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The Fisher paradox: A primer

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Non-technical summary

Research Question

Can central banks increase inflation in the short run by increasing nominal interest rates? Yes, according to the neo-Fisherian view. For central bankers this sounds odd, since they widely believe that cutting interest rates increases inflation.

Contribution

In this primer we illustrate that the neo-Fisherian view critically hinges on equilibrium selection and, more generally, on the formation of expectations. The neo-Fisherian view is thus not compelling.

Results

We show, first, that the prototypical new-Keynesian model for monetary policy analysis with a temporary fixed interest rate generates multiple equilibrium paths: some of them are consistent with the neo-Fisherian view, others are not. Second, the unique optimal monetary policy at the lower bound on interest rates, which can be implemented in the model with interest rate rules and *state-contingent* forward guidance, does not result in a paradoxical interest-inflation channel. Third, if one relaxes the assumption of perfect foresight or rational expectations, the new-Keynesian model produces an equilibrium that is not consistent with the new-Fisherian view.

Nichttechnische Zusammenfassung

Fragestellung

Können Notenbanken in der kurzen Frist die Inflation erhöhen, wenn sie den gelpolitischen Nominalzins anheben? Die neo-Fishersche Sicht (neo-Fisherian view), bejaht dies. Dies sollte Notenbanker verblüffen, denn überwiegend wird eine Zinssenkung zur Erhöhung der Inflation als zielführend gesehen.

Beitrag

In dieser Einführung zeigen wir, dass die neo-Fishersche Sichtweise wesentlich von der Gleichgewichtsselektion abhängt und – allgemeiner – von der Erwartungsbildung der Akteure. Wir erachten die neo-Fishersche Sichtweise daher nicht als zwingend.

Ergebnisse

Wir zeigen als erstes Resultat, dass das prototypische neu-Keynesiansche Modell mit einem temporär fixierten Zinssatz (z.B. an der Zinsuntergrenze) ein Kontinuum an Gleichgewichten impliziert. Manche dieser Gleichgewichte sind mit der neo-Fisherschen Sichtweise kompatibel, andere sind es nicht. Zweitens, die eindeutig bestimmte optimale Geldpolitik an der Zinsuntergrenze, implementiert durch eine Zinsregel und zustandsabhängiger Forward-Guidance, weist keinen paradoxen Zins-Inflationskanal auf. Drittens, weicht man von der üblichen Annahme der perfekten Voraussicht oder rationaler Erwartungen ab, ergibt sich im neu-Keynesianischen Modell ein Gleichgewicht, das nicht der neo-Fischerschen Sichtweise entspricht.

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The Fisher paradox: A primer[1]

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Abstract

The *neo-Fisherian view* does not consider a negative interest rate gap a prerequisite for boosting inflation. Instead, a negative interest rate gap is said to *lower* inflation. We discuss this counterintuitive response – known as the Fisher paradox – in a prototypical new-Keynesian model. We draw the following conclusions. First, with a temporarily pegged nominal rate during a liquidity trap (given an otherwise standard Taylor rule) the model generally produces multiple equilibrium paths: some of these paths are consistent with the neo-Fisherian view, others are not. Second, the unique optimal monetary policy at the lower bound on interest rates, which can be implemented in the model with interest rate rules and *state-contingent* forward guidance, does not result in a paradox. Third, if the assumption of perfect foresight or rational expectations is relaxed, the model produces an equilibrium that is not consistent with the neo-Fisherian view.

Keywords: Neo-Fisherian; Interest Rates; Inflation; Multiple Equilibria; Rational Expectations

JEL-Classification: E31; E43; E52

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1 Introduction

It is widely accepted that a central bank should cut its policy rate rapidly and sharply in response to an adverse shock. If the shock is large enough, the policy rate, in extreme circumstances, will be brought down to the lower bound (which, for the sake of simplicity, is set at zero). "Sooner or later", this kind of monetary policy (conditional on how it is implemented) should lead to an upturn in aggregate demand and thus drive up inflation.

However, despite the very expansionary monetary policy stance, particularly in the euro area but also in the United States, no self-supporting rise in inflation can be identified to date. It is commonly argued that this is because of a (temporarily) low natural rate of interest.[2] The argument holds that the natural rate of interest fell so sharply during the Great Recession that even the current low real interest rates are still having a contractionary effect, and that there is therefore a positive interest rate gap, ie $i(t)-\pi(t)-r(t)>0$, where $r(t)$ denotes the natural interest rate.

An alternative explanation, which will be our main focus in this paper, is that a negative interest rate gap is by no means a prerequisite for driving up inflation. Instead, according to this view, a negative interest rate gap *lowers* inflation, thus reversing the sign vis-à-vis the conventional interest rate channel. Such a counterintuitive response from the inflation rate following a fall in the policy rate is known as Fisher paradox or neo-Fisherianism.

The neo-Fisherian view holds that the long period of low interest rates itself contributed to the fall in inflation and thus to the current low inflation rates. Thus the forward guidance that policy rates would remain low for an extended period of time should have steered inflation expectations towards deflation or at least disinflation. Conversely, under the neo-Fisherian view the central banks should raise interest rates in order to increase inflation. Prominent advocates of this view include John Cochrane and Stephen Williamson.[3]

2 A low natural interest rate compared with the pre-crisis level.

3 Alongside the academic discussion, the neo-Fisherian view is also hotly debated on some blogs. See http://johncochrane.blogspot.com and http://newmonetarism.blogspot.com.

We discuss the neo-Fisherian view in a prototypical new-Keynesian model and come to the following conclusions. First, with a temporarily pegged nominal rate during a liquidity trap (given an otherwise standard Taylor rule) the model generally produces multiple equilibrium paths: some of these paths are consistent with the neo-Fisherian view, others are not. Second, the optimal monetary policy at the lower bound on interest rates, which can be implemented in the model with interest rate rules and *state-contingent* forward guidance, does not result in a paradox (see Duarte, 2016). Third, if the assumption of perfect foresight or rational expectations is relaxed (either along the lines of García-Schmidt and Woodford, 2015 or Gabaix, 2016), the model produces an equilibrium that is not consistent with the neo-Fisherian view.

In related work Garín, Lester and Sims (2016) document how reducing the inherently forward-looking nature of the standard new-Keynesian model also helps to escape the Fisher paradox.

