Introduction: Computing Runtimes

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Algorithmic Design and Techniques
Algorithms and Data Structures

Learning Objectives

- Describe some of the issues involved with computing the runtime of an actual program.
- Understand why finding exact runtimes is a problem.

Outline

1. Revisit Fibonacci

2. Other Things to Consider

Runtime Analysis

Function FibList(n)

```
create an array F[0...n]
F[0] ← 0
F[1] ← 1
for i from 2 to n:
    F[i] ← F[i-1] + F[i-2]
return F[n]
```

Runtime Analysis

Function FibList(n)

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create an array F[0...n]
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for i from 2 to n:
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```

$2n + 2$ lines of code. Does this really describe the runtime of the algorithm?

Individual Lines

```
Function FibList(n)
create an array F[0...n]
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```

Depends on memory management system.

Individual Lines

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Assignment.

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Increment, comparison, branch.

Individual Lines

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Lookup, assignment, addition of big integers.

Individual Lines

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```

Lookup, return.

Outline

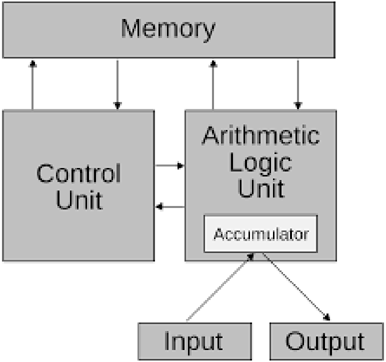
1. Revisit Fibonacci
2. Other Things to Consider

Computing Runtime

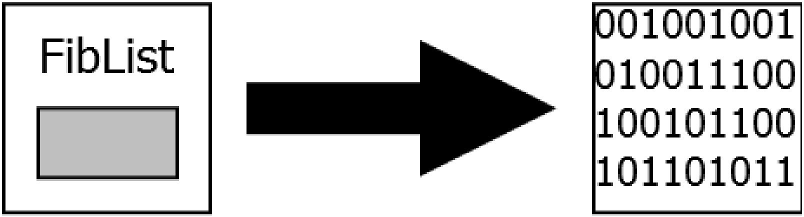
To figure out how long this simple program would actually take to run on a real computer, we would also need to know things like:

Speed of the Computer

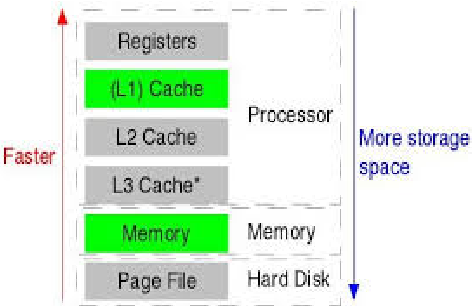
The System Architecture



The Compiler Being Used



Details of the Memory Hierarchy



Problem

- Figuring out accurate runtime is a huge mess

Problem

- Figuring out accurate runtime is a huge mess
- In practice, you might not even know some of these details

Goal

Want to:

- Measure runtime without knowing these details.
- Get results that work for large inputs.

Introduction: Asymptotic Notation

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Algorithms and Data Structures

Learning Objectives

- Understand the basic idea behind asymptotic runtimes.
- Describe some of the advantages to using asymptotic runtimes.

Last Time

Computing Runtimes Hard

- Depends on fine details of program.
- Depends on details of computer.

Idea

All of these issues can multiply runtimes by (large) constant.

