Gaussian probability theory is completely positive

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September 25, 2025

When applied to the †-compact closed category $(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$, Selinger's *CPM construction* [1] provides an abstract setting for mixed finite dimensional quantum theory. Alternatively, following Carette et al., $\mathsf{CPM}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$ is equivalent to the †-compact closed category $\mathsf{Disc}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$ obtained by *freely discarding isometries* in $\mathsf{Mat}_{\mathbb{C}}$ in a way that is compatible with the symmetric monoidal structure [2]. Therefore, because both the discard and CPM construction coincide, one says that $(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$ has *enough isometries*.

We show that compact closed †-symmetric monoidal, and †-biproduct monoidal categories have enough isometries in case they admit a †-symmetric monoidal notion of singular value decomposition. For example, we show that the †-symmetric monoidal category of complex matrices, *under the direct sum*, $(\mathsf{Mat}_{\mathbb{C}}, \oplus, 1, \dagger)$ has enough isometries, $\mathsf{Disc}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger) \cong \mathsf{CPM}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$ is equivalent to the category of circularly symmetric complex Gaussian transformations.

1 Introduction

In this article, we investigate a fundamental, abstract connection between the categorical semantics of probability theory and quantum theory; establishing a formal connection between the process of adding classical noise (mixed states) to pure quantum theory, and the process of adding a source of Gaussian noise to linear transformations between finite dimensional vector spaces. We establish this connection by categorifying singular value decomposition to the setting of †-symmetric monoidal categories. In well-behaved settings which posses such a notion of singular value decomposition, we show that Selinger's CPM construction [1, Definition 4.8] and Carette et al.'s discard construction [2] coincide.

The discard construction

Given a †-symmetric monoidal category $(\mathscr{C}, \otimes, I, \dagger)$, Carette et al.'s *discard construction* [2] produces a symmetric monoidal category $(\mathscr{C}, \otimes, I, \dagger) \to \mathsf{Disc}(\mathscr{C}, \otimes, I, \dagger)$ by coherently adding a supply of effects, $d_X : X \to I$ to $(\mathscr{C}, \otimes, I, \dagger)$ for all objects X such that such that $d_X \otimes d_Y = d_{X \otimes Y}$, $d_I = 1_I : I \to I$, and for all isometries $U : X \to Y$, $U; d_Y = d_X$. Because we have only assumed that our category is †-symmetric monoidal, we can alternatively add a supply of codiscarding *states*, which codiscard coisometries via the codiscard constructructions $(\mathscr{C}, \otimes, I, \dagger) \to (\mathsf{Disc}(\mathscr{C}, \otimes, I, \dagger)^{\mathsf{op}})^{\mathsf{op}}$. In the case when $(\mathscr{C}, \otimes, I, \dagger)^{\mathsf{op}} \cong$

*This work has been partially funded by the European Union through the MSCA SE project QCOMICAL and within the framework of "Plan France 2030", under the research projects EPIQ ANR-22-PETQ-0007 and HQI-R&D ANR-22-PNCQ-0002.

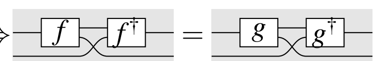
$(\mathscr{C}, \otimes, I, \dagger)$ is self-dual as a $\dagger$-symmetric monoidal category, then the discard and codiscard constructions are antiequivalent.

When the codiscard construction is applied to $(\mathsf{Mat}_{\mathbb{R}}, \oplus, 0, \dagger)$. This adds a supply of states which codiscard rotations and the terminal morphisms. This is equivalent to the symmetric monoidal category of Gaussian linear transformations between finite dimensional real vector spaces described in [3, Section 6], [4, Section 5.2] and [5, Section 6]. In this setting where the codiscard samples noise from the standard Gaussian distribution.

Similarly, when the codiscard construction is applied to $(\mathsf{Mat}_{\mathbb{C}}, \oplus, 0, \dagger)$.This adds a supply of states which codiscard unitaries and the terminal morphisms. This is equivalent to the symmetric monoidal category of circularly symmetric Gaussian linear transformations between finite dimensional complex vector spaces; where the codiscard samples noise from the standard complex Gaussian distribution.

The CPM construction

Similarly, given a $\dagger$-symmetric monoidal category $(\mathscr{C}, \otimes, I, \dagger)$, Selinger's *CPM construction* [1, Definition 4.8], $(\mathscr{C}, \otimes, I, \dagger) \to \mathsf{CPM}(\mathscr{C}, \otimes, I, \dagger)$ is constructed by adding a supply of effects, $\mathrm{Tr}_X : X \to I$ to $(\mathscr{C}, \otimes, I, \dagger)$ for all objects X, denoted graphically by $\dashv$. We again impose the equations $\mathrm{Tr}_X \otimes \mathrm{Tr}_Y = \mathrm{Tr}_{X \otimes Y}$ and $\mathrm{Tr}_I = 1_I : I \to I$ in addition to the equations imposing, for all $f : A \to S \otimes B$ and $g : A \to T \otimes B$ in $(\mathscr{C}, \otimes, I, \dagger)$:

 in $\mathsf{CPM}(\mathscr{C}, \otimes, I, \dagger) \Longleftrightarrow$  in $(\mathscr{C}, \otimes, I, \dagger)$

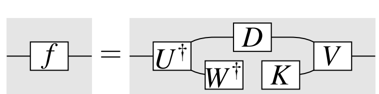
The CPM construction was introduced by Selinger under the assumption that the base category is $\dagger$-compact closed. When applied to the $\dagger$-compact closed category $(\mathsf{FHilb}, \otimes, \mathbb{C}, \dagger) \cong (\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$, this yields a categorical semantics for finite dimensional mixed quantum theory where the trace $\mathrm{Tr}_{\mathscr{H}} : \mathscr{H} \to \mathbb{C}$ is interpreted as exposing the system represented by $\mathscr{H}$ to the classical world, whose adjoint $\mathrm{Tr}^{\dagger}_{\mathscr{H}} : \mathbb{C} \to \mathscr{H}$ is interpreted as introducing a source of completely uncorrelated classical noise.

Having enough isometries

Carette et al. [2] proved there is an isomorphism of $\dagger$-compact closed categories $\mathsf{CPM}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger) \cong \mathsf{Disc}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$. In such a case when the CPM and discard construction coincide, one says that the base $\dagger$-symmetric monoidal category has *enough isometries*, manifestly making the CPM construction produce a well-defined symmetric monoidal category, even when the base category is not $\dagger$-compact closed.

Singular value decomposition and having enough isometries

In this article, we present two general sufficient conditions which guarantee that a $\dagger$-symmetric monoidal category has enough isometries, using an abstract $\dagger$-symmetric monoidal notion of singular value decomposition. Both sufficient conditions require that for map f in the base category admits what we call a *PGCSVD*, read *purifiable generalized compact singular value decomposition*. That is to say that there exist unitaries U, V, isometries W, K and an isomorphism D with the appropriate types such that:



In order to prove that a category which PCSVDs has enough isometries, we impose two essentially disjoint requirements. Either the base †-symmetric monoidal category: is *†-compact closed*; or has a monoidal structure induced by *finite †-biproducts*. To expose why these conditions are sufficient to have enough isometries, we review Cockett and Lemay's treatment of *Moore-Penrose †-categories*, and their †-categorified notions of abstract singular value decomposition [6], and then augment these notions to be coherent with †-*symmetric monoidal* structure.

The two †-symmetric monoidal structures of complex matrices

The †-compact closed categories $(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$ has PCSVDs; and thus, enough isometries, such that:

$$(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger) \longrightarrow \mathsf{CPM}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger) \xrightarrow{\cong} \mathsf{Disc}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$$

Therefore the discard and CPM constructions applied to $(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$ can be interpreted as adding a source of uncorrelated classical noise to finite dimensional quantum mechanics.

By changing the †-monoidal structure to be induced by finite †-biproducts (the direct sum and 0), the †-symmetric monoidal category $(\mathsf{Mat}_{\dagger}, \oplus, 0, \dagger)$ also has enough (co)isometries, thus:

$$(\mathsf{Mat}_{\mathbb{C}}, \oplus, 0, \dagger) \to \mathsf{CPM}((\mathsf{Mat}_{\mathbb{C}}, \oplus, 0, \dagger)^{\mathsf{op}})^{\mathsf{op}} \cong \mathsf{Disc}((\mathsf{Mat}_{\mathbb{C}}, \oplus, 0, \dagger)^{\mathsf{op}})^{\mathsf{op}}$$

These constructions can be interpreted as adding a source of noise to real matrices, sampled from the standard Gaussian distribution. The contravariance is an artifact of having added a supply of effects rather than states, which are anti-equivalent because $(\mathsf{Mat}_{\mathbb{C}}, \oplus, 0, \dagger)^{\mathsf{op}} \cong (\mathsf{Mat}_{\mathbb{C}}, \oplus, 0, \dagger)$.

That is to say, that the choice of monoidal structure used for the CPM or discard construction is the difference between categorical semantics for:

- quantum mechanics, via the tensor product,
- probability theory, via the direct sum.

