Constructing the block anti-diagonal matrix over the determinant

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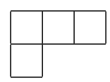
Introduction

We start with a n-dimensional complex vector space V with basis $\{x_1, \dots, x_n\}$, and let $A = \mathbb{C}[x_1, \dots, x_n]$. Consider the permutation group $G = S_n$ acting on V such that for $g \in G$ the action $g.(c_1x_1 + \dots + c_nx_n) = c_1x_{g(1)} + \dots + c_nx_{g(n)}$, let $B = A^G$ be the subring of A fixed by the action of S_n, thus $B = \mathbb{C}[f_1, \dots, f_n]$ where $\{f_1, \dots, f_n\}$ are the n symmetric polynomials of A. Define $B_+ = (f_1, \dots., f_n)$, by the Chevallay-Sheppard-Todd theorem we have that A/B_+ is a finitely generated B-module, since S_n is generated by reflections, therefore $A = B \otimes_{\mathbb{C}} (A/B_+)$.

Next we will discuss how to find such basis for (A/B_+), we start with $\lambda = (p_1, \dots, p_k)$ a partition of n, we may assume that $p_1 \geqslant p_i$ for $i > 1$. Define a **Young Diagram** to be a diagram with exactly n-rows and p_i columns, as an example take the partition $(3, 1)$ of 4, the young diagram of this partition would be as such.



We can define then a **Young Tableau**, or tableau, to be a bijection of $\{1, \dots, n\}$ onto the entries of a young diagram, we call a tableau T **standard** if it for every column c and row r we have that if $i < j$ then $r_i < r_j$ and $c_i < c_j$. For any partition λ, let $(c_1, \dots, c_d)$ be the columns of the young diagram, note that $(\mid c_1 \mid, \dots, \mid c_k \mid)$ is also a partition of n, where $\mid c_i \mid$ is the size of the column c_i, this partition is called the conjugate partition of λ, denoted λ'. We can define for any tableau T of λ the conjugate tableau T' in which the columns of T are the rows of T'. Below is an example of a standard young tableau on the partition $\lambda = (3, 1)$, and its conjugate tableau on the partition $\lambda' = (2, 1, 1)$, note that the conjugate tableau is also standard.

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad T_1' = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Recall that the representations of S_n are determined by the partitions of n, that is, for each partition λ of n we can define V_λ to be the irreducible S_n representation correspondent to λ. The dimension of V_λ can be determined from the Young Diagram of λ, where $\dim(V_\lambda) = \frac{n!}{\Pi_{i,j} h(i,j)}$ where $h(i, j)$ is the hook length of the (i, j)-position of the diagram, here i is the column index and j is the row index.

Let λ be a partition of n, we denote $\mathrm{ST}(\lambda)$ to be the set of all standard tableaux of shape λ. We can define a group action of S_n on a given young tableaux t, where for any $g \in S_n$ then $g.t(i, j) = g(t(i, j))$, where i is the row and j the column, therefore define C_t and R_t to be the column and row stabilizer of t under this action. Recall the group algebra $\mathbb{C}S_n = \{ \sum\limits_{g \in S_n} c_g g \mid c_i \in \mathbb{C} \text{ for } i \in S_n\}$, we will define for $t \in \mathrm{ST}(\lambda)$, the following idempotents in $\mathbb{C}S_n$

$$\epsilon_t = \frac{\dim(V_\lambda)}{n!} \sum_{c \in C_t} \sum_{r \in R_t} \mathrm{sgn}(c) cr \quad \text{and} \quad \sigma_t = \frac{\dim(V_\lambda)}{n!} \sum_{c \in C_t} \sum_{r \in R_t} \mathrm{sgn}(c) rc$$

As an example, if we pick $T \in \mathrm{ST}((2, 1))$ as shown bellow, we obtain that $C_T = \{e, (1, 3)\}$ and $R_T = \{e, (1, 2)\}$, thus ϵ_t and σ_t are shown bellow

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \qquad \begin{aligned}\epsilon_T &= () + (12) - (13) - (123) \\ \sigma_T &= () + (12) - (13) - (132)\end{aligned}$$

Next we define for a standard tableau T, the charge $i(T)$, let $\{r_1, \ldots, r_k\}$ be the rows of T, we define $w(T) = \{r_{k_1}, \ldots, r_{k_{|r_k|}}, \ldots, r_{k_n}\}$ to be the tuple obtained by concatenating the rows of T from the bottom upwards, for every $1 \leqslant j \leqslant n$ define $1 \leqslant k_j \leqslant n$ such that $w(T)_{k_j} = j$, now we build $w(i(t))$ recursively where $i(T)_{k_1} = 0$ and for $j > 1$ if $k_{j-1} < k_j$ then $i(T)_{k_j} = i(t)_{k_{j-1}}$, if $k_{j-1} > k_j$ then $i(t)_{k_j} = i(t)_{k_{j-1}} + 1$. With the same tableau as the previous example below are the $w(T)$ and $i(T)$.

