

1. Consider the following integral:

$$\int_a^b (7 - x^2) dx.$$

While this is a relatively straightforward integral – split the difference and use power rule for integration – we can approximate the area under this curve using Riemann sums. Below, I describe four rules for computing Riemann sums using rectangles of width δ_x .

- Left Rule uses the left points to create the rectangles.

$$\text{Area} = \delta_x (f(a) + f(a + \delta_x) + f(a + 2\delta_x) + \cdots + f(b - \delta_x))$$

- Right Rule uses the right points to create the rectangles.

$$\text{Area} = \delta_x (f(a + \delta_x) + f(a + 2\delta_x) + \cdots + f(b - \delta_x) + f(b))$$

- Midpoint Rule uses the midpoints between the left and right points to create the rectangles.

$$\text{Area} = \delta_x \left(f\left(a + \frac{\delta_x}{2}\right) + f\left(a + \frac{3\delta_x}{2}\right) + \cdots + f\left(b - \frac{\delta_x}{2}\right) \right)$$

- Trapezoidal Rule averages the rectangles created using left and right endpoints, which results in areas of trapezoids.

$$\text{Area} = \frac{1}{2} \delta_x (f(a) + 2f(a + \delta_x) + 2f(a + 2\delta_x) + \cdots + f(b))$$

The first step, is to create a function that computes the integrand.

```
integrand <- function(x){
  f <- 7 - 2 * x^2
  return(f)
}
```

Next, I set up an example choice of a , b , and the number of rectangles.

```
a <- 0
b <- 2
n.rect <- 100
(delta.x <- (b-a)/n.rect)

## [1] 0.02
```

We can compute the area using the Left Rule as follows.

```
left.points <- a + 0:99*(delta.x)
(left.area <- sum(delta.x*(integrand(left.points))))

## [1] 8.7464
```

We can compute the area using the Right Rule as follows.

```
right.points <- a + 1:100*(delta.x)
(right.area <- sum(delta.x*(integrand(right.points))))

## [1] 8.5864
```

We can compute the area using the Midpoint Rule as follows.

```
mid.points <- (left.points+right.points)/2
(mid.area <- sum(delta.x*(integrand(mid.points))))

## [1] 8.6668
```

- (a) Write code that computes the area using the Trapezoidal Rule.

Solution:

```
trap.midpoints = a + 1:99*(delta.x) #all the points excluding the two endpoints a and b
trap.area = sum(0.5 * delta.x * (integrand(a) + 2 * sum(integrand(trap.midpoints)) + integrand(b))) #trapezoidal rule formula
```

- (b) Write a function that takes **a**, **b**, and number of rectangles (**n.rect**) in as input and returns the Trapezoidal Rule by default, but can return left, right, or midpoint when necessary. Use the skeleton below. When you are done, remove **eval=FALSE** to show that the function provides the expected result for the example above.

Solution:

```
riemann.sums <- function(fnct,                # function to integrate
                        a,                    # lower bound of integral
                        b,                    # upper bound of integral
                        n.rect,               # number of bound of integral
                        method = "Trapezoidal"){ # method to use (trap by default)

  #####
  # Check Input
  #####
  if(!is.numeric(a)){ # if a is not numeric
    stop("The lower bound of the integral (a) must be numeric.")
  }
  if(!is.numeric(b)){ # if b is not numeric
    stop("The lower bound of the integral (a) must be numeric.")
  }
  if(!is.numeric(n.rect) | (n.rect%%1!=0)){ # if n.rect is not a whole number
    stop("The number of rectangles must be a positive whole number.")
  }
  #####
  # Compute Area
  #####
  if(method == "Left"){
    (delta.x <- (b-a)/n.rect)
    left.points <- a + 0:(n.rect-1)*(delta.x)
    (area <- sum(delta.x*(fnct(left.points))))
  }else if(method == "Right"){
    (delta.x <- (b-a)/n.rect)
    right.points <- a + 1:(n.rect)*(delta.x)
    (area <- sum(delta.x*(fnct(right.points))))
  }else if(method == "Midpoint"){
    (delta.x <- (b-a)/n.rect)
    left.points <- a + 0:(n.rect-1)*(delta.x)
    right.points <- a + 1:(n.rect)*(delta.x)
    mid.points <- (left.points+right.points)/2
    (area <- sum(delta.x*(fnct(mid.points))))
  }else if(method == "Trapezoidal"){
    (delta.x <- (b-a)/n.rect)
    trap.midpoints = a + 1:(n.rect-1)*(delta.x) #all the points excluding the two endpoints a and b
    area = sum(0.5 * delta.x * (fnct(a) + 2 * sum(fnct(trap.midpoints)) + fnct(b))) #formula for trapezoidal rule
  }else{
    stop("Please select a valid method (e.g., 'Left', 'Right', 'Midpoint', 'Trapezoidal')")
  }
  #####
  # Return the area
  #####
  return(area)
}

#####
# Test the function
#####
riemann.sums(fnct = integrand,
             a = 0,
             b = 2,
             n.rect = 100,
             method="Trapezoidal")

## [1] 8.6664

#####
# Compare to numerical integral
#####
integrate(f = integrand, # integrate() is an R function
         lower = 0,      # that completes numerical
         upper = 2)      # integration

## 8.666667 with absolute error < 9.8e-14
```