Symmetry in Motion: Discovering Platonic Solids

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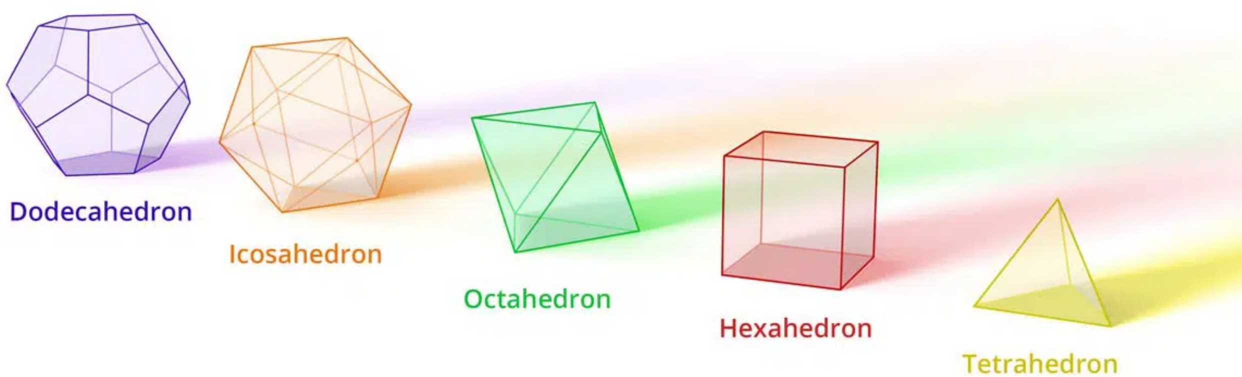


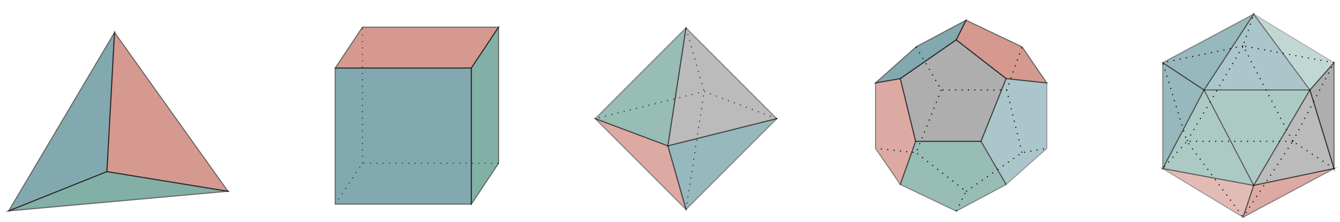
Contents

1 Chapter 1 (Introduction)

In this paper, we delve into the realm of symmetry, focusing particularly on the rotational symmetries exhibited by one of the most iconic Platonic solids: the cube. Central to our exploration is the notion of the orbit-stabilizer theorem, a powerful tool in the study of group actions and symmetries. This theorem provides a systematic framework for analyzing the rotational symmetries of the cube and illuminates the intricate relationship between its geometric properties and its symmetrical transformations. Join us as we embark on a journey through the symmetrical wonders of the Platonic solids.

1.1 Platonic Solids Definition

The Platonic solids are a group of five convex polyhedra. Each of their faces are identical regular polygons, and the same number of faces meet at each vertex. These shapes include following:



Tetrahedron Hexahedron (Cube) Octahedron Dodecahedron Icosahedron

Each of these Platonic solid follows the Euler's formula which states that:

$$V - E + F = 2$$

Where V is the number of vertices, E is the number of edges, and F is the number of faces. We can also assign a pair of integers {p,q}, where p is the number of edges of each face and q is the number of faces that meet at each vertex. Also known as the Schläfli symbol. Because every edge joins two vertices and has two adjacent faces we also have the equation:

$$pF = 2E = qV$$

Combining the equations from above we get:

$$\frac{2E}{q} - E + \frac{2E}{p} = 2$$

Simplifying we get:

$$\frac{1}{q} + \frac{1}{p} = \frac{1}{2} + \frac{1}{E}$$

Because $E \geq 6$ we know:

$$\frac{1}{q} + \frac{1}{p} > \frac{1}{2}$$

Having established the foundational characteristics of the Platonic solids, lets take a look at their counterparts the dual polyhedra. Every polyhedra has a dual structure where the vertices of one correspond to the faces of the other, and the edges of pairs of vertices correspond to the edges between pairs of faces.