$$w(T) = (3, 1, 2) \text{ and } i(T) = (1, 0, 0)$$

Now we will relate the standard tableaux with our original ring A, fix a partition λ of n, for any $T, S \in \mathrm{ST}(\lambda)$ define the monomial $x_T^S = x_{w(T)_1}^{i(S)_1} \ldots x_{w(T)_n}^{i(S)_n}$, with this we may define the **higher specht polynomials** as $F_T^S = \epsilon_T.x_T^S$. As an example let T be the same as the previous example, we compute x_T^T, firstly $w(T) = (3, 1, 2)$ and $i(T) = (1, 0, 0)$ therefore $x_T^T = x_3^1 x_1^0 x_2^0 = x_3$ and we can produce the polynomial F_T^T

$$F_T^T = \epsilon_T.x_T^T = \frac{1}{3}(() + (12) - (13) - (123)).x_3 = \frac{2}{3}(x_3 - x_1)$$

Recall that as a representation of S_n we know that $\mathbb{C}S_n \cong \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus dim(V_\lambda)}$, we also know that $A = B \otimes_{\mathbb{C}} A/B_+ \cong B \otimes_{\mathbb{C}} \mathbb{C}S_n$ as a $\mathbb{C}S_n$ module. With this notation we can define the isotypical components of A correspondent to V_λ as $A_\lambda = \hom_{\mathbb{C}S_n}(V_\lambda, A) \otimes V_\lambda$, furthermore the set $F_\lambda = \{F_t^s\}_{s,t \in \mathrm{ST}(\lambda)}$ forms a basis for A_λ in A over B, This allows us to write $A \cong \langle F_p \rangle_{\lambda \vdash n}$ over B. Consider the element $z = F_{alt}^{alt}$ where $alt = \sum_{1<i<n} 1$ is the alternating partition of n, we may write $z = \Pi_{i<j}(x_i - x_j)$ the map $z : A \to A$ defined by $z : x \mapsto zx$ is a linear map, thus we may write it as a matrix $M_z : B^{n!} \to B^{n!}$ since A may be considered a free B module. We wish to find a basis for F such that M_z is block anti-diagonal. Note that z is a relative invariant under S_n, therefore the map $z : A_\lambda \to A_{\lambda'}$ for any partition λ of n, this means that for any $S, T \in \mathrm{ST}(\lambda)$ we have the following.

$$zF_T^S = \sum_{V \in \mathrm{ST}(\lambda')} \sum_{W \in \mathrm{ST}(\lambda')} g_{V,T}^{W,S} F_V^W \qquad \text{where } g_{V,T}^{W,S} \in B$$

This means that for every partition of n, we can define $U_\lambda = [g_{V,T}^{W,S}]_{S,T,W',V' \in \mathrm{ST}(\lambda)} \in B^{\dim(V_\lambda)^2 \times \dim(V_\lambda)^2}$ which has the propriety that $U_\lambda U_{\lambda'} = z^2 I_{\dim(V_\lambda)^2} = \Delta I_{\dim(V_\lambda)^2}$, which is a matrix factorization of Δ. Note that A_λ contains $\dim(V_\lambda)$ copies of A_λ, in the next section we will construct a basis which will further reduce the size of the matrix factorization to $\dim(V_\lambda)$.

New Basis of A over B

We will now discuss a new basis for A over B, for any two standard young tableaux T and V of shape λ, define the following

$$\sigma_T = \frac{n!}{\dim(V_\lambda)} \sum_{c \in C_T} \sum_{r \in R_T} sng(c)rc = n_T \sum_{c \in C_T} \sum_{r \in R_T} sng(c)rc$$

$$H_T^V = \sigma_T(\varepsilon_T . x_T^V) = \sigma_T(F_T^V)$$

In order to show that $\{H_T^V\}_{T,V \in ST(\lambda)}$ are a basis for A_λ, we define a bilinear form on A over B, let $f, g \in A$,

$$\langle f, g \rangle = \sum_{x \in S_n} \text{sgn}(x) x.(fg)/\Pi_{i<j}(x_i - x_j)$$

We now have three known results about this bilinear form,

Lemma 1: For any $g \in S_n$ and $f_1, f_2 \in A$ we have that $\langle g.f_1, f_2 \rangle = sng(g)\langle f_1, g^{-1}.f_2 \rangle$

As a consequence we have that for any tableaux T and its conjugate T' the following holds, $\langle \varepsilon_T . f_1, f_2 \rangle = \langle f_1, \varepsilon_{T'} . f_2 \rangle$ and $\langle \sigma_T . f_1, f_2 \rangle = \langle f_1, \sigma_{T'} . f_2 \rangle$.

