A ZX-calculus for continous-variable Gaussian quantum circuits

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Stabiliser quantum mechanics

Stabiliser states

Definition
Pauli groups $\mathcal{P}_p^{\otimes n}$ generated by:

$$\mathcal{X}\,|k\rangle = |k+1\rangle \quad \text{and} \quad \mathcal{Z}\,|k\rangle = e^{i\frac{2\pi}{p}k}\,|k\rangle$$

Definition
Each maximal Abelian subgroup $S \subseteq \mathcal{P}_p^{\otimes n}$ uniquely determines a pure quantum state $|S\rangle : \mathcal{H}_p^{\otimes n}$ such that for all $U \in S$, $U\,|S\rangle = |S\rangle$.

S is called the **stabiliser group** associated to the **stabiliser state** $|S\rangle$ and vice-versa.

- If S were not maximal then $|S\rangle$ would not be pure.
- If S were not Abelian then $|S\rangle$ would not be well-defined.

Geometry of stabiliser states

Lemma

The stabiliser group of an n-quopit stabiliser state is represented by an affine $S \subseteq \mathbb{F}_p^{2n}$.

$$\text{Each} \quad (\vec{z}, \vec{x}) \in \mathbb{F}_p^{2n} \quad \text{is identified with} \quad \bigotimes_{j=0}^{n-1} \mathcal{Z}^{z_j} \mathcal{X}^{x_j} : \mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}$$

Lemma

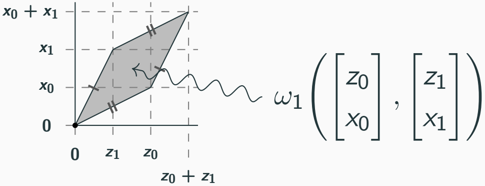
Conversely, every n dimensional affine subspace $S \subseteq \mathbb{F}_p^{2n}$ corresponds to a stabiliser state precisely when all the corresponding Pauli operators commute.

$$\text{That is when, for all} \quad (\vec{z}, \vec{x}), (\vec{q}, \vec{p}) \in \mathbb{F}_p^{n} : \quad \omega_n((\vec{z}, \vec{x}), (\vec{q}, \vec{p})) = 0.$$

$$\text{Where} \quad \omega_n : \mathbb{F}_p^{2n} \oplus \mathbb{F}_p^{2n} \to \mathbb{F}_p; \quad ((\vec{z}, \vec{x}), (\vec{q}, \vec{p})) \mapsto \vec{z} \cdot \vec{p} - \vec{x} \cdot \vec{q}.$$

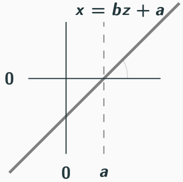
Geometry of single quopit stabiliser states

For single quopits, $\omega_1((z_0, x_0), (z_1, x_1)) = (z_0, x_0) \wedge (z_1, x_1)$ is the volume form!



Therefore, stabiliser states are in bijection with lines in $\mathbb{F}_p^2$.

The points on the lines are the stabilisers:



Unitary transformations of stabilisers

Definition
The **Clifford group** $\mathcal{C}_p^n$ for n quopits is the normaliser of $\mathcal{P}_p^{\otimes n}$ in the group of unitaries on $\mathcal{H}_p^{\otimes n}$: $U \in \mathcal{C}_p^n$ if and only if $UPU^\dagger \in \mathcal{P}_p^{\otimes n}$ for all $P \in \mathcal{P}_p^{\otimes n}$.

Lemma
If S is a stabiliser group, and $C \in \mathcal{C}_p^n$, then $CSC^\dagger$ is the stabiliser group of the stabiliser state $\left|CSC^\dagger\right\rangle = C\left|S\right\rangle$

Corollary
The group of affine isomorphisms $\mathbb{F}_p^{2n} \to \mathbb{F}_p^{2n}$ which preserve ω_n is isomorphic to $\mathcal{C}_p^n$ modulo phase.

For single quopits, these are the affine isomorphisms that preserve the volume.

