

STRATIFICATION OF DISCRIMINANTS AND IRREDUCIBLE REPRESENTATIONS

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ABSTRACT. not very

Some facts about stratification of discriminants

Should define what a stratum is!

Essentially: Let X be an algebraic variety of dimension d , start with a filtration of $X = X_d \supseteq X_{d-1} \supseteq \cdots \supseteq X_1 \supseteq X_0$, such that each $X_i - X_{i-1}$ is a smooth open subvariety of X_i or empty. (Each X_i is a subvariety of dimension i of X) Then the connected components Z_α of $X_i - X_{i-1}$ are called the strata. Note that the Z_α are smooth but their closure $\overline{Z_\alpha}$ not necessarily. Note that X can be written as a disjoint union of its strata.

Some basics about reflection groups and discriminants: let G be a complex reflection group acting on the k -vectorspace V , $\dim(V) = n$. We denote by $S = \text{Sym}_k(V) \cong k[x_1, \dots, x_n]$ and $R = S^G$. By the Chevalley–Shephard–Todd theorem REF $R \cong k[f_1, \dots, f_n]$, where the basic invariants f_i are algebraically independent and homogeneous. They are not unique but their degrees $d_i = \deg(f_i)$ are.

Denote by $\pi : V \rightarrow V/G$ the canonical projection. The quotient space V/G is $\text{Spec}(R)$ and smooth of dimension $\dim(V)$. The image of the reflection arrangement \mathcal{A} in V is the discriminant $V(\Delta)$ in V/G .

1. STRATIFICATION OF THE DISCRIMINANT OF S_n

Usually we restrict to the invariant hyperplane $V(x_1 + \cdots + x_n)$ in V , so that S_n acts on k^{n-1} . By the theorem of Chevalley–Shephard–Todd the invariant ring $R = S^G$ is isomorphic to a polynomial ring in n (or $n - 1$) variables, generated by invariant polynomials f_1, \dots, f_n , where each $f_i \in S$ can be chosen homogeneous of degree i . Note that the f_i are not unique but their degrees are.

Popular choices for the f_i are the p_i or the symmetric polynomials $s_i := \sum_{k=1}^n x_k^i$.

For the case $G = S_n$ we have $\mathcal{A} = V(\prod_{i < j} (x_i - x_j))$ and Δ can be identified with the discriminant in a versal deformation of an A_n -singularity (see Arnold’s paper REF).

This paragraph is a mess! More generally: Arnold proved this for all Coxeter groups with simply laced Dynkin diagrams, i.e. $A_n, D_n, E_{6,7,8}$: discriminant of the reflection group is diffeomorphic to discriminant of versal deformations (there called singularities of wave fronts - the corresponding hyperplane arrangement is called caustic or bifurcation set

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in the dynamical systems speech). Later extended to crystallographic Coxeter groups: B_n, C_n, F_4, G_2 by WEYL?! see Book Arnold-Varchenko-Gusein-Zade:1 and finally also to H_2, H_3, H_4 by Lyashko, cf. [Lja79, Lya83] (see paper by Shcherbak and its Mathsinet review by Janeczko for more detail)).

In particular, for any Coxeter group G one gets a natural stratification of the discriminant relating the strata to subgraphs of the Coxeter graph (see Section 6 of [Shc88]).

In the case for $G = S_n$ the quotient V/G can be identified with the space of polynomials

$$F_n = x_n + \lambda_1 x^{n-1} + \dots + \lambda_{n-1}$$

and the stratification of this space is given by the multiplicities of the roots of F_n . One can encode them into Young diagrams with n boxes and relate a subgraph of the Coxeter graph to each Young diagram (explained in [Shc88, p.185]).

This might not be standard convention but following Shcherbak We let $n \geq k_1 \geq k_2 \geq \dots \geq k_s \geq 1$ be a partition of n (i.e., $\sum_{i=1}^s k_i = n$), and the corresponding Young diagram be given with s columns, where the i -th column consists of k_i boxes. (i.e., the trivial representation corresponds to the partition $(1, \dots, 1)$).

Let us now prove an interesting fact that is mentioned in [Shc88, p.185] without proof:

Lem:rectangular

Lemma 1.1. *Consider $G = S_n$ acting on V . Then the closure of the stratum corresponding to a rectangular Young diagram is a smooth subvariety of $V(\Delta)$. Should be if and only if!*

Proof. In order to have a rectangular Young diagram, n must be decomposable, say $n = lk$. Consider the partition (k, \dots, k) corresponding to the rectangular Young diagram with $l \times k$ boxes. This means that a point in the corresponding stratum of V/G is the image of a point $p \in V$ of the form

$$p = (\underbrace{\sigma_1, \dots, \sigma_1}_k, \dots, \underbrace{\sigma_l, \dots, \sigma_l}_k),$$

where $\sigma_i \neq \sigma_j$ for $i \neq j$ (to be precise: the whole S_n -orbit of p projects to the same point in V/G). This means that p lies in the intersection of $\binom{l}{2} = \frac{n(k-1)}{2}$ hyperplanes in the reflection arrangement. **Note: we are not restricting to the hyperplane $x_1 + \dots + x_n$ yet.** Take the power sums $s_m = \sum_{i=1}^n x_i^m$ as basic invariants of R , i.e., $R = S^n = k[s_1, \dots, s_n]$. We can restrict to the hyperplane by setting $s_1 = 0$. Note in particular that the s_m are algebraically independent for $m = 1, \dots, n$.

