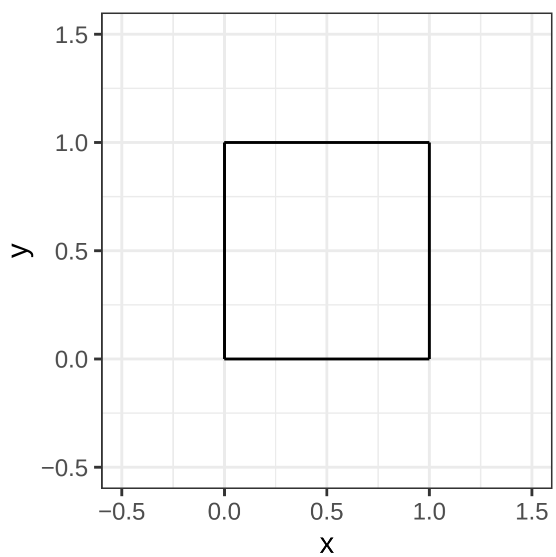
Lab Three – Matrices and Data Frames in R

- Complete the tasks below. Make sure to start your solutions in on a new line that starts with "**Solution**:".
- Make sure to use the Quarto Cheatsheet. This will make completing and writing up the lab *much* easier.

Consider the unit square depicted in Figure 1.

```r
p0 <- c(0, 0)
p1 <- c(1, 0)
p2 <- c(1, 1)
p3 <- c(0, 1)
ggplot() +
  geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2])) +
  geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2])) +
  geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2])) +
  geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2])) +
  theme_bw() +
  xlim(-0.5, 1.5) +
  ylim(-0.5, 1.5) +
  labs(x = "x", y = "y")
```

Figure 1: The unit square.



The unit square can be defined by two basis vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From these vectors, we can compute the corner vertices.

$$\begin{aligned}
p_0 &= (0, 0) = (x_0, y_0) && \text{(The origin)} \\
p_1 &= v_1 = (x_1, y_1) = (1, 0) && \text{(The first basis vector)} \\
p_2 &= v_1 + v_2 = (x_2, y_2) = (1, 1) && \text{(The sum of basis vectors)} \\
p_3 &= v_2 = (x_3, y_3) = (0, 1) && \text{(The second basis vector)}
\end{aligned}$$

Note that the segments that form the boundary go from p_0 to p_1 to p_2 to p_3 and back to p_0.

Consider the matrix M to be the matrix we get from binding the two basis vectors at the column,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1 Question 1

Let's transform $\mathbf{M}$ to demonstrate an interesting linear algebra property.

1.1 Part a

Create the matrix `A` in `R`.

$$\mathbf{A} = \begin{bmatrix} 1 & -7 \\ 5 & 9 \end{bmatrix}$$

Solution

```
(A = matrix(data = c(1,-7,5,9), byrow = T, nrow = 2, ncol = 2))
```

```
     [,1] [,2]
[1,]    1   -7
[2,]    5    9
```

1.2 Part b

Compute the product below in `R` and store it in an object `T`.

$$\mathbf{T} = \mathbf{AM}.$$

Solution

```
M = matrix(data = c(1,0,0,1), byrow = T, nrow = 2, ncol = 2)
(T = A %*% M)
```

```
     [,1] [,2]
[1,]    1   -7
[2,]    5    9
```

1.3 Part c

Create `basis.vector.1` (first column of `T`) and `basis.vector.2` (second column of `T`) in `R`.

Solution

```
(basis.vector.1 = c(1,5))
```

```
[1] 1 5
```

```
(basis.vector.2 = c(-7,9))
```

```
[1] -7  9
```

1.4 Part d

Compute the points p_0, p_1, p_2, and p_3.

Solution

```
p0 = c(0,0)
p1 = basis.vector.1
p2 = basis.vector.1 + basis.vector.2
p3 = basis.vector.2
```

1.5 Part e

Copy and paste the code for the unit square. Edit the code to draw the segments that form the boundary based on the points in Part d.