2 The Fisher paradox under perfect foresight and its resolution

2.1 The problem of multiple equilibria

The academic debate about the neo-Fisherian view is based on the prototypical (linearised) new-Keynesian model, which is composed of an IS and a Phillips curve,

$$\begin{aligned} \dot{x}(t) &= \sigma\big(i(t) - r(t) - \pi(t)\big), \\ \dot{\pi}(t) &= \rho\pi(t) - \kappa x(t), \end{aligned}$$

where $x(t)$ and $\pi(t)$ denote the output gap and inflation, $r(t)$ is the natural interest rate and $i(t)$ represents the nominal interest rate (the new-Keynesian model is shown here in time-continuous form).[4] In this model framework, the natural interest rate is assumed to be exogenous and therefore independent of nominal interest rates (according to conventional wisdom, the natural interest rate is independent of nominal factors).

If the zero lower bound is binding, the Taylor rule $i(t) = \phi_\pi \pi(t) + \phi_x x(t) + r(t)$ is temporarily disabled; for a finite period the nominal interest rate becomes thus a fixed

[4] For more information, see the mathematical annex in Chapter 5. The parameter σ represents the interest elasticity (and $1/\sigma$ the elasticity of intertemporal substitution), ρ denotes the discount rate of the agents and κ the degree of price rigidity.

variable (interest rate peg). Without an active Taylor rule (active implies $\phi_{\pi} + (\rho / \kappa)\phi_{x} > 1$), be it permanent or only temporary, the determinate stable equilibrium is "lost", ie the prototypical two-equation new-Keynesian model then generally produces multiple equilibrium paths.[5] As is illustrated below, the differing points of view on the aforementioned interest rate channel and thus the neo-Fisherian view are closely connected with the existence of multiple equilibria.

Yet in the long run, at least on the basis of the new-Keynesian model, there is no discord between the two points of view, as the long-term equilibrium level of inflation – for both the conventional and the neo-Fisherian views – can be described using the Fisher equation (which, in turn, can be derived from the IS curve):

$$\pi(t) \quad = \quad i(t) - r(t).$$

According to this, a lasting increase in nominal interest rates – for a given natural interest rate – raises inflation. In the long term, *Fisher neutrality* therefore applies: $d\pi(\infty) = di(\infty)$.

Although "exact" Fisher neutrality cannot be observed empirically, higher nominal interest rates are often accompanied by higher inflation (eg Evans, 1998). For the long run, the neo-Fisherian view is therefore uncontroversial (eg Gabaix, 2016). However, there is some dispute over whether the positive relationship between the level of nominal interest rates and the level of inflation already applies in the short term.[6] This "wrong" sign vis-à-vis the conventional policy rate channel formally describes the Fisher paradox.

Where does the Fisher paradox – and thus the difference of opinion regarding the short term – originate? As mentioned above, the prototypical new-Keynesian model with a temporary interest peg (given an otherwise applicable Taylor rule) generally produces multiple equilibrium paths. Some of these paths are compatible with the paradox, ie they model the neo-Fisherian view; others are not and thus model the conventional policy rate channel. This means that the choice of equilibrium is critical to the existence

[5] If the Taylor rule *always holds* and there is no lower bound on interest rates, there is a locally determinate stable equilibrium (Duarte, 2016, p 23 and García-Schmidt and Woodford, 2015, p 16).

[6] See the abstract in Cochrane (2016a: "*If the Fed raises nominal interest rates, the [conventional new-Keynesian] model predicts that inflation will smoothly rise, both in the short run and long run. [My] paper presents a series of failed attempts to escape this prediction.*"

and thus the plausibility of the Fisher paradox. However, until it is "clarified" beyond doubt what economic criteria should be applied when choosing from the many equilibrium paths in the prototypical new-Keynesian model, the question of what happens after an, say, interest rate hike cannot be answered conclusively.

From a mathematical point of view, the multiple equilibria in the present model framework are the result of having "too many" degrees of freedom; in our prototypical model these are two free constants, which are part of the solution to the two differential equations of the prototypical new-Keynesian model.[7] Depending on how these constants are chosen, the model generates an equilibrium path that is either compatible or incompatible with the Fisher paradox. Below, we therefore present a series of alternatives for selecting equilibria (see also the mathematical annex) and explain the implications that this selection has for the occurrence of the paradox.

There appears to be no consensus in the literature about the economic criteria for the choice of "free" constants (see Cochrane, 2011). This explains why some prominent economists consider the Fisher paradox to be plausible while others do not.

