Hochschild (co)homology of exterior algebras¹

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Abstract: The minimal projective bimodule resolutions of the exterior algebras are explicitly constructed. They are applied to calculate the Hochschild (co)homology of the exterior algebras. Thus the cyclic homology of the exterior algebras can be calculated in case the underlying field is of characteristic zero. Moreover, the Hochschild cohomology rings of the exterior algebras are determined by generators and relations.

Keyword: Hochschild (co)homology, exterior algebra, minimal projective resolution

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0. Introduction

Fix a field k. Let Λ be a finite-dimensional k-algebra (associative with identity). Denote by Λ^e the enveloping algebra of Λ , i.e., the tensor product $\Lambda \otimes_k \Lambda^{op}$ of the algebra Λ and its opposite Λ^{op} . The Hochschild homology and cohomology of Λ are defined by

$$HH_m(\Lambda) = \operatorname{Tor}_m^{\Lambda^e}(\Lambda, \Lambda)$$
 and $HH^m(\Lambda) = \operatorname{Ext}_{\Lambda^e}^m(\Lambda, \Lambda)$

respectively [27]. The Hochschild (co)homology of an algebra have played a fundamental role in representation theory of artin algebras: Hochschild cohomology is closely related to simple connectedness, separability and deformation theory [31, 1, 15]; Hochschild homology is closely related to the oriented cycle and the global dimension of algebras [21, 24, 3, 18, 23].

Though Hochschild (co)homology is theoretically computable for a concrete algebra via derived functors, actual calculation for a class of algebras is still very convenient, important and difficult. So far the Hochschild co-homology was calculated for hereditary algebras [8, 20], incidence algebras

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[7, 16], algebras with narrow quivers [20, 8], radical square zero algebras [9], monomial algebras [10], truncated quiver algebras [11, 34, 25], special biserial algebras as well as their trivial extensions [32, 19], and so on. The Hochschild homology was calculated for truncated algebras [24, 35, 30], quasi-hereditary algebras [33], monomial algebras [18], and so on.

The exterior algebras play extremely important roles in many mathematical branches such as algebraic geometry, commutative algebra, differential geometry. It is well-known that the exterior algebras can be viewed as $\mathbb{Z}/2$ -graded algebras [22]. The graded exterior algebras (called *Grass*mann algebras as well [13]) have applications in physics and their Hochschild (co)homology and cyclic (co)homology were known (cf. [22, 12, 13, 14]). However, up to now the Hochschild (co)homology of the ungraded exterior algebras (called parity-free Grassmann algebras as well [13]) are still unknown. In this paper we shall deal with this problem. Our method is purely algebraic and combinatorial. From now on all the exterior algebras are ungraded. The content of this paper is organized as follows: In section 1, we shall provide the minimal projective bimodule resolutions of the exterior algebras. In section 2, we shall apply these minimal projective bimodule resolutions to calculate the Hochschild homology of the exterior algebras and their cyclic homology in case the underlying field is of characteristic zero. In section 3, we shall apply these minimal projective bimodule resolutions to calculate the Hochschild cohomology of the exterior algebras. In section 4, we shall determine the Hochschild cohomology rings of the exterior algebras by generators and relations.

1 Minimal projective bimodule resolutions

Let Q be the quiver given by one point 1 and n-loops $x_1, x_2, ..., x_n$ with $n \geq 2$. Denote by I the ideal of the path algebra kQ generated by $R := \{x_i^2 | 1 \leq i \leq n\} \cup \{x_i x_j + x_j x_i | 1 \leq i < j \leq n\}$. For the knowledge on quiver we refer to [2]. Set $\Lambda = kQ/I$. Then Λ is just the exterior algebra over k (cf. [28]). Order the paths in Q by left length lexicographic order by choosing $1 < x_1 < x_2 < \cdots < x_n$, namely, $y_1 \cdots y_s < z_1 \cdots z_t$ with y_i and z_i being arrows if s < t or if s = t, for some $1 \leq r \leq s$, $y_i = z_i$ for $1 \leq i < r$ and $y_r < z_r$. Then Λ has a basis $\mathcal{B} = \bigcup_{i=0}^n \mathcal{B}_i$, where $\mathcal{B}_i = \{x_{t_1} x_{t_2} \cdots x_{t_i} | 1 \leq t_1 < t_2 < \cdots < t_i \leq n\}$. So $\dim_k \Lambda = 2^n$. It is well-known that Λ is a Koszul algebra and its quadratic dual is just the algebra of polynomials $k[x_1, ..., x_n]$ (cf. [4]).

Now we construct a minimal projective bimodule resolution $(P_{\bullet}, \delta_{\bullet})$ of Λ . Denote by $k\langle x_1, ..., x_n \rangle$ the noncommutative free associative algebra over k with free generators $x_1, ..., x_n$. Denote by $k\langle x_1, ..., x_n \rangle_m$ the k-subspace

of $k\langle x_1,...,x_n\rangle$ generated by all monomials of degree m. For each $m\geq 0$, we firstly construct elements $\{f_{1^{i_1}2^{i_2}...n^{i_n}}^m|i_1+i_2+\cdots+i_n=m,(i_1,...,i_n)\in\mathbb{N}^n\}\subseteq k\langle x_1,...,x_n\rangle_m$: Let $f_0^0=1, f_1^1=x_1, f_2^1=x_2,..., f_n^1=x_n$. Define $f_{1^{i_1}2^{i_2}...n^{i_n}}^m$ for all $m\geq 2$ inductively by $f_{1^{i_1}2^{i_2}...n^{i_n}}^m=\sum_{h=1}^n f_{1^{i_1}...h^{i_h-1}...n^{i_n}}^{m-1}x_h$, where $i_1+i_2+\cdots+i_n=m,(i_1,...,i_n)\in\mathbb{N}^n$ and $f_{1^{i_1}...h^{-1}...n^{i_n}}^{m-1}=0$ for all $1\leq h\leq n$. It is well-known that the number of non-negative integral solutions of the equation $i_1+i_2+\cdots+i_n=m$ on $i_1,...,i_n$ is $\binom{n+m-1}{n-1}$ for any given positive integers n and m.

Denote $\otimes := \otimes_k$. Let $P_m := \coprod_{i_1+i_2+\dots+i_n=m} \Lambda \otimes f^m_{1^{i_1}2^{i_2}\dots n^{i_n}} \otimes \Lambda \subseteq \Lambda \otimes k\langle x_1,\dots,x_n\rangle_m \otimes \Lambda$ for $m\geq 0$, and let $\widetilde{f}^m_{1^{i_1}2^{i_2}\dots n^{i_n}} := 1\otimes f^m_{1^{i_1}2^{i_2}\dots n^{i_n}} \otimes 1$ for $m\geq 1$ and $\widetilde{f}^0_0=1\otimes 1$. Note that we identify P_0 with $\Lambda\otimes\Lambda$. Define $\delta_m:P_m\to P_{m-1}$ by setting

$$\delta_m(\widetilde{f}_{1^{i_1}2^{i_2}\cdots n^{i_n}}^m) = \sum_{h=1}^n (x_h \widetilde{f}_{1^{i_1}\cdots h^{i_h-1}\cdots n^{i_n}}^{m-1} + (-1)^m \widetilde{f}_{1^{i_1}\cdots h^{i_h-1}\cdots n^{i_n}}^{m-1} x_h).$$

Theorem 1. The complex $(P_{\bullet}, \delta_{\bullet})$

$$\cdots \to P_{m+1} \xrightarrow{\delta_{m+1}} P_m \xrightarrow{\delta_m} \cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \longrightarrow 0$$

is a minimal projective bimodule resolution of the exterior algebra $\Lambda = kQ/I$.

Proof. For Koszul algebra Λ we can construct its Koszul complex which is a minimal projective resolution of the trivial Λ -module k (cf. [4, Section 2.6]). Furthermore we can use the Koszul complex to construct the bimodule Koszul complex which is a minimal projective bimodule resolution of Λ (cf. [6, Section 9]). Let $X = k\{x_1, x_2, \cdots, x_n\}$. We show that $\{f_{1^{i_1}2^{i_2}...n^{i_n}}^m|i_1+i_2+\cdots+i_n=m, (i_1,...,i_n)\in\mathbb{N}^n\}$ is a k-basis of the k-vector space $K_m:=\bigcap_{p+q=m-2}X^pRX^q$ for all $m\geq 2$: Firstly, we verify $f_{1^{i_1}2^{i_2}...n^{i_n}}^m\in K_m$: It is

clear that $f_{1^{i_1}2^{i_2}\cdots n^{i_n}}^m=\sum_{h=1}^n f_{1^{i_1}\cdots h^{i_h-1}\cdots n^{i_n}}^{m-1}x_h=\sum_{h=1}^n x_h f_{1^{i_1}\cdots h^{i_h-1}\cdots n^{i_n}}^{m-1}$. Thus the assertion follows by induction on m. Secondly, $\{f_{1^{i_1}2^{i_2}\cdots n^{i_n}}^m|i_1+i_2+\cdots+i_n=m,(i_1,\ldots,i_n)\in\mathbb{N}^n\}$ is k-linearly independent: Indeed, by induction, one can show that each monomial in $f_{1^{i_1}2^{i_2}\cdots n^{i_n}}^m$ contains just i_1 x_1 's, i_2 x_2 's, \dots , i_n x_n 's. Thirdly, we have $\dim_k K_m = \binom{n+m-1}{n-1}$: The quadratic dual of the Koszul algebra Λ is just the algebra of polynomials $k[x_1,\ldots,x_n]$ which is isomorphic to the Yoneda algebra $E(\Lambda) = \coprod_{m\geq 0} \operatorname{Ext}_{\Lambda}^m(k,k)$ of Λ (cf. [4, Theorem 2.10.1]).