Solution

```
p0
```

```
[1] 0 0
```

```
p1
```

```
[1] 1 5
```

```
p2
```

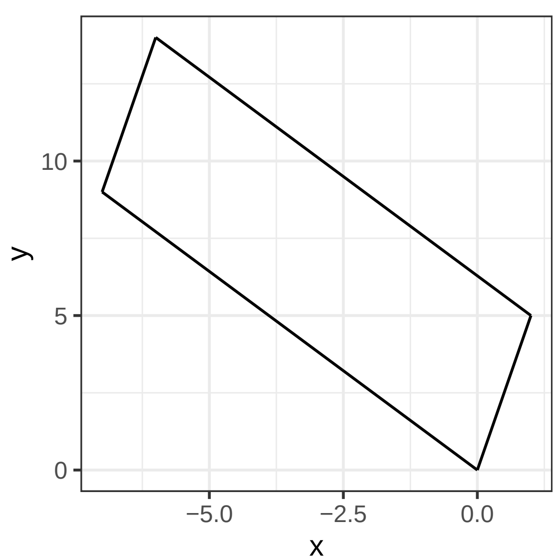
```
[1] -6 14
```

```
p3
```

```
[1] -7  9
```

```
ggplot() +
  geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2])) +
  geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2])) +
  geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2])) +
  geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2])) +
  theme_bw() +
  # xlim(-8, 1.5) +
  # ylim(-0.5, 1.5) +
  labs(x = "x", y = "y")
```

Figure 2: The unit square stretched.



1.6 Part f

Interestingly, the determinant of **A** gives us the area of the shape plotted in Part e. Find the determinant of **A**.

Solution

```
(det(A))
```

```
[1] 44
```

2 Question 2

Recomplete Question 1 using a new matrix for **A**. This matrix is not as well behaved.

2.1 Part a

Create the matrix A in R.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Solution

```r
(A = matrix(data = c(1,2,2,4), byrow = T, nrow = 2, ncol = 2))
```

```
     [,1] [,2]
[1,]    1    2
[2,]    2    4
```

2.2 Part b

Compute the product below in R and store it in an object T.

$$\mathbf{T} = \mathbf{AM}.$$

Solution

```r
M = matrix(data = c(1,0,0,1), byrow = T, nrow = 2, ncol = 2)
(T = A %*% M)
```

```
     [,1] [,2]
[1,]    1    2
[2,]    2    4
```

2.3 Part c

Create `basis.vector.1` (first column of T) and `basis.vector.2` (second column of T) in R.

Solution

```r
(basis.vector.1 = c(1,2))
```

```
[1] 1 2
```

```r
(basis.vector.2 = c(2,4))
```

```
[1] 2 4
```

2.4 Part d

Compute the points p_0, p_1, p_2, and p_3.

Solution

```r
p0 = c(0,0)
p1 = basis.vector.1
p2 = basis.vector.1 + basis.vector.2
p3 = basis.vector.2
```

2.5 Part e

Copy and paste the code for plotting the unit square. Edit the code to draw the segments that form the boundary based on the points in Part d.

Note: As you continue to plot this here and through question 2, you will find that the x and y limits need to change. Try `xlim(-15,15)` and `ylim(-5, 25)`.

Solution

```r
p0
```

```
[1] 0 0
```

```r
p1
```

```
[1] 1 2
```

```r
p2
```

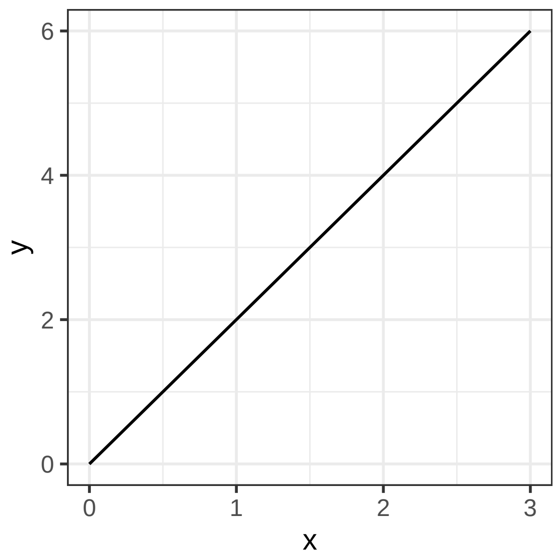
```
[1] 3 6
```

```r
p3
```

```
[1] 2 4
```

```r
ggplot() +
  geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2])) +
  geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2])) +
  geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2])) +
  geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2])) +
  theme_bw() +
  # xlim(-8, 1.5) +
  # ylim(-0.5, 1.5) +
  labs(x = "x", y = "y")
```

Figure 3: The unit square transformed into a line.



2.6 Part f

Interestingly, the determinant of **A** gives us the area of the shape plotted in Part e. Find the determinant of **A**.