Stabiliser effects

Lemma
The postselected projection of a stabiliser state $|\psi\rangle : \mathcal{H}_p^{\otimes(m+n)}$ *onto a stabiliser state* $|\varphi\rangle : \mathcal{H}_p^{\otimes m}$ *is a stabiliser state:* $(\langle\psi| \otimes I_n)\,|\varphi\rangle : \mathcal{H}_p^{\otimes n}$.

How can this be connected to the $\mathbb{F}_p$-affine picture in a clean way?????

Definition
The **prop of affine Lagrangian relations** $\mathsf{AffLagRel}_{\mathbb{K}}$ has:

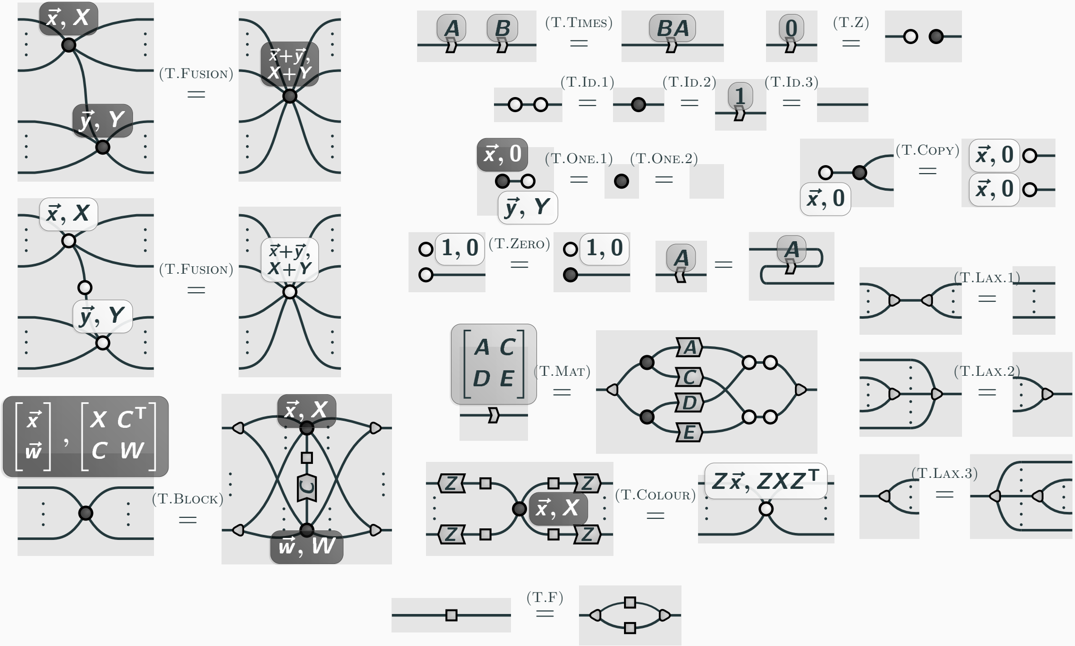
- **Arrows** $n \to m$ are given by maximally commuting affine subspaces of $S \subseteq \mathbb{K}^{2n} \oplus \mathbb{K}^{2m}$.
- **Composition** is given by relational composition.
  That is for $S \subseteq \mathbb{K}^{2n} \oplus \mathbb{K}^{2m}$ and $R \subseteq \mathbb{K}^{2m} \oplus \mathbb{K}^{2k}$:

$$R \circ S := \{(\vec{x}, \vec{z}) \in \mathbb{K}^{2n} \oplus \mathbb{K}^{2k} \mid \exists \vec{y} \in \mathbb{K}^{2m} : (\vec{x}, \vec{y}) \in S, (\vec{y}, \vec{z}) \in R\}$$

- **The monoidal structure** is given by the direct sum.

