September 3rd

Goal

Calculus is the study of functions $f : [a, b] \to R$ and how they change.

To study calculus we must develop the language of real analysis.

1

1.1 Completeness Axiom

Definition 1.1

Let A be every nonempty subset of $\mathbb{R}$ or $(\emptyset \neq A \subseteq \mathbb{R})$.

If A has a largest element we denote it with $\max(A)$ and the min is $\min(A)$.

This leads to a problem.

Example

$A = [0, 1)$

$\min(A) = 0$

$\max(A) = \text{DNE}$

A has a least upper bound.

Definition 1.2

$\emptyset \neq A \subseteq \mathbb{R}$

We say A is bounded above iff there exists $m \in \mathbb{R}$ such that $a \leq m$ for all $a \in A$ we call such an m an upper bound for A.

Similarly we define bounded below as lower bound.

A is bounded if it's both bounded below and above.

Axiom 1.3 **Completeness**

if $\emptyset \neq A \subseteq \mathbb{R}$ is bounded above it has a least upper bound, called the supremum of A, $\sup(A)$

Example

$A = [0, 1)$ then $\sup(A) = 1$.

Proposition 1.4

if $\emptyset \neq A \subseteq \mathbb{R}$ is bounded below then A has a greatest lower bound, called the infimum of A, $\inf(A)$

Proof: Assume $m \in \mathbb{R}$ such that $a \geq m$ for all $a \in A$. Then, $-a \leq m, \forall a \in A$.

Hence, $-A = \{-a : a \in A\}$ is bounded above.

By the completeness axiom $L = \sup(-A)$ exists.

Claim

$\inf(A) = -L$

For all $a \in A, -a \leq L \Rightarrow a \geq -L \Rightarrow -L$ is a lower bound for A.

If $N \in \mathbb{R}$ is a lower bound for A then for all $a \in A, N \leq a$
$$\begin{aligned} &\Rightarrow -a \leq -N \\ &\Rightarrow L \leq -N (\text{def of sup}) \\ &\Rightarrow -L \leq N \\ &\Rightarrow -L \leq \inf(A) \end{aligned}$$
□

Remark

- $\sup(A) = L = -\inf(A)$

Example

$\emptyset \neq A, B \subseteq \mathbb{R}$ is bounded above.

Consider, $A + B = \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup(A) + \sup(B)$

Proof: if $a \in A$ and $b \in B$ then $a \leq \sup(A)$ and $b \leq \sup(B)$.

This implies $a + b \leq \sup(A) + \sup(B)$.

Hence $\sup(A + B) \leq \sup(A) + \sup(B)$

Let $\varepsilon < 0$ be given.

Observe that there exists an $a \in A, b \in B$ such that

$$\sup(A) - \frac{\varepsilon}{2} < a$$
$$\sup(B) - \frac{\varepsilon}{2} < b$$

This implies $\sup(A) + \sup(B) - \varepsilon \leq a + b \leq \sup(A + B)$.

Since $\varepsilon > 0$ was arbitrary, $\sup(A) + \sup(B) \leq \sup(A + B)$ □

September 5th

Example

$\emptyset \neq A \subseteq R$ is bounded above, let $\alpha > 0$ be given. $\alpha A = \{\alpha a : a \in A\}$

Prove that $\sup(\alpha A) = \alpha \sup(A)$

Proof: for $a \in A, a \leq \sup(A)$

$$\begin{aligned} &\Rightarrow \alpha a \leq \alpha \sup(A) \\ &\Rightarrow \sup(\alpha A) \leq \alpha \sup(A) \text{ (by the def of sup)} \end{aligned}$$

for $a \in A$

$$
\begin{aligned}
\alpha a &\leq \sup(\alpha A) \\
\Rightarrow a &\leq \frac{1}{\alpha} \sup(\alpha a) \\
\Rightarrow \sup(A) &\leq \frac{1}{\alpha} \sup(\alpha a) \\
\Rightarrow \alpha \sup(A) &\leq \sup(\alpha A)
\end{aligned}
$$

□

Exercise 1.5

What if $\alpha < 0$?

$$
\begin{aligned}
\sup(\alpha A) &= \sup(-(-\alpha)A) \\
&= -\inf(-\alpha A) \text{ (did last lecture)} \\
&= -(-\alpha)\inf(A) \text{ (for homework)} \\
&= \alpha \inf(A)
\end{aligned}
$$

Remark

$\emptyset \neq A \subseteq \mathbb{R}$

If A is not bounded above, we write $\sup(A) = \infty$.

