Let k be a field and let $q_{ij} \in k$ for $1 \leq i < j \leq n$ and for convenience, when q_{ij} $neq 0$, we set $q_{ji} = q_{ij}^{-1}$ and $q_{ii} = 1$ and write $Q = (q_{ij})$ be the multiplicately skew-symmetric matrix. Let $S = k\langle x_1, \ldots, x_n \rangle$. Let I be the two sided ideal of S generated by

$$x_j x_i - q_{ij} x_i x_j \qquad 1 \leq i < j \leq n.$$

We let $A = A_Q$ be S/I. Note that A has a k-basis of ordered monomials and so we have

$$A \simeq \bigoplus_{i_1 \geq 0, \cdots, i_n \geq 0} k x_1^{i_1} \cdots x_n^{i_n}$$

as vector spaces.

We will consider first order deformations of $A = S/I$. We consider $a_{ij} \in A$ with $1 \leq i < j \leq n$. Let $k_1 = k[\varepsilon]/(\varepsilon^2)$ and write $\alpha_{ij} = \varepsilon a_{ij}$.

$$x_j x_i - q_{ij} x_i x_j - \alpha_{ij} \qquad 1 \leq i < j \leq n.$$

We obtain the following family of algebras

$$A_1 = k_1 \langle x_1, \ldots, x_n \rangle / I_1.$$

Proposition 1 *The algebra A_1 is a first order deformation of A if and only one of the three equivalent conditions hold.*

1. *A_1 is a flat k_1 algebra with a fixed isomorphism $A_1/(\varepsilon) \simeq A$.*
2. *$\varepsilon A_1 \simeq A$ as k-vector spaces.*
3. *(a_{ij}) are a Hocschild cocyle giving a class in $HH^2(A)$.*
4. *If $k > j > i$ when we reduce $(x_k x_j) x_i = x_k (x_j x_i)$ both ways to ordered monomials, we get the same answer.*

We begin by computing the fourth condition above.

$$\begin{aligned}
x_k(x_j x_i) &= x_k(q_{ij} x_i x_j + \alpha_{ij}) = q_{ij} x_k x_i x_j + x_k \alpha_{ij} \\
&= q_{ij}(q_{ik} x_i x_k + \alpha_{ik}) x_j + x \alpha_{ij} \\
&= q_{ij} q_{ik} x_i x_k x_j + q_{ij} \alpha_{ik} x_j + x_k \alpha_{ij} \\
&= q_{ij} q_{ik} q_{jk} x_i x_j x_k + q_{ik} q_{ik} x_i \alpha_{jk} + q_{ij} \alpha_{ik} x_j + x_k \alpha_{ij}
\end{aligned}$$

$$\begin{aligned}
(x_k x_j) x_i = (q_{jk} x_j x_k + \alpha_{jk}) x_k = q_{jk} x_j x_k x_i + \alpha_{jk} x_i \\
= q_{jk} x_j (q_{ik} x_i x_k + \alpha_{ik}) + \alpha_{jk} x_i \\
= q_{jk} q_{ik} x_j x_i x_k + q_{jk} x_j \alpha_{ik} + \alpha_{jk} x_i \\
= q_{jk} q_{ik} q_{ij} x_i x_j x_k + q_{jk} q_{ik} \alpha_{ij} x_k + q_{jk} x_j \alpha_{ik} \alpha_{jk} x_i
\end{aligned}$$

Hence we have a first order deformation if and only if

$$q_{ij} q_{ik} x_i \alpha_{jk} + q_{ij} \alpha_{ik} x_j + x_k \alpha_{ij} = q_{jk} q_{ik} \alpha_{ij} x_k + q_{jk} x_j \alpha_{ik} + \alpha_{jk} x_i$$

We will solve these equations for graded deformations, i.e. when $\alpha_{ij} \in \varepsilon A_2$ are quadratic. This could be done for choices of degree other than two.

