ME 263Z Project 1

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Abstract

Point vortices can be used to approximate vortex wakes of airplanes under certain loading and wing configurations. Sets of point vortices are used in this project to model the wakes of planes under 4 distinct conditions; wings with inboard flaps deployed, a wing with tip flaps deployed, during takeoff and landing, as well as a wing under elliptical loading conditions. In order to model the movement of the vortices generated by these conditions, a fourth order Runge-Kutta scheme is used. The simulations conducted using this method suggest that the runge-kutta model used is viable only to a certain simulated time beyond what is necessary to observe the characterizing behaviour of the vortex systems provided, and that increasing relative strengths of vortices in a system has an effect on the path and frequency of repetition in point vortex systems.

1 Problem Description

In order to characterize the motion of a system of n point vortices in an incompressible, inviscid flow on a 2D plane, we can use the concept of superposition of flows to sum the total fluid velocity of each vortex. The equation for the velocity induced at point (x, y) by a single point vortex is

$$\mathbf{U} = \frac{\Gamma_j}{2\pi[(x - x_j)^2 + (y - y_j)^2]}[-(y - y_j), (x - x_j)] \tag{1}$$

to find the motion of a single vortex, since we know that vortices move with the surrounding fluid due to Helmholtz's laws, we can state the velocity of the ith vortex in a system of n vortices as

$$\mathbf{U}_i = (\frac{dx_i}{dt}, \frac{dy_i}{dt}) = \sum_{j=1, j\neq i}^{n} \frac{\Gamma_j}{2\pi[(x_i - x_j)^2 + (y_i - y_j)^2]}[-(y_i - y_j), (x_i - x_j)] \tag{2}$$

This is a system of $2 * n$ ordinary differential equations, all of which must be integrated simultaneously. In order to avoid this ugly affair, a numerical approach must be taken to determine the paths of the vortices.

2 Formulation

2.1 Fourth Order Runge-Kutta Scheme

In order to simultaneously solve the ODEs, a Fourth Order Runge-Kutta Scheme can be used. The time frame being analyzed will be broken into a number of steps N, each covering Δt seconds. Given a system of n differential equations, each dependent on n variables

$$x_1' = f(x1, x2, \dots, xn), x_2' = f(x1, x2, \dots, xn), \dots, x_n' = f(x1, x2, \dots, xn)$$

The value of each parameter at the next time step $t + 1$

$$(x_n')_{t+1} = (x_n')_t + \frac{1}{6}(k_{1,n} + 2k_{2,n} + 2k_{3,n} + k_{4,n})$$

where the value of each of the constants k_1, k_2, k_3, k_4 for the ith variable is

$$k_{1,i} = x_i'(x_1, x_2, \dots, x_n)\Delta t$$

$$k_{2,i} = x_i'(x_1 + \frac{1}{2}k_{1,1}, x_2 + \frac{1}{2}k_{1,2}, \dots, x_n + \frac{1}{2}k_{1,n})\Delta t$$

$$k_{3,i} = x_i'(x_1 + \frac{1}{2}k_{2,1}, x_2 + \frac{1}{2}k_{2,2}, \dots, x_n + \frac{1}{2}k_{2,n})\Delta t$$

$$k_{4,i} = x_i'(x_1 + k_{3,1}, x_2 + k_{3,2}, \dots, x_n + k_{3,n})\Delta t$$

Taking all values $x_1, x_2, \dots, x_n$ to be the values at time-step t

As stated by equation 2, there are two differential equations for the velocity of each point i, which may be expressed as a sum of the induced velocities

$$\frac{dx_i}{dt} = \sum_{j=1, j\neq i}^{n} \frac{\Gamma_j}{2\pi[(x_i - x_j)^2 + (y_i - y_j)^2]}(y_j - y_i) = \sum_{j=1, j\neq i}^{n} f_{x_i}(x_i, y_i, x_j, y_j) \tag{3}$$

$$\frac{dy_i}{dt} = \sum_{j=1, j\neq i}^{n} \frac{\Gamma_j}{2\pi[(x_i - x_j)^2 + (y_i - y_j)^2]}(x_i - x_j) = \sum_{j=1, j\neq i}^{n} f_{y_i}(x_i, y_i, x_j, y_j) \tag{4}$$