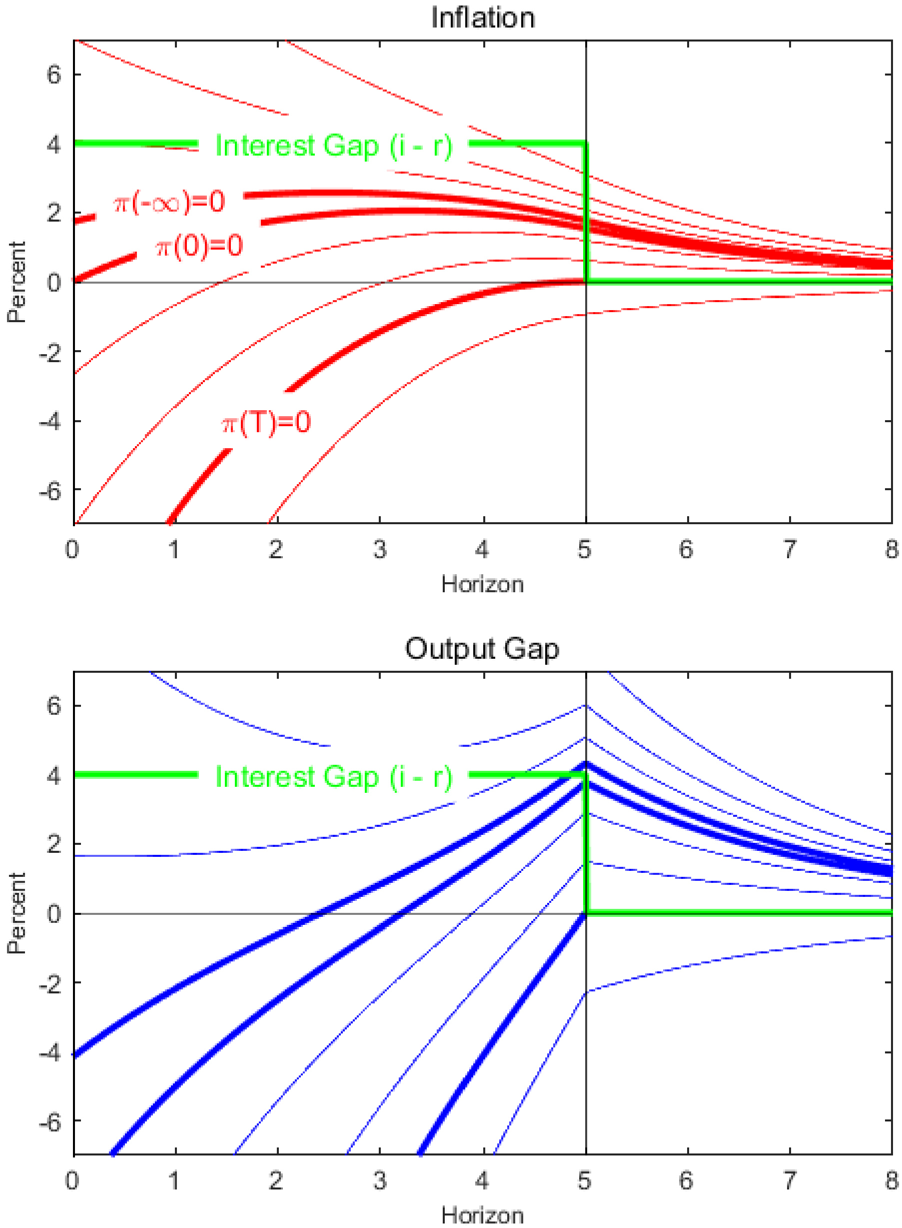
By way of example, Figure 1 shows nine equilibrium paths from a continuum of equilibria which are all consistent with the same interest rate path. For the first five years, a liquidity trap with a positive interest rate gap is assumed, $i(t) - r(t) = 4\%$ (for the sake of simplicity, the inflation target is set at zero) where the nominal interest rate $i(t) = 0\%$ and the natural interest rate $r(t) = -4\%$. Following the liquidity trap, the natural interest rate returns to the steady state and, accordingly, the central bank is able to close the interest rate gap, meaning that $i(t) = r(t)$ applies following the liquidity trap. For each of the stable equilibria modelled here, one of the free constants is calculated with an additional condition for $\pi(t)$ at a given time t.[8] The equilibrium accompanying the conventional interest rate channel – indexed by $\pi(T) = 0$ in Figure 1 – assumes that, at the end of the liquidity trap, the central bank is aiming for and achieves the steady

[7] See the mathematical annex in Chapter 5 for details on this solution and a discussion of the "free" constants.

[8] The mathematical annex in Chapter 5 details how a free constant is calculated using such an additional condition. Alternatively, an additional condition could be formulated for $x(t)$. The second free constant can be calculated comparatively "uncontroversially" provided that, as is usual in the literature, the analysis is restricted to those paths which are locally stable, ie the equilibria which are stable going forwards in time and thus return to the steady state.

state with $\pi(T) = x(T) = 0$. During the liquidity trap, this commonly discussed equilibrium (eg Werning, 2012) is accompanied by a deep recession and deflation. Such

Figure 1: Multiple equilibrium paths given perfect foresight



Notes: By way of example, Figure 1 shows nine stable equilibrium paths. The economy falls into a liquidity trap with $i(t) = 0\%$ and $r(t) = -4\%$ between $t = 0$ and $t = T = 5$; for $t > T$ the interest rate gap is closed, ie $i(t) = r(t) = 4\%$ (along similar lines to Cochrane, 2016b). The prototypical two-equation model is calibrated as in Galí (2015): $\rho = 0.01$, $\sigma = 1$ and $\kappa = 0.17$. As in Cochrane (2016b), we show only those paths from the continuum of equilibria that are stable going forwards in time.

deflation is not observed in the equilibria favoured by Cochrane (2016b), indexed by $\pi(-\infty)=0$ or $\pi(0)=0$.[9] For both equilibrium paths, the inflation rates remain in or above the steady state throughout and thus reflect the neo-Fisherian view. For these two paths, the recession is less sharp and is followed by a boom at the end of the liquidity trap. Cochrane (2016a) "justifies" these equilibria in part by pointing out that sharp deflation was not observed in the United States after the onset of the crisis, thus concluding that the conventional equilibrium and its implications were not plausible.[10]

2.2 One resolution of the paradox

One logical option for solving the problem of having multiple equilibria which is not considered by Cochrane (2016a, 2016b) is to determine the optimal monetary policy – ie the one that maximises utility – because the optimal equilibrium is determinate (eg Werning, 2012 and Duarte, 2016). For the continuous-time model, the initial conditions that are consistent with the optimal monetary policy can be determined using the maximum principle.

Direct implementation of the optimal monetary policy at the zero bound is not possible, however, as knowledge of the optimal monetary policy does not *per se* provide any indication of how it can be implemented in practice (a case in point is the *targeting rule under commitment*, which basically only formulates how output and inflation can be optimally managed). In other words, knowledge of the optimal and determinate equilibrium – characterised by a certain output/inflation path – does not initially provide any indication as to how this can be implemented via a monetary policy rule and forward guidance.

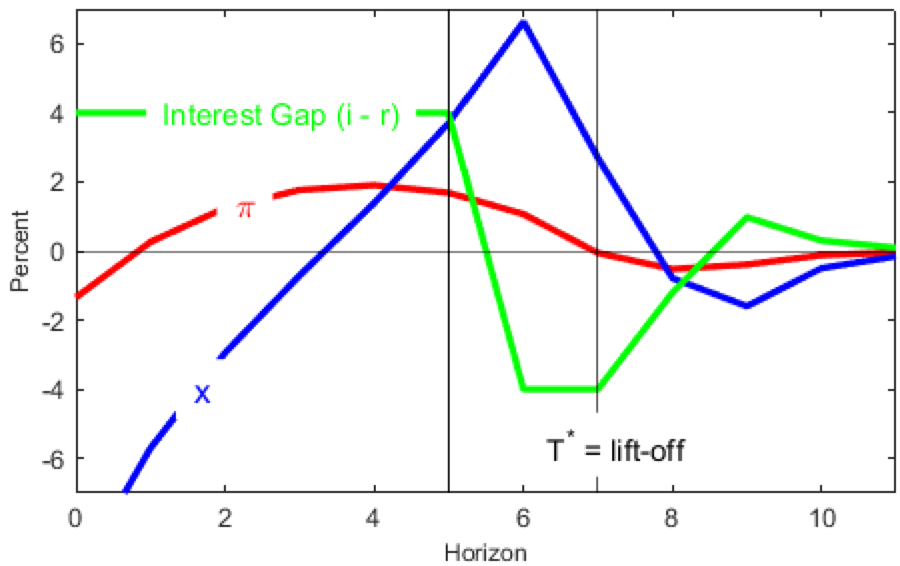
If the central bank combines a monetary policy rule with calendar-based forward guidance, it is not possible to implement the optimal monetary policy in a determinate manner (Duarte, 2016, Proposition 1). The problem of selection and thus the existence of multiple equilibrium paths therefore remains if forward guidance is calendar-based. By contrast, if a monetary policy rule with state-contingent forward guidance is implemented, it is possible to apply the optimal monetary policy as a determinate