Thus $\dim_k K_m = \dim_k \operatorname{Ext}_{\Lambda}^m(k,k) = \binom{n+m-1}{n-1}$. Hence $\{f_{1i_1 2i_2 \dots n^{i_n}}^m | i_1 + i_2 + \dots + i_n \}$

 $i_n = m$ } is a k-basis of K_m . Therefore P_{\bullet} are just those projective bimodules in the bimodule Koszul complex of Λ (cf. [6, Section 9]). Furthermore, δ_{\bullet} are just those differentials in the bimodule Koszul complex of Λ (cf. [6, p. 354]).

2 Hochschild homology

In this section we calculate the k-dimensions of Hochschild homology groups and cyclic homology groups (in case char k = 0) of the exterior algebras.

Applying the functor $\Lambda \otimes_{\Lambda^e}$ – to the minimal projective bimodule resolution $(P_{\bullet}, \delta_{\bullet})$, we have $\Lambda \otimes_{\Lambda^e} (P_{\bullet}, \delta_{\bullet}) = (M_{\bullet}, \tau_{\bullet})$ where $M_m = \coprod_{i_1 + i_2 + \dots + i_n = m} \Lambda \otimes I_{i_1 + i_2 + \dots + i_n = m} \Lambda \otimes I_{i_1 + i_2 + \dots + i_n = m}$

 $f_{1^{i_1}2^{i_2}\cdots n^{i_n}}^m$, $\tau_m(\lambda\otimes f_{1^{i_1}2^{i_2}\cdots n^{i_n}}^m)=\sum\limits_{h=1}^n(\lambda x_h\otimes f_{1^{i_1}\cdots h^{i_h-1}\cdots n^{i_n}}^{m-1}+(-1)^mx_h\lambda\otimes f_{1^{i_1}\cdots h^{i_h-1}\cdots n^{i_n}}^{m-1})$ for $m\geq 1$, and $M_0=\Lambda\otimes k$. Throughout we assume that the combinatorial number $\binom{0}{0}=1$ and $\binom{i}{j}=0$ if i< j. Write p(j)=p(m) if j and m are of the same parity.

Lemma 1. For $m \geq 1$, we have

$$\operatorname{rank} \tau_m = \begin{cases} \sum_{i=1}^n \binom{n}{i} \sum_{\substack{j=0 \\ p(j)=p(m)}}^{i-1} \binom{j+m-1}{i-1} \binom{i-1}{j}, & \text{if char } k \neq 2; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Denote by $k[x_1, ..., x_n]_m$ the subspace of $k[x_1, ..., x_n]$ generated by all monomials of degree m. Obviously the complex $(M_{\bullet}, \tau_{\bullet})$ is isomorphic to the complex $(N_{\bullet}, \sigma_{\bullet})$ which is defined by $N_m := \Lambda \otimes k[x_1, ..., x_n]_m$ and $\sigma_m : N_m \to N_{m-1}, \lambda \otimes x_1^{i_1} \cdots x_n^{i_n} \mapsto \sum_{h=1}^n (\lambda x_h \otimes x_1^{i_1} \cdots x_h^{i_h-1} \cdots x_n^{i_n} + (-1)^m x_h \lambda \otimes x_1^{i_1} \cdots x_h^{i_h-1} \cdots x_n^{i_n}).$

For any $\lambda = x_{t_1} \cdots x_{t_j} \in \mathcal{B}$, define $\mu_{\lambda}(h) := |\{t_l | t_l < h, 1 \le l \le j\}|$. Then

$$\sigma_m(\lambda \otimes x_1^{i_1} \cdots x_n^{i_n}) = ((-1)^j + (-1)^m) \sum_{h=1}^n (-1)^{\mu_\lambda(h)} N(\lambda x_h) \otimes x_1^{i_1} \cdots x_h^{i_h-1} \cdots x_n^{i_n}$$

where $N(\lambda x_h) := x_{t_1} \cdots x_{t_{\mu_{\lambda}(h)}} x_h x_{t_{\mu_{\lambda}(h)+1}} \cdots x_{t_j}$, called the *normal form* of λx_h in Λ . Note that the meaning of the normal form above is completely different from that in the Gröbner basis theory (cf. [17]). It follows from the formula above that $\sigma_m(\lambda \otimes x_1^{i_1} \cdots x_n^{i_n}) = 0$ if $p(j) \neq p(m)$.

Clearly, N_m has a basis $\mathcal{N}_m := \{\lambda \otimes x_1^{i_1} \cdots x_n^{i_n} \mid \lambda \in \mathcal{B}, i_1 + i_2 + \cdots + i_n = m\}$. If $\lambda = x_{t_1} \cdots x_{t_j} \in \mathcal{B}_j$ then j is called the degree of $\lambda \otimes x_1^{i_1} \cdots x_n^{i_n}$,

and $j + \sum_{h=1}^{n} i_h$ is called the *total degree* of $\lambda \otimes x_1^{i_1} \cdots x_n^{i_n}$. If, viewed as a monomial in $k[x_1, ..., x_n]$, $\lambda x_1^{i_1} \cdots x_n^{i_n}$ can be written as $x_1^{j_1} \cdots x_n^{j_n}$, then the set $\{l|j_l \neq 0, 1 \leq l \leq n\} = \{t_1, \cdots, t_j\} \cup \{l|i_l \neq 0, 1 \leq l \leq n\}$ is called the *support* of $\lambda \otimes x_1^{i_1} \cdots x_n^{i_n}$ and the cardinal number of the support $|\{l|j_l \neq 0, 1 \leq l \leq n\}|$ is called the *grade* of $\lambda \otimes x_1^{i_1} \cdots x_n^{i_n}$.

Firstly, denote by $N_m(i)$ the subspace of N_m generated by all elements in \mathcal{N}_m of grade i. Since σ_m keeps the grade of the elements, we have rank $\sigma_m = \sum_{i=1}^n \operatorname{rank}(\sigma_m|N_m(i))$.

Secondly, denote by $N_m(x_{s_1} \cdots x_{s_i})$ the subspace of $N_m(i)$ generated by all elements $\lambda \otimes x_1^{i_1} \cdots x_n^{i_n}$ in \mathcal{N}_m with support $\{s_1, \cdots, s_i\}$. Here $1 \leq s_1 < s_2 < \cdots < s_i \leq n$. The restrictions of σ_m to $N_m(x_{s_1} \cdots x_{s_i})$ and $N_m(x_{s_1'} \cdots x_{s_i'})$ are given by the same matrices if adequate bases are chosen, thus rank $\sigma_m = \sum_{i=1}^n \binom{n}{i} \operatorname{rank}(\sigma_m | N_m(x_1 \cdots x_i))$.

Thirdly, denote by $N_m(x_1\cdots x_i,j)$ the subspace of $N_m(x_1\cdots x_i)$ generated by all elements $\lambda\otimes x_1^{l_1}\cdots x_i^{l_i}$ in \mathcal{N}_m of degree j (equivalently, of total degree j+m) with $0\leq j\leq i$. Moreover, denote by $N_m(x_1^{j_1}\cdots x_i^{j_i},j)$ the subspace of $N_m(x_1\cdots x_i,j)$ generated by all elements $\lambda\otimes x_1^{l_1}\cdots x_i^{l_i}$ in \mathcal{N}_m with $\lambda x_1^{l_1}\cdots x_i^{l_i}=x_1^{j_1}\cdots x_i^{j_i},j_1,...,j_i\geq 1$ and $\sum_{l=1}^i j_l=j+m$. Consider the basis $\mathcal{N}_m(x_1^{j_1}\cdots x_i^{j_i},j):=\mathcal{N}_m\cap N_m(x_1^{j_1}\cdots x_i^{j_i},j)$ of the vector space $N_m(x_1^{j_1}\cdots x_i^{j_i},j)$. Order the elements $\lambda\otimes x_1^{l_1}\cdots x_i^{l_i}$ in $\mathcal{N}_m(x_1^{j_1}\cdots x_i^{j_i},j)$ by the left lexicographic order on λ . Obviously σ_m maps $N_m(x_1^{j_1}\cdots x_i^{j_i},j)$ into $N_{m-1}(x_1^{j_1}\cdots x_i^{j_i},j+1)$ for $0\leq j\leq i-1$ and into 0 for j=i. Let $\alpha:N_m(x_1^{j_1}\cdots x_i^{j_i},j)\to N_{m-1}(x_1^{j_1}\cdots x_i^{j_i},j+1)$ with $0\leq j\leq i-1$ be the restriction σ_m . Written as a matrix under the basis $\mathcal{N}_m(x_1^{j_1}\cdots x_i^{j_i},j)$ of $N_m(x_1^{j_1}\cdots x_i^{j_i},j)$ and $\mathcal{N}_{m-1}(x_1^{j_1}\cdots x_i^{j_i},j+1)$ of $N_{m-1}(x_1^{j_1}\cdots x_i^{j_i},j+1)$, α is a $\binom{j+1}{j+1}\times\binom{j}{j}$ matrix. Partition the elements $\lambda\otimes x_1^{l_1}\cdots x_i^{l_i}$ in $\mathcal{N}_m(x_1^{j_1}\cdots x_i^{j_i},j)$ and $\mathcal{N}_{m-1}(x_1^{j_1}\cdots x_i^{j_i},j+1)$ according to whether λ contains x_1 or not. In this way, neglected the sign $(-1)^j+(-1)^m$, this matrix is partitioned into a 2×2 partitioned matrix $\begin{bmatrix} A & I \\ i+1 \end{pmatrix}$ where A is an $\binom{i-1}{j}\times\binom{i-1}{j-1}$ matrix, B is an $\binom{i-1}{j+1}\times\binom{i-1}{j}$ matrix and I is an $\binom{i-1}{j}\times\binom{i-1}{j}$ identity matrix.

