

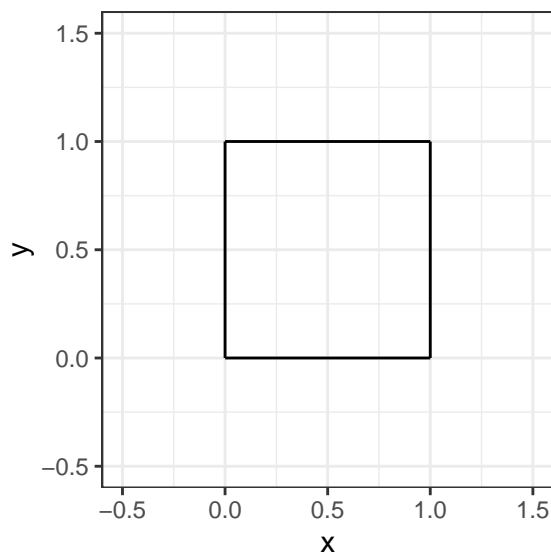
## Lab Three – Matrices and Data Frames in R

- Complete the tasks below. Make sure to start your solutions in on a new line that starts with “**Solution:**”.
- Make sure to use the Quarto Cheatsheet. This will make completing and writing up the lab *much* easier.

Consider the unit square depicted in Figure 1.

```
1 ggplot() +  
2   geom_segment(aes(x = 0, xend = 0,  
3                     y=0, yend = 1))+  
4   geom_segment(aes(x = 0, xend = 1,  
5                     y=1, yend = 1))+  
6   geom_segment(aes(x = 1, xend = 1,  
7                     y=0, yend = 1))+  
8   geom_segment(aes(x = 0, xend = 1,  
9                     y=0, yend = 0)) +  
10  theme_bw() +  
11  xlim(-0.5, 1.5)+  
12  ylim(-0.5, 1.5)+  
13  labs(x = "x",  
14        y = "y")
```

Figure 1: The unit square.



The unit square can be defined by two basis vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From these vectors, we can compute the corner vertices.

$p_0 = (0, 0) = (x_0, y_0)$	(The origin)
$p_1 = v_1 = (x_1, y_1) = (1, 0)$	(The first basis vector)
$p_2 = v_1 + v_2 = (x_2, y_2) = (1, 1)$	(The sum of basis vectors)
$p_3 = v_2 = (x_3, y_3) = (0, 1)$	(The second basis vector)

Note that the segments that form the boundary go from  $p_0$  to  $p_1$  to  $p_2$  to  $p_3$  and back to  $p_0$ .

Consider the matrix  $M$  to be the matrix we get from binding the two basis vectors at the column,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

## 1 Question 1

Let's transform  $\mathbf{M}$  to demonstrate an interesting linear algebra property.

### 1.1 Part a

Create the matrix  $\mathbf{A}$  in R.

$$\mathbf{A} = \begin{bmatrix} 1 & -7 \\ 5 & 9 \end{bmatrix}$$

**Solution:** This is Matrix  $\mathbf{A}$ .

```
1 (A = matrix(data=c(1, -7, 5, 9), nrow = 2, ncol = 2, byrow = TRUE))
```

```
      [,1] [,2]  
[1,]     1  -7  
[2,]     5   9
```

### 1.2 Part b

Compute the product below in R and store it in an object  $\mathbf{T}$ .

$$\mathbf{T} = \mathbf{A}\mathbf{M}.$$

**Solution:** This is Matrix  $\mathbf{T} = \mathbf{A}\mathbf{M}$ .

```
1 M = matrix(data=c(1, 0, 0, 1), nrow = 2, ncol = 2, byrow = TRUE)  
2 (T = A %*% M)
```

```
      [,1] [,2]  
[1,]     1  -7  
[2,]     5   9
```

### 1.3 Part c

Create `basis.vector.1` (first column of  $\mathbf{T}$ ) and `basis.vector.2` (second column of  $\mathbf{T}$ ) in R.