If A is not bounded below, we write $\inf(A) = -\infty$.

Note

As a convention in this class we say

$\sup(\emptyset) = -\infty$ (the smallest of everything)
$\inf(\emptyset) = \infty$ (the largest of everything)

Proposition 1.6 **Density of $\mathbb{Q}$**

For all real numbers $a < b, \exists q \in Q$ such that $a < q < b$.

Proof: Choose $n \in \mathbb{N}$ large enough such that $n(b - a) > 1$.

Take $k \in \mathbb{N}$ such that $-k < an < bn < k$.

Consider $K = [-k, k] \cap \mathbb{Z}$ and such that $m = \min\{j \in K : an < j\}$. Thus $-K < an < m$ and so $m - 1 \in K$.

By minimality $an \geq m - 1$. Hence $m \leq an + 1 < bn$ and $an < m < bn$, this implies that $a < \frac{m}{n} < b$. □

2 Sequences

2.1 Limit of Sequences

September 8th

Definition 2.1

A sequence is an infinite list of real numbers:

$$(a_n)_{n=1}^{\infty} = (a_n) = (a_1, a_2, \ldots)$$

We can view a sequence as a function $f : \mathbb{N} \to \mathbb{R}$ via the correspond $(a_1, a_2, \ldots) \leftrightarrow f : n \mapsto a_n$

Notation

Let (a_n) be a sequence such that $a_n \in A$ for all $n \in \mathbb{N}$ we write $(a_n) \subseteq A$.

Note **Big Idea**

$(a_n) \subseteq \mathbb{R}, a \in \mathbb{R}$

We say "(a_n) converges to a" iff no matter how close you wish, eventually in (a_n) the terms are that close to a.

Definition 2.2

$(a_n) \subseteq \mathbb{R}, a \in \mathbb{R}$

We say (a_n) converges to a, $a \to a$ iff for all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$.

We call a limit of (a_n) and write $\lim\limits_{n\to\infty} a_n = a$ or $\lim a_n = a$.

Definition 2.3

if (a_n) does not converge to any $a \in \mathbb{R}$ we say it diverges.

Remark

By taking the next highest natural number, we may assume $N \in \mathbb{N}$. Symbolically:

$$a_n \to a \iff \forall \varepsilon > 0, \exists N \in \mathbb{R}, (n \geq N \implies |a_n - a| < \varepsilon)$$

Example

- $a_n = \frac{1}{n}$

Claim: $a_n \to 0$

Proof: Let $\varepsilon > 0$ be given and take $N = \frac{1}{\varepsilon} + 1$.

We see that $a_n - 0| < \varepsilon \iff \frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n$.

For $n \geq N, n > \frac{1}{\varepsilon}$ and so $|a_n - 0| < \varepsilon$. □asdads

-

2.2

2.3

September 22th

Theorem 2.4 **Balzard-Weierstaw**

Every bounded sequence, $(a_n) \subseteq \mathbb{R}$ has a convergent subsequence.

2.4 Completeness of $\mathbb{R}$

Definition 2.5

We say $(a_n) \subseteq \mathbb{R}$ is cauchy iff $\forall \varepsilon > 0, \exists N \in \mathbb{N}, n, m \geq N \Rightarrow |a_n - a_m| < \varepsilon$.

Proposition 2.6

If (a_n) is convergent then (a_n) is cauchy.

Proof: Suppose $a_n \to a$.

Let $\varepsilon > 0$ be given and take $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$.

