

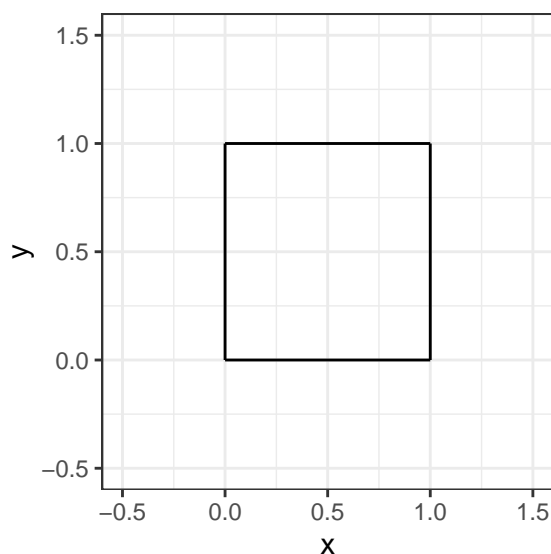
## Lab Three – Matrices and Data Frames in R

- Complete the tasks below. Make sure to start your solutions in on a new line that starts with “**Solution:**”.
- Make sure to use the Quarto Cheatsheet. This will make completing and writing up the lab *much* easier.

Consider the unit square depicted in Figure 1.

```
1 ggplot() +  
2   geom_segment(aes(x = 0, xend = 0,  
3                     y=0, yend = 1))+  
4   geom_segment(aes(x = 0, xend = 1,  
5                     y=1, yend = 1))+  
6   geom_segment(aes(x = 1, xend = 1,  
7                     y=0, yend = 1))+  
8   geom_segment(aes(x = 0, xend = 1,  
9                     y=0, yend = 0)) +  
10  theme_bw() +  
11  xlim(-0.5, 1.5)+  
12  ylim(-0.5, 1.5)+  
13  labs(x = "x",  
14        y = "y")
```

Figure 1: The unit square.



The unit square can be defined by two basis vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From these vectors, we can compute the corner vertices.

$p_0 = (0, 0) = (x_0, y_0)$	(The origin)
$p_1 = v_1 = (x_1, y_1)$	(The first basis vector)
$p_2 = v_1 + v_2 = (x_2, y_2)$	(The sum of basis vectors)
$p_3 = v_1 = (x_3, y_3)$	(The second basis vector)

Note that the segments that form the boundary go from  $p_0$  to  $p_1$  to  $p_2$  to  $p_3$ .

Consider the matrix  $M$  to be the matrix we get from binding the two basis vectors at the column,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

## 1 Question 1

Let's transform  $\mathbf{M}$  to demonstrate an interesting linear algebra property.

### Solution

We first need to make Matrix M:

```
1 # I'm going to annoy you with using only by column
2 (M = matrix(c(1,0,0,1),2,2))
```

```
      [,1] [,2]
[1,]     1     0
[2,]     0     1
```

### 1.1 Part a

Create the matrix A in R.

$$\mathbf{A} = \begin{bmatrix} 1 & -7 \\ 5 & 9 \end{bmatrix}$$

**Solution** Matrix A:

```
1 (A = matrix(c(1,5,-7,9),2,2))
```

```
      [,1] [,2]
[1,]     1    -7
[2,]     5     9
```

### 1.2 Part b

Compute the product below in R and store it in an object T.

$$\mathbf{T} = \mathbf{AM}.$$

### Solution

```
1 (T = A%*%M)
```

```
      [,1] [,2]
[1,]     1    -7
[2,]     5     9
```

### 1.3 Part c

Create `basis.vector.1` (first column of T) and `basis.vector.2` (second column of T) in R.

```
1 (basis.vector.1 = T[,1])
```

```
[1] 1 5
```

```
1 (basis.vector.2 = T[,2])
```

```
[1] -7 9
```

### 1.4 Part d

Compute the points  $p_0$ ,  $p_1$ ,  $p_2$ , and  $p_3$ .

```
1 (p0 = c(0,0))
```

```
[1] 0 0
```

```

1 (p1 = basis.vector.1)

[1] 1 5
1 (p2 = basis.vector.1 + basis.vector.2)

[1] -6 14
1 (p3 = basis.vector.2)

[1] -7 9

```

## 1.5 Part e

Copy and paste the code for the unit square. Edit the code to draw the segments that form the boundary based on the points in Part d.

