STRATIFICATION OF DISCRIMINANTS AND IRREDUCIBLE REPRESENTATIONS

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ABSTRACT. not very

Some facts about stratification of discriminants

Should define what a stratum is!
Essentially: Let X be an algebraic variety of dimension d, start with a filtration of $X = X_d \supseteq X_{d-1} \supseteq \cdots \supseteq X_1 \supseteq X_0$, such that each $X_i - X_{i-1}$ is a smooth open subvariety of X_i or empty. (Each X_i is a subvariety of dimension i of X) Then the connected components Z_α of $X_i - X_{i-1}$ are called the strata. Note that the Z_α are smooth but their closure $\overline{Z_\alpha}$ not necessarily. Note that X can be written as a disjoint union of its strata.

Some basics about reflection groups and discriminants: let G be a complex reflection group acting on the k-vectorspace V, $\dim(V) = n$. We denote by $S = \mathrm{Sym}_k(V) \cong k[x_1, \ldots, x_n]$ and $R = S^G$. By the Chevalley–Shephard–Todd theorem REF $R \cong k[f_1, \ldots, f_n]$, where the basic invariants f_i are algebraically independent and homogeneous. They are not unique but their degrees $d_i = \deg(f_i)$ are.

Denote by $\pi : V \to V/G$ the canonical projection. The quotient space V/G is $\mathrm{Spec}(R)$ and smooth of dimension $\dim(V)$. The image of the reflection arrangement $\mathcal{A}$ in V is the discriminant $V(\Delta)$ in V/G.

1. STRATIFICATION OF THE DISCRIMINANT OF S_n

Usually we restrict to the invariant hyperplane $V(x_1 + \cdots + x_n)$ in V, so that S_n acts on k^{n-1}. By the theorem of Chevalley–Shephard–Todd the invariant ring $R = S^G$ is isomorphic to a polynomial ring in n (or $n-1$) variables, generated by invariant polynomials $f_1, \ldots, f_n$, where each $f_i \in S$ can be chosen homogeneous of degree i. Note that the f_i are not unique but their degrees are.
Popular choices for the f_i are the p_i or the symmetric polynomials $s_i := \sum_{k=1}^n x_k^i$.

For the case $G = S_n$ we have $\mathcal{A} = V(\prod_{i<j}(x_i - x_j))$ and Δ can be identified with the discriminant in a versal deformation of an A_n-singularity (see Arnold's paper REF).

This paragraph is a mess!More generally: Arnold proved this for all Coxeter groups with simply laced Dynkin diagrams, i.e. A_n, D_n, $E_{6,7,8}$: discriminant of the reflection group is difeomorphic to discriminant of versal deformations (there called singularities of wave fronts - the corresponding hyperplane arrangement is called caustic or bifurcation set

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in the dynamical systems speech). Later extended to crystallographic Coxeter groups: B_n, C_n, F_4, G_2 by WEYL?! see Book Arnold-Varchenko-Gusein-Zade:1 and finally also to H_2, H_3, H_4 by Lyashko, cf. [?, ?] (see paper by Shcherbak and its Mathsinet review by Janeczko for more detail)).

In particular, for any Coxeter group G one gets a natural stratification of the discriminant relating the strata to subgraphs of the Coxeter graph (see Section 6 of [?]).
In the case for $G = S_n$ the quotient V/G can be identified with the space of polynomials

$$F_n = x_n + \lambda_1 x^{n-1} + \ldots + \lambda_{n-1}$$

and the stratification of this space is given by the multiplicities of the roots of F_n. One can encode them into Young diagrams with n boxes and relate a subgraph of the Coxeter graph to each Young diagram (explained in [?, p.185]).

This might not be standard convention but following Shcherbak We let $n \geq k_1 \geq k_2 \geq \cdots \geq k_s \geq 1$ be a partition of n (i.e., $\sum_{i=1}^{s} k_i = n$), and the corresponding Young diagram be given with s columns, where the i-th column consists of k_i boxes. (i.e., the trivial representation corresponds to the partition $(1, \ldots, 1)$).

Let us now prove an interesting fact that is mentioned in [?, p.185] without proof:

Lem:rectangular **Lemma 1.1.** *Consider $G = S_n$ acting on V. Then the closure of the stratum corresponding to a rectangular Young diagram is a smooth subvariety of $V(\Delta)$. Should be if and only if!*

Proof. In order to have a rectangular Young diagram, n must be decomposable, say $n = lk$. Consider the partition $(k, \ldots, k)$ corresponding to the rectangular Young diagram with $l \times k$ boxes. This means that a point in the corresponding stratum of V/G is the image of a point $p \in V$ of the form

$$p = (\underbrace{\sigma_1, \ldots, \sigma_1}_{k}, \ldots, \underbrace{\sigma_l, \ldots, \sigma_l}_{k}),$$

where $\sigma_i \neq \sigma_j$ for $i \neq j$ (to be precise: the whole S_n-orbit of p projects to the same point in V/G). This means that p lies in the intersection of $l\binom{k}{2} = \frac{n(k-1)}{2}$ hyperplanes in the reflection arrangement. Note: we are not restricting to the hyperplane $x_1 + \cdots + x_n$ yet. Take the power sums $s_m = \sum_{i=1}^{n} x_i^m$ as basic invariants of R, i.e., $R = S^{S_n} = k[s_1, \ldots, s_n]$. We can restrict to the hyperplane by setting $s_1 = 0$. Note in particular that the s_m are algebraically independent for $m = 1, \ldots, n$.

Now $\pi(p) = (s_1(p), \ldots, s_n(p))$ evaluates to

$$(ks_1(\sigma_1, \ldots, \sigma_l), \ldots, ks_l(\sigma_1, \ldots, \sigma_l), \ldots, ks_n(\sigma_1, \ldots, \sigma_l)).$$

These are polynomials in the l distinct roots $\sigma_1, \ldots, \sigma_l$ Here this is a bit subtle: we denote also by s_m the power sums of the l variables, strictly speaking we should maybe denote them by $\tilde{s}_m$. Now since $\tilde{s}_1, \ldots, \tilde{s}_l$ are algebraically independent (they are the basic invariants for S_l), it follows that $\tilde{s}_i(\sigma_1, \ldots, \sigma_l)$ for $i = l+1, \ldots, n$ is a polynomial P_i in the $\tilde{s}_j$ for $j = 1, \ldots, l$. Hence the image of p can be written in the coordinates $\tilde{s}_i$ as

$$(k\tilde{s}_1, \ldots, k\tilde{s}_l, kP_{l+1}(\tilde{s}_1, \ldots, \tilde{s}_l), \ldots, kP_n(\tilde{s}_1, \ldots, \tilde{s}_l)).$$

This parametrizes an l-dimensional subvariety of V/G. The closure of this image has equations, now denoting the generators of S^{S_n} by $y_1, \ldots, y_n$:

$$y_{l+1} - kP_{l+1}(\frac{y_1}{k}, \ldots, \frac{y_l}{k}), \ldots, y_n - kP_n(\frac{y_1}{k}, \ldots, \frac{y_l}{k}).$$

Denote the ideal generated by these $n-l$ equations by I. Then $V(I)$ is a complete intersection subvariety of $V/G = \mathrm{Spec}(k[y_1, \ldots, y_n])$ of codimension $n-l$, i.e., $V(I)$ has dimension l. Clearly (Jacobian criterion! - since $k \geq 2$, there is a $l \times l$-minor of the Jacobian matrix of I that is the identity matrix $\mathbb{1}_l$) this subvariety is smooth and is isomorphic to $\mathrm{Spec}(k[y_1, \ldots, y_l])$. □

–Simon–

Lemma 1.2. *Consider $G = S_n$ acting on V. Then the closure of the stratum corresponding to the partition $(j, n-j)$ where $0 < j < n$ and $j \neq \frac{n}{2}$ is a subvariety of $V(\Delta)$ isomorphic to a cusp singularity.*

Proof. We note that when $j = \frac{n}{2}$, then the closure of the stratum corresponding to the partition $(\frac{n}{2}, \frac{n}{2})$ is smooth.

Again we take the powers sums $s_m = \sum_{i=1}^{n} x_i^m$ as basic invariants of R.

$$p = (\underbrace{\sigma_1, \ldots, \sigma_1}_{j}, \underbrace{\sigma_2, \ldots, \sigma_2}_{n-j}),$$

Now $\pi(p) = (s_1(p), \ldots, s_n(p))$ evaluates to

$$(j\sigma_1 + (n-j)\sigma_2, j\sigma_1^2 + (n-j)\sigma_2^2, ..., j\sigma_1^n + (n-j)\sigma_2^n)$$

Now restricting to the hyperplane $s_1 = 0$ we get the relation $-j\sigma_1 = (n-j)\sigma_2$, rearranging to get $\sigma_2 = -\frac{j}{(n-j)}\sigma_1$. After this restriction, the projection becomes:

$$\pi(p) = (s_1(p), \ldots, s_n(p)) = (0, \frac{j(n-j) + (-j)^2}{n-j})\sigma_1^2, ..., \frac{j(n-j)^{n-1} + (-j)^n}{(n-j)^{n-1}})\sigma_1^n).$$

Again if $j = \frac{n}{2}$ then only we see that $s_i(p) = 0$ for all odd i - leading to a parabola.

For cleanliness sake, Let $a_i := \frac{j(n-j)^{i-1} + (-j)^i}{(n-j)^{i-1}}$, the above becomes:

$$\pi(p) = (s_1(p), \ldots, s_n(p)) = (0, a_1\sigma_1^2, a_2\sigma_1^3, ..., a_{n-1}\sigma_1^n).$$

The closure of the image of π is a subvariety of $V(\Delta)$ given by the equations:

$$s_1, a_1^3 s_3^2 - a_2^2 s_2^3.$$

and let m be an integer such that $3 < m$ and $m = 2m_1 + 3$, then we get the relations:

$$a_1^{m_1} a_2 s_m - a_{m-1} s_2^{m_1} s_3$$

Which is isomorphic (unneeded?) a cusp singularity.

□

2. Questions

1. Description of the other strata? In particular: which singularities arise (always the same ones?)? Find all strata for the S_5-discriminant
2. Similar statement as Lemma 1.1 for other complex reflection groups? At least for the crystallographic Coxeter groups?
3. Find the fitting ideals for the MCM-modules coming from the irreducible representations of G: do they correspond to certain strata (closures of them)? Which ones? Work this out for S_5! Also interesting: B_3
4. What about connection between discriminants of deformations and reflection groups for non-Coxeter groups: do we at least get some statement for the true reflection groups? Haven't really thought about this, but the literature is all about real reflection groups - in Orlik–Terao is also stratification of discriminant for any complex reflection group via fitting ideals, but this is too coarse

3. General facts about discriminants

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