Section summary:

In section 2, we review the CoPara construction which freely adds a supply of effects, compatible with the monoidal structure. Using this, in section 3, we review the CPM construction and its application to the categorical semantics of density matrices in finite-dimensional quantum mechanics. In section 4, we recall the discard construction and its relation the categorical semantics for Gaussian linear transformations in finite-dimensional probability theory. We generalize what it known about real Gaussian linear transformations to circularly symmetric complex Gaussian transformations. In section section 5 we recall examples from the literature in which the CPM construction and discard construction are known to coincide. In section 6, we review the categorical semantics of abstract singular value decompositions, taking categorical semantics in Moore-Penrose †-categories. Section 7, consists of original research; where we discuss two refinements of Moore-Penrose †-categories, and their associated notions of abstract singular value decomposition which ensure that one has enough isometries. Here we prove that real and complex matrices have enough isometries with respect to both tensor products. We discuss future work, and sketch the deeper connection to semiclassical quantum mechanics via phase-space in section 8.

Notation and assumed background:

We assume familiarity with the notions of †-symmetric monoidal categories, †-compact closed categories, biproducts and kernels.

We use diagrammatic notation for composition:

$$\frac{f : A \to B, \quad g : B \to C}{f;g : A \to C}$$

We denote symmetric monoidal categories and compact closed categories as triples $(\mathscr{C}, \otimes, I)$ where $\mathscr{C}$ is the underlying category, $\otimes$ is the monoidal product and I is the monoidal unit. Similarly, we denote †-monoidal and †-compact closed categories by quadruples $(\mathscr{C}, \otimes, I, \dagger)$ where †-denotes the †-functor. We suppress coherence isomorphisms from these tuples for notational convenience. We denote the components of the monoidal symmetry by $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$; the left and right unitors by $u^L_X : I \otimes X \to X$ and $u^R_X : X \otimes I \to X$; the associator by $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$. Denote the unit of the compact structure by $\eta_X : I \to X^* \otimes X$ and the counit by $\varepsilon_X : X \otimes X^* \to I$.

We use the special notation $(\mathscr{C}, \oplus, 0)$ for monoidal categories whose monoidal structure is induced by finite biproducts. Denote the pairing and copairing morphisms by $\Delta_X : X \to X \oplus X$ and $\nabla_X : X \oplus X \to X$. Similarly denote the initial and terminal morphisms by $?_X : 0 \to X$ and $!_X : X \to 0$.

We denote the categories of real and complex matrices between natural numbers by $\mathsf{Mat}_{\mathbb{R}}$ and $\mathsf{Mat}_{\mathbb{C}}$. The dagger functor on the real matrices is the transpose, denoted $\top$. The dagger functor on the complex matrices is given by the complex conjugate transpose, denoted †. We denote the category of finite dimensional Hilbert spaces by FHilb, where the dagger functor is the Hermitian adjoint, denoted †. By convention, we denote the Kronecker product of matrices and the bilinear tensor product of Hilbert spaces by $\otimes$.

Acknowledgments:

We thank Titouan Carette and Robert Booth for their deep insight, without which, this paper would not be possible.

2 The CoPara construction: adding a family of effects

We recall the following abstract construction, which allows us to freely add a supply of effects to a symmetric monoidal category coherent with the symmetric monoidal structure. Quotienting the result of this construction by a congruence relation will later yield the CPM and discard constructions.

A symmetric monoidal category $(\mathscr{C}, \otimes, I)$, acts on itself as a strong monad graded by its commutative monoid of objects $(\mathsf{Ob}(\mathscr{C}), \otimes, \mathsf{I})$. The Kleisli category of this graded monad yields the following construction:

Definition 1. *Given a symmetric monoidal category $(\mathscr{C}, \otimes, I)$, there is a symmetric monoidal category of coparametrized maps $\mathsf{CoPara}(\mathscr{C}, \otimes, I)$ with:*

Objects: *same as $\mathscr{C}$.*

Morphisms: *pairs $(S, f) : A \to B$ where S is an object of $\mathscr{C}$ and $f : A \to S \otimes B$ is a morphism in $\mathscr{C}$.*

Identities: *given by the inverse left unitors $1_A := (I, (u^A_L)^{-1}) : A \to A$.*

Composition: *Given some* $(S, f) : A \to B$ *and* $(T, f) : B \to C$*,* $(S, f);(T, g)$ *is defined by:*

$$(S \otimes T,\ A \xrightarrow{f} S \otimes B \xrightarrow{1_S \otimes g} S \otimes (T \otimes B) \xrightarrow{\alpha^{-1}_{S,T,B}} (S \otimes T) \otimes B\ ) : A \to C$$

Monoidal structure *The tensor product of* $(S, f) : A \to B$ *and* $(T, g) : C \to D$ *is given by*

$$(S, f) \otimes (T, g) \coloneqq (S \otimes T, h : A \otimes C \to B \otimes C)$$

where

$$\begin{tikzcd}
A \otimes C \arrow[r, "f \otimes g"] \arrow[dr, dashed, "h"'] & (S \otimes C) \otimes (T \otimes D) \arrow[r, "\alpha^{-1}_{S \otimes C,T,D}"] & ((S \otimes C) \otimes T) \otimes D \arrow[r, "\alpha_{S,T,C} \otimes 1_T"] & (S \otimes (C \otimes T)) \otimes D \arrow[d, "(1_S \otimes \sigma_{C,T}) \otimes 1_D"] \\
& (S \otimes T) \otimes (C \otimes D) & ((S \otimes T) \otimes C) \otimes D \arrow[l, "\alpha_{S \otimes T,C,D}"] & (S \otimes (T \otimes C)) \otimes D \arrow[l, "\alpha^{-1}_{S,T,C} \otimes 1_D"]
\end{tikzcd}$$

The tensor unit is given by the tensor unit in $\mathscr{C}$*. The unitors, associators, symmetry are all induced by those of* $\mathscr{C}$*. Given such a coherence isomorphism* $\phi : A \to B$ *in* $\mathscr{C}$*, the induced coherence morphism in* $\mathsf{CPM}(\mathscr{C})$ *is given by postcomposition with the inverse left unitor at B:*

$$(I,\ A \xrightarrow{\phi} B \xrightarrow{(u^L_B)^{-1}} I \otimes B\ ) : A \to B$$

In case the monoidal category posesses †-compact closed structure $(\mathscr{C}, \otimes, I, \dagger)$*, then* $\mathsf{CoPara}(\mathscr{C}, \otimes, I, \dagger)$ *has induced †-compact structure:*

Dagger structure: *Given some* $(S, f) : A \to B$*, the dagger is defined to be:*

$$(S^*,\ B \xrightarrow{(u^R_B)^{-1}} I \otimes B \xrightarrow{\eta_S \otimes 1_B} (S^* \otimes S) \otimes B \xrightarrow{\alpha_{S^*,S,B}} S^* \otimes (S \otimes B) \xrightarrow{1^*_S \otimes f^\dagger} S^* \otimes A\ ) : B \to A$$

Compact structure: *again this is given by postcomposition by the inverse left unitor:*

$$\eta_X \coloneqq (I,\ I \xrightarrow{\eta_X} X^* \otimes X \xrightarrow{(u^L_{X^* \otimes X})^{-1}} I \otimes (B^* \otimes B)\ ) : I \to B^* \otimes B$$

$$\varepsilon_X \coloneqq (I,\ B \otimes B^* \xrightarrow{\eta_X} I \xrightarrow{(u^L_I)^{-1}} I \otimes (I \otimes I)\ ) : B \otimes B^* \to I$$

When $(\mathscr{C}, \otimes, I)$ *is symmetric monoidal, there is a faithful, identity on objects strong symmetric monoidal functor* $(\mathscr{C}, \otimes, I) \hookrightarrow \mathsf{CoPara}(\mathscr{C}, \otimes, I)$*. In the case when* $(\mathscr{C}, \otimes, I, \dagger)$ *is †-compact closed then the embedding* $(\mathscr{C}, \otimes, I, \dagger) \hookrightarrow \mathsf{CoPara}(\mathscr{C}, \otimes, I, \dagger)$ *also preserves the †-structure (and thus the †-compact closed structure). This embedding sends morphisms* $f : A \to B$ *in* $\mathscr{C}$ *to their postcompositon by the inverse left unitor:*

$$(I,\ A \xrightarrow{f} B \xrightarrow{(u^L_B)^{-1}} I \otimes B\ )$$

In addition to the morphisms in the image of this embedding, for every object A in $\mathscr{C}$ *there are distinguished* ***hiding effects****:*

$$(B,\ A \xrightarrow{f} B \xrightarrow{(u^R_B)^{-1}} B \otimes I\ )$$

These two classes of morphisms generate $\mathsf{CoPara}(\mathscr{C}, \otimes, I)$ *as a symmetric monoidal category.*

The CoPara construction is just a formal way to freely add effects to a symmetric monoidal category in way which is compatible with the symmetric monoidal structure. We could have equivalently constructed $\mathsf{CoPara}(\mathscr{C},\otimes,I)$ by first *choosing* a symmetric monoidal strictification $(\mathscr{D},\otimes,I)$ for our base symmetric monoidal category $(\mathscr{C},\otimes,I)$. Next, for each object $X \in \mathscr{D}$, adding a supply of generators $X \to I$ to the symmetric monoidal equational theory for $(\mathscr{D},\otimes,I)$, thus generating a strict symmetric monoidal category which is equivalent as a strong symmetric monoidal category to $\mathsf{CoPara}(\mathscr{C},\otimes,I)$. However, the CoPara construction is the unbiased definition because it avoids choosing a symmetric monoidal strictification.