For $n, m \geq N$,

$$\begin{aligned}
&|a_n - a_m| \\
&= |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \mathrm{E}.
\end{aligned}$$

□

Proposition 2.7 **$(a_n) \subseteq \mathbb{R}$ cauchy**

Suppose $\left(a_{n_k}\right)$ is a subsequence such that $a_{n_k} \to a$ then, $a_n \to a$.

Proof: Let $\varepsilon > 0$ be given.

We know, $\exists N_1 \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\varepsilon}{2}$ for $n, m \geq N_1$.

There exists $N_2 \geq N_1$ such that $\left|a_{n_k} - a\right| < \frac{\varepsilon}{2}$ for $k \geq N_2$

For $n \geq N_1$

$$\begin{aligned}
|a_n - a| &= \left|a_n - a_{n_{N_2}} + a_{n_{N_2}} - a\right| \\
&\leq \left|a_n - a_{n_{N_2}}\right| + \left|a_{n_{N_2}} - a\right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

September 24th

Definition 2.8

We say $A \subseteq \mathbb{R}$ is complete iff whenever $(a_n) \subseteq A$ is cauchy then $a_n \to a$ for some $a \in A$.

Theorem 2.9

$\mathbb{R}$ is complete.

Proof: Let $(a_n) \subseteq \mathbb{R}$ be cauchy. Then (a_n) is bounded.

By the B-W thm, $\exists \left(a_{n_k}\right)$ such that $a_{n_k} \to a \in \mathbb{R}$.

From before, $a_n \to a$ $\square$

Example

$A = (0, 1]$

$a_n = \frac{1}{n}$, then $a_n \to 0 \Rightarrow (a_n) \subseteq A$ is cauchy since $0 \notin A$, then A is not complete.

Definition 2.10

We say $C \subseteq \mathbb{R}$ is closed iff $(x_n) \subseteq C$ such that $x_n \to x \in R$ then $x \in C$.

Example

- $(0, 1]$ is not closed
- $\mathbb{R}$ is not closed
- Assignment 2 implies $[a, b]$ is closed.
- $\mathbb{Z}$ is closed.

Proposition 2.11

For $A \subseteq \mathbb{R}$, A is closed iff A is complete.

Proof: Assume A is closed. Let $(a_n) \subseteq A$ be cauchy.

Since $\mathbb{R}$ is complete, $a_n \to a \in \mathbb{R}$ for some a.

However A is closed and so $a \in A$.

Hence, A is complete.

Assume A is complete.

Let $(a_n) \subseteq A$ such that $a_n \to a \in \mathbb{R}$.

Since (a_n) is cauchy and A is complete, we know $a_n \to a \in A$.

Hence, A is closed. $\square$

2.5 lim sup, lim inf

2.6 Typology

September 29th

Definition 2.12

We say that $U \subset \mathbb{R}$ is open if and only if $\forall x \in U, \exists \varepsilon > 0, (x - \varepsilon, x + \varepsilon) \subseteq U$

Example

- (a, b) is open
- $[0, 1]$ is neither open or closed
- $\forall \varepsilon > 0, \exists q \in \mathbb{Q}, q \in (\pi - \varepsilon, \pi + \varepsilon) \subseteq \mathbb{R} \setminus \mathbb{Q}$

Proposition 2.13

A set $C \subseteq \mathbb{R}$ is closed iff $\mathbb{R} \setminus C$ is open

Proof: Assume C is closed.

Let $x \in \mathbb{R} \setminus C$.

For a contradiction, suppose $\forall \varepsilon > 0, \exists c \in C$ such that $c \in (x - \varepsilon, x + \varepsilon)$.

Thus $\forall n \in \mathbb{N}, \exists c_n \in C$ such that $c_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$. Then $|c_n - x| < \frac{1}{n} \to 0$ and so $c_n \to x$.

Since C is closed, $x \in C$, Contradiction!.

Assume $\mathbb{R} \setminus C$ is open.

Let $(c_n) \subseteq C$ such that $c_n \to x \in \mathbb{R}$.