Since these functions can be expressed as the sums of other functions of four variables, the implementation of the Runge-Kutta method can be made much simpler by expressing each k value as a function of these "sub-functions". For example k_2 for an arbitrary x variable x_i can be expressed as

$$k_{2,x_i} = \sum_{j=1, j\neq i}^{n} f_{x_i}(x_i + \frac{1}{2}k_{1,x_i}, y_i + \frac{1}{2}k_{1,y_i}, x_j + \frac{1}{2}k_{1,x_j}, y_j + \frac{1}{2}k_{1,y_j})\Delta t$$

for visualization purposes, the loop for determining k_2 in Matlab is pictured below.

```
% k2 Values
for s = 1:n                 % looping the vortex being analyzed
    for r = 1:n             % looping the vortex inducing velocity
        if s ~=r            % ensuring the sum doesnt consider i = j
            % x differential k
            k2(s,1) = dx(...
                xn(h,s) + k1(s,1)*dt/2,...
                xn(h,r) + k1(r,1)*dt/2,...
                yn(h,s) + k1(s,2)*dt/2,...
                yn(h,r) + k1(r,2)*dt/2,...
                S_n(r)) + k2(s,1);
            % y differential k
            k2(s,2) = dy(...
                xn(h,s) + k1(s,1)*dt/2,...
                xn(h,r) + k1(r,1)*dt/2,...
                yn(h,s) + k1(s,2)*dt/2,...
                yn(h,r) + k1(r,2)*dt/2,...
                S_n(r)) + k2(s,2);
        end
    end
end
```

Figure 1: Matlab Code used to find values for $k2$. dx() and dy() are functions which are equivalent to equations 3 and 4.

2.2 Variation of Vortex Parameters

The magnitude of the circulation Γ and the distance between vortices both have significant effects on how quickly each vortex moves relative to other vortices, however, the way they affect the system is different. Increasing Γ increases the velocity of the vortices, allowing them to go through more cycles and travel a farther distance, as demonstrated in figure 2.

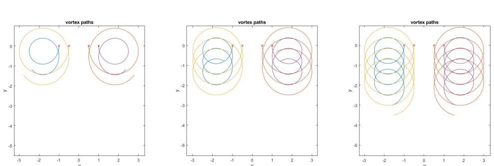

Figure 2: Path lines of vortices in a system with the only varied parameter being circulation magnitude. $\Gamma = 2$ for the leftmost figure, $\Gamma = 4$ for the middle figure and $\Gamma = 6$ for the rightmost.

Increasing the distance between points without changing the general configuration or the circulation strength slows the flow and frequency of path line loops, since the induced velocity at a point is inversely proportional to distance from the vortex. Path lines follow the same shape, but are scaled up. Figure 3 shows a set of vortices placed on a line with their x-positions from the y axis being $x = [-b/2, -b/4, b/4, b/2]$ with three different values for b, listed in the figure.

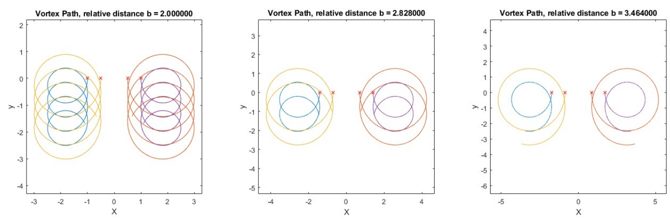

Figure 3: Path lines for a series of vortices placed on the x axis at $x = [-b/2, -b/4, b/4, b/2]$ for varying values of b

After conducting a set of simulations, it was determined that the results of the runge-kutta simulation would degenerate into asymmetry after a period of time which was in excess of what was needed to observe the cyclic nature of the vortices' movement when $b \approx \Gamma < 10$, which was between 10 and 100 seconds. Beyond this point in the most susceptible cases, asymmetry began developing across the y axis as shown in figure 4. This is likely either a limitation of Runge-Kutta scheme or a limitation of my own programming skills.

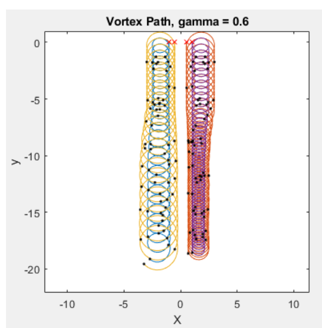

Figure 4: Vortex paths of a system with a total simulation time of 320s

Ultimately $b = \Gamma = 2$ with a total simulation time of between 5 and 20 seconds and a time-step value of $\Delta t = 0.001s$ was chosen, mostly to keep simulation time down to avoid the aforementioned errors and increase the resolution of the data

3 Results and Discussion

3.1 Test Cases

All test cases behaved as expected, with the first case, two vortices with x, y and circulation values of $[(-b/2, 0, -\Gamma), (b/2, 0, \gamma\Gamma)]$ moving downward at a constant velocity, and all points placed on equidistant locations on a circle rotating around the circle's center. Plots for paths of each test case are below. Red Xs denote starting locations for each vortex, and black dots when present indicate 200 time-steps.