Owing to its canonical nature, the CoPara construction and its dual, the Para construction exist throughout the literature in various levels of generality by various names. The dual version of the CoPara construction we use here is exposed for example in the work of Paquette and Saville [7, 2nd example in page 3].

3 The CPM construction and classical noise in quantum mechanics

In this section, we review the CPM construction, and its quantum mechanical interpretation.

Using the CoPara construction, we can impose extra equations which govern the interaction of these hiding effects with the morphisms in the original category. The following definition reproduces the construction orignally introduced by Selinger [1]:

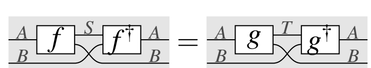
Definition 2. *Given a †-compact closed category $(\mathscr{C},\otimes,I,\dagger)$, the* ***category of abstract completely positive maps****, $\mathsf{CPM}(\mathscr{C},\otimes,I,\dagger)$ is the quotient of $\mathsf{CoPara}(\mathscr{C},\otimes,I,\dagger)$ by the congruence relation denoted*

$$(S,f) \overset{\mathsf{CPM}}{\sim} (T,g) : A \to B$$

if and only if the following diagram commutes:

$$\begin{tikzcd}
A\otimes B \arrow[r, "f\otimes 1_B"] \arrow[d, equal] & (S\otimes B)\otimes B \arrow[r, "\alpha_{S,B,B}"] & S\otimes(B\otimes B) \arrow[r, "1_S\otimes\sigma_{B,B}"] & S\otimes(B\otimes B) \arrow[r, "\alpha^{-1}_{S,B,B}"] & (S\otimes B)\otimes B \arrow[r, "f^\dagger\otimes 1_B"] & A\otimes B \arrow[d, equal] \\
A\otimes B \arrow[r, "g\otimes 1_B"] & (T\otimes B)\otimes B \arrow[r, "\alpha_{T,B,B}"] & T\otimes(B\otimes B) \arrow[r, "1_T\otimes\sigma_{B,B}"] & T\otimes(B\otimes B) \arrow[r, "\alpha^{-1}_{T,B,B}"] & (T\otimes B)\otimes B \arrow[r, "g^\dagger\otimes 1_B"] & A\otimes B
\end{tikzcd}$$

Graphically, this equation takes the following form:



The congruence class of the hiding effect at A is called the ***trace****, denoted:*

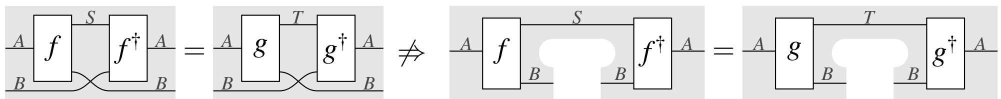
$$\mathrm{Tr}_A \coloneqq (A,(u_R^A)^{-1}) : A \to U$$

A map $(S,f) : A \to B$ is ***trace-preserving*** *when $(S,f);\mathrm{Tr}_B = \mathrm{Tr}_A$. That is to say $(S,f) : A \to B$ is trace preserving if and only if f is an isometry in $\mathscr{C}$.*

Remark 1. *When $(\mathscr{C},\otimes,I,\dagger)$ is compact closed, then then $\mathsf{CPM}(\mathscr{C},\otimes,I,\dagger)$ can be regarded as a †-compact closed category; inheriting this structure from $\mathsf{CoPara}(\mathscr{C},\otimes,I,\dagger)$.*

However, when $(\mathscr{C},\otimes,I,\dagger)$ is only †-symmetric monoidal, there is no guarantee that $\overset{\mathsf{CPM}}{\sim}$ will be a congruence with respect to postcomposition. If it happens to be a congruence, then $\mathsf{CPM}(\mathscr{C},\otimes,I,\dagger)$ will inherit tsymmetric monoidal category structure from $\mathsf{CoPara}(\mathscr{C},\otimes,I,\dagger)$. Therefore, in order to apply the CPM construction to a specific †-symmetric monoidal category, one must first prove that $\overset{\mathsf{CPM}}{\sim}$ is actually a congruence.

The problem is subtle. Given a symmetric monoidal equational theory for a symmetric monoidal category; the coherence theorem only allows one to rewrite within convex regions of the string diagram with globular boundary. However, if we try to treat the symmetry in the equivalence relation as a hole, this region of the string diagram is no longer convex, so that:



This observation was noted by Coecke and Heunen in the published version of their article [8, Remark 8] after an earlier preprint of their paper mistakenly assumed that $\overset{\mathsf{CPM}}{\sim}$ was a congurence when applied to arbitrary †-symmetric monoidal categories [9, Proposition 2]. Diagrams for †-symmetric monoidal categories with holes are themselves well-defined, but they are not in general given by the CPM-construction. Instead these diagrams are naturally formalized using pointed profunctors; and they form a symmetric polycategory rather than a symmetric monoidal category [10].

We very briefly recall the how the CPM-construction is used to represent quantum channels...

Finite dimensional quantum mechanics via the CPM construction

The CPM construction was originally motivated by Selinger in terms of its application in quantum mechanics [1]. The †-compact closed category $\mathsf{CPM}(\mathsf{FHilb}, \otimes, \mathbb{C}, \dagger) \cong \mathsf{CPM}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$ is a natural synthetic setting for quantum mechanics. For example, the purely quantum processes which are not exposed to the classical environment can be represented in terms of isometries in $\mathsf{Mat}_{\mathbb{C}}$, up to global phase. That is to say, the image of the diagram of the symmetric monoidal functor:

$$\mathsf{Isom}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger) \rightarrowtail (\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger) \to \mathsf{CPM}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$$

Where $\mathsf{Isom}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$ denotes the symmetric monoidal subcategory of isometries in $(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$. A pure quantum process which is also a state, is called a pure state.

On the other hand, quantum processes which may moreover interact with the classical environment can be represented by the trace-perserving morphisms in $\mathsf{CPM}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$. In quantum theory, these are known as quantum channels, or completely positive trace-preserving maps (hence the name CPM). The trace $\mathrm{Tr}_n : n \to 1$ in $\mathsf{CPM}(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger)$ is interpreted as the quantum channel which discards a quantum system into the classical environment. Moreover, the representatives in the equivalence classes of trace-preserving maps are called Kraus operators.

Trace preserving states are called mixed states; which are equivalently generated in terms of convex combinations of orthonormal bases of pure states. The coefficients of the pure states are interpreted as the probability of measuring the pure state with respect to that basis. In particular, the adjoint of the trace is uniform convex combination of pure states called the maximally mixed state. Measuring the maximally mixed state with respect to any basis yields the each outcome with the same probability. That is to say, it is the classical state of completely uncorrelated noise.

4 The discard construction and Gaussian probability

In this section we define an alternative equivalence relation to be compared with $\overset{\mathsf{CPM}}{\sim}$; allowing us to produce symmetric monoidal categories from mere †-symmetric monoidal categories.

First, recall the observation of Huot and Staton [11] that the category of affine symmetric monoidal categories (where the monoidal unit is terminal) is a reflective subcategory of the category of the category of symmetric monoidal categories. Taking the unit of this adjunction yields the following construction:

Definition 3. *Given a symmetric monoidal category $(\mathscr{C},\otimes,I)$ its **affine completion** $(\mathscr{C},\otimes,I)^!$ is the symmetric monoidal category given by freely making the monoidal unit I terminal.*

We immediately have the following more concrete description of the affine completion:

Lemma 1. *$(\mathscr{C},\otimes,I)^!$ is strong symmetric monoidally equivalent to the symmetric monoidal category given by quotienting the morphisms in $\mathsf{CoPara}(\mathscr{C},\otimes,I)$ by the transitive closure of the symmetric, reflexive relation $\overset{!}{\sim}$ on morphisms such that $(S,f)\overset{!}{\sim}(T,g):A\to B$ if there exist an object R and morphisms $h:S\to R$ and $k:T\to R$ such that $f;(h\otimes 1_B)=g;(k\otimes 1_B)$.*

For this article, we only want to make the tensor unit terminal on the symmetric monoidal subcategory of isometries. Recall the following definition from Carette et al. [2], which was originally stated in the strict symmetric monoidal setting:

Definition 4. *Given a †-symmetric monoidal category $(\mathscr{C},\otimes,I,\dagger)$, the **discard construction** (or the affine completion of isometries), $\mathsf{Disc}(\mathscr{C},\otimes,I,\dagger)$ is the symmetric monoidal category given by the following pushout in the category of symmetric monoidal categories and strong symmetric monoidal functors:*

$$\begin{tikzcd}
\mathsf{Disc}(\mathscr{C},\otimes,I,\dagger) \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Isom}(\mathscr{C},\otimes,I,\dagger)^! \arrow[l, tail] \\
(\mathscr{C},\otimes,I) \arrow[u] & \mathsf{Isom}(\mathscr{C},\otimes,I,\dagger) \arrow[l, tail] \arrow[u]
\end{tikzcd}$$

Where $\mathsf{Isom}(\mathscr{C},\otimes,I)$ denotes the symmetric monoidal subcategory of isometries in $(\mathscr{C},\otimes,I)$.