[9] For the equilibrium indexed by $\pi(-\infty)=0$, the free constant is set to zero. See the mathematical annex in Chapter 5.1.

[10] Another attempt to support the neo-Fisherian view is the price puzzle, a characteristic of many empirical VAR studies (see Cochrane, 2016a).

equilibrium (Duarte, 2016, Chapter 5.2 and 5.3 shows interest rate rules of this kind).[11] The optimal monetary policy has two characteristics. First, it implies that the policy rate must be kept at the lower bound for longer than the actual shock lasts (see Figure 2). This means that the interest rate gap for $T^* > t > T$ is negative, as the central bank keeps the policy rate at the zero bound until T^* despite the natural interest rate being positive. *Only afterwards* is the interest rate gap closed. The optimal time T^* for the policy *"lift-off"* is state-contingent and is not given *a priori*.

Figure 2: Optimal determinate equilibrium under commitment



Notes: Figure 2 shows the optimal equilibrium path. Much like in Figure 1, between $t=0$ and $t=T=5$ the economy falls into a liquidity trap with $i(t)=0$ and $r(t)=-4\%$; for $T^* > t > T$ the interest rate gap is negative, as the central bank keeps the policy rate at the zero bound until T^* despite the natural interest rate being positive. *Only afterwards* is the interest rate gap closed (ie the nominal rate is adjusted to the natural rate). The prototypical two-equation model is calibrated as in Figure 1.

In addition, the initial response of inflation given the optimal monetary policy has the "right" sign, ie a positive interest rate gap initially dampens the inflation rate. The Fisher paradox thus cannot be observed (ironically, given the optimal monetary policy, a positive inflation rate can also be observed in some phases of the liquidity trap.

[11] For the new-Keynesian model in discrete time, see Eggertsson and Woodford, 2003, and Jung, Teranishi and Watanabe, 2005.

However, this is not a general feature of the optimal policy but is due to the specific calibration.)[12]

3 Fisher paradox vanishes without perfect foresight

An alternative option for solving the problem of equilibrium selection and thus avoiding multiple equilibria is to modify the present underlying analytical framework, especially the assumption of a perfect foresight equilibrium (PFE) or the "softening" of rational expectations. A proposal of this sort was put forward by García-Schmidt and Woodford (2015) (henceforth GSW). A different way of resolving the Fisher paradox focusses on behavioural economic considerations, which we will briefly outline following GSW.

The foundation of the considerations of GSW is a new equilibrium concept which is no longer based on the assumption of rational expectations but has an underlying "process of reflection" (roughly, a way of learning); agents iteratively adapt their initial beliefs until a PFE has (ideally) been reached. The underlying equilibrium concept is called reflective equilibrium (see Woodford, 2013). The iterative process can, under certain circumstances, converge towards one of the multiple equilibria. In this context, the process of reflection thus takes on the equilibrium selection (the features of an equilibrium selected in this way largely depend on the duration of the iteration process).

In the prototypical new-Keynesian two-equation model with a (temporary or permanent) fixed nominal interest rate, the process of reflection converges at a slower pace than in cases where the Taylor rule always applies. If the interest rate has been fixed for a sufficient length of time, the reflective equilibrium does not at all converge towards one of the multiple "rational" equilibria. It is therefore impossible to make an unambiguous statement about the features of the equilibrium "determined" using the iterative process because the features of the equilibrium determined in this way hinge on how long the iterative process is permitted to last.

According to GSW, it is nevertheless possible to assess the equilibrium paths in qualitative terms, if these were determined using the iterative process. Hence, according to GSW, a *temporarily* constant interest rate generally leads to higher demand, which,

[12] The calibration of the prototypical two equation model is based on Galí (2015). The calibration is merely for illustrative purposes and does not purport to bring the model "as close as" possible to the data.

in turn, increases output and inflation.[13] The reflective equilibrium thus does not run into the Fisher paradox, which may be regarded as irrelevant in the context of this approach; this is because of the fact that the PFE equilibrium paths compatible with the paradox cannot be achieved iteratively. Similarly, if the interest rate is fixed for a *very long period*, the Fisher paradox cannot be observed either (see GSW, chart 5 and proposition 6).

An alternative way of resolving the paradox, which represents a somewhat less radical deviation from the paradigm of perfect foresight or rational expectations, is based on behavioural economics, but does not require a new equilibrium concept. In this type of models agents are not completely rational and are "inattentive" to macroeconomic indicators (bounded rationality).[14] Based on this assumption, Gabaix (2016) derives a modified IS and the Phillips curve

$$\begin{aligned} \dot{x}(t) &= \delta x(t) + \sigma\left(i(t) - r(t) - \pi(t)\right), \\ \dot{\pi}(t) &= \tilde{\rho}\pi(t) - \kappa x(t), \end{aligned}$$