For a contradiction assume $x \notin C$.

Since $\mathbb{R} \setminus C$ is open, $\exists \varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq \mathbb{R} \setminus C$.

Using $c_n \to x, \exists N$ such that $c_N \in (x - \varepsilon, x + \varepsilon) \subseteq \mathbb{R} \setminus C$, Contradiction!. □

Remark

$U \in \mathbb{R}$ is open $\Leftrightarrow \mathbb{R} \setminus (\mathbb{R} \setminus U)$ is open $\Leftrightarrow \mathbb{R} \setminus U$ is closed.

Proposition 2.14

Let I be the index set

1. If $u_i \subseteq \mathbb{R}, i \in I$ are open, then $\cup_{i \in I} u_i$ is open.
2. If $c_i \subseteq \mathbb{R}, i \in I$ are closed, then $\cap_{i \in I} c_i$ is closed.
3. If $u, v \in \mathbb{R}$ are open, then $u \cap v$ is open.
4. If $C, D \in \mathbb{R}$ are closed, then $C \cup D$ is closed.

Proof:

1. Let $x \in \cup u_i \Rightarrow \exists i \in I, x \in u_i \Rightarrow \exists \varepsilon > 0, (x - \varepsilon, x + \varepsilon) \subseteq u_i \subseteq U u_i$
2. $\mathbb{R} \setminus \cap c_i = \cup (\mathbb{R} \setminus c_i)$ is open $\Rightarrow \cap c_i$ is closed.
3. Let $x \in U \cap V$, such that

$$\begin{aligned}
&\exists \varepsilon_1 > 0, (x - \varepsilon_1, x + \varepsilon_1) \subseteq U \\
&\exists \varepsilon_2 > 0 (x - \varepsilon_2, x + \varepsilon_2) \subseteq V \\
&\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}
\end{aligned}$$

   This implies $(x - \varepsilon, x + \varepsilon) \subseteq U \cap V$.
4. $\mathbb{R} \setminus (C \cap D) = (\mathbb{R} \setminus C) \cap (\mathbb{R} \setminus D)$ open $\Rightarrow C \cup D$ closed.

□

Example

- $\underbrace{\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)}_{\text{open}} = \underbrace{\{0\}}_{\text{not open}}$
- $\underbrace{\bigcup_{n=1}^{\infty}\left[0,1-\frac{1}{n}\right]}_{\text{closed}} = \underbrace{[0,1)}_{\text{not closed}}$

Definition 2.15

$A \subseteq \mathbb{R}$

1. The closure of A is

$$\overline{A} = \bigcap_{A \subseteq C} C, \text{where } C \text{ is closed.}$$

2. The interior of A is

$$\operatorname{int}(A) = \bigcup_{U \subseteq A} U, \text{where } U \text{ is open.}$$

Remark

1. $\overline{A}$ is the smallest closed set containing A
2. $\operatorname{int}(A)$ is the largest open set contained in A

Example

$$\begin{aligned} A &= [0,1) \\ \operatorname{int}(A) &= (0,1) \\ \overline{A} &= [0,1] \end{aligned}$$

Definition 2.16

$A \subseteq \mathbb{R}$

1. We say $x \in \mathbb{R}$ is a limit point of A iff $\exists (a_n) \subseteq A$ such that $a_n \to x$.
2. We say $x \in \mathbb{R}$ is an interior point of A iff $\exists \varepsilon > 0, (x-\varepsilon, x+\varepsilon) \subseteq A$.

Proposition 2.17

1. $\overline{A} = \{\text{limit pts of A}\}$
2. $\operatorname{int}(A) = \{\text{interior pts of A}\}$

October 1st

Proof:

1. Let $L = \{x : x \text{ limit point of } A\}$

Let $x \in \overline{A}$. If $x \notin L$ then, $\exists \varepsilon > 0, (x-\varepsilon, x+\varepsilon) \subseteq \mathbb{R} \setminus A$. This implies $\underbrace{\mathbb{R} \setminus (x-\varepsilon, x+\varepsilon)}_{\text{closed}} \supseteq A$

Since $x \in \overline{A}, \Rightarrow x \in \mathbb{R} \setminus (x-\varepsilon, x+\varepsilon)$, Contradiction!.