Lemma 2: For any standard tableaux T, V, S, W we have that $\langle F_T^V, F_S^W \rangle = c \in \mathbb{C}^*$ if and only if $T = S'$ and $V = W'$ where T and V are of the same shape, otherwise $\langle F_T^V, F_S^W \rangle = 0$.

Corollary 1: The determinant of the gramian matrix with respect to the higher Specht polynomials and the bilinear form above is a non-0 constant in $\mathbb{C}$. This would show that the higher Specht polynomials indeed form a basis for A over B.

We will now use this bilinear form to show that the new defined polynomials form a basis for A over B.

Lemma 3: The set $\{H_T^S\}_{T,S \in ST(\lambda), \lambda \vdash n}$ is a basis for A over B.

Proof: Let T and S be standard tableaux of shape λ_1, and let V and W be standard tableaux of shape λ_2 then we have two cases, if $\lambda_1' \neq \lambda_2$ then we have that

$$\begin{aligned}
\langle H_T^S, H_V^W \rangle &= \langle \sigma_T . F_T^S, \sigma_V . F_V^W \rangle \\
&= \langle F_T^S, \sigma_{T'} \sigma_V . F_V^W \rangle \\
&= \langle F_T^S, 0.F_V^S \rangle && \text{Since } T \text{ and } V \text{ are not of the same shape} \\
&= \langle F_T^S, 0 \rangle = 0
\end{aligned}$$

The second case where $\lambda_1' = \lambda_2$, then there are three possibilities, if $T' \neq V$ then $\sigma_{T'}\sigma_V = 0$ and thus we have that $\langle H_T^S, H_V^W \rangle = 0$ by the same computation above. If $T' = V$ and $S' \neq W$ then firstly note that $\langle F_T^S, F_V^W \rangle = 0$ since $S' \neq W$ and secondly

$$\begin{aligned}
H_V^W &= \sigma_V . F_V^W \\
&= n_V \sum_{c \in C_V} \sum_{r \in R_V} \text{sgn}(c) rc . F_V^W \\
&= n_V \sum_{r,c} \text{sgn}(c) F_{rc.V}^W
\end{aligned}$$

With this we can write the following if $T' = V$ and $S' \neq W$,

$$\begin{aligned}
\langle H_T^S, H_V^W \rangle &= \langle \sigma_T . F_T^S, \sigma_V . F_V^W \rangle \\
&= \langle F_T^S, \sigma_{T'} \sigma_V . F_V^W \rangle \\
&= \langle F_T^S, \sigma_V . F_V^W \rangle && \text{since } T' = V \\
&= \langle F_T^S, \sum_{r,c} \operatorname{sgn}(c) F_{rc.V}^W \rangle && \text{by the computation above} \\
&= \sum_{r,c} \operatorname{sgn}(c) \langle F_T^S, F_{rc.V}^W \rangle \\
&= \sum_{r,c} (0) && \text{since } S' \neq W \\
&= 0
\end{aligned}$$

Lastly if $T' = V$ and $S' = W$ then consider the sets $a_T = \sum_{r \in R_T} r$ and $b_T = \sum_{c \in C_t} \operatorname{sgn}(c) c$ then $\varepsilon_T = n_T b_T a_T$ and $\sigma_T = n_T a_T b_T$, by the way they are defined its easy to see that $b_T^2 =\mid C_T \mid b_T$, therefore $\sigma_T \varepsilon_T = n_T^2 a_T b_T^2 a_T =\mid C_T \mid n_T^2 a_T b_T a_T$ which gives us a easy way to write the fact that $\sigma_T F_T^V =\mid C_T \mid n_T \sum_{r \in R_T} F_{r.T}^V$.