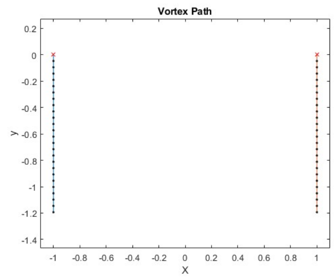

Figure 5: Test Case 1: two points with opposing circulation strengths $\Gamma, -\Gamma$

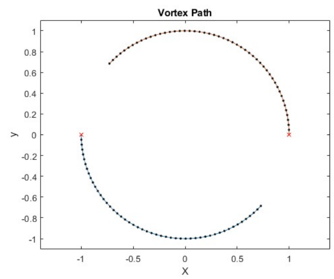

Figure 6: Test Case 1: two points with identical circulation strengths Γ

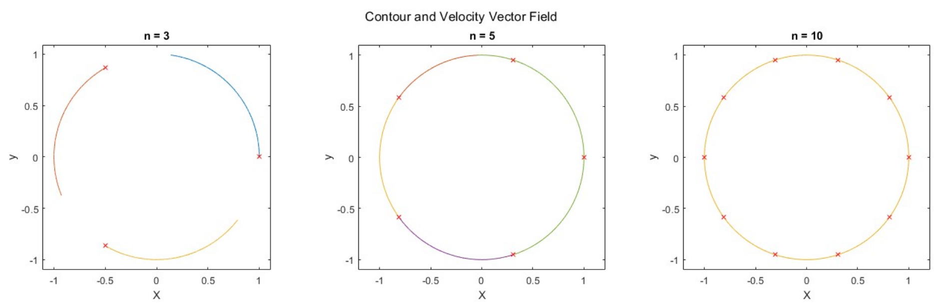

Figure 7: Test Case 1: $n = [3, 5, 10]$ points equidistant across a circle's circumference with identical circulation strengths Γ

3.2 Case 1 - Symmetric Co-rotating Vortex Pairs

This case, meant to model the wake of a wing with inboard flaps deployed, is given as 4 vortices with x positions, y positions and circulation strengths given by $[(-b/2, 0, -\Gamma), (-b/4, 0, -\gamma\Gamma), (b/4, 0, \gamma\Gamma), (b/2, 0, \Gamma)]$. γ was given values of 0.2, 0.4 and 0.6, as it will be for the following two cases. The vortices on either side rotate around each other as they move downwards. As γ increases, the vortices move faster and with a greater frequency, and the exterior, stronger vortices form loops with their paths.

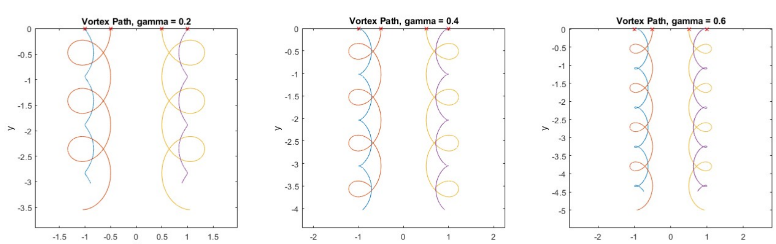

Figure 8: Case 1 Vortex paths for varying values of γ, stated in the figure.

3.3 Case 2 - Symmetric Counter-rotating Vortex Pairs

This case, meant to model the wake of a wing with wing tip flaps deployed, is given as 4 vortices with x positions, y positions and circulation strengths given by $[(-b/2, 0, -\Gamma), (-b/4, 0, \gamma\Gamma), (b/4, 0, -\gamma\Gamma), (b/2, 0, \Gamma)]$.

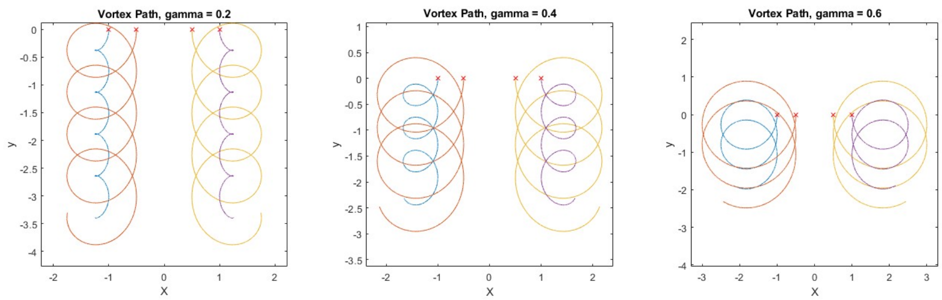

Figure 9: Case 2 Vortex paths for varying values of γ, stated in the figure.

increasing γ in this case impedes the downward motion of the vortices, and increases the radius of the loops created by the paths of each vortex. The frequency of the loops is also decreased when γ increases.