So we let

$$\alpha_{ij} = \sum_{1 \leq \ell \leq m \leq n} \alpha_{ij}^{\ell m} x_\ell x_m$$

be arbitrary quadratic elements of A. We look at the above condition for these α. So we have

$$\sum (q_{ij} q_{ik} x_i \alpha_{jk}^{\ell m} x_\ell x_m + q_{ij} \alpha_{ik}^{\ell m} x_\ell x_m x_j + x_k \alpha_{ij}^{\ell m} x_\ell x_m) = \sum (q_{jk} q_{ik} \alpha_{ij}^{\ell m} x_\ell x_m x_k + q_{jk} x_j \alpha_{ik}^{\ell m} x_\ell x_m + \alpha_{jk}^{\ell m} x_\ell x_m x_i)$$

This gives cubic expressions in A, which we can separate in to different equations, for example:

$$(q_{ij} q_{ik} \alpha_{jk}^{ii} - \alpha_{jk}^{ii}) x_i^3 = 0.$$

On examining the coefficients of different monomials, we obtain the following equations:

$$(q_{ij} q_{ik} - q_{i\ell}) \alpha_{jk}^{i\ell} = 0 \quad \{j,k\} \cap \{i,\ell\} = \emptyset, \quad j < k, \quad i \leq \ell$$

$$q_{ij}(q_{ik} - qi\ell) \alpha_{jk}^{j\ell} + q_{ij}(q_{j\ell} - q_{jk}) \alpha_{ik}^{i\ell} = 0 \qquad |\{i,j,k\}| = 3.$$

We note that these conditions are mostly independent. The first set of equations is completely independent from each other and the second set of equations. The second set of equations decomposes into sets of three equations for each unordered pair k, ℓ with $k \neq \ell$. Hence we can combine them into a matrix equation for $i < j < k$ we have

$$\begin{pmatrix} q_{jk} & & \\ & q_{ik} & \\ & & q_{ij} \end{pmatrix} \begin{pmatrix} & q_{k\ell} - q_{kk} & q_{j\ell} - q_{jk} \\ q_{kk} - q_{k\ell} & 0 & q_{i\ell} - q_{ik} \\ q_{jk} - q_{j\ell} & q_{ik} - q_{i\ell} & 0 \end{pmatrix}$$

We first study the deformations for generic $Q = (q_{ij})$. We will determine later that the exact conditions we need are:

$$q_{ij} \neq 0 \qquad 1 \leq i < j \leq n.$$

$$q_{ij} q_{ik} \neq q_{i\ell} \qquad \{j,k\} \cap \{i,\ell\} = \emptyset$$

$$Q \notin V(q)$$

Next, we will discuss infinitesimal automorphisms of first order deformations. Hochschild cohomology captures infinitesimal deformation up infitesimal isomorphism. Let $d_i \in A$ for $1 \leq i \leq n$ and write $\delta_i = \varepsilon d_i$. We can carry out an infinitesimal change of coordinates that will change our presentation of A_1. Consider

$$x_i \mapsto x_i + \delta_i.$$

We can associate a derivation δ of S to this information by

$$x_i \mapsto d_i$$

and extending linearly and by the Leibniz rule. The effect of an infinitesimal change of coordinates on relations is determined by

$$r_{ij} \mapsto \delta(r_{ij}).$$

In particular for our case, we have

$$x_j x_i - q_{ij} x_i x_j \mapsto \delta_j x_i + x_j \delta_i - q_{ij}(\delta_i x_j + x_i \delta_j).$$

This changes the relations in the following way:

Proposition 2 *Two first order deformations A_1 and A_1' determined by (a_{ij}) and (a_{ij}') are infinitesimally isomorphic if and only if*

1. *There is a commutative diagram*

$$A_1 \to AA_1' \to A$$

2. *There are $d_i \in A$ so that have*

$$a_{ij} = a_{ij}' + d_j x_i + x_j d_i - q_{ij}(d_i x_j + x_i d_j).$$