Claim. BA = 0.

Proof of the Claim. Take any row of B. Assume that it corresponds to the element $x_{q_1}\cdots x_{q_{j+1}}^{j_1}\otimes x_1^{j_1}\cdots x_{q_1}^{j_{q_1}-1}\cdots x_{q_{j+1}}^{j_{q_{j+1}}-1}\cdots x_i^{j_i}\in \mathcal{N}_{m-1}(x_1^{j_1}\cdots x_i^{j_i},j+1)$. Then, according to the definition of σ_m , this row is just the vector whose components corresponding to $x_{q_1}\cdots x_{q_j}\otimes x_1^{j_1}\cdots x_{q_1}^{j_{q_1}-1}\cdots x_{q_j}^{j_{q_j}-1}\cdots x_i^{j_i},\ldots,$

 $x_{q_2}\cdots x_{q_{j+1}}\otimes x_1^{j_1}\cdots x_{q_2}^{j_{q_2}-1}\cdots x_{q_{j+1}}^{j_{q_{j+1}}-1}\cdots x_i^{j_i} \text{ are } (-1)^j,\dots,(-1)^0 \text{ respectively,}$ and other components are 0. Take any column of A. Assume that it corresponds to the element $x_1x_{q_1}\cdots \hat{x}_{q_a}\cdots \hat{x}_{q_b}\cdots x_{q_{j+1}}^{j_{q_{j+1}}-1}\cdots x_{q_a}^{j_{q_1}-1}\cdots x_{q_a}^{j_{q_a}}\cdots x_{q_b}^{j_{q_{j+1}}-1}\cdots x_{q_i}^{j_i}\cdots x_i^{j_{q_a}}\cdots x_{q_b}^{j_{q_{j+1}}-1}\cdots x_i^{j_i}\in \mathcal{N}_m(x_1^{j_1}\cdots x_i^{j_i},j).$ Here \hat{x} means x is deleted. This column is just the vector whose components corresponding to $x_1x_{q_1}\cdots x_{q_a}\cdots \hat{x}_{q_b}\cdots x_{q_{j+1}}\otimes x_1^{j_1-1}\cdots x_{q_1}^{j_{q_1}-1}\cdots x_{q_a}^{j_{q_a}-1}\cdots x_{q_b}^{j_{q_b}}\cdots x_{q_{j+1}}^{j_{q_{j+1}}-1}\cdots x_i^{j_i} \text{ and } x_1x_{q_1}\cdots \hat{x}_{q_a}\cdots \hat{x}_{q_a}\cdots x_{q_b}\cdots x_{q_{j+1}}\otimes x_1^{j_1-1}\cdots x_{q_1}^{j_{q_1}-1}\cdots x_{q_a}^{j_{q_a}}\cdots x_{q_b}^{j_{q_b}-1}\cdots x_{q_{j+1}}^{j_{q_{j+1}}-1}\cdots x_i^{j_i} \text{ are } (-1)^a$ and $(-1)^{b-1}$ respectively, and other components are 0. Thus the inner product of this row of B and this column of A is just $(-1)^{b-1}(-1)^a+(-1)^{a-1}(-1)^{b-1}=0$. Hence the claim holds.

Note that under the chosen basis α is the matrix $((-1)^j + (-1)^m) \begin{bmatrix} A & I \\ 0 & B \end{bmatrix}$. Multiply $\begin{bmatrix} A & I \\ 0 & B \end{bmatrix}$ on the left by the invertible matrix $\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}$, we obtain $\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} \begin{bmatrix} A & I \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & I \\ -BA & 0 \end{bmatrix} = \begin{bmatrix} A & I \\ 0 & 0 \end{bmatrix}$. Hence $\operatorname{rank} \begin{bmatrix} A & I \\ 0 & B \end{bmatrix} = \begin{pmatrix} i-1 \\ j \end{pmatrix}$. Thus

$$\operatorname{rank}(\sigma_m|N_m(x_1^{j_1}\cdots x_i^{j_i},j)) = \operatorname{rank}\alpha$$

$$= \begin{cases} \binom{i-1}{j}, & \text{if } p(j) = p(m) \text{ and char } k \neq 2; \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$\operatorname{rank}(\sigma_{m}|N_{m}(x_{1}\cdots x_{i})) = \sum_{j=0}^{i-1} \sum_{\substack{j_{1}+\cdots+j_{i}=j+m\\j_{1},\dots,j_{i}\geq 1}} \operatorname{rank}(\sigma_{m}|N_{m}(x_{1}^{j_{1}}\cdots x_{i}^{j_{i}},j))$$

$$= \begin{cases} \sum_{\substack{j=0\\p(j)=p(m)\\0,\\0,\\0,\\0}} {j+m-1\choose i-1} {i-1\choose j}, & \text{if char } k\neq 2; \\ 0, & \text{otherwise.} \end{cases}$$

we have

$$\operatorname{rank} \sigma_{m} = \sum_{i=1}^{n} \binom{n}{i} \operatorname{rank} (\sigma_{m} | N_{m}(x_{1} \cdots x_{i}))$$

$$= \begin{cases} \sum_{i=1}^{n} \binom{n}{i} \sum_{\substack{j=0 \ p(j)=p(m)}}^{i-1} \binom{j+m-1}{i-1} \binom{i-1}{j}, & \text{if char } k \neq 2; \\ 0, & \text{otherwise.} \end{cases}$$

The lemma follows from the fact that $\operatorname{rank}\tau_m = \operatorname{rank}\sigma_m$.

Lemma 2.
$$\sum_{i=j+1}^{n} {n \choose i} {j+m-1 \choose i-1} {i-1 \choose j} = \sum_{i=1}^{n-j} {n-i \choose j} {m+n-i-1 \choose n-i}$$
.

Proof. For $0 \le j \le n-1$, $m \ge 1-j$ and $0 \le r \le i-1$, define $S_{m,r} = \sum_{i=j+1}^{n} \binom{n}{i} \binom{j+m-1}{i-1-r} \binom{i-1}{j}$, and $T_{m,r} = \sum_{i=1}^{n-j} \binom{n-i}{j} \binom{m+n-i-1}{n-i-r}$. We shall show that $S_{m,r} = T_{m,r}$ for all $0 \le r \le n-1$ and $m \ge 1-i$.

Firstly, we have $S_{m,n-1} = T_{m,n-1}$ for all $m \ge 1 - j$: This follows from $S_{m,n-1} = \sum_{i=j+1}^{n} \binom{n}{i} \binom{j+m-1}{i-n} \binom{i-1}{j} = \binom{n}{n} \binom{j+m-1}{0} \binom{n-1}{j} = \binom{n-1}{j}$ and $T_{m,n-1} = \binom{n-j}{n-j}$

$$\sum_{i=1}^{n-j} \binom{n-i}{j} \binom{m+n-i-1}{1-i} = \binom{n-1}{j}.$$

Secondly, we have $S_{1-j,r} = T_{1-j,r}$ for all $0 \le r \le n-1$: Note that $S_{1-j,r} = \sum_{i=j+1}^{n} \binom{n}{i} \binom{0}{i-1-r} \binom{i-1}{j}$ and $T_{1-j,r} = \sum_{i=1}^{n-j} \binom{n-i}{j} \binom{n-i-j}{n-i-r}$. If r < i then $S_{1-j,r} = 0 = \sum_{i=j+1}^{n-j} \binom{n-i}{i} \binom{n-i-j}{n-i-r}$.

$$T_{1-j,r}$$
. If $r \ge i$ then $S_{1-j,r} = \binom{n}{r+1} \binom{r}{j}$ and $T_{1-j,r} = \sum_{i=1}^{n-j} \binom{n-i}{r} \binom{r}{i} = \binom{n}{r+1} \binom{r}{j}$.

Thirdly, we have $S_{m,r} = T_{m,r}$ for all $0 \le r \le n-1$ and $m \ge 1-j$: Since $S_{m,r} = S_{m-1,r+1} + S_{m-1,r}$ and $T_{m,r} = T_{m-1,r+1} + T_{m-1,r}$, we have

$$S_{m,r} = S_{m-1,r+1} + S_{m-2,r+1} + \dots + S_{1-j,r+1} + S_{1-j,r}$$

and

$$T_{m,r} = T_{m-1,r+1} + T_{m-2,r+1} + \dots + T_{1-j,r+1} + T_{1-j,r}.$$

Thus

$$S_{m,n-2} = S_{m-1,n-1} + S_{m-2,n-1} + \dots + S_{1-j,n-1} + S_{1-j,n-2}$$

= $T_{m-1,n-1} + T_{m-2,n-1} + \dots + T_{1-j,n-1} + T_{1-j,n-2}$
= $T_{m,n-2}$

for all $m \geq 1-j$. Similarly, by induction, we have $S_{m,r} = T_{m,r}$ for all $0 \leq r \leq n-1$ and $m \geq 1-j$.

Fourthly and finally, Lemma 2 follows from $S_{m,0} = T_{m,0}$.

Lemma 3. If char $k \neq 2$ and $m \geq 1$ then

rank
$$\tau_m = \sum_{i=1}^{n-1} 2^{i-1} \binom{m+i-1}{i} + \begin{cases} 1, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. By Lemma 2, we have

$$\operatorname{rank} \tau_{m} = \sum_{\substack{j=0 \ p(j)=p(m)}}^{n-1} \sum_{i=j+1}^{n} \binom{n}{i} \binom{j+m-1}{i-1} \binom{i-1}{j}$$

$$= \sum_{\substack{j=0 \ p(j)=p(m)}}^{n-1} \sum_{i=1}^{n-j} \binom{n-i}{j} \binom{m+n-i-1}{n-i}$$

$$= \sum_{i=1}^{n} \sum_{\substack{j=0 \ p(j)=p(m)}}^{n-i} \binom{n-i}{j} \binom{m+n-i-1}{n-i}$$

$$= \sum_{i=1}^{n} \binom{m+n-i-1}{n-i} \sum_{\substack{j=0 \ p(j)=p(m)}}^{n-i} \binom{n-i}{j}$$

$$= \sum_{i=1}^{n-1} 2^{n-i-1} \binom{m+n-i-1}{n-i} + \begin{cases} 1, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

$$= \sum_{i=1}^{n-1} 2^{i-1} \binom{m+i-1}{i} + \begin{cases} 1, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

In the last step we replace n-i with i.