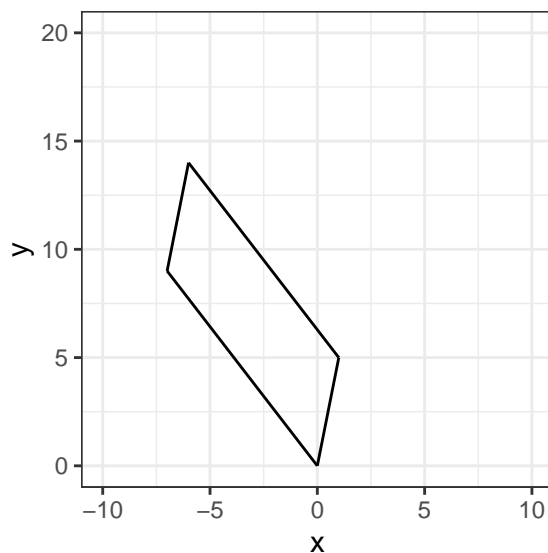
### Solution

```

1 ggplot() +
2   geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2])) +
3   geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2])) +
4   geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2])) +
5   geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2])) +
6   theme_bw() +
7   xlim(-10, 10)+
8   ylim(0, 20)+
9   labs(x = "x",
10        y = "y")

```

Figure 2: Matrix T on a Graph.



## 1.6 Part f

Interestingly, the determinant of  $\mathbf{A}$  gives us the area of the shape plotted in Part e. Find the determinant of  $\mathbf{A}$ .

### Solution

```

1 (det(A))

[1] 44

```

## 2 Question 2

Recomplete Question 1 using a new matrix for  $\mathbf{A}$ . This matrix is not as well behaved.

## 2.1 Part a

Create the matrix **A** in **R**.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

### Solution

```
1 (A = matrix(c(1,2,2,4),2,2))
```

```
      [,1] [,2]  
[1,]     1     2  
[2,]     2     4
```

## 2.2 Part b

Compute the product below in **R** and store it in an object **T**.

$$\mathbf{T} = \mathbf{A}\mathbf{M}.$$

### Solution

```
1 (T = A%%M)
```

```
      [,1] [,2]  
[1,]     1     2  
[2,]     2     4
```

## 2.3 Part c

Create **basis.vector.1** (first column of **T**) and **basis.vector.2** (second column of **T**) in **R**.

### Solution

```
1 (basis.vector.1 = T[,1])
```

```
[1] 1 2
```

```
1 (basis.vector.2 = T[,2])
```

```
[1] 2 4
```

## 2.4 Part d

Compute the points  $p_0$ ,  $p_1$ ,  $p_2$ , and  $p_3$ .

### Solution

```
1 (p0 = c(0,0))
```

```
[1] 0 0
```

```
1 (p1 = basis.vector.1)
```

```
[1] 1 2
```

```
1 (p2 = basis.vector.1+basis.vector.2)
```

```
[1] 3 6
```

```
1 (p3 = basis.vector.2)
```

```
[1] 2 4
```

## 2.5 Part e

Copy and paste the code for plotting the unit square. Edit the code to draw the segments that form the boundary based on the points in Part d.

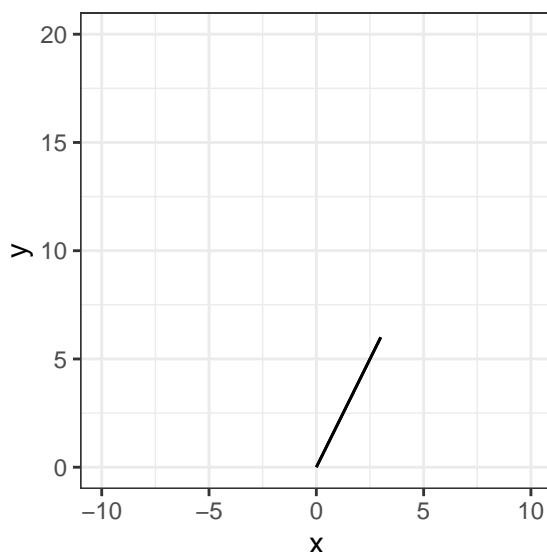
### Solution

```

1 ggplot() +
2   geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2])) +
3   geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2])) +
4   geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2])) +
5   geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2])) +
6   theme_bw() +
7   xlim(-10, 10)+
8   ylim(0, 20)+
9   labs(x = "x",
10        y = "y")