$(\Longleftarrow)$

Let $x \in L$ □

Proposition 2.18

1. $\overline{\mathbb{R} \setminus A} = \mathbb{R} \setminus \operatorname{int}(A)$
2. $\operatorname{int}(\mathbb{R} \setminus A) = \mathbb{R} \setminus \overline{A}$

Proof: □

Example

- $\overline{\mathbb{Q}} = \mathbb{R}$
  $\operatorname{int}(\mathbb{Q}) = \emptyset$
-

3 Continuity

October 6th

Recall recall

if $f : A \to \mathbb{R}$ is cst at $a \in A$ iff $f(a_n) \to f(a)$ whenever $(a_n) \subseteq A, a_n \to a$

Proposition 3.1

$f : A \to \mathbb{R}$

Then, f is cts at $a \in A$ iff $\forall \varepsilon > 0, \exists \delta > 0, x \in A, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Proof: $(\Rightarrow)$ Suppose f is cts at $a \in A$. Let $\varepsilon > 0$ be given.
Suppose no such $\delta > 0$ exists. Thus for all $n \in \mathbb{N}$, $\exists a_n \in A$ such that $|a_n - a| < \frac{1}{n}$ but $|f(a_n) - f(a) \geq \varepsilon$.

$N a_n$, $a_n \to a$ and so $f(a_n) \to f(a)$ by cty at a.

For a large N, $|f(a_N) - f(a)| < \varepsilon$. Contradiction!.

$(\Longleftarrow)$ Suppose f satisfies the ε, δ Condition at a.

Assume $(a_n) \subseteq A$ such that $a_n \to a$. □

Claim

$f(a_n) \to f(a)$

Proof: Let $\varepsilon > 0$ be given.

There exists $\delta > 0$ such that $x \in A, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Also, $\exists N \in \mathbb{N}$ such that $n \geq |a_n - a| < \delta$.

For $n \geq N, |f(a_n) - f(a)| < \varepsilon$. □

Proposition 3.2

$f : A \to \mathbb{R}$

Then, f is cts iff $f^-1(u)$ is relatively open in A, whenever $U \subseteq \mathbb{R}$ is open.

Notation

$f : X \to Y, B \subseteq Y$.

The pre-image of B under f is $f^-1(B) = \{x \in X : f(x) \in B\}$.

Proof: Assume f is cts.

Assume there exists an open $U \subseteq \mathbb{R}$ such that $f^-1(u)$ is not relatively open in A.

Thus $\exists x \in f^{-1}(u)$ such that $\forall n \in N, \exists a_n \in A$ with, $|a_n - x| < \frac{1}{n}$ and $a_n \notin f^{-1}(u)$.

$\therefore a_n \to x$ and so, $f(a_n) \to f(x)$ by cty.

Since U is open, $\exists \varepsilon > 0, (f(x) - \varepsilon, f(x) + \varepsilon) \subseteq U$

However, $f(a_n) \to f(x)$ and so for large N, $f(a_n) \in (f(x) - \varepsilon, f(x) + \varepsilon) \subseteq u$. This implies $a_N \in f^{-1}(u)$, Contradiction!.

$\Longleftarrow$ Assume the pre-image Condition.

Let $(a_n) \subseteq A$ such that $a_n \to a \in A$.

Claim

$f(a_n) \to f(a)$

Let $\varepsilon > 0$ be given.

Consider $u = (f(a) - \varepsilon, f(a) + \varepsilon)$.

By assumption, $f^{-1}(u)$ is relatively open in A.

Since $a \in f^{-1}(u), \exists \delta > 0$ such that. $(a - \delta, a + \delta) \cap A \subseteq f^{-1}(u)$.

There exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |a_n - a| < \delta$.

Hence, $a \geq N \Rightarrow a_n \in (a - \delta, a + \delta) \cap A, \subset f^{-1}(u)$.

