Practice Set 14

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ELEMENTARY SET

1. Use Lagrange multipliers to find the absolute maximum of the function

$$f(x, y) = y^2 - 4xy + x^2 - 12x$$

on the circle $x^2 + y^2 = 2$.

2. Find the absolute minimum for the function $f(x, y) = e^{xy}$ on the ellipse $x^2 + 4y^2 = 2$.

FUNDAMENTAL SET

1. Find the points on the surface $y^2 = 9 + xz$ closest to the origin.

2. Find the dimensions of the box with the largest volume among all boxes with a diagonal of length $9m$.

3. The total production P of a certain product depends on the amount L of labor used and the amount K of capital investment.

   The Cobb-Douglas model states that $P = bL^aK^{1-a}$ where a and b are constants satisfying $0 < a < 1$ and $0 < b$.

   If, additionally the cost of a unit of labor is m and the cost of a unit of capital is n, and the company can only spend p dollars, determine the absolute value of production given the constraint $mL + nK = p$.

4. Find the maximum and minimum values (and the points where they are achieved) for the function

$$f(x, y) = \frac{xy}{y^2 + 1} \text{ with the constraint } x^2 + y^2 = 1.$$

5. Determine the maximum and the minimum of the function

$$f(x, y) = x^2 - 4xy + 9y^2 + 2x$$

in the region bounded by $y \geq 0$, $x \leq 4$ and $2y \leq x$.

6. Determine the absolute maximum for $f(x, y) = xy + 6y$ on the region defined by $x^2 + y^2 \leq 8$ and $y \geq x$.

STRENGTHENING SET

1. Find the dimensions of the triangle with the largest area amongst all triangles with a perimeter of 24.

   *Recall that the area of a triangle with sides of length a, b and c can be found using **Heron's formula**:*

$$A = \sqrt{s(s-a)(s-b)(s-c)} \text{ with } s = \frac{a+b+c}{2}, s \text{ is called the semiperimeter.}$$

2. The arithmetic mean-geometric mean inequality states that if $a_1, a_2, \ldots, a_n$ are non-negative numbers then

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}.$$

And that equality happens exactly when $a_1 = a_2 = \ldots = a_n$.

Show that the maximum of the function $P(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$, with $x_1, x_2, \ldots, x_n \geq 0$ is $\frac{L}{n}$ under the constraint $x_1 + x_2 + \ldots + x_n = L$, and that it happens when $x_1 = x_2 = \ldots = x_n = \frac{L}{n}$.

3. Determine the maximum and the minimum of the function

$$f(x, y) = x^3 + 4xy^2 + 5y^2 - 12x$$

in the region bounded by $x^2 + 4y^2 \leq 8$ and $x \leq 2$.

4. Consider the function $f(x, y, z) = xy^2 + yz^2 + zx^2$ and the constraint $2xz + y^2 = 3$.

   (a) Compute the gradients $\nabla f(x, y, z)$ and $\nabla g(x, y, z)$, where $g(x, y, z) = 2xz + y^2 - 3$.

   (b) Determine the singular points of $\nabla f(x, y, z)$ and $\nabla g(x, y, z)$. Show that those singular points do NOT satisfy the constraint.

   (c) Determine the points that satisfy the equation $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, for some $\lambda \neq 0$, and compute the values of $f(x, y, z)$ at those points.

   (d) Consider your responses to the previous parts to answer the following question: "What are the minimum and maximum values of $f(x, y, z)$ on the constraint?" [***Remark****: Before you lock in an answer compute $f(11, 5, -1)$ and think about how this relates to your process.*]

Remark: It is a good idea to create an animation for this problem on Grapher. I think it would strengthen your understanding of the problem. Follow the steps decribed below:

- Your first equation should be $xz + y^2 = 3$. The surface you just graphed is the constraint. What do you notice about it? [*I recommend you use the inspector so that the color of the surface is a clear one, I recommend "Sky" and so that the opacity is low, below* 50%].
- Introduce a time parameter by creating a new equation $T = 0$.
- We will now introduce the level surface to $f(x, y, z)$ at the T level. Create a new equation $xy^2 + yz^2 + zx^2 = T$. Notice that the level surface at 0 and the constraint intersect, this means that the function $f(x, y, z)$ achieves the value of 0 in the constraint, (at multiple points actually!!). [*Use the inspector so that the color of the surface is also a clear one with low opacity, try "Flora" and* 45%.]
- Select the equation "T=0" and go to the *Equation* menu, click on it and select "Animate parameter".

- Click on the symbol with the two check marks. This opens the options for the animation. On *Value Range*, type -10 for Minimum Value, 10 for Maximum Value, on "Steps" type 41, (so the animation will highlight the curves at the values -10, -9.5, -9, and so on until you get to 10). Also select the boxes next to "Continuous Range" and "Loop Back and Forth" (this will made the animation run smoothly and go on a loop).
- Click the play button. Notice that if the value displayed on the box "$T = k$" means that the level surface being graphed is the one at the k level, i.e. $xy^2 + yz^2 + zx^2 = k$. It is relevant to realize that as T takes all values between -10 and 10 the level surface being graphed and the constraint always intersect, and they do so on multiple points!!! This means that the function $f(x, y, z) = xy^2 + yz^2 + zx^2$ achieves all the values between -10 and 10 in the constraint!