

# Lab 08 – MATH 240 – Computational Statistics

Cristian Palmer  
Student  
Mathematics  
cpalmer@colgate.edu

## 1 Introduction

Lab 8 is a continuation of the work which we began during lab 7. In lab 7, we were tasked with computing the population moments for four distinct cases of the beta distribution and graphically comparing the different cases. In lab 8 we continued to build on our understanding of the beta distribution by modeling country death rates worldwide with the distribution. Our end goal with this lab was to be able to describe the beta distribution. Particularly, this write up aims to provide answers to questions such as: What is the beta distribution? What does it look like? What is it used for? What are its properties? And, what additional information do we gain from the simulations and real data analysis?

## 2 Density Functions and Parameters

To begin, the beta distribution's probability density function (PDF) is given by:

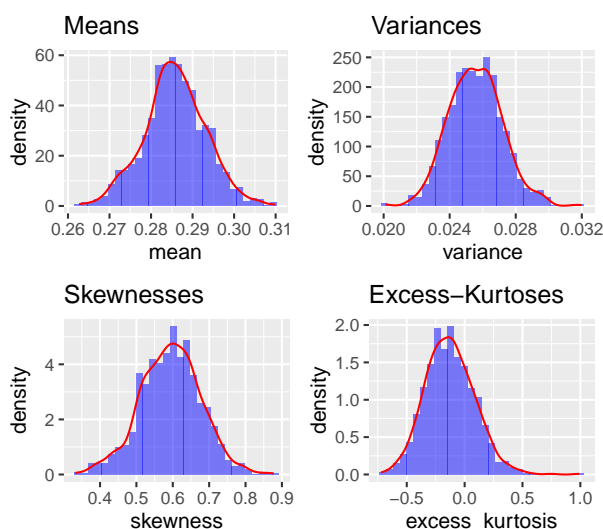
$$f_X(x | \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} I(x \in [0, 1])$$

This PDF is expressed using the gamma function and involves the random variable  $x$  along with parameters  $\alpha$  and  $\beta$ . The domain of the beta distribution is restricted to the interval  $[0, 1]$ , meaning  $0 \leq x \leq 1$ . Additionally, both shape parameters  $\alpha$  and  $\beta$  must be strictly positive for the distribution to be properly defined.

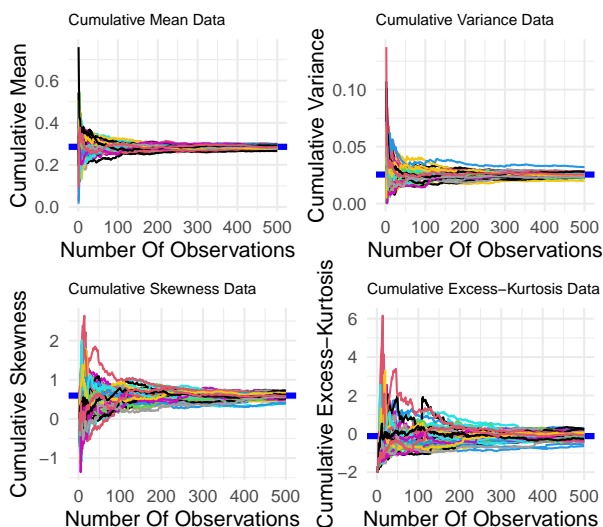
## 3 Properties

During lab 7 multiple of our tasks saw us calculating certain properties of the beta distribution for our alternate  $\alpha$  and  $\beta$  values. Since the distribution's shape is defined by its parameters, the distributions properties, notably its mean, variance, skewness, and excess kurtosis are also described by the  $\alpha$  and  $\beta$  parameters.

Pictured at the top of the second column is a 2x2 grid of histograms depicting the respective distributions and density curves for the beta distribution with an  $\alpha$  of 2, and a  $\beta$  of 5. We iterated over this specific beta distribution 1000 times, each time generating 500 values for each property which we saved to a tibble and graphed below. We utilized the `patchwork` (Pedersen, 2024) library in order to orient the four graphs in this 2x2 grid fashion.



