Denotational semantics for stabiliser quantum programs

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Abstract

The stabiliser fragment of quantum theory is a foundational building block for quantum error correction and the fault-tolerant compilation of quantum programs. In this article, we develop a sound, universal and complete denotational semantics for stabiliser operations which include measurement and classical control and in which quantum error-correcting codes are first-class objects. The operations are interpreted as certain *affine relations*, offering a significantly simpler alternative to the standard operator-algebraic semantics of quantum programs.

We demonstrate the power of the resulting semantics by describing a small, proof-of-concept assembly language for stabiliser programs with fully-abstract denotational semantics. These results pave the way to formally verified fault-tolerant quantum compilers based on the stabiliser fragment.

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1 Introduction

The problem of compiling quantum algorithms into fault-tolerant hardware-level instructions is a central challenge in the design of scalable quantum systems [10, 3, 32]. Although the stabiliser fragment of quantum mechanics can be simulated efficiently on a classical computer [2, 28], it nevertheless forms the theoretical basis for much of quantum error correction (QEC) [22]. In particular, using techniques such as magic state injection [7], the stabiliser fragment can be bootstrapped to enable universal quantum computation, exploiting the error-correcting properties of the stabiliser fragment to guarantee fault-tolerance. For fault-tolerant compilation to scale, we need a better understanding of the compositional structure of fault-tolerance, and therefore of the stabiliser fragment.

In this article, we develop a *denotational semantics* for quantum programs built from stabiliser operations, including Clifford operators, Pauli errors, Pauli measurement and classically-controlled Pauli operators. In other words, we introduce a formal setting to reason about the fault-tolerance properties of quantum programs. This framework draws from two lines of work: the categorical semantics of quantum programming languages and quantum computing [41, 40, 39, 27]; and the symplectic representation of pure stabiliser circuits [24, 31, 26, 15, 6]. Ultimately, our goal is to support the development of formally verified fault-tolerant quantum compilers based on the stabiliser fragment, integrating current approaches to compilation [17, 16, 25, 34] and verification [37, 12, 29, 45, 43, 19, 33, 36]. We hope that our compositional framework for fault-tolerance can be integrated into more computationally expressive quantum programming languages [23, 20].

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The categorical semantics of quantum theory builds on the mathematical semantics of finite-dimensional quantum processes with measurement and classical control. These semantics can be formally stated in the language of operator algebras [8], and are built in three stages of increasing expressivity:

1. *Pure quantum mechanics* via finite-dimensional Hilbert spaces;
2. *Mixed quantum mechanics* via completely-positive maps between matrix algebras;
3. *Quantum measurements and classical control* via completely-positive maps between finite-dimensional C^*-algebras.

These increasing stages of expressivity can be restated by applying the following functorial constructions to the †-compact-closed category, FHilb, of finite-dimensional Hilbert spaces and linear maps:

$$\textit{pure}\ \mathrm{QM} \xrightarrow{\text{CPM construction [39]}} \textit{mixed}\ \mathrm{QM} \xrightarrow{\text{Splitting }\dagger\text{-idempotents [40]}} \mathrm{QM\ w/}\ \textit{measurements}$$

Finite-dimensional quantum mechanics can therefore be understood in purely categorical terms, agnostic to the theory of operator algebras. This point of view is highly amenable to generalisation and specialisation: simply replace FHilb with any other †-compact-closed category, and apply these constructions to add abstract notions of mixing and measurement.

In this article, we work with †-compact-closed categories specifically tailored to the stabiliser fragment. The first semantics is obtained directly by restricting FHilb to the stabiliser fragment; whereas, the second semantics is given by the symplectic representation of stabiliser maps. Specifically, we work throughout with odd-prime-dimensional quantum systems, which ensures that the symplectic representation is well-behaved. Whilst the full symplectic semantics breaks down in even characteristic, we can nevertheless recover the theory of CSS codes in the qubit case [9, 42, 14, 30].

Outline. Section 2 begins with a review of the stabiliser formalism and its symplectic formulation. We then describe novel denotational semantics for mixing in section 3 and measurement in section 4. In each section, we develop the standard operator-theoretic semantics given by restriction, and the corresponding symplectic representation, proving their equivalence:

$$\begin{tikzcd}
\text{pure stabiliser theory} \arrow[r, leftrightarrow, "\text{[31, Chapter 9],[15],(Section 2)}"] \arrow[d, "\text{CPM construction}"'] & \text{affine Lagrangian relations} \arrow[d, "\text{CPM construction}"] \\
\text{mixed stabiliser theory} \arrow[r, leftrightarrow, "\text{Section 3}"] \arrow[d, "\text{Splitting }\dagger\text{-idempotents}"'] & \text{affine coisotropic relations} \arrow[d, "\text{Splitting }\dagger\text{-idempotents}"] \\
\text{stabiliser theory with measurements} \arrow[r, leftrightarrow, "\text{Section 4}"] & \text{affine relations with symplectic types}
\end{tikzcd}$$

Finally, in section 5, we define a simple imperative language for stabiliser quantum programs, including Pauli measurement and classically-controlled Pauli operators, equipped with a fully abstract denotational semantics derived from section 4.

Contributions. We present several novel contributions:

- Corollary 23: a symplectic, relational semantics for completely positive stabiliser maps;
- Theorem 39: we model stabiliser quantum measurements and classical control as affine relations augmented with a modality to represent quantum data;
- Propositions 27,40: we prove that the physically-realisable stabiliser programs, i.e. *stabiliser quantum channels*, are represented by the total relations;
- Theorem 45: we interpret a toy programming language in this relational semantics, and prove full abstraction.

We do not merely apply the CPM construction and split †-idempotents to obtain our denotational semantics. We construct finely-tuned, yet equivalent, categories of relations which offer a significantly simpler alternative to the standard operator-theoretic semantics, whilst supporting concrete computational tools, unlike the naïve categorical semantics.

Notation. Throughout, p denotes an *odd* prime, so that $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ is the field of integers modulo p. Let FHilb denote the category of finite-dimensional Hilbert spaces and linear maps. The inner product is denoted by $\langle - | = \rangle$, the outer product by $|-\rangle\langle =|$, vectors by $|\varphi\rangle$, and their Hermitian adjoints by $\langle\varphi| := |\varphi\rangle^\dagger$. Denote the internal hom of linear endomorphisms on a finite-dimensional Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H}) \cong \mathcal{H}^* \otimes \mathcal{H}$.

We assume familiarity with †-symmetric monoidal categories (†-SMCs), †-compact-closed categories (†-CCCs), as well as a basic understanding of their string diagrams. See Selinger's survey article for reference [38]. Given a monoidal category $\mathcal{C}$ with an endomorphism *group* of scalars, let $\mathrm{Proj}(\mathcal{C})$ be the quotient of $\mathcal{C}$ by invertible scalars.

2 Preliminaries: the stabiliser theory

We review the elements of the stabiliser theory, and its representation with symplectic linear algebra. Explicitly, we define two †-CCCs for the pure stabiliser theory:

1. a *concrete* †-CCC, Stab_p, given by restricting FHilb;
2. an *abstract* †-CCC, $\mathsf{AffLagRel}_{\mathbb{F}_p}$, described in terms of symplectic linear algebra.

These two †-CCCss are known to be equivalent up to nonzero scalars [31, 15]. We will take $\mathsf{AffLagRel}_{\mathbb{F}_p}$ to serve as the basis from which we build our abstract denotational semantics.

2.1 The Hilbert space picture

Consider the p-dimensional complex Hilbert space $\mathcal{H}_p := \mathbb{C}^p$, equipped with the canonical orthonormal basis $\{|x\rangle \mid x \in \mathbb{F}_p\}$. $\mathcal{H}_p$ describes the state space of a p-dimensional quantum system, or **qupit**. Let $\chi(x) := \exp(i2\pi x/p)$, then the **Pauli operators** on $\mathcal{H}_p$ are generated by $Z\,|x\rangle := \chi(x)\,|x\rangle$ and $X\,|x\rangle := |x+1\rangle$ and assemble into the qupit **Pauli group** $\mathcal{P}_p := \{\chi(y)X^xZ^z \mid x,y,z \in \mathbb{F}_p\}$. The n-qupit Pauli group is defined to be the n-fold tensor product $\mathcal{P}_p^{\otimes n}$, so that an arbitrary Pauli operator takes the form $\chi(y)\prod_{j=1}^n X_j^{x_j} Z_j^{z_j}$ for some vectors $\mathbf{x}, \mathbf{z} \in \mathbb{F}_p^n$ and $y \in \mathbb{F}_p$. The **Clifford group** is the unitary normaliser of $\mathcal{P}_p^{\otimes n}$: $\mathcal{C}\ell_p^n := \{U \in \mathcal{U}(\mathcal{H}_p^{\otimes n}) \mid \forall P \in \mathcal{P}_p^{\otimes n}, UPU^\dagger \in \mathcal{P}_p^{\otimes n}\}$.