**Solution:** This is basis vector 1 and 2 in order.

```
1 (basis.vector.1 = T[,1])
```

```
[1] 1 5
```

```
1 (basis.vector.2 = T[,2])
```

```
[1] -7 9
```

### 1.4 Part d

Compute the points  $p_0$ ,  $p_1$ ,  $p_2$ , and  $p_3$ .

**Solution:** This is  $p_0$ ,  $p_1$ ,  $p_2$ , and  $p_3$  in order.

```
1 (p0 = c(0,0))
```

```
[1] 0 0
```

```

1 (p1 = basis.vector.1)

[1] 1 5
1 (p2 = basis.vector.1 + basis.vector.2)

[1] -6 14
1 (p3 = basis.vector.2)

[1] -7 9

```

## 1.5 Part e

Copy and paste the code for the unit square. Edit the code to draw the segments that form the boundary based on the points in Part d.

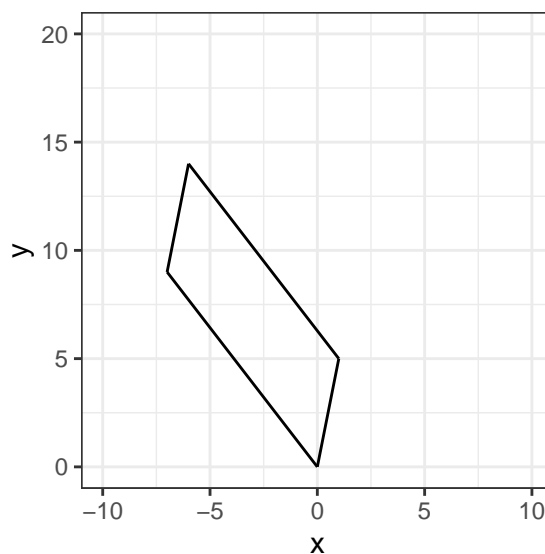
### Solution

```

1 ggplot() +
2   geom_segment(aes(x = 0, xend = 1,
3                     y=0, yend = 5))+
4   geom_segment(aes(x = -6, xend = 1,
5                     y=14, yend = 5))+
6   geom_segment(aes(x = -7, xend = -6,
7                     y=9, yend = 14))+
8   geom_segment(aes(x = -7, xend = 0,
9                     y=9, yend = 0)) +
10  theme_bw() +
11  xlim(-10, 10) +
12  ylim(0, 20) +
13  labs(x = "x",
14       y = "y")

```

Figure 2: The T Shape.



## 1.6 Part f

Interestingly, the determinant of  $\mathbf{A}$  gives us the area of the shape plotted in Part e. Find the determinant of  $\mathbf{A}$ .

**Solution:** The area is equal to the determinant calculated by running `det(A)`

```

1 (det(A))

[1] 44

```

## 2 Question 2

Recomplete Question 1 using a new matrix for **A**. This matrix is not as well behaved.

### 2.1 Part a

Create the matrix **A** in R.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

**Solution:** This is the new Matrix **A**.

```
1 (A = matrix(data=c(1, 2, 2, 4), nrow = 2, ncol = 2, byrow = TRUE))
```

```
      [,1] [,2]  
[1,]     1     2  
[2,]     2     4
```

### 2.2 Part b

Compute the product below in R and store it in an object **T**.

$$\mathbf{T} = \mathbf{A}\mathbf{M}.$$

**Solution:** This is Matrix  $T = AM$ .

```
1 M = matrix(data=c(1, 0, 0, 1), nrow = 2, ncol = 2, byrow = TRUE)  
2 (T = A %*% M)
```

```
      [,1] [,2]  
[1,]     1     2  
[2,]     2     4
```

### 2.3 Part c

Create **basis.vector.1** (first column of **T**) and **basis.vector.2** (second column of **T**) in R.

**Solution:** This is basis vector 1 and 2 in order.

```
1 (basis.vector.1 = T[,1])
```

```
[1] 1 2
```

```
1 (basis.vector.2 = T[,2])
```

```
[1] 2 4
```

### 2.4 Part d

Compute the points  $p_0$ ,  $p_1$ ,  $p_2$ , and  $p_3$ .