With that, if $r \in R_T$ and $r \notin \operatorname{Stab}(F_T^S)$ then $\langle F_{r.T}^V, F_{T'}^{V'} \rangle = 0$. Therefore, let $\langle F_T^S, F_{T'}^{S'} \rangle = h \in \mathbb{C}^*$, then we can write the following.

$$\begin{aligned}
\langle H_T^S, H_{T'}^{S'} \rangle &= \langle \sigma_T . F_T^S, \sigma_{T'} . F_{T'}^{S'} \rangle \\
&= \langle \sigma_T F_T^S, F_{T'}^{S'} \rangle \\
&= \langle \mid C_T \mid n_T \sum_{r \in R_T} F_{r.T}^S, F_{T'}^{S'} \rangle && \text{by the argument above} \\
&=\mid C_T \mid n_T \sum_{r \in R_T} \langle F_{r.T}^S, F_{T'}^{S'} \rangle && \text{since } \langle \rangle \text{ is bilinear} \\
&=\mid C_T \mid n_T \sum_{r \in R_T \cap \operatorname{Stab}(F_t^S)} \langle F_{r.T}^S, F_{T'}^{S'} \rangle \\
&=\mid C_T \mid n_T \sum_{r \in R_T \cap \operatorname{Stab}(F_t^S)} (h) \neq 0 && \text{since } e \in R_T \cap \operatorname{Stab}(F_t^S)
\end{aligned}$$

Therefore $0 \neq \langle H_T^S, H_{T'}^{S'} \rangle \in \mathbb{C}$, and we may conclude that for standard tableaux T, V, S, W $\langle H_T^S, H_V^W \rangle = c \in \mathbb{C}^*$ if $V = T'$ and $S' = W$, and 0 otherwise, meaning the determinant of the gramian according to this basis is a non-0 constant in $\mathbb{C}$, meaning it the set makes a basis for A over B. □

Lemma 4: For given standard tableaux T and S of shape λ, we have that $H_T^S \in \langle F_V^S \rangle_{V \in \mathrm{ST}(\lambda)}$.

Proof: Note that for a Young Tableaux W and a standard tableaux S of the same shape λ, then $F_T^S \in \langle F_V^S \rangle_{V \in \mathrm{ST}(\lambda)}$, and by the proof of the previous lemma we have that for $T, S \in \mathrm{ST}(\lambda)$, then we have that $H_T^S = \sigma_T F_T^S =\mid C_T \mid n_T \sum_{r \in R_T} F_{r.T}^S \in \langle F_T^S \rangle$

The immediate corollary we get is that $\{H_T^S\}_{T,S \in \mathrm{ST}(\lambda)}$ is a basis for A_λ over B.

Matrix Factorization of Δ

Now we will use the new basis of A_λ over B to produce a matrix factorization $MN = I\Delta$ where $M, N \in B^{\dim(V_\lambda)\times\dim(V_\lambda)}$. Recall that z is a relative invariant and consider the following for some fixed $T, S \in \mathrm{ST}(\lambda)$, since $H_T^S \in A_\lambda$ we know then multiplication by $zH_T^S \in A_{\lambda'}$. For the remainder of this section we will define for $T \in \mathrm{ST}(\lambda)$ the free modules of dimension $\dim(V_\lambda)$ over B as $FA_T = \langle F_T^S\rangle_{S\in\mathrm{ST}(\lambda)}$ and $HA_T = \langle H_T^S\rangle_{S\in\mathrm{ST}(\lambda)}$.

Lemma 1: For $f \in HA_T$ and $g \in FA_T$ we have that $zf \in FA_{T'}$ and $zg \in HA_{T'}$

Proof: To show this we will irst show that for each $S \in \mathrm{ST}(\lambda)$ we may write zH_T^S in terms of the basis of $FA_{T'}$, note that since $zH_T^S \in A_{\lambda'}$ then we can write it the following way

$$zH_T^S = \sum_{V\in\mathrm{ST}(\lambda')}\sum_{W\in\mathrm{ST}(\lambda')} g_{V,S}^{W,T} F_V^W \quad \text{where } g_{V,T}^{W,S} \in B$$

Now lets see what happens when we apply $\varepsilon_{T'}$ to both sides,

$$\begin{aligned}
\varepsilon_{T'}(zH_T^S) &= \sum_{c\in C_{T'}}\sum_{r\in R_{T'}} sg(c)cr(zH_T^S) && \text{by definition}\\
&= z(\sum_{c\in C_{T'}}\sum_{r\in R_{T'}} sg(r)crH_T^S) && \text{since } g.z = sg(g)z\\
&= z(\sum_{c\in R_T}\sum_{r\in C_T} sg(r)crH_T^S) && \text{since } R_{T'} = C_T \text{ and } C_{T'} = R_t\\
&= z(\sigma_T H_t^S) = zH_T^S && \text{by definition of } H_T^S
\end{aligned}$$

