The Fundamental Theorem of LA in action:

- Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \end{pmatrix}$ so $A: \mathbb{R}^3 \to \mathbb{R}^2$

1) To find $\ker(A)$, we solve:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \leftarrow \text{2 l.i. rows}$$

So $\vec{x} \in \ker(A) \to x_1 + x_3 = 0,\ x_2 = 0$

So $\ker(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

Moreover, we [redacted] see $\text{rank}(A) = 3 - \dim(\ker(A))$
$= 3 - 1 = 2.$

2) To find $\text{rng}(A^T)$, we use the fact that $\text{rng}(A^T) \perp \ker(A)$

So if $\vec{x} \in \text{rng}(A^T) \to \left\langle \vec{x}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle = 0$

$$\to x_1 - x_3 = 0 \to \vec{x} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

So $\text{rng}(A^T) = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$

Now we see $\mathbb{R}^3 = \underbrace{\text{Span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}}_{=\ker(A)} \oplus \underbrace{\text{Span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}}_{=\text{rng}(A^T)}$ ②

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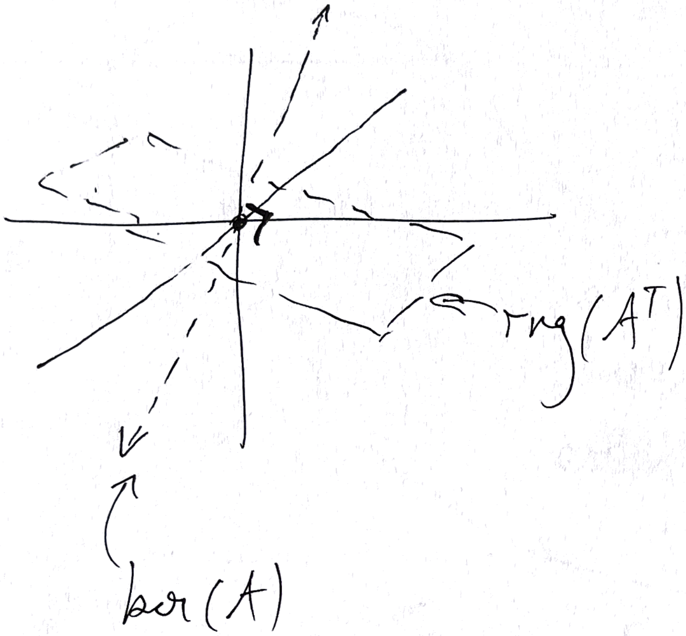


As for $\mathbb{R}^2 = \ker(A^T) \oplus \text{rng}(A)$

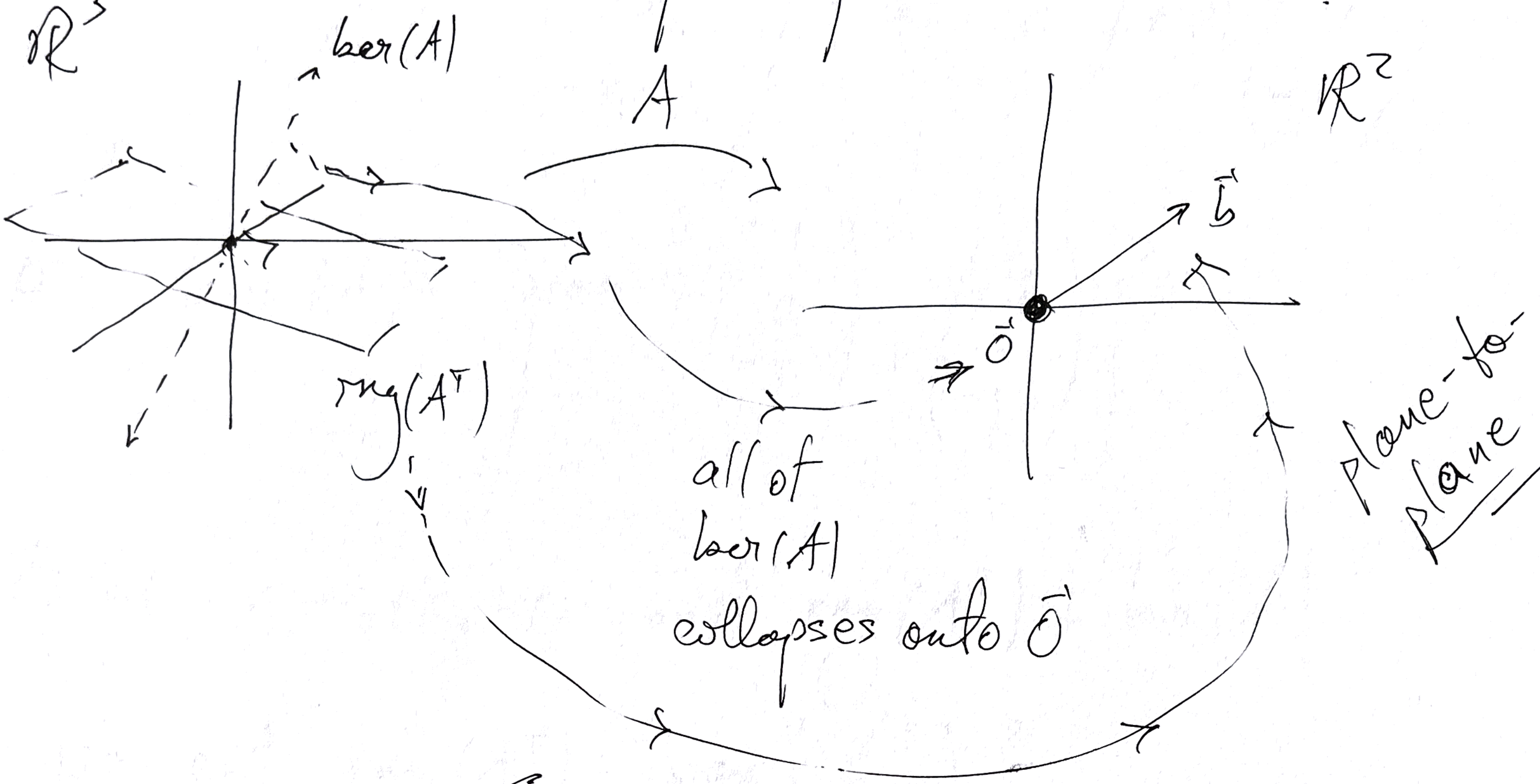
3) Find $\ker(A^T)$ $\to$ $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

So $y_1 = y_2 = 0$ or $\ker(A^T) = \{\vec{0}\}$.

(for $\vec{y} \in \ker(A^T)$)

So $\mathbb{R}^2 = \text{rng}(A) \Rightarrow$ ~~So~~ we see again $\text{rank}(A) = \dim(\text{rng}(A)) = 2$

So we now have a complete picture of A:



So any $\vec{b} \in \mathbb{R}^2$ has a pre-image in rng(A^T).

i.e. $A\vec{x} = \vec{b}$ has a solution $\forall \vec{b} \in \mathbb{R}^2$

- Now let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$

1) ker $(A) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

So $\vec{x} \in$ ker $(A) \to x_1 + x_2 + x_3 = 0$ or $x_3 = -x_1 - x_2$

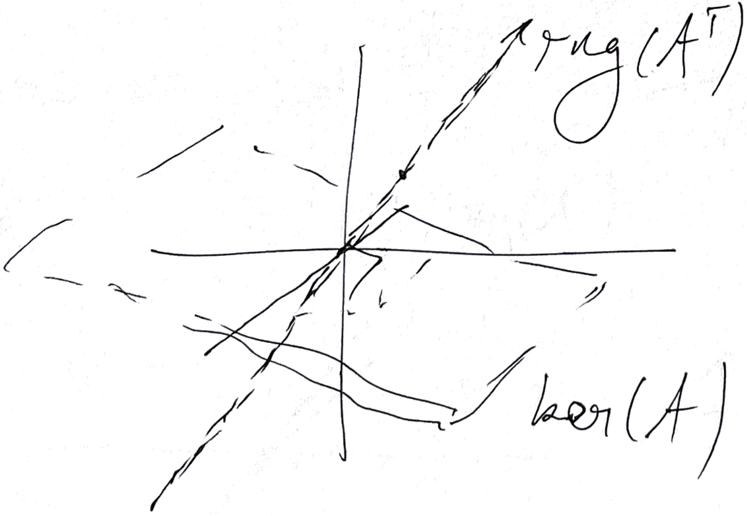
So $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -x_1 - x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

or $\ker(A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

From $x_1 + x_2 + x_3 = 0$ and $\operatorname{rng}(A^T) \perp \ker(A)$

we get $\operatorname{rng}(A^T) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

and $\operatorname{rank}(A) = 1$.



As for $\ker(A^T) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ ⑤

So, $y_1 + 2y_2 = 0$ or $\ker(A^T) = \text{Span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$

Given: $\text{rng}(A) \perp \ker(A^T)$, then

$$\vec{y} \in \text{rng}(A) \Rightarrow \left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\rangle = 0$$

$$\text{or} \quad -2y_1 + y_2 = 0$$

So $\text{rng}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$

Now we see: