Invent. math. 106, 335–388 (1991)

Inventiones mathematicae

Modules over regular algebras of dimension 3

M. Artin[1], J. Tate[2], and M. Van den Bergh[3]

[1] MIT, Department of Mathematics, Cambridge, MA 02139, USA
[2] University of Texas, Austin, TX 78712, USA
[3] Universitaire Instelling Antwerpen, B-2610 Wilrijk, Belgium

Oblatum 15-VII-1990

Table of contents

1 Introduction

Let k be a field. In a previous paper [ATV] (see also [OF]) some graded k-algebras A, regular algebras of dimension 3, were constructed from certain automorphisms σ of elliptic curves or of more general one-dimensional schemes E with arithmetic genus 1, which are embedded as cubics in $\mathbb{P}^2$ or as divisors of bidegree (2, 2) in $\mathbb{P}^1 \times \mathbb{P}^1$. In this correspondence, the points of the scheme E were shown to parametrize certain A-modules called *point modules*. A point module N is a graded right A-module with these properties:

(1.1) (i) $N_0 = k$,
(ii) N_0 generates N, and
(iii) $\dim_k N_i = 1$ for all $n \geqq 0$.

The structure of these point modules is related in a nice way to the geometry of the scheme E and its automorphism σ. For example, if $N = N_p$ is the module corresponding to a point p of E, then the normalized shift N^+, defined by

$$N_i^+ = \begin{cases} N_{i+1} & \text{if } i \geqq 0 \\ 0 & \text{if } i < 0 \end{cases}, \tag{1.2}$$

is the point module which corresponds to the point σp. The object of this paper is to study point modules and their relation to the geometry of E. The main results were announced in [VdB].

To fix ideas, let us consider the case that our algebra A corresponds to a cubic curve E in the plane. In this case, A is a non-commutative analogue of a polynomial ring in 3 variables. There is a normalizing element g of degree 3 in A which is unique up to constant factor. It is the analogue of the cubic equation defining the curve, and the ring $B = A/gA$ is the analogue of the homogeneous coordinate ring of E, defined explicitly by $B = \oplus H^0(E, \mathscr{L} \otimes \mathscr{L}^\sigma \otimes \ldots \otimes \mathscr{L}^{\sigma^{n-1}})$, where $\mathscr{L} = \mathscr{O}_E(1)$ (see [ATV]).

If R is a graded k-algebra, then by analogy with the commutative case, we imagine Proj R to be defined and to have a geometric meaning, and we think of it as the non-commutative analogue of a projective scheme. Thus Proj A is a non-commutative (or "quantum") analogue of the projective plane $\mathbb{P}^2$. We call two A-modules *equivalent* if they are isomorphic modulo $\mathfrak{m}$-torsion, i.e., if they correspond to the same imagined sheaf on Proj A (see (6.5)).

Again by analogy, if $B = A/gA$ as above, then Proj A contains Proj B as a "closed subscheme". And though the structure of Proj A is somewhat obscure, that of Proj B is well understood. The category of graded left (or of right) B-modules modulo torsion is equivalent to the category of quasi-coherent sheaves on the cubic curve E, just as in the commutative case when σ is the identity (see [AV]). The new feature comes into the shift operation on graded B-modules. In the commutative case, the corresponding operation on sheaves is $\mathscr{F} \rightsquigarrow \mathscr{L} \otimes_{\mathscr{O}} \mathscr{F}$, where $\mathscr{L} = \mathscr{O}_E(1)$. Here this operation is replaced by the operation $\mathscr{F} \rightsquigarrow \mathscr{L} \otimes_{\mathscr{O}} \mathscr{F}^\sigma$.

In addition to A and B, we will consider the $\mathbb{Z}$-graded ring $\Lambda = A[g^{-1}]$ obtained by adjoining the inverse of the normalizing element g, and its subring Λ_0 of elements of degree zero. Intuitively, the non-commutative affine scheme Spec Λ_0 plays the role of the "open complement" of Proj B in Proj A. It is clear that the structures of A and of Λ_0 are closely related. For the ring Λ_0, we have the following rather strong dichotomy (see (7.3)).

Theorem I *Let s denote the order of the σ-orbit of the class $[\mathscr{L}]$ of $\mathscr{L} = \mathscr{O}_E(1)$ in the Picard group of E. Then if $s < \infty$, Λ_0 is an Azumaya algebra of rank s^2 over its center, while if $s = \infty$, Λ_0 is a simple ring.*

We are also able to show (7.18) in the elliptic case that if σ itself is of finite order, then some power of the normalizing element g is in the center of A. Using this fact, we derive the result which is one of our main goals (see (7.1)):

Theorem II *A regular algebra of dimension 3 is a finite module over its center if and only if the automorphism σ has finite order.*

It is quite easy to exhibit the center of the associated algebra B explicitly, so Theorem II is easy to prove in the linear case [ATV, 8.5]. But since we don't have a conceptual description of the algebra A in terms of its triple $(E, \sigma, \mathscr{L})$ in the elliptic case, we aren't able to exhibit the center of an elliptic algebra A explicitly. Instead, we construct a family of graded A-modules of gk-dimension 1 and fixed multiplicity, such that the intersection of their annihilators is zero. This is the main step, because it proves that A is a polynomial identity ring [SSW].

The space A_1 of elements of degree 1 in A has dimension 3, and there is an interplay between the geometry of Proj A and of the ordinary projective space $\mathbb{P}^2 = \mathbb{P}(A_1)$. (We use Grothendieck's notation: $\mathbb{P}(V)$ denotes Proj $(S(V))$, where $S(V)$ is the symmetric algebra on a vector space V. Thus points of $\mathbb{P}(A_1)$ are in bijective correspondence with one-dimensional subspaces of A_1.) For example, right modules of the form $M = A/aA$, where a is a non-zero element of A_1, are in canonical bijective correspondence with lines ℓ in the projective space $\mathbb{P}^2$. We call these modules *line modules*.

Let M be the line module corresponding to a line ℓ. The point modules which are quotients of M correspond to points of intersection of E with ℓ (6.23). This is not very surprising. A less intuitive fact is that *any* critical module N of gk-dimension 1 (such a module may be thought of as corresponding to a closed point of Proj A) is equivalent to a quotient of some line module M, which we interpret intuitively as saying that N is supported on the line corresponding to M (6.7). This gives us a start towards the construction of the modules of dimension 1 which we use for the proofs of Theorems I and II. These considerations are carried out in Sects. 5, 6, and 7.

Section 2 reviews standard material about Hilbert series, and it contains an important characterization of line modules (2.43). In Sect. 3 we prove that noetherian regular graded algebras of dimension at most 4 are domains. The duality relating left and right A-modules is described in Sect. 4.

In Sect. 8, we describe a process of twisting a graded algebra A by an automorphism τ to obtain a new algebra A_τ. This twisted algebra can be quite different from A, but it should be considered as having the same Proj. We then determine explicitly those regular algebras which correspond to non-reduced divisors E, by showing that they are all twists of a few special types. The corresponding algebras A_0 are unchanged by twisting, and can be determined completely. They are closely related to the Weyl algebra.

Acknowledgement. We thank the referee for suggesting an improvement of our original version of Proposition (3.4).

2 Modules over regular algebras

This section reviews well-known properties of graded modules over regular noetherian graded algebras. The only results which may be new are at the end of the section, beginning with Proposition 2.41. Except when the contrary is stated explicitly, our algebras will be assumed to be finitely generated graded k-algebras of the form $A = k + A_1 + A_2 + \dots$. Such an algebra is called *regular* if it has the following properties:

(2.1) (i) A has finite global dimension d,
(ii) polynomial growth, and
(iii) is Gorenstein.

(See [ArSch, ATV].) When not otherwise specified, the symbol A will denote a regular algebra, which in addition is left and right noetherian. These properties of A are equivalent to the following:

(2.2) (i) The left module ${}_A k$ has a minimal graded resolution

$$0 \to P^d \to \ldots \to P^1 \to P^0 \to {}_A k \to 0$$

of length d by projectives of finite type.

(ii) (the Gorenstein condition) The transpose of this resolution is a resolution of a right module isomorphic to $k_A(c)$, the shift of k_A to some degree c:

$$0 \leftarrow k_A(c) \leftarrow P^{d*} \leftarrow \cdots \leftarrow P^{0*} \leftarrow 0 .$$

(iii) There are positive constants a, b such that $\dim_k A_n \leqq bn^a$ for all $n > 0$.

(iv) Every finite graded A-module M has a graded resolution (which will be of length at most d) by projectives of finite type.

Remarks. We conjecture that all regular algebras are noetherian domains. Also, in all examples which we know, the integer d is equal to the gk-dimension δ of A. By definition, δ is 1 more than the minimal a in (iii) (see below).

In later sections, we will be concerned mainly with the regular algebras of dimension 3 which are generated in degree 1. These are the algebras which were studied in [ArSch] and [ATV], and they are noetherian [ATV, 8.1]. We recall that there are two basic possibilities for such an algebra A: It will have r generators and r defining relations of degree $5 - r$, where

$$r = 2 \text{ or } 3. \tag{2.3}$$

This number will be denoted by r throughout. In order to shorten the phrase, let us agree that by *regular algebra of dimension 3* we will mean one which is generated in degree 1, unless we mention the contrary.

By A-module, we will mean a graded left or right module over A. We often use the term *finite A-module* to mean finitely generated graded A-module. The symbol $\mathrm{Hom}_A(M, N)$ will denote the graded group whose component of degree v consists of the degree-preserving homomorphisms $M \to N(v)$, where $N(v)$ denotes the *shifted module* defined by $N(v)_n = N_{v+n}$. The notation $\mathrm{Ext}^q_A(M, N)$ is to be interpreted as the derived functor of the graded $\mathrm{Hom}_A(M, N)$ in the category of graded modules. There are enough projectives and injectives in that category [NV, Ch. A]. Note that $\mathrm{Hom}_A(A, N) = N$ is true in the graded category. It follows that if M is a finite module, then $\mathrm{Ext}^q_A(M, N)$ agrees with the ungraded Ext.