Furthermore, the `cumstats` (Erdely and Castillo, 2017) package allowed us to compute the cumulative numerical summaries for each property. For 50 iterations of the  $\alpha = 2$  and  $\beta = 5$  beta distribution, we ran the `cumstats` functions on a sample size of 500 to generate a set of 50 line plots for each property. For each property, the plot below shows how with a large enough sample size, all the properties all average out to their true value. Plotted below is another 2x2 grid made using `patchwork`, however this grid depicts four line graphs, each showing 50 different iterations in which each property averages out to their respective true value as the sample size increases.



## 4 Estimators

In lab 8 we were tasked with computing the Method of Moments (MOM), and Maximum Likelihood (MLE) estimators for the beta distribution which we used to model country death rates worldwide in 2022. The Method of Moments estimator estimates alpha and beta in the beta distribution by equating the sample moments, such as the sample mean and variance, to the theoretical moments of the distribution. The Maximum Likelihood Estimator estimates the parameters of the Beta distribution by maximizing the Beta distribution's likelihood function, which represents the probability of observing the given data as a function of the distribution's parameters.

## 5 Example

Computing the MOM and MLE estimators gave us estimates of the true values for alpha and beta for the beta distribution which can model country death rates worldwide in 2022. The first graph in the appendix shows both the MLE estimator and the MOM estimator superimposed onto the actual data. Although each estimator slightly underestimates the peak of the distribution, they each very closely follow it and are unsurprisingly nearly identical to each other.

The final part of lab 8 tasked us with determining which estimator is better in this case, the MOM estimator or MLE estimator. To do this, we set  $\alpha = 8$  and  $\beta = 950$ , estimates determined from when we computed the MOM and MLE estimates. Setting the number of observations for each estimation to 266, we iterated through the MLE and MOM calculations 1000 times, providing us with 1000 method of moments estimates for alpha and beta, and maximum likelihood estimates for alpha and beta. The second graph in the appendix is another 2x2 grid made with `patchwork` which shows four density curves, each being attributed to either alpha or beta, and either calculated via the MOM estimator or the MLE. Looking at the graphs themselves, it is hard to determine which estimator is better in this case. Each of the alpha plots are nearly indistinguishable from each other, and so are the beta plots.

To better determine which estimator is more useful for this distribution, we calculated the bias, precision, and mean

squared error for the estimates. The table can be found as the third entry in the appendix. Looking at the table, it becomes apparent that for both alpha and beta, the MOM estimator is slightly more biased, less precise, and has a greater mean squared error. Therefore, in the case of using the beta distribution to model country death rates worldwide in 2022, the MLE estimator is a little bit better at estimating the true values of alpha and beta. This result makes sense because the MLE estimator is typically more consistent with larger sample sizes when compared to the MOM estimator. As a result, the MLE is better suited for providing more reliable estimates of alpha and beta when modeling complex data like country death rates worldwide in 2022.

## 6 Summary and Elaboration

The Beta distribution is a continuous probability distribution defined on the interval  $[0, 1]$ , characterized by two shape parameters, alpha and beta. It is often used to model random variables that represent proportions or probabilities, such as country death rates as we looked at in this lab. The distribution's shape is highly sensitive to these parameters, and it can take a variety of forms. Figure four in the appendix shows the shapes the distribution took on for the values of alpha and beta we were using during lab 7. The beta distribution's properties include mean, variance, skewness, and excess kurtosis, all of which can be derived from alpha and beta. To find alpha and beta, one can use either the MOM or MLE estimators, however we found that in the lab the MLE estimator was better. By simulating data and analyzing real-world data such as we did in this lab, we can better understand how the Beta distribution behaves under different conditions, revealing key insights about the distribution's role in modeling diverse data sets.

