Introduction: Fibonacci Numbers

Daniel Kane

Department of Computer Science and Engineering
University of California, San Diego

Algorithmic Design and Techniques
Algorithms and Data Structures

Learning Objectives

- Understand the definition of the Fibonacci numbers.
- Show that the naive algorithm for computing them is slow.
- Efficiently compute large Fibonacci numbers.

Outline

1. Problem Overview
2. Naive Algorithm
3. Efficient Algorithm

Definition

$$F_n = \begin{cases} 0, & n = 0\,, \\ 1, & n = 1\,, \\ F_{n-1} + F_{n-2}, & n > 1\,. \end{cases}$$

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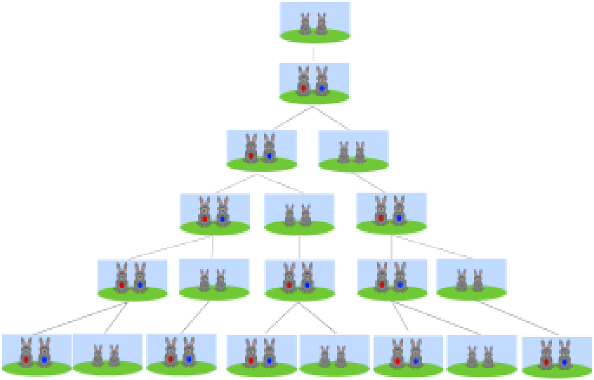
$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Developed to Study Rabbit Populations



Rapid Growth

Lemma

$F_n \geq 2^{n/2}$ for $n \geq 6$.

Rapid Growth

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Proof

By induction

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Base case: $n = 6, 7$ (by direct computation).

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Base case: $n = 6, 7$ (by direct computation).

Inductive step:

$$F_n = F_{n-1} + F_{n-2} \geq 2^{(n-1)/2} + 2^{(n-2)/2} \geq 2 \cdot 2^{(n-2)/2} = 2^{n/2}. \qquad \square$$

Formula

Theorem

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

Example

$$F_{20} = 6765$$

Example

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\begin{aligned}
F_{20} &= 6765 \\
F_{50} &= 12586269025
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Computing Fibonacci numbers

Compute F_n

Input: An integer $n \geq 0$.

Output: F_n.

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Algorithm

FibRecurs(n)

```
if n ≤ 1:
  return n
```

Algorithm

FibRecurs(n)