1.2 Dual Polytopes

The Tetrahedron is a dual of itself (self-dual). The Square and the Octahedron form a dual pair. Lastly the Dodecahedron and the Icosahedron from a dual pair. If a polyhedra has Schläfli symbol $\{p, q\}$, then the dual of it has the symbol $\{q, p\}$. You can create the dual polyhedron by putting a vertex at the center of each face and then connecting the vertices of adjacent faces.

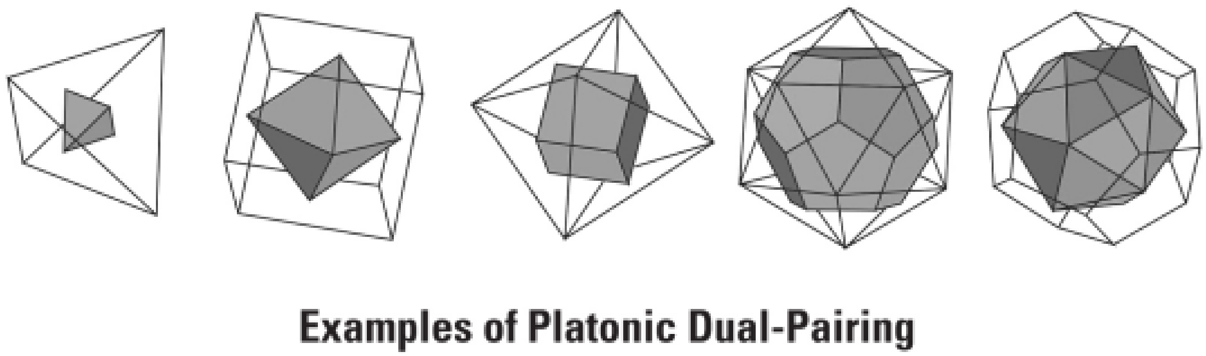


Figure 1: Each of the five platonic solids depicted with their duals.

As we explore the realm of symmetry we can find a profound link between platonic solids and the very abstract realm of group theory. A fundamental principle used to understand the symmetries of these shapes include the Orbit Stabilizer Theorem.

1.3 Orbit-Stabilizer

Definition 1.1 (Orbit, Stabilizer)**.** Given a group G acting on X the **orbit** of $x \in X$ is the set

$$G \cdot x = \{gx : x \in G\} \subset X$$

. We define the **stabilizer** of $x \in X$ to be the subgroup

$$\text{Stab}(x) = \{g \in G : gx = x\} \leq G.$$

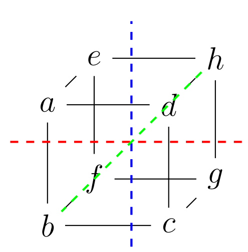
For a finite group G acting on X, by fixing an element $x \in X$ we have the following theorem relating the size of the orbit and stabilizer to the size of G.

Theorem 1.2 (Orbit-Stabilizer Theorem)**.** *If G is a finite group acting on a set X. Consider a elemenet $x \in X$. Then*

$$|G| = |G \cdot x||Stab(x)|$$

that is the size of the group G is the size orbit times the size of the stabilizer.

Example 1.3. For example lets find out what the orbit of a cubes face is. Consider rotating the cube around the following axes.



We can move the top face $\begin{matrix} e & - & h \\ | & & | \\ a & - & d \end{matrix}$ to the bottom face $\begin{matrix} f & - & g \\ | & & | \\ b & - & c \end{matrix}$, right face $\begin{matrix} d & - & h \\ | & & | \\ c & - & g \end{matrix}$, or left face $\begin{matrix} a & - & e \\ | & & | \\ b & - & f \end{matrix}$ by rotating around the green axis. To acquire the other two front face $\begin{matrix} a & - & d \\ | & & | \\ b & - & c \end{matrix}$, and back faces $\begin{matrix} e & - & h \\ | & & | \\ f & - & g \end{matrix}$ we just have to rotate around the red axis. Thus we can move any face of the cube to any other face concluding that the size of the orbit of a cube's face is 6. Having explored the orbit of a cubes face under different rotations, we now pivot our attention to the first of the Platonic solids the Tetrahedron.

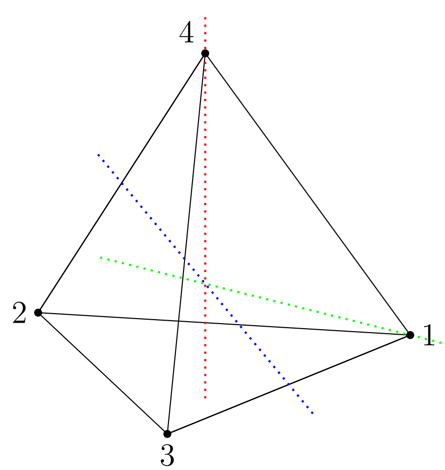
2 Chapter 2 (Tetrahedron)

2.1 Definition

The Tetrahedron is made up of four equilateral triangular faces, four vertices, and six edges. It is the simplest of the Platonic solids and is the only self-dual polyhedra. See Figure 1.