```

Figure 3: Matrix T (ver. 2) on a Graph.



## 2.6 Part f

Interestingly, the determinant of  $\mathbf{A}$  gives us the area of the shape plotted in Part e. Find the determinant of  $\mathbf{A}$ .

### Solution

```
1 (det(A))
```

```
[1] 0
```

## 3 Question 3

We can conduct a shear transformation that keeps the area of the shape the same but horizontally or vertically slants the object.

In fact, this is one of the ways artists can make something look like it's leaning or being pushed. Other applications where this is useful include fluid dynamics and crystallography, where the area of a deformed object must remain constant, or in photography where perspective in photos can be adjusted.

A shear transformation is implemented by altering the basis vectors we start with. For vertical slanting,

$$\mathbf{M}_v = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

and for horizontal slanting

$$\mathbf{M}_h = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$

In both cases,  $k \in \mathbb{R}$  is called a shear factor and its sign determines the direction of the slant and its magnitude determines the severity of the slant.

### 3.1 Part a

Use the following to generate a new shape. Compare the shape and its area to your answer in Question 1.

$$\mathbf{M}_v = \begin{bmatrix} 1 & k = 0.5 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & -7 \\ 5 & 9 \end{bmatrix}$$

#### Solution

```
1 (Mv = matrix(c(1,0,0.5,1),2,2))

      [,1] [,2]
[1,]    1  0.5
[2,]    0  1.0

1 (A = matrix(c(1,5,-7,9),2,2))

      [,1] [,2]
[1,]    1  -7
[2,]    5   9

1 (T = A%/%Mv)

      [,1] [,2]
[1,]    1 -6.5
[2,]    5 11.5

1 (p0 = c(0,0))

[1] 0 0

1 (p1 = T[,1])

[1] 1 5

1 (p2 = T[,1]+T[,2])

[1] -5.5 16.5

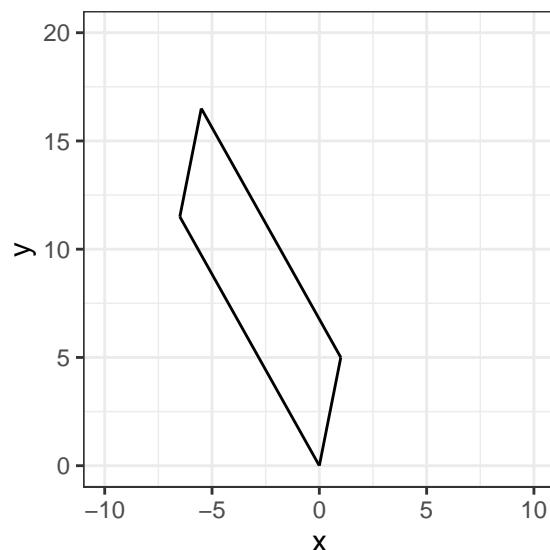
1 (p3 = T[,2])

[1] -6.5 11.5

1 ggplot() +
2   geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2]))+
3   geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2]))+
4   geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2]))+
5   geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2]))+
6   theme_bw() +
7   xlim(-10, 10)+
8   ylim(0, 20)+
9   labs(x = "x",
10        y = "y")
11
12 (det(A))

[1] 44
```

Figure 4: Matrix 3.1 on a Graph.



Here we see that the area is the same, as the determinant remains as 44. However, we see that the shape is pulled vertically up by a significant amount. As it is pulled up it seems it narrows in width.