## References

- Erdelyi, A. and Castillo, I. (2017). *cumstats: Cumulative Descriptive Statistics*. R package version 1.0.
- Pedersen, T. L. (2024). *patchwork: The Composer of Plots*. R package version 1.3.0.
- Wickham, H., Averick, M., Bryan, J., Chang, W., McGowan, L. D., François, R., Grolemond, G., Hayes, A., Henry, L., Hester, J., Kuhn, M., Pedersen, T. L., Miller, E., Bache, S. M., Müller, K., Ooms, J., Robinson, D., Seidel, D. P., Spinu, V., Takahashi, K., Vaughan, D., Wilke, C., Woo, K., and Yutani, H. (2019). Welcome to the tidyverse. *Journal of Open Source Software*, 4(43):1686.

## 7 Appendix

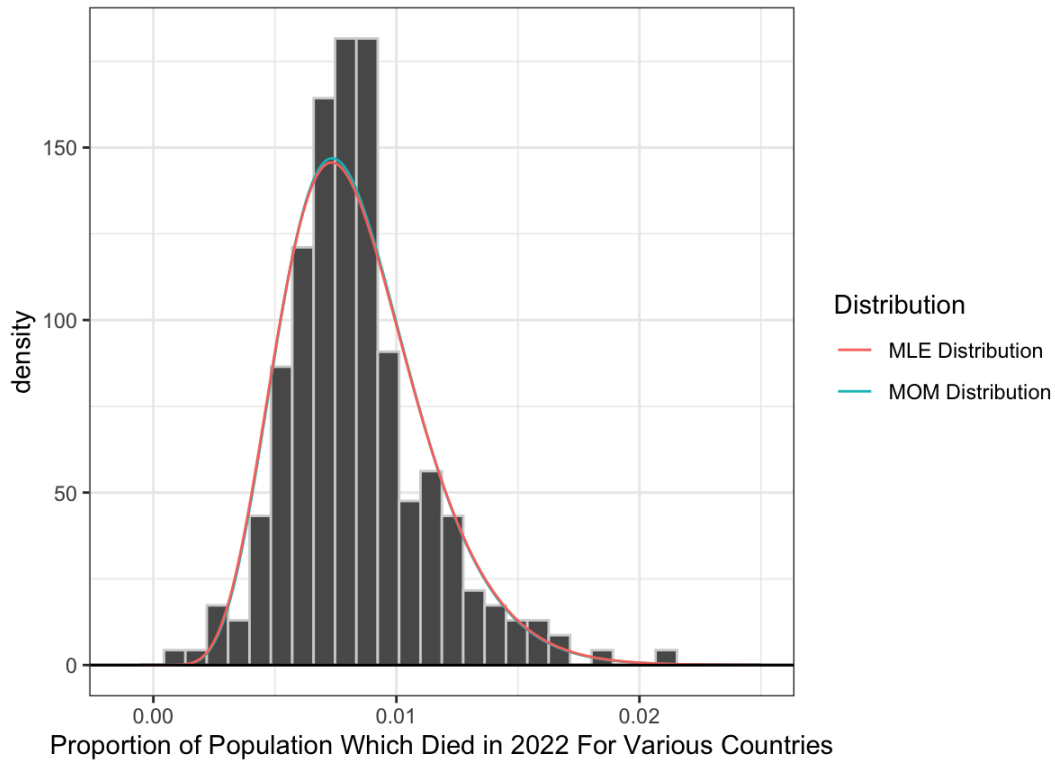


Figure 1

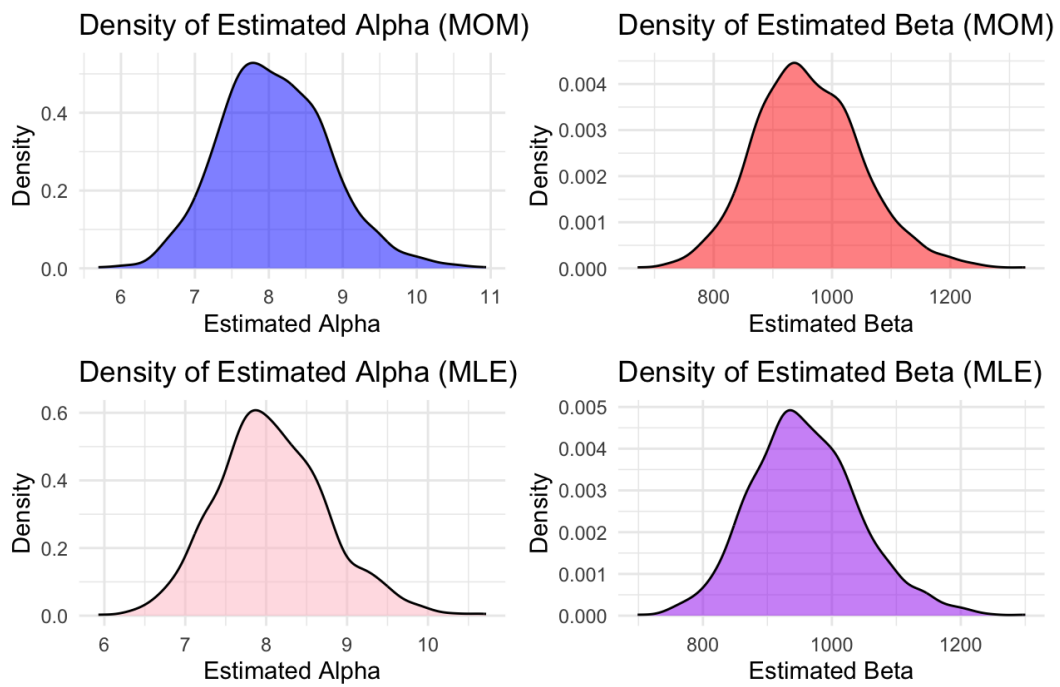


Figure 2

Table 1: Estimation Metrics

Parameter	Bias	Precision	MSE
Alpha MOM	0.08	1.83	0.55
Beta MOM	10.29	0.00	8288.46
Alpha MLE	0.07	2.13	0.48
Beta MLE	9.11	0.00	7132.70

Figure 3

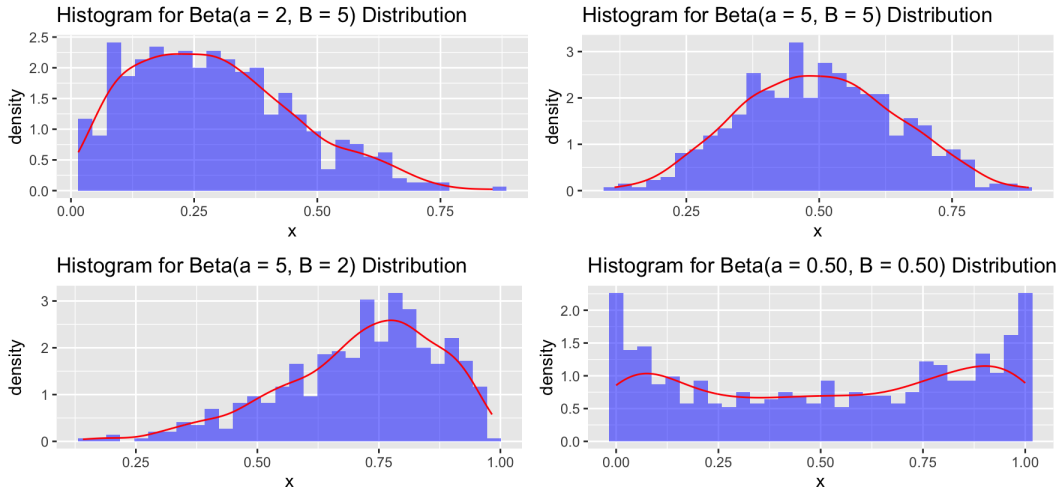


Figure 4