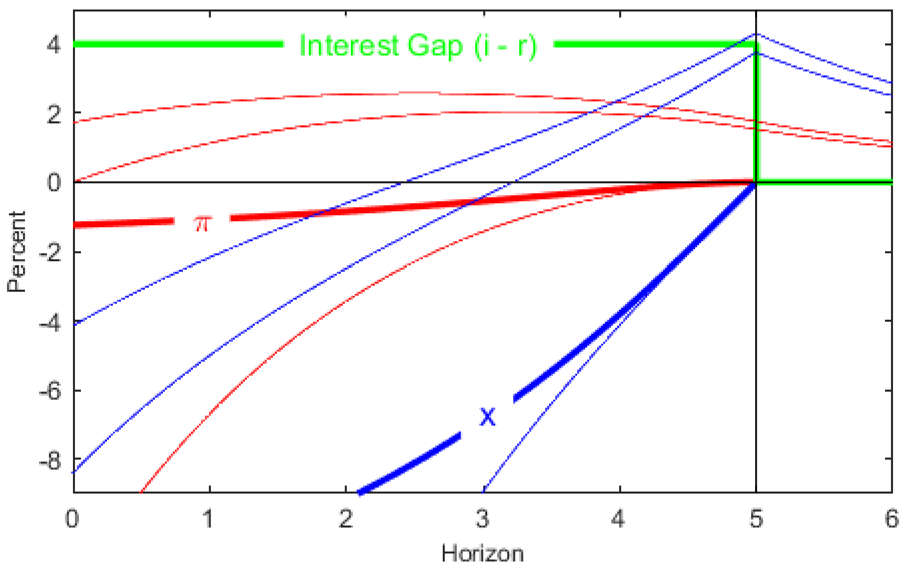
where merely the discount factor $0 < \delta \leq 1$ is added to the IS equation as a model parameter (in this set-up, the standard discount factor in the Phillips curve also changes; "$\tilde{\rho}$" is therefore used to make the distinction). If agents are sufficiently myopic, in other words, if they have a high preference for the present (which is controlled via discount factor δ), the solution of the above system of differential equation is stable and unique. This solution contradicts the neo-Fisherian view (see Figure 3). In mathematical terms, this is owed to the fact that all free constants can now be eliminated by considering stable equilibria only.[15] The Fisher paradox is thus also resolved in the behavioural model of Gabaix (2016).

[13] If the iterative process lasts long enough, the reflective equilibrium converges towards a unique PFE (García-Schmidt and Woodford, 2015, proposition 4). This PFE is not compatible with the Fisher paradox.

[14] Empirical evidence for households' and corporations' "inattention" to macroeconomic indicators can be found in Coibion and Gorodnichenko (2015) and Taubinsky and Rees-Jones (2015) as well as Caplin, Dean and Martin (2011).

[15] For the detailed derivations, see the mathematical annex (Chapter 5.2).

Figure 3: Behavioural model – unique equilibrium



Notes: As in Figure 1, the economy falls into a liquidity trap between $t = 0$ and $t = T = 5$, while the interest rate gap is closed for $t > T$. The prototypical two-equation model is calibrated as in Galí (2015): $\sigma = 1$ and $\kappa = 0.17$. The discount factor in the IS curve is set at $\delta = 0.8$, leading to $\tilde{\rho} = 0.73$ in the Phillips curve. The paths (thin lines) highlighted in Figure 1 and discussed in the main text are shown for comparison.

4 Assessment

This short note discusses the Fisher paradox based on the prototypical new-Keynesian model using the liquidity trap as an example. The explanations in this paper are intended to illustrate that the neo-Fisherian view is not an imperative implication of the new-Keynesian model, but is related to the issue of equilibrium selection.

What are the advantages of selecting an equilibrium path that is compatible with the neo-Fisherian view? According to Cochrane (2016b), it is not least the equilibrium path's implications at the zero lower bound on interest rates that represent an advantage, as it does not exhibit a deep and long-lasting recession or a pronounced deflation in the liquidity trap. On the contrary, the inflation rates are consistently at or above the steady state. In this sense, such equilibrium better describes the evidence than the "traditional" equilibrium of the new-Keynesian model. However, the optimal monetary policy shows similar implications for model dynamics at the zero lower bound without following the logic of the neo-Fisherian view. As such, selecting the equilibrium which features the Fisher paradox is anything but imperative.

Another argument against the paradox may be its lack of robustness. The work by both GSW and Gabaix (2016) (among others) shows that the paradox disappears if the model framework is adjusted, especially if the assumption of PFE or rational expectations is abandoned or behavioural economic considerations are used as a basis in an otherwise standard new-Keynesian model.

This yields two general insights, which go beyond the issues discussed here. First, any analysis based on the new-Keynesian model and conducted at the zero lower bound should, as a rule, be carefully and critically examined. Second, the profession already has begun to (seriously) question one key assumptions of the new-Keynesian model – the assumption of rational expectations.

5 Mathematical appendix

5.1 The prototypical new-Keynesian model

In its condensed form, the prototypical linearised[16] new-Keynesian model comprises an IS curve and a Phillips curve, and models the dynamic interaction between the output gap (x) and inflation (π):

$$\begin{aligned} x_t &= E_t x_{t+1} - \sigma\left(i_t - r_t - E_t \pi_{t+1}\right) \\ \pi_t &= \beta E_t \pi_{t+1} + \kappa x_t . \end{aligned}$$

The parameters $\sigma > 0$, $0 < \beta < 1$ and $\kappa > 0$ denote interest rate elasticity, the discount factor and the degree of price stickiness, respectively. The remaining variables are the nominal interest rate (i) and the natural interest rate (r), the paths of which are discussed below.

The formula, as depicted above, shows the discrete-time form with the expectation operator E_t as is usually found in the literature. However, in this paper we have chosen to use the deterministic (ie with perfect foresight) continuous-time version of the prototypical new-Keynesian model for our analysis. This version of the model enables a comparatively simple analytical solution to be found, especially when the focus is on what is taking place at the zero lower bound with a fixed nominal interest rate.

When transposed into the deterministic continuous-time version, this results in the following differential equations:

(1)
$$\begin{aligned} \dot{x}(t) &= \sigma\left(i(t) - r(t) - \pi(t)\right), \\ \dot{\pi}(t) &= \rho\pi(t) - \kappa x(t), \end{aligned}$$

where, in a departure from the discrete-time presentation, the discount factor now amounts to $\rho = 1 - \beta$, and $\dot{x}(t)$ and $\dot{\pi}(t)$ represent the first derivatives of the output gap and inflation with respect to time.