▶ **Lemma 1** (Stabiliser states). *Take a maximal Abelian subgroup $S \subseteq \mathcal{P}_p^{\otimes n}$ such that $\chi(x)1_{\mathcal{H}_p^{\otimes n}} \in S$ if and only if $x = 0$. Then S uniquely determines a normalised quantum state $|S\rangle \in \mathcal{H}_p^{\otimes n}$, up to a global phase $\exp(2\pi i\theta)$ where $\theta \in [0,1]$. The equivalence class $[\exp(2\pi i\theta)\,|S\rangle]_\theta$ is called the **stabiliser state** associated to the **stabiliser group** S.*

▶ **Definition 2.** *The †-CCC Stab_p of qupit **stabiliser maps** is the †-compact-closed subcategory of FHilb generated by the qupit stabiliser states and Clifford operators as well as the scalars $1/\sqrt{p}$ and $\sqrt{p}$ under tensor product, composition and the Hermitian adjoint.*

2.2 The symplectic picture

We recall how the (pure) qupit stabiliser theory can be restated in purely symplectic terms by taking the notion of a stabiliser group, and their symplectic representation, as fundamental. The following notion will serve as the objects in the "symplectic" category $\mathsf{AffLagRel}_{\mathbb{F}_p}$:

▶ **Definition 3.** *A* ***symplectic vector space*** (V, ω_V) *is a (finite-dimensional)* $\mathbb{F}_p$*-vector space* V *equipped with a non-degenerate, alternating, bilinear form* $\omega_V : V \oplus V \to \mathbb{F}_p$.

One can always choose the following concrete symplectic form:

▶ **Example 4** (Standard symplectic form)**.** Given any $n \in \mathbb{N}$, $(\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$ is a symplectic vector space where $\omega_n\big((\mathbf{x}, \mathbf{z}), (\mathbf{a}, \mathbf{b})\big) := \mathbf{x} \cdot \mathbf{b} - \mathbf{z} \cdot \mathbf{a}$.

The Pauli group $\mathcal{P}_p^{\otimes n}$ is a central extension of the Abelian group $\mathbb{F}_p^n \oplus \mathbb{F}_p^n$ by $\mathbb{F}_p$ [26, § 6.2.3]. This means that there is a function which parametrises Pauli operators

$$\pi : \mathbb{F}_p \oplus \mathbb{F}_p^n \oplus \mathbb{F}_p^n \longrightarrow \mathcal{P}_p^{\otimes n} : (y, \mathbf{x}, \mathbf{z}) \longmapsto \chi(y)\chi(2^{-1}\mathbf{x} \cdot \mathbf{z}) \prod_{k=1}^{n} X_k^{x_k} Z_k^{z_k}, \tag{1}$$

chosen such that $\pi(y, \mathbf{x}, \mathbf{z})\pi(c, \mathbf{a}, \mathbf{b}) = \pi(y + c + 2^{-1}\mathbf{x} \cdot \mathbf{a}, \mathbf{x} + \mathbf{a}, \mathbf{z} + \mathbf{b})$. This allows us to work with Pauli operators purely in terms of symplectic data; for example:

▶ **Lemma 5.** *Two Paulis* $\pi(y, \mathbf{x}, \mathbf{z})$ *and* $\pi(c, \mathbf{a}, \mathbf{b})$ *commute if and only if* $\omega_n((\mathbf{x}, \mathbf{z}), (\mathbf{a}, \mathbf{b})) = 0$.

Recall that the commutation of Paulis was needed to define stabiliser groups. Therefore, the representation π allows us to define stabiliser groups purely at the symplectic level:

▶ **Definition 6.** *Given a linear subspace* S *of a symplectic vector space* (X, ω)*, its* ***symplectic complement*** *is* $S^\omega := \{\mathbf{x} \in X \mid \forall \mathbf{s} \in S, \omega_n(\mathbf{x}, \mathbf{s}) = 0\}$. *The subspace* S *is:*

- ***isotropic*** *if* $S \subseteq S^\omega$;
- ***coisotropic*** *if* $S^\omega \subseteq S$;
- ***Lagrangian*** *if it is maximally isotropic, i.e.* $S = S^\omega$.

An affine subspace is isotropic / coisotropic / Lagrangian if its linear part[1] *is. By convention, empty subspaces are affine Lagrangian.*

The isotropic condition on a subspace is equivalent to the commutation of the corresponding Pauli operators. The additional Lagrangian condition imposes that these subgroups must be maximal. The affine component accounts for the phase factor $\chi(a)$.

▶ **Proposition 7** ([24])**.** *The mapping* (1) *lifts to a bijection between the set of non-empty affine Lagrangian subspaces of* $(\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$ *and the set of stabiliser subgroups of* $\mathcal{P}_p^{\otimes n}$ *(i.e., the set of stabiliser states of* n *qupits up to a phase* $\chi(a)$*.).*

Symplectic vector spaces admit a notion of basis compatible with the symplectic form:

▶ **Proposition 8.** *Every symplectic vector space* (X, ω) *admits a basis of the form* $\{e_j, f_j \mid 1 \geqslant j \geqslant n\}$ *for some* $n \in \mathbb{N}$ *and for which, for any* j, k,

$$\omega(e_j, e_k) = 0, \quad \omega(f_j, f_k) = 0 \quad \textit{and} \quad \omega(e_j, f_k) = \begin{cases} 1 & \textit{if} \quad j = k; \\ 0 & \textit{otherwise.} \end{cases} \tag{2}$$

Such a basis is called a ***symplectic basis****. Furthermore, if* S *is an isotropic subspace of* X*, every basis of* S *extends to a symplectic basis of* X *via a symplectic Gram-Schmidt process.*

A symplectic basis for X is equivalent to a structure-preserving isomorphism $X \cong \mathbb{F}_p^n \oplus \mathbb{F}_p^n$:

[1] The **linear part** of an affine subspace $A \subseteq \mathbb{F}_p^n$ is the subspace of $\mathbb{F}_p^n$ given by $\{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in A\}$.

▶ **Definition 9.** *A* ***symplectomorphism*** $\varphi : (X, \omega_X) \to (Y, \omega_Y)$ *is a linear isomorphism such that for all* $\mathbf{x}, \mathbf{y} \in X$*,* $\omega_Y(\varphi(\mathbf{x}), \varphi(\mathbf{y})) = \omega_X(\mathbf{x}, \mathbf{y})$*. An* ***affine symplectomorphism*** *is an affine map* $X \to Y$ *whose linear part*[2] *is a symplectomorphism.*

All symplectic vector spaces (X, ω_X) are even-dimensional and symplectomorphic to a standard symplectic vector space $(\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$ for $n := \dim(X)/2$. In particular:

▶ **Proposition 10.** *There is an isomorphism between the Clifford group* $\mathcal{C}\ell_p^n$*, modulo global phase, and the group of affine symplectomorphisms on* $(\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$*.*

Affine Lagrangian relations

Affine Lagrangian subspaces are *the* fundamental notion in the symplectic representation:

1. the *graph* of an affine symplectomorphism $\varphi : (X, \omega_X) \to (Y, \omega_Y)$ is an affine Lagrangian subspace $\mathrm{Gr}(S) := \{(\mathbf{x}, \varphi(\mathbf{x})) \mid x \in X \subseteq X \oplus Y\} \subseteq (X \oplus Y, -\omega_X \oplus \omega_Y)$;
2. the composition of affine symplectomorphisms, and thus Clifford operators, is compatible with the relational composition of their graphs $\mathrm{Gr}(\varphi); \mathrm{Gr}(\psi) = \mathrm{Gr}(\psi \circ \varphi)$;
3. the action of Clifford operators on stabiliser states is compatible with the relational composition of their corresponding affine Lagrangian subspaces.

In other words, following Weinstein [44], we adopt the motto:

Everything is an affine Lagrangian relation!

▶ **Definition 11.** *The* ***category*** $\mathsf{AffLagRel}_{\mathbb{F}_p}$ ***of affine Lagrangian relations*** *has:*
- ***objects****: symplectic vector spaces* (V, ω_V)*;*
- ***morphisms*** $(V, \omega_V) \to (W, \omega_W)$*: affine Lagrangian subspaces of* $(V \oplus W, -\omega_V \oplus \omega_W)$*;*
- ***monoidal product****: given by the direct sum* $(V \oplus W, \omega_V \oplus \omega_W)$*;*
- ***monoidal unit****: given by the trivial symplectic vector space* $I := (\mathbb{F}_p^0, 1) \cong (\mathbb{F}_p^0 \oplus \mathbb{F}_p^0, \omega_0)$*;*
- ***dagger****: given by the relational converse,* $(\mathbf{x}, \mathbf{y}) \in L^\dagger$ *if and only if* $(\mathbf{y}, \mathbf{x}) \in L$*;*
- ***†-compact structure****: given by cups* $\eta_{(V,\omega_V)} := \{(0, (\mathbf{x}, \mathbf{x}))\} : I \to (V, \omega_V)^* \oplus (V, \omega_V)$*, where* $(V, \omega_V)^* := (V, -\omega_V)$*.*

▶ **Theorem 12** ([15, 31])**.** *There is an essentially surjective and full †-compact-closed functor* $\mathrm{Rel} : \mathsf{Stab}_p \to \mathsf{AffLagRel}_{\mathbb{F}_p}$*. This restricts to a †-compact-closed equivalence when quotienting by invertible scalars:* $\mathrm{Proj}(\mathsf{Stab}_p) \simeq \mathsf{AffLagRel}_{\mathbb{F}_p}$*.*

For convenience, denote $\mathsf{AffLagRel}_{\mathbb{F}_p}(n, m) := \mathsf{AffLagRel}_{\mathbb{F}_p}((\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n), (\mathbb{F}_p^m \oplus \mathbb{F}_p^m, \omega_m))$, which via theorem 12 and equation (1) are the concrete affine Lagrangian relations representing stabiliser maps $\mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes m}$. These concrete relations represent the basic operations of the stabiliser theory. However, the full stabiliser quantum theory is *much richer* than what we have described. For example, *stabiliser codes* are fundamental in quantum error correction, but they are not described by pure states. Similarly, the measurement and classical control of stabiliser circuits, which we have not yet discussed, are essential for error correction.

