:

$$\frac{d}{dt}\vec{\mathbf{x}} = A\vec{\mathbf{x}} + \vec{\mathbf{f}} \quad \text{where } A = \begin{bmatrix} 0 & -2\pi \\ 2\pi & 0 \end{bmatrix} \text{ and } \vec{\mathbf{f}} = c \cdot \begin{bmatrix} -\sin(\omega_V t) \\ +\cos(\omega_V t) \end{bmatrix}$$ represents the forcing term.

Derivation of the System: Let Earth and Venus be at their closest approach at time 0, so that Venus is between the Sun and the Earth and all three are collinear. We can represent the motion of each planet as:

$$\overset{\textbf{Earth}}{\vec{\mathbf{x}}_e = R_e \cdot \begin{bmatrix} \cos \omega_e t \\ \sin \omega_e t \end{bmatrix}} \quad \text{and} \quad \overset{\textbf{Venus}}{\vec{\mathbf{x}}_v = R_v \cdot \begin{bmatrix} \cos \omega_v t \\ \sin \omega_v t \end{bmatrix}}$$

The location of Venus as observed from the Earth is:

$$\vec{\mathbf{z}}(t) = \vec{\mathbf{x}}_v - \vec{\mathbf{x}}_e = \begin{bmatrix} R_v \cos \omega_v t - R_e \cos \omega_e t \\ R_v \sin \omega_v t - R_e \sin \omega_e t \end{bmatrix}$$

Now take the time derivative:

$$\frac{d}{dt}\vec{\mathbf{z}}(t) = \begin{bmatrix} -R_v\omega_v \sin \omega_v t + R_e\omega_e \sin \omega_e t \\ +R_v\omega_v \cos \omega_v t - R_e\omega_e \cos \omega_e t \end{bmatrix} = \omega_e \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{\mathbf{z}}(t) + \begin{bmatrix} -R_v\omega_v \sin \omega_v t + \omega_e R_v \sin \omega_v t \\ +R_v\omega_v \cos \omega_v t - \omega_e R_v \cos \omega_v t \end{bmatrix}$$

which simplifies to:

$$\frac{d}{dt}\vec{\mathbf{z}}(t) = \omega_e \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{\mathbf{z}}(t) + R_v(\omega_v - \omega_e) \begin{bmatrix} -\sin \omega_v t \\ +\cos \omega_v t \end{bmatrix}$$

You can see this agrees with the DE given above with $c = (\omega_V - \omega_E) \cdot R_V = 2.827$.

Here again is our Rose of Venus equation in matrix form:

$\frac{d}{dt}\vec{\mathbf{x}} = A\vec{\mathbf{x}} + \vec{\mathbf{f}}$ where $A = \begin{bmatrix} 0 & -2\pi \\ 2\pi & 0 \end{bmatrix}$ and $\vec{\mathbf{f}} = c \cdot \begin{bmatrix} -\sin(\omega_V t) \\ +\cos(\omega_V t) \end{bmatrix}$ represents the forcing term.

We defined $c = (\omega_V - \omega_E) \cdot R_V = 2.8274$ and $\omega_V = 2\pi \cdot \frac{13}{8}$ and the initial condition $\vec{\mathbf{x}}(0) = \begin{bmatrix} -0.28 \\ 0.00 \end{bmatrix}$

We have seen the solution can be obtained all at once using: $\vec{\mathbf{X}}(s) = (sI - A)^{-1}\left[\vec{\mathbf{x}}(0) + \vec{\mathbf{F}}(s)\right]$
where $\vec{\mathbf{F}}(s)$ is the laplace transform of the vector $\vec{\mathbf{f}}(t) = c \cdot \begin{bmatrix} -\sin(\omega_V t) \\ +\cos(\omega_V t) \end{bmatrix}$.

Enter the matrix A and the forcing vector $\vec{\mathbf{f}}$. Also enter the initial condition $\vec{\mathbf{x}}(0) = \begin{bmatrix} -0.28 \\ 0.00 \end{bmatrix}$

a. Find the **resolvent** $R(s) = (sI - A)^{-1}$ for the Rose of Venus. A common factor and the first row has been given for you. Express answers here using s and π.

$$R(s) = (sI - A)^{-1} = \frac{1}{s^2 + 4\pi^2} \cdot \begin{bmatrix} s & -2\pi \\ ? & ? \end{bmatrix}$$

b. Find the Laplace transform $\vec{\mathbf{F}}(s)$ of the forcing vector $\vec{\mathbf{f}}(t) = c \cdot \begin{bmatrix} -\sin(\omega_V t) \\ +\cos(\omega_V t) \end{bmatrix}$

For now, treat the constants c and ω_V as symbols. >> syms c wv
Note a common factor and one component has been given for free. Express answers here using s and/or ω_V.

$$\vec{\mathbf{F}}(s) = \frac{c}{s^2 + \omega_V^2} \cdot \begin{bmatrix} -\omega_V \\ ? \end{bmatrix}$$

c. Find the **zero-input** solution $\vec{\mathbf{X}}zin(s)$ (in the s domain) using $\vec{\mathbf{X}}zin(s) = (sI - A)^{-1}\,\vec{\mathbf{x}}(0)$ where $\vec{\mathbf{x}}(0) = \begin{bmatrix} -0.28 \\ 0.00 \end{bmatrix}$ Note a common factor has been given for free. Note MATLAB will express –0.28 as –7/25.