Solution

```r
det(A)
```

```
[1] 0
```

3 Question 3

We can conduct a shear transformation that keeps the area of the shape the same but horizontally or vertically slants the object.

In fact, this is one of the ways artists can make something look like it's leaning or being pushed. Other applications where this is useful include fluid dynamics and crystallography, where the area of a deformed object must remain constant, or in photography where perspective in photos can be adjusted.

A shear transformation is implemented by altering the basis vectors we start with. For vertical slanting,

$$\mathbf{M}_v = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

and for horizontal slanting

$$\mathbf{M}_h = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$

In both cases, $k \in \mathbb{R}$ is called a shear factor and its sign determines the direction of the slant and its magnitude determines the severity of the slant.

3.1 Part a

Use the following to generate a new shape. Compare the shape and its area to your answer in Question 1.

$$\mathbf{M}_v = \begin{bmatrix} 1 & k = 0.5 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & -7 \\ 5 & 9 \end{bmatrix}$$

Solution

```
# Create A

(A = matrix(data = c(1,-7,5,9), byrow = T, nrow = 2, ncol = 2))
```

```
     [,1] [,2]
[1,]    1   -7
[2,]    5    9
```

```
# Create M and calculate T

M = matrix(data = c(1,0.5,0,1), byrow = T, nrow = 2, ncol = 2)
(T = A %*% M)
```

```
     [,1] [,2]
[1,]    1 -6.5
[2,]    5 11.5
```

```
# Create basis vectors (generalized setup)
(basis.vector.1 = T[,1])
```

```
[1] 1 5
```

```
(basis.vector.2 = T[,2])
```

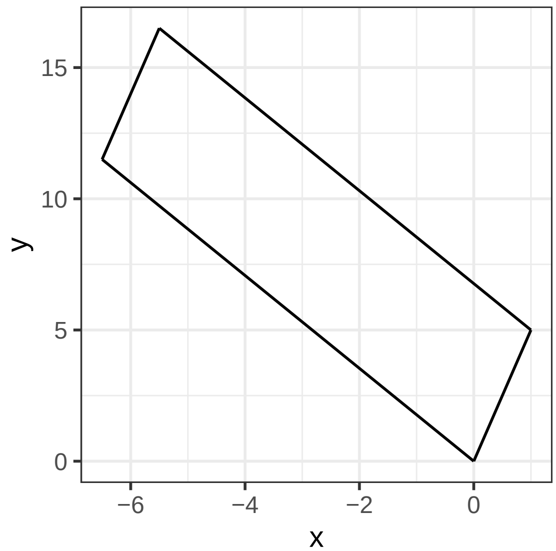
```
[1] -6.5 11.5
```

```
# Find corners from basis vectors

p0 = c(0,0)
p1 = basis.vector.1
p2 = basis.vector.1 + basis.vector.2
p3 = basis.vector.2
```

```
ggplot() +
  geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2])) +
  geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2])) +
  geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2])) +
  geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2])) +
  theme_bw() +
  # xlim(-8, 1.5) +
  # ylim(-0.5, 1.5) +
  labs(x = "x", y = "y")
```

Figure 4: The unit square sheared vertically.




```
# Find area

cat("The area of the shape is",  det(A))
```

```
The area of the shape is 44
```

The shape appears to be less sheared than in Question 1; this is probably because adding a vertical slant coefficient of 0.5 “undid” the vertical slant inherently contained in the A matrix. The area is the same, and the general form of the shape is similar to that in Question 1.

3.2 Part b

Use the following to generate a new shape. Compare the shape and its area to your answer in Questions 1 and the vertical shear transform in part (a).

$$\mathbf{M}_h = \begin{bmatrix} 1 & 0 \\ k = 0.5 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & -7 \\ 5 & 9 \end{bmatrix}$$

Solution

```
# Create A

(A = matrix(data = c(1,-7,5,9), byrow = T, nrow = 2, ncol = 2))
```

```
     [,1] [,2]
[1,]    1   -7
[2,]    5    9
```

```
# Create M and calculate T

M = matrix(data = c(1,0,0.5,1), byrow = T, nrow = 2, ncol = 2)
(T = A %*% M)
```

```
     [,1] [,2]
[1,] -2.5   -7
[2,]  9.5    9
```

```r
# Create basis vectors (generalized setup)
(basis.vector.1 = T[,1])
```

```
[1] -2.5  9.5
```

```r
(basis.vector.2 = T[,2])
```

```
[1] -7  9
```

```r
# Find corners from basis vectors

p0 = c(0,0)
p1 = basis.vector.1
p2 = basis.vector.1 + basis.vector.2
p3 = basis.vector.2


# Plot Shape

#| label: fig-unitslanthorizontal
#| fig-cap: "The unit square sheared horizontally."
#| fig-width: 3
#| fig-height: 3
#| scale: "80%"
#| fig-align: "center"
#| size: "scriptsize"
p0
```

```
[1] 0 0
```

```r
p1
```

```
[1] -2.5  9.5
```