Therefore $\varepsilon_{T'}(zH_T^S) = zH_T^S$, and the right hand side would then reduce the following way,

$$\begin{aligned}
\varepsilon_{T'}(\sum_{q\in\mathrm{ST}(p')}\sum_{w\in\mathrm{ST}(p')} g_{V,T}^{W,S} F_V^W) &= \sum_{q\in\mathrm{ST}(p')}\sum_{w\in\mathrm{ST}(p')} g_{V,T}^{W,S}(\varepsilon_{T'}\varepsilon_V.x_V^W)\\
&= \sum_{W\in\mathrm{ST}(p')} g_{T',T}^{W,S} F_{T'}^W && \text{since } e_{T'}e_V = 0 \text{ if } V \neq T'
\end{aligned}$$

Therefore we have that $zH_T^S = \sum_{W\in\mathrm{ST}(p')} g_{T',T}^{W,S} F_{T'}^W \in FA_{T'}$, therefore this proves that $zf \in FA_{T'}$ for $f \in HA_T$, to prove the second part is identical to the computation above where

$$\sigma_{T'}(zF_T^S) = \sum_{W\in\mathrm{ST}(p')} h_{T',T}^{W,S} H_{T'}^W \text{ where } h_{T',T}^{W,S} \in B$$

Which concludes the proof □

With the lemma above we can write multiplication by z in terms of the basis for HA_T and $FA_{T'}$, with the same notation of the lemma for each $T \in \mathrm{ST}(\lambda)$ we write $M_T = [g_{T',T}^{W,S}]_{T',S\in\mathrm{ST}(\lambda')}$ and $N_T = [h_{T',T}^{W,S}]_{T',S\in\mathrm{ST}(\lambda')}$ where $M_T, N_T \in B^{\dim(V_\lambda)\times\dim(V_\lambda)}$, clearly $M_T : HA_T \to FA_T$ and $N_T : FA_T \to HA_T$. Finally this yields the fact that $M_T N_{T'} = N_T M_{T'} = \Delta I_{\dim(V_\lambda)}$. Now we have the following,

Lemma 2: If $T, V \in \mathrm{ST}(\lambda)$ with $\pi \in S_n$ such that $\pi.V = T$ then $M_T = \mathrm{sgn}(\pi)M_V$, similarly $N_T = \mathrm{sgn}(\pi)N_V$

Proof: First let $c = \mathrm{sgn}(\pi) = \mathrm{sgn}(\pi^{-1})$ and $S \in \mathrm{ST}(\lambda)$, we can write $H_T^S = \pi H_V^S$ which gives the following,

$$zH_T^S = z(\pi H_V^S) = \pi(czH_V^S)$$

By Lemma 1 we have the following

$$zH_T^S = \sum_{W \in \mathrm{ST}(\lambda')} g_{T',T}^{W,S} F_{T'}^W \tag{1}$$
$$zH_V^S = \sum_{W \in \mathrm{ST}(\lambda')} g_{V',V}^{W,S} F_{V'}^W \in FA_{T'} \tag{2}$$

We can use it two write the following

$$\begin{aligned}
\sum_{W \in \mathrm{ST}(\lambda')} g_{T',T}^{W,S} F_{T'}^W &= \sum_{W \in \mathrm{ST}(\lambda')} g_{T',T}^{W,S} (\pi F_V^S) \\
&= \pi(\sum_{W \in \mathrm{ST}(\lambda')} g_{T',T}^{W,S} F_{V'}^W) \\
\pi(czH_V^S) &= \pi(\sum_{W \in \mathrm{ST}(\lambda')} cg_{V',V}^{W,S} F_{V'}^W)
\end{aligned}$$

Now since $zH_T^S = \pi(czH_V^S)$ this would give us that $\pi(\sum_{W \in \mathrm{ST}(\lambda')} g_{T',T}^{W,S} F_{V'}^W) = \pi(\sum_{W \in \mathrm{ST}(\lambda')} cg_{V',V}^{W,S} F_{V'}^W)$ and applying π^{-1} to both sides we have that $\sum_{W \in \mathrm{ST}(\lambda')} (g_{T',T}^{W,S} - cg_{V',V}^{W,S}) F_{V'}^W = 0$, implying that $g_{T',T}^{W,S} = cg_{V',V}^{W,S}) F_{V'}^W$. Therefore we can show the following, $M_T = [g_{T',T}^{W,S}]_{T',S \in \mathrm{ST}(\lambda')} = [cg_{V',V}^{W,S}]_{V',S \in \mathrm{ST}(\lambda')} = cM_V$. The case for N_T can be computed in a similar manner to this. $\square$

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