$$\vec{\mathbf{X}}zin(s) = \frac{1}{s^2 + 4\pi^2} \cdot \begin{bmatrix} -\dfrac{7s}{25} \\ ? \end{bmatrix} = \frac{1}{s^2 + 4\pi^2} \cdot \begin{bmatrix} -0.28\, s \\ ? \end{bmatrix}$$

d. Find the **zero-state** solution $\vec{\mathbf{X}}zs(s)$ (in the s domain) using $\vec{\mathbf{X}}zs(s) = (sI - A)^{-1}\,\vec{\mathbf{F}}(s)$
The top component ia given for free. Use **simplify** or **simplifyFraction**, to see the form asked for. As before, leave the constants c and ω_V as <u>symbols</u>. >> syms c wv

$$\vec{\mathbf{X}}zs(s) = \frac{c}{(s^2 + 4\pi^2)(s^2 + \omega_V{}^2)} \cdot \begin{bmatrix} -(2\pi + \omega_V) \cdot s \\ ? \end{bmatrix}$$

Combine the zero-state solution and the zero-input solution to find the **<u>total</u>** solution $\vec{\mathbf{X}}(s)$.
Then take the inverse transform to find the solution in the time domain $\vec{\mathbf{x}}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

The **total solution** for the **Rose of Venus** in the time domain has the form shown below.

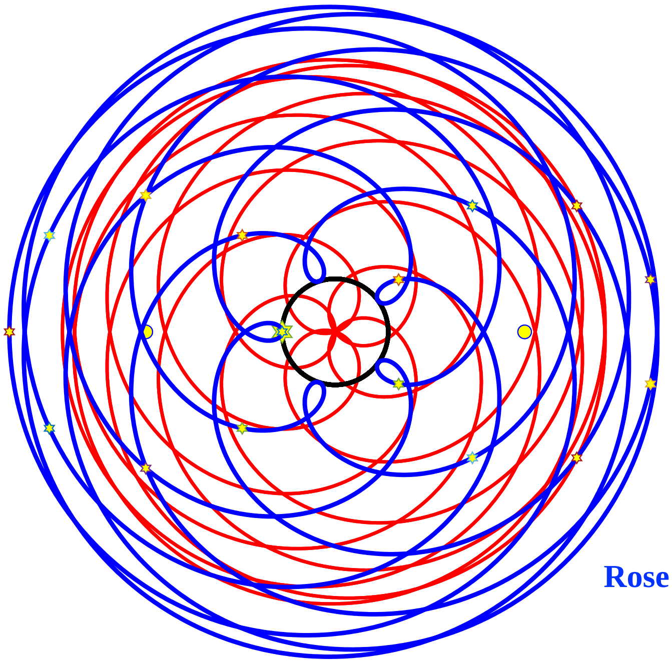
$$\vec{\mathbf{x}}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \dfrac{18}{25} \cdot \cos\left(\dfrac{13}{4}\pi t\right) & - \cos 2\pi t \\ \dfrac{18}{25} \cdot \sin\left(\dfrac{13}{4}\pi t\right) & - \sin 2\pi t \end{bmatrix}$$

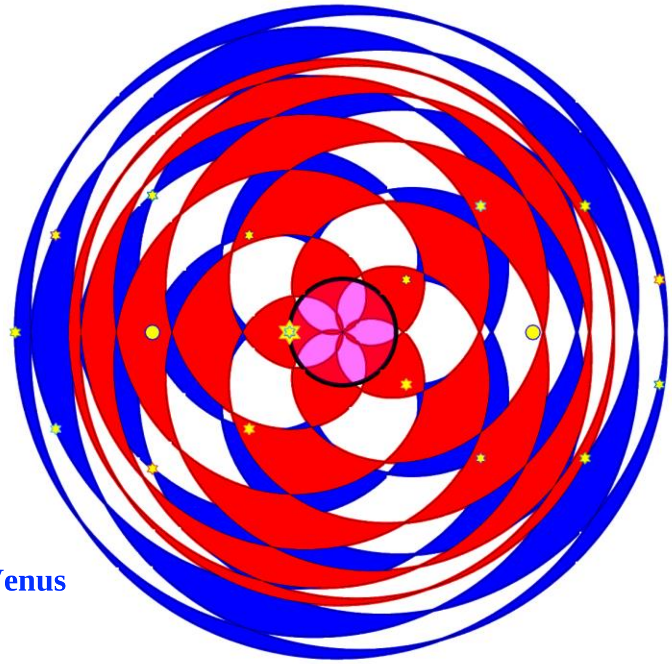
e. Find the value of $\vec{\mathbf{x}}(4)$, half-way thru the rose. **Hint:** Both entrees are real numbers.

$$\vec{\mathbf{x}}(4) = \begin{bmatrix} ? \\ 0.00 \end{bmatrix}$$

This figure is not required! Here are two views of the Rose of Venus. The small **black** circle is the **zero-input solution**. The **red** flower inside with five-fold symmetry is the **zero-state solution**. The **blue** curve is the **total solution** which you should have produced in section **B**. The view on the right uses **fill** to create a more colorful rose. The yellow stars show the relative location of Venus once every six months. You can see these trace out two circles. There is hidden beauty inside many differential equations. Enjoy!

So the color scheme is different here than in the previous graphs which was done for artistic effect.





Rose of Venus

Ready to Submit?

Be sure all ten questions are answered. When your lab is complete, be sure to submit three files:

1. Your **completed Answer Template** as a PDF file
2. A copy of your **MATLAB Live Script**
3. A **PDF** copy of your **MATLAB Live Script** (Save-Export to PDF…)

The due date is the day after your lab section by **11:59pm** to receive full credit. You have one more day, to submit the lab (but with a small penalty), and then the window closes for good and your grade will be zero.