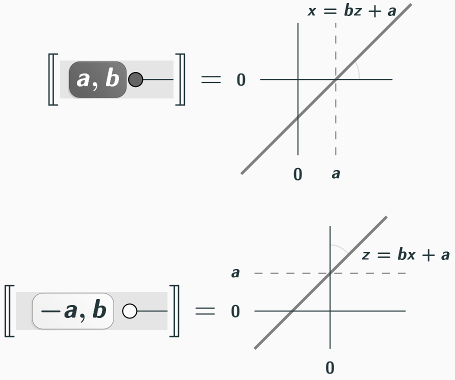
Theorem (Comfort and Kissinger [CK22])
There is a projective equivalence between the quopit stabiliser theory and $\mathsf{AffLagRel}_{\mathbb{F}_p}$

The stabiliser ZX-calculus (Booth et al. [BCC24b]/Poór et al. [Poó+23])



Geometric interpretation of 1-dimensional spiders

1-dimensional spiders with labels $(a, b) \in \mathbb{K}^2$ represent lines in $\mathbb{K}^2$ with origin $\pm a$ and slope b:

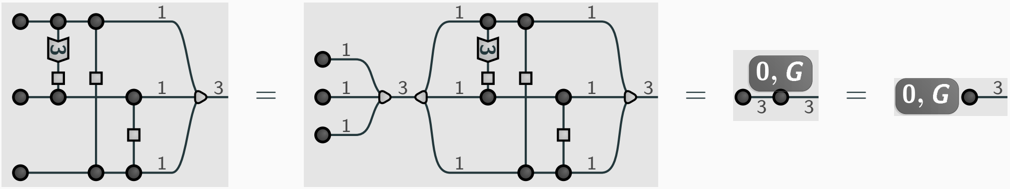


$\mathcal{Z}^z \left|+\right\rangle$ is a grey spider with label $(z, 0)$.

$\mathcal{X}^x \left|0\right\rangle$ is a grey spider with label $(-x, 0)$.

n-dimensional spiders labelled by $(0, G) \in \mathbb{K}^n \times \mathrm{Sym}_n(\mathbb{F}_p)$ represent graph states.

For example taking $G = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then:



Gaussian quantum mechanics

The shortest introduction to continous-variable quantum computing

The state space of a quantum particle in free 1D space, a **qumode**, is the Hilbert space of complex square-integrable functions:

$$L^2(\mathbb{R}) := \left\{ \varphi : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{\mathbb{R}} |\varphi(x)|^2 \,\mathrm{d}x < \infty \right\}$$

This space is equipped with **position** and **momentum** observables $L^2(\mathbb{R}) \to L^2(\mathbb{R})$:

$$\hat{q}f(x) := xf(x) \quad \text{and} \quad \hat{p}f := \frac{\partial f}{\partial x}$$

and **displacement** operators (in analogy to Paulis) $L^2(\mathbb{R}) \to L^2(\mathbb{R})$:

$$\hat{X}(s)\varphi(x) := \varphi(x - s) \quad \text{and} \quad \hat{Z}(t)\varphi(x) := e^{i2\pi tx}\varphi(x)$$

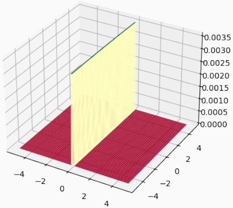
Lemma
Displacement operators can be obtained by exponentiating position/momentum:

$$\hat{X}(s) = \exp(is\hat{p}) \quad \textit{and} \quad \hat{Z}(t) = \exp(it\hat{q})$$

The problem with continuous-variable stabiliser theory

- Naïvely, one might try to associate maximally commuting subspaces of $\mathbb{R}^{2n}$ to CV stabiliser states on $L^2(\mathbb{R}^n)$.
- However, the induced map $\mathbb{C} \to L^2(\mathbb{R}^n)$ is no longer continuous!
- Such CV stabilliser states are called "infinitely squeezed" by physicists because they have infinite energy.
  The canonical such example is the Dirac delta distribution.
- We can resolve this problem by applying Gaussian convolution to $\mathrm{AffLagRel}_{\mathbb{R}}$:

Dirac delta distribution

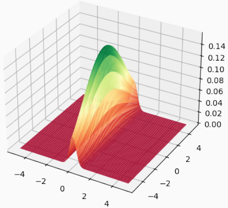


convolution by

$$\mathcal{N}\left(\begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$\longmapsto$$

Gaussian density function



Gaussian quantum mechanics

Quantum states obstained by the Gaussian convolution of CV stabilisers have the following form:

Definition
An n-qumode **Gaussian state** $\varphi \in L^2(\mathbb{R}^n)$ is given by:

$$\varphi(\vec{x}) = \exp(i\alpha)\exp\left(i\vec{s}^{\mathsf{T}}\vec{x}\right)\sqrt[4]{\det(\mathrm{Im}(\Phi))/\pi^n}\exp\left(i(\vec{x}-\vec{t})^{\mathsf{T}}\Phi(\vec{x}-\vec{t})/2\right)$$

For some $\alpha \in [0, 2\pi)$, $\vec{s}, \vec{t} \in \mathbb{R}^n$, and $\Phi \in \mathrm{Sym}_n(\mathbb{C})$ with $\mathrm{Im}(\Phi) \succ 0$.

