Assignment # 2 Commutative Algebra
MATH 4107, MATH 5001
Due Oct. 29, 2020.

All rings are commutative with unity.
Recall that for an ideal I of a ring R we have:

$$V(I) = \{p \in \operatorname{Spec} R \,|\, p \supseteq I\}.$$

1. Let R be a ring and let $e \in R$ be an idempotent. Show that Re and $R(1-e)$ are non-unital subrings of R, i.e. they are subrings but their units are not necessarily 1_R. Show that R is a product of these rings $R = Re \times R(1-e)$.

2. Let
$$\mathbb{Z}_{(2)} = \{a/b \,|\, a, b \in Z \text{ and } b \notin 2\mathbb{Z}\}$$
$$\mathbb{C}[x]_{(x)} = \{f/g \,|\, f, g \in \mathbb{C}[x] \text{ and } g(0) \neq 0\}.$$
Show that these are local rings.

3. Find the ideal in $\mathbb{R}[x, y]$ corresponding to the variety that is the parabola $y = x^2$ union the point $(-1, -1)$ in $\mathbb{R}^2$.

4. Describe the ideal in $\mathbb{R}[x, y, z]$ corresponding to the variety that is the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the cone $x^2 + y^2 = z^2$.

5. Let $\nu : \mathbb{C}^2 \to \mathbb{C}^4$ be the polynomial mapping given by
$$\nu(x, y) = (x^3, x^2y, xy^2, y^3).$$
Find the ideal of the image of ν, i.e. $I(\nu(\mathbb{C}^2))$. Let $\sigma = xy + yz + xz$. We define a map $p : \mathbb{C}^3 \to \mathbb{C}^4$ as
$$p(x, y, z) = (x\sigma, y\sigma, z\sigma, -xyz).$$
Find the ideal of the image.

6. Recall that for an ideal I of a ring R we have:
$$V(I) = \{p \in \operatorname{Spec} R \,|\, p \supseteq I\}.$$
Let R be a ring and let I, J be ideals of R. Show that
   (a) $V(IJ) = V(I \cap J) = V(I) \cup V(J)$
   (b) $V(I + J) = V(I) \cap V(J)$.
   (c) $V(\sqrt{I}) = V(I)$.
   (d) Assume $I = \sqrt{I}$, show that $V(I : J) = \overline{V(I) \setminus V(J)}$

7. Let R, S be rings. Write $\coprod$ for disjoint union. Show that $\operatorname{Spec}(R \times S) = \operatorname{Spec} R \coprod \operatorname{Spec} S$ as topological spaces.

8. Let $\phi^* : R \to S$ be a ring homomorphism with induced map of spectra $\phi : \operatorname{Spec} S \to \operatorname{Spec} R$. Let I be an ideal of R. Show that $V(\phi^*(I)S) = \phi^{-1}(V(I))$.

9. Let $\pi^* : \mathbb{C}[x] \to \mathbb{C}[x, y]/(y^2 - x^3 - x)$ be the natural inclusion of $\mathbb{C}$-algebras. and write $\pi : \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^3 - x) \to \operatorname{Spec} \mathbb{C}[x]$ be the induced map of spectra.

(a) Suppose we have a point $(a, b) \in \mathbb{C}^2$ that satisfies the equation y^2-x^3-x. Let $m_{(a,b)} = (x-a, y-b)$. Describe $\pi(m_{(a,b)})$.

(b) For $a \in \mathbb{C}$ let $m_a = (x-a)$ be the corresponding maximal ideal of $\mathbb{C}[x]$. Relate the ideal $m_a\mathbb{C}[x, y]/(y^2-x^3-x)$ to the fibre of the map $\pi^{-1}(m_a)$ on spectra. Note the different behaviour for certain values of a.

10. Let R be a ring and let $a \in R$. Let M be an R-module.
    (a) Show that $\{m \in M \mid am = 0\}$ is a submodule of M.
    (b) Show that $\mathrm{Hom}_R(R/aR, M) = \{m \in M \mid am = 0\}$

11. Find and state the structure theorem that describes all finitely generated modules over a PID. Let us call this theorem FPID. We will use this theorem in the next questions.
    (a) Derive the fundamental theorem of finite abelian groups from FPID.
    (b) Let k be a field. Show that a $k[x]$ module is given by a k-vector space V with a linear transformation $T = \rho_x$ where $\rho_x : V \to V$ is right multiplication by x.
    (c) Suppose k is an algebraically closed field and let V be a finite dimensional $k[x]$ module. What does FPID say about V as a $k[x]$ module? Use FPID to find a basis for V where the linear transformation $T = \rho_x$ has Jordan canonical form.
    (d) Repeat the above where k is not necessarily algebraically closed to derive the rational canonical form theorem.

12. Let k be a field. Prove that the category of finite dimensional vector spaces $\mathbf{Vect}_k^{\mathrm{fin}}$ is equivalent to the category $\mathbf{Mat}_k$ as described in Lecture 10.

13. Find the 'free' functor defined to be the left adjoint of the following forgetful functors
    (a) $\mathbf{Top} \to \mathbf{Set}$.
    (b) $\mathbf{Rings} \to \mathbf{Ab}$ by $R \mapsto (R, +)$.
    (c) $\mathbf{Rings} \to \mathbf{Ab}$ by $R \mapsto R^*$.

14. A morphism $\phi : R \to S$ in a category is an epi if for all $g_1, g_2 : S \to T$ we have $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.
    (a) Show that an epi in the category of sets is a surjection.
    (b) Show that the ring inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epi in the category of rings.