The projective dimension of a module M will be denoted by $pd(M)$. Consideration of a minimal projective resolution for M shows that for $M \neq 0$, $pd\, M$ is the largest integer i such that $\mathrm{Ext}^i_A(M, k) \neq 0$, and hence also the largest integer such that $\mathrm{Ext}^i_A(M, A) \neq 0$.

We begin by reviewing standard material about the growth properties of finite (graded) modules over noetherian regular algebras. A good general reference for this material is [Stan]. The *Hilbert series* of a module or a graded k-vector space M is, by definition, the series

$$h_M(t) = \sum_n (\dim_k M_n)\, t^n . \tag{2.4}$$

This is an additive function on the Grothendieck group of finite A-modules M. The resolution (2.2i) provides a recursion relation which allows us to compute the Hilbert series

$$h_A(t) = \sum a_n t^n . \tag{2.5}$$

of A. A finitely generated projective module P is a sum of shifts of A. (To see this, choose a minimal surjection $\bigoplus_i A(v_i) \to P$. Minimality means that $\varphi \otimes k$ is bijective. Since P is projective this map splits, and the Nakayama lemma [ATV1, Proposition 2.2] shows that it is bijective.) So we may write

(2.6) $$P^i = \bigoplus_{j=1}^{r_i} A(-\ell_{ij}) ,$$

for suitable non-negative integers r_i and ℓ_{ij}. Of course, $P^0 = A$.

The *characteristic polynomial* of A is defined to be

(2.7) $$p_A(t) = \sum_{i=0}^{d} (-1)^i \sum_{j=0}^{r_i} t^{\ell_{ij}} = 1 + \ldots + (-1)^d t^c ,$$

with c and d as in (2.2).

For a regular algebra of dimension 3, the resolution (2.2i) has the form

$$0 \to A(-s-1) \to A(-s)^r \to A(-1)^r \to A \to {}_A k \to 0 ,$$

where $s = 5 - r$. Hence the characteristic polynomial of such an algebra is

(2.8) $$p_A(t) = \begin{cases} 1 - 3t + 3t^2 - t^3 = (1-t)^3 & \text{if } r = 3 , \\ 1 - 2t + 2t^3 - t^4 = (1-t)^2(1-t^2) & \text{if } r = 2 . \end{cases}$$

Proposition 2.9 *With the above notation,*

$$h_A(t) p_A(t) = 1 .$$

Proof. The coefficient of t^n in this product is

(2.10) $$\sum_{i=0}^{d} (-1)^i \sum_{j=1}^{r_i} a_{n-\ell_{ij}} = \sum_i (-1)^i \dim (P^i)_n .$$

This coefficient is 0 if $n \neq 0$ and 1 if $n = 0$ because the sequence (2.2i) is exact. □

We factor p_A in $\mathbb{C}[t]$, writing

(2.11) $$p_A(t) = \prod_{\nu} (1 - \alpha_\nu t) ,$$

and calling α_ν the *characteristic roots* of A. Then

(2.12) $$h_A(t) = \prod_{\nu} (1 - \alpha_\nu t)^{-1} = \prod_{\nu} (1 + \alpha_\nu t + \alpha_\nu^2 t^2 + \ldots) .$$

This product expansion implies the following proposition:

Proposition 2.13 *Let A be a graded algebra satisfying* (2.2i). *With the above notation, the following are equivalent*:

(i) *A has polynomial growth,*
(ii) *$h_A(t)$ converges for $t < 1$,*
(iii) *the characteristic roots α_ν have absolute value* $\leqq 1$.

Next, we note that the Gorenstein condition (2.2ii) yields a functional equation for the Hilbert function:

Proposition 2.14 *Let A be a graded algebra satisfying* (2.2i) *and* (ii). *Then*

(i) $$p_A(t) = (-1)^d t^c p_A(t^{-1}),$$

where c is as in (2.2ii), *and is also the degree of p_A.*

(ii) *The product of the characteristic roots of A is $(-1)^{d+c}$.*

(iii) *If A has polynomial growth, then the characteristic roots of A are roots of unity.*

Proof. The integers ℓ_{ij} appearing in (2.6) are determined by the formula

(2.15) $$\mathrm{Tor}_i^A(k_A, {}_A k) = \bigoplus_j k(-\ell_{ij}),$$

so they are unchanged if the sequence (2.2i) is replaced by a resolution of the right module k_A. Moreover,

$$P^{i*} = \mathrm{Hom}_A(P^i, A) = \sum_j A(\ell_{ij}).$$

The functional equation (i) follows immediately from this equation, and the fact that c is the degree of p_A is clear from its shape. Since the constant term of p_A is 1, the functional equation shows that its leading coefficient is $(-1)^d$. Finally, if A has polynomial growth, then the characteristic roots have absolute value $\leqq 1$ (2.13), and their product is ± 1. Thus $|\alpha| = 1$ for each characteristic root α. Since p_A is a polynomial with integer coefficients and leading coefficient ± 1, it follows that α is an algebraic integer all of whose conjugates have absolute value 1. Therefore α is a root of unity [BS, Ch. 2, Thm. 2.]. □

We now turn to Hilbert series of arbitrary finite modules. It will be convenient to work in the derived category $D_\ell^b(A)$ of bounded complexes of finite left A-modules. It follows from (2.2iv) that every such complex is isomorphic in $D_\ell^b(A)$ to a finite complex of projectives. Since every projective is a sum of modules $A(v)$, we can compute the Hilbert series of an arbitrary module M or of an element of $D_\ell^b(A)$ in terms of that of A. Given a resolution

(2.16) $$0 \to P^r \to \cdots \to P^1 \to P^0 \to M \to 0$$

of a module M, we have

(2.17) $$h_M = \sum (-1)^i h_{P^i}.$$

The Hilbert series of $A(-v)$ is $t^v/p_A(t)$. So if we write $P^i = \sum_j A(-v_{ij})$, we obtain the formula

(2.18) $$h_M = q_M(t)/p_A(t), \text{ or } h_M/h_A = q_M(t),$$

where

(2.19) $$q_M(t) = \sum_{i,j} (-1)^i t^{v_{ij}} \in \mathbb{Z}[t, t^{-1}].$$

Similarly, the Hilbert series of an arbitrary bounded complex M of modules is defined by the same formula. It satisfies the rule

(2.20) $$h_M = \sum (-1)^i h_{H^i(M)},$$

$H^i(M)$ denoting the cohomology of the complex M.

Proposition 2.21 *Let $h_M = \sum m_n t^n$ be the Hilbert series of a finite A-module M. Then*

(i) *The order p of pole of h at $t = 1$ is the maximum order of pole at points $t \neq 0$.*

(ii) *The order of growth of the coefficients m_n is as a polynomial of degree $p - 1$ in n. More precisely, if $p = 0$, then $m_n = 0$ for sufficiently large n. If $p > 0$, then $m_n = O(n^{p-1})$ as $n \to \infty$, but $m_n \neq O(n^{p-1-\delta})$ if $\delta > 0$.*

(iii) *The leading coefficient $e(M)$ of the series expansion of h_M in powers of $1 - t$, called the multiplicity of M, is positive, and it is an integer multiple of the multiplicity $e(A)$ of A.*

For convenience, we set

$$\iota := e(A)^{-1} . \tag{2.22}$$

Part (iii) of the proposition asserts that, for a module over one of these algebras,

$$\varepsilon(M) := \iota e(M) = e(M)/e(A) \tag{2.23}$$

is an integer. It is often convenient to work with $\varepsilon(M)$ rather than with $e(M)$.

If we expand $q_M(t)$ in powers of $1 - t$:

$$q_M(t) = q_0 + q_1(1-t) + q_2(1-t)^2 + \dots , \tag{2.24}$$

where $q_0 = q_M(1)$, $q_1 = -q'_M(1)/1!$, etc., then formula (2.18) tells us that $\varepsilon(M)$ is the first non-vanishing coefficient q_i.

For a regular algebra of dimension 3 we have

$$\iota = \begin{cases} 1 & \text{if } r = 3 \\ 2 & \text{if } r = 2 \end{cases}, \text{ and } \varepsilon(M) = \begin{cases} e(M) & \text{if } r = 3 \\ 2e(M) & \text{if } r = 2 \end{cases}. \tag{2.25}$$

Proof of Proposition 2.21 We have seen that the characteristic roots of A are roots of unity; say they are powers of a primitive N-th root of unity ζ. Let p be the highest order of pole of h_M at the characteristic roots of A. Then h_M has a partial fraction expansion

$$h_M(t) = \sum_{i,j} c_{ij}/(1 - \zeta^i t)^j + f(t) , \tag{2.26}$$

where $i = 0, \dots, N - 1, j = 1, \dots, p$, and $f(t) \in \mathbb{Z}[t, t^{-1}]$. The binomial expansion of $1/(1 - t)^j$ shows that, for large n,

$$\begin{aligned} m_n &= \sum_{i,j} c_{ij} \binom{n+j-1}{j-1} \zeta^{in} \\ &= \left(\sum_i c_i \zeta^{in} \right) n^{p-1}/(p-1)! + (\text{terms of lower degree in } n) , \end{aligned}$$

where $c_i = c_{ip}$. This function cycles through N polynomials, according to the congruence class (modulo N) [Stan]. If $p = 0$, i.e., if $h_M(t) = f(t)$, then (i) and (ii) are obvious. Suppose $p > 0$. Then by assumption the c_i are not all zero. So the leading coefficients $\sum c_i \zeta^{in}$ are not all zero (Vandermonde). This proves (ii). Also, since $m_n \geqq 0$, $\sum c_i \zeta^{in} \geqq 0$ for all n. Summing n from 0 to $N-1$, we find

$$0 < \sum_{n,i=0}^{N-1} c_i \zeta^{in} = N c_0 .$$

Therefore $c_0 > 0$. This proves (i), and also shows that

$$c_0 = e(M), \text{ if } p > 0 . \tag{2.27}$$

If $p = 0$, then $h_M(t) \in \mathbb{Z}[t, t^{-1}]$ has coefficients $\geqq 0$, and $e_M = h_M(1)$, so $e_M > 0$ for all $M \neq 0$. Since $h_M h_A^{-1}$ is a polynomial with integer coefficients in all cases, $\varepsilon(M) \in \mathbb{Z}$. □

This proposition allows us to define the *gk-dimension* $gk(M)$ of a non-zero module M to be the order of pole of $h_M(t)$ at $t = 1$. Equivalently, the gk-dimension measures the order of growth of $\dim_k M_n$. One can define the gk-dimension of a module more generally [KL], but in our case, the dimensions which arise are non-negative integers.