3 Stabiliser codes and mixed states

The stabiliser theory presented in Section 2 is, from the perspective of quantum computation, fundamentally limited: it is efficiently simulatable on a classical computer [21, 2]. Nevertheless,

[2] The **linear part** of an affine map $\varphi : \mathbb{F}_p^n \to \mathbb{F}_p^m$ is the linear map $\mathbf{x} \mapsto \varphi(\mathbf{x}) - \varphi(\mathbf{0})$.

the algebraic structure of the theory—specifically, the concept of stabiliser subgroups—provides the foundation for quantum error correction (QEC). QEC leverages these elements to encode and manipulate information in a way that supports universal quantum computation, while retaining structural features which permit fault tolerance.

A **stabiliser code** is a *non-maximal* stabiliser group, i.e. an Abelian subgroup S of $\mathcal{P}_p^{\otimes n}$ such that $\chi(x) \cdot 1 \in S$ if and only if $x = 0$. Whereas a maximal stabiliser group uniquely determines a pure stabiliser state $|S\rangle$—a one-dimensional subspace of the Hilbert space $\mathcal{H}_p^{\otimes n}$—a stabiliser code determines a higher-dimensional subspace, the *codespace*:

$$\mathcal{H}_S := \{|\varphi\rangle \in \mathcal{H}_p^{\otimes n} \mid s\,|\varphi\rangle = |\varphi\rangle \text{ for all } s \in S\} \subseteq \mathcal{H}_p^{\otimes n} \tag{3}$$

Elements of the stabiliser group impose linear constraints on the codespace. Relaxing the number of constraints therefore yields a larger subspace, while still enforcing sufficient symmetry to make it possible to detect Pauli errors.

Semantically, it is natural to view a stabiliser code not just as a subspace but as the completely mixed state on that subspace. Concretely, let $\Pi_S : \mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}$ be the projection onto $\mathcal{H}_S$. The normalised map $\rho_S := \Pi_S / \operatorname{Tr}(\Pi_S)$ is the *mixed state* for the uniform mixed state on $\mathcal{H}_S$. This interpretation allows stabiliser codes to be treated within the framework of mixed-state quantum mechanics, and more importantly, to obtain denotational semantics for stabiliser codes via categorical constructions of mixed state quantum theory.

In this section, we introduce two semantics for mixed stabiliser quantum mechanics by:

1. *restricting* the mixed processes to those built out stabiliser maps;
2. *generalising* the symplectic representation to affine coisotropic relations.

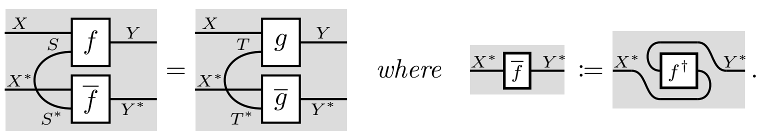
Because the first semantics is given by restriction, it is hard to grapple with. On the other hand, the second semantics is novel, and much more apt to reason about stabiliser codes.

3.1 Completely-positive maps between matrix algebras

Selinger's CPM construction builds a category of mixed processes CPM(C) out of a †-CCC C by adding a notion of discarding that respects the dagger structure [39]. This plays a similar role to how the Kleisli categories Set_P and Meas_G, respectively over the power-set and Giry monads, are categorical semantics for nondeterministic and probabilistic computations:

▶ **Definition 13** ([39]). *Given a †-CCC* C*, the †-CCC* CPM(C) *has:*

- ***objects:*** *same as* C.
- ***morphisms*** $[f, S] : X \to Y$***:*** *are equivalence classes of pairs* (f, S)*, where* S *is an object of* C *and* $f : X \otimes S \to Y$ *in* C*, modulo the equivalence relation*

$(f, S) \sim (g, T) \iff$ 

- ***all other †-compact-closed structure:*** *inherited from* C.

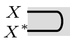
There is a canonical "doubling" functor $\iota : \mathsf{C} \to \mathrm{CPM}(\mathsf{C})$ which sends morphisms $f : X \to Y$ to $(u_X^R; f, I) : X \to Y$. This means that the CPM construction is adding a new morphism that connects both halves of this doubling. For the example of $\mathsf{C} := \mathsf{FHilb}$, this is interpreted as adding the maximally mixed state:

▶ **Theorem 14** ([39, Ex. 4.21]). CPM(FHilb) *is equivalent to the †-CCC of* completely-positive *(CP) maps between matrix algebras* $\mathcal{B}(\mathcal{H})$*, for all* $\mathcal{H} \in$ FHilb.

▶ **Corollary 15.** *There is a faithful †-compact functor* $\mathrm{CPM}(\mathsf{Stab}_p) \rightarrowtail \mathrm{CPM}(\mathsf{FHilb})$.

The projection $\Pi_S : \mathcal{H}_p^{\otimes n} \to \mathcal{H}_p^{\otimes n}$ onto the codespace of a stabiliser code S is a state in $\mathrm{CPM}(\mathsf{Stab}_p)$. Moreover, Clifford operators C in $\mathcal{C}\ell_p^n \to \mathsf{Stab}_p \to \mathrm{CPM}(\mathsf{Stab}_p)$ act on these projectors by conjugation $C^\dagger \Pi_S C = \Pi_{C^\dagger SC}$ as expected. In particular, for a stabilizer state $|S\rangle$, we have $C^\dagger \, |S\rangle\!\langle S| \, C = \left|C^\dagger SC\right\rangle\!\left\langle C^\dagger SC\right|$.

To restrict physical processes, we impose the additional normalisation constraint.

▶ **Definition 16.** *Given any* $X \in \mathsf{C}$*, let* Tr_X *denote the morphism* $X \to I \in \mathrm{CPM}(\mathsf{C})$ *given by the equivalence class* . *A morphism* $[f, S] : X \to Y$ *in* $\mathrm{CPM}(\mathsf{C})$ *is* ***causal*** *if and only if* $\mathrm{Tr}_Y[f, S] = \mathrm{Tr}_X$. *Denote the symmetric monoidal subcategory of causal morphisms in* $\mathrm{CPM}(\mathsf{C})$ *by* $\mathrm{Caus}(\mathrm{CPM}(\mathsf{C}))$.

Concretely, the morphisms $\mathrm{Tr}_{\mathcal{H}}$ in $\mathrm{CPM}(\mathsf{FHilb})$, are given by the linear-algebraic trace in FHilb. In the language of operator algebras:

▶ **Corollary 17.** $\mathrm{Caus}(\mathrm{CPM}(\mathsf{FHilb}))$ *is equivalent to the SMC of completely-positive* trace-preserving *(CPTP) maps between matrix algebras* $\mathcal{B}(\mathcal{H}) \cong \mathcal{H}^* \otimes \mathcal{H}$ *for all* $\mathcal{H} \in \mathsf{FHilb}$.

In other words, these are CP maps which preserve the trace norm, in analogy to how Markov processes preserve the L^1 norm.

▶ **Example 18.** The normalisation $\rho_S = \Pi_S / \mathrm{Tr}(\Pi_S)$ of the projector Π_S onto the code space of a stabiliser code S is completely positive and trace preserving; whereas without the normalisation factor, the projector Π_S is only trace-preserving when $\mathrm{Tr}(\Pi_S) = 1$.

3.2 Stabiliser codes as affine coisotropic relations

Let's apply the CPM construction to the †-CCC of affine Lagrangian relations, obtaining a relational semantics for $\mathrm{CPM}(\mathsf{Stab}_p)$. In particular, we show that this produces the poset-enriched †-CCC of affine *coisotropic* relations, relaxing the dimensionality requirement for Lagrangian relations.

▶ **Definition 19.** *The †-CCC* $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ *of* ***affine coisotropic relations*** *has the same structure as* $\mathsf{AffLagRel}_{\mathbb{F}_p}$*, where now the morphisms* $(V, \omega_V) \to (W, \omega_W)$ *are affine coisotropic subspaces of* $(V \oplus W, -\omega_V \oplus \omega_W)$.

It is well-understood that "phaseless" stabiliser codes are in bijection with isotropic subspaces [24], and hence also with coisotropic subspaces via the symplectic complement. However, once Pauli phases are introduced, this second bijection breaks down: phased stabiliser codes correspond exactly to affine coisotropic subspaces but *not* to affine isotropic subspaces. Thus, affine coisotropic relations are the correct algebraic setting for the stabiliser theory with non-maximal stabiliser groups.