Now $\pi(p) = (s_1(p), \dots, s_n(p))$ evaluates to

$$(ks_1(\sigma_1, \dots, \sigma_l), \dots, ks_l(\sigma_1, \dots, \sigma_l), \dots, ks_n(\sigma_1, \dots, \sigma_l)).$$

These are polynomials in the l distinct roots $\sigma_1, \dots, \sigma_l$. **Here this is a bit subtle: we denote also by s_m the power sums of the l variables, strictly speaking we should maybe denote them by \tilde{s}_m .** Now since $\tilde{s}_1, \dots, \tilde{s}_l$ are algebraically independent (they are the basic invariants for S_l), it follows that $\tilde{s}_i(\sigma_1, \dots, \sigma_l)$ for $i = l+1, \dots, n$ is a polynomial P_i in the \tilde{s}_j for $j = 1, \dots, l$. Hence the image of p can be written in the coordinates \tilde{s}_i as

$$(k\tilde{s}_1, \dots, k\tilde{s}_l, kP_{l+1}(\tilde{s}_1, \dots, \tilde{s}_l), \dots, kP_n(\tilde{s}_1, \dots, \tilde{s}_l)).$$

This parametrizes an l -dimensional subvariety of V/G . The closure of this image has equations, now denoting the generators of S^n by y_1, \dots, y_n :

$$y_{l+1} - kP_{l+1}(\frac{y_1}{k}, \dots, \frac{y_l}{k}), \dots, y_n - kP_n(\frac{y_1}{k}, \dots, \frac{y_l}{k}).$$

Denote the ideal generated by these $n - l$ equations by I . Then $V(I)$ is a complete intersection subvariety of $V/G = \text{Spec}(k[y_1, \dots, y_n])$ of codimension $n - l$, i.e., $V(I)$ has dimension l . Clearly (Jacobian criterion! - since $k \geq 2$, there is a $l \times l$ -minor of the Jacobian matrix of I that is the identity matrix $\mathbb{1}_l$) this subvariety is smooth and is isomorphic to $\text{Spec}(k[y_1, \dots, y_l])$. \square

–Simon–

Lemma 1.2. *Consider $G = S_n$ acting on V . Then the closure of the stratum corresponding to the partition $(j, n - j)$ where $0 < j < n$ and $j \neq \frac{n}{2}$ is a subvariety of $V(\Delta)$ isomorphic to a cusp singularity.*

Proof. We note that when $j = \frac{n}{2}$, then the closure of the stratum corresponding to the partition $(\frac{n}{2}, \frac{n}{2})$ is smooth.

Again we take the powers sums $s_m = \sum_{i=1}^n x_i^m$ as basic invariants of R .

$$p = (\underbrace{\sigma_1, \dots, \sigma_1}_j, \underbrace{\sigma_2, \dots, \sigma_2}_{n-j}),$$

Now $\pi(p) = (s_1(p), \dots, s_n(p))$ evaluates to

$$(j\sigma_1 + (n-j)\sigma_2, j\sigma_1^2 + (n-j)\sigma_2^2, \dots, j\sigma_1^n + (n-j)\sigma_2^n)$$

Now restricting to the hyperplane $s_1 = 0$ we get the relation $-j\sigma_1 = (n-j)\sigma_2$, rearranging to get $\sigma_2 = -\frac{j}{(n-j)}\sigma_1$. After this restriction, the projection becomes:

$$\pi(p) = (s_1(p), \dots, s_n(p)) = (0, \frac{j(n-j) + (-j)^2}{n-j}\sigma_1^2, \dots, \frac{j(n-j)^{n-1} + (-j)^n}{(n-j)^{n-1}}\sigma_1^n).$$

Again if $j = \frac{n}{2}$ then only we see that $s_i(p) = 0$ for all odd i - leading to a parabola.

For cleanliness sake, Let $a_i := \frac{j(n-j)^{i-1} + (-j)^i}{(n-j)^{i-1}}$, the above becomes:

$$\pi(p) = (s_1(p), \dots, s_n(p)) = (0, a_1\sigma_1^2, a_2\sigma_1^3, \dots, a_{n-1}\sigma_1^n).$$

The closure of the image of π is a subvariety of $V(\Delta)$ given by the equations:

$$s_1, a_1^3s_3^2 - a_2^2s_2^3.$$

and let m be an integer such that $3 < m$ and $m = 2m_1 + 3$, then we get the relations:

$$a_1^{m_1}a_2s_m - a_{m-1}s_2^{m_1}s_3$$

Which is isomorphic (unneded?) a cusp singularity.

\square

2. QUESTIONS

1. Description of the other strata? In particular: which singularities arise (always the same ones?)? Find all strata for the S_5 -discriminant
2. Similar statement as Lemma 1.1 for other complex reflection groups? At least for the crystallographic Coxeter groups?
3. Find the fitting ideals for the MCM-modules coming from the irreducible representations of G : do they correspond to certain strata (closures of them)? Which ones? Work this out for S_5 ! Also interesting: B_3
4. What about connection between discriminants of deformations and reflection groups for non-Coxeter groups: do we at least get some statement for the true reflection groups? Haven't really thought about this, but the literature is all about real reflection groups - in Orlik–Terao is also stratification of discriminant for any complex reflection group via fitting ideals, but this is too coarse

3. GENERAL FACTS ABOUT DISCRIMINANTS

REFERENCES

- [Lja79] O. V. Ljaško. The geometry of bifurcation diagrams. *Uspekhi Mat. Nauk*, 34(3(207)):205–206, 1979. 2
- [Lya83] O. V. Lyashko. Geometry of bifurcation diagrams. In *Current problems in mathematics, Vol. 22*, Itogi Nauki i Tekhniki, pages 94–129. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1983. 2
- [Shc88] O. P. Shcherbak. Wave fronts and reflection groups. *Uspekhi Mat. Nauk*, 43(3(261)):125–160, 271, 272, 1988. 2

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