**Solution:** This is  $p_0$ ,  $p_1$ ,  $p_2$ , and  $p_3$  in order.

```
1 (p0 = c(0,0))
```

```
[1] 0 0
```

```
1 (p1 = basis.vector.1)
```

```
[1] 1 2
```

```
1 (p2 = basis.vector.1 + basis.vector.2)
```

```
[1] 3 6
```

```
1 (p3 = basis.vector.2)
```

```
[1] 2 4
```

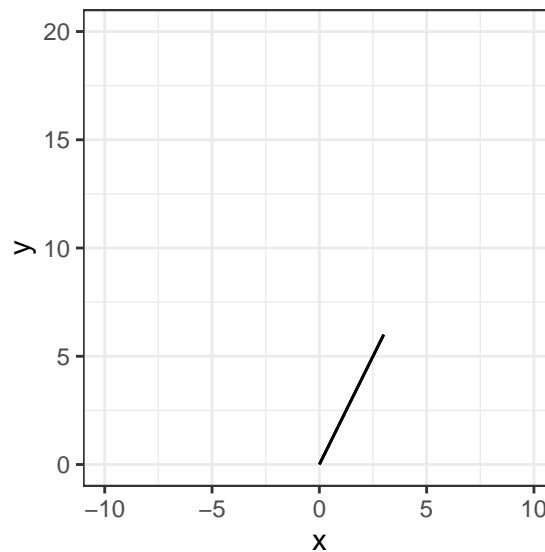
## 2.5 Part e

Copy and paste the code for plotting the unit square. Edit the code to draw the segments that form the boundary based on the points in Part d.

### Solution

```
1 p0 <- c(0, 0)
2 p1 <- c(1, 2)
3 p2 <- c(3, 6)
4 p3 <- c(2, 4)
5 ggplot() +
6   geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2])) +
7   geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2])) +
8   geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2])) +
9   geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2])) +
10  theme_bw() +
11  xlim(-10, 10) +
12  ylim(0, 20) +
13  labs(x = "x", y = "y")
```

Figure 3: The T2 Shape.



## 2.6 Part f

Interestingly, the determinant of  $\mathbf{A}$  gives us the area of the shape plotted in Part e. Find the determinant of  $\mathbf{A}$ .

**Solution:** The area is equal to the determinant calculated by running `det(A)`

```
1 det(A)
```

```
[1] 0
```

## 3 Question 3

We can conduct a shear transformation that keeps the area of the shape the same but horizontally or vertically slants the object.

In fact, this is one of the ways artists can make something look like it's leaning or being pushed. Other applications where this is useful include fluid dynamics and crystallography, where the area of a deformed object must remain constant, or in photography where perspective in photos can be adjusted.

A shear transformation is implemented by altering the basis vectors we start with. For vertical slanting,

$$\mathbf{M}_v = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

and for horizontal slanting

$$\mathbf{M}_h = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$

In both cases,  $k \in \mathbb{R}$  is called a shear factor and its sign determines the direction of the slant and its magnitude determines the severity of the slant.

### 3.1 Part a

Use the following to generate a new shape. Compare the shape and its area to your answer in Question 1.

$$\mathbf{M}_v = \begin{bmatrix} 1 & k = 0.5 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & -7 \\ 5 & 9 \end{bmatrix}$$

#### Solution

This is Matrix T.

```
1 A = matrix(data=c(1, -7, 5, 9), nrow = 2, ncol = 2, byrow = TRUE)
2 M = matrix(data=c(1, 0.5, 0, 1), nrow = 2, ncol = 2, byrow = TRUE)
3 (T = A %*% M)
```

```
      [,1] [,2]
[1,]      1 -6.5
[2,]      5 11.5
```