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if n ≤ 1:
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  return FibRecurs(n − 1) + FibRecurs(n − 2)
```

Running time

Let $T(n)$ denote the number of lines of code executed by `FibRecurs`(n).

If $n \leq 1$

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if n ≤ 1:
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$T(n) = 2$.

If $n \geq 2$

FibRecurs(n)

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if n ≤ 1:
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else:
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If $n \geq 2$

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$T(n) = 3$

If $n \geq 2$

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$$T(n) = 3 + T(n-1) + T(n-2).$$

Running Time

$$T(n) = \begin{cases} 2 & \text{if } n \leq 1 \\ T(n-1) + T(n-2) + 3 & \text{else.} \end{cases}$$

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$T(100) \approx 1.77 \cdot 10^{21}$ (1.77 sextillion)

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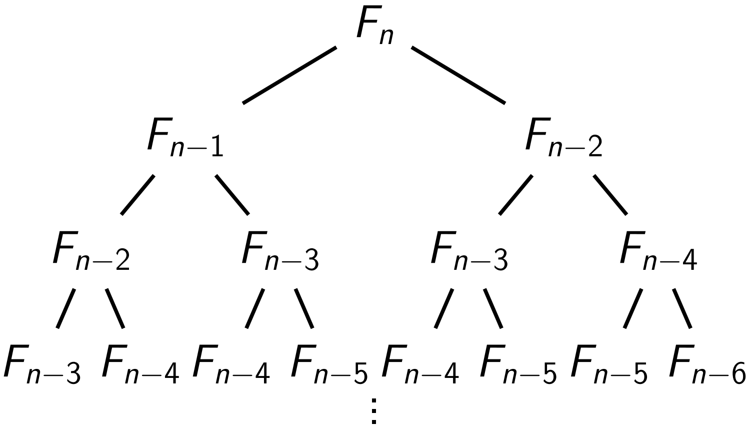
Takes 56,000 years at 1GHz.

Why so slow?



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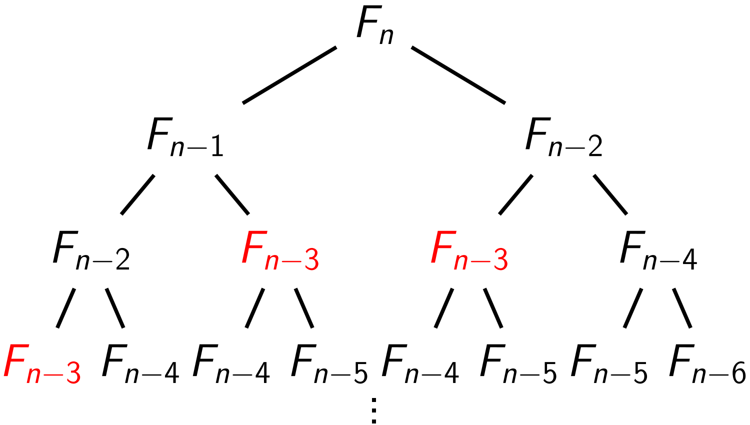

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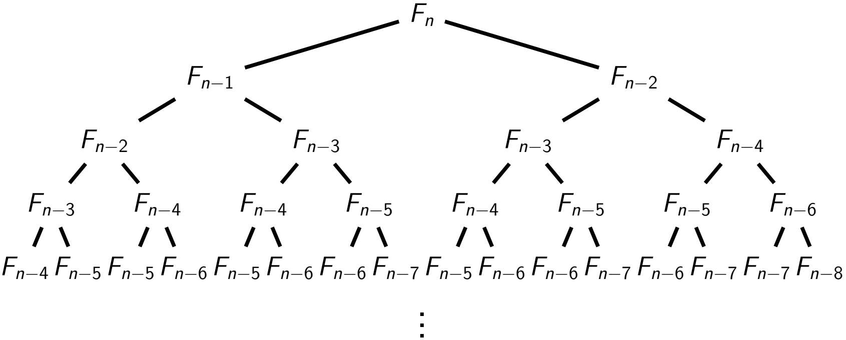
F_n
F_{n-1}
F_{n-2}
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F_{n-4}
F_{n-5}
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$\vdots$

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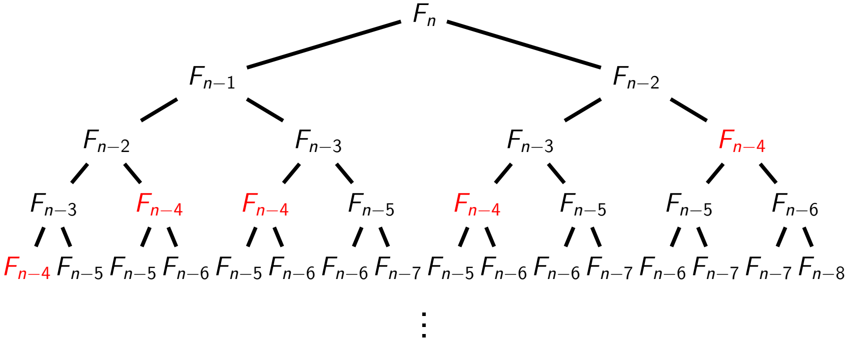

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Outline

1. Problem Overview
2. Naive Algorithm
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Another Algorithm

Imitate hand computation:

0, 1

Another Algorithm

Imitate hand computation:

0, 1, 1

$$0 + 1 = 1$$

Another Algorithm

Imitate hand computation:

0, 1, 1, 2

$$0 + 1 = 1$$
$$1 + 1 = 2$$

Another Algorithm

Imitate hand computation:

0, 1, 1, 2, 3

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Another Algorithm

Imitate hand computation:

0, 1, 1, 2, 3, 5

$0 + 1 = 1$

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$1 + 2 = 3$

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Imitate hand computation:

0, 1, 1, 2, 3, 5, 8

$0 + 1 = 1$

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$2 + 3 = 5$

$3 + 5 = 8$

New Algorithm

FibList(n)

```
create an array F[0...n]
F[0] ← 0
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for i from 2 to n:
    F[i] ← F[i − 1] + F[i − 2]
return F[n]
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- $T(n) = 2n + 2$. So $T(100) = 202$.
- Easy to compute.

Summary

- Introduced Fibonacci numbers.
- Naive algorithm takes ridiculously long time on small examples.
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Introduction: Fibonacci Numbers I

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Algorithmic Design and Techniques
Algorithms and Data Structures

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- Understand the definition of the Fibonacci numbers.
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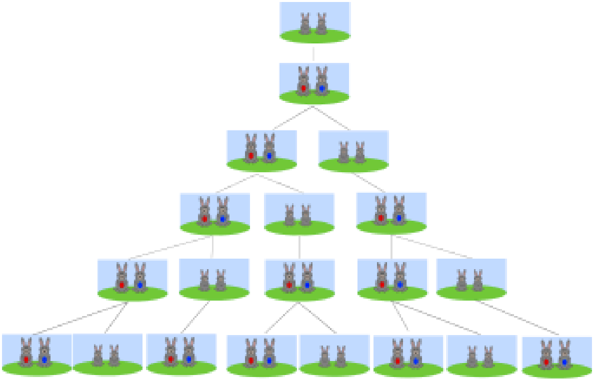

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Computing Fibonacci numbers

Compute F_n

Input: An integer $n \geq 0$.

Output: F_n.

Introduction: Fibonacci Numbers II

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Algorithmic Design and Techniques
Algorithms and Data Structures

Learning Objectives

- Produce a simple algorithm to compute Fibonacci numbers.
- Show that this algorithm is very slow.