Note that, by its definition, $gk(M)$ depends only on M as a graded k-module, and does not depend on the A-module structure, although if M is a finite left or right A-module, then $gk(M) \leqq gk(A)$.

We obtain an additive function e_ρ on the Grothendieck group of modules of gk-dimension $\leqq \rho$, by putting $e_\rho(M) = e(M)$ if $gk(M) = \rho$ and $e_\rho(M) = 0$ if $gk(M) < \rho$. We can also define the order of pole and multiplicity of an arbitrary bounded complex formally, but the alternating sign may cause cancellation. Therefore the order of pole need not reflect the growth of the cohomology modules, though we do have the following trivial fact:

Corollary 2.28 *Let M be a bounded complex of A-modules. Assume that the order of pole of $h_M(t)$ at $t = 1$ is ρ, and that there is an integer i such that $gk(H^\nu(M)) < \rho$ if $\nu \neq i$. Then $gk(H^i(M)) = \rho$, and $e(H^i(M)) = (-1)^i e(M) = (-1)^i e_\rho(M)$.*

The following proposition is standard [KL].

Proposition 2.29 *Let A be a noetherian regular algebra, and let M be a finite left or right A-module of gk-dimension m.*

(i) *The sum M_ν of all submodules of M of gk-dimension $\leqq \nu$ is a characteristic submodule of M, $gk(M_\nu) \leqq \nu$, and if $m = gk(M)$, then*

$$M = M_m \supset M_{m-1} \supset \ldots \supset M_1 \supset M_0 .$$

(ii) *The quotient module $M_\nu / M_{\nu-1}$ is pure ν-dimensional. That is, all of its non-zero submodules have gk-dimension ν.*

(iii) *If M is a bimodule which is finite as left and as right module, then M_ν is a two-sided submodule, independent of choice of left or right in the definition.*

(iv) *For all finite right modules N_A and all q, the graded vector space $\mathrm{Tor}_q^A(N, M)$ has gk-dimension $\leqq m$.*

We also need to recall the definition of critical module. An A-module M is *critical* if it is not zero and if every proper quotient has lower gk-dimension. Note that a critical module is pure. Some other key facts about critical modules are as follows:

Proposition 2.30 (i) *Every non-zero finite module contains a critical submodule. In fact, every finite module M contains an essential submodule which is a direct sum of critical modules. (A submodule is called essential if it is not a direct summand of a strictly larger submodule.)*

(ii) *A finite module has a finite filtration whose successive quotients are critical.*

(iii) *If M is a module of gk-dimension v, the successive quotients in such a filtration are of gk-dimension $\leqq v$, and the number whose gk-dimension is equal to v is independent of the filtration. It will be called the v-length of M.*

(iv) *If M is pure v-dimensional then it has a filtration such that the successive quotients are critical and of gk-dimension v.*

(v) *If A is a prime ring of gk-dimension δ with left ring of fractions K, then A is a pure A-module, and a left module M has gk-dimension $<\delta$ if and only if $K \otimes_A M = 0$. If A is a domain, then A is a critical A-module.*

(vi) *Let M be a finite critical A-module. The annihilator P of M is a prime ideal, and P is also the annihilator of each non-zero submodule of M.*

(vii) *Suppose that k is algebraically closed. Then the only degree-zero endomorphisms of a critical module are scalars.*

Since these results are standard, we will content ourselves with a proof of (vi). Let M' be a non-zero submodule, and let P' be its annihilator. Tensoring the exact sequence

$$0 \to M' \to M \to M/M' \to 0$$

by A/P' yields an exact sequence $\mathrm{Tor}_1^A(A/P', M/M') \to M' \to M/P'M$. Since $gk(\mathrm{Tor}_1^A(A/P', M/M')) < gk(M) = gk(M')$ by the previous proposition, we have $gk(M') = gk(M/P'M)$. Since M is critical, $P'M = 0$. This shows that $P' = P$. To show that P is a prime ideal, suppose that $P \supset IJ$ but that $JM \neq 0$. We set $M' = JM$ and apply what has been shown. □

If M is a finite module or an element of $D_\ell^b(A)$, we denote by M^D its dual $\mathrm{RHom}_A(M, A)$, which is an element of the derived category $D_r^b(A)$ of bounded complexes of finite right modules. When M is represented by a finite complex of projectives

$$0 \to P^k \to \cdots \to P^1 \to P^0 \to 0\,, \tag{2.31}$$

for example by means of projective resolution if M is a module, then M^D is represented by the transpose sequence of right modules

$$0 \leftarrow P^{k*} \leftarrow \cdots \leftarrow P^{1*} \leftarrow P^{0*} \leftarrow 0\,, \tag{2.32}$$

where $P^* = \mathrm{Hom}_A(P, A)$. The q-th cohomology of this complex is $\mathrm{Ext}_A^q(M, A)$.

Clearly, there is a "biduality" isomorphism $M \to M^{DD}$, which expresses itself on Ext by a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_A^p(\mathrm{Ext}_A^{-q}(M, A), A) \Rightarrow M.$$

In order to put this spectral sequence into the standard first quadrant form, we reindex, writing it as

$$E_2^{p,q} = \mathrm{Ext}_A^p(\mathrm{Ext}_A^{d-q}(M, A), A) \Rightarrow M_{[d]}\,. \tag{2.33}$$

where $M_{[d]}$ denotes the shift of position by d in the complex M.

The Hilbert series of the dual M^D can be computed directly in terms of that of M. Write $h_M = q_M(t)/p_A(t)$ as before. Then

$$q_{M^D}(t) = q_M(t^{-1}) . \tag{2.34}$$

Combining this with (2.14), we obtain the formula

$$h_{M^D}(t) = (-1)^d t^c h_M(t^{-1}) , \tag{2.35}$$

which gives us the following corollary:

Proposition 2.36 *Let M be a bounded complex of A-modules. Then*

(i) *The order m of pole of h_M* at $t = 1$ *is equal to the order of pole of $h(M^D)$ at $t = 1$.*

(ii) $e(M^D) = (-1)^{d-m} e(M)$.

The last general property of Hilbert series which we will review is their behavior with respect to tensor products. If M, N are bounded complexes of finite right and left A-modules, we denote by $M \overset{L}{\otimes} N$ the tensor product in the derived category. It is represented by the tensor product complex, provided that one of the complexes is replaced by a bounded complex of projectives. Thus $M \overset{L}{\otimes} N$ is a complex of graded vector spaces, and as such it has a Hilbert series, which we denote by $h_{M \overset{L}{\otimes} N}(t)$. We have

$$h_{M \overset{L}{\otimes} N}(t) = \sum_i (-1)^i h_{\mathrm{Tor}_i(M,N)}(t) . \tag{2.37}$$

Direct computation of this Hilbert series yields the following

Proposition 2.38 *Let M, N be bounded complexes of right and left A-modules respectively. Then the Hilbert series of $M \overset{L}{\otimes} N$ has the form*

$$h_{M \overset{L}{\otimes} N}(t) = q_M(t) q_N(t)/p_A(t) ,$$

where q_M and q_N are the "numerators" of the Hilbert series' for M and N which appear in (2.18), *and p_A is the characteristic polynomial of A.* □

Copying the definition from commutative algebra, we will say that a finite module M is a *k-th syzygy* if there is an exact sequence of the form

$$0 \to M \to P^1 \to P^2 \to \cdots \to P^k , \tag{2.39}$$

where P^i are finitely generated projective modules.

Proposition 2.40 (i) *If A has global dimension d and M is a k-th syzygy, then the projective dimension $pd(M)$ is at most $max\{0, d - k\}$.*

(ii) *A module M is a first syzygy if and only if the map $M \to M^{**}$ from M to its bidual is injective, and M is a second syzygy if and only if this map is bijective, i.e., if and only if M* is *reflexive.*

The first assertion is trivial, and the second results from the consideration of a projective resolution of M^* (see [EG, Thm. 3.6]).

Proposition 2.41 *Let A be a regular noetherian algebra of gk-dimension n. Let M be a non-zero A-module such that $pd(M) \leqq 1$, and let*

$$0 \to \sum_i A(-i)^{b_i} \xrightarrow{f} \sum_i A(-i)^{a_i} \to M \to 0 \tag{2.42}$$

be a minimal resolution of M. Then $gk(M) \geqq n-1$, and $gk(M) = n-1$ if and only if $\sum_i a_i = \sum_i b_i$, in which case we have

$$\sum_i b_i \leqq \sum_i i(b_i - a_i) = \varepsilon(M) .$$

Corollary 2.43 *The following properties of a graded A-module M with $pd(M) = 1$ are equivalent:*

(i) *$gk(M) = n - 1$, and $\varepsilon(M) = 1$,*

(ii) *M is isomorphic to a shift of a module of the form A/aA, where a is a left regular element of A_1.*

Proof of the corollary, assuming the proposition. Suppose that (i) holds. Then $\sum_i a_i = \sum_i b_i \leqq 1$. Hence $a_i = 1$ for a single index i, say $i = i_1$, and is zero otherwise. Similarly, $b_i = 1$ for some $i = i_2$, and $b_i = 0$ otherwise. The equation $i_2 - i_1 = \sum_i i(b_i - a_i) = 1$ shows that the minimal resolution of M is

$$0 \to A(-i_1 - 1) \to A(-i_1) \to M \to 0 , \tag{2.44}$$

i.e., that M is isomorphic to $(A/aA)(-i_1)$, where a is a left regular element of degree 1. Thus (ii) holds.