These are the basis vectors.

```
1 (basis.vector.1 = T[,1])
```

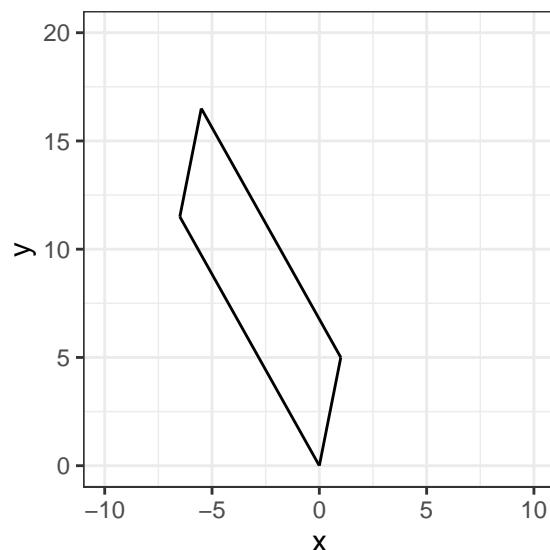
```
[1] 1 5
```

```
1 (basis.vector.2 = T[,2])
```

```
[1] -6.5 11.5
```

```
1 p0 = c(0,0)
2 p1 = basis.vector.1
3 p2 = basis.vector.1 + basis.vector.2
4 p3 = basis.vector.2
5 ggplot() +
6   geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2])) +
7   geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2])) +
8   geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2])) +
9   geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2])) +
10  theme_bw() +
11  xlim(-10, 10) +
12  ylim(0, 20) +
13  labs(x = "x", y = "y")
```

Figure 4: The Q3a Shape.



Area

```
1 det(A)
```

```
[1] 44
```

The shape is thin as it stretches vertically higher on the y-axis. The area stays the same however.

### 3.2 Part b

Use the following to generate a new shape. Compare the shape and its area to your answer in Questions 1 and the vertical shear transform in part (a).

$$\mathbf{M}_h = \begin{bmatrix} 1 & 0 \\ k = 0.5 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & -7 \\ 5 & 9 \end{bmatrix}$$

#### Solution

This is Matrix T.

```
1 A = matrix(data=c(1, -7, 5, 9), nrow = 2, ncol = 2, byrow = TRUE)
2 M = matrix(data=c(1, 0, 0.5, 1), nrow = 2, ncol = 2, byrow = TRUE)
3 (T = A %*% M)
```

```
      [,1] [,2]
[1,] -2.5  -7
[2,]  9.5   9
```

These are the basis vectors.

```
1 (basis.vector.1 = T[,1])
```

```
[1] -2.5  9.5
```

```
1 (basis.vector.2 = T[,2])
```

```
[1] -7  9
```

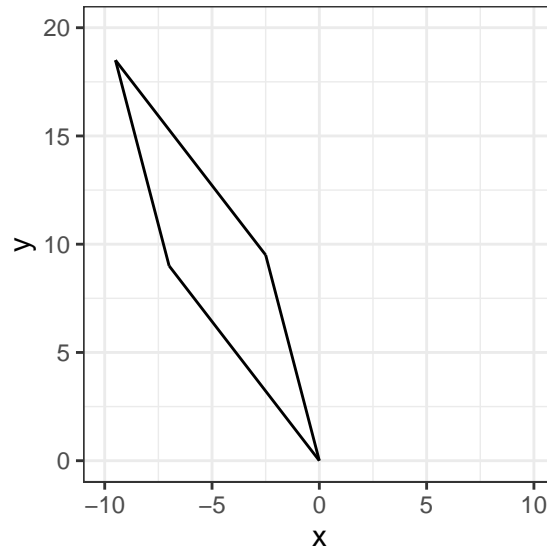
```
1 p0 = c(0,0)
2 p1 = basis.vector.1
3 p2 = basis.vector.1 + basis.vector.2
4 p3 = basis.vector.2
5 ggplot() +
```

```

6 geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2])) +
7 geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2])) +
8 geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2])) +
9 geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2])) +
10 theme_bw() +
11 xlim(-10, 10) +
12 ylim(0, 20) +
13 labs(x = "x", y = "y")

```

Figure 5: The Q3b Shape.



Area

```

1 det(A)

```

```

[1] 44

```

The shape has the same area, however, it stretches horizontally. It is clearly a different shape leaning more to the left like a rhombus.