### 3.2 Part b

Use the following to generate a new shape. Compare the shape and its area to your answer in Questions 1 and the vertical shear transform in part (a).

$$\mathbf{M}_h = \begin{bmatrix} 1 & 0 \\ k = 0.5 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & -7 \\ 5 & 9 \end{bmatrix}$$

#### Solution

```
1 (Mh = matrix(c(1,0.5,0,1),2,2))

      [,1] [,2]
[1,]  1.0   0
[2,]  0.5   1
1 (A = matrix(c(1,5,-7,9),2,2))

      [,1] [,2]
[1,]    1  -7
[2,]    5   9
1 (T = A%*%Mh)

      [,1] [,2]
[1,] -2.5  -7
[2,]  9.5   9
1 (p0 = c(0,0))

[1] 0 0
1 (p1 = T[,1])

[1] -2.5  9.5
1 (p2 = T[,1]+T[,2])

[1] -9.5 18.5
1 (p3 = T[,2])

[1] -7  9
1 ggplot() +
2   geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2]))+
3   geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2]))+
4   geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2]))+
```

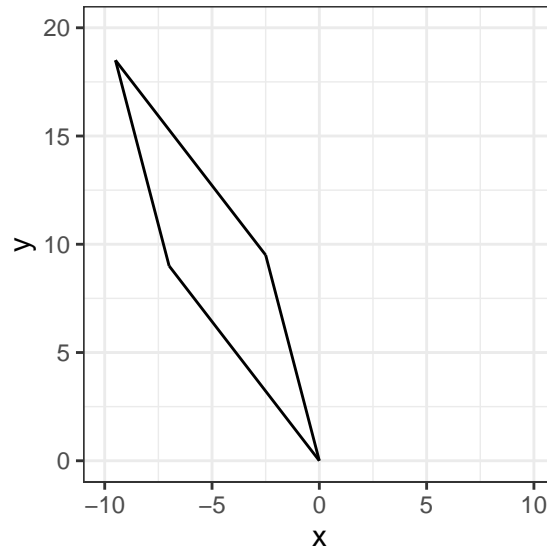
```

5 geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2]))+
6 theme_bw() +
7 xlim(-10, 10)+
8 ylim(0, 20)+
9 labs(x = "x",
10      y = "y")
11
12 (det(A))

```

[1] 44

Figure 5: Matrix 3.2 on a Graph.



Here, we once again see the area to be the same, but as compared to the shape in Question 1 and in Part 3a, the parallelogram is much skinnier, almost resembling a rhombus. The shape has been squished more horizontally, which has also made it increase in height as well relative to the Question 1 shape.

### 3.3 Part c

Can we shear vertically and horizontally at the same time? Use the following to generate a new shape. Compare the shape and its area to your answers in parts (a) and (b).

$$\mathbf{M}_{vh} = \begin{bmatrix} 1 & k = 0.5 \\ k = 0.5 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & -7 \\ 5 & 9 \end{bmatrix}$$

#### Solution

```

1 (Mvh = matrix(c(1,0.5,0.5,1),2,2))

```

```

      [,1] [,2]
[1,]  1.0  0.5
[2,]  0.5  1.0

```

```

1 (A = matrix(c(1,5,-7,9),2,2))

```

```

      [,1] [,2]
[1,]    1  -7
[2,]    5   9

```

```

1 (T = A%*%Mvh)

```

```

      [,1] [,2]
[1,] -2.5 -6.5
[2,]  9.5 11.5

```

```

1 (p0 = c(0,0))