Conversely, assume (ii). Then the minimal resolution is of the form (2.44) for some integer i_1. Hence $h_M(t) = t^{i_1} h_A(t)(1-t)$, from which we find $gk(M) = gk(A)$ and $e(M) = e(A)$, i.e., that (i) holds. □

Proof of Proposition 2.41 The difference $n - gk(M)$ is the order of zero at $t = 1$ of the function

$$q_M(t) = \frac{h_M(t)}{h_A(t)} = \sum_i (a_i - b_i) t^i = \sum_\nu q_\nu (1-t)^\nu , \tag{2.45}$$

where

$$q_0 = \sum_i (a_i - b_i), \quad q_1 = \sum_i i(b_i - a_i), \quad q_2 = \sum_i \binom{i}{2}(a_i - b_i), \text{ etc} \ldots$$

(see (2.24)). Thus $gk(M) < n$ if and only if $q_0 = 0$, i.e., $\sum a_i = \sum b_i$. Suppose that this is the case. To finish the proof we must prove the inequality $\sum b_i \leqq \sum i(b_i - a_i)$ $(= q_1)$. Then, since $M \neq 0$ implies $\sum b_i > 0$, it will follow that $q_1 \neq 0$, hence that $gk(M) = n - 1$, and $e(M) = q_1 e(A)$, as required (see (2.21iii)).

Since (2.42) is minimal, the matrix entries of the map f all have positive degree. It follows from this fact that for each integer j the image by f of the module $\sum_{i \leqq j} A(-i)^{b_i}$ is contained in $\sum_{i<j} A(-i)^{a_i}$.

Let X be the quotient module and let h_x be its Hilbert function. Since f is injective,

$$\frac{h_X(t)}{h_A(t)} = \sum_{i<j} a_i t^i - \sum_{i \leqq j} b_i t^i .$$

Letting t approach 1 from below shows that

$$\sum_{i<j} a_i - \sum_{i \leqq j} b_i \geqq 0 .$$

We write this inequality in the form

$$b_j \leqq \sum_{i<j} (a_i - b_i)$$

and sum over j for $j \leqq m$, where m is a fixed integer large enough so that $a_\mu = b_\mu = 0$ for all $\mu \geqq m$. Since $q_0 = 0$, we find

$$\sum_j b_j = \sum_{j \leqq m} b_j \leqq \sum_{j \leqq m} \sum_{i<j} (a_i - b_i) = \sum_{i<m} (m-i)(a_i - b_i) = mq_0 + q_1 = q_1 ,$$

as was to be shown. □

Proposition 2.46 *Let A be a noetherian regular algebra of global dimension d, and let M be a finite A-module. Let $\mathfrak{m} = A_1 + A_2 + \ldots$ be the augmentation ideal of A.*

(i) *If $pd(M) < d$ then the socle* $\mathrm{Hom}_A(k, M)$ *of M is zero. The converse is true if* $d > 0$.

(ii) *Let T denote the $\mathfrak{m}$-torsion submodule of M. Then* $\mathrm{Ext}_A^d(M, A) \approx \mathrm{Ext}_A^d(T, A)$. *In particular,* $\mathrm{Ext}_A^d(M, A)$ *is a finite-dimensional k-vector space of the same dimension as T.*

(iii) *Let $\bar{M} = M/T$. Then* $\mathrm{Ext}^q(M, A) \approx \mathrm{Ext}^q(\bar{M}, A)$ *for all $q < d$.*

Proof. Note that $T \neq 0$ if and only if $\mathrm{Hom}_A(k, M) \neq 0$. Moreover, since T has finite length, the fact that A is Gorenstein shows that $\mathrm{Ext}_A^i(T, A) = 0$ if $i < d$, and that $\mathrm{Ext}_A^d(T, A)$ is dual to T. Since A has global dimension d, we obtain an exact sequence

$$\mathrm{Ext}_A^d(\bar{M}, A) \to \mathrm{Ext}_A^d(M, A) \to \mathrm{Ext}_A^d(T, A) \to 0 .$$

Thus $pd(M) = d$ if the socle of M is non-zero. Since the socle of $\bar{M}$ is zero, this sequence shows that (ii) follows from (i), applied to $\bar{M}$. Also, (iii) follows from (ii) and from the Ext sequence associated to the exact sequence

$$0 \to T \to M \to \bar{M} \to 0 .$$

To complete the proof of (i), we may assume that $pd\, M = d > 0$. Let

$$0 \to P^d \to P^{d-1} \to \cdots \to P^0 \to M \to 0$$

be a minimal resolution. The boundary maps in this complex carry P^j to $\mathfrak{m}P^{j-1}$ for each j, and it follows that the maps $\mathrm{Ext}_A^d(k, P^i) \to \mathrm{Ext}_A^d(k, P^{i-1})$ vanish for all i, in particular for $i = d$. Set $M^0 = M$ and $M^i = \mathrm{im}(P^i \to P^{i-1}) = \ker(P^{i-1} \to P^{i-2})$, so that there are exact sequences

$$0 \to M^i \to P^{i-1} \to M^{i-1} \to 0$$

for $i = 1, \ldots, d-1$. These sequences induce isomorphisms

$$\mathrm{Hom}_A(k, M^0) \approx \mathrm{Ext}_A^1(k, M^1) \approx \cdots \approx \mathrm{Ext}_A^{d-1}(k, M^{d-1}) .$$

We also have an exact sequence

$$0 \to P^d \to P^{d-1} \to M^{d-1} \to 0 ,$$

which gives rise to an exact sequence

$$0 \to \mathrm{Ext}_A^{d-1}(k, M^{d-1}) \to \mathrm{Ext}_A^d(k, P^d) \to \mathrm{Ext}_A^d(k, P^{d-1}) .$$

As we remarked, the right hand map in this sequence is zero. This shows that $\mathrm{Hom}_A(k, M) = 0$ if and only if $P^d = 0$, as required. □

3 Proof that regular algebras of dimension at most 4 are domains

Throughout this section, the term *module* will mean finite graded A-module, and *bimodule* will mean graded A-bimodule which is finite as left and as right module. As we have noted, the characteristic filtration (2.29) is the same, whether we view M as a left or as a right module, and it consists of two-sided submodules. In particular, if M is pure ν-dimensional as a left module, it is so as a right module as well, and vice versa.

Our goal in this section is to prove that regular noetherian algebras are domains if their dimension is at most 4. This is an old result of Ramras [R] for rings of dimension 2 and it has also been proved by Snider [Sn] for rings of dimension 3. Before proving the theorem, we will collect some elementary facts about bimodules over regular noetherian algebras, which will then be applied to study the characteristic filtration (2.29) of A.

Proposition 3.1 *Let A be a regular noetherian algebra, and let M be an A-bimodule. Let I be the right (or left) annihilator of M. Then $gk(A/I) = gk(M)$.*

Proof. The inequality $gk(A/I) \geqq gk(M)$ holds because M is a finite right A/I-module. To prove the other inequality, choose generators $x_1, \ldots, x_k$ for M as left module, and let I_ν be the right annihilator of x_ν, so that $x_\nu A \approx A/I_\nu$, and $I = \cap I_\nu$. Then $gk(A/I) \leqq \max\{gk(A/I_\nu)\} = \max\{gk(x_\nu A)\} \leqq gk(M)$. □

Proposition 3.2 *Let B be a quotient of gk-dimension k of a regular noetherian algebra A, and let N be a pure k-dimensional B-bimodule. Then every regular element of B is left N-regular.*

Proof. Let u be a left regular element of B, and let $N' = \ker \lambda$, where $\lambda = \lambda_u$ denotes left multiplication by u on N. This is a right B-module. To show that $N' = 0$, it suffices to show $gk(N') < gk(B)$. The sequence $0 \to B \to B \to B/uB \to 0$ shows that $gk(B/uB) < gk(B)$ and also that $N' = \mathrm{Tor}_1^B(B/uB, N)$. Therefore $gk(N') \leqq gk(B/uB) < gk(B)$ (2.29iv). □

Proposition 3.3 *Let M be critical as an A-bimodule, and let P be the left annihilator of M. Then P is a prime ideal, and $gk(A/P) = gk(M)$.*

Proof. Let Q denote the annihilator of M in $A \otimes A^0$. This is a prime ideal (2.30vi). Since $A \otimes A^0$ is centrally generated over A, the intersection of Q with A, which is P, is prime too. The last assertion is a special case of (3.1). □

Proposition 3.4 *Let $A = k + A_1 + A_2 + \ldots$ be a noetherian graded k-algebra, and let $P_1, \ldots, P_r$ be a finite set of graded prime ideals of A, not including the augmentation ideal $\mathfrak{m} = A_1 + A_2 + \ldots$. There is a homogeneous element $x \in A$ of positive degree whose residue in $B_i = A/P_i$ is a regular element for each i.*

Proof. The proof is similar to the one given in the ungraded case by Stafford [Staf, Prop. 2.4]. We order the prime ideals P_j in such a way that P_r does not contain P_j for all $j < r$. By induction, we may assume that there exists a homogeneous element b which is regular in B_j for all $j < r$. We set $I = P_1 \cap \ldots \cap P_{r-1}$. It suffices to find a homogeneous element $d \in I$ such that, for some k, $b^k + d$ is homogeneous and has a regular image in B_r. From our chosen ordering of the prime ideals, it follows that the image of I is a nonzero two-sided ideal of B_r. Reducing modulo P_r, we see that it suffices to prove the following lemma.

