MAT300 HW#3

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1. Find the product of polynomial and piecewise polynomial as a sum of truncated power functions $(t-\mathrm{c})_+^{\mathrm{k}}$:
$(t-1)_+^3(t^2-3t+5) = (t-1)_+^3((t^2-2t+1)-(\mathrm{t}-4))$
$= (t-1)_+^3(t-1)_+^2 - t(t-1)_+^3 + 4\,(t-1)_+^3$
$= (t-1)_+^5 - t(t-1)_+^3 + 4\,(t-1)_+^3$

2. Write the piecewise polynomial function f(t) as a sum of truncated power functions $(t-\mathrm{c})_+^{\mathrm{k}}$:

$$\mathrm{f(t)} = \begin{cases} 0, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ 0, & 2 \le t \le 3 \end{cases}$$

$Interval[0,1), f(t) = 0:$
  $0 = a_0 + a_1 t + zero\ terms$
  $a_0 = 0,\ a_1 = 0$
$Interval[1,2), f(t) = t-1:$
  $t-1 = a_2(t-1)_+^1 + a_3(t-1)_+^0 + zero\ terms$
  $a_2 = 1,\ a_3 = 0$
$Interval[2,3), f(t) = 0:$
  $0 = 0 + 1(t-1)_+^1 + 0 + a_4(t-2)_+^1 + a_5(t-2)_+^0 + zero\ terms$
  $a_4 = -1,\ a_5 = -1$

  $f(t) = (t-1)_+^1 - (t-2)_+^1 - (t-2)_+^0$

3. This problem constructs a spline with divided differences.
 (a) Show that if c < 0 is a constant, then $[0,1,2,3](t-\mathrm{c})_+^2 = 0$

0	$(0-c)_+^2$	$(1-\mathrm{c})_+^2-(0-\mathrm{c})_+^2$	$\frac{(2-\mathrm{c})_+^2-2(1-\mathrm{c})_+^2+(0-\mathrm{c})_+^2}{2}$	$\frac{(3-\mathrm{c})_+^2-3(2-\mathrm{c})_+^2+3(1-\mathrm{c})_+^2-(0-\mathrm{c})_+^2}{6}$
1	$(1-\mathrm{c})_+^2$	$(2-\mathrm{c})_+^2-(1-\mathrm{c})_+^2$	$\frac{(3-\mathrm{c})_+^2-2(2-\mathrm{c})_+^2+(1-\mathrm{c})_+^2}{2}$	
2	$(2-\mathrm{c})_+^2$	$(3-\mathrm{c})_+^2-(2-\mathrm{c})_+^2$		
3	$(3-\mathrm{c})_+^2$			

$\frac{(3-\mathrm{c})_+^2-3(2-\mathrm{c})_+^2+3(1-\mathrm{c})_+^2-(0-\mathrm{c})_+^2}{6} = \frac{(9-6c+c^2-3(4-4c+c^2)+3(1-2c+c^2)-c^2}{6}$
$= \frac{(9-12+3)+(-6+12-6)c+(1-3+3-1)c^2}{6} = 0$

 (b) Show that if c > 3 is a constant, then $[0,1,2,3](t-\mathrm{c})_+^2 = 0$
 If(c>t) then $(t-\mathrm{c})_+^{\mathrm{k}} = 0$

0	0	0	0	0
1	0	0	0	
2	0	0		
3	0			

$p(t) = 0 + 0t + 0t^2 + 0t^3 = 0$

 (c) Let c be in one of the intervals [0, 1), [1, 2), or [2, 3]. In each of these three cases, work out the divided difference table to compute $[0,1,2,3](t-\mathrm{c})_+^2$ treating c as a constant.

0	0	0	$\frac{(4-4c+c^2)}{2}$	$\frac{1+2c-c^2\ -\ (4-4c+c^2)}{6}$ $=\frac{-3+6c-2c^2}{6}$
1	0	$(2-c)^2$ $= (4-4c+c^2)$	$\frac{(5-2c)-(4-4c+c^2)}{2}$ $=\frac{1+2c-c^2}{2}$	
2	$(2-c)^2$	$(3-c)^2-(2-c)^2$ $= (9-6c+c^2)-(4-4c+c^2)$ $= 5-2c$		
3	$(3-c)^2$			

$$p(t) = \frac{-3 + 6c - 2c^2}{6} t^3 + \frac{(4 - 4c + c^2)}{2} t^2$$

(d) Graph the function f(x) = $[0, 1, 2, 3](t - x)_+^2$ for all real numbers x. Note: the divided difference is computed with x as a constant and with the function $(t - x)_+^2$ as a function of t.

0	x^2	1-2x	1	0
1	1-2x+x^2	3-2x	1	
2	4-4x+x^2	5-2x		
3	9-6x+x^2			

$x^2 + (1 - 2x)t + t^2$

4. Find the osculating polynomial $p(t)$ for the data values [0,0,1,1] with $g(0) = 2$, $g'(0) = -3$, $g(1) = -5$, and $g'(1) = -11$. Recompute the polynomial $p(t)$ with the sequence [1,1,0,0]. Write both answers in standard basis to check that they are the same.

0	2	-3	-4	0
0	2	-7	-4	
1	-5	-11		
1	-5			

$p(t) = 2 - 3(t - 0) - 4(t - 0)^2$
$p(t) = 2 - 3t - 4t^2 \; with \; p'(t) = -3 - 8t$

1	-5	-11	-4	0
1	-5	-7	-4	
0	2	-3		
0	2			

$p(t) = -5 - 11(t - 1) - 4(t - 1)^2 + 0(t - 1)^2(t - 0) = -5 + 11 - 11t - 4t^2 + 8t - 4$
$p(t) = 2 - 3t - 4t^2 \; with \; p'(t) = -3 - 8t$

5. Let $g(t) = \cos\left(\frac{\pi}{2} t\right)$. In each part find the quadratic osculating polynomial $p(t)$ which agrees with the function $g(t)$ for each of the data sequences: (Also, use the values and derivatives to check your work.) Sketch three graphs, one for each polynomial together with the function g(t).
   i) [0,0,1]

0	1	0	-1
0	1	-1	
1	0		

$p(t) = 1 + 0(t - 0) - 1(t - 0)^2 = 1 - t^2$

ii) [0,1,1]

0	1	-1	0
1	0	-1	
1	0		

$p(t) = 1 - (t-0) + 0(t-0)(t-1) = 1 - t$

iii) [0,0,0]

0	1	0	-1
0	1	0	
0	1		

$p(t) = 1 + 0(t-0) - 1(t-0)^2 = 1 - t^2$

6. Find a polynomial $p(t)$ of degree 6 which has a zero of multiplicity 2 at t= 1 and a zero of multiplicity 3 at t = 2, and also satisfies: p(0) = 2 and p'(0) = 1. What is the other root of $p(t)$? Do this problem three different ways:
   i) Write p(t) in factored form with a free coefficient a and one free root b.
   Solve for a and b using the conditions on $p(0)$ and $p'(0)$.
   $p(t) = 2 + t + ?\,(t-1)^2(t-2)^3 = 2 + t + ?\,(t^5 - 8t^4 + 25t^3 - 38t^2 + 28t - 8)$
   ii) Use a divided difference table for osculating polynomial. Find a and b from the Newton form. (Hint: It is easiest if you arrange the values t_i to have lots of zeros along the top of the triangle.)
   iii) Write a linear system involving coefficients for the standard basis as the variables, using the derivative definition of multiplicity. Solve it with a symbolic algebra tool.