### 3.3 Part c

Can we shear vertically and horizontally at the same time? Use the following to generate a new shape. Compare the shape and its area to your answers in parts (a) and (b).

$$\mathbf{M}_{vh} = \begin{bmatrix} 1 & k = 0.5 \\ k = 0.5 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & -7 \\ 5 & 9 \end{bmatrix}$$

#### Solution

This is Matrix T.

```

1 A = matrix(data=c(1, -7, 5, 9), nrow = 2, ncol = 2, byrow = TRUE)
2 M = matrix(data=c(1, 0.5, 0.5, 1), nrow = 2, ncol = 2, byrow = TRUE)
3 (T = A %*% M)

```

```

      [,1] [,2]
[1,] -2.5 -6.5
[2,]  9.5 11.5

```

These are the basis vectors.



```

1 (basis.vector.1 = T[,1])

[1] -2.5  9.5

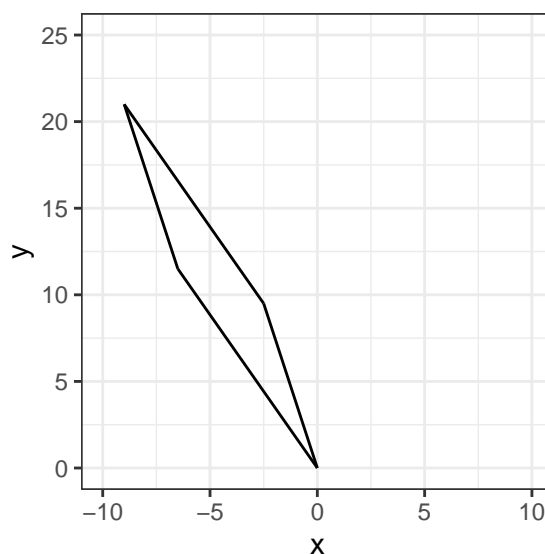
1 (basis.vector.2 = T[,2])

[1] -6.5 11.5

1 p0 = c(0,0)
2 p1 = basis.vector.1
3 p2 = basis.vector.1 + basis.vector.2
4 p3 = basis.vector.2
5 ggplot() +
6   geom_segment(aes(x = p0[1], y = p0[2], xend = p1[1], yend = p1[2])) +
7   geom_segment(aes(x = p1[1], y = p1[2], xend = p2[1], yend = p2[2])) +
8   geom_segment(aes(x = p2[1], y = p2[2], xend = p3[1], yend = p3[2])) +
9   geom_segment(aes(x = p3[1], y = p3[2], xend = p0[1], yend = p0[2])) +
10  theme_bw() +
11  xlim(-10, 10) +
12  ylim(0, 25) +
13  labs(x = "x", y = "y")

```

Figure 6: The Q3c Shape.



Area

```

1 det(A)

```

```

[1] 44

```

The shape has the same area again. However, it reaches higher on the y-axis, while also leaning horizontally further left. It has a similar rhombus shape from part c.

## 4 Question 4

### 4.1 Part a

Create a data frame `ladder` with the following data. Note your data frame should have three columns; I split it to save vertical space here.

x	y	group	x	y	group
0	0	1	1	8	6
0	20	1	0	10	7

x	y	group	x	y	group
1	0	2	1	10	7
1	20	2	0	12	8
0	2	3	1	12	8
1	2	3	0	14	9
0	4	4	1	14	9
1	4	4	0	16	10
0	6	5	1	16	10
1	6	5	0	18	11
0	8	6	1	18	11

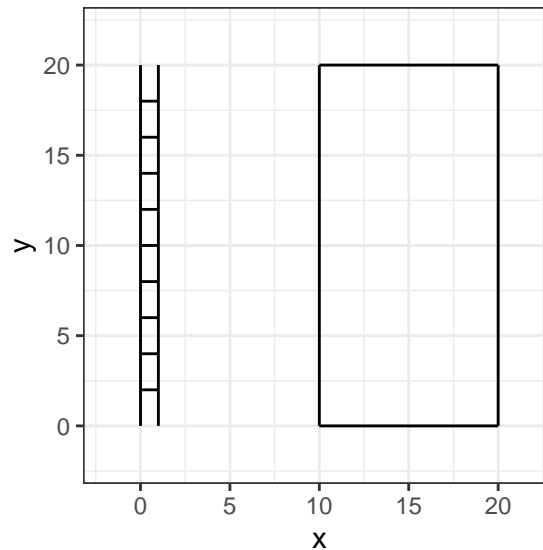
## Solution

```
1 ladder = data.frame( x = c(0,0,1,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,1), y = c(0,20,0,20,2,2,4,4,6,6,8,8,10,10,12,12,14,14,16,16))
```