*The congruence class of the hiding effect at S is $((u_S^R)^{-1},S):S\to I$ is called the **discard map** on S.*

When $(\mathscr{C},\otimes,I,\dagger)$ is †-compact closed, $\mathsf{Disc}(\mathscr{C},\otimes,I,\dagger)$ inherits the †-compact closed structure from the quotient $\mathsf{CoPara}(\mathscr{C},\otimes,I,\dagger)\to\mathsf{Disc}(\mathscr{C},\otimes,I,\dagger)$.

Following Carette et al. [2], albeit working nonstrictly, we find that the discard construction admits a concrete description in terms of a quotient of the CoPara construction:

Lemma 2. *Given a †-symmetric monoidal category $(\mathscr{C},\otimes,I,\dagger)$, $\mathsf{Disc}(\mathscr{C},\otimes,I,\dagger)$ is strong symmetric monoidally equivalent to the symmetric monoidal category given by quotienting the morphisms in $\mathsf{CoPara}(\mathscr{C},\otimes,I,\dagger)$ by the transitive closure $\overset{\mathsf{Isom}^*}{\sim}$ of the reflexive, symmetric relation which identifies*

$$(S,f:A\to S\otimes B)\overset{\mathsf{Isom}}{\sim}(T,g:A\to T\otimes B)$$

if and only if there exists an object R, and isometries $U:S\to R$, $V:T\to R$ such that

$$f;(U\otimes 1_B)=g;(V\otimes 1_B)$$

As concretely as possible:

$$(S,f:A\to S\otimes B)\overset{\mathsf{Isom}^*}{\sim}(T,g:A\to T\otimes B)$$

if and only if there exists a natural number $n\geq 1$, sequences of objects

$$\{S_j\}_{j\in\mathbb{Z}/n\mathbb{Z}} \quad \text{and} \quad \{R_j\}_{j\in\mathbb{Z}/(n-1)\mathbb{Z}}$$

as well as a sequence of morphisms

$$\{f_j : A \to S_j \otimes B\}_{j \in \mathbb{Z}/n\mathbb{Z}}$$

and sequences of isometries

$$\{U_j : S_j \to R_j\}_{j \in \mathbb{Z}/n\mathbb{Z}} \quad \textit{and} \quad \{V_j : S_{j+1} \to R_j\}_{j \in \mathbb{Z}/(n-1)\mathbb{Z}}$$

such that $S_0 = S$, $S_{n-1} = T$, $f_0 = f$, $f_{n-1} = g$ *and for all* $j \in \mathbb{Z}/n\mathbb{Z}$, $f_j;(U_j \otimes 1_B) = g_j;(V_j \otimes 1_B)$.

Notice, that unlike $\underset{\sim}{\mathsf{CPM}}$, the equivalence relation $\underset{\sim}{\mathsf{Isom}^*}$ is a symmetric monoidal congruence given any base †-symmetric monoidal category. Both the CPM and discard construction can be regarded as quotients of the CoPara construction. In the former case, the hiding effects are identified with the traces; whereas in the latter case, they are identified with the discard maps.

Gaussian probability theory via the discard construction

We review basic Gaussian probability theory in order to give our first example of the discard construction:

Definition 5. *A* ***centered Gaussian distribution*** *on* $\mathbb{R}^n$*, denoted* $\mathcal{N}(\Sigma)$ *is given by a positive semidefinite real* $n \times n$ *matrix* $\Sigma = N^\top;N$*. Given* $j,k \in \{0,\cdots,n-1\}$*, the entry* $\Sigma_{j,k}$ *is interpreted as the* ***covariance*** *between the* j *and* k*th components of* $\mathbb{R}^n$*. In the cases when* Σ *is strictly positive definite, so that* Σ *is invertible, then the associated* ***probability density function*** $\Pr[(-) \in \mathcal{N}(\Sigma)] : \mathbb{R}^n \to [0,1]$ *is given by the measurable function:*

$$\Pr[\vec{x} \in \mathcal{N}(\Sigma)] = (2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp\Big(-1/2 \cdot \vec{x};\Sigma^{-1};\vec{x}^\top\Big)$$

Which is indeed a probability density function as there is a resolution of unity:

$$\int \Pr[\vec{x} \in \mathcal{N}(\Sigma)]\, d\vec{x} = 1$$

The ***standard probability distribution*** *on* $\mathbb{R}^n$ *is given by* $\mathcal{N}(I_n)$.

Remark 2. $n \times n$ *positive semidefinite matrices* Σ *with* k *nonzero eigenvalues parametrize* k*-dimensional elipsoids suspended in* $\mathbb{R}^n$ *with points* $\{\vec{x} \in \mathbb{R}^n \mid \vec{x};\Sigma;\vec{x}^\top = 0\}$*. Therefore the covariance matrix* I_n *for the standard probabillity distribution parametrizes the* n*-dimensional unit hypersphere in* $\mathbb{R}^n$.

Recall the following construction which is apparently due to Golubtsov [3, Section 6] and [4, Section 5.2]. See the review article of Fritz for a discussion in English [5, Section 6].

Definition 6. *The strict symmetric monoidal category of (real)* ***Gaussian matrices****,* $\mathsf{GMat}_\mathbb{R}$ *has:*

Objects: *Natural numbers:*

Morphisms: *are pairs* $(M,\Sigma) : n \to m$ *where* M *is a real* $m \times n$ *matrix and* Σ *is a real* $m \times m$ *positive semidefinite matrix.*

Identity: *Given by the identity paired with the zero matrix* $1_n := (I_n, 0)$.

Composition: *given* $(M,\Sigma) : n \to m$ *and* $(N,\Gamma) : m \to k$:

$$(M,\Sigma);(N,\Gamma) := (M;N, N^\top;\Sigma;N + \Gamma)$$

Monoidal product: *Given pointwise by the direct sum:*

$$(M,\Sigma)\otimes(N,\Gamma) := (M\oplus N, \Sigma\oplus\Gamma)$$

Monoidal unit *given by the object* 0*.*

Symmetry: *Given by the symmetry of matrices paired with the zero matrix.*

$\mathsf{GMat}_{\mathbb{R}}$ augments $\mathsf{Mat}_{\mathbb{R}}$ which Gaussian noise. When one computes the composition, the matrices are multiplied, and the Gaussian distribution is computed via convolution.

Lemma 3. *There is a faithful an identity on objects, strict symmetric monoidal functor* $(\mathsf{Mat}_{\mathbb{R}},\oplus,0,\top)\rightarrowtail\mathsf{GMat}_{\mathbb{R}}$ *sending* $M\mapsto(M,0)$*. The symmetric monoidal category* $\mathsf{GMat}_{\mathbb{R}}$ *is generated by image of this embedding, in addition to the morphism* $(0,1):0\to 1$*, picking out the standard Gaussian distribution on* $\mathbb{R}^1$*.*

Proof. Consider an abritrary Gaussian transformation $(M,\Sigma):n\to m$. By assumption Σ is positive semidefinite, meaning that there exists a matrix N such that $\Sigma = N^\top;N$. Therefore

$$\begin{aligned}(((0,1)^{\oplus n};(N,0))\otimes(M,0));(I_m\oplus I_m,0) &= (((0,I_n);(N,0))\otimes(M,0));(I_m\oplus I_m,0)\\ &= ((0,\Sigma)\otimes(M,0));(I_m\oplus I_m,0)\\ &= (0\oplus M,\Sigma\oplus 0);(I_m\oplus I_m,0)\\ &= (M,\Sigma)\end{aligned}$$

□

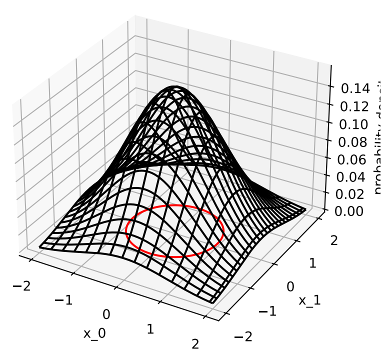
States in $\mathsf{GMat}_{\mathbb{R}}$ can be interpreted as Gaussian probability distributions. Therefore this lemma means that $\mathsf{GMat}_{\mathbb{R}}$ can be interpreted as augmenting real matrices with a single state $(0,1):0\to 1$ which samples from the standard Gaussian distribution on $\mathbb{R}$.