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All of these issues can multiply runtimes by (large) constant. So measure runtime in a way that ignores constant multiples.

Problem

Unfortunately, 1 second, 1 hour, 1 year only differ by constant multiples.

Solution

Consider asymptotic runtimes. How does runtime scale with input size.

Approximate Runtimes

	n	$n \log n$	n^2	2^n
$n = 20$	1 sec	1 sec	1 sec	1 sec
$n = 50$	1 sec	1 sec	1 sec	13 day
$n = 10^2$	1 sec	1 sec	1 sec	$4 \cdot 10^{13}$ year
$n = 10^6$	1 sec	1 sec	17 min	
$n = 10^9$	1 sec	30 sec	30 year	
max n	10^9	$10^{7.5}$	$10^{4.5}$	30

$$\log n \prec \sqrt{n} \prec n \prec n \log n \prec n^2 \prec 2^n$$

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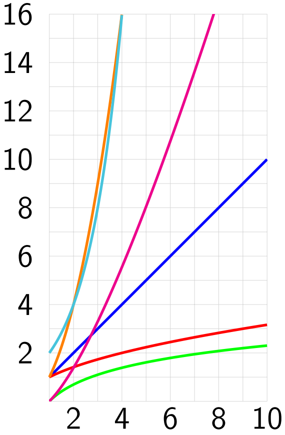

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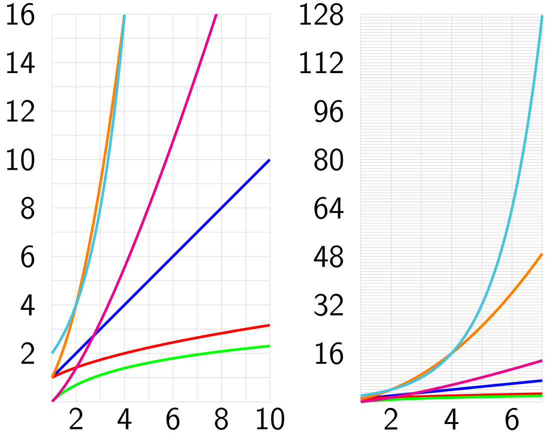

Introduction: Big-O Notation

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Algorithmic Design and Techniques
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Learning Objectives

- Understand the meaning of Big-O notation.
- Describe some of the advantages and disadvantages of using Big-O notation.

Big-O Notation

Definition

$f(n) = O(g(n))$ (f is Big-O of g) or $f \preceq g$ if there exist constants N and c so that for all $n \geq N$, $f(n) \leq c \cdot g(n)$.

Big-O Notation

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f is bounded above by some constant multiple of g.

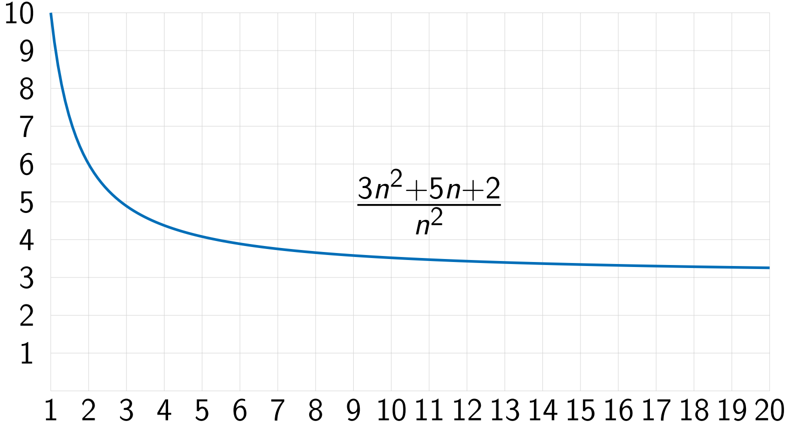
Big-O Notation

Example

$3n^2 + 5n + 2 = O(n^2)$ since if $n \geq 1$,
$3n^2 + 5n + 2 \leq 3n^2 + 5n^2 + 2n^2 = 10n^2$.

Growth Rate

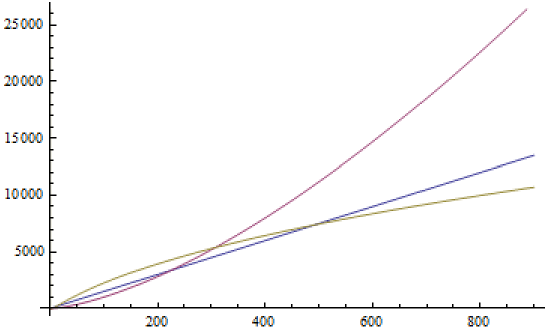
$3n^2 + 5n + 2$ has the same growth rate as n^2



Using Big-O

We will use Big-O notation to report algorithm runtimes. This has several advantages.

Clarifies Growth Rate



Cleans up Notation

- $O(n^2)$ vs. $3n^2 + 5n + 2$.
- $O(n)$ vs. $n + \log_2(n) + \sin(n)$.

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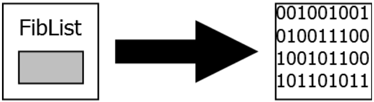
- $O(n^2)$ vs. $3n^2 + 5n + 2$.
- $O(n)$ vs. $n + \log_2(n) + \sin(n)$.
- $O(n \log(n))$ vs. $4n \log_2(n) + 7$.
    - Note: $\log_2(n), \log_3(n), \log_x(n)$ differ by constant multiples, don't need to specify which.

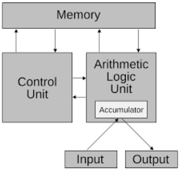
Cleans up Notation

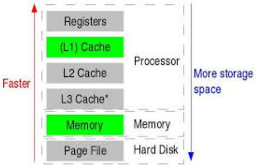
- $O(n^2)$ vs. $3n^2 + 5n + 2$.
- $O(n)$ vs. $n + \log_2(n) + \sin(n)$.
- $O(n \log(n))$ vs. $4n \log_2(n) + 7$.
  - Note: $\log_2(n), \log_3(n), \log_x(n)$ differ by constant multiples, don't need to specify which.
- Makes algebra easier.

Can Ignore Complicated Details