Lemma 3.5 *Let B be a graded noetherian prime ring, let I be a nonzero two-sided ideal of B, and let b an arbitrary homogeneous element of B. There exists an element $d \in I$ such that, for some k, $b^k + d$ is homogeneous and regular.*

Proof. Since $I \neq 0$ and B is prime, I is an essential ideal. We set $b = b_0$. Replacing b_0 by a power as necessary, we may assume that $\operatorname{ann}(b_0) = \operatorname{ann}(b_0^n)$ for every $n > 0$, where ann denotes the left annihilator.

We follow the proof of the graded Goldie theorem [NV, Theorem C.I.1.6]. (When $r = 2$, the lemma follows directly from this theorem.) If $\operatorname{ann}(b_0) \neq 0$, then since I is essential, there exists an element b_1 in $I \cap \operatorname{ann}(b_0)$ which is not nilpotent [NV, Lemma C.I.1.4]. (Note: The word "semisimple" in the statement of this lemma should read "semiprime".) Replacing b_1 by a power, we may assume that $\operatorname{ann}(b_1) = \operatorname{ann}(b_1^n)$ for every $n > 0$. If $\operatorname{ann}(b_0) \cap \operatorname{ann}(b_1) \neq 0$, we choose a non-nilpotent element $b_2 \in I \cap \operatorname{ann}(b_0) \cap \operatorname{ann}(b_1)$, and we replace it by a power as necessary, so that $\operatorname{ann}(b_2) = \operatorname{ann}(b_2^n)$ for every $n > 0$. This procedure can be repeated so long as $\operatorname{ann}(b_0) \cap \ldots \cap \operatorname{ann}(b_s) \neq 0$, and it yields a sequence of nonzero elements $b_0, b_1, b_2, \ldots$

From the choice of b_i, it follows that the sum of right ideals $b_0 A + b_1 A + b_2 A + \ldots + b_s A$ is a direct sum. So since B is noetherian, the procedure must stop, at which time $\operatorname{ann}(b_0) \cap \ldots \cap \operatorname{ann}(b_s) = 0$.

Choose k_i so that $x = b_0^{k_0} + b_1^{k_1} + \ldots + b_s^{k_s}$ is homogeneous. Then from the above direct sum decomposition, we find $\operatorname{ann}(x) \subset \operatorname{ann}(b_0) \cap \ldots \cap \operatorname{ann}(b_s) = 0$. Hence x is regular. Furthermore, by construction, $d = b_1^{k_1} + \ldots + b_s^{k_s}$ is an element of I. $\square$

We now return to our regular noetherian algebra A. Let M be an A-bimodule, and consider the characteristic filtration (2.29)

$$M = M_r \supset \ldots \supset M_0\,, \tag{3.6}$$

in which M_ν has gk-dimension ν, and where $M_\nu / M_{\nu-1} = N_\nu$ is pure ν-dimensional. We may choose a filtration of each of the modules N_ν, whose successive quotients are critical ν-dimensional bimodules. Each of these quotients will have a left annihilator P which, by Proposition 3.3, is a prime ideal. Let $P_1, \ldots, P_s$ be the set of these prime ideals. It is natural to call them (the) *associated primes* of M.

Corollary 3.7 *Let M be an A-bimodule whose socle is trivial. There exists a homogeneous element $x \in A$ of positive degree which is M-regular.*

This is clear from the preceeding remarks and from Propositions 3.2 and 3.4. $\square$

Proposition 3.8 *Let A be a regular noetherian algebra of global dimension d. For any bimodule M, the gk-dimension of* $\operatorname{Ext}_A^{d-1}(M, A)$ *is at most* 1.

Proof. We may also assume that the socle of M is zero (cf. Proposition 2.46 (iii)). Then Corollary 3.7 tells us that there is a homogeneous element $x \in A$ of positive degree v which is not a right zero-divisor in M. Let $N = M/Mx$, and let $E = \mathrm{Ext}_A^{d-1}(M, A)$. Right multiplication by x on M induces an exact sequence

$$E \xrightarrow{\lambda_x} E \to \mathrm{Ext}_A^d(N, A),$$

where λ_x is left multiplication on the left module E. Since $\mathrm{Ext}^d(N, A)$ has finite length over k (2.46), it follows that λ_x is surjective in large degree. Thus $\dim E_n \geqq \dim E_{n+v}$ if $n \gg 0$, which shows that $gk(E) \leqq 1$, as required. □

We are now ready to prove the main result of the section:

Theorem 3.9 *A regular noetherian algebra of global dimension and gk-dimension $d \leqq 4$ is a domain.*

Our proof is arranged as a sequence of lemmas, some of which are true in arbitrary dimension. We denote the global dimension of our regular noetherian algebra A by d, and its gk-dimension by d'.

Lemma 3.10 *The d'-length of A is* 1.

Proof. Every finite A-module has a finite resolution by finite sums of modules $A(v)$. Since d'-length is an additive function on the Grothendieck group of finite modules, the d'-length of any module is an integer multiple of the d'-length of A. On the other hand, every critical module of gk-dimension d' has d'-length 1. □

For the rest of this section, we denote by N the largest ideal of A of gk-dimension $< d'$. The next lemma shows that it suffices to prove $N = 0$.

Lemma 3.11 *A/N is a domain.*

Proof. By the definition of N, the module $\bar{A} = A/N$ is pure d'-dimensional. Let K be the left annihilator of an element $b \in \bar{A}$, so that $\bar{A}b \approx \bar{A}/K$. By the previous lemma, one of the two left ideals K or $\bar{A}b$ has gk-dimension $< d'$, and is therefore zero. □

Lemma 3.12 *Let I be the right annihilator of the ideal N. Then $gk(I) = gk(A)$, hence there is an element $b \in I$ which is not in N.*

This follows from Proposition 3.1 and the definition of N. □

Lemma 3.13 *N is a reflexive A-module, and $pd(N) \leqq \max\{0, d-2\}$.*

Proof. Lemma 3.12 tells us that there is an element b in the annihilator of N, but not in N. Therefore the kernel of right multiplication $\rho_b = \rho$ by b on A contains N, while right multiplication by b on the domain $\bar{A} = A/N$ is injective. It follows that $\ker \rho = N$, and that the sequence

(3.14) $$0 \to N \to A \xrightarrow{\rho} A \to A/Ab \to 0$$

is exact. So N is a second syzygy, and hence is reflexive and of projective dimension $\leqq \max(0, d-2)$ by (2.40). □

Lemma 3.15 *Theorem* (3.9) *is true if the global dimension d is* $\leqq 2$.

Proof. If the global dimension d of A is at most 2, then N is projective, by the previous lemma, hence it is a sum of shifts of A. Since $gk(N) < gk(A)$, it follows that $N = 0$, and by Lemma 3.11 that A is a domain. $\square$

Note that we did not use the hypothesis $d = d'$ here. In fact, the structure of regular graded algebras of global dimension $\leqq 2$ is known (see [ATV, 3.14] for the case of algebras of dimension 2 generated in degree 1) and from the structure theorem it is easy to see that they are noetherian domains with $d = d'$.

We assume from now on that $d = d' \geqq 3$.

Lemma 3.16 *N contains no non-zero submodule of gk-dimension* $\leqq 1$.

Proof. Since N is reflexive, $pd(N) \leqq d-2$. Also, the Gorenstein condition implies that the socle of N is trivial (2.46). Let N_1 be the characteristic submodule of N of gk-dimension $\leqq 1$. We want to show that $N_1 = 0$. Now the bimodule $M = N_1 \oplus N \oplus (N/N_1)$ has trivial socle, and so Corollary 3.7 tells us that there is an element x in A of positive degree which is M-regular. Note that N_1 has gk-dimension $\leqq 1$. If $N_1 \neq 0$, it follows that N_1/xN_1 is a non-zero module of finite length. Since x is (N/N_1)-regular, multiplication by x in the exact sequence

$$0 \to N_1 \to N \to (N/N_1) \to 0$$

shows that N/xN has a non-zero socle, which implies that $\mathrm{pd}(N/xN) = d$ (2.46). This contradicts the facts that x is N-regular and that $pd(N) \leqq d-2$. $\square$

We now proceed with the proof of Theorem 3.9. To simplify notation, we will write

$$E^q(M) := \mathrm{Ext}^q_A(M, A)\,.$$

Suppose $d = d' = 3$ and $N \neq 0$. Then by Lemma 3.13, $pd(N) \leqq 1$, hence by Proposition 2.41, $gk(N) \geqq 2$. By the definition of N, $gk(N) \leqq 2$, hence $gk(N) = 2$. Proposition 2.36 tells us that $e(N^D) = -e(N)$. On the other hand, the only homology modules of the complex N^D are $N^* = E^0(N)$ and $E^1(N)$. Moreover, since N is a second syzygy, $E^1(N) = E^3(M)$ for some M, hence $E^1(N)$ is a finite length module. Therefore N^* has gk-dimension 2, and $e(N^*) = e(N^D) = -e(N)$. This contradicts the fact that $e(N^*) \geqq 0$, and completes the proof in the case of global dimension 3.

From now on we assume that $d = d' = 4$. The proof is harder in this case, when we know only that $pd(N) \leqq 2$.

Lemma 3.17 *Let M be a reflexive A-bimodule. Then* $gk(E^1(M)) \leqq 1$.