We will perform some calculations involving states in $\mathsf{GMat}_{\mathbb{R}}$ in order to give intuition for what is going on:

Example 1. *Consider the state* $(0,1)\otimes(0,1)=(0,I_2):0\to 2$ *which picks out the standard Gaussian distribution on* $\mathbb{R}^2$*. The density function is given by:*

$$\Pr[\vec{x}\in\mathcal{N}(I_2)] = \frac{1}{2\pi}\exp\left(\frac{-(x_0^2+x_1^2)}{2}\right)$$

We can visualize the density function as a 3-dimensional surface, where the value of the vertical axis at a point $\vec{x}\in\mathbb{R}^2$ *on the plane denotes the probability density at* $\vec{x}$*. The red circle denotes covariance*[1]*:*



[1] vizualized using the python packages matplotlib, numpy and scipy

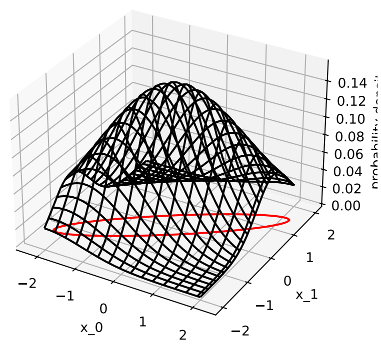
Post composing $(0, I_2) : 0 \to 2$ *with a shearing operation*

$$\left(M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, 0\right)$$

yields the state

$$(0, I_2); (M, 0) = (0, M^\top; I_2; M) = \left(0, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

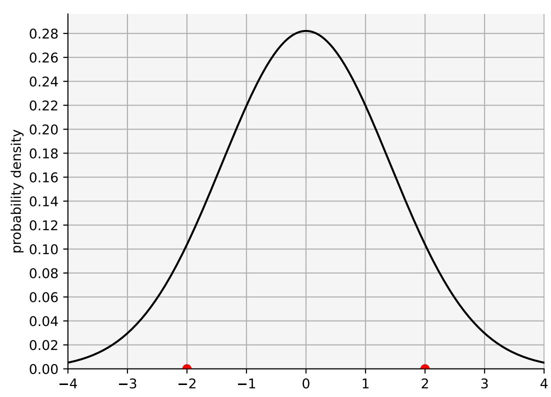
Notice that the probability density function is deformed, where the covariance is now an elipse:



Postcomposing with the projection onto the second component $(0, \begin{bmatrix} 0 & 1 \end{bmatrix}) : 2 \to 1$ *produces the morphism:*

$$\left(0, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right); (\begin{bmatrix} 0 & 1 \end{bmatrix}, 0) = (0, 2) : 0 \to 1$$

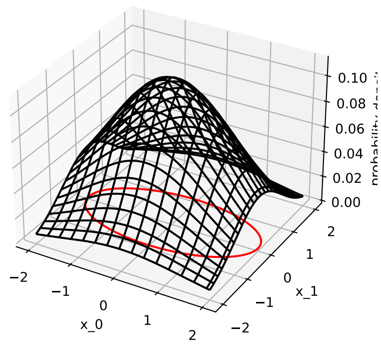
Which can be vizualized as:



We can take the monoidal product of this state with the state picking out the standard Gaussian distribution on $\mathbb{R}^1$*, yielding*

$$(0, 2) \otimes (0, 1) = \left(0, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) : 0 \to 2$$

Which can be vizualized as:



Of course, it is harder to visualize morphisms which are not states, higher dimensional Gaussian distributions, or Gaussian distributions which do not admit density functions. But the point is that $\mathsf{GMat}_{\mathbb{R}}$ provides the formal framework for working with such things.

Remark 3. *The standard Gaussian distribution $\mathcal{N}(I_n)$ on $\mathbb{R}^n$ is the unique Gaussian distribution whose covariance matrix both has determinant* 1 *(and thus no zero eigenvalues) and is invariant under the action of the group of rotations $\mathsf{SO}(n)$. This is because interpreting positive semidefinite matrices as elipsoids, the matrix I_n represents the n-dimensional unit hypersphere in $\mathbb{R}^n$.*

This invariance can be regarded universal property of the standard Gaussian distributions in the following sense (which is an immediate corollary of [12, Theorem 4.4.]):

Proposition 1. *There is a strict symmetric monoidal isomorphism of strict symmetric monoidal categories:*

$$\mathsf{Disc}(\mathsf{Mat}_{\mathbb{R}}, 0, \oplus, \dagger) \cong \mathsf{GMat}_{\mathbb{R}}^{\mathsf{op}}$$

Proof. We reprove this result for the sake of exposition. Take an arbitrary Gaussian transformation (M,Σ), factoring it as $(((0,I_n);(N,0))\otimes(M,0));(I_m \oplus I_m, 0)$ as above for some matrix N such that $\Sigma = N^\top;N$. Given a singular value decomposition (U,V,D) of N, we have the following equation in $\mathsf{GMat}_{\mathbb{R}}$:

$$(((0,I_n);(N,0)) = (((0,I_n);(U^\top;D;V,0)) = (0,(U^\top;D;V)^\top;U^\top;D;V) = (0,V^\top;D^\top;D;V)$$

Which follows syntactically by codiscarding the coisometry $U^\top$.

Given another factorization $(((0,I_n);(T,0))\otimes(S,0));(I_m \oplus I_m, 0) = (M,\Sigma)$, we have that $S = M$, so we can ignore the matrix component and focus only on the covariance matrix, yielding the equation:

$$(0,I_n);(T,0) = (0,T^\top;T) = (0,\Sigma) = (0,N^\top;N) = (0,I_n);(N,0) = (0,V^\top;D^\top;D;V)$$

Therefore, T has a singular value decomposition (W,V,D), so that in $\mathsf{GMat}_{\mathbb{R}}$:

$$(((0,I_n);(T,0)) = (0,(W^\dagger;D;V)^\top;W^\top;D;V) = (0,V^\top;D^\top;D;V) = (((0,I_n);(N,0))$$

Which follows syntactically by codiscarding the coisometry $W^\top$, yielding the desired result. $\square$

Because the unitaries in $\mathsf{CoPara}(\mathsf{Mat}_{\mathbb{R}}, \oplus, 0, \top)$ are given by the rotation matrices, and the (co)isometries are given by the unitaries and zero morphisms, that is to say that $\mathsf{GMat}_{\mathbb{R}}$ is antiequivalent to the quotient of $\mathsf{CoPara}(\mathsf{Mat}_{\mathbb{R}}, \oplus, 0, \top)$ imposing that the parameter hiding morphisms discard zero matrices and rotation matrices.

Remark 4. *Note that there is an isomorphism of †-symmetric monoidal categories* $(\mathsf{Mat}_{\mathbb{R}}, 0, \oplus, \top)^{\mathsf{op}} \cong (\mathsf{Mat}_{\mathbb{R}}, 0, \oplus, \top)$*, so this contravariance is merely an artifact of the way which the discard construction is defined in terms or discarding tensor factors of the codomain rather than codiscarding tensor factors of the domain.*

The discard construction and circularly symmetric complex Gaussian probability

The whole story with real matrices, the discard construction and the CPM construction can be adapted to the complex setting. First, we recall the relatively obscure definition:

Definition 7. *An $n \times n$ complex matrix Σ is **Hermitian positive semidefinite** in case there exists a complex matrix N such that $N^\dagger ; N = \Sigma$. Such a matrix is strictly Hermitian positive definite in case it is moreover invertible.*

Definition 8. *A **circularly symmetric Gaussian distribution** on $\mathbb{C}^n$, denoted $\mathscr{CN}(\Sigma)$, consists of an $n \times n$ Hermitian semidefinite matrix Σ, interpreted as the **complex covariance matrix**.*

In case the complex covariance matrix is strictly Hermitian positive definite, the probability density function $\Pr[(-) \in \mathscr{CN}(\Sigma)] : \mathbb{C}^n \to [0,1]$ is given by the measurable function:

$$\Pr[(\vec{z}) \in \mathscr{CN}(\Sigma)] = \det(\pi\Sigma)^{-1} \exp\left(-1/2 \cdot z ; \Sigma^{-1} ; \overline{\vec{z}}\right)$$

Again, the standard (circularly symmetric) complex Gaussian distribution on $\mathbb{C}^n$ is given by $\mathscr{CN}(I_n)$.

The term circularly symmetric means that the probability distribution is invariant under the action of $\exp(2\pi i\theta)$ on the complex hyperplane for all $\theta \in [0,1]$. In particular, when the complex covariance matrix Σ is strictly Hermitian positive definite, this means that:

$$\Pr[(- \cdot \exp(2\pi i\theta)) \in \mathscr{CN}(\Sigma)] = \Pr[(-) \in \mathscr{CN}(\Sigma)]$$

The more general notion of complex Gaussian distribution refers to a Gaussian distribution on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ which does not enjoy this nice property.

In analogy to $\mathsf{GMat}_{\mathbb{R}}$ in the real setting:

Definition 9. *The strict symmetric monoidal category of (complex) **circularly symmetric Gaussian matrices**, $\mathsf{CsGMat}_{\mathbb{C}}$, has:*

Objects: *Natural numbers:*

Morphisms: *are pairs $(M, \Sigma) : n \to m$ where M is a complex $m \times n$ matrix and Σ is a complex $m \times m$ Hermitian positive semidefinite matrix.*

Identity: *Given by the identity paired with the zero matrix $1_n := (I_n, 0)$.*

Composition: *given $(M, \Sigma) : n \to m$ and $(N, \Gamma) : m \to k$:*

$$(M, \Sigma) ; (N, \Gamma) := (M ; N, N^\dagger ; \Sigma ; N + \Gamma)$$

Monoidal product: *Given pointwise by the direct sum:*

$$(M, \Sigma) \otimes (N, \Gamma) := (M \oplus N, \Sigma \oplus \Gamma)$$

Monoidal unit *given by the object* 0*.*

Symmetry: *Given by the symmetry of matrices paired with the zero matrix.*

By the same argument a before, we have:

Proposition 2. *There is a strict symmetric monoidal isomorphism of strict symmetric monoidal categories:*

$$\mathsf{Disc}(\mathsf{Mat}_{\mathbb{C}}, \oplus, 0, \dagger) \cong \mathsf{CsGMat}_{\mathbb{C}}^{\mathsf{op}}$$

The theory is completely analogous to the real setting, the only substantive difference being that things are higher dimensional, and thus harder to visualize.