## Part b Remove `#|eval: false` in the code below. The result should look like a ladder and a very rectangular house if part a is correct.

```
1 ggplot() +
2   geom_path(data = ladder,
3             aes(x = x, y = y, group = group)) +
4     geom_segment(aes(x = 10, xend = 10,
5                     y = 0, yend = 20))+
6   geom_segment(aes(x = 10, xend = 20,
7                     y = 20, yend = 20))+
8   geom_segment(aes(x = 20, xend = 20,
9                     y = 0, yend = 20))+
10  geom_segment(aes(x = 10, xend = 20,
11                  y = 0, yend = 0)) +
12  theme_bw() +
13  xlim(-2,22) +
14  ylim(-2,22)
```

Figure 7: A ladder.



## 4.2 Part c

Use `as.matrix()` to create a  $22 \times 2$  matrix  $\mathbf{A}$  containing the `x` and `y` observations from the `ladder` data frame. Use matrix multiplication to compute

$$\mathbf{T} = \mathbf{A}\mathbf{M}_h.$$

Recall,

$$\mathbf{M}_h = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

and use  $k = 0.5$ .

**Solution:** This is Matrix  $T$  after multiplying  $AM$ .

```
1 A = as.matrix(ladder[, c(1,2)])
2 M = matrix(data=c(1, 0, 0.5, 1), nrow = 2, ncol = 2, byrow = TRUE)
3 (T = A %*% M)
```

	[,1]	[,2]
[1,]	0	0
[2,]	10	20
[3,]	1	0
[4,]	11	20
[5,]	1	2
[6,]	2	2
[7,]	2	4
[8,]	3	4
[9,]	3	6
[10,]	4	6
[11,]	4	8
[12,]	5	8
[13,]	5	10
[14,]	6	10
[15,]	6	12
[16,]	7	12
[17,]	7	14
[18,]	8	14
[19,]	8	16
[20,]	9	16
[21,]	9	18
[22,]	10	18

### 4.3 Part d

Create a data frame `leaning.ladder` with the `x` and `y` equal to the first and second column of  $T$ , respectively, and the same values of `group` from `ladder`.

**Solution**

```
1 (leaning.ladder = data.frame(x = T[,1], y = T[,2], group = ladder[,3]))
```

	x	y	group
1	0	0	1
2	10	20	1
3	1	0	2
4	11	20	2
5	1	2	3
6	2	2	3
7	2	4	4
8	3	4	4
9	3	6	5
10	4	6	5
11	4	8	6
12	5	8	6
13	5	10	7
14	6	10	7
15	6	12	8
16	7	12	8
17	7	14	9
18	8	14	9

```

19 8 16    10
20 9 16    10
21 9 18    11
22 10 18   11

```

#### 4.4 Part e

Remove `eval: false` in the code below. The result should look like a ladder and a very rectangular house if part a is correct. The new leaning ladder should look like it is leaning on the house.

```

1 ggplot() +
2   geom_path(data = ladder,
3             aes(x = x, y = y, group = group, color = "Starting Ladder")) +
4   geom_path(data = leaning.ladder,
5             aes(x = x, y = y, group = group, color = "Leaning Ladder")) +
6   geom_segment(aes(x = 10, xend = 10,
7                    y=0, yend = 20))+
8   geom_segment(aes(x = 10, xend = 20,
9                    y=20, yend = 20))+
10  geom_segment(aes(x = 20, xend = 20,
11                  y=0, yend = 20))+
12  geom_segment(aes(x = 10, xend = 20,
13                  y=0, yend = 0)) +
14  theme_bw() +
15  xlim(-2,22) +
16  ylim(-2,22) +
17  scale_color_manual("", values=c("darkred", "grey"))

```

Figure 8: A ladder and a leaning ladder.