Definition
Gaussian unitaries are those unitaries that "preserve Gaussianity."

Lemma
Post-selecting a Gaussian state on a Gaussian effect is Gaussian.

The nullifier formalism
(folklore, but exposed in Gu et al. [Gu+09], Menicucci et al. [MFL11])

Definition
A **nullifier** of a CV quantum state φ is a $+0$-eigenvector $\hat{K}\varphi = \mathbf{0}$

In general, CV stabiliser "states" are nullified by real affine combinations of $\hat{p}$ and $\hat{q}$:

Example
The Dirac delta is nullified by $\hat{q}$, and thus stabilised by $\exp(i\hat{p}) = \hat{X}$.

...whereas, Gaussian states are nullified by *complex* affine combinations:

Example
The **vacuum state** on one qumode (defined by $s = t = 0$ and $\Phi = i$)
is nullified by the **annihilation** operator $\hat{a} = \hat{q} - i\hat{p}$.

One Gaussian state can have many different nullifiers...

Geometry of nullifier space

The space of nullifiers of a Gaussian state can be captured in terms an affine Lagrangian subspace *over the complex numbers*:

Definition
A complex affine Lagrangian subspace $(S, \vec{a}) \subseteq \mathbb{C}^{2n}$ is **positive** when:

- $\vec{a} \in \mathbb{R}^{2n}$;
- for all $\vec{v} \in S$, then $i\omega_n\left(\overline{\vec{v}}, \vec{v}\right) \geqslant 0$.

Lemma (Booth et al. [BCC24a])
Positive affine Lagrangian subspaces represent the nullifiers of both Gaussian states and CV stabiliser states:

- *When S restricts to a real subspace, these represent precisely CV stabiliser states.*
- *Otherwise, these represent precisely Gaussian states.*

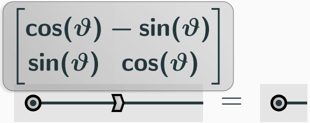
The nullifier formalism, compositionally

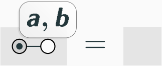
We take the same approach as for stabilisers, and represent states/unitaries/postselections in terms of affine subspaces:

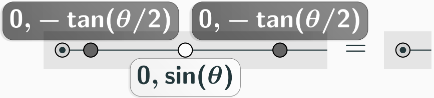
Definition
Let $\mathsf{AffLagRel}^{+}_{\mathbb{C}}$ denote the sub-prop of $\mathsf{AffLagRel}_{\mathbb{C}}$ of *positive* affine Lagrangian relations.

Theorem (Booth et al. [BCC24a])
$\mathsf{AffLagRel}^{+}_{\mathbb{C}}$ *captures the nullifier theory for Gaussian quantum mechanics, formally extended with CV stabilisers.*

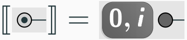
Theorem (Booth et al. [BCC24a])
We extend the graphical language for $\mathsf{AffLagRel}_{\mathbb{R}}$ *by a single state*  *(interpreted as the vacuum state) to obtain a presentation for* $\mathsf{AffLagRel}^{+}_{\mathbb{C}}$*, so that for all* $\vartheta \in [0, 2\pi)$ *and* $\theta \in (-\pi, \pi)$*:*







where

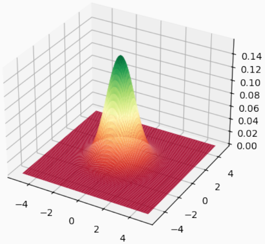
Corollary
Because $\mathsf{AffLagRel}^{+}_{\mathbb{C}} \hookrightarrow \mathsf{AffLagRel}_{\mathbb{C}}$ *it's sound to make the identification*


and use the equations of $\mathsf{AffLagRel}_{\mathbb{C}}$*.*

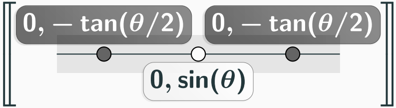
Rotational invariance of the vacuum state

The vacuum state is the unique Gaussian distribution satisfying the Heisenberg uncertainty principle which is invariant under rotation.