Proof. The dual module $M^* = E^0(M)$ is also a reflexive bimodule, so replacing M by M^* shows that it suffices to prove $gk(E^1(E^0(M))) \leqq 1$. We consider the spectral sequence (2.33) $E_2^{pq} = E^p(E^{4-q}(M)) \Rightarrow M_{[4]}$. Denoting $E^i(E^j(M))$ by E^iE^j, we have $E^1E^0 = E_2^{14}$. Since the abutment of the spectral sequence is $M_{[4]}$, we know that $E_\infty^{14} = 0$. The non-zero coboundary maps involving E^{14} are $d_2\colon E_2^{14} \to E_2^{33} = E^3E^1$ and $d_3\colon E_3^{14} \to E_3^{42}$, where E_3^{42} is a quotient of $E_2^{42} = E^4E^2$. Proposition 3.8 tells us that $gk(E^3E^1) \leqq 1$, and $E^4(L)$ has finite length for every finite module L. It follows that $gk(E^1E^0) \leqq 1$, as required. $\square$

Lemma 3.18 *N is pure 2-dimensional.*

Proof. By definition, $gk(N) \leqq 3$, and by Lemma 3.16, N contains no submodule of gk-dimension $\leqq 1$. On the other hand, if $gk(N) = 3$, then we can argue as above: Proposition 2.36 tells us that $e(N^D) = -e(N)$. On the other hand, $E^q(N) = 0$ except for $q = 0, 1, 2$. Moreover, $E^2(N) = E^4(M)$ for some M, and so $E^2(N)$ is a finite length module, while $E^1(N)$ has gk-dimension $\leqq 1$ by the preceeding proposition. Therefore N^* has gk-dimension 3 by Proposition 2.36, and $e(N^*) = e(N^D)$, by Corollary 2.28. This contradicts $e(N^D) = -e(N)$. □

Lemma 3.19 *Let M be a right module such that $pd(M) \leqq 2$. Then* $\mathrm{Tor}_i(M, N) = 0$ *if $i > 0$.*

Proof. Since N is a second syzygy, there is a module N' such that $\mathrm{Tor}_i(M, N) = \mathrm{Tor}_{i+2}(M, N')$. Since $pd(M) \leqq 2$, $\mathrm{Tor}_{i+2}(M, N') = 0$ if $i > 0$. □

Lemma 3.20 *Suppose that $N \neq 0$ and that M is a non-zero module of projective dimension $\leqq 2$. Then $gk(M) \geqq 2$.*

Proof. Suppose $gk(M) \leqq 1$, and that M is a right module. Let $q_M(t)$ denote the numerator of the Hilbert series $h_M(t)$, defined as in (2.18). Since $gk(A) = 4$, the order of zero of $q_M(t)$ at $t = 1$ is at least 3. Also, since $gk(N) = 2$, the numerator $q_N(t)$ has a zero of order at least 2. By the previous lemma and by Proposition 2.38, the Hilbert series of the tensor product module $M \otimes N$ has the form $h_{M \otimes N}(t) = q_M(t) q_N(t)/p_A(t)$. The numerator of this series has a zero of order at least 5 at $t = 1$. Therefore $h_{M \otimes N}$ vanishes at $t = 1$. This happens only for the zero module. But since M, N are not zero, neither is $M \otimes N$. □

Lemma 3.21 *Assume that $N \neq 0$. Let M be a module of gk-dimension $\leqq 1$. Then* $\mathrm{pd}(E^1(M)) \leqq 2$.

Proof. We examine the spectral sequence (2.33) $E_2^{pq} = E^p(E^{4-q}(M)) \Rightarrow M_{[4]}$ again. The assertion of the lemma is that $E_2^{p3} = E^p(E^1(M)) = 0$ when $p > 2$. Since N is pure 2-dimensional and $gk(M) \leqq 1$, $M^* = E^0(M) = 0$, and so $E_2^{i4} = E^i(M^*) = 0$. And, since the abutment of the spectral sequence is $M_{[4]}$, the terms E_∞^{33} and E_∞^{43} vanish. This implies that $E_2^{33} = E_2^{43} = 0$ too, as required. □

Lemma 3.22 $N = 0$.

Proof. Assume $N \neq 0$. There is an N-regular element x, (3.7). Dualizing the exact sequence

$$0 \to N \xrightarrow{x} N \to N/xN \to 0$$

gives an exact sequence

$$0 \to N^* \xrightarrow{x} N^* \to E \to E^1(N)\,,$$

where $E = E^1(N/xN)$. Since $gk(N) = 2$, it follows that $gk(N/xN) \leqq 1$, hence $\mathrm{pd}(E) \leqq 2$ by the last lemma. Therefore $gk(E) \geqq 2$, by Lemma 3.20. On the other hand, $gk(E^1(N)) \leqq 1$ by Lemma 3.17. Therefore $gk(N^*) \geqq 3$. Since N is reflexive, the left annihilator of N is the right annihilator of N^*, and so Proposition 3.1 shows that $gk(N) = gk(N^*)$. This is a contradiction, which completes the proof of the Lemma and of Theorem 3.9. □

4 Dimensions of the dual modules

In this section we estimate the gk-dimensions of the modules $E^q(M) := \mathrm{Ext}^q_A(M, A)$ when A is a noetherian regular ring of global dimension 3. As in Sect. 3, we will use the facts that $gk(A) = 3$ and that A is noetherian, but will make no other use of our assumption that A is generated in degree 1. The results are summed up in the following theorem.

Theorem 4.1 *Let A be a regular algebra of dimension 3, and let $M \neq 0$ be a finite left A-module of gk-dimension m. Let $E^j(M) = \mathrm{Ext}^j_A(M, A)$, and denote $E^{3-m}(M)$ by $M^\vee$. Then*
(i) $E^j(M) = 0$ *if* $j < 3 - m$.
(ii) $gk(M^\vee) = m$, *and* $e(M^\vee) = e(M)$.
(iii) $gk(E^j(M)) \leqq 3 - j$ *for all* j. *Moreover, the following assertions are equivalent:*
 (a) $gk(E^j(M)) = 3 - j$,
 (b) $E^j(E^j(M)) \neq 0$,
 (c) *M contains a non-zero submodule of gk-dimension* $3 - j$.

The next corollary describes the duality between left and right modules given by $M \rightsquigarrow M^\vee$. A module M is called *Cohen-Macaulay* if $E^q(M) = 0$ for all $q \neq 3 - gk(M)$, or equivalently if $\mathrm{pd}(M) = 3 - gk(M)$.

Corollary 4.2 *With the notations of the previous theorem,*
(i) *There is a canonical map $\mu = \mu_M \colon M \to M^{\vee\vee}$, which is an isomorphism if M is Cohen-Macaulay.*
(ii) *If $m < 3$, $M^\vee$ is Cohen-Macaulay.*
(iii) *$M^\vee$ is pure m-dimensional.*
(iv) ker *μ is the maximal submodule of M which has gk-dimension $< m$, and* $gk(\operatorname{coker}\mu) \leqq m - 2$.

Needless to say, (4.1) and (4.2) are true for right modules as well.

Note. The referee remarks that (4.1i, iii) implies that A is Auslander regular in the sense of Björk [Bj]. Moreover, (4.2iii) implies that the filtration defined by the spectral sequence (2.33) is the same as the filtration (2.29i) by gk-dimension.

We note the following corollary to Theorem 3.9:

Lemma 4.3 *Let A be a regular algebra of dimension 3 and let M be a finite A-module. Then*
(i) *A is a critical A-module, and for every non-zero $a \in A$ of positive degree,* $gk(A/Aa) = 2$.
(ii) $E^0(M) = M^* = 0$ *if and only if* $gk(M) < 3$.
(iii) *For every* $q > 0$, $gk(E^q(M)) < 3$.

The proof of this lemma is routine, part (iii) being a consequence of the fact that A is a Goldie domain whose field of fractions is semi-simple. Alternatively, it suffices to prove (iii) for $M = A/L$, where L is a non-zero left ideal of A. Let a be a non-zero element of L. Since $gk(L/Aa) \leqq 2$ by (i), we have $(L/Aa)^* = 0$ by (ii). Hence $E^1(M)$ injects into $E^1(A/Aa) = A/aA$, which has gk-dimension equal to 2 unless it is zero. □

Note that this lemma holds for a graded regular noetherian domain of arbitrary dimension d', if we replace 3 and 2 by d' and $d' - 1$.

The proofs of Theorem 4.1 and Corollary 4.2 are based on an analysis of the spectral sequence (2.33). Taking into account the previous lemma, the fact that A is Gorenstein, and the fact that the abutment of the spectral sequence is in degree 3, produces zeros in the E_2^{pq} terms of this spectral sequence as indicated below:

$$
\begin{array}{cccc}
E^0E^0 & E^1E^0 & 0 & 0 \\
0 & E^1E^1 & E^2E^1 & E^3E^1 \\
0 & E^1E^2 & E^2E^2 & E^3E^2 \\
0 & 0 & 0 & E^3E^3
\end{array}, \tag{4.4}
$$

where E^iE^j stands for $E^i(E^j(M))$. Lemma 4.7 below tells us that $E^1E^2 = 0$ too.

Lemma 4.5 *Let M be a finite module of gk-dimension < 3. If $E^1(M) \neq 0$, then $pd(E^1(M)) = 1$ and $gk(E^1(M)) = 2$.*

Proof. Since $gk(M) < 3$, the previous lemma tells us that $E^0(M) = 0$. This produces some more zeros in the spectral sequence (4.4), as is indicated below:

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & E^1E^1 & 0 & 0 \\
0 & E^1E^2 & E^2E^2 & E^3E^2 \\
0 & 0 & 0 & E^3E^3
\end{array}, \tag{4.6}
$$

The second row from the top shows that $pd(E^1(M)) \leqq 1$. By Proposition 2.41, $gk(E^1(M)) \geqq 2$. On the other hand, $gk(E^1(M)) < 3$ by (4.3). Thus $gk(E^1(M)) = 2$. □

Lemma 4.7 *For any finite module M, $E^1E^2(M) = 0$.*

Proof. The fact that the abutment of the spectral sequence (4.4) is concentrated in degree 3 shows that the coboundary map $E^1E^2 \to E^3E^3$ is injective. Since $gk(E^2(M)) < 3$, the previous lemma applies, to show that either $E^1E^2 = 0$, or else $gk(E^1E^2) = 2$. On the other hand, E^3 has finite length for every finite module. So $E^1E^2 = 0$. □

Lemma 4.8 *Let M be a finite module of gk-dimension $\leqq 1$. Then $E^j(M) = 0$ for $j = 0, 1$.*

Proof. We already know that $E^0(M) = 0$, so the spectral sequence (4.6) gives us an exact sequence $M \to E^1E^1 \to E^3E^2$. Since E^3 has finite length, it follows that $gk(E^1E^1) \leqq 1$. Lemma 4.5 shows that $E^1E^1 = 0$, so by (4.6), that $E^iE^1 = 0$ for all i. This implies that $(E^1)^D = 0$, hence that $E^1 = 0$. □

We now proceed with the proof of Theorem 4.1. As we have noted before (2.46), the theorem follows in the case that $gk(M) = 0$ from the fact that A is Gorenstein. So we assume from now on that $gk(M) = m > 0$.