No longer need to worry about:







Warning

- Using Big-O loses important information about constant multiples.
- Big-O is *only* asymptotic.

Introduction: Using Big-O

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Learning Objectives

- Manipulate expressions involving Big-O and other asymptotic notation.
- Compute algorithm runtimes in terms of Big-O.

Big-O Notation

Definition

$f(n) = O(g(n))$ (f is Big-O of g) or $f \preceq g$ if there exist constants N and c so that for all $n \geq N$, $f(n) \leq c \cdot g(n)$.

Common Rules

Multiplicative constants can be omitted:

$$7n^3 = O(n^3),\ \frac{n^2}{3} = O(n^2)$$

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$n^a \prec b^n$ $(a > 0, b > 1)$:

$$n^5 = O(\sqrt{2}^n),\ n^{100} = O(1.1^n)$$

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$(\log n)^a \prec n^b$ $(a, b > 0)$:

$$(\log n)^3 = O(\sqrt{n}),\ n \log n = O(n^2)$$

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$(\log n)^a \prec n^b$ $(a, b > 0)$:

$$(\log n)^3 = O(\sqrt{n}),\ n \log n = O(n^2)$$

Smaller terms can be omitted :

$$n^2 + n = O(n^2),\ 2^n + n^9 = O(2^n)$$

Recall Algorithm

Function FibList(n)

create an array $F[0 \ldots n]$
$F[0] \leftarrow 0$
$F[1] \leftarrow 1$
for i from 2 to n:
  $F[i] \leftarrow F[i-1] + F[i-2]$
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Big-O in Practice

Operation	Runtime

Big-O in Practice

Operation	Runtime
create an array $F[0 \ldots n]$	$O(n)$

Big-O in Practice

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Big-O in Practice

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for i from 2 to n:	Loop $O(n)$ times

Big-O in Practice

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$F[0] \leftarrow 0$	$O(1)$
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for i from 2 to n:	Loop $O(n)$ times
$\quad F[i] \leftarrow F[i-1] + F[i-2]$	$O(n)$
return $F[n]$	$O(1)$

Total:

$$O(n)+O(1)+O(1)+O(n)\cdot O(n)+O(1) = O(n^2).$$

Other Notation

Definition

For functions $f, g : \mathbb{N} \to \mathbb{R}^+$ we say that:

- $f(n) = \Omega(g(n))$ or $f \succeq g$ if for some c, $f(n) \geq c \cdot g(n)$ (f grows no slower than g).
- $f(n) = \Theta(g(n))$ or $f \asymp g$ if $f = O(g)$ and $f = \Omega(g)$ (f grows at the same rate as g).

Other Notation

Definition

For functions $f, g : \mathbb{N} \to \mathbb{R}^+$ we say that:

- $f(n) = o(g(n))$ or $f \prec g$ if $f(n)/g(n) \to 0$ as $n \to \infty$ (f grows slower than g).

Asymptotic Notation

- Lets us ignore messy details in analysis.
- Produces clean answers.
- Throws away a lot of practically useful information.