In other words, it has uniform covariance in all dimensions. For a single qumode:



Where



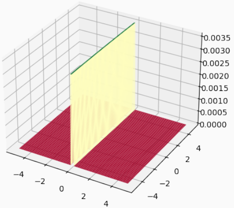
$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$ rotates $\mathbb{R}^2$ by θ.

The second equation is a higher dimensional generalization of this property.

Picturing Gaussian convolution

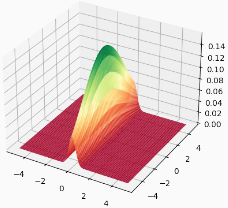
Dirac delta distribution

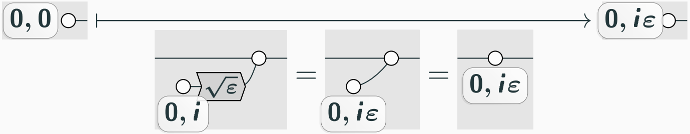
Gaussian density function



convolution by

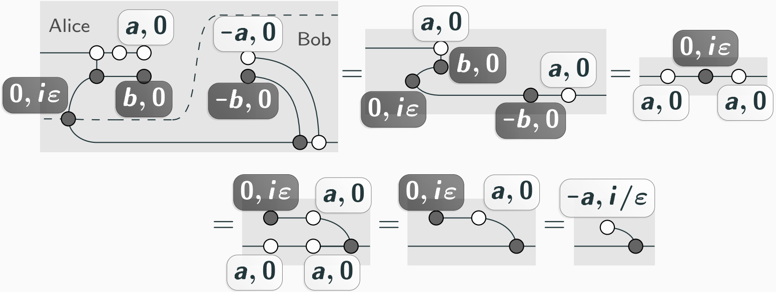
$$\mathcal{N}\left(\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{bmatrix}, \begin{bmatrix}0\\ 0\end{bmatrix}\right)$$





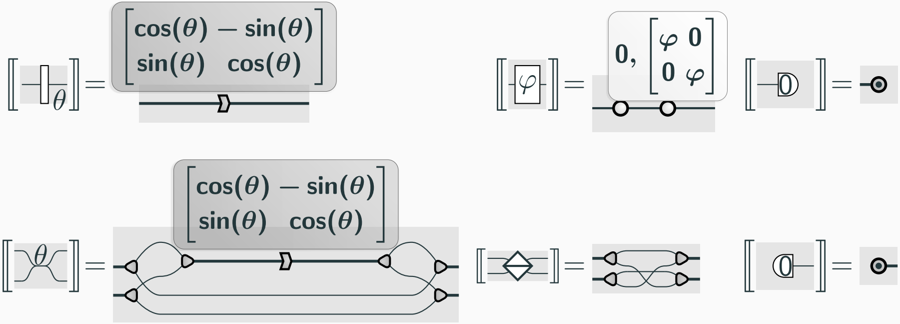
Picturing continuous-variable quantum teleportation

We can interpret the original quantum teleportation algorithm of Braunstein and Kimble [BK98]:



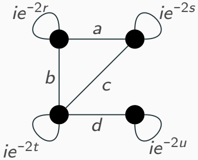
Interpreting the LOv-calculus

We can interpret the LOv-calculus of Clément et al. [Clé+22]:

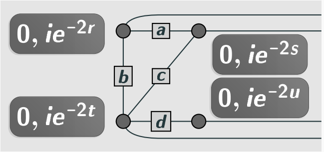


Interpreting CV cluster states

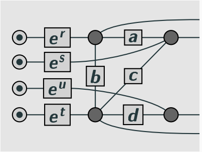
The graph-theoretic representation for Gaussian states of Menicucci et al. [MFL11] can be directly translated into our calculus:



$\longmapsto$



$\longmapsto$



The graph transformation rules all follow by completeness!

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