Lemma 4. If char $k \neq 2$ and $m \geq 1$ then

rank
$$\tau_m + \text{rank } \tau_{m+1} = 2^{n-1} \binom{m+n-1}{n-1}.$$

Proof. Denote $U_n^m := \sum_{i=1}^{n-1} 2^{i-1} {m+i-1 \choose i}$. Then $U_n^{m+1} = \sum_{i=1}^{n-1} 2^{i-1} {m+i \choose i}$ and $2U_n^{m+1} = \sum_{i=1}^{n-1} 2^i {m+i \choose i}$. So $-U_n^{m+1} = {m+1 \choose 1} + \sum_{i=2}^{n-1} 2^{i-1} {m+i-1 \choose i} - 2^{n-1} {m+n-1 \choose n-1} = U_n^m - 2^{n-1} {m+n-1 \choose n-1} + 1$. Thus $U_n^m + U_n^{m+1} = 2^{n-1} {m+n-1 \choose n-1} - 1$. By Lemma 3, we have rank $\tau_m = U_n^m + \begin{cases} 1, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$ Thus rank $\tau_m + \text{rank } \tau_{m+1} = U_n^m + \begin{cases} 1, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd.} \end{cases} + U_n^{m+1} + \begin{cases} 1, & \text{if } m+1 \text{ is even;} \\ 0, & \text{if } m+1 \text{ is odd.} \end{cases} = 2^{n-1} {m+n-1 \choose n-1}.$

Now we can calculate the Hochschild homology of the exterior algebra:

Theorem 2. Let $\Lambda = kQ/I$ be the exterior algebra. Then

$$\dim_k HH_m(\Lambda) = \begin{cases} 2^n \binom{n+m-1}{n-1}, & \text{if char } k = 2. \\ 2^{n-1} + 1, & \text{if } m = 0 \text{ and char } k \neq 2; \\ 2^{n-1} \binom{n+m-1}{n-1}, & \text{if } m \geq 1 \text{ and char } k \neq 2; \end{cases}$$

Proof. Case char $k \neq 2$: If $m \geq 1$ then, by Lemma 4,

$$\dim_k HH_m(\Lambda) = \dim_k \operatorname{Ker} \tau_m - \dim_k \operatorname{Im} \tau_{m+1}$$

$$= \dim_k P_m - \dim_k \operatorname{Im} \tau_m - \dim_k \operatorname{Im} \tau_{m+1}$$

$$= 2^n \binom{n+m-1}{n-1} - \operatorname{rank} \tau_m - \operatorname{rank} \tau_{m+1}$$

$$= 2^{n-1} \binom{n+m-1}{n-1}$$

If m=0 then $HH_0(\Lambda)=\Lambda/[\Lambda,\Lambda]$, where $[\Lambda,\Lambda]=\{\lambda_1\lambda_2-\lambda_2\lambda_1|\lambda_1,\lambda_2\in\Lambda\}$. Note that as a vector space $[\Lambda,\Lambda]$ is generated by all commutators $x_{i_1}\cdots x_{i_s}x_{j_1}\cdots x_{j_t}-x_{j_1}\cdots x_{j_t}x_{i_1}\cdots x_{i_s}=(1+(-1)^{st+1})x_{i_1}\cdots x_{i_s}x_{j_1}\cdots x_{j_t}$ for all $x_{i_1}\cdots x_{i_s},x_{j_1}\cdots x_{j_t}\in\mathcal{B}$. If both s and t are odd then $x_{i_1}\cdots x_{i_s}x_{j_1}\cdots x_{j_t}\in[\Lambda,\Lambda]$. If $r\geq 2$ is even then r=(r-1)+1. Thus $x_1\cdots x_r\in[\Lambda,\Lambda]$. This implies that $[\Lambda,\Lambda]$ has a basis consisting of the images of all paths of even length (≥ 2) . Therefore $\Lambda/[\Lambda,\Lambda]$ has a basis consisting of the images of identity and all paths of odd length. Hence $\dim_k HH_0(\Lambda)=1+\sum_{\substack{i=1\\i\text{ odd}}}^n \binom{n}{i}=2^{n-1}+1$.

Case char k=2: In this case all maps in the complex $(M_{\bullet}, \tau_{\bullet})$ are zero. Thus $HH_m(\Lambda) \cong \Lambda^{\binom{n+m-1}{n-1}}$ and $\dim_k HH_m(\Lambda) = 2^n \binom{n+m-1}{n-1}$ for $m \geq 1$. Clearly $\dim_k HH_0(\Lambda) = \dim_k \Lambda = 2^n$.

Denote by $HC_m(\Lambda)$ the *m*-th cyclic homology group of A (cf. [26]). Let $hc_m(\Lambda) := \dim_k HC_m(\Lambda)$ and $hh_m(\Lambda) := \dim_k HH_m(\Lambda)$

Corollary 1. Let $\Lambda = kQ/I$ be the exterior algebra and char k = 0. Then $hc_m(\Lambda) = \sum_{i=0}^{m} (-1)^{m-i} 2^{n-1} {n+i-1 \choose n-1} + \begin{cases} 1, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$

Proof. By [26, Theorem 4.1.13], we have

$$(hc_m(\Lambda) - hc_m(k)) = -(hc_{m-1}(\Lambda) - hc_{m-1}(k)) + (hh_m(\Lambda) - hh_m(k)).$$

Thus $(hc_m(\Lambda) - hc_m(k)) = \sum_{i=0}^{m} (-1)^{m-i} (hh_i(\Lambda) - hh_i(k))$. It is well-known that

$$hh_i(k) = \begin{cases} 1, & \text{if } i = 0; \\ 0, & \text{if } i \ge 1. \end{cases}$$
 and $hc_m(k) = \begin{cases} 1, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$ By Theorem 2,

we have
$$hc_m(\Lambda) = \sum_{i=0}^{m} (-1)^{m-i} 2^{n-1} {n+i-1 \choose n-1} + \begin{cases} 1, & \text{if } m \text{ is even;} \\ 0, & \text{if } m \text{ is odd.} \end{cases}$$

3 Hochschild cohomology

In this section we calculate the k-dimensions of the Hochschild cohomological groups of the exterior algebras.

Applying the functor $\operatorname{Hom}_{\Lambda^e}(-,\Lambda)$ to the minimal projective bimodule resolution $(P_{\bullet},\delta_{\bullet})$, we have $\operatorname{Hom}_{\Lambda^e}((P_{\bullet},\delta_{\bullet}),\Lambda)=(P_{\bullet}^*,\delta_{\bullet}^*)$ where $P_m^*=\operatorname{Hom}_{\Lambda^e}(P_m,\Lambda)$ and $\delta_m^*(\phi)=\phi\delta_m$ for any $\phi\in P_m^*$. As k-vector spaces, P_m^* is canonically isomorphic to $M^m:=\Lambda^{\binom{n+m-1}{n-1}}$. Let $\{\phi_{1^{i_1}\cdots n^{i_n}}^m|i_1+\cdots+i_n=m\}$ be a basis of the k-vector space P_m^* defined by $\phi_{1^{i_1}\cdots n^{i_n}}^m(1\otimes f_{1^{i_1}\cdots n^{i_n}}^m\otimes 1)=1$ and $\phi_{1^{i_1}\cdots n^{i_n}}^m(1\otimes f_{1^{j_1}\cdots n^{j_n}}^m\otimes 1)=0$ for $(j_1,\cdots,j_n)\neq (i_1,\cdots,i_n)$. Let $e_{1^{i_1}\cdots n^{i_n}}^m$ be the image of $\phi_{1^{i_1}\cdots n^{i_n}}^m$ under the canonical isomorphism $P_m^*\cong M^m$. Then the complex $(P_{\bullet}^*,\delta_{\bullet}^*)$ is isomorphic to the complex $(M^{\bullet},\tau^{\bullet})$ where $\tau^{m+1}:M^m\to M^{m+1},\lambda e_{1^{i_1}\cdots n^{i_n}}^m\mapsto \sum_{h=1}^n (x_h\lambda+(-1)^{m+1}\lambda x_h)e_{1^{i_1}\cdots h^{i_h+1}\cdots n^{i_n}}^m=(1+(-1)^{m+j+1})\sum_{h=1}^n (-1)^{\mu_{\lambda}(h)}N(x_h\lambda)e_{1^{i_1}\cdots h^{i_h+1}\cdots n^{i_n}}^{m+1}$ for $\lambda\in\mathcal{B}_j$.