```

[1] 0 0



```

1 (p1 = T[,1])

[1] -2.5  9.5
1 (p2 = T[,1]+T[,2])

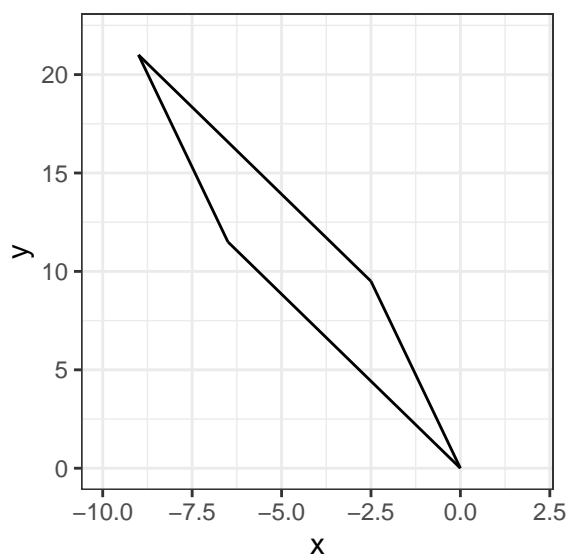
[1] -9 21
1 (p3 = T[,2])

[1] -6.5 11.5
1 ggplot() +
2   geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2]))+
3   geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2]))+
4   geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2]))+
5   geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2]))+
6   theme_bw() +
7   xlim(-10, 2)+
8   ylim(0, 22)+
9   labs(x = "x",
10        y = "y")
12 (det(A))

[1] 44

```

Figure 6: Matrix 3.3 on a Graph.



Here, the area is again the same, due to the invariant determinant. Comparing to parts a and b, this shape is much more tilted to the left, and is squished much more into a narrower shape. The left lean is much more prominent and we can see the shape being both squished horizontally while also being stretched in the vertical direction.

## 4 Question 4

### 4.1 Part a

Create a data frame `ladder` with the following data. Note your data frame should have three columns; I split it to save vertical space here.

x	y	group	x	y	group
0	0	1	1	8	6
0	20	1	0	10	7
1	0	2	1	10	7
1	20	2	0	12	8

x	y	group	x	y	group
0	2	3	1	12	8
1	2	3	0	14	9
0	4	4	1	14	9
1	4	4	0	16	10
0	6	5	1	16	10
1	6	5	0	18	11
0	8	6	1	18	11

## Solution

```

1 (x=c(0,0,1,1),rep(c(0,1),times=9)))

[1] 0 0 1 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1

1 (y=c(0,20,0,20),rep(seq(from=2,to=18,by=2),each=2)))

[1] 0 20 0 20 2 2 4 4 6 6 8 8 10 10 12 12 14 14 16 16 18 18

1 (group = rep(seq(from=1,to=11),each=2))

[1] 1 1 2 2 3 3 4 4 5 5 6 6 7 7 8 8 9 9 10 10 11 11

1 ladder = data.frame("x"= x, "y"=y, "group" = group)
2 (ladder)

   x  y group
1  0  0     1
2  0 20     1
3  1  0     2
4  1 20     2
5  0  2     3
6  1  2     3
7  0  4     4
8  1  4     4
9  0  6     5
10 1  6     5
11 0  8     6
12 1  8     6
13 0 10     7
14 1 10     7
15 0 12     8
16 1 12     8
17 0 14     9
18 1 14     9
19 0 16    10
20 1 16    10
21 0 18    11
22 1 18    11

```

## 4.2 Part b

Remove `#|eval: false` in the code below. The result should look like a ladder and a very rectangular house if part a is correct.

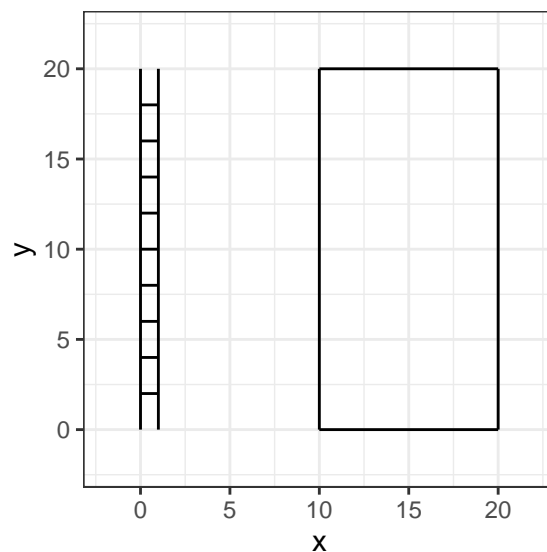
## Solution

```

1 ggplot() +
2   geom_path(data = ladder,
3             aes(x = x, y = y, group = group)) +
4     geom_segment(aes(x = 10, xend = 10,
5                       y=0, yend = 20))+
6     geom_segment(aes(x = 10, xend = 20,
7                       y=20, yend = 20))+
8     geom_segment(aes(x = 20, xend = 20,
9                       y=0, yend = 20))+
10    geom_segment(aes(x = 10, xend = 20,
11                     y=0, yend = 0)) +
12    theme_bw() +
13    xlim(-2,22) +
14    ylim(-2,22)

```

Figure 7: A ladder.