$\therefore n \geq f(a_n) \in U \Rightarrow |f(a_n) - f(a)| < \varepsilon$. □

Corollary 3.3

$f : A \to \mathbb{R}$

Then, f is cts iff $f^{-1}(c)$ is rel closed in A whenever $C \subseteq \mathbb{R}$ is closed.

Proof: $\Rightarrow$ Let f be cts, and let $C \subseteq \mathbb{R}$ be closed.

Then $\mathbb{R} \setminus C$ is open and so $f^{-1}(\mathbb{R} \setminus C$ is rel open in A.

For homework check, $f^{-1}(\mathbb{R} \setminus C) = f^{-1}(\mathbb{R}) \setminus f - 1(C) = A \setminus f^{-1}(c) \Rightarrow f^-1(C)$ rel closed in A.

$\Longleftarrow$ Identical, do for homework if time. □

October 8th

Proposition 3.4

$f, g : A \to \mathbb{R}, a \in A$ is cts at $a \in A$

Proof: Then, $f + g, \alpha f, fg$, where $\frac{f}{g} g(0) \neq 0$ are all cts at a.

$(a_n) \subseteq A, a_n \to a, (f+g)(a_n) = f(a_n) + g(a_n) \to f(a) + g(a)$, etc. □

Proposition 3.5

If g is cts at a, and f is cts at (a), then $f \circ g$ is cts at a.

Note **why**

$(a_n) \subseteq \text{dom}(g) < a_n \to a, g \text{ cts} \Rightarrow g(a_n) \to g(a)\ f \text{ cts} \Rightarrow f(g(a_n)) \to f(g(a)) \Rightarrow (f \circ g)(a_n) \to (f \circ g)(a)$.

Example

- $f, g : A \to \mathbb{R}$ are cts

  $X = \{x \in A : f(x) < g(x)\}$

  Prove that X is relatively open in A.

  $X = (g - f)^{-1}(0, \infty)$

  - $f, g : A \to \mathbb{R}$ is cts

    Prove that $\max\{f, g\}$ is cts.

    $\max\{f, g\} = \frac{(f+g)+|f-g|}{2}$

Proposition 3.6

$K \subseteq \mathbb{R}$ is compact

$f : K \to \mathbb{R}$ is cts

Then, $f(k)$ is compact.

Note **Notation**

$f : X \to Y, A \subseteq X$ the image of A is $f(A) = \{f(a) : a \in A$.

Proof: Let $(y_n) \subseteq f(k)$ and say $y_n = f(x_n), x_n \in K$.

Since $(x_n) \subseteq K$ and K is compact, $\exists a$ subsequence $X_{n_k} \to x \in K$.

Using continuity of f, we have $f\left(x_{n_k}\right) \to f(x)$, and so $y_{n_k} \to f(x) \in f(K)$ □

Example

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \arctan(x) = \tan^{-1}(x)$,

$$\underbrace{f(\mathbb{R})}_{\text{closed}} = \underbrace{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}_{\text{not closed}}$$

END OF MIDTERM MATERIAL

Midterm Exam Wed, Oct, 22, 7:00 - 8:30 PM. (not 8:50). DWE 1501

Every question has a theme

1. Suprema

    1. [5] From class (either a example problem or proof)
    2. [5] New, (new homework type problem)
2. Sequence Conv
    1. [3] New (a is separate)
    2. [3] From Class (b, c, d) are all part of the same idea
    3. [2] Stat a thm
    4. [2] From Class
3. Cty + topology
    1. [5] From class
    2. [5] New
4. Assignment Probs (read the assignment solutions)
    1. [5]
    2. [5]
5. Short Answer/Computation 5 x [2] = [10]

Out of 50 points

The density of $\mathbb{Q}$ proof is not on the midterm.

