Matrix Factorization of the Determinant over G(m,1,n)

Consider a finite reflection group $G = G(m, 1, n)$ acting on $A = \mathbb{C}[x_1, \ldots, x_n]$ by permuting the elements, then by the theorem of Chevalley-Sheppard Todd there exists algebraic independent $f_1, \ldots, f_n \in A$ such that $B = A^G = \mathbb{C}[f_1, \ldots, f_n]$ and A is a finite dimensional free module over B. For $\lambda = (\lambda_1, \ldots, \lambda_m) \in \text{type}(n_1, \ldots, n_m)$ for a partition $(n_1, \ldots, n_m) \vdash n$ we consider the Specht module S_λ and the isotypical components of A over λ as

$$A_\lambda = \hom_{\mathbb{C}G}(S_\lambda, A)$$

Consider the set of reflecting hyperplane $\mathfrak{A}(G)$ to be the set of hyperplanes in $\mathbb{C}^n$ which are fixed for some reflection in G. For some reflecting hyperplane $h \in \mathfrak{A}(G)$ then the subgroup $\text{stab}(h) \leqslant G$ is cyclic with order d_h, furthermore we could describe each hyperplane as a linear equation $e_h = 0$ where $e_h \in A$. We then define the following polynomials

$$z = \Pi_{h \in \mathfrak{A}(G)} e_h \text{ and } j = \Pi_{h \in \mathfrak{A}(G)} e_h^{d_h}$$

With this we define the discriminant of G to be $\Delta = zj$. Note that we can define the linear maps $z, j : A \to A$ by multiplication by z and j respectively, and since A is a free module over B these maps are linear maps from A to itself as a B-module, which we will denote ρ_z and ρ_j respectively. It is known that z and j belongs to the determinial representation *det* so that multiplication by z yields the following map $z : A_\lambda \to A_{\lambda'}$ where $\lambda' = (\lambda_m, \lambda_1, \ldots, \lambda_{m-1})$ while multiplication by j gives $j : A_\lambda \to A_{\lambda^{-1}}$ where $\lambda^{-1} = (\lambda_2, \ldots, \lambda_m, \lambda_1)$. Knowing that $\Delta \in B$ we know that $\rho_z \circ \rho_j = \Delta I$ where I is the identity map from A to A as a module over B. This gives us a matrix factorization of Δ given by multiplication by z and j respectively

$$A_\lambda \xrightarrow{z} A_{\lambda'} \xrightarrow{j} A_\lambda$$

Recall that $\text{ST}(\lambda)$ is the set of m-tableaux $(T_1, \ldots, T_m)$ such that T_i is a standard tableaux of shape λ_i, thus if $T \in \text{ST}(\lambda)$ then for each $1 \leqslant i \leqslant m$ we can build the following idempotents $\varepsilon_T = \sum_{c \in C(T_i)} \sum_{r \in R(T_i)} \text{sgn}(c) cr$ and $\sigma_T = \sum_{c \in C(T_i)} \sum_{r \in R(T_i)} \text{sgn}(r) cr$, where $C(T_i)$ and $R(T_i)$ are the column and row stabilizer respectively. Given $T, V \in \text{ST}(\lambda)$ we define the following polynomials

$$F_T^V = (\varepsilon_T . x_T^V)(\mu_T) \text{ and } H_T^V = (\sigma_T \varepsilon_T . x_T^V)(\mu_T)$$

Where $\mu_T = \Pi_{i=1}^m \Pi_{j \in T_i} (x_j)^i$. With this we also define the following sets $F_T = \{F_T^S \mid S \in \text{ST}(\lambda)\}$ and $F^T = \{F_S^T \mid S \in \text{ST}(\lambda)\}$, similarly we define similar sets for H_T and H^T. Lastly if we index by λ, so $F_\lambda = \{F_T^V \mid T, V \in \text{ST}(\lambda)\}$

With the basis F_p and H_p we can now express the linear maps ρ_z and ρ_j as a matrix over B, where both matrices given are of size $\dim(S_\lambda)^2$. However we wish to decompose these matrix factorization further, to begin let us examine using H_p as a basis for A_p and $F_{p'}$ as a basis for $A_{p'}$ and restricting $\rho_z : H_p \to F_{p'}$. We obtain the following.

