1. Consider the following integral:

$$\int_a^b (7 - x^2)dx.$$

While this is a relatively straightforward integral – split the difference and use power rule for integration – we can approximate the area under this curve using Riemann sums. Below, I describe four rules for computing Riemann sums using rectangles of width δ_x.

- Left Rule uses the left points to create the rectangles.

$$\text{Area} = \delta_x(f(a) + f(a + \delta_x) + f(a + 2\delta_x) + \cdots + f(b - \delta_x))$$

- Right Rule uses the right points to create the rectangles.

$$\text{Area} = \delta_x(f(a + \delta_x) + f(a + 2\delta_x) + \cdots + f(b - \delta_x) + f(b))$$

- Midpoint Rule uses the midpoints between the left and right points to create the rectangles.

$$\text{Area} = \delta_x \left(f\left(a + \frac{\delta_x}{2}\right) + f\left(a + \frac{3\delta_x}{2}\right) + \cdots + f\left(b - \frac{\delta_x}{2}\right) \right)$$

- Trapezoidial Rule averages the rectangles created using left and right endpoints, which results in areas of trapezoids.

$$\text{Area} = \frac{1}{2}\delta_x \left(f(a) + 2f(a + \delta_x) + 2f(a + 2\delta_x) + \cdots + f(b)\right)$$

The first step, is to create a function that computes the integrand.

```
integrand <- function(x){
  f <- 7 - 2 * x^2
  return(f)
}
```

Next, I set up an example choice of a, b, and the number of rectangles.

```
a <- 0
b <- 2
n.rect <- 100
(delta.x <- (b-a)/n.rect)


## [1] 0.02
```

We can compute the area using the Left Rule as follows.

```
left.points <- a + 0:99*(delta.x)
(left.area <- sum(delta.x*(integrand(left.points))))


## [1] 8.7464
```

We can compute the area using the Right Rule as follows.

```
right.points <- a + 1:100*(delta.x)
(right.area <- sum(delta.x*(integrand(right.points))))


## [1] 8.5864
```

We can compute the area using the Midpoint Rule as follows.

```
mid.points <- (left.points+right.points)/2
(mid.area <- sum(delta.x*(integrand(mid.points))))


## [1] 8.6668
```

(a) Write code that computes the area using the Trapezoidial Rule.
Solution:

```
trap.points <- a + 1:99*(delta.x)
(trap.area <- 1/2*delta.x*(integrand(a)) + sum(1/2*delta.x*(2*integrand(trap.points))) + 1/2*delta.x*(integrand(b)))

## [1] 8.6664
```

(b) Write a function that takes `a`, `b`, and number of rectangles (`n.rect`) in as input and returns the Trapezoidial Rule by default, but can return left, right, or midpoint when necessary. Use the skeleton below. When you are done, remove `eval=FALSE` to show that the function provides the expected result for the example above.
Solution:

```
riemann.sums <- function(fnct,                          # function to integrate
                         a,                             # lower bound of integral
                         b,                             # upper bound of integral
                         n.rect,                        # number of  bound of integral
                         method = "Trapezoidial"){      # method to use (trap by default)
  ######################################
  # Check Input
  ######################################
  if(!is.numeric(a)){ # if a is not numeric
    stop("The lower bound of the integral (a) must be numeric.")
  }
  if(!is.numeric(b)){ # if b is not numeric
    stop("The lower bound of the integral (a) must be numeric.")
  }
  if(!(is.numeric(n.rect)) | (n.rect%%1!=0)){ # if n.rect is not a whole number
    stop("The number of rectangles must be a positive whole number.")
  }
  ######################################
  # Compute Area
  ######################################
  area <- 0
  delta.x <- (b-a)/n.rect
  end.point <- (n.rect-1)
  if(method == "Left"){
    left.points <- a + 0:end.point*(delta.x)
    area <- sum(delta.x*(fnct(left.points)))
  }else if(method == "Right"){
    right.points <- a + 1:n.rect*(delta.x)
    area <- sum(delta.x*(fnct(right.points)))
  }else if(method == "Midpoint"){
    mid.points <- (left.points+right.points)/2
    area <- sum(delta.x*(fnct(mid.points)))
  }else if(method == "Trapezoidial"){
    trap.points <- a + 1:end.point*(delta.x)
    area <- 1/2*delta.x*(fnct(a)) + sum(1/2*delta.x*(2*fnct(trap.points))) + 1/2*delta.x*(fnct(b))
  }else{
    stop("Please select a valid method (e.g., 'Left', 'Right', 'Midpoint', 'Trapezoidial')")
  }
  ######################################
  # Return the area
  ######################################
  return(area)
}
######################################
# Test the function
######################################
riemann.sums(fnct = integrand,
             a = 0,
             b = 2,
             n.rect = 100)

## [1] 8.6664

######################################
# Compare to numerical integral
######################################
integrate(f = integrand, # integrate() is an R function
          lower = 0,     # that completes numerical
          upper = 2)     # integration

## 8.666667 with absolute error < 9.8e-14
```