

1. Consider the following integral:

$$\int_a^b (7 - 2x^2)dx.$$

While this is a relatively straightforward integral – split the difference and use power rule for integration – we can approximate the area under this curve using Riemann sums. Below, I describe four rules for computing Riemann sums using rectangles of width  $\delta_x$ .

- Left Rule uses the left points to create the rectangles.

$$\text{Area} = \delta_x (f(a) + f(a + \delta_x) + f(a + 2\delta_x) + \cdots + f(b - \delta_x))$$

- Right Rule uses the right points to create the rectangles.

$$\text{Area} = \delta_x (f(a + \delta_x) + f(a + 2\delta_x) + \cdots + f(b - \delta_x) + f(b))$$

- Midpoint Rule uses the midpoints between the left and right points to create the rectangles.

$$\text{Area} = \delta_x \left( f\left(a + \frac{\delta_x}{2}\right) + f\left(a + \frac{3\delta_x}{2}\right) + \cdots + f\left(b - \frac{\delta_x}{2}\right) \right)$$

- Trapezoidal Rule averages the rectangles created using left and right endpoints, which results in areas of trapezoids.

$$\text{Area} = \frac{1}{2} \delta_x (f(a) + 2f(a + \delta_x) + 2f(a + 2\delta_x) + \cdots + f(b))$$

The first step, is to create a function that computes the integrand.

```
integrand <- function(x){
  f <- 7 - 2 * x^2
  return(f)
}
```

Next, I set up an example choice of  $a$ ,  $b$ , and the number of rectangles.

```
a <- 0
b <- 2
n.rect <- 100
(delta.x <- (b-a)/n.rect)

## [1] 0.02
```

We can compute the area using the Left Rule as follows.

```
left.points <- a + 0:99*(delta.x)
(left.area <- sum(delta.x*(integrand(left.points))))

## [1] 8.7464
```

We can compute the area using the Right Rule as follows.

```
right.points <- a + 1:100*(delta.x)
(right.area <- sum(delta.x*(integrand(right.points))))

## [1] 8.5864
```

We can compute the area using the Midpoint Rule as follows.

```
mid.points <- (left.points+right.points)/2
(mid.area <- sum(delta.x*(integrand(mid.points))))

## [1] 8.6668
```

- (a) Write code that computes the area using the Trapezoidal Rule.

**Solution:**

```
trapezoid.middle <- left.points[2:100]

(total <- (sum(2*(integrand(trapezoid.middle))) + (integrand(left.points[1])) + (integrand(right.points[100]))))
  *delta.x /2)

## [1] 8.6664
```

- (b) Write a function that takes `a`, `b`, and number of rectangles (`n.rect`) in as input and returns the Trapezoidal Rule by default, but can return left, right, or midpoint when necessary. Use the skeleton below. When you are done, remove `eval=FALSE` to show that the function provides the expected result for the example above.

**Solution:**

```
riemann.sums <- function(fnct,                # function to integrate
  a,                                          # lower bound of integral
  b,                                          # upper bound of integral
  n.rect,                                    # number of bound of integral
  method = "Trapezoidal"){                  # method to use (trap by default)

  #####
  # Check Input
  #####
  if(!is.numeric(a)){ # if a is not numeric
    stop("The lower bound of the integral (a) must be numeric.")
  }
  if(!is.numeric(b)){ # if b is not numeric
    stop("The lower bound of the integral (a) must be numeric.")
  }
  if(!(is.numeric(n.rect)) | (n.rect%%1!=0)){ # if n.rect is not a whole number
    stop("The number of rectangles must be a positive whole number.")
  }
  if(a > b){
    stop("bounds invalid")
  }
  #####
  # Compute Area
  #####
  delta.x <- (b-a)/n.rect
  left.sides <- a + 0:(n.rect-1)*(delta.x)
  right.sides <- a + 1:(n.rect)*(delta.x)

  if(method == "Left"){
    area <- sum(delta.x*(integrand(left.sides)))
  }else if(method == "Right"){
    area <- sum(delta.x*(integrand(right.sides)))
  }else if(method == "Midpoint"){
    area <- sum(delta.x*(integrand((left.points+right.points)/2)))
  }else if(method == "Trapezoidal"){
    trapezoid.middle <- left.sides[2:100]
    area <- (sum(2*(integrand(trapezoid.middle))) + (integrand(left.sides[1])) + (integrand(right.sides[100])))*delta.x /2
  }else{
    stop("Please select a valid method (e.g., 'Left', 'Right', 'Midpoint', 'Trapezoidal')")
  }
  #####
  # Return the area
  #####
  return(area)
}

#####
# Test the function
#####
riemann.sums(fnct = integrand,
  a = 0,
  b = 2,
  n.rect = 100)

## [1] 8.6664

#####
# Compare to numerical integral
#####
integrate(f = integrand, # integrate() is an R function
  lower = 0,             # that completes numerical
  upper = 2)             # integration

## 8.666667 with absolute error < 9.8e-14
```