▶ **Example 20.** The total subspace can be regarded as an affine coisotropic relation:

$$\mathrm{Im}_{(V,\omega_V)} \coloneqq \{(0, \mathbf{v}) \mid \forall \mathbf{v} \in V\} : I \to (V, \omega_V) \tag{4}$$

We use the name $\mathrm{Im}_{(V,\omega_V)}$ because postcomposition with an affine Lagrangian, or affine coisotropic relation $R : (V, \omega_v) \to (W, \omega_W)$ is identified with the set-theoretic image:

$$\mathrm{Im}_V ; R = \{(0, \mathbf{w}) \mid \exists \mathbf{v} : (\mathbf{v}, \mathbf{w}) \in R\} = \mathbb{F}_p^0 \oplus \mathrm{Im}(R) \cong \mathrm{Im}(R) \tag{5}$$

Adding the image as a generator to $\mathsf{AffLagRel}_{\mathbb{F}_p}$ yields $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$:

▶ **Proposition 21.** *Every non-empty affine coisotropic subspace of $\mathbb{F}_p^n \oplus \mathbb{F}_p^n$ of dimension $n+m$ is the image of an affine Lagrangian coisometry $\mathbb{F}_p^m \oplus \mathbb{F}_p^m \to \mathbb{F}_p^n \oplus \mathbb{F}_p^n$.*

Proof. We prove the claim for Lagrangian and coisotropic linear subspaces, after which the affine generalisation follows immediately.

Consider a basis of S and extend it to a symplectic basis of (V, ω). This yields a symplectomorphism $\varphi : (V, \omega) \to (\mathbb{F}_p^n \oplus \mathbb{F}_p^n, \omega_n)$ that takes this symplectic basis of V to the standard basis, and such that $\varphi(S) = \{(\mathbf{x}, \mathbf{0}_{2n-k}) \mid \mathbf{x} \in \mathbb{F}_p^k\}$. Note that the subspace $D \coloneqq \{(\mathbf{x}, \mathbf{0}_k) \mid x \in \mathbb{F}_p^k\}$ is Lagrangian in $(\mathbb{F}_p^k \oplus \mathbb{F}_p^k, \omega_k)$ so that $\varphi(S) = D \oplus \{\mathbf{0}_{2(n-k)}\}$. It follows that $C \coloneqq \mathrm{Gr}(\varphi); (D \oplus 1_{2(n-k)})$ is precisely a Lagrangian relation for which $\ker C = C^{\mathsf{R}}(\{\mathbf{0}_{2(n-k)}\}) = S$. Finally, C is a composition of coisometries, and thus a coisometry. ◀

Identifying the dual of the image with the trace:

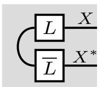
▶ **Theorem 22.** *There is a †-compact isomorphism* $\mathrm{CPM}(\mathsf{LagRel}_{\mathbb{F}_p}) \cong \mathsf{CoisotRel}_{\mathbb{F}_p}$ *sending:*

$$[f, (S, \omega_S)] : (X, \omega_S) \to (Y, \omega_Y) \quad \longmapsto \quad (1_{(X,\omega_X)} \oplus \mathrm{Im}_{(S,\omega_S)}); f : (X, \omega_S) \to (Y, \omega_Y)$$

Proof. We prove the proposition for Lagrangian and coisotropic linear relations, after which the affine extension follows immediately.

This assignment is clearly functorial and identity-on-objects, and preserves the †-compact-closed structure. Moreover, since both $\mathrm{CPM}(\mathsf{LagRel}_{\mathbb{F}_p})$ and $\mathsf{CoisotRel}_{\mathbb{F}_p}$ are compact-closed, it suffices to prove that the states in both categories are in canonical bijection. We already have surjectivity by proposition 21, so that all we need to prove is injectivity.

Given Lagrangian relations $L : S \to X$ and $M : T \to X$ such that $\mathrm{Im}(L) \neq \mathrm{Im}(M)$, then

$\mathbf{x} \in \mathrm{Im}(L)$ if and only if $\begin{bmatrix}\mathbf{x}\\ \mathbf{x}\end{bmatrix} \in$ 

$= \left\{ \begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix} \middle| \; \exists \mathbf{z} : \begin{array}{c}(\mathbf{z}, \mathbf{x}) \in L \\ (\mathbf{z}, \mathbf{y}) \in \overline{L}\end{array} \right\}. \qquad (6)$

But by assumption there is some $\mathbf{x}$ such that $\mathbf{x} \in \mathrm{Im}(L)$ and $\mathbf{x} \notin \mathrm{Im}(M)$, and it is therefore immediate that $[L, (S, \omega_S)] \neq [M, (T, \omega_T)]$. ◀

▶ **Corollary 23.** *There is a †-compact functor* $\mathrm{Rel} : \mathrm{CPM}(\mathsf{Stab}_p) \to \mathsf{AffCoisotRel}_{\mathbb{F}_p}$*, which restricts to an equivalence when quotienting by scalars* $\mathsf{AffCoisotRel}_{\mathbb{F}_p} \simeq \mathrm{Proj}(\mathrm{CPM}(\mathsf{Stab}_p))$.

Proof. This follows immediately from the equivalence $\mathsf{AffCoisotRel}_{\mathbb{F}_p} \cong \mathrm{CPM}(\mathsf{AffLagRel}_{\mathbb{F}_p}) \simeq \mathrm{CPM}(\mathrm{Proj}(\mathsf{Stab}_p))$ and observing that, in the case of Stab_p, we obtain the same category if we quotient by scalars before or after applying the CPM construction. ◀

To include mixed states and stabiliser codes in our semantics, we update our motto:

Everything is an affine coisotropic relation!

▶ **Example 24.** The following quantum channels are represented in $\mathrm{Rel}(\mathrm{CPM}(\mathsf{Stab}_p))$:
- the *maximally mixed state* by $\mathrm{Im}_{(V,\omega_V)}$;
- the *quantum trace* by its relational converse $\mathrm{Im}^{\dagger}_{(V,\omega_V)}$;
- the *completely depolarising channel* by $\mathrm{Im}^{\dagger}_{(V,\omega_V)}; \mathrm{Im}_{(V,\omega_V)}$;
- the *Z-flip channel* by $\mathscr{E}_X \coloneqq \{((z, x), (z', x)) \mid x, z, z' \in \mathbb{F}_p\}$.

The category $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ carries a natural *poset-enrichment*, where morphisms are ordered by subspace inclusion. Operationally, $R \subseteq S$ means that the process S can be obtained from

R by *discarding* quantum information. Regarding $\mathrm{CPM}(\mathsf{FHilb})$ as poset-enriched category under the Löwner order, $\mathrm{CPM}(\mathsf{FHilb}) \to \mathsf{AffCoisotRel}_{\mathbb{F}_p}$ preserves this ordering of maps.

By contrast, the poset-enrichment collapses in $\mathsf{AffLagRel}_{\mathbb{F}_p}$: Lagrangian relations are exactly the coisotropic relations of *minimal* dimension, so none is strictly contained in another. Hence the passage $\mathsf{AffLagRel}_{\mathbb{F}_p} \to \mathsf{AffCoisotRel}_{\mathbb{F}_p}$ not only adds mixedness but also equips the semantics with a quantitative notion of "being more pure".

▶ **Example 25.** A state in $\mathsf{AffCoisotRel}_{\mathbb{F}_p}(0, n)$ corresponds to a stabiliser code in $\mathcal{H}_p^{\otimes n}$, which, as described above, we can think of as a projector on $\mathcal{H}_p^{\otimes n}$. A pure stabiliser channel $f \in \mathsf{AffLagRel}_{\mathbb{F}_p}(m, n)$ respects choices of stabiliser codes $A \subseteq \mathbb{F}_p^m \oplus \mathbb{F}_p^m$ and $B \subseteq \mathbb{F}_p^n \oplus \mathbb{F}_p^n$ if $f \circ A = f(A) \subseteq B$. By the correspondence with the Löwner order on projectors, this guarantees that f always maps states stabilised by A to states stabilised by B.

The converse inclusion is also of interest. We can read $B \subseteq f \circ A$ as a *promise*: that f either projects onto the stabilisers of A, or forwards them as a *demand* B for subsequent morphisms. As an elementary example, consider an encoder/decoder pair $k \xrightarrow{E} n \xrightarrow{D} k$ for a code $A \subseteq \mathbb{F}_p^n \oplus \mathbb{F}_p^n$. The encoder E produces the stabiliser code A as a constraint, and D consumes A to recover the trivial code.

Both of these properties are important in the context of fault-tolerant compilation. The compositional structure of fault-tolerance can therefore be studied in the coslice category of $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ under the object 0. We leave a detailed investigation of this structure to future work.