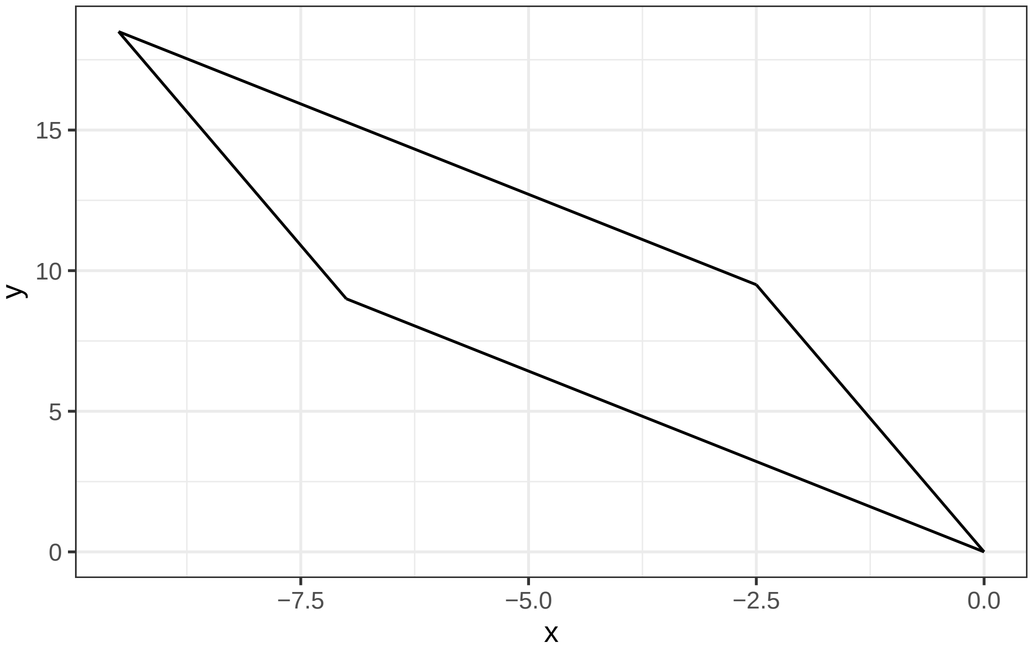
```r
p2
```

```
[1] -9.5 18.5
```

```r
p3
```

```
[1] -7  9
```

```r
ggplot() +
  geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2])) +
  geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2])) +
  geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2])) +
  geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2])) +
  theme_bw() +
  # xlim(-8, 1.5) +
  # ylim(-0.5, 1.5) +
  labs(x = "x", y = "y")
```




```
# Find area

cat("The area of the shape is",  det(A))
```

```
The area of the shape is 44
```

This result seems to confirm our hypothesis in part 3a. The area is the same as that of the shape in Question 1 and the sape in part 3a, but there is a lot of vertical shear (probably the same amount as in Question 1), and more horizontal shear.

3.3 Part c

Can we shear vertically and horizontally at the same time? Use the following to generate a new shape. Compare the shape and its area to your answers in parts (a) and (b).

$$\mathbf{M}_{vh} = \begin{bmatrix} 1 & k = 0.5 \\ k = 0.5 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{bmatrix} 1 & -7 \\ 5 & 9 \end{bmatrix}$$