3.2 EVT + IVT

> **Recall**
>
> $\emptyset \neq A \subseteq \mathbb{R}$ bd
>
> $\exists (a_n), (b_n) \subseteq A$ such that
>
> $$a_{n_>} \sup(A) \in \overline{A}$$
>
> $$b_{n_>} \inf(A) \in \overline{A}$$

> **Theorem 3.7** — **Exterme Value Thm**
>
> If $K \subseteq \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is cts, then $\exists a, b \in K$ such that $f(a) = \max f(k)$ and $f(b) = \min f(k)$

Proof: Since $f(k)$ is compact

$\sup f(k) \in \overline{f(k)} = f(k)$ and $\inf f(k) \in \overline{f(k)} = f(k)$. □

October 10th

> **Theorem 3.8** — **Intermediate Value Theorem**
>
> If $f : [a, b] \to \mathbb{R}$ is cts, then $f([a, b])$ is a compact interval.

Proof: Let $y_1, y_2 \in f([a, b])$ and let $z \in \mathbb{R}$ such that $y_1 < z < y_2$

Say $y_i = f(x_i), x_i \in [a, b]$

Case 1: $x_1 < x_2$ Let

$$x_0 = \sup\{x \in [x_1, x_2] : f(x) < z\}.$$

For every $n \in \mathbb{N}, \exists a_n \in [x_1, x_2]$ s.t. $x_0 - \frac{1}{n} < a_n \leq x_0$ and $f(a_n) < z$.

Then, we have $a_n \to x_0$ and so $f(a_n) \to f(x_0) < z$.

Hence by limits preserve order, $f(x_0) \leq z$.

Let $t_n = \min\{x_0 + \frac{1}{n}, x_2\}$

$$\therefore x_0 \leq t_n < x_0 + \frac{1}{n} \Rightarrow t_n \to x_0 \Rightarrow f(t_n) \to f(x_0)$$

However, $f(t_n) \geq z$ and so $f(x_0) \geq z$

$\therefore z = f(x_0)$.

Case 2: $x_2 < x_1$

Similar, so we are done. □

> **Example**
>
> - $K \subseteq \mathbb{R}$ compact
>
> $f : K \to K$ is cts. $\forall x \neq y, |f(x) - f(y)| < |x - y|$
>
>   Prove that $\exists x \in K$ such that $f(x) = x$.

Proof: Consider $g : K \to \mathbb{R}$, $g(x) = |f(x) - x|$. This is a comp of cts functions which implies g is cts.

Using EVT, Let $g(x_0) = \min g(K)$.

> **Claim**
>
> $f(x_0) = x_0$

Suppose for a contradiction $f(x_0) \neq x_0$.

$\therefore |f(f(x_0)) - f(x_0)| < |f(x_0) - x_0| \Rightarrow g(f(x_0)) < g(x_0)$.

By minimally $g(x_0)$ must the smallest value, Condition! □

> **Example**
>
> $f : [0, 1] \to [0, 1]$ cts
>
> Prove that $x \in [0, 1]$ such that $f(x) = x$.
>
> > **Note** why
> >
> > $g(x) = f(x) - 1$
> > $g(0) = f(0) \geq 0$
> > $g(1) = f(1)01 \leq 1$
>
> By the IVT, $g([0, 1]$ is an interval. $g(1) \leq 0 \leq g(0)$
>
> $\therefore \exists x \in [0, 1]$ such that $g(x) = 0$.

Example

Prove $\exists x \in \mathbb{R}$ such that $\cos x = \frac{1}{x}$.

$f(x) = \cos(x) - \frac{1}{x}$ is cts.

$f\left(\frac{\varphi}{2}\right) = 0 - \frac{2}{\pi} < 0$

$f(2\pi) = 1 - \frac{1}{2}\pi > 0$

Since $f : \left[\frac{\pi}{2}, 2\pi\right] \to \mathbb{R}$ is cts, by the IVT, $f\left(\frac{\pi}{2}, 2p\right)$.

Hence we have, $f\left(\frac{\pi}{2}\right) < 0 < f(2\pi)$.

Thus $\exists x \in \left[\frac{\pi}{2}, 2\pi\right]$ such that $f(x) = 0$.