Lemma 5. For $m \ge 0$ we have

$$\operatorname{rank} \tau^{m+1} = \begin{cases} \sum_{i=1}^{n} {n \choose i} \sum_{\substack{j=0 \ p(j) = p(n+m)}}^{i-1} {j+m \choose i-1} {i-1 \choose j}, & \text{if char } k \neq 2; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Obviously the complex $(M^{\bullet}, \tau^{\bullet})$ is isomorphic to the complex $(N^{\bullet}, \sigma^{\bullet})$ which is defined by $N^m := \Lambda \otimes k[x_1, ..., x_n]_m$ and $\sigma^{m+1} : N^m \to N^{m+1}, \lambda \otimes x_1^{i_1} \cdots x_n^{i_n} \mapsto \sum_{h=1}^n (x_h \lambda \otimes x_1^{i_1} \cdots x_h^{i_h+1} \cdots x_n^{i_n} + (-1)^{m+1} \lambda x_h \otimes x_1^{i_1} \cdots x_h^{i_h+1} \cdots x_n^{i_n})$. For any $\lambda = x_{t_1} \cdots x_{t_j} \in \mathcal{B}$, we have $\sigma^{m+1}(\lambda \otimes x_1^{i_1} \cdots x_n^{i_n}) = (1 + (-1)^{m+j+1}) \sum_{h=1}^n (-1)^{\mu_{\lambda}(h)} N(\lambda x_h) \otimes x_1^{i_1} \cdots x_h^{i_h+1} \cdots x_n^{i_n}$.

Clearly, N^m has a basis $\mathcal{N}^m := \{\lambda \otimes x_1^{i_1} \cdots x_n^{i_n} \mid \lambda \in \mathcal{B}, i_1 + i_2 + \cdots + i_n = m\}$. If, viewed as a monomial in $k[x_1, ..., x_n, x_1^{-1}, ..., x_n^{-1}], \lambda x_1^{-i_1} \cdots x_n^{-i_n}$ can be written as $x_1^{j_1} \cdots x_n^{j_n}$, then the sum i (resp. -j) of all the positive (resp. negative) j_l is called the *positive* (resp. negative) degree of $\lambda \otimes x_1^{i_1} \cdots x_n^{i_n}$. Note that the positive j_l must be 1. Moreover, $\{l|j_l>0, 1\leq l\leq n\}$ (resp. $\{l|j_l<0, 1\leq l\leq n\}$) is called the positive (resp. negative) support of $\lambda \otimes x_1^{i_1} \cdots x_n^{i_n}$.

Firstly, denote by $N^m(i)$ the subspace of N^m generated by all elements in \mathcal{N}^m of positive degree i. Since σ^{m+1} keeps the positive degree of the elements, we have rank $\sigma^{m+1} = \sum_{i=1}^n \operatorname{rank}(\sigma^{m+1}|N^m(i))$.

Secondly, denote by $N^m(x_{s_1} \cdots x_{s_i})$ the subspace of $N^m(i)$ generated by all elements $\lambda \otimes x_1^{i_1} \cdots x_n^{i_n}$ in \mathcal{N}^m with positive support $\{s_1, \cdots, s_i\}$. Here $1 \leq s_1 < s_2 < \cdots < s_i \leq n$.

Thirdly, denote by $N^m(x_{s_1}\cdots x_{s_i}, x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n})$, where $0 \le i \le n, 0 \le j \le m, j_{i+1}+\cdots+j_n=j, \{s_{i+1},...,s_n\}=\{1,...,n\}\setminus\{s_1,...,s_i\}$ and $s_{i+1}<\cdots< s_n,$

the subspace of N^m generated by all elements $N(x_{s_1} \cdots x_{s_i} x_{s_{t_1}} \cdots x_{s_{t_{m-i}}}) \otimes$ $x_{s_{t_1}} \cdots x_{s_{t_{m-i}}} x_{s_{i+1}}^{j_{i+1}} \cdots x_{s_n}^{j_n}$ of positive degree i and negative degree j in \mathcal{N}^m with $\{s_{t_1}, ..., s_{t_{m-j}}\}\subseteq \{s_{i+1}, ..., s_n\}$ and $s_{t_1} < \cdots < s_{t_{m-j}}$. Clearly this subspace is 0 unless $0 \le m - j \le n - i$. Consider the basis $\mathcal{N}^m(x_{s_1} \cdots x_{s_i})$ $x_{s_{i+1}}^{j_{i+1}} \cdots x_{s_n}^{j_n}$:= $\mathcal{N}^m \cap N^m(x_{s_1} \cdots x_{s_i}, x_{s_{i+1}}^{j_{i+1}} \cdots x_{s_n}^{j_n})$ of the vector space $N^m(x_{s_1}\cdots x_{s_i},x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n}).$ Order the elements $\lambda\otimes x_{s_{i+1}}^{l_{i+1}}\cdots x_{s_n}^{l_n}$ in $\mathcal{N}^m(x_{s_1}\cdots x_{s_i}, x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n})$ by the left lexicographic order on λ . Obviously σ^{m+1} maps $N^m(x_{s_1}\cdots x_{s_i}, x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n})$ into $N^{m+1}(x_{s_1}\cdots x_{s_i}, x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n})$ if $0 \le i \le n-1$, and into 0 if i = n. Let the map $\alpha : N^m(x_{s_1} \cdots x_{s_i}, x_{s_{i+1}}^{j_{i+1}} \cdots x_{s_n}^{j_n})$ $\to N^{m+1}(x_{s_1}\cdots x_{s_i},x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n})$ be the restriction of σ^{m+1} . Written as a matrix under the basis $\mathcal{N}^m(x_{s_1}\cdots x_{s_i}, x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n})$ of $N^m(x_{s_1}\cdots x_{s_i}, x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n})$ and $\mathcal{N}^{m+1}(x_{s_1}\cdots x_{s_i}, x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n})$ of $N^{m+1}(x_{s_1}\cdots x_{s_i}, x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n})$, α is an $\binom{n-i}{m-j+1}\times\binom{n-i}{m-j}$ matrix. Partition the elements $\lambda\otimes x_{s_{i+1}}^{l_{i+1}}\cdots x_{s_n}^{l_n}$ in $\mathcal{N}^m(x_{s_1}\cdots x_{s_i}, x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n})$ and $\mathcal{N}^{m+1}(x_{s_1}\cdots x_{s_i}, x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n})$ according to whether λ contains $x_{s_{i+1}}$ or not. In this way, neglected the sign $a_{s_{i+1}} = a_{s_{i+1}} =$ matrix.

Claim. BA = 0.

Proof of the Claim. Take any row of B. Assume that it corresponds to the element $N(x_{s_1}\cdots x_{s_i}x_{sq_1}\cdots x_{sq_{j+1}})\otimes x_{s_{i+1}}^{j_{i+1}}\cdots x_{sq_1}^{j_{q_1}+1}\cdots x_{sq_{j+1}}^{j_{q_{j+1}}+1}\cdots x_{s_n}^{j_n}\in \mathcal{N}^{m+1}(x_{s_1}\cdots x_{s_i},x_{s_{i+1}}^{j_{i+1}}\cdots x_{s_n}^{j_n})$. Then, according to the definition of σ^{m+1} , this row is just the vector whose components corresponding to $N(x_{s_1}\cdots x_{s_i}x_{sq_1}\cdots x_{sq_1}^{j_{q_1}}\cdots x_{sq_j}^{j_{q_1}}\cdots x_{sq_j}^{j_{n_1}}\cdots x$

 $(-1)^{\mu_{x_{s_{1}}\cdots x_{s_{i}}}(s_{q_{b}})+b-1}(-1)^{\mu_{x_{s_{1}}\cdots x_{s_{i}}}(s_{q_{a}})+a}+(-1)^{\mu_{x_{s_{1}}\cdots x_{s_{i}}}(s_{q_{a}})+a-1}(-1)^{\mu_{x_{s_{1}}\cdots x_{s_{i}}}(s_{q_{b}})+b-1}=0.$ Hence the claim holds.

Under the chosen basis, α is the matrix $(1+(-1)^{i-j+1})\begin{bmatrix} A & (-1)^{\mu_{x_{s_1}\cdots x_{s_i}}(s_{i+1})}I \\ 0 & B \end{bmatrix}$. Thus we have rank $\alpha = \begin{cases} \binom{n-i-1}{m-j}, & \text{if } p(i) \neq p(j) \text{ and char } k \neq 2; \\ 0, & \text{otherwise.} \end{cases}$ Since σ^{m+1} preserves both positive degree and negative degree, we have

$$\begin{aligned} & \operatorname{rank} \sigma^{m+1} \\ &= \sum_{i=0}^{n} \operatorname{rank}(\sigma^{m+1}|N^{m}(i)) \\ &= \sum_{i=0}^{n} \sum_{\{s_{1}, \dots, s_{i}\} \subseteq \{1, \dots, n\}} \operatorname{rank}(\sigma^{m+1}|N^{m}(x_{s_{1}} \cdots x_{s_{i}})) \\ &= \sum_{i=0}^{n} \sum_{\{s_{1}, \dots, s_{i}\} \subseteq \{1, \dots, n\}} \sum_{j=m-n+i}^{m} \sum_{j=m-n+i} \operatorname{rank}(\sigma^{m+1}|N^{m}(x_{s_{1}} \cdots x_{s_{i}}, x_{s_{i+1}}^{j_{i+1}} \cdots x_{s_{n}}^{j_{n}})) \\ &= \begin{cases} \sum_{i=0}^{n-1} \binom{n}{i} \sum_{\substack{j=m-n+i \\ p(j) \neq p(i)}}^{m} \binom{n-i+j-1}{n-i-1} \binom{n-i-1}{m-j}, & \text{if char } k \neq 2; \\ 0, & \text{otherwise.} \end{cases}$$

If char $k \neq 2$ then we have

$$\operatorname{rank}\sigma^{m+1} = \sum_{i=0}^{n-1} \binom{n}{i} \sum_{\substack{j=m-n+i\\p(j)\neq p(i)}}^{m} \binom{n-i+j-1}{n-i-1} \binom{n-i-1}{m-j}$$

$$= \sum_{i=1}^{n} \binom{n}{i} \sum_{\substack{j=0\\p(m-j)\neq p(n-i)}}^{i} \binom{m-j+i-1}{i-1} \binom{i-1}{j}$$

$$= \sum_{i=1}^{n} \binom{n}{i} \sum_{\substack{j=0\\p(j)\neq p(n+m)\\p(j)=p(n+m)}}^{i-1} \binom{m+j-1}{i-1} \binom{i-1}{j}.$$

The lemma follows from the fact that $\operatorname{rank} \tau^{m+1} = \operatorname{rank} \sigma^{m+1}$.