Solution

```
# Create A

(A = matrix(data = c(1,-7,5,9), byrow = T, nrow = 2, ncol = 2))
```

```
     [,1] [,2]
[1,]    1   -7
[2,]    5    9
```

```
# Create M and calculate T

M = matrix(data = c(1,0.5,0.5,1), byrow = T, nrow = 2, ncol = 2)
(T = A %*% M)
```

```
     [,1] [,2]
[1,] -2.5 -6.5
[2,]  9.5 11.5
```

```r
# Create basis vectors (generalized setup)
(basis.vector.1 = T[,1])
```

```
[1] -2.5  9.5
```

```r
(basis.vector.2 = T[,2])
```

```
[1] -6.5 11.5
```

```r
# Find corners from basis vectors

p0 = c(0,0)
p1 = basis.vector.1
p2 = basis.vector.1 + basis.vector.2
p3 = basis.vector.2


# Plot Shape

#| label: fig-unitslantboth
#| fig-cap: "The unit square sheared both horizontally and vertically."
#| fig-width: 3
#| fig-height: 3
#| scale: "80%"
#| fig-align: "center"
#| size: "scriptsize"
p0
```

```
[1] 0 0
```

```r
p1
```

```
[1] -2.5  9.5
```

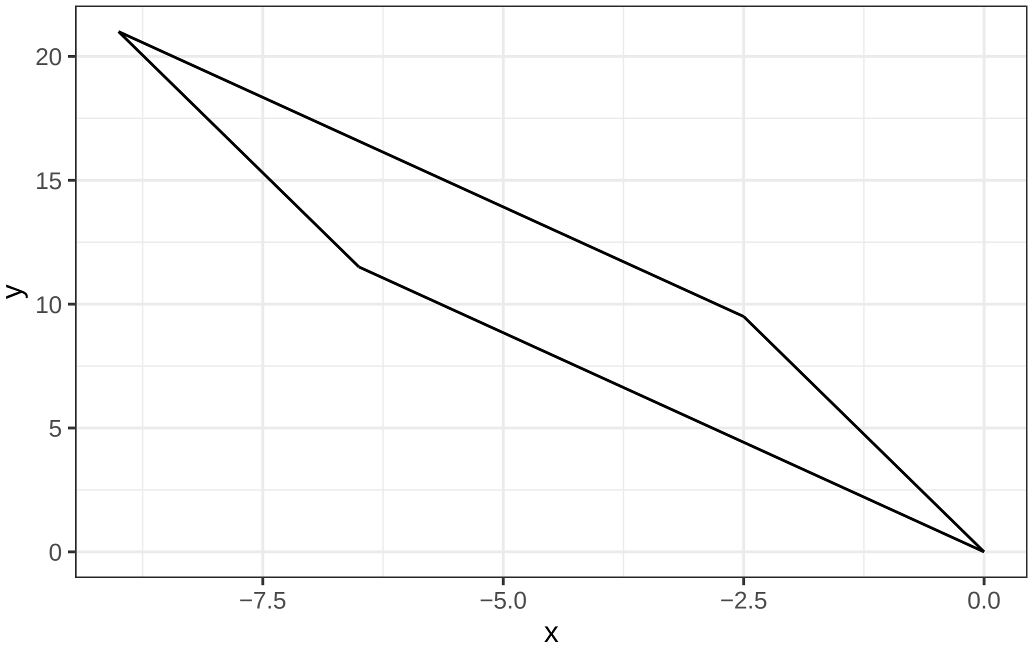
```r
p2
```

```
[1] -9 21
```

```r
p3
```

```
[1] -6.5 11.5
```

```r
ggplot() +
  geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2])) +
  geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2])) +
  geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2])) +
  geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2])) +
  theme_bw() +
  # xlim(-8, 1.5) +
  # ylim(-0.5, 1.5) +
  labs(x = "x", y = "y")
```




```
# Find area

cat("The area of the shape is",  det(A))
```

```
The area of the shape is 44
```

The determinant (and hence the area) is the same for all three shapes. This shape is more distorted, and seems to approach a "flipped" version of the diagonal line in part 1e. I believe that if we increase both k-values (maybe as they approach 1?), the shape will approach a diagonal line. (I tested this and it is true!)

4 Question 4

4.1 Part a

Create a data frame `ladder` with the following data. Note your data frame should have three columns; I split it to save vertical space here.

x	y	group	x	y	group
0	0	1	1	8	6
0	20	1	0	10	7
1	0	2	1	10	7
1	20	2	0	12	8
0	2	3	1	12	8
1	2	3	0	14	9
0	4	4	1	14	9
1	4	4	0	16	10
0	6	5	1	16	10
1	6	5	0	18	11
0	8	6	1	18	11