▶ **Definition 26.** *A relation $R : X \to Y$ with converse $R^\dagger$ is **total** when $\mathrm{Im}(R^\dagger) = X$. Given a category C of relations, let $\mathrm{Total}(\mathsf{C})$ denote the subcategory of total maps.*

By restricting the mixed stabiliser theory to the trace-preserving maps (the physical processes), the functor $\mathrm{CPM}(\mathsf{Stab}_p) \to \mathsf{AffCoisotRel}_{\mathbb{F}_p}$ restricts to an equivalence on the nose, without quotienting by scalars:

▶ **Proposition 27.** *The functor $\mathrm{Rel} : \mathrm{CPM}(\mathsf{Stab}_p) \to \mathsf{AffCoisotRel}_{\mathbb{F}_p}$ restricts to a symmetric monoidal equivalence $\mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p)) \simeq \mathrm{Total}(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Total}(\mathsf{AffCoisotRel}_{\mathbb{F}_p}) \arrow[rr, tail] \arrow[d, phantom, "\wr"] & & \mathsf{AffCoisotRel}_{\mathbb{F}_p} \arrow[d, phantom, "\wr"] \\
\mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p)) \arrow[r, tail] \arrow[d, tail] & \mathrm{CPM}(\mathsf{Stab}_p) \arrow[r, two heads] \arrow[d, tail] & \mathrm{Proj}(\mathrm{CPM}(\mathsf{Stab}_p)) \arrow[d, tail] \\
\begin{array}{c}\textit{CPTP maps between}\\ \textit{matrix algebras}\end{array} \arrow[r, tail] & \begin{array}{c}\textit{CP maps between}\\ \textit{matrix algebras}\end{array} \arrow[r, two heads] & \mathrm{Proj}\left(\begin{array}{c}\textit{CP maps between}\\ \textit{matrix algebras}\end{array}\right)
\end{tikzcd}$$

Proof. It is immediate that $\mathrm{Rel} : \mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p)) \to \mathrm{Total}(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ is an essentially surjective, full, monoidal functor, making the diagram commute. It remains to prove faithfulness. Take two maps $[f, S], [g, T] : X \to Y$ in $\mathrm{CPM}(\mathsf{Stab})$ such that $[g, T] = \lambda \cdot [f, S]$ some $\lambda \neq 0$. Then $\mathrm{Tr}_Y[g, T] = \lambda \, \mathrm{Tr}_Y[f, S] = \lambda \, \mathrm{Tr}_X$. Therefore $[g, T]$ is causal iff $\lambda = 1$ i.e. $[g, T] = [f, S]$, thus, each projective equivalence class of morphisms $\mathrm{Caus}(\mathrm{CPM}(\mathsf{Stab}_p))$ contains at most one representative. Therefore, the equivalence $\mathsf{AffCoisotRel}_{\mathbb{F}_p} \simeq \mathrm{Proj}(\mathrm{CPM}(\mathsf{Stab}_p))$ uniquely lifts along Proj on causal maps. ◀

4 Measurement and classical types

In the previous section, we saw how the CPM construction provides an abstract setting for:

1. general mixed state quantum mechanics when applied to FHilb;

2. more specifically, stabiliser codes when applied to $\mathsf{AffLagRel}_{\mathbb{F}_p} \simeq \mathsf{Stab}_p$.

Stabilizer codes are used to detect and correct errors on faulty quantum channels: encoding quantum information redundantly as a mixed state. However, to actually detect errors, one has to measure part of the code space; and to correct errors, one must apply operations to the code space conditional on the measurement outcomes.

Given an indexed orthonormal basis $B = \{|\lambda_1\rangle, \cdots, |\lambda_n\rangle\}$ for a finite-dimensional Hilbert space $\mathcal{H}$, the measurement in this basis is represented by the projector $\mathcal{E}_B : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ which sends pure states to probabilistic mixtures of pure states according to the *Born rule*:

$$\mathcal{E}_B(|\varphi\rangle\langle\varphi|) := \sum_{j=1}^{n} |\lambda_j\rangle\langle\lambda_j| \, |\varphi\rangle\langle\varphi| \, |\lambda_j\rangle\langle\lambda_j| = \sum_{j=1}^{n} |\langle\lambda_j|\varphi\rangle|^2 \, |\lambda_j\rangle\langle\lambda_j| \tag{7}$$

The indices of the basis are interpreted as the measurement outcomes occurring with probability $|\langle\lambda_j|\varphi\rangle|^2$. This projector $\mathcal{E}_B$ is an endomorphism on $\mathcal{H}$ in CPM(FHilb). In particular, a Pauli-X basis measurement is an endomorphism on $\mathcal{H}_p$ in CPM(Stab_p). Therefore, in some sense, $\mathcal{E}_B$ is the "classical" subobject of the "quantum" object $\mathcal{H}$ which has been measured in the basis B. By promoting these subobjects to objects, we obtain a categorical semantics for quantum theory with classical and quantum types; reproducing the usual setting for finite-dimensional quantum mechanics. We perform an analogous construction to stabiliser circuits to obtain a fully relational semantics.

4.1 Adding classical types by splitting dagger-idempotents

We review the †-idempotent completion of a †-CCC, recalling:

▶ **Definition 28.** *A* ***†-idempotent*** *in a †-SMC is a map f such that $f^\dagger = f$ and $f; f = 1$.*

In the setting of finite-dimensional, mixed quantum theory:

▶ **Example 29** ([27, Thm. 2.5], [13, Prop. 3.5])**.** The †-idempotents on $\mathcal{H}$ in CPM(FHilb) are in bijection with C^*-subalgebras of the matrix algebra $\mathcal{B}(H) \cong \mathcal{H}^* \otimes \mathcal{H}$.

The identity on $\mathcal{H}$ is an idempotent and corresponds to the trivial C^*-subalgebra $\mathcal{B}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$. On the other hand, projectors onto subspaces induced by measurement, such as measurement onto a basis $\mathcal{E}_B : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ correspond to commutative C^*-subalgebras of $\mathcal{B}(\mathcal{H})$. We promote these subobjects to objects:

▶ **Definition 30** ([40, Def. 3.13])**.** *Given a †-CCC C, the †-idempotent completion,* $\mathrm{Split}^\dagger(\mathsf{C})$*, is the †-CCC with:*

- ***objects:*** *pairs (A, a) where A is an object of C and $a : A \to A$ is a †-idempotent;*
- ***morphisms:*** *$f : (A, a) \to (B, b)$ are morphisms $f : A \to B$ in C such that $a; f; b = f$;*
- ***identities:*** *$1_{(A,a)} := a$;*
- ***rest of †-compact structure*** *given pointwise in C.*

There is a canonical embedding $\mathcal{C} \to \mathrm{Split}^\dagger(\mathcal{C})$ sending objects $A \mapsto (A, 1_A)$ and acting as the identity on morphisms. When applied to CPM(FHilb), the †-idempotent completion reproduces the standard setting for finite-dimensional quantum mechanics:

▶ **Theorem 31** ([27, Thm. 2.5], [13, Prop. 3.5])**.** $\mathrm{Split}^\dagger$(CPM(FHilb)) *is equivalent to the †-CCC of completely-positive maps between finite-dimensional C^*-algebras.*

The objects of the form $(\mathcal{H}, 1_{\mathcal{H}})$ represent the matrix algebras, interpreted as the purely quantum systems in $\mathrm{CPM}(\mathsf{FHilb})$. On the other hand, the new objects added by †-idempotent completion correspond to non-matrix C^*-algebras, interpreted as being more classical. For the example of a quantum measurement induced by an orthonormal basis B, the object $(\mathcal{H}, \mathcal{E}_B)$ is interpreted as a classical system measured according to the basis B. The canonical map $\mathcal{E}_B : (\mathcal{H}, 1_{\mathcal{H}}) \to (\mathcal{H}, \mathcal{E}_B)$ is interpreted as the measurement induced by B; whereas the map $\mathcal{E}_B : (\mathcal{H}, \mathcal{E}_B) \to (\mathcal{H}, 1_{\mathcal{H}})$ is interpreted as the state preparation induced by B. Measurement followed by state preparation yields the quantum system projected onto the measurement basis; whereas, state preparation followed by measurement yields the identity on the classical system:

Measuring then preparing:
$$\begin{tikzcd} (\mathcal{H}, 1_{\mathcal{H}}) \arrow[d, two heads, "\mathcal{E}_B"'] \arrow[dr, "\mathcal{E}_B"] & \\ (\mathcal{H}, \mathcal{E}_B) \arrow[r, tail, "\mathcal{E}_B"'] & (\mathcal{H}, 1_{\mathcal{H}}) \end{tikzcd}$$

Preparing then measuring:
$$\begin{tikzcd} (\mathcal{H}, \mathcal{E}_B) \arrow[r, tail, "\mathcal{E}_B"] \arrow[dr, equal, "\mathcal{E}_B"'] & (\mathcal{H}, 1_{\mathcal{H}}) \arrow[d, two heads, "\mathcal{E}_B"] \\ & (\mathcal{H}, \mathcal{E}_B) \end{tikzcd}$$

Arbitrary completely-positive maps between C^*-algebras cannot be physically implemented. Just as in the previous section, we must impose an additional constraint:

▶ **Definition 32.** *A morphism* $[f, S] : (X, x) \to (Y, y)$ *in* $\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{C}))$ *is* ***causal*** *if and only if* $\mathrm{Tr}_Y[f, S] = \mathrm{Tr}_X : (X, x) \to (I, 1_I)$. *We denote* $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{C})))$ *the symmetric monoidal subcategory of causal morphisms in* $\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{C}))$.

In the setting of finite-dimensional quantum theory; this reproduces the usual operator algebraic setting for finite-dimensional quantum mechanics:

▶ **Corollary 33.** $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb})))$ *is equivalent to the symmetric monoidal category of completely-positive trace-preserving (CPTP) maps between finite-dimensional* C^**-algebras.*

In other words, the morphisms in $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb})))$ correspond to finite-dimensional **quantum channels**, and the states correspond to **density matrices**. Importantly, the state preparation and measurement maps are quantum channels.