The method used here, to solve this system of differential equations separately for $x(t)$ and $\pi(t)$, follows Gandolfo (1997, Chapter 18). The Phillips curve in (1) is then differentiated once again with respect to time t, ie

[16] Ie linearised around a steady state with zero inflation and a closed output gap.

$$\ddot{\pi}(t) = \rho\dot{\pi}(t) - \kappa\dot{\mathrm{x}}(\mathrm{t}) ,$$

and subsequently combined with the IS equation into a second-order differential equation in $\pi(t)$:

(2) $$\ddot{\pi}(t) - \rho\dot{\pi}(t) - \kappa\sigma\pi(t) = -z(t)$$

with the "*forcing variable*" or disturbance function $z(t) = \kappa\sigma\big(i(t) - r(t)\big)$. Equation (2) can alternatively be expressed in operator form:

(3) $$\left(\frac{d}{dt} - \lambda_1\right)\left(\frac{d}{dt} + \lambda_2\right)\pi(t) = -z(t) ,$$

with the eigenvalues λ_1 and λ_2, where

(4) $$\lambda_1 = \frac{1}{2}\left(\sqrt{\rho^2 + 4\kappa\sigma} + \rho\right) > 0 \text{ and } \lambda_2 = \frac{1}{2}\left(\sqrt{\rho^2 + 4\kappa\sigma} - \rho\right) > 0 ,$$

as well as $\rho = \lambda_1 - \lambda_2$ and $\kappa\sigma = \lambda_1\lambda_2$ hold true.

To invert (3), ie to find a solution for $\pi(t)$, we will first take a separate look at the two bracketed terms on the left-hand side. The first-order differential equation

$$\left(\frac{d}{dt} - \lambda_1\right)\pi(t) = -z(t)$$

has a standard solution (eg Borelli and Coleman, 2004, Chapter 2.2) of

(5) $$\pi(t) = C_1 e^{\lambda_1 t} + e^{\lambda_1 t}\int_{s=t}^{\infty} e^{-\lambda_1 s} z(s)ds$$

where the first part – with the "free" constant C_1 – describes the dynamics stemming from the "initial condition" (homogeneous solution). The second part – with the integration factor $e^{\lambda_1 t}$ – characterises the dynamics that arise due to the forcing variable $z(t)$ (particular solution). The second first-order differential equation

$$\left(\frac{d}{dt} + \lambda_2\right)\pi(t) = -z(t)$$

has the following standard solution:

(6) $$\pi(t) = C_2 e^{-\lambda_2 t} - e^{-\lambda_2 t}\int_{s=-\infty}^{t} e^{\lambda_2 s} z(s)ds .$$

It should be noted here that the differing signs in (5) and (6) arise as a result of the signs for λ_1 and λ_2. The signs in (3) and (4) also determine whether the solution is "forward-looking" – as in (5) – or "backward-looking" – as in (6).[17]

Equations (5) and (6) generally supply all the components required to invert (3) and solve for $\pi(t)$, ie:

(7) $$\pi(t) = \frac{-1}{\left(\frac{d}{dt} - \lambda_1\right)\left(\frac{d}{dt} + \lambda_2\right)} z(t)$$

However, the product in the denominator makes it difficult to solve the equation. It is thus a good idea to transform the expression above into a sum by means of partial fraction decomposition:

$$\frac{-1}{\left(\frac{d}{dt} - \lambda_1\right)\left(\frac{d}{dt} + \lambda_2\right)} = \frac{A}{\left(\frac{d}{dt} - \lambda_1\right)} + \frac{B}{\left(\frac{d}{dt} + \lambda_2\right)},$$

where the parameters A and B can be determined in two stages. In the first stage, the denominators are taken "up" a level and the brackets are solved:

$$-1 = A\left(\frac{d}{dt} + \lambda_2\right) + B\left(\frac{d}{dt} - \lambda_1\right) = A\frac{d}{dt} + A\lambda_2 + B\frac{d}{dt} - B\lambda_1 .$$

In the second stage, the values are ordered in accordance with the same powers, yielding

$$\left.\begin{aligned} -1 &= A\lambda_2 - B\lambda_1 \\ 0 &= A\frac{d}{dt} + B\frac{d}{dt} \end{aligned}\right\} \Rightarrow A = -\frac{1}{\lambda_1 + \lambda_2} \text{ and } B = \frac{1}{\lambda_1 + \lambda}.$$

If the expressions for A and B are then inserted, the solution for $\pi(t)$ in (7) is:

[17] The best way to illustrate the difference between forward and backward-looking is to take a short digression into the discrete-time world. If we look at the transformation of the following discrete AR(1) processes in their continuous-time form

$$\pi_t - (\lambda_1 - 1)\pi_{t-1} = u_{1,t} \Rightarrow \Delta\pi_t - \lambda_1\pi_{t-1} = u_{1,t} \Rightarrow \left(\frac{d}{dt} - \lambda_1\right)\pi(t) = u_1(t)$$

and

$$\pi_t - (\lambda_2 + 1)\pi_{t+1} = u_{2,t} \Rightarrow \Delta\pi_{t+1} + \lambda_2\pi_{t+1} = -u_{2,t} \Rightarrow \left(\frac{d}{dt} + \lambda_2\right)\pi(t) = -u_2(t)$$

it is clear that the former is iterated forward and the latter backward.

(8) $$\pi(t)=\frac{1}{\lambda_1+\lambda_2}\left[\frac{1}{\frac{d}{dt}+\lambda_2}-\frac{1}{\frac{d}{dt}-\lambda_1}\right]z(t)\,.$$

Because of the positive eigenvalue in $C_1e^{\lambda_1 t}$, the "forward"-looking path of $\pi(t)$ in (5) is explosive if $t\to\infty$. The literature usually only considers stable forward-looking paths (unless the focus is on so-called "bubble" solutions): the "free" constant is thus set to zero ($C_1=0$). The "backward"-looking path of $\pi(t)$ in (6) explodes if $t\to-\infty$. If $t=0$, the "free" constant C_2 defines the initial condition $\pi(0)$. As a rule, the free constant C_2 (for $C_1=0$) thus indexes the various equilibrium paths which are all consistent with the same forcing variable $z(t)$. In line with the literature, we assume that $C_1=0$ (in other words, we observe only forward-stable paths) while C_2 initially remains a variable that can be "freely" selected.