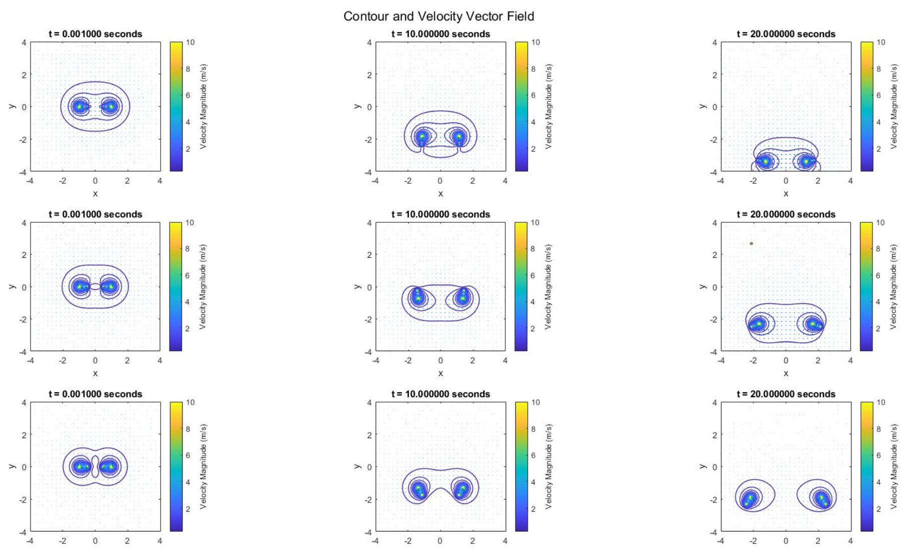

Figure 10: Case 2 Velocity Field and Contour at $t = 0s, 10s, 20s$ for increasing values of γ. Row 1 is $\gamma = 0.2$, row 2 is $\gamma = 0.4$ and row 3 is $\gamma = 0.6$

The velocity approaches zero in the space between the two vortex pairs, and is only lowered as γ increases. This limits the effect of the wake in comparison to case 1, where opposing co-rotating pairs accelerate fluid downward creating a much faster and larger disruption in their wake.

3.4 Case 3 - co-rotating vortex pairs over the ground

This case was meant to model the vortex wake during takeoff and landing, and consists of modeling four vortices at $[(-b/2, b/2, -\Gamma), (-b/4, b/2, -\gamma\Gamma), (b/4, b/2, \gamma\Gamma), (b/2, b/2, \Gamma)]$, as well as a set of four vortices at opposite y values and with opposing circulations.

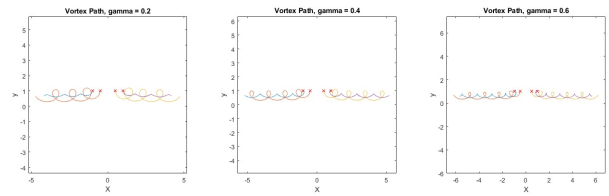

Figure 11: Case 3 Vortex paths for varying values of γ, stated in the figure.

The path of the vortices is largely linear following the ground in either direction, with increasing values of γ increase the speed at which the vortices travel as well as the frequency of loops. Vortex pathlines do not create an identical pattern as they move, but rather seem to be "leveling out" to a pattern which runs along the x-axis from one that ran across the y-axis, or at least would have if the vortices had started higher above the ground.

3.5 Case 4 - Vortex Sheet Roll Up

Case 4 is meant to simulate the vortex wake of an airplane while the airfoils are under elliptical loading conditions. The addition of extra points seemingly increases the magnitude of roll-ups as well as their frequency. Smaller vortices orbit larger ones and form two symmetric "clusters". Below is a figure depicting the contour surrounding analogous "vortex clusters" at $t = 5s$. The roll-ups become more and more obvious and increase in number as the number of nodes increase.

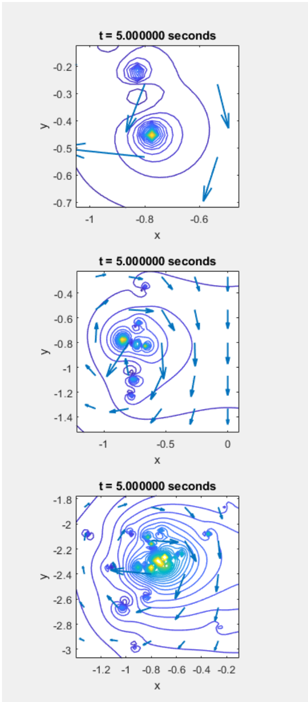

Figure 12: Vortex Sheet velocity field and contour plot at $t = 5s$ for increasing number of vortices. The top plot, row 1 is $n = 10$, plot 2 is $n = 20$ and plot 3 is $n = 50$

Additionally, adding more nodes increased the speed at which the vortices rotated and the speed the system of vortices moved at, as can be seen by the y positions of the vortices in each case at 5 seconds.