One important thing to note is that whereas where before the unitaries in real matrices were rotation matrices; now they are actually unitary complex matrices. So the codiscard rotation adds a state which discards unitary matrices and the terminal morphism.

To our knowledge, $\mathsf{CsGMat}_{\mathbb{C}}$, does not appear in the literature. However, it is the obvious generalization of $\mathsf{GMat}_{\mathbb{R}}$. In particular we have that the following diagram of strict symmetric monoidal categories and strong symmetric monoidal functors commutes, where the horizontal inclusions are given by scalar extension, and its obvious liftings:

$$\begin{tikzcd}
(\mathsf{Mat}_{\mathbb{R}}, \oplus, 0, \top) \arrow[r, tail] \arrow[d, tail, "\mathsf{Disc}(-)"'] & (\mathsf{Mat}_{\mathbb{C}}, \oplus, 0, \dagger) \arrow[d, tail, "\mathsf{Disc}(-)"] \\
\mathsf{Disc}(\mathsf{Mat}_{\mathbb{R}}, \oplus, 0, \top) \arrow[r, tail] \arrow[d, "\cong"'] & \mathsf{Disc}(\mathsf{Mat}_{\mathbb{C}}, \oplus, 0, \dagger) \arrow[d, "\cong"] \\
\mathsf{GMat}_{\mathbb{R}} \arrow[r, tail] & \mathsf{CsGMat}_{\mathbb{C}}
\end{tikzcd}$$

5 Having enough isometries

In many cases the discard construction and CPM-construction agree. This coincidence is so desirable, that Carette et al gave it a name [2]:

Definition 10. *A* $\dagger$*-symmetric monoidal category* $(\mathscr{C}, \otimes, I, \dagger)$ *has* ***enough isometries*** *in case there is an equivalence of* $\dagger$*-symmetric monoidal categories* $\mathsf{CPM}(\mathscr{C}, \otimes, I, \dagger) \cong \mathsf{Disc}(\mathscr{C}, \otimes, I, \dagger)$*.*

Therefore, when a $\dagger$-symmetric monoidal category has enough isometries, $\overset{\mathsf{CPM}}{\sim}$ is a congruence thus making the CPM construction is well defined. Moreover, Carette et al. [2] showed that:

Example 2. *The following* $\dagger$*-compact closed categories have enough isometries:*

- $(\mathsf{FHilb}, \otimes, 1, \dagger)$
- *The full subcategory of qubit complex matrices* $((\mathsf{Mat}_{\mathbb{C}})_2, \otimes, 1, \dagger)$
- *The subcategory of stabilizer circuits between qubit Hilbert spaces* $(\mathsf{Stab}_2, \otimes, 1, \dagger)$

The category of affine Lagrangian relations over an arbitrary field has also been shown to have enough isometries with the dagger induced by the symplectic conjugate [12, Section 5]. To the best of our knowledge these are the only $\dagger$-symmetric monoidal categories which have proven to have enough isometries in the literature.

Without using the techniques which we later develop in this article, it is quite involved to verify when when has enough isometries. Knowing when a fragment, or toy model of quantum mechanics has enough isometries is useful for the following reason:

Remark 5. *The* CPM *construction is a natural categorical construction with which one can add mixing to fragments of pure quantum mechanics. On the other hand, the discard construction lends itself naturally to graphical languages. If one has a complete and universal graphical languages for* $(\mathscr{C},\otimes,I,\dagger)$ *and a generating set of isometries, then to obtain a complete and universal graphical language for* $\mathsf{Disc}(\mathscr{C},\otimes,I,\dagger)$ *one simply has to introduce a supply of effects for the trace, as well as for each generating isometry, an equation imposing it is trace preserving, in a way which is compatible with the symmetric monoidal structure.*

Therefore, because both the full subcategory of qubit quantum circuits, and qubit stabilizer circuits both have enough isometries, their mixed versions both enjoy complete and universal presentations via the discard construction [2].

On the other hand, it was observed by Carette et al. [2] that qubit Clifford+T circuits, which admit a ZX-calculus presentation [13], do not possess enough isometries. In this setting, the CPM construction adds mixing, but there is no evident physical interpretation of the discard construction. This means that there is an obstruction to easily constructing a graphical language for mixed qubit Clifford+T circuits.

Outside of quantum mechanics, we show in Example 7, Section 7 that $(\mathsf{Mat}_{\mathbb{R}},\oplus,0,\top)$ and $(\mathsf{Mat}_{\mathbb{C}},\oplus,0,\dagger)$ have enough isometries, so that applying the CPM-construction yield $\mathsf{GMat}_{\mathbb{R}}$ and $\mathsf{CsGMat}_{\mathbb{C}}$. We give considerable exposition and to prove this fact in the most abstract way possible.

6 Moore Penrose Inverse categories and †-categorical notions of singular value decomposition

The condition of a †-compact closed category having enough isometries is hard to verify in general. Moreover, further complicating the matter, the CPM construction is not even guaranteed to be defined in general on †-symmetric monoidal categories. To say the least, showing that $(\mathsf{Mat}_{\mathbb{R}},\oplus,0,\top)$ has enough isometries will take some work. To this end, in this section we expose the theory of Moore-Penrose †-categories developed by Cockett and Lemay [6]:

Definition 11. *A* ***Moore-Penrose †-category*** *(MP-†-category) is a †-category* $(\mathscr{C},\dagger)$*, such that for every morphism* $f: X \to Y$*, there exists a* ***Moore-Penrose inverse*** $f^\circ : Y \to X$*, such that:*

$$f;f^\circ;f = f; \quad f^\circ;f;f^\circ = f^\circ; \quad (f;f^\circ)^\dagger = f;f^\circ; \quad (f^\circ;f)^\dagger = f^\circ;f.$$

Recall the following result of Cockett and Lemay [6, Lemma 2.4]:

Lemma 4. *In a †-category, Moore-Penrose inverses are unique.*

Example 3. *In the †-category* $(\mathsf{FHilb},\dagger) \cong (\mathsf{Mat}_{\mathbb{C}},\dagger)$*, the Moore-Penrose inverse of* f *is given by* $f^\dagger;(f;f^\dagger)^{-1}$*. Similarly, the Moore-Penrose inverse of a real matrix* f *in* $(\mathsf{Mat}_{\mathbb{R}},\top)$ *is given by* $f^\top;(f;f^\top)^{-1}$.

In their article, Cockett and Lemay [6] are interested in the case when individual maps possess properties such as having Moore-Penrose inverses. However, because we use this theory to establish sufficient conditions for having enough isometries, which is a property of entire †-categories and not their individual maps, we adapt their exposition in asking that all morphisms in a †-category have the required structure. We include citations to the relevant morphism-related definitions and results in such cases.

The theory Moore-Penrose inverses in intimately connected with the notion of singular value decomposition. In the subsequent section, we show that two different classes of Moore-Penrose †-symmetric monoidal categories admit notions of singular value decompostion which ensure that they have enough isometries. However we must first expose the theory of Moore-Penrose †-categories. Recalling the following definitions form Selinger [14]:

Definition 12. *A **†-idempotent** is a map $e : A \to A$ in a †-category such that, $e;e = e$ and $e^\dagger = e$. A †-idempotent $e : A \to A$ **†-splits** in case there is an object B and an isometry $U : B \to A$ such that $U^\dagger;U = e$.*

Example 4. *In the †-categories $(\mathsf{FHilb}, \dagger) \cong (\mathsf{Mat}_{\mathbb{C}}, \dagger)$ or $(\mathsf{Mat}_{\mathbb{R}}, \top)$ the †-idempotents are precisely the orthogonal projectors, which †-split through the kernel.*

Following Cockett and Lemay [6, Definition 3.2]:

Definition 13. *A **Moore-Penrose complete †-category** (MPC-†-category) is a Moore-Penrose †-category such that for all morphisms $f : X \to Y$, the †-idempotent $f^\circ;f$ †-splits.*

Note that a Moore-Penrose †-category always embeds into a Moore-Penrose complete category by splitting the †-idempotents $f^\circ;f$ in the sense of Selinger [14].

Recall the following definition from Cockett and Lemay [6, Definition 3.5]:

Definition 14. *Given a morphism $f : X \to Y$ in a †-category, a **generalized compact singular value decomposition** for f (GCSVD), if it exists, is a tuple (E,F,U,V,D) where E and F are objects $U : E \to X$ and $V : F \to Y$ are isometries, and $D : E \to F$ is an isomorphism such that $f = U^\dagger;D;V$.*

Recalling the following result of Cockett and Lemay [6, Corollary 3.11]:

Proposition 3. *A †-category is Moore-Penrose complete if and only if it has GCSVDs.*

In particular, given a morphism f with a GCSVD (E,F,U,V,D), its Moore-Penrose inverse f° has a GCSVD (F,E,V,U,D^{-1}).