2.2 Symmetry Group Sym_4

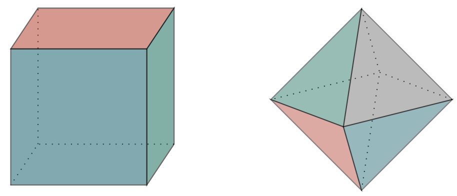
Consider a regular Tetrahedron. First we will focus on the rotational symmetries. The figure below shows two axes one of which that passes through a vertex and the center of the opposite face well make this red. The other which passes through the midpoints of opposite edges and we will make this blue. There are four potential red axes and two rotations along each of them. As for the blue axis there are three potential axes but only one rotation which sends the tetrahedron to itself. Adding the identity we have 12 symmetries. For a more detailed explanation of this please read Armstrong's Groups and Symmetry book [1]



3 Chapter 3 (Cubes and Octahedrons)

3.1 Definition

The Cube is made up of six square faces, twelve edges, and eight vertices. The Octahedron is made up of eight triangular faces, twelve edges, and six vertices. They are duals of each other as explained in Chapter 1 1.



Hexahedron (Cube) Octahedron

3.2 Order via Orbit-Stabilizer

The cube has 48 symmetries, 24 of which are rotational and the other being 24 from products of reflections and products of rotations and the antipodal map. If we rotate the cube around the same axes as we did in Example 1.3 we can find all 24 symmetries of the cube using the orbit stabilizer theorem.

Lemma 3.1. *The stabilizer of a face under the group of rotational symmetries of a cube has order 4.*

Proof. Rotations stabilize a line through the origin in $\mathbb{R}^3$. If you stabilize the front face by a rotation you need to stabilize it through the center. So any rotation stabilizing the front face, fixes the line through the front face and origin, meaning it fixes the plane parallel to it. If we were to rotate the cube around a different type of axis, for example one that connects opposite edges. This kind of rotation actually swaps adjacent faces thus any rotation along these axes don't stabilize the face of a cube. The only types of rotations that send vertices to vertices in the front face are rotations by multiples of $\pi/2$ of which there are 4.

$\square$

See Figure 1.3 from Example 1.3 to see the 3 different axes that pass through the center of opposite faces.

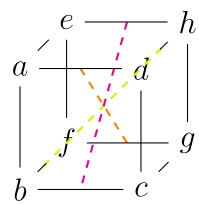
Theorem 3.2. *The rotational symmetry group of the cube has order 24.*

Proof. In the lemma 3.1 above we have showed that the only way to stabilize a cubes face is by rotating around an axis that goes through the center of that face. That of which the cube has 4 rotations along. Using the previous knowledge from Example 1.3. We know that the orbit of a cubes face is 6. To see the full group of rotational symmetries of a cube we will use the Orbit Stabilizer Theorem 1.2 from Chapter 1. Multiplying the orbit of a cubes face by the stabilizer gives us the size of our group. That is $6 \cdot 4 = 24 = |G|$.

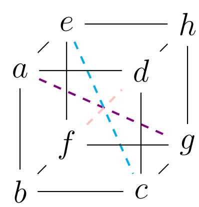
$\square$

3.3 Rotational Symmetries of a Cube

Lets talk about the rotational symmetries of a Cube. As proven before in Theorem 3.2 the Cube has 24 rotational symmetries and they form a group that is isomorphic to S_4 (the group of permutations of four objects). Consider the rotations along the axes that go through the center of opposite faces from Example 1.3. We showed earlier that there are 3 non-identity rotations we can do along these axes by 90°, 180°, 270°. Lets look at the rotations along different kinds of axes. We know that there are 12 edges in a cube thus we can form 6 distinct axes that go through the midpoint of opposite edges. If we rotate the cube around any of these axes we can find that there exists only 1 non-identity rotation by 180°.



Lastly lets look at rotating the cube around axes that go through opposite vertices. We know that the cube has 8 vertices thus there exists 4 such axes. Rotating around these axes we find that there are 2 non-identity rotations once by 120° and once by 240°.