Lemma 6. If char $k \neq 2$ and $m \geq 1$ then

$$\operatorname{rank} \tau^m + \operatorname{rank} \tau^{m+1} = 2^{n-1} \binom{n+m-1}{n-1}.$$

Proof. Note that

$$\begin{aligned} \operatorname{rank} \tau^{m+1} &= \sum_{i=1}^{n} \binom{n}{i} \sum_{\substack{j=0 \\ p(j) = p(n+m)}}^{i-1} \binom{j+m}{i-1} \binom{i-1}{j} \\ &= \sum_{\substack{j=0 \\ p(j) = p(n+m)}}^{n-1} \sum_{i=j+1}^{n} \binom{n}{i} \binom{j+m}{i-1} \binom{i-1}{j} \\ &= \sum_{\substack{j=0 \\ p(j) = p(n+m)}}^{n-1} \sum_{i=1}^{n-j} \binom{n-i}{j} \binom{m+n-i}{n-i} \\ &= \sum_{i=1}^{n} \sum_{\substack{j=0 \\ p(j) = p(n+m)}}^{n-i} \binom{n-i}{j} \binom{m+n-i}{n-i} \\ &= \sum_{i=1}^{n} \binom{m+n-i}{n-i} \sum_{\substack{j=0 \\ p(j) = p(n+m)}}^{n-i} \binom{n-i}{j} \\ &= \sum_{i=1}^{n-1} 2^{n-i-1} \binom{m+n-i}{n-i} + \left\{ \begin{array}{c} 1, & \text{if } n+m \text{ is even;} \\ 0, & \text{if } n+m \text{ is odd.} \\ &= \sum_{i=1}^{n-1} 2^{i-1} \binom{m+i}{i} + \left\{ \begin{array}{c} 1, & \text{if } n+m \text{ is even;} \\ 0, & \text{if } n+m \text{ is odd.} \end{array} \right. \end{aligned}$$

where we apply Lemma 5 and Lemma 2 in the first and the third steps respectively. By the proof of Lemma 4, we have $\operatorname{rank} \tau^m + \operatorname{rank} \tau^{m+1} = U_n^m + U_n^{m+1} + 1 = 2^{n-1} \binom{n+m-1}{n-1}$.

Theorem 3. Let $\Lambda = kQ/I$ be the exterior algebra. Then

$$\dim_k HH^m(\Lambda) = \begin{cases} 2^n \binom{n+m-1}{n-1}, & \text{if char } k = 2; \\ 2^{n-1} + 1, & \text{if } m = 0, n \text{ is odd and char } k \neq 2; \\ 2^{n-1} \binom{n+m-1}{n-1}, & \text{otherwise.} \end{cases}$$

Proof. Case char $k \neq 2$: If $m \geq 1$ then, by Lemma 6,

$$\dim_k HH^m(\Lambda) = \dim_k \operatorname{Ker} \tau^m - \dim_k \operatorname{Im} \tau^{m+1}$$

$$= \dim_k P_m^* - \dim_k \operatorname{Im} \tau^m - \dim_k \operatorname{Im} \tau^{m+1}$$

$$= 2^n \binom{n+m-1}{n-1} - \operatorname{rank} \tau^m - \operatorname{rank} \tau^{m+1}$$

$$= 2^{n-1} \binom{n+m-1}{n-1}.$$

If m=0 then $HH^0(\Lambda)=Z(\Lambda)$ which is the center of Λ . Since Λ is gradable, $\lambda \in Z(\Lambda)$ if and only if all its components belong to $Z(\Lambda)$. Thus it is enough to consider $\mathcal{B} \cap Z(\Lambda)$. Note that $x_{t_1} \cdots x_{t_i} \in Z(\Lambda)$ if and only if $x_{t_1} \cdots x_{t_i} x_j = x_j x_{t_1} \cdots x_{t_i}$ for all $1 \leq j \leq n$, if and only if i=n or i is

even. Therefore $\dim_k HH^0(\Lambda) = \dim_k Z(\Lambda) = \sum_{i=0 \atop i \text{ even}}^n \binom{n}{i} = 2^{n-1}$ if n is even, and $\dim_k HH^0(\Lambda) = 2^{n-1} + 1$ if n is odd.

Case char k=2: In these case all maps in the complex $(M^{\bullet}, \tau^{\bullet})$ are zero. If $m \geq 1$ then $\dim_k HH^m(\Lambda) = \dim_k \Lambda^{\binom{n+m-1}{n-1}} = 2^n \binom{n+m-1}{n-1}$. If m=0 then $\dim_k HH^0(\Lambda) = \dim_k \operatorname{Hom}_{\Lambda^e}(\Lambda^e, \Lambda) = \dim_k \Lambda = 2^n$.

Remark 1. The case n = 2 was obtained in [5].

Corollary 2. The Hilbert series of the exterior algebra Λ

$$\sum_{m=0}^{\infty} \dim_k HH^m(\Lambda) t^m = \begin{cases} \frac{2^{n-1}}{(1-t)^n}, & \text{if } n \text{ is even and char } k \neq 2; \\ \frac{2^{n-1}}{(1-t)^n} + 1, & \text{if } n \text{ is odd and char } k \neq 2; \\ \frac{2^n}{(1-t)^n}, & \text{if char } k = 2. \end{cases}$$

Proof. Note that
$$\sum_{m=0}^{\infty} {n+m-1 \choose n-1} t^m = \frac{1}{(1-t)^n}.$$

4 Hochschild cohomology rings

In this section we shall determine the Hochschild cohomology rings of the exterior algebras by generators and relations.

Now we construct another minimal projective bimodule resolution $(P'_{\bullet}, \delta'_{\bullet})$ of Λ . For each $m \geq 0$, we firstly construct elements $\{g^m_{1^{i_1}2^{i_2}\cdots n^{i_n}} \in \Lambda^{\otimes m}|i_1+\cdots+i_n=m \text{ with } (i_1,\ldots,i_n)\in\mathbb{N}^n\}$: Let $g^0_0=1,g^1_1=x_1,g^1_2=x_2,\ldots,g^1_n=x_n$. Define $g^m_{1^{i_1}2^{i_2}\cdots n^{i_n}}$ for all $m\geq 2$ inductively by $g^m_{1^{i_1}2^{i_2}\cdots n^{i_n}}=\sum\limits_{h=1}^n g^{m-1}_{1^{i_1}\cdots h^{i_h-1}\cdots n^{i_n}}\otimes x_h$, where $(i_1,\ldots,i_n)\in\mathbb{N}^n,i_1+i_2+\cdots+i_n=m$ and $g^{m-1}_{1^{i_1}2^{i_2}\cdots n^{i_n}}=0$ for all $1\leq h\leq n$. It is easy to see that $g^m_{1^{i_1}2^{i_2}\cdots n^{i_n}}=\sum\limits_{h=1}^n x_h\otimes g^{m-1}_{1^{i_1}\cdots h^{i_h-1}\cdots n^{i_n}}$. Let $P'_m:=\coprod\limits_{i_1+i_2+\cdots+i_n=m}\Lambda\otimes g^m_{1^{i_1}2^{i_2}\cdots n^{i_n}}\otimes \Lambda\subseteq \Lambda^{\otimes m+2}$ for $m\geq 0$, and let $\widetilde{g}^m_{1^{i_1}2^{i_2}\cdots n^{i_n}}:=1\otimes g^m_{1^{i_1}2^{i_2}\cdots n^{i_n}}\otimes 1$ for $m\geq 1$ and $\widetilde{g}^0_0=1\otimes 1$. Note that we identify P'_0 with $\Lambda\otimes\Lambda$. Define $\delta'_m:P'_m\to P'_{m-1}$ by setting $\delta'_m(\widetilde{g}^m_{1^{i_1}2^{i_2}\cdots n^{i_n}})=\sum\limits_{h=1}^n (x_h\widetilde{g}^{m-1}_{1^{i_1}\cdots h^{i_h-1}\cdots n^{i_h}}+(-1)^m\widetilde{g}^{m-1}_{1^{i_1}\cdots h^{i_h-1}\cdots n^{i_n}}x_h)$.

Lemma 7. The complex $\mathbb{P} := (P'_{\bullet}, \delta'_{\bullet})$:

$$\cdots \to P'_{m+1} \xrightarrow{\delta'_{m+1}} P'_m \xrightarrow{\delta'_m} \cdots \xrightarrow{\delta'_3} P'_2 \xrightarrow{\delta'_2} P'_1 \xrightarrow{\delta'_1} P'_0 \longrightarrow 0$$

is a minimal projective bimodule resolution of the exterior algebra $\Lambda = kQ/I$.

Proof. Clearly the complex $\mathbb{P} = (P'_{\bullet}, \delta'_{\bullet})$ is isomorphic to the complex $(P_{\bullet}, \delta_{\bullet})$.

Applying the functor $\operatorname{Hom}_{\Lambda^e}(-,\Lambda)$, we have $\mathbb{P}^* = (P'_{\bullet},\delta'_{\bullet})^* \cong (P_{\bullet},\delta_{\bullet})^* = (P^*_{\bullet},\delta^*_{\bullet}) \cong (M^{\bullet},\tau^{\bullet})$. Thus every element in P'^*_m can be represented as a linear combination of the elements in $\{\lambda e^m_{1^{i_1}2^{i_2}\dots n^{i_n}} | \lambda \in \mathcal{B}, i_1+i_2+\dots+i_n=m\} \subseteq M^m$. Throughout we do not distinguish an element in $\operatorname{Ker} \delta'^*_{m+1} \subseteq P'^*_m$ with its equivalent class in $HH^m(\Lambda)$.

