1. This week's Problem of the Week in Math is described as follows:

    > *There are thirty positive integers less than 100 that share a certain property. Your friend, Blake, wrote them down in the table to the left. But Blake made a mistake! One of the numbers listed is wrong and should be replaced with another. Which number is incorrect, what should it be replaced with, and why?*

    The numbers are listed below.

6	10	14	15	21
22	26	33	34	35
38	39	46	51	55
57	58	62	65	69
75	77	82	85	86
87	91	93	94	95

Use the fact that the "certain" property is that these numbers are all supposed to be the product of *unique* prime numbers to find and fix the mistake that Blake made.
Reminder: Code your solution in an `R` script and copy it over to this `.Rnw` file.
Hint: You may find the `%in%` operator and the `setdiff()` function to be helpful.

Solution:

```
# install the package and load the library for numbers
# provides number-theoretic functions for factorization, prime numbers, etc.
library(numbers)

# define the numbers given from Blake
blake_numbers <- c(6, 10, 14, 15, 21, 22, 26, 33, 34, 35, 38, 39, 46, 51, 55,
                   57, 58, 62, 65, 69, 75, 77, 82, 85, 86, 87, 91, 93, 94, 95)

# check if the number is a product of unique prime numbers
# return true if
is_product_of_unique_primes <- function(n) {
  factors <- primeFactors(n)
  return(length(factors) == 2 && length(factors) == length(unique(factors)))
}

# find valid unique prime products less than 100 (numbers 1-99)
# sapply(1:99, is_product_of_unique_primes) returns a logical vector
# which extracts the indicies where the function returns TRUE (valid #'s)
# the result is a vector of numbers that are products of unique prime factors

valid_numbers <- which(sapply(1:99, is_product_of_unique_primes))

# identify the incorrect number
(incorrect_number <- setdiff(blake_numbers,valid_numbers))

## [1] 75

# identify missing numbers
(missing_number <- setdiff(valid_numbers, blake_numbers))

## [1] 74
```

Blake's list of numbers is supposed to contain only numbers that are products of unique prime factors (i.e., products of two distinct prime numbers). However, Blake made a mistake by including an incorrect number in the list. Our goal is to identify which number is incorrect and find the correct number to replace it. To do this, we implemented four key steps. First, we defined a function to check if a number is a product of unique primes. We used the `primeFactors(n)` function from the `numbers` package (Borchers, 2022) to find the prime factors of a given number. The function `is_product_of_unique_primes` ensures that the number has exactly two prime factors and that the factors are distinct. Next, we generated a list of valid numbers that meet the criteria. Using `sapply(1:99, is_product_of_unique_primes)`, we applied the function to numbers 1 to 99 to see which satisfy our conditions. The `which()` function extracted the numbers where the function returned `TRUE`, giving us our list of `valid_numbers`. Next, we found the incorrect number in Blake's list. We compared `blake_numbers` to `valid_numbers` using `setdiff()`. This found which number was in Blake's list but not in the valid numbers, meaning that this was the incorrect number. Finally, we found the missing number that should be in the list. This was similar to finding the incorrect number, but now we used `setdiff(valid_numbers, blake_numbers)` to determine which number was in the valid list and not Blake's, meaning this was the missing number. Through this analysis, we found that the incorrect number was 75 and the missing number was 74. This makes sense, because the prime factorization of 75 is $3 \times 5 \times 5$, which is not the product of *distinct* prime numbers. On the other hand, the prime factorization of 74 is 2×37, which is a product of distinct prime numbers and fits in the list between 69 and 77.

References

Borchers, H. W. (2022). *numbers: Number-Theoretic Functions.* R package version 0.8-5.