4.2 The stabiliser theory with affine nondeterministic classical control

In the previous subsection, we recalled how the symmetric monoidal category $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb})))$ is equivalent to the standard setting for quantum channels. That is to say, the finite-dimensional quantum circuits with measurement and classical control. In this subsection, by applying the same constructions to $\mathrm{CPM}(\mathsf{Stab}_p) \rightarrowtail \mathrm{CPM}(\mathsf{FHilb})$; we show that the canonical setting for stabilizer quantum mechanics with Pauli measurement and Pauli state preparation admits a concise, entirely relational description. To this end:

▶ **Definition 34.** *The †-compact-closed category* $\mathsf{AffRel}_{\mathbb{F}_p}$ *of* ***affine relations*** *has finite-dimensional* $\mathbb{F}_p$*-vector spaces as objects and affine subspaces as morphisms. Composition is given by relational composition, whilst the identity and compact-closed structure are given by the diagonal relation. The dagger is given by the relational converse.*

▶ **Lemma 35.** *There is a faithful †-compact-closed functor* $Q : \mathsf{AffLagRel}_{\mathbb{F}_p} \to \mathsf{AffRel}_{\mathbb{F}_p}$ *which forgets symplectic structure.*

Instead of forming $\mathrm{Split}^\dagger(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ on the nose, we can add additional affine relations to the image of $Q : \mathsf{AffCoisotRel}_{\mathbb{F}_p} \to \mathsf{AffRel}_{\mathbb{F}_p}$ wich †-split †-idempotents:

▶ **Proposition 36.** *The †-idempotents in* $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ *†-split through the forgetful functor* $\mathsf{AffCoisotRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$. *In particular, Pauli-X measurement splits through the relations:*

$$\mu_X := \left\{ \left(\begin{bmatrix} x \\ z \end{bmatrix}, x \right) \in \mathbb{F}_p^2 \oplus \mathbb{F}_p \right\} : Q(\mathbb{F}_p^2, \omega_2) \to \mathbb{F}_p, \;\; \eta_X := \left\{ \left(x, \begin{bmatrix} x \\ z \end{bmatrix} \right) \in \mathbb{Z}_p \oplus \mathbb{F}_p^2 \right\} : \mathbb{F}_p \to Q(\mathbb{F}_p^2, \omega_2)$$

Proof. Consider the relation in $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ corresponding to the Z-flip channel:

$$\mathscr{E}_X := \mathrm{Rel}(\mathcal{E}_X) = \left\{ \left(\begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} x \\ z' \end{bmatrix} \right) \in \mathbb{F}_p^2 \oplus \mathbb{F}_p^2 \right\} : (\mathbb{F}_p^2, \omega_2) \to (\mathbb{F}_p^2, \omega_2)$$

Any †-idempotent in $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ is affine symplectomorphic to $\mathscr{E}_X^{\oplus n} \oplus 1_{(\mathbb{F}_p^m \otimes \mathbb{F}_p^m, \omega_m)}$ for some $n, m \in \mathbb{N}$. Moreover, $Q(\mathscr{E}_X) = \mu_X ; \eta_X$ splits as $\eta_X ; \mu_X = 1_{\mathbb{F}_p}$. ◀

The process of †-splitting †-idempotents through $Q : \mathsf{AffCoisotRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$ adds a "quantum" modality Q to $\mathsf{AffRel}_{\mathbb{F}_p}$; imposing compatibility with the symplectic structure:

▶ **Definition 37.** *Let $\mathsf{AffRel}^Q_{\mathbb{F}_p}$ denote the †-CCC with:*

- ***Objects:*** *Generated by finite direct sums of finite dimensional symplectic vector spaces $Q(V, \omega_V) \in Q(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$, and finite dimensional vector spaces $W \in \mathsf{AffRel}_{\mathbb{F}_p}$;*
- ***Morphisms:*** *Generated by $Q(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$ in addition to $\mu_X : Q(\mathbb{F}_p^2, \omega_2) \to \mathbb{F}_p$ and $\nu_X : \mathbb{F}_p \to Q(\mathbb{F}_p^2, \omega_2)$ under the direct sum and relational composition;*
- ***†-compact structure:*** *Given pointwise in $Q(\mathsf{AffCoisotRel}_{\mathbb{F}_p})$, $\mathsf{AffRel}_{\mathbb{F}_p}$, where $\mu_X^\dagger := \nu_X$.*

By restricting to either class of objects, it is immediate that:

▶ **Lemma 38.** $\mathsf{AffCoisotRel}_{\mathbb{F}_p}$ *and* $\mathsf{AffRel}_{\mathbb{F}_p}$ *are full †-compact closed subcategories of* $\mathsf{AffRel}^Q_{\mathbb{F}_p}$.

Moreover, because $Q : \mathsf{AffCoisotRel}_{\mathbb{F}_p} \rightarrowtail \mathsf{AffRel}_{\mathbb{F}_p}$ is faithful, it is immediate that:

▶ **Theorem 39.** *There is a †-compact closed equivalence* $\mathrm{Split}^\dagger(\mathsf{AffCoisotRel}_{\mathbb{F}_p}) \simeq \mathsf{AffRel}^Q_{\mathbb{F}_p}$.

In other words, this category is obtained by glueing together the †-CCCs $\mathsf{AffLagRel}_{\mathbb{F}_p}$ and $\mathsf{AffRel}_{\mathbb{F}_p}$ along the map μ_X which projects onto the X subspace and its transpose. We interpret the symplectic objects $Q(V, \omega_V)$ as the quantum types; the objects W with no symplectic structure as the classical types; μ_X as the measurement in the Pauli-X basis; and ν_X as state preparation in the Pauli-X basis.

Finally, our moto becomes:

Everything is an affine relation, with quantum data captured by a symplectic modality!

which, admittedly, is not *quite* as catchy as the previous motos.

However, just as for $\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{FHilb}))$; the category $\mathrm{Proj}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p)) \cong \mathsf{AffRel}^Q_{\mathbb{F}_p}$ has morphisms which do not correspond to operations which can be physically implemented. We restrict ourselves to the completely-positive maps:

▶ **Proposition 40.** *The induced functor* $\mathrm{Rel} : \mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p)) \to \mathsf{AffRel}^Q_{\mathbb{F}_p}$ *restricts to a symmetric monoidal equivalence* $\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p))) \simeq \mathrm{Total}(\mathsf{AffRel}^Q_{\mathbb{F}_p})$ *making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Total}(\mathsf{AffRel}^Q_{\mathbb{F}_p}) \arrow[rr, tail] & & \mathsf{AffRel}^Q_{\mathbb{F}_p} \\
\mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p))) \arrow[u, sloped, "\sim"] \arrow[r, tail] \arrow[d, tail] & \mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p)) \arrow[r, two heads] \arrow[d, tail] & \mathrm{Proj}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p))) \arrow[u, sloped, "\sim"] \arrow[d, tail] \\
\begin{matrix} \textit{CPTP maps between} \\ \textit{f.d. } C^*\textit{-algebras} \end{matrix} \arrow[r, tail] & \begin{matrix} \textit{CP maps between} \\ \textit{f.d. } C^*\textit{-algebras} \end{matrix} \arrow[r, two heads] & \mathrm{Proj}\begin{pmatrix} \textit{CP maps between} \\ \textit{f.d. } C^*\textit{-algebras} \end{pmatrix}
\end{tikzcd}$$

Proof. This follows from essentially the same argument as for proposition 27. ◀

In other words, the *stabiliser quantum channels* and *stabiliser density matrices* are fully and faithfully reprented in $\mathrm{Total}(\mathsf{AffRel}^Q_{\mathbb{F}_p})$. Note that classically-controlled Pauli operators can be represented in $\mathrm{Total}(\mathsf{AffRel}^Q_{\mathbb{F}_p})$ because they can be constructed with Clifford operators as well as Pauli state preparation and measurements.

$$\frac{\Gamma \vdash c \triangleright \Delta \quad \Delta \vdash d \triangleright \Sigma}{\Gamma \vdash c \,\mathbin{;}\, d \triangleright \Sigma} \qquad \frac{}{\Gamma \vdash \textbf{init}\ \underline{x} \triangleright \underline{x} : \mathtt{pit}, \Gamma} \qquad \frac{}{\Gamma \vdash \textbf{qinit}\ \underline{x} \triangleright \underline{x} : \mathtt{qpit}, \Gamma}$$

$$\frac{}{\underline{\mathbf{x}} : \mathtt{pit}^n, \Gamma \vdash \underline{\mathbf{y}} = A * \underline{\mathbf{x}} \triangleright \underline{\mathbf{x}} : \mathtt{pit}^n, \underline{\mathbf{y}} : \mathtt{pit}^m, \Gamma} \qquad \frac{}{\underline{x} : \mathtt{qpit}, \Gamma \vdash \textbf{meas}\ \underline{x} \triangleright \underline{x} : \mathtt{pit}, \Gamma}$$

$$\frac{}{\Gamma \vdash \textbf{skip} \triangleright \Gamma} \qquad \frac{}{\underline{x} : \mathtt{qpit}, \Gamma \vdash \textbf{disc}\ \underline{x} \triangleright \Gamma} \qquad \frac{}{\underline{\mathbf{x}} : \mathtt{qpit}^n, \Gamma \vdash \underline{\mathbf{x}} *\!= U \triangleright \underline{\mathbf{x}} : \mathtt{qpit}^n, \Gamma}$$

$$\frac{}{\underline{x} : \mathtt{pit}, \underline{y} : \mathtt{qpit}, \Gamma \vdash \textbf{ctrl}_P\ \underline{x}\ \underline{y} \triangleright \underline{x} : \mathtt{pit}, \underline{y} : \mathtt{qpit}, \Gamma}$$

Figure 1 Formation rules for SPL. $n \in \mathbb{N}^{>0}$ and $\tau \in \mathsf{Ty}$, $\underline{\mathbf{x}} : \tau^n$ is shorthand for $\{\underline{x_1} : \tau, \cdots, \underline{x_n} : \tau\}$ such that $\underline{\mathbf{x}} = (\underline{x_1}, \cdots, \underline{x_n}) \in \mathsf{Reg}^n$. New variables are always assumed to be fresh.