Inserting (5) and (6) into (8) gives us the following solution:

(9) $$\pi(t)=C_2e^{-\lambda_2 t}+\frac{1}{\lambda_1+\lambda_2}\left[\int_{s=-\infty}^{t}e^{-\lambda_2(t-s)}z(s)ds+\int_{s=t}^{\infty}e^{-\lambda_1(s-t)}z(s)ds\right].$$

$x(t)$ can be solved by differentiating the solution for $\pi(t)$ with respect to t, that is

$$\dot{\pi}(t)=-\lambda_2C_2e^{-\lambda_2 t}+\frac{1}{\lambda_1+\lambda_2}\left[-\lambda_2\int_{s=-\infty}^{t}e^{-\lambda_2(t-s)}z(s)ds+\lambda_1\int_{s=t}^{\infty}e^{-\lambda_1(s-t)}z(s)ds\right],$$

and by plugging this expression back into the Phillips curve in (1). After rearranging, using the link $\rho=\lambda_1-\lambda_2$ between the discount factor and the eigenvalues, we get

(10) $$x(t)=\frac{\lambda_1}{\kappa}C_2e^{-\lambda_2 t}+\frac{1}{\kappa\left(\lambda_1+\lambda_2\right)}\left[\lambda_1\int_{s=-\infty}^{t}e^{-\lambda_2(t-s)}z(s)ds-\lambda_2\int_{s=t}^{\infty}e^{-\lambda_1(s-t)}z(s)ds\right].$$

Liquidity trap

We now assume that the economy "falls" into a liquidity trap between $t=0$ and $t=T$ with a nominal interest rate set at the zero lower bound $i(t)=\underline{i}=0$ and a negative natural interest rate $r(t)=\underline{r}<0$. As a result, $z=\kappa\sigma(\underline{i}-\underline{r})$ holds true for $0<t<T$ and $z=0$ for $t>T$. Calculating the integrals in (9) and (10) yields:

(11) $$\pi(t)=C_2e^{-\lambda_2 t}+\frac{\kappa\sigma(\underline{i}-\underline{r})}{\lambda_1+\lambda_2}w(t)\ \text{ and }\ x(t)=\frac{\lambda_1}{\kappa}C_2e^{-\lambda_2 t}+\frac{\sigma(\underline{i}-\underline{r})}{\lambda_1+\lambda_2}v(t)\,,$$

where

$$0<t<T \quad : \quad w(t)=\frac{1}{\lambda_2}\left(1-e^{-\lambda_2 t}\right)+\frac{1}{\lambda_1}\left(1-e^{-\lambda_1(T-t)}\right)$$
$$t>T \quad : \quad w(t)=\frac{1}{\lambda_2}\left(e^{-\lambda_2(t-T)}-e^{-\lambda_2 t}\right)$$

and

$$0<t<T \quad : \quad v(t)=\frac{\lambda_1}{\lambda_2}\left(1-e^{-\lambda_2 t}\right)-\frac{\lambda_2}{\lambda_1}\left(1-e^{-\lambda_1(T-t)}\right)$$
$$t>T \quad : \quad v(t)=\frac{\lambda_1}{\lambda_2}\left(e^{-\lambda_2(t-T)}-e^{-\lambda_2 t}\right).$$

We can solve C_2 for the equilibria shown in Figure 1 as follows:

$$\pi(-\infty)=0 \quad : \quad \lim_{t\to-\infty} C_2 e^{-\lambda_2 t}=0 \quad \Rightarrow \quad C_2=0$$
$$\pi(0)=0 \quad : \quad C_2+\frac{\kappa\sigma(\underline{i}-\underline{r})}{\lambda_1+\lambda_2}\frac{1}{\lambda_1}\left(1-e^{-\lambda_1 T}\right)=0$$
$$\pi(T)=0 \quad : \quad C_2 e^{-\lambda_2 T}+\frac{\kappa\sigma(\underline{i}-\underline{r})}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}\left(1-e^{-\lambda_2 T}\right)=0$$

Liquidity trap and forward guidance

We extend the considerations above by a period τ, in which we assume that the economy has left the liquidity trap ($r(t)=\overline{r}>0$ for $t>T$), but the central bank leaves the nominal interest rate deliberately at the zero lower bound ($i(t)=0$ for $0<t<T+\tau$ and $i(t)=\overline{r}$ for $t>T+\tau$). This results in $z=\kappa\sigma(\underline{i}-\underline{r})$ for $0<t<T$, $z=\kappa\sigma(\underline{i}-\overline{r})$ for $T<t<T+\tau$ and $z=0$ for $t>T+\tau$. Calculating the integrals in (9) and (10) then yields:

(12) $$\pi(t)=C_2 e^{-\lambda_2 t}+\frac{\kappa\sigma w(t)}{\lambda_1+\lambda_2} \text{ and } x(t)=\frac{\lambda_1}{\kappa}C_2 e^{-\lambda_2 t}+\frac{\sigma v(t)}{\lambda_1+\lambda_2}$$

where

$$0<t<T \quad : \quad w(t)=\frac{(\underline{i}-\underline{r})}{\lambda_2}\left(1-e^{-\lambda_2 t}\right)+\frac{(\underline{i}-\underline{r})}{\lambda_1}\left(1-e^{-\lambda_1(T-t)}\right)+\frac{(\underline{i}-\overline{r})}{\lambda_1}\left(1-e^{-\lambda_1\tau}\right)e^{\lambda_1(t-T)}$$
$$T<t<T+\tau \quad : \quad w(t)=\frac{(\underline{i}-\underline{r})}{\lambda_2}\left(e^{-\lambda_2 T}-1\right)e^{-\lambda_2 t}+\frac{(\underline{i}-\overline{r})}{\lambda_2}\left(1-e^{-\lambda_2(t-T)}\right)+\frac{(\underline{i}-\overline{r})}{\lambda_1}\left(1-e^{-\lambda_1(T+\tau-t)}\right)$$
$$t>T+\tau \quad : \quad w(t)=\frac{(\underline{i}-\underline{r})}{\lambda_2}\left(e^{-\lambda_2 T}-1\right)e^{-\lambda_2 t}+\frac{(\underline{i}-\overline{r})}{\lambda_2}\left(e^{\lambda_2\tau}-1\right)e^{-\lambda_2(t-T)}$$

and

$$\begin{aligned}
0<t<T \quad &: \quad v(t)=\frac{\lambda_1(\underline{i}-\underline{r})}{\lambda_2}\left(1-e^{-\lambda_2 t}\right)-\frac{\lambda_2(\underline{i}-\underline{r})}{\lambda_1}\left(1-e^{-\lambda_1(T-t)}\right)-\frac{\lambda_2(\underline{i}-\overline{r})}{\lambda_1}\left(1-e^{-\lambda_1\tau}\right)e^{\lambda_1(t-T)}\\
T<t<T+\tau \quad &: \quad v(t)=\frac{\lambda_1(\underline{i}-\underline{r})}{\lambda_2}\left(e^{-\lambda_2 T}-1\right)e^{-\lambda_2 t}+\frac{\lambda_2(\underline{i}-\overline{r})}{\lambda_2}\left(1-e^{-\lambda_2(t-T)}\right)-\frac{\lambda_2(\underline{i}-\overline{r})}{\lambda_1}\left(1-e^{-\lambda_1(T+\tau-t)}\right)\\
t>T+\tau \quad &: \quad v(t)=\frac{\lambda_1(\underline{i}-\underline{r})}{\lambda_2}\left(e^{-\lambda_2 T}-1\right)e^{-\lambda_2 t}+\frac{\lambda_1(\underline{i}-\overline{r})}{\lambda_2}\left(e^{\lambda_2\tau}-1\right)e^{-\lambda_2(t-T)}.
\end{aligned}$$