### 4.3 Part c

Use `as.matrix()` to create a  $22 \times 2$  matrix  $\mathbf{A}$  containing the `x` and `y` observations from the `ladder` data frame. Use matrix multiplication to compute

$$\mathbf{T} = \mathbf{A}\mathbf{M}_h.$$

Recall,

$$\mathbf{M}_h = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

and use  $k = 0.5$ .

#### Solution

```
1 A = as.matrix(ladder[,c("x","y")])
2 (A)
```

```
      x  y
[1,] 0  0
[2,] 0 20
[3,] 1  0
[4,] 1 20
[5,] 0  2
[6,] 1  2
[7,] 0  4
[8,] 1  4
[9,] 0  6
[10,] 1  6
[11,] 0  8
[12,] 1  8
[13,] 0 10
[14,] 1 10
[15,] 0 12
[16,] 1 12
[17,] 0 14
[18,] 1 14
[19,] 0 16
[20,] 1 16
[21,] 0 18
[22,] 1 18
```

```
1 Mh = matrix(c(1,0.5,0,1),2,2)
2 (Mh)
```

```
      [,1] [,2]
[1,]  1.0  0
[2,]  0.5  1
```

```
1 T = A%Mh
2 (T)
```

	[,1]	[,2]
[1,]	0	0
[2,]	10	20
[3,]	1	0
[4,]	11	20
[5,]	1	2
[6,]	2	2
[7,]	2	4
[8,]	3	4
[9,]	3	6
[10,]	4	6
[11,]	4	8
[12,]	5	8
[13,]	5	10
[14,]	6	10
[15,]	6	12
[16,]	7	12
[17,]	7	14
[18,]	8	14
[19,]	8	16
[20,]	9	16
[21,]	9	18
[22,]	10	18

## 4.4 Part d

Create a data frame `leaning.ladder` with the `x` and `y` equal to the first and second column of `T`, respectively, and the same values of `group` from `ladder`.

### Solution

```
1 leaning.ladder=data.frame("x" = T[,1], "y" = T[,2], "group" = ladder[,c("group")])
2 (leaning.ladder)
```

	x	y	group
1	0	0	1
2	10	20	1
3	1	0	2
4	11	20	2
5	1	2	3
6	2	2	3
7	2	4	4
8	3	4	4
9	3	6	5
10	4	6	5
11	4	8	6
12	5	8	6
13	5	10	7
14	6	10	7
15	6	12	8
16	7	12	8
17	7	14	9
18	8	14	9
19	8	16	10
20	9	16	10
21	9	18	11
22	10	18	11

## 4.5 Part e

Remove `eval: false` in the code below. The result should look like a ladder and a very rectangular house if part a is correct. The new leaning ladder should look like it is leaning on the house.

```
1 ggplot() +
2   geom_path(data = ladder,
3             aes(x = x, y = y, group = group, color = "Starting Ladder")) +
4   geom_path(data = leaning.ladder,
5             aes(x = x, y = y, group = group, color = "Leaning Ladder")) +
6   geom_segment(aes(x = 10, xend = 10,
7                    y=0, yend = 20))+
8   geom_segment(aes(x = 10, xend = 20,
9                    y=20, yend = 20))+
10  geom_segment(aes(x = 20, xend = 20,
11                  y=0, yend = 20))+
12  geom_segment(aes(x = 10, xend = 20,
13                  y=0, yend = 0)) +
14  theme_bw() +
15  xlim(-2,22) +
16  ylim(-2,22) +
17  scale_color_manual("", values=c("darkred", "grey"))
```