5 Case study: a small imperative language for stabiliser QEC

In this section, we introduce a minimal imperative language SPL (Stabiliser Programming Language) for stabiliser quantum channels. In other words, this is a language for quantum error correction, including measurements and classical control. This language is strongly inspired by the language QPL [41], but restricted to stabiliser operations and total, non-deterministic, *affine* classical operations.

We give SPL small-step operational semantics on pairs $[C|\rho]$ of terms acting on density operators as CPTP maps (similar to that of Ying [46, Section 3.2]), and a fully abstract denotational semantics in the SMC $\mathrm{Total}(\mathsf{AffRel}^M_{\mathbb{F}_p})$. This case study serves as a proof-of-concept to demonstrate that our symplectic semantics can be used as the foundation of a quantum compilation stack whose target code is fault-tolerant by construction and with a denotational semantics amenable to formal verification. The purpose of SPL is to show that $\mathrm{Total}(\mathsf{AffRel}^M_{\mathbb{F}_p})$ serves as a denotational semantics for a stabiliser quantum programs, and is to be contrasted with more powerful, and computationally expressive languages such as Quipper [23] and Proto-Quipper [20] which are not specifically tailored to the stabiliser fragment.

5.1 Syntax

SPL has quantum and classical types $\mathsf{Ty} ::= \mathtt{pit} \mid \mathtt{qpit}$. The terms are generated from the following grammar with respect to some fixed, linearly ordered set Reg indexing registers:

$$c, d \ ::= \ c \,\mathbin{;}\, d \mid \textbf{init}\ \underline{x} \mid \underline{\mathbf{y}} = A * \underline{\mathbf{x}} \mid \textbf{disc}\ \underline{x} \mid \textbf{qinit}\ \underline{x} \mid \underline{\mathbf{x}} *\!= U \mid \textbf{meas}\ \underline{x} \mid \textbf{ctrl}_P\ \underline{x}\ \underline{y} \mid \textbf{skip}.$$

for all $n, m \in \mathbb{N}$, $\alpha \in \mathbb{F}_p, U \in \mathcal{C}\ell_p^n, P \in \mathcal{P}_p^{\otimes n}$, $\mathbb{F}_p$-affine transformations $A : \mathbb{F}_p^n \to \mathbb{F}_p^m$, and $\underline{x}, \underline{y} \in \mathsf{Reg}$, $\underline{\mathbf{x}} \in \mathsf{Reg}^n$, $\underline{\mathbf{y}} \in \mathsf{Reg}^m$.

The term $c \,\mathbin{;}\, d$ represents the sequential composition of subterms; **init** $\underline{x}$ represents the initialisation of $\underline{x}$ as the p-ary digit 0; $\underline{\mathbf{y}} = A * \underline{\mathbf{x}}$ applies the affine transformation A to $\underline{\mathbf{x}}$ and stores the result on $\underline{\mathbf{y}}$; **disc** $\underline{x}$ takes the trace of $\underline{x}$; **qinit** $\underline{x}$ represents initialisation of $\underline{x}$ as the qu$p$it $|0\rangle$; $\underline{\mathbf{x}} *\!= U$ applies the Clifford operator U on $\underline{\mathbf{x}}$; **meas** $\underline{x}$ represents the Pauli-X measurement on $\underline{x}$; $\textbf{ctrl}_P\ \underline{x}\ \underline{y}$ applies the Pauli operator P on $\underline{y}$, classically controlled by $\underline{x}$ in the Pauli-X basis; and **skip** represents the identity.

SPL is equipped with an *environment-transforming* type system, which enforces linear usage of quantum data. Typed environments are partial functions $\Gamma : \mathsf{Reg} \to \mathsf{Ty}$ which bind registers to be either qupits or pits, and which we sometimes represent as $\{\underline{x} : \tau,\ \underline{y} : \sigma,\ \underline{z} : \mu,\ \cdots\}$ for $\underline{x}, \underline{y}, \underline{z} \cdots \in \mathsf{Reg}$ and $\tau, \sigma, \mu, \cdots \in \mathsf{Ty}$. We impose that the domain $\mathrm{dom}(\Gamma)$ of Γ, i.e.

the set of bound registers $\{\underline{x}, \underline{y}, \underline{z}, \cdots\}$, is *finite*. Judgments are triples $\Gamma \vdash t \triangleright \Delta$ consisting of a term t and typed environments Γ, Δ. A judgment $\Gamma \vdash t \triangleright \Delta$ is **well-formed** if it is derivable from the formation rules given in figure 1.

5.2 Operational semantics

In this subsection we define a structured operational semantics for SPL, which is strongly inspired by Ying's operational semantics for quantum programs [46, Section 3.2].

For notational convenience, let $\mathsf{StabChan}_p := \mathrm{Caus}(\mathrm{Split}^\dagger(\mathrm{CPM}(\mathsf{Stab}_p)))$ denote the category of quantum channels built out of stabiliser maps and Pauli measurements. We interpret typed environments as objects in $\mathsf{StabChan}_p$:

▶ **Definition 41.** *Given a typed environment* Γ*, let* $\langle\Gamma\rangle$ *be the dependent tensor product:*

$$\langle\Gamma\rangle := \bigotimes_{\underline{x}\in\mathrm{dom}(\Gamma)} \Big\{ (\mathcal{H}_p, 1_{\mathcal{H}_p}) \quad \textit{if} \quad \Gamma(\underline{x}) = \mathtt{qpit}; \quad \textit{else} \quad (\mathcal{H}_p, \mathcal{E}_X) \quad \textit{if} \quad \Gamma(\underline{x}) = \mathtt{pit}$$

and $\mathcal{D}(\Gamma) := \mathsf{StabChan}_p(I, \langle\Gamma\rangle)$ *be the set of density operators on* $\langle\Gamma\rangle$*.*

To give our operational semantics, we establish notation to represent stabiliser quantum channels acting on subspaces of a larger ambient space. Take typed environments Γ, Δ and ordered subsets (lists) $\underline{\mathbf{x}} \subseteq \mathrm{dom}(\Gamma)$ and $\underline{\mathbf{y}} \subseteq \mathrm{dom}(\Delta)$, where moreover, $\mathrm{dom}(\Gamma) \setminus \underline{\mathbf{x}} = \mathrm{dom}(\Delta) \setminus \underline{\mathbf{y}}$. Given a stabiliser quantum channel $\mathcal{C} : \langle\Gamma|_{\underline{\mathbf{x}}}\rangle \to \langle\Delta|_{\underline{\mathbf{y}}}\rangle$ let $\mathcal{C}_{\underline{\mathbf{x}},\underline{\mathbf{y}}} : \langle\Gamma\rangle \to \langle\Gamma'\rangle$ be the stabiliser quantum channel acting as $\mathcal{C}$ on the subspace $\langle\Delta\rangle \subseteq \langle\Gamma\rangle$ and trivially on its orthogonal complement $\langle\Gamma \setminus \Delta\rangle \subseteq \langle\Gamma\rangle$.

▶ **Definition 42.** *A* ***configuration*** *is a pair consisting of a well-formed judgement* $\Gamma \vdash t \triangleright \Sigma$ *and a density operator* $\rho \in \mathcal{D}(\Gamma)$*, denoted* $[\Gamma \vdash t \triangleright \Sigma \mid \rho \in \mathcal{D}(\Gamma)]$*, or* $[t|\rho]$ *for short.*

The **small-step operational semantics** of SPL is defined by the following reduction rules, where the typed environments are omitted for notational convenience:

$$[\mathbf{skip} \,⨾\, t|\rho] \rightsquigarrow [t|\rho] \quad [(\mathbf{init}\,\underline{x}) \,⨾\, t|\rho] \rightsquigarrow [t|\rho; \iota(|0\rangle)_{\underline{x},\underline{x}}] \quad [(\underline{\mathbf{y}} = A * \underline{\mathbf{x}}) \,⨾\, t|\rho] \rightsquigarrow [t|\rho; \iota(\mathcal{M}^A)_{\underline{\mathbf{x}},\underline{\mathbf{y}}}]$$
$$[(\mathbf{disc}\,\underline{x}) \,⨾\, t|\rho] \rightsquigarrow [t|\rho; (\mathrm{Tr}_{\mathcal{H}_p})_{\underline{x},\underline{x}}] \quad [(\mathbf{qinit}\,\underline{x}) \,⨾\, t|\rho] \rightsquigarrow [t|\rho; \iota(|0\rangle)_{\underline{x},\underline{x}}] \quad [(\underline{\mathbf{x}} \mathrel{*}= U) \,⨾\, t|\rho] \rightsquigarrow [t|\rho; \iota(U)_{\underline{\mathbf{x}},\underline{\mathbf{x}}}]$$
$$[(\mathbf{meas}\,\underline{x}) \,⨾\, t|\rho] \rightsquigarrow [t|\rho; (\mathcal{E}_X)_{\underline{x},\underline{x}}] \quad [(\mathbf{ctrl}_P\,\underline{x}\,\underline{y}) \,⨾\, t|\rho] \rightsquigarrow [t|\rho; CP_{(\underline{x},\underline{y}),(\underline{x},\underline{y})}],$$