Proof of Theorem 4.1 (i) The case $m = 1$ was treated in Lemma (4.8). Also, if $m < 3$, then $E^0(M) = 0$ because A is a domain. This settles the case $m = 2$, and the case $m = 3$ is trivial. □

Proof of Theorem 4.1 (ii)

Case m = 1 Here $M^\vee = E^2(M)$. Lemma 4.8 shows that $E^j(M) = 0$ unless $j = 2, 3$. Also, $E^3(M)$ has finite length. Thus Corollary 2.28 applies. It shows that M and $M^\vee$ have the same gk-dimension and the same multiplicity.

Case m = 2 Here $M^\vee = E^1(M)$. In this case formula (2.36) reads $e(M^D) = -e(M)$. Since $gk(E^j(M)) \leqq 2$ for all j, we have $e(M^D) = \sum(-1)^j e_j$, where $e_j = e(E^j(M))$ if $gk(E^j(M)) = 2$ and is zero otherwise. Since $E^0(M)$ is zero and $E^3(M)$ has finite length, we find $-e(M) = e(M^D) = -e_1 + e_2$. This shows that $e_1 > 0$, hence that $gk(M^\vee) = 2$. To show that $e(M^\vee) = e(M)$, we must show that $e_2 = 0$, i.e., that $gk(E^2(M)) < 2$. Suppose that $gk(E^2(M)) = 2$. We substitute $E^2(M)$ for M into what was just shown, to conclude that $gk(E^1E^2(M)) = 2$. But by Lemma 4.7, $E^1E^2 = 0$.

Case m = 3 This case follows in the same way, from (2.28) and (4.3). □

Proof of Theorem 4.1 (iii) We have seen (4.4), (4.7) that $E^iE^j = 0$ if $i < j$. Part (ii) of the theorem shows that $gk(E^j(M)) \leqq 3 - j$ and that assertions (a) and (b) are equivalent. Moreover, it shows that $gk(E^jE^j) = 3 - j$ if and only if $E^jE^j \neq 0$.

To prove that assertion (b) implies (c), we examine the gk-dimensions of the non-zero terms E^iE^j of the spectral sequence (4.4), using the fact just proved, that $gk(E^iE^j) \leqq 3 - i$. If $gk(E^jE^j) = 3 - j$, we conclude that the corresponding term in E_∞, which is $E_\infty^{j\,3-j}$, has the same gk-dimension. Then the filtration of M whose associated graded module is $\bigoplus E_\infty^{j\,3-j}$ supplies a non-zero submodule having gk-dimension $3 - j$, whenever $gk(E^jE^j) = 3 - j$, i.e., whenever $E^jE^j \neq 0$.

Finally, let us show that (c) implies (a). We assume that M contains a submodule of gk-dimension $3 - j$, and we let N denote the largest such submodule. We denote the module M/N by $\bar{M}$. Since $E^{j-1}(N) = 0$ by (i), the exact sequence

$$0 \to N \to M \to \bar{M} \to 0$$

gives us an exact sequence

$$0 \to E^j(\bar{M}) \to E^j(M) \to E^j(N) \to E^{j+1}(\bar{M}) .$$

We know that $gk(E^j(N)) = 3 - j$, and that $gk(E^{j+1}(\bar{M})) \leqq 3 - j - 1$. Thus the image D of $E^j(M)$ in $E^j(N)$ has gk-dimension $3 - j$. Taking Ext once more, we obtain a sequence

$$E^{j-1}E^j(\bar{M}) \to E^j(D) \to E^jE^j(M) .$$

The left hand term is zero and $E^j(D) \neq 0$. Therefore $E^jE^j(M) \neq 0$, as required. This completes the proof of Theorem 4.1. □

Proof of Corollary 4.2 We examine the spectral sequence (4.4) once more. Let $p = 3 - m$. Using the previous theorem, we find that $E^jE^j = 0$ for $j < p$. The spectral sequence now provides the canonical map $\mu_M : M \to E^pE^p(M) = M^{\vee\vee}$. Part (ii) follows by inspection of the spectral sequence (4.6), and it shows that μ_M is bijective if M is Cohen-Macaulay. Part (iii) is true if $m = 3$, because in that case $M^\vee = M^*$ and A is a domain. If $m < 3$, it follows from part (i) and (4.1iii). To prove (iv), we note that the cokernel of the map μ is controlled by the images of the coboundary maps of the spectral sequence (4.4), which are maps $E^pE^p \to E^{p+k+1}E^{p+k}$. Assertion (iii) of the corollary and Theorem 4.1(iii) tell us that $gk(E^{p+k+1}E^{p+k}) \leqq m-2$, hence $gk(\operatorname{coker}\mu) \leqq m-2$. Since (4.1ii) $M, M^\vee$, and $M^{\vee\vee}$ have the same multiplicity, $gk(\ker\mu) < m$, and since $M^{\vee\vee}$ is pure, $\ker\mu$ is the maximal submodule of gk-dimension $<m$. □

5 Some more preliminary considerations

For the rest of the paper, we restrict our attention to regular algebras of dimension 3 generated in degree 1, and we will review some notation before going on. Recall [ATV] that such an algebra A defines a regular triple $\mathcal{T} = (E, \sigma, \mathcal{L})$, where σ is an automorphism of the scheme E and $\mathcal{L}$ is an invertible sheaf on E, and that $A_1 = H^0(E, \mathcal{L})$. There are four possibilities for the triple:

(5.1) *the elliptic case*:

(a) $r=3$, E is a cubic divisor in $\mathbb{P}^2$, and $\mathcal{L} = \mathcal{O}_E(1)$,
(b) $r=2$, E is a divisor of bidegree (2, 2) in $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{L} = pr_1^*\mathcal{O}_{\mathbb{P}^1}(1)$,

the linear case:

(a) $r=3$, $E = \mathbb{P}^2$, and $\mathcal{L} \approx \mathcal{O}_{\mathbb{P}^2}(1)$,
(b) $r=2$, $E = \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{L} \approx pr_1^*\mathcal{O}_{\mathbb{P}^1}(1)$.

The elliptic case is the more interesting one.

If $r=2$, the automorphism σ has the form [ATV, 4.5]

(5.2) $$\sigma(p_1, p_2) = (p_2, f(p_1, p_2)) .$$

In order to be regular, i.e., to define a regular algebra, the automorphism σ must be related to the invertible sheaf $\mathcal{L}$ in the following way [ATV, 4.8]:

(5.3) $$\mathcal{L}^{(\sigma-1)(\sigma^\iota-1)} \approx \mathcal{O}_E ,$$

where $\iota = 4 - r$ as in (2.25). When (5.3) holds the triple $\mathcal{T}$ determines the algebra $A = A(\mathcal{T})$. (n.b. The statement that $\mathcal{T}$ is a regular triple does not imply that the divisor E is smooth!).

Let us denote by $[\mathcal{L}]$ the class of the invertible sheaf $\mathcal{L}$ in Pic E. The covariant operation of σ on Pic E is defined by: $\sigma[\mathcal{L}] = [\mathcal{L}^{\sigma^{-1}}]$, and (5.3) amounts to $(\sigma-1)(\sigma^\iota-1)[\mathcal{L}] = 0$. We will often denote by $\mathcal{Q}$ the invertible sheaf $\mathcal{L}^{1-\sigma^{-\iota}}$, so that $[\mathcal{Q}] = (1-\sigma^\iota)[\mathcal{L}]$. Then our condition (5.3) says that $[\mathcal{Q}]$ is σ-invariant:

(5.4) $$\mathcal{Q}^\sigma \approx \mathcal{Q} .$$

Moreover, the triple is linear if and only if $[\mathcal{Q}] = 0$, i.e., if and only if $[\mathcal{L}]$ is σ^ι-invariant [ATV, 4.8′].

Let S be a scheme, and let $\pi: E_S \to S$ be a family of divisors of degree 3 in $\mathbb{P}^2$ or of bidegree (2, 2) in $\mathbb{P}^1 \times \mathbb{P}^1$, parametrized by S. Let $Pic^0 E_S/S$ be the subscheme of the relative Picard scheme $Pic\, E_S/S$ of classes of invertible sheaves whose restriction to each irreducible component of each geometric fibre of E_S/S has degree zero. Corollary (5.7) below describes an operation of the algebraic group scheme $Pic^0 E_S/S$ on the scheme E_S, which in the case that E_S is smooth is the usual action by translation on an elliptic curve. If $p\colon S \to E_S$ is a section, we denote by p also its image, by $\mathcal{I}_p \subset \mathcal{O}_{E_S}$ the ideal sheaf of the image, and by $\mathcal{O}_p$ its structure sheaf.