Lemma 8. Let $\eta = \sum_{i_1+i_2+\dots+i_n=s} \lambda_{i_1i_2\dots i_n} e^s_{1^{i_1}2^{i_2}\dots n^{i_n}} \in HH^s(\Lambda)$ and $\theta = \sum_{j_1+j_2+\dots+j_n=t} \lambda'_{j_1j_2\dots j_n} e^t_{1^{j_1}2^{j_2}\dots n^{j_n}} \in HH^t(\Lambda)$. Then the cup product of η and θ in $HH^{s+t}(\Lambda)$:

$$\eta * \theta = \sum_{\substack{i_l + j_l = h_l \\ 1 \le l \le n}} \lambda_{i_1 i_2 \dots i_n} \lambda'_{j_1 j_2 \dots j_n} e^{s+t}_{1^{i_1 + j_1} 2^{i_2 + j_2} \dots n^{i_n + j_n}}.$$

Consider the diagonal map $\Delta: \mathbb{B} \to \mathbb{B} \otimes_{\Lambda} \mathbb{B}$ given by $\Delta(\lambda_0 \otimes \cdots \otimes \lambda_{m+1}) = \sum_{i=0}^{m} (\lambda_0 \otimes \cdots \otimes \lambda_i \otimes 1) \otimes_{\Lambda} (1 \otimes \lambda_{i+1} \otimes \cdots \otimes \lambda_{m+1})$. The cup product $\eta' \cup \theta'$ in the Hochschild cohomology ring $HH^*(\Lambda)$ of two cycles η' and θ' from $\text{Hom}_{\Lambda^e}(\mathbb{B}, \Lambda)$ is given by the composition $\mathbb{B} \xrightarrow{\Delta} \mathbb{B} \otimes_{\Lambda} \mathbb{B} \xrightarrow{\eta' \otimes \theta'} \Lambda \otimes_{\Lambda} \Lambda \xrightarrow{\nu} \Lambda$ where $\nu: \Lambda \otimes_{\Lambda} \Lambda \to \Lambda$ is the multiplication in Λ (cf. [29]).

Let $\mu: \mathbb{P} \to \mathbb{B}$ be the natural inclusion and let $\pi: \mathbb{B} \to \mathbb{P}$ be a chain map such that $\pi\mu = 1_{\mathbb{P}}$. We have $\Delta(\mu\mathbb{P}) \subseteq \mu\mathbb{P} \otimes_{\Lambda} \mu\mathbb{P} \subseteq \mathbb{B} \otimes_{\Lambda} \mathbb{B}$: Indeed, fix an s with $0 \leq s \leq m$, it is easy to see that $g_{1^{h_1}2^{h_2}...n^{h_n}}^m = \sum_{\substack{i_l+j_l=h_l\\1\leq l\leq n}} g_{1^{i_1}2^{i_2}...n^{i_n}}^s g_{1^{j_1}2^{j_2}...n^{j_n}}^{m-s}$, here we neglect g_0^0 if s=0 or m. Thus we can

infer that $\Delta(\widetilde{g}^m_{1^{h_1}2^{h_2}\cdots n^{h_n}}) = \sum_{s=0}^m \sum_{\substack{i_l+j_l=h_l\\1\leq l\leq n}} \widetilde{g}^s_{1^{i_1}2^{i_2}\cdots n^{i_n}} \widetilde{g}^{m-s}_{1^{j_1}2^{j_2}\cdots n^{j_n}}$ which implies that

 $\Delta(\mu\mathbb{P}) \subseteq \mu\mathbb{P} \otimes_{\Lambda} \mu\mathbb{P}.$

Let Δ' be the restriction of Δ on \mathbb{P} . Then $\Delta \mu = (\mu \otimes \mu)\Delta'$. Viewed as elements in $P_s'^*$ and $P_t'^*$, η and θ can be represented by $\eta \pi_s$ and $\theta \pi_t$ by the bar resolution respectively. Therefore

$$\eta * \theta = (\eta * \theta) \pi_{s+t} \mu_{s+t}
= (\eta \pi_s \cup \theta \pi_t) \mu_{s+t}
= \nu (\eta \pi_s \otimes \theta \pi_t) \Delta \mu_{s+t}
= \nu (\eta \pi_s \otimes \theta \pi_t) (\mu_s \otimes \mu_t) \Delta'
= \nu (\eta \otimes \theta) \Delta'.$$

Since $\eta(\widetilde{g}_{1^{i_1}2^{i_2}\cdots n^{i_n}}^s) = \lambda_{i_1i_2\cdots i_n}$ and $\theta(\widetilde{g}_{1^{j_1}2^{j_2}\cdots n^{j_n}}^t) = \lambda'_{j_1j_2\cdots j_n}$, we have

$$(\eta * \theta)(\widetilde{g}_{1^{h_1}2^{h_2}\cdots n^{h_n}}^{s+t}) = \sum_{\substack{i_l+j_l=h_l\\1\leq l\leq n}} \lambda_{i_1i_2\cdots i_n} \lambda'_{j_1j_2\cdots j_n}$$

for all $(h_1, ..., h_n) \in \mathbb{N}^n$ with $h_1 + \cdots + h_n = s + t$. Viewed as an element in M^{s+t} ,

$$\eta * \theta = \sum_{\substack{i_l + j_l = h_l \\ 1 < l < n}} \lambda_{i_1 i_2 \cdots i_n} \lambda'_{j_1 j_2 \cdots j_n} e^{s+t}_{1^{i_1} + j_1 2^{i_2} + j_2 \cdots n^{i_n + j_n}}.$$

Lemma 9. Let $\Lambda = kQ/I$ be the exterior algebra and chark $\neq 2$. Then the k-vector space $HH^m(\Lambda)$ has a basis $\{\lambda e^m_{1^{i_1}2^{i_2}\cdots n^{i_n}}|\lambda\in\mathcal{B}_i, p(i)=p(m), 1\leq i\leq n\}$.

Proof. If $\lambda \in \mathcal{B}_i$ and p(i) = p(m) then $\tau^{m+1}(\lambda e^m_{1^{i_1}2^{i_2}\dots n^{i_n}}) = 0$. Thus we have $\{\lambda e^m_{1^{i_1}2^{i_2}\dots n^{i_n}}|\lambda \in \mathcal{B}_i, p(i) = p(m), 1 \leq i \leq n\} \subseteq \operatorname{Ker} \tau^{m+1}$. Clearly $\operatorname{Im} \tau^m$ is contained in the subspace of $\operatorname{Ker} \tau^{m+1}$ generated by $\{\lambda e^m_{1^{i_1}2^{i_2}\dots n^{i_n}}|\lambda \in \mathcal{B}_i, p(i) \neq p(m), 1 \leq i \leq n\}$. By Theorem 3, we have $\dim_k HH^m(\Lambda) = 2^{n-1}\binom{n+m-1}{n-1}$. Thus the Lemma holds.

It follows from Lemma 9 that, as a k-algebra, $HH^*(\Lambda)$ is generated by $\{x_ix_j|1\leq i< j\leq n\}\cup\{x_pe_q^1|1\leq p,q\leq n\}\cup\{e_{st}^2|1\leq s\leq t\leq n\}$ which satisfy all relations in the following Table H.

Table H

$$\begin{array}{llll} & (H1.1) & (x_ix_j)(x_sx_t) = (x_sx_t)(x_ix_j) & \text{if } i < j \text{ and } s < t \\ & (H1.2) & (x_ix_j)(x_sx_t) = 0 & \text{if } \{i,j\} \cap \{s,t\} \neq \emptyset \\ & (H1.3) & (x_ix_j)(x_sx_t) = -(x_ix_s)(x_jx_t) & \text{if } i < s < j < t \\ & (H1.4) & (x_ix_j)(x_sx_t) = (x_ix_s)(x_tx_j) & \text{if } i < s < t < j \\ & (H1.5) & (x_ix_j)(x_sx_t) = -(x_sx_i)(x_tx_j) & \text{if } s < i < t < j \\ & (H1.6) & (x_ix_j)(x_sx_t) = -(x_sx_i)(x_jx_t) & \text{if } s < i < t < j \\ & (H2.1) & (x_ix_j)(x_se_t^1) = (x_se_t^1)(x_ix_j) & \text{if } i < j \\ & (H2.2) & (x_ix_j)(x_se_t^1) = 0 & \text{if } s \in \{i,j\} \\ & (H2.3) & (x_ix_j)(x_se_t^1) = (x_sx_i)(x_je_t^1) & \text{if } i < s < j \\ & (H2.4) & (x_ix_j)(x_se_t^1) = -(x_ix_s)(x_je_t^1) & \text{if } i < s < j \\ & (H3.1) & (x_ix_j)e_{st}^2 = e_{st}^2(x_ix_j) & \text{if } i < s \text{ and } j \leq t \\ & (H4.1) & (x_ie_j^1)(x_se_t^1) = 0 & \text{if } i = s \\ & (H4.2) & (x_ie_j^1)(x_se_t^1) = (x_ix_s)e_{ij}^2 & \text{if } i < s \text{ and } j \leq t \\ & (H4.3) & (x_ie_j^1)(x_se_t^1) = (x_ix_s)e_{ij}^2 & \text{if } i < s \text{ and } t \leq j \\ & (H4.4) & (x_ie_j^1)(x_se_t^1) = -(x_sx_i)e_{ij}^2 & \text{if } s < i \text{ and } t \leq j \\ & (H4.5) & (x_ie_j^1)(x_se_t^1) = -(x_sx_i)e_{ij}^2 & \text{if } s < i \text{ and } t \leq j \\ & (H5.1) & (x_ie_j^1)e_{st}^2 = e_{st}^2(x_ie_j^1) & \text{if } s \leq t \\ & (H5.2) & (x_ie_j^1)e_{st}^2 = (x_ie_s^1)e_{ij}^2 & \text{if } s < t \leq j \\ & (H5.3) & (x_ie_j^1)e_{st}^2 = (x_ie_s^1)e_{ij}^2 & \text{if } s < t \leq j \\ & (H6.1) & e_{ij}^2e_{st}^2 = e_{is}^2e_{ij}^2 & \text{if } i \leq s \leq j \leq t \\ & (H6.2) & e_{ij}^2e_{st}^2 = e_{is}^2e_{ij}^2 & \text{if } i \leq s \leq t \leq j \\ & (H6.4) & e_{ij}^2e_{st}^2 = e_{is}^2e_{ij}^2 & \text{if } i \leq s \leq t \leq j \\ & (H6.4) & e_{ij}^2e_{st}^2 = e_{is}^2e_{ij}^2 & \text{if } i \leq s \leq t \leq j \\ & (H6.4) & e_{ij}^2e_{st}^2 = e_{is}^2e_{ij}^2 & \text{if } i \leq s \leq t \leq j \\ & (H6.5) & e_{ij}^2e_{st}^2 = e_{is}^2e_{ij}^2 & \text{if } i \leq s \leq t \leq j \\ \end{pmatrix}$$

