Math 596: Homework One (Theory)

1. Show that a unitary matrix, say $\mathbf{U}$ where $\mathbf{U}^* = \mathbf{U}^{-1}$, is an isometry in the 2-norm, i.e.
$$||\mathbf{Ux}||_2 = ||\mathbf{x}||_2$$
Note, the easiest way to do this is to use the fact that
$$||\mathbf{x}||_2^2 = \langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^* \mathbf{x}$$
and to remember that for any matrix $\mathbf{A}$
$$\langle \mathbf{Ax}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{A}^* \mathbf{x} \rangle$$

2. Describe in words the previous result. In particular, what does this tell us about the effect of using unitary matrices as mappings in vector spaces?

3. Given that any matrix $\mathbf{A}$ has an SVD, say
$$\mathbf{A} = \mathbf{U\Sigma V}^*$$
explain briefly how any matrix $\mathbf{A}$ maps a vector from the domain vector space to the range vector space.

4. Show that if for matrix $\mathbf{A}$ we use the full SVD such that $\mathbf{A} = \mathbf{U\Sigma V}^*$, then
   - $\mathbf{A}^*\mathbf{A} = \mathbf{V}\Sigma^2\mathbf{V}^*$
   - $\mathbf{A}\mathbf{A}^* = \mathbf{U}\Sigma^2\mathbf{U}^*$

   How do these results possibly change if we use the economy-SVD?

5. (Graduate/Extra Credit Problem) When using the economy SVD, only one of the products $\mathbf{U}^*\mathbf{U}$ and $\mathbf{U}\mathbf{U}^*$ is guaranteed to be the identity, while the other is a projection. Which is which and why?

6. (Graduate/Extra Credit Problem) In the Frobenius norm $||\cdot||_F$ where
$$||\mathbf{A}||_F^2 = \mathrm{tr}\left(\mathbf{A}^\dagger \mathbf{A}\right)$$

if using the SVD we have $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\dagger}$, show that

$$||\mathbf{A}||_F = \left(\sum_{j=1}^{N} \sigma_j^2 \right)^{1/2}$$

where $\sigma_j \geq 0$ are the singular values of $\mathbf{A}$.