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Grows rapidly.

Computing Fibonacci numbers

Compute F_n

Input: An integer $n \geq 0$.

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Algorithm

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$T(n) = 2.$

If $n \geq 2$

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If $n \geq 2$

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FibRecurs(n)

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$T(n) = 3$

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$$T(n) = 3 + T(n-1) + T(n-2).$$

Running Time

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Takes 56,000 years at 1GHz.

Why so slow?

$$F_n$$

$$F_{n-1} \qquad F_{n-2}$$

$$F_{n-2} \quad F_{n-3} \quad F_{n-3} \quad F_{n-4}$$

$$F_{n-3} \; F_{n-4} \; F_{n-4} \; F_{n-5} \; F_{n-4} \; F_{n-5} \; F_{n-5} \; F_{n-6}$$

$$\vdots$$

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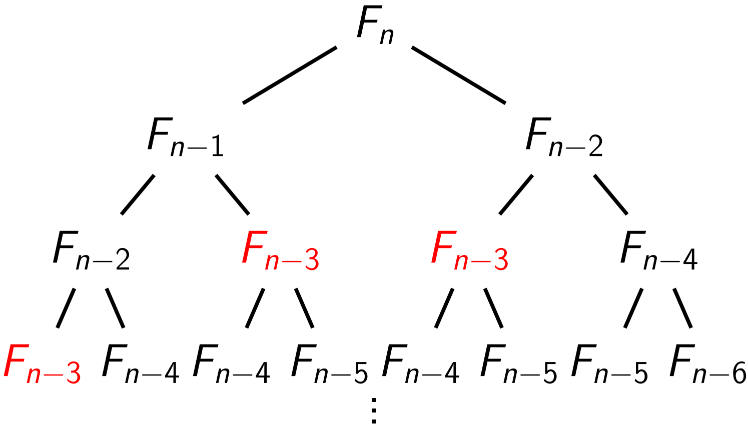

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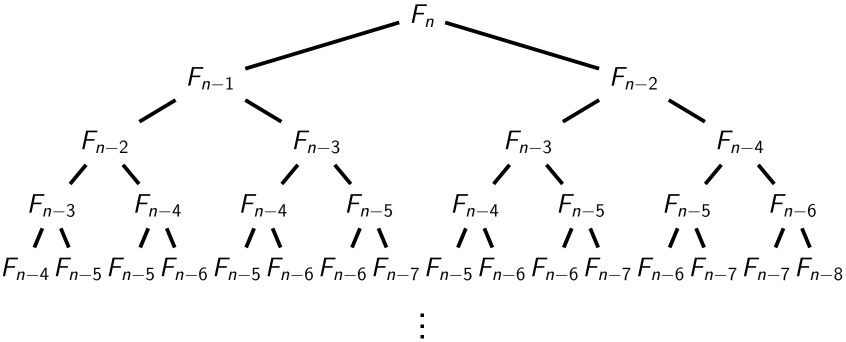
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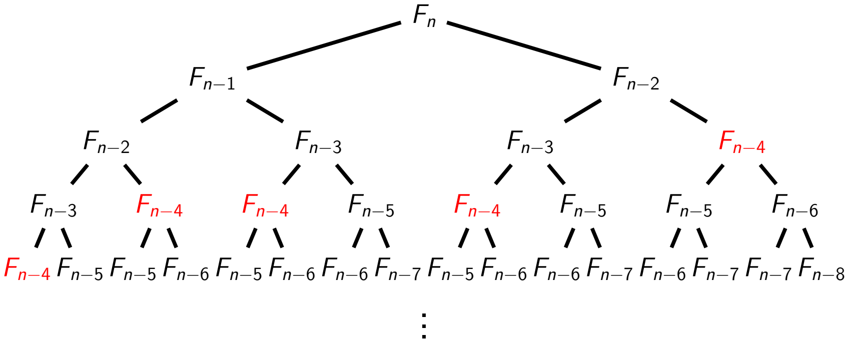
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Introduction: Fibonacci Numbers III

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Algorithmic Design and Techniques
Algorithms and Data Structures

Learning Objectives

- Compute Fibonacci numbers efficiently.

Definition

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Algorithm

FibRecurs(n)

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```

Too slow!

Another Algorithm

Imitate hand computation:

0, 1

Another Algorithm

Imitate hand computation:

0, 1, 1

$$0 + 1 = 1$$

Another Algorithm

Imitate hand computation:

0, 1, 1, 2

$0 + 1 = 1$

$1 + 1 = 2$

Another Algorithm

Imitate hand computation:

0, 1, 1, 2, 3

$$0 + 1 = 1$$
$$1 + 1 = 2$$
$$1 + 2 = 3$$

Another Algorithm

Imitate hand computation:

0, 1, 1, 2, 3, 5

$0 + 1 = 1$

$1 + 1 = 2$

$1 + 2 = 3$

$2 + 3 = 5$

Another Algorithm

Imitate hand computation:

0, 1, 1, 2, 3, 5, 8

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New Algorithm

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