Proposition 5.5 *Let $\mathcal{Q}$ be an invertible sheaf on E_S whose restriction to each irreducible component of each geometric fibre of E_S/S has degree zero, and let p be a section of E_S over S.*
(i) *$R^1\pi_*(\mathcal{Q} \otimes \mathcal{I}_p)$ is a locally free $\mathcal{O}_S$-module of rank 1, and $R^q\pi_*(\mathcal{Q} \otimes \mathcal{I}_p) = 0$ if $q \neq 1$.*
(ii) *$\pi_*(Hom(\mathcal{Q} \otimes \mathcal{I}_p, \mathcal{O}_{E_S}))$ is a locally free $\mathcal{O}_S$-module of rank 1, and $R^q\pi_*(Hom(\mathcal{Q} \otimes \mathcal{I}_p, \mathcal{O}_{E_S})) = 0$ if $q \neq 0$.*
(iii) *There is a unique section $p'\colon S \to E_S$ whose ideal sheaf $\mathcal{I}_{p'}$ is, locally over S, isomorphic to $\mathcal{Q} \otimes \mathcal{I}_p$.*

Proof. (i) We note that $\mathcal{Q} \otimes \mathcal{I}_p$ is $\mathcal{O}_S$-flat. In view of this, standard considerations show that it suffices to prove the assertion in the case that S is the spectrum of a field K. Then what must be proved is that $H^1(E_S, \mathcal{Q} \otimes \mathcal{I}_p)$ has dimension 1 and that $H^q(E_S, \mathcal{Q} \otimes \mathcal{I}_p) = 0$ if $q \neq 1$. A consideration of Euler characteristics reduces us to showing that $H^0(E_S, \mathcal{Q} \otimes \mathcal{I}_p) = 0$. By [ATV, 7.12], $\mathcal{Q}$ is tame. If $\mathcal{Q}$ is not isomorphic to $\mathcal{O}_E$, then $H^0(E_S, \mathcal{Q}) = 0$ [ATV, 7.10], hence $H^0(E_S, \mathcal{Q} \otimes \mathcal{I}_p) = 0$ too. If $\mathcal{Q} \approx \mathcal{O}$, then since $H^0(E_S, \mathcal{O}) = k$, $H^0(E_S, \mathcal{Q} \otimes \mathcal{I}_p) = H^0(E_S, \mathcal{I}_p) = 0$ as well.

(ii) This follows from (i) and the Grothendieck duality isomorphism

$$R\pi_*(RHom(\mathcal{Q} \otimes \mathcal{I}_p, \omega_{E_S/S})) \approx RHom(R\pi_*(\mathcal{Q} \otimes \mathcal{I}_p), \mathcal{O}_S). \tag{5.6}$$

Using (i), the right side reduces to $Hom(R^1\pi_*(\mathcal{Q} \otimes \mathcal{I}_p), \mathcal{O}_S)$, and this sheaf is locally free of rank one. Also, the sheaf $\omega_{E_S/S}$ is, locally over S, isomorphic to $\mathcal{O}_{E_S}$. The *Ext* sequence associated to the exact sequence

$$0 \to \mathcal{Q} \otimes \mathcal{I}_p \to \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{O}_p \to 0$$

shows that $Ext^q(\mathcal{Q} \otimes \mathcal{I}_p, \omega_{E_S/S}) = 0$ for $q > 0$. Therefore the left side of (5.6) is locally isomorphic to $R\pi_*(Hom(\mathcal{Q} \otimes \mathcal{I}_p, \mathcal{O}_{E_S}))$, and assertion (ii) follows.

(iii) Let $f \in \mathrm{Hom}(\mathcal{Q} \otimes \mathcal{I}_p, \mathcal{O}_{E_S})$ be a local generator for $\pi_*(Hom(\mathcal{Q} \otimes \mathcal{I}_p, \mathcal{O}_{E_S}))$, and let J be its image, an ideal in $\mathcal{O} = \mathcal{O}_{E_S}$. We claim that f is injective, and that $\mathcal{O}/J$ is the structure sheaf of a section of E_S/S. Then J will be the ideal sheaf of the required section p'. To show this, it is enough to treat the case that $S = \operatorname{Spec} R$, where R is an artinian ring. In that case, induction on the nilradical reduces us to the case that R is a field again. Moreover, it suffices to show that f is injective. The fact that coker f has dimension one will follow from a consideration of degrees. Tensoring with $\mathcal{Q}^*$, we interpret f as a map $\mathcal{I}_p \to \mathcal{Q}^*$. Suppose that $\ker f = J$ is not zero. Since $\mathcal{I}_p$ has no embedded component, f must vanish on some component of E. Let A be the largest divisor $< E$ on which f restricts to zero, and let $A + B = E$. Then f defines a map $\mathcal{I}_p\mathcal{O}_B \to \mathcal{Q}^*$, which we denote by the same symbol. Since $\operatorname{Pic} B$ is discrete and since $\mathcal{Q}^*$ has degree 0 on each component, $\mathcal{Q}^* \otimes \mathcal{O}_B \approx \omega_E$. Thus $Hom_{\mathcal{O}_E}(\mathcal{I}_p\mathcal{O}_B, \mathcal{Q}^*) \approx Hom_{\mathcal{O}_E}(\mathcal{I}_p\mathcal{O}_B, \omega_E)$. By Serre duality, $\mathrm{Hom}(\mathcal{I}_p\mathcal{O}_B, \mathcal{Q}^*)$ is dual to $H^1(E, \mathcal{I}_p\mathcal{O}_B) = 0$. This shows that $f = 0$, contrary to assumption.

The uniqueness of the section p' follows easily. If p'' is another section, an isomorphism $\mathcal{I}_{p''} \approx \mathcal{Q} \otimes \mathcal{I}_p$ defines a map $\mathcal{Q} \otimes \mathcal{I}_p \to \mathcal{O}$. By what has been proved, this map is a multiple of f, hence $p'' \supset f(S) = p'$, which implies that $p' = p''$. $\square$

The existence of the operation of the algebraic group scheme $Pic^0 E_S/S$ on the scheme E_S follows immediately from the previous proposition.

Corollary 5.7 *With the above notation, there is an operation of $Pic^0 E_S/S$ on E_S, which is compatible with base change and which has the following property: Let $p: S \to E_S$ be a section, and let $\mathscr{Q}$ denote an invertible sheaf on E_S whose class q is a section of $Pic^0 E_S/S$. Then $qp = p'$ is the point determined as in Proposition* 5.5(iii).

We return to the case that the base scheme S is the spectrum of an algebraically closed field k. Given an invertible sheaf $\mathscr{Q}$ whose class q is in $\mathrm{Pic}^0 E$, we often use the notation

$$\eta = \eta_{\mathscr{Q}} \tag{5.8}$$

for the automorphism of translation of E given by the action of $-q$ described in Corollary 5.7.

Proposition 5.9 *Let $\eta = \eta_{\mathscr{Q}}$ be as above.*

(i) *If p is a smooth point of E, then ηp is the unique smooth point with the property that $\mathscr{O}_E(\eta p) \approx \mathscr{Q}(p)$. If p is a singular point of E, then $\eta p = p$.*

(ii) *The irreducible components of E are stabilized by η.*

(iii) *If $E = 3C$ is a triple line in $\mathbb{P}^2$, then η restricts to the identity on $2C$.*

Proof. (i) If p is a simple point of E, then $\mathscr{I}_p = \mathscr{O}_E(-p)$ is an invertible sheaf, and $\mathrm{Hom}(\mathscr{O}_E(-p), \mathscr{Q}) \approx \mathrm{Hom}(\mathscr{O}_E, \mathscr{Q}(p)) \approx H^0(E, \mathscr{Q}(p))$. A non-zero section of $\mathscr{Q}(p)$ vanishes at the point indicated. If p is a singular point, then p is the unique point at which $\mathscr{I}_p$ fails to be locally free. Since $\mathscr{Q}$ is locally free everywhere, no map $\mathscr{Q}^* \otimes \mathscr{I}_p \to \mathscr{O}_E$ can be an isomorphism at p. Hence $p = \eta(p)$.

(ii) This follows by continuity from the fact that $Pic^0 E$ is connected and that η is the identity when $\mathscr{Q} \approx \mathscr{O}_E$.

(iii) To show this, it suffices to show that η acts trivially on points with values in $k[\varepsilon]/(\varepsilon^2)$ which are transversal to C. We do this by a local calculation, choosing local coordinates so that the relevant completion becomes $\hat{\mathscr{O}} \approx k[[x, y]]/(y^3)$, and that the point p_ε in question is $x = 0$, $y = \varepsilon$. We denote by p_0 the underlying point with values in k. We also choose a local isomorphism $\hat{\mathscr{O}} \approx \hat{\mathscr{Q}}$. The ideal of p_ε is $I = (x, y - \varepsilon)\, \hat{\mathscr{O}}[\varepsilon]$. Then the generator for $\mathrm{Hom}(\mathscr{Q}^* \otimes I, \hat{\mathscr{O}})$ can be viewed as an injective map $f: I \to \hat{\mathscr{O}}[\varepsilon]$. Since x is not a zero divisor in $\hat{\mathscr{O}}[\varepsilon]$, f is determined by the image $f(x)$. We have $f(y - \varepsilon) = x^{-1}(y - \varepsilon) f(x)$, and this element must lie in $\hat{\mathscr{O}}[\varepsilon]$. We write $f(x) = \sum a_{ij} y^i \varepsilon^j$, where $0 \leqq i \leqq 2$, $0 \leqq j \leqq 1$, and where $a_{ij} \in k[[x]]$. Since p_0 is a singular point of E, it is fixed by η. Therefore $f(x) = ux$ (modulo (y, ε)), where u is a unit in $k[[x]]$. In other words, $a_{00} = ux$. We may adjust $f(x)$ by a unit factor in $\hat{\mathscr{O}}[\varepsilon]$ to make $a_{00} = x$, and $a_{ij} \in k$, if $i, j \neq 0, 0$. When this is done, we have

$$x^{-1}(y - \varepsilon) f(x) = y - \varepsilon + x^{-1}(a_{10} y^2 + (a_{01} - a_{10}) y\varepsilon + (a_{11} - a_{20})\, y^2 \varepsilon)\,.$$

Hence $a_{10} = a_{01} = 0$, and $a_{11} = a_{20}$. Thus f