Solution

```
# Create column vectors

x = c(0,0,1,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1)
```

```r
y = c(0,20,0,20,2,2,4,4,6,6,8,8,10,10,12,12,14,14,16,16,18,18)
group = rep(c(1,2,3,4,5,6,7,8,9,10,11), each = 2)

# Create data frame

(ladder = data.frame(x,y,group))
```

```
   x  y group
1  0  0     1
2  0 20     1
3  1  0     2
4  1 20     2
5  0  2     3
6  1  2     3
7  0  4     4
8  1  4     4
9  0  6     5
10 1  6     5
11 0  8     6
12 1  8     6
13 0 10     7
14 1 10     7
15 0 12     8
16 1 12     8
17 0 14     9
18 1 14     9
19 0 16    10
20 1 16    10
21 0 18    11
22 1 18    11
```

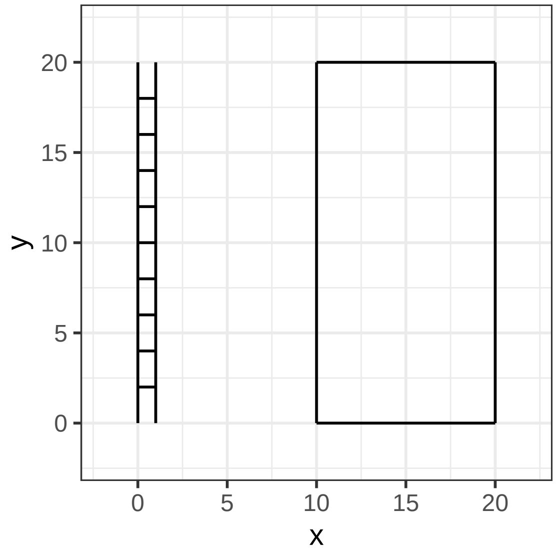
4.2 Part b

Remove `#|eval: false` in the code below. The result should look like a ladder and a very rectangular house if part a is correct.

Solution

```r
ggplot() +
  geom_path(data = ladder,
            aes(x = x, y = y, group = group)) +
                geom_segment(aes(x = 10, xend = 10,
                 y=0, yend = 20))+
  geom_segment(aes(x = 10, xend = 20,
                 y=20, yend = 20))+
  geom_segment(aes(x = 20, xend = 20,
                 y=0, yend = 20))+
  geom_segment(aes(x = 10, xend = 20,
                 y=0, yend = 0)) +
  theme_bw() +
  xlim(-2,22) +
  ylim(-2,22)
```

Figure 5: A ladder.



4.3 Part c

Use `as.matrix()` to create a 22×2 matrix $\mathbf{A}$ containing the `x` and `y` observations from the `ladder` data frame. Use matrix multiplication to compute

$$\mathbf{T} = \mathbf{A}\mathbf{M}_h.$$

Recall,

$$\mathbf{M}_h = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

and use $k = 0.5$.

Solution

```
A = as.matrix(ladder[,c("x","y")])
M = matrix(data = c(1,0,0.5,1), byrow = T, nrow = 2, ncol = 2)
T = A %*% M
head(T)
```

```
     [,1] [,2]
[1,]    0    0
[2,]   10   20
[3,]    1    0
[4,]   11   20
[5,]    1    2
[6,]    2    2
```

4.4 Part d

Create a data frame `leaning.ladder` with the `x` and `y` equal to the first and second column of $\mathbf{T}$, respectively, and the same values of `group` from `ladder`.

Solution

```
x = T[,1]
y = T[,2]
leaning.ladder = data.frame(x,y,group)
head(leaning.ladder)
```

```
   x  y group
1  0  0     1
2 10 20     1
3  1  0     2
4 11 20     2
5  1  2     3
6  2  2     3
```

4.5 Part e

Remove `eval: false` in the code below. The result should look like a ladder and a very rectangular house if part a is correct. The new leaning ladder should look like it is leaning on the house.

```
ggplot() +
  geom_path(data = ladder,
            aes(x = x, y = y, group = group, color = "Starting Ladder")) +
  geom_path(data = leaning.ladder,
            aes(x = x, y = y, group = group, color = "Leaning Ladder")) +
  geom_segment(aes(x = 10, xend = 10,
                   y=0, yend = 20))+
  geom_segment(aes(x = 10, xend = 20,
                   y=20, yend = 20))+
  geom_segment(aes(x = 20, xend = 20,
                   y=0, yend = 20))+
  geom_segment(aes(x = 10, xend = 20,
                   y=0, yend = 0)) +
  theme_bw() +
  xlim(-2,22) +
  ylim(-2,22) +
  scale_color_manual("", values=c("darkred", "grey"))
```

Figure 6: A ladder and a leaning ladder.