where

- $\iota : \mathsf{Stab}_p \to \mathrm{CPM}(\mathsf{Stab}_p)$ takes pure stabilizer maps to stabiliser quantum channels;
- $\mathcal{E}_X : (\mathcal{H}_p, 1_{\mathcal{H}_p}) \to (\mathcal{H}_p, \mathcal{E}_X)$ denotes the Pauli-X measurement;
- $\mathrm{Tr}_{\mathcal{H}_p} : (\mathcal{H}_p, 1_{\mathcal{H}_p}) \to I$ denotes the trace;
- $\mathcal{M}^A := \sum_{\mathbf{x}\in\mathbb{F}_p^m} |A\mathbf{x}\rangle\langle\mathbf{x}| : (\mathcal{H}_p, \mathcal{E}_x)^{\otimes m} \to (\mathcal{H}_p, \mathcal{E}_x)^{\otimes n}$;
- $CP : (\mathcal{H}_p, \mathcal{E}_X) \otimes (\mathcal{H}_p, 1_{\mathcal{H}_p}) \to (\mathcal{H}_p, \mathcal{E}_X) \otimes (\mathcal{H}_p, 1_{\mathcal{H}_p})$ is the classically controlled $P \in \mathcal{P}_p$.

The types of density operators can be inferred from the typed environments. For example, reduction rules for pit vs qupit initialisation produce different density operators:

$$[\Gamma \vdash (\mathbf{init}\,\underline{x}) \,⨾\, t \triangleright \underline{x} : \mathtt{pit}, \Delta \mid \rho \in \mathcal{D}(\Gamma)] \rightsquigarrow [\underline{x} : \mathtt{pit}, \Gamma \vdash t \triangleright \Delta \mid \rho; \iota(|0\rangle)_{\underline{x},\underline{x}} \in \mathcal{D}(\underline{x} : \mathtt{pit}, \Gamma)]$$
$$[\Gamma \vdash (\mathbf{qinit}\,\underline{x}) \,⨾\, t \triangleright \underline{x} : \mathtt{qpit}, \Delta \mid \rho \in \mathcal{D}(\Gamma)] \rightsquigarrow [\underline{x} : \mathtt{qpit}, \Gamma \vdash t \triangleright \Delta \mid \rho; \iota(|0\rangle)_{\underline{x},\underline{x}} \in \mathcal{D}(\underline{x} : \mathtt{qpit}, \Gamma)]$$

▶ **Definition 43.** *Two quantum channels are* ***observationally equivalent*** *if they produce the same measurement statistics according to the Born rule when acting on all density matrices.*

▶ **Theorem 44.** *The operational semantics* $\rightsquigarrow^*$ *for* SPL *is sound, complete, and universal for the observational equivalence of stabiliser quantum channels.*

Proof. It is straightforward to see that given any configuration $[t|\rho]$, there is a unique ρ' such that $[t|\rho] \rightsquigarrow^* [\mathbf{skip}|\rho']$. Therefore, given any well-formed judgement t, there is a unique stabiliser quantum channel $[t|-]$. The observational equivalence of well-formed judgements c and d under $\rightsquigarrow^*$ therefore amounts to equality as stabiliser quantum channels, $[c|-] = [d|-]$, and thus, equality as quantum channels. ◀

5.3 Denotational semantics

We give SPL a denotational semantics in $\mathrm{Total}(\mathsf{AffRel}^Q_{\mathbb{F}_p})$. On types, let $[\![\texttt{pit}]\!] \coloneqq \mathbb{F}_p$ and $[\![\texttt{qpit}]\!] \coloneqq Q(\mathbb{F}_p^2, \omega_2)$. Define the denotation of a typed environment to be the dependent direct sum $[\![\Gamma]\!] \coloneqq \bigoplus_{\underline{x} \in \mathrm{dom}(\Gamma)} [\![\Gamma(\underline{x})]\!]$.

The denotation of well-formed judgments $\Gamma \vdash t \rhd \Delta$ is given by the maps $\mathsf{AffRel}^Q_{\mathbb{F}_p}([\![\Gamma]\!], [\![\Delta]\!])$ defined inductively from the denotation of generating terms. As before, we need to establish notation to represent affine relations acting on a subset of the registers of the context. Take ordered subsets $\underline{\mathbf{x}} \subseteq \mathrm{dom}(\Gamma)$ and $\underline{\mathbf{y}} \subseteq \mathrm{dom}(\Delta)$, where moreover, $\mathrm{dom}(\Gamma) \setminus \underline{\mathbf{x}} = \mathrm{dom}(\Delta) \setminus \underline{\mathbf{y}}$. Given a relation $S : [\![\Gamma|_{\underline{\mathbf{x}}}]\!] \to \left[\!\!\left[\Delta|_{\underline{\mathbf{y}}}\right]\!\!\right]$ let $S_{\underline{\mathbf{x}},\underline{\mathbf{y}}} : [\![\Gamma]\!] \to [\![\Gamma']\!]$ denote the relation acting as S on the subset $[\![\Delta]\!] \subseteq [\![\Gamma]\!]$ and trivially everywhere else $[\![\Gamma \setminus \Delta]\!] \subseteq [\![\Gamma]\!]$. The denotation of terms is defined inductively:

$$[\![c \,⨾\, d]\!] \coloneqq [\![c]\!] \,;\, [\![d]\!] \qquad [\![\mathbf{skip}]\!] \coloneqq 1_{[\![\Gamma]\!]} \qquad [\![\mathbf{init}\ \underline{x}]\!] \coloneqq (\{(0,0)\} : \mathbb{F}_p^0 \to \mathbb{F}_p\})_{\varnothing,\underline{x}}$$

$$[\![\underline{\mathbf{y}} = A * \underline{\mathbf{x}}]\!] \coloneqq (\mathrm{Gr}(A))_{\underline{\mathbf{x}},\underline{\mathbf{y}}} \qquad [\![\underline{\mathbf{x}} *\!= U]\!] \coloneqq (\mathrm{Rel}(U))_{\underline{\mathbf{x}},\underline{\mathbf{x}}} \qquad [\![\mathbf{disc}\ \underline{x}]\!] \coloneqq \{(x,0) \mid x \in \mathbb{F}_p^2\}_{\underline{x},\varnothing}$$

$$[\![\mathbf{meas}\ \underline{x}]\!] \coloneqq (\mathrm{Gr}(\pi_1))_{\underline{x},\underline{x}} \qquad [\![\mathbf{qinit}\ \underline{x}]\!] \coloneqq (\{(0,0)\} : \mathbb{F}_p^0 \to \mathbb{F}_p\}; \eta_X)_{\varnothing,\underline{x}}$$

$$[\![\mathbf{ctrl}_P\ \underline{x}\ \underline{y}\ ]\!] \coloneqq \left\{ \left(\left(s, \begin{bmatrix} t \\ u \end{bmatrix} \right), \left(s, \begin{bmatrix} t + s \cdot a \\ u + s \cdot b \end{bmatrix} \right) \right) \;\middle|\; s, t, u \in \mathbb{F}_p \right\}_{\underline{x} \sqcup \underline{y}, \underline{x} \sqcup \underline{y}}$$

where π_k is the direct-sum projection onto the k-th component, and $P = \pi(0, a, b) \in \mathcal{P}_p$. We have omitted the typed contexts, which are given in figure 1.

▶ **Theorem 45** (Full abstraction)**.** *Well-formed judgements c and d are observationally equivalent if and only if $[\![c]\!] = [\![d]\!]$.*

Proof. Since all generating judgements are stabiliser, it follows from a straightforward induction that $[c|-]$ lies in the embedding of $\mathsf{StabChan}_p$ into the category of CPTP maps between finite-dimensional C^*-algebras. By construction $[\![c]\!] = \mathrm{Rel}[c|-]$, therefore the claim follows from proposition 40. ◀

6 Conclusion

We have developed a denotational semantics for stabiliser quantum programs which allows for the manipulation of stabiliser codes, Pauli measurements, and classical control. We demonstrated the power of this semantics by giving a fully abstract denotational semantics to a toy imperative stabiliser language.

In the case of qubits, the affine, symplectic representation of stabiliser maps breaks down so that $\mathrm{Proj}(\mathsf{Stab}_2) \not\simeq \mathsf{AffLagRel}_{\mathbb{F}_2}$ [15]. By restricting the unitary operations to be generated by the controlled-not gate, the Pauli group and the swap gate, we obtain the maximal subcategory of qubit stabiliser maps on which the symplectic representation still holds [14, p. 156]. This is the natural setting for CSS codes [9, 42], which are widely used in QEC [1].

It is also future work to explore denotational semantics for stabiliser quantum programs using their graphical calculus. There is a complete ZX-calculus for affine Lagrangian relations [6], which is equivalent to the qupit ZX-calculus [5, 35] modulo scalars. This is an interesting direction for future work because the ZX-calculus has already been successful for constructing fault-tolerant quantum circuits [4], and the design and verification of QEC codes [11, 18].

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