The notion of GCSVD is motivated by the following examples:

Example 5. *Because the †-categories of real and complex matrices are Moore-Penrose complete, they have GCSVDs. This coincides with the usual notion of singular value decomposition.*

It is naive to believe that having GCSVDs implies that one has enough isometries: for one thing, there is no compatibility required with the monoidal structure. However, the special case when the †-symmetric monoidal structure is induced by finite †-biproducts is investigated by Cockett and Lemay [6, Section 4]. We expose the basic theory of †-limits to expose their work:

Definition 15. *A †-category has **†-finite biproducts** in case it has finite biproducts (ie. a binary biproduct and zero object) such that the pairing map is adjoint to the copairing $\Delta_X^\dagger = \nabla_X$ and the terminal morphism is adjoint to the initial morphism $!_X^\dagger = ?_X$.*

Definition 16. *A †-category has **†-kernels** in case it has kernels, such that for every map $f : A \to B$, the canonical injection of the kernel into the domain $\ker f \to A$ is an isometry.*

Definition 17. *Recall that having finite biproducts yields finite **sums**. That is to say, for all objects A and B, $\hom(A,B)$ is endowed with the structure of a commutative monoid*

$$(+ : \hom(A,B) \times \hom(A,B) \to \hom(A,B), 0 : \{\bullet\} \to \hom(A,B))$$

The binary sum sends parallel morphisms $f,g : A \to B$ to $f + g := \Delta_A;(f \oplus g);\nabla_B$. Where $\Delta_A : A \to A \oplus A$ denotes the pairing map, and $\nabla_B : B \oplus B \to B$ denotes the copairing map. The additive identity on an object picks out the unique map $0 : A \to B$ factoring through the zero object.

*A category with finite biproducts has **negatives** in case the finite sum commutative monoid is an Abelian group (ie, there are additive inverses of morphisms).*

Recalling the following definition of Cockett and Lemay [6, Definition 3.5]:

Definition 18. *Given a morphism $f : X \to Y$ in a †-category with finite †-biproducts, a* ***generalized singular value decomposition*** *(GSVD) for f, if it exists, is a tuple (E,F,G,H,U,V,D) where E,F,G and H are objects, $U : E \oplus G \to X$ and $V : F \oplus H \to Y$ are unitaries, and $D : E \to F$ is an isomorphism such that $f = U^\dagger;(D \oplus 0);V$.*

Recalling the following result of Cockett and Lemay [6, Proposition 4.4]:

Remark 6. *A morphism with a GSVD (E,F,G,H,U,V,D) has a GCSVD $(E,F,(1_E \oplus ?_G);U,(1_F \oplus ?_H);V,D)$*

Recalling the following result of Cockett and Lemay [6, Corollary 4.7]:

Proposition 4. *A †-category with finite †-biproducts and negatives has GSVDs if and only if it is MP-split and has †-kernels.*

Example 6. *We see the case of real and complex matrices have this structure; and the GSVD coincides with their usual notions of SVD.*

7 Abstract singular value decompositions compatible with †-symmetric monoidal structure yield enough isometries

The previous sections were exposition. This section contains entirely original research.

Asking for GSVDs is too strong to capture all of our examples. As far has having enough isometries is concerned, the factorization using finite †-biproduct structure priori not relevant for our analysis of matrices with respect to to the tensor product. However, asking for GCSVDs is almost certainly too weak to ensure that one has enough isometries when the monoidal structure is not given by †-biproducts. To this end, we will impose the following condition:

Definition 19. *A †-monoidal category admits* ***unitary purification of isometries*** *in case for every isometry $V : X \to Y$ there exists an object S, an isometry $W : I \to S$ and a unitary $U : X \otimes S \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd} X \otimes I \arrow[r, tail, "1_X \otimes W"] \arrow[d, "u_X^R"', "\cong"] & X \otimes S \arrow[d, "U", "\cong"'] \\ X \arrow[r, tail, "V"'] & Y \end{tikzcd}$$

This condition comes up all the time in quantum foundations, but it would be nice to connect it more firmly to †-categories:

Recall that †-categories being Moore-Penrose complete, having †-biproducts, kernels, †-kernels and negatives is equivalent to having GSVDs. Therefore, because the canonical injection into the †-biproduct is an isometry, it follows immediately that:

Lemma 5. *Moore-Penrose complete †-categories with finite †-biproducts, kernels, †-kernels and negatives admit unitary purification of isometries with respect to the symmetric monoidal structure induced by the finite †-biproducts.*

Using this purification assumption, we refine the notion of GCSVD in a way which mimics the GSVD, expect in a way which is compatible with a †-symmetric monoidal structure which need not come from having finite †-biproducts:

Definition 20. *Given a morphism $f : X \to Y$ in a †-symmetric monoidal category,* ***purified generalized compact singular value decomposition*** *(PGCSVD) for f, if it exists, is a tuple (E,F,G,H,U,V,D,W,K) where E,F,G and H are objects, $U : E \oplus G \to X$ and $V : F \oplus H \to Y$ are unitaries, $W : I \to G$ and $K : I \to H$ are isometries, and $D : E \to F$ is an isomorphism such that $f = U^\dagger;(D \oplus (W^\dagger;K));V$.*

Remark 7. *A morphism with GSVD (E,F,G,H,U,V,D) also has a PGCSVD $(E,F,G,H,U,V,D,?_G,?_H)$ with repsect to the symmetric monoidal structure induced by the †-biproducts.*

Therefore, it follows immediately that:

Lemma 6. *A †-symmetric monoidal with unitary purification of isometries has PCSVDs if and only if it is Moore-Penrose complete.*

In this setting, we see that our two equivalence relations $\overset{\mathsf{CPM}}{\sim}$ and $\overset{\mathsf{Isom}^*}{\sim}$ agree on the homsets $\mathsf{CoPara}(\mathscr{C},\otimes,I)(X,I)$ for all objects X in $\mathscr{C}$:

Proposition 5. *Consider an MP-complete †-symmetric monoidal category with unitary purification of isometries. Take two morphisms $f : A \to S \otimes I$ and $g : A \to T \otimes I$. Then*

$$(S,f;(u^R_S)^{-1}) \overset{\mathsf{CPM}}{\sim} (T,g;(u^R_T)^{-1})) \quad \textit{if and only if} \quad (S,f;(u^R_S)^{-1}) \overset{\mathsf{Isom}^*}{\sim} (T,g;(u^R_T)^{-1})$$

Proof. Consider an MP-complete †-symmetric monoidal category with unitary purification of isometries. Take two maps $f : A \to S$ and $g : A \to T$. If $(S,f;(u^R_S)^{-1}) \overset{\mathsf{Isom}^*}{\sim} (T,g;(u^R_T)^{-1})$ it is immediate that $(S,f;(u^R_S)^{-1}) \overset{\mathsf{CPM}}{\sim} (T,g;(u^R_T)^{-1})$. For the converse, suppose that $(S,f;(u^R_S)^{-1}) \overset{\mathsf{CPM}}{\sim} (T,g;(u^R_T)^{-1})$. By MP-completeness and purifications of isometries, f and g admit PGCSVDs

$$(E_f,F_f,G_f,H_f,f_l^\dagger,f_r,d_f,U_f^\dagger,V_f) \quad \text{and} \quad (E_g,F_g,G_g,H_g,g_l^\dagger,g_r,d_g,U_g^\dagger,V_g) \tag{1}$$

It follows immediately that the morphisms $f^\dagger;f$ and $g^\dagger;g$, has a PCGSVDs, respectively given by:

$$(E_f,E_f,G_f,G_f,f_l,f_l,f_d;f_d^\dagger,U_f,U_f) \quad \text{and} \quad (E_g,E_g,G_g,G_g,g_l,g_l,g_d;g_d^\dagger,U_g,U_g) \tag{2}$$

Together with the fact that $(S,f) \overset{\mathsf{CPM}}{\sim} (T,g)$, equation 2 implies that

$$g;g^\dagger = f;f^\dagger \tag{3}$$

By the uniqueness of MP-inverses, $(f;f^\dagger)^\circ = (g;g^\dagger)^\circ$; moreover, the two GCSVDs of $f;f^\dagger = g;g^\dagger$ induce two GCSVDs for $(f;f^\dagger)^\circ = (g;g^\dagger)^\circ$

$$(E_f,E_f,G_f,G_f,f_l,f_l,(f_d;f_d^\dagger)^{-1},U_g,U_g) \quad \text{and} \quad (E_g,E_g,G_g,G_g,g_l,g_l,(g_d;g_d^\dagger)^{-1},U_g,U_g) \tag{4}$$

It is an immediate consequence of equation 4, that the following map is an isometry:

$$h := (u^R_{F_g})^{-1};(f_d^\dagger \otimes U_f);(f_l^\dagger;g_l);((g_d^{-1})^\dagger \otimes U_g^\dagger);u^R_{F_f} : F_f \to F_g$$

We can now prove the desired result:

$$
\begin{aligned}
(S,f;(u^R_S)^{-1}) &\overset{\mathsf{Isom}}{\sim}(F_f\otimes H_f,f;f_r^{\dagger};(u^R_{F_f\otimes H_f})^{-1}) && f_r^{\dagger}\text{ is unitary}\\
&=(F_f\oplus H_f,f_l;(1_{F_f}\otimes U_f^{\dagger});(D_f\oplus I);(1_{F_f}\otimes V_f);f_r;f_r^{\dagger};(u^R_{F_f\otimes G_f})^{-1}) && \text{PGCSVD of } f\\
&=(F_f\oplus H_f,f_l;(f_d\otimes(U_f;U_f^{\dagger}));(u^R_{F_f\otimes G_f})^{-1}) && f_r\text{ is unitary}\\
&\overset{\mathsf{Isom}}{\sim}(F_f,f_l;(f_d\otimes U_f)) && U_f^{\dagger}\text{ is an isometry}\\
&\overset{\mathsf{Isom}}{\sim}(F_g,f_l;(f_d\otimes U_f);(h\otimes I)) && h\text{ is an isometry}\\
&=(F_g;f;f^{\dagger};g_l;((g_d^{-1})^{\dagger}\otimes U_g)) && \text{definition of } h\\
&=(F_g;g;g^{\dagger};g_l;((g_d^{-1})^{\dagger}\otimes U_g) && \text{equation 3}\\
&=(F_g,g_l;(g_d\otimes U_g)) && \text{canceling inverses}\\
&\overset{\mathsf{Isom}^*}{\sim}(T,g;(u^R_T)^{-1}) && \text{same argument as for } f
\end{aligned}
$$

Therefore, $(S,f;(u^R_S)^{-1})\overset{\mathsf{Isom}^*}{\sim}(T,g;(u^R_T)^{-1})$. □

Proposition 5 shows that having PGCSVD means that one has enough isometries so that effects in the CoPara construction are identified by both equivalence relations.