Optimal monetary policy

At time $t=0$, the point in time when the economy "steps" into the liquidity trap ($i(t)=0$ and $r(t)<0$), the central bank chooses the optimal path for inflation and the output gap in line with the following standard optimisation problem:

$$\text{(13)} \qquad \mathbb{W} \equiv \min_{\{x(t),\pi(t),i(t)\}} \frac{1}{2}\int_{t=0}^{\infty} e^{-\rho t}\left(x(t)^2+\vartheta\pi(t)^2\right)dt$$

subject to (12) and with the free initial conditions $x(0)$, $\pi(0)$ (or, as an equivalent alternative, the free constants C_1 and C_2). The central bank thus keeps any deviations from zero (ie from the steady state) at a minimum for inflation and the output gap. The parameter ϑ controls the relative weight between the two targets.[18]

The minimisation problem stated above can be solved numerically using Pontryagin's "Maximum Principle" (see Gandolfo, 1997, Chapter 22.1 and Werning, 2012).

5.2 The new-Keynesian "behavioural model"

Under the assumption that households and firms are not completely rational, Gabaix (2016) derives the following modified IS and Phillips curves:

$$\begin{aligned}
x_t &= ME_t x_{t+1}-\sigma\left(i_t-r_t-E_t\pi_{t+1}\right)\\
\pi_t &= \beta M^f E_t\pi_{t+1}+\kappa x_t.
\end{aligned}$$

New additions to the formula are the discount factors M and M^f, where M^f is a function of M and other structural parameters of the new-Keynesian model.

Transposed into the deterministic continuous-time version, this results in slight modifications to the differential equations compared with (1)

[18] In a prototypical new-Keynesian model, the quadratic loss function in (13) can be derived from a second-order approximation of households' utility function (see Galí, 2015, Chapter 4).

(14) $$\begin{aligned}\dot{x}(t) &= \delta x(t) + \sigma\big(i(t) - r(t) - \pi(t)\big),\\ \dot{\pi}(t) &= \tilde{\rho}\pi(t) - \kappa x(t),\end{aligned}$$

where the discount factors are now defined by $\delta = 1 - M$ and $\tilde{\rho} = 1 - \beta M^f$ in a departure from the discrete-time version.

Using the same calculation rules as in the prototypical model, the following two eigenvalues can be derived:

(15) $$\begin{aligned}\lambda_1 &= \frac{1}{2}\Big(\sqrt{(\tilde{\rho}+\delta)^2 + 4(\kappa\sigma - \tilde{\rho}\delta)} + (\tilde{\rho}+\delta)\Big),\\ \lambda_2 &= \frac{1}{2}\Big(\sqrt{(\tilde{\rho}+\delta)^2 + 4(\kappa\sigma - \tilde{\rho}\delta)} - (\tilde{\rho}+\delta)\Big).\end{aligned}$$

In a departure from (4), the second eigenvalue can actually be negative, $\lambda_2 < 0$, for instance, if

(16) $$\frac{\tilde{\rho}\delta}{\kappa\sigma} > 1.$$

In such cases, the sign in the second bracket term in (3) changes "implicitly" and the solution follows suit. This is no longer backward-looking, but now forward-looking just as in (5). The counterpart to (6) in the "behavioural model" is thus as follows:

(17) $$\pi(t) = C_2 e^{-\lambda_2 t} + e^{-\lambda_2 t}\int_{s=t}^{\infty} e^{\lambda_2 s} z(s)ds.$$

The main difference is therefore that the "second" free constant C_2 now also implies forward-explosive dynamics. As in the prototypical model, excluding these solutions yields $C_2 = 0$. Thus the differential equation system is uniquely determined by the given forcing variable $z(t)$ and shows no multiple forward-stable equilibria.

The solution is found in the same way as in the prototypical model, in other words by inserting (5) and (17) into (8):

(18) $$\pi(t) = \frac{1}{\lambda_1 + \lambda_2}\left[\int_{s=t}^{\infty} e^{-\lambda_1(s-t)} z(s)ds - \int_{s=t}^{\infty} e^{\lambda_2(s-t)} z(s)ds\right],$$

differentiating the solution with respect to t and inserting it into the Phillips curve (14)

(19) $$x(t) = \frac{1}{\kappa\big(\lambda_1 + \lambda_2\big)}\left[\lambda_2\int_{s=t}^{\infty} e^{-\lambda_1(s-t)} z(s)ds - \lambda_1\int_{s=t}^{\infty} e^{\lambda_2(s-t)} z(s)ds\right].$$

The integrals are calculated the same way as in (11) and yield:

(20) $$\pi(t) = \frac{\kappa\sigma(\underline{i} - \underline{r})}{\lambda_1 + \lambda_2} w(t) \quad \text{and} \quad x(t) = \frac{\sigma(\underline{i} - \underline{r})}{\lambda_1 + \lambda_2} v(t),$$

where

$$\begin{aligned} 0 < t < T \quad &: \quad w(t) = \frac{1}{\lambda_2}\left(1 - e^{\lambda_2 (T-t)}\right) + \frac{1}{\lambda_1}\left(1 - e^{-\lambda_1 (T-t)}\right) \\ t > T \quad &: \quad w(t) = 0; \end{aligned}$$

and

$$\begin{aligned} 0 < t < T \quad &: \quad v(t) = \frac{\lambda_1}{\lambda_2}\left(1 - e^{\lambda_2 (T-t)}\right) - \frac{\lambda_2}{\lambda_1}\left(1 - e^{-\lambda_1 (T-t)}\right) \\ t > T \quad &: \quad v(t) = 0. \end{aligned}$$

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