$$zH_T^V = \sum_{U, W \in \text{ST}(\lambda')} g_{U,T}^{W,V} F_U^W \tag{1}$$

Where $g_{U,T}^{W,V} \in B$ are the entries in this matrix. Recall that for $T, V \in \text{ST}(\lambda)$ we have that $F_T^V = \varepsilon_T x_T^V(\mu_T)$ and $H_T^V = \sigma_T \varepsilon_T x_T^V(\mu_T)$ where $\mu_T = \Pi_{i=1}^m \Pi_{j \in T_i} (x_j)^i$. First consider the group $P = \text{Perm}(T_i)$ for some $1 \leqslant i \leqslant m$ to be the group permuting the entries of T_i of m-tableaux T, then it is easy to see that

$$\begin{aligned} g.F_T^V &= g(\varepsilon_T.x_T^V)(\mu_T) \\ &= (g.\varepsilon_T.x_T^V)(g.\mu_T) \\ &= (g.\varepsilon_T.x_T^V)(g.\Pi_{i=1}^m \Pi_{j\in T_i}(x_j)^i) \\ &= (g.\varepsilon_T.x_T^V)(\Pi_{i=1}^m \Pi_{j\in T_i}(x_{g(j)})^i) \\ &= (g.\varepsilon_T.x_T^V)(\mu_T) \end{aligned}$$

Therefore applying the idempotent $\varepsilon_{T_i}.F_T^V = F_T^V$ since all the terms in ε_{T_i} belong to permutations in T_i and $\varepsilon_{T_i}\varepsilon_T = \varepsilon_T$, a similar argument can be made and we may show that $\sigma_{T_i}.H_T^V = H_T^V$. Using this we can decompose the matrix factorizations we obtained from ρ_z and ρ_j into smaller ones by analyzing the $g_{U,T}^{W,V}$ coefficients in equation (1)

Theorem 1: given $T \in \mathrm{ST}(\lambda)$ we have that multiplication by z will induce the map $z : H_T \to F_{T'}$ and $z : F_{T'} \to H_{T'}$

Proof: First consider $H_T^V \in H_T$, and recall the equation obtained by applying ρ_z to H_T^V we have

$$zH_T^V = \sum_{U,W\in \mathrm{ST}(\lambda')} g_{U,T}^{W,V} F_U^W$$

Recall that since $z \in A_{det}$ then for any $g \in S_n$ we have that $g.z = \mathrm{sgn}(g)z$, and that the row stabilizer $R(T_i)$ is the column stabilizer $C(T_i')$ of its conjugate tableaux, similarly $C(T_i) = R(T_i')$, with this we obtain the following

$$\begin{aligned} \varepsilon_{T_i'}(zH_T^V) &= \sum_{c\in C(T_i'),r\in R(T_i')} \mathrm{sgn}(c)cr(zH_T^V) \\ &= \sum_{c\in C(T_i'),r\in R(T_i')} z(\mathrm{sgn}(r)(cr(H_T^V))) \\ &= z(\sigma_{T_i}(H_T^V)) \\ &= z(\sigma_{T_i}(\sigma_{T_1}\cdots\sigma_{T_m}).x_T^V) \\ &= z(H_T^V) \end{aligned}$$

Thus $z(H_T^V)$ is invariant under the action of ε_{T_i} for any i. Now we look at the right hand side of the equation

$$\begin{aligned} \varepsilon_{T_i'}\Big(\sum_{U,W\in \mathrm{ST}(\lambda')} g_{U,T}^{W,V} F_U^W\Big) &= \sum_{U,W\in \mathrm{ST}(\lambda')} g_{U,T}^{W,V} \varepsilon_{T_i'} F_U^W \\ &= \sum_{U,W\in \mathrm{ST}(\lambda') \text{ and } U_j = T_i'} g_{U,T}^{W,V} F_U^W \end{aligned}$$

Therefore by applying $\varepsilon_{T_i'}$ we kill any higher Specht polynomial F_U^W of type λ' if $U = (U_1,\ldots,U_m)$ does not match one of it's j'th component with T_i'. From this we can conclude that by applying $\varepsilon_{T'}$ we would be left with term $F_{T'}^W$.

$$zH_T = \varepsilon_T(zH_T) = \varepsilon_T\Big(\sum_{U,W\in \mathrm{ST}(\lambda')} g_{U,T}^{W,V} F_U^W\Big) = \sum_{W\in \mathrm{ST}(\lambda')} g_{U,T}^{W,V} F_{T'}^W \in F_{T'}$$

□

With this theorem we obtain that each $T \in \mathrm{ST}(\lambda)$ we have a matrix M_T such that $M_T : H_T \to F_{T'}$ given by the map ρ_z. Note that we can define a similar matrix N_T for the map ρ_j, which together will form the matrix factorization of $M_T N_{T'} = N_{T'} M_T = \Delta I$.