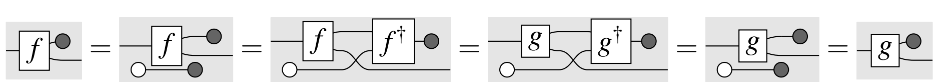
We must make additional assumptions on our base †-symmetric monoidal category to ensure that one has enough isometries to make both equivalence relations coincide in general, making the CPM construction well-defined. We assume either that our that our †-symmetric monoidal category has †-monoidal structure induced by †-biproducts (theorem 1), or is compact closed (theorem 2). For the first case:

Theorem 1. *†-idempotent complete †-categories with finite †-biproducts, †-kernels and negatives have enough isometries with respect to the †-symmetric monoidal structure induced by finite †-biproducts.*

Proof. Suppose that $\mathscr{C}$ is a †-idempotent complete category $\mathscr{C}$ with finite †-biproducts, †-kernels and negatives. We prove that the two equivalence relations coincide. As an immediate corollary of this fact it follows simultaneously that $\mathsf{CPM}(\mathscr{C},\oplus,0,\dagger)$ is a symmetric monoidal category, and that $\mathsf{CPM}(\mathscr{C},\oplus,0,\dagger)\cong\mathsf{Disc}(\mathscr{C},\oplus,0,\dagger)$.

For the first inclusion, it is immediate that $\overset{\mathsf{Isom}^*}{\sim}\subseteq\overset{\mathsf{CPM}}{\sim}$.

For the converse, suppose that $(S,f:A\to S\oplus B)\overset{\mathsf{CPM}}{\sim}(T,g:A\to T\oplus B)$. We appeal to the graphical calculus for symmetric monoidal categories which makes the proof substantially easier. Denote the initial morphisms by white states and the terminal morphisms by grey effects. We push the terminal map through $f^{\dagger}$ and then make use of the equation given by the equivalence $(S,f:A\to S\oplus B)\overset{\mathsf{CPM}}{\sim}(T,g:A\to T\oplus B)$:



Moreover, by proposition 5, we have $(S,f;(1_S\oplus !_B);u^R_S)\overset{\mathsf{Isom}^*}{\sim}(T,g;(1_T\oplus !_B);u^R_T)$.

That is to say that both projections are equivalent so that:

$$
\begin{aligned}
(S,f) &=(S,\delta_A;((f;(1_S\oplus !_B);u^R_S)\oplus(f;(!_S\oplus 1_B);u^L_B)))\\
&=(S,\delta_A;((f;(1_S\oplus !_B);u^R_S)\oplus(g;(!_T\oplus 1_B);u^L_B)))\\
&\overset{\mathsf{Isom}^*}{\sim}(S,\delta_A;((g;(1_T\oplus !_B);u^R_T)\oplus(g;(!_T\oplus 1_B);u^L_B)))\\
&=(T,g)
\end{aligned}
$$

□

This is true for our running example:

Example 7. *Both* $\mathsf{Mat}_{\mathbb{C}} \cong \mathsf{FHilb}$ *and* $\mathsf{Mat}_{\mathbb{R}}$ *satisfy this criteria, and thus have enough isometries with respect to the †-symmetric monoidal structure induced by their finite †-biproducts. Therefore, the symmetric monoidal category of abstract completely positive maps is well-defined* with respect to the direct sum*, and equivalent to the discard construction:*

$$\mathsf{CPM}(\mathsf{Mat}_{\mathbb{R}}, \oplus, 0, \top) \cong \mathsf{Disc}(\mathsf{Mat}_{\mathbb{R}}, \oplus, 0, \top) \cong \mathsf{GMat}_{\mathbb{R}}^{\mathsf{op}}$$
$$\mathsf{CPM}(\mathsf{Mat}_{\mathbb{C}}, \oplus, 0, \dagger) \cong \mathsf{Disc}(\mathsf{Mat}_{\mathbb{C}}, \oplus, 0, \dagger) \cong \mathsf{CsGMat}_{\mathbb{C}}^{\mathsf{op}}$$

Similarly, instead of asking for finite †-biproducts, asking for compact closure also suffices.

Theorem 2. *Moore-Penrose complete compact closed †-symmetric monoidal categories with unitary purification have enough isometries.*

Proof. Consider a Moore-Penrose complete compact closed †-symmetric monoidal category $(\mathscr{C}, \otimes, I, \dagger)$. Suppose that

$$(S, f : A \to S \otimes B) \overset{\mathsf{Isom}^*}{\sim} (T, f : A \to T \otimes B)$$

It follows immediately that

$$(S, f : A \to S \otimes B) \overset{\mathsf{CPM}}{\sim} (T, f : A \to T \otimes B)$$

Conversely, suppose that $(S, f : A \to S \otimes B) \overset{\mathsf{CPM}}{\sim} (T, g : A \to T \otimes B)$. By compact closure, f and g can be curried into maps $f' : A \otimes B^* \to S$ and $g' : A \otimes B^* \to S$. Because $\overset{\mathsf{CPM}}{\sim}$ is a congruence, it follows that

$$(S, f'; (u_S^R)^{-1}) \overset{\mathsf{CPM}}{\sim} (T, f'; (u_T^R)^{-1})$$

Therefore by proposition 5,

$$(S, f'; (u_S^R)^{-1}) \overset{\mathsf{Isom}^*}{\sim} (T, f'; (u_T^R)^{-1})$$

Because $\overset{\mathsf{Isom}^*}{\sim}$ is a congruence, it follows that

$$(S, f : A \to S \otimes B) \overset{\mathsf{Isom}^*}{\sim} (T, f : A \to T \otimes B)$$

□

Note that we do not require that the compact closed †-symmetric monoidal category be †-compact closed, so that the compact closed structure need not be compatible with the †-functor.

This is also true for all of our compact closed examples:

Example 8. *Both* $(\mathsf{Mat}_{\mathbb{C}}, \otimes, 1, \dagger) \cong (\mathsf{FHilb}, \otimes, \mathbb{C}, \dagger)$ *and* $(\mathsf{Mat}_{\mathbb{R}}, \otimes, 1, \top)$ *are Moore-Penrose complete †-compact closed categories admitting unitary purification of isometries. Similarly for* $((\mathsf{Mat}_{\mathbb{C}})_2, \otimes, \mathbb{C}, \dagger)$ *and* $(\mathsf{Stab}_2, \otimes, \mathbb{C}, \dagger)$. *Therefore, this gives a uniform conceptual explanation for why these categories have enough isometries.*

Similarly,

Example 9. *Affine Lagrangian relations form a Moore-Penrose complete †-compact closed category admitting unitary purification, with respect to the aforementioned dagger.*

This also gives us a counterexample:

Example 10. *Because qubit Clifford+T quantum circuits are †-compact closed with unitary purification of isometries, it does not have enough isometries precisely because it is not a Moore-Penrose complete category.*

Asking for compact closure with respect to monoidal structure induced by finite †-biproducts is extremely degenerate, so these really do give two interesting, essentially disjoint, classes of examples of †-symmetric monoidal categories with enough isometries. Moreover, when applied to the two different †-symmetric monoidal structures of complex matrices, the CPM/discard construction does not produce the same category, even when ones forgets the symmetric monoidal structure. We already know that $\mathsf{Mat}_{\mathbb{C}}$ embeds into $\mathsf{CsGMat}_{\mathbb{C}}$; whereas the induced morphism of $\mathsf{Mat}_{\mathbb{C}}$ into $\mathsf{CPM}(\mathsf{Mat}_{\mathbb{C}}, \otimes 1, \dagger)$ quotients global phase.

Remark 8. *These two extra conditions appearing in theorems 1, and 2 are very close to the conditions used by Hefford and Comfort two prove the equivalence between extensional and extensional combs [10]. Up to our invocation of Proposition 5, the proof of theorem 1 is essentially the same as [10, proposition 4]; and the proof of theorem 2 is essentially the same as [10, proposition 3]. It is a simple consequence of [10, section 3], that under both of our additional assumptions on top of having PGCSVDs, the* CPM *construction is not only equivalent to the discard construction, but also to their the so-called †-optic and †-comb constructions which are two other quotients of the* CoPara *construction.*

8 Conclusion

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