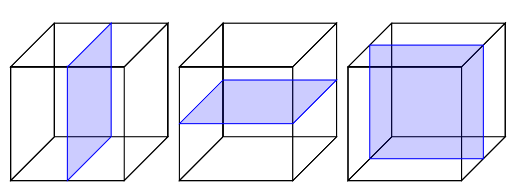
Adding up all of the possible rotations along all the different axes gives us $(3\cdot3)+(6\cdot1)+(4\cdot2) = 23$ symmetries of the cube. The last one is the identity where there are no rotations involved. This is where all 24 symmetries of the cube come from. But where do the other 24 symmetries come from? Some of them come from products of reflections also known as the Antipodal map.

3.4 Antipodal Map

The Antipodal Map is similar to a reflection but it takes points on the polyhedron and sends them to their opposite point through the origin on the shape. For example, consider the cube centered at the origin. The antipodal map sends each point to its negative. As a permutation of the labeled vertices in the cube from Example 1.3 it sends

$$a \longleftrightarrow g, \quad b \longleftrightarrow h, \quad c \longleftrightarrow e, \quad d \longleftrightarrow f$$

This map can also be described as the composition of reflections in the xy-plane, yz-plane, and xz-plane.



Furthermore, the antipodal map takes the front face $\begin{matrix} a & - & d \\ | & & | \\ b & - & c \end{matrix}$, reading the vertices counterclockwise starting at a to the face $\begin{matrix} e & - & h \\ | & & | \\ f & - & g \end{matrix}$, reading the vertices counterclockwise starting at g. That is, the front face gets reflected to the back face and then rotated by π. This doesnt get us all 48 symmetries of the cube however. The rest of the group comes from products of rotations and the antiopdal map.

3.5 Full Group As a Product of Rotations and Antipodal Map

To obtain all symmetries of the cube we work with compositions of rotations and the antipodal map. Specifically,

Theorem 3.3. *Every symmetry of the cube can be described as a rotation or a rotation composed with the antipodal map.*

Proof. We showed in the Rotational Symmetries of a Cube that we are able to rotate the cube about the 3 different x, y, and z axes creating 9 different symmetries. We also showed that the Antipodal map swaps points opposite one another through the origin of the cube. This also creates symmetries of the cube. Thus composing rotations with the antipodal map creates another symmetry of the cube. In doing so we can find all 48 symmetries of the cube. □

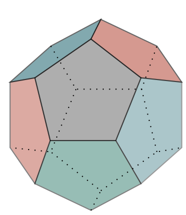
Note that the antipodal map commutes with every rotation so that we have the following result as a direct consequence of the above Theorem 3.3

Corollary 3.4. *The full group of symmetries of a cube is isomorphic to* $\mathsf{Sym}_4 \times \mathbb{Z}_2$.

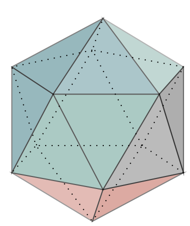
4 Chapter 4 (Dodecahedrons and Icosahedrons)

4.1 Definition

The Dodecahedron is made up of twelve pentagonal faces, thirty edges, and twenty vertices. The Icosahedron is made up of twenty equilateral triangular faces, thirty edges, and twelve vertices. They are duals of each other as explained in Chapter 1 1.



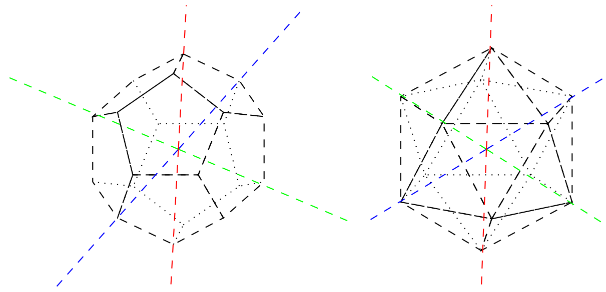
Dodecahedron



Icosahedron

4.2 Symmetry Group

We can once again find the orbit of the face of Dodecahedrons and Icosahedrons by rotating them around different axes.



In doing so we find that the Dodecahedron and Icosahedron both have a total of 120 symmetries. 60 of which come from rotations and form a group that is isomorphic to A_5 the other 60 from products of reflections as well as products of rotations and reflections. For more interested readers please read Chapter 8 of Armstrong's Groups and Symmetry [1]

5 Conclusion

After traversing the complicated landscape of geometry and group theory through the symmetries of the Platonic Solids. From the fundamental simplicity of the tetrahedron to the intricate symmetries of the cube. The orbit stabilizer theorem shines through as a guiding light to help us understand the symmetrical transformations of these shapes.

As we conclude our journey let us come together and appreciate the Platonic solids extends far beyond their aesthetic appeal. May our exploration inspire further inquiry and curiosity.

References

[1] M. A. Armstrong. *Groups and symmetry*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1988.