Replace $x_i x_j, x_p e_q^1, e_{st}^2$ with u_{ij}, v_{pq}, w_{st} respectively, we have the following Table F.

Table F

$$(F1.1) \quad u_{ij}u_{st} = u_{st}u_{ij} \quad \text{if } i < j \text{ and } s < t \\ (F1.2) \quad u_{ij}u_{st} = 0 \quad \text{if } \{i,j\} \cap \{s,t\} \neq \emptyset \\ (F1.3) \quad u_{ij}u_{st} = -u_{is}u_{jt} \quad \text{if } i < s < j < t \\ (F1.4) \quad u_{ij}u_{st} = u_{is}u_{tj} \quad \text{if } i < s < t < j \\ (F1.5) \quad u_{ij}u_{st} = -u_{si}u_{tj} \quad \text{if } s < i < t < j \\ (F1.6) \quad u_{ij}u_{st} = u_{si}u_{jt} \quad \text{if } s < i < t < j \\ (F1.6) \quad u_{ij}u_{st} = u_{si}u_{jt} \quad \text{if } s < i < j < t \\ (F2.1) \quad u_{ij}v_{st} = v_{st}u_{ij} \quad \text{if } i < j \\ (F2.2) \quad u_{ij}v_{st} = 0 \quad \text{if } s \in \{i,j\} \\ (F2.3) \quad u_{ij}v_{st} = u_{si}v_{jt} \quad \text{if } i < s < j \\ (F2.4) \quad u_{ij}v_{st} = -u_{is}v_{jt} \quad \text{if } i < s < j \\ (F3.1) \quad u_{ij}w_{st} = w_{st}u_{ij} \quad \text{if } i < s \text{ and } j \leq t \\ (F4.2) \quad v_{ij}v_{st} = u_{is}w_{jt} \quad \text{if } i < s \text{ and } t \leq j \\ (F4.3) \quad v_{ij}v_{st} = u_{is}w_{tj} \quad \text{if } i < s \text{ and } t \leq j \\ (F4.4) \quad v_{ij}v_{st} = -u_{si}w_{tj} \quad \text{if } s < i \text{ and } t \leq j \\ (F4.5) \quad v_{ij}v_{st} = -u_{si}w_{tj} \quad \text{if } s < i \text{ and } t \leq j \\ (F5.1) \quad v_{ij}w_{st} = w_{st}v_{ij} \quad \text{if } s < t \leq j \leq t \\ (F5.2) \quad v_{ij}w_{st} = v_{is}w_{tj} \quad \text{if } s < t \leq j \leq t \\ (F6.1) \quad w_{ij}w_{st} = w_{is}w_{tj} \quad \text{if } i \leq s \leq j \leq t \\ (F6.2) \quad w_{ij}w_{st} = w_{is}w_{tj} \quad \text{if } i \leq s \leq t \leq j \\ (F6.4) \quad w_{ij}w_{st} = w_{si}w_{tj} \quad \text{if } s \leq i \leq t \leq j \\ (F6.5) \quad w_{ij}w_{st} = w_{si}w_{tj} \quad \text{if } s \leq i \leq j \leq t \\ (F6.5) \quad w_{ij}w_{st} = w_{si}w_{tj} \quad \text{if } s \leq i \leq j \leq t \\ (F6.5) \quad w_{ij}w_{st} = w_{si}w_{tj} \quad \text{if } s \leq i \leq j \leq t \\ (F6.5) \quad w_{ij}w_{st} = w_{si}w_{ij} \quad \text{if } s \leq i \leq j \leq t \\ (F6.5) \quad w_{ij}w_{st} = w_{si}w_{ij} \quad \text{if } s \leq i \leq j \leq t \\ (F6.5) \quad w_{ij}w_{st} = w_{si}w_{ij} \quad \text{if } s \leq i \leq j \leq t \\ (F6.5) \quad w_{ij}w_{st} = w_{si}w_{ij} \quad \text{if } s \leq i \leq j \leq t \\ (F6.5) \quad w_{ij}w_{st} = w_{si}w_{ij} \quad \text{if } s \leq i \leq j \leq t \\ (F6.5) \quad w_{ij}w_{st} = w_{si}w_{ij} \quad \text{if } s \leq i \leq j \leq t \\ (F6.5) \quad w_{ij}w_{st} = w_{si}w_{ij} \quad \text{if } s \leq i \leq j \leq t \\ (F6.5) \quad w_{ij}w_{st} = w_{si}w_{ij} \quad \text{if } s \leq i \leq j \leq t \\ (F6.5) \quad w_{ij}w_{st} = w_{si}w_{ij} \quad \text{if } s \leq$$

Theorem 4. Let Q' be the quiver with one vertex 1 and $2n^2$ loops $\{u_{ij}|1 \le i < j \le n\} \cup \{v_{pq}|1 \le p, q \le n\} \cup \{w_{st}|1 \le s \le t \le n\}$. Let I' be the ideal of kQ' generated by all relations in Table F. Then the Hochschild cohomology ring of the exterior algebra $\Lambda = kQ/I$,

$$HH^*(\Lambda) = \begin{cases} kQ'/I', & \text{if char } k \neq 2; \\ \Lambda[z_1, ..., z_n], & \text{if char } k = 2. \end{cases}$$

Proof. Case char $k \neq 2$: Firstly, as a k-algebra $HH^*(\Lambda)$ is generated by $\{x_ix_j|1 \leq i < j \leq n\} \cup \{x_pe_q^1|1 \leq p, q \leq n\} \cup \{e_{st}^2|1 \leq s \leq t \leq n\}$.

Thus we have an epimorphism of k-algebras $\psi: kQ' \to HH^*(\Lambda)$ which maps u_{ij}, v_{pq}, w_{st} to $x_i x_j, x_p e_q^1, e_{st}^2$ respectively. Comparing the relations in Table H with those in Table F, we have $I' \subseteq \text{Ker}\psi$. Hence ψ induces an epimorphism $\varphi: kQ'/I' \to HH^*(\Lambda)$.

Apply the relations in Table H one by one, it is not difficult to see that the set $\{(x_{i_1}x_{i_2})\cdots(x_{i_{2l-1}}x_{i_{2l}})e^2_{j_1j_2}\cdots e^2_{j_{2r-1}j_{2r}}|1< i_1< i_2<\cdots< i_{2l-1}< i_{2l}, j_1\leq j_2\leq\cdots\leq j_{2r}\}\cup\{(x_{i_1}x_{i_2})\cdots(x_{i_{2l-1}}x_{i_{2l}})(x_{2l+1}e^1_{j_1})e^2_{j_2j_3}\cdots e^2_{j_{2r}j_{2r+1}}|1< i_1< i_2<\cdots< i_{2l-1}< i_{2l}< i_{2l+1}, j_1\leq j_2\leq\cdots\leq j_{2r}\leq j_{2r+1}\}$ is a k-basis of $HH^*(\Lambda)$.

Apply the relations in Table F one by one, it is not difficult to see that the set $\{u_{i_1i_2}\cdots u_{i_{2l-1}i_{2l}}w_{j_1j_2}\cdots w_{j_{2r-1}j_{2r}}|1< i_1< i_2<\cdots< i_{2l-1}< i_{2l}, j_1\leq j_2\leq\cdots\leq j_{2r}\}\cup\{u_{i_1i_2}\cdots u_{i_{2l-1}i_{2l}}v_{(2l+1)j_1}w_{j_2j_3}\cdots w_{j_{2r}j_{2r+1}}|1< i_1< i_2<\cdots< i_{2l-1}< i_{2l}< i_{2l+1}, j_1\leq j_2\leq\cdots\leq j_{2r}\leq j_{2r+1}\}$ is a k-basis of kQ'/I'.

Since φ maps a basis element to a basis element, φ is also a monomorphism. Hence it is an isomorphism.

Case char k=2: The differential in the complex $\operatorname{Hom}_{\Lambda^e}(\mathbb{P},\Lambda)$ is 0. All maps in $\operatorname{Hom}_{\Lambda^e}(\mathbb{P},\Lambda)$ represent nonzero elements in $HH^*(\Lambda)$. Let $\phi:\Lambda[z_1,...,z_n]\to HH^*(\Lambda), z_i\mapsto e_i^1$. Then ϕ is surjective and thus it is injective restricted to each degree. Hence it is an isomorphism.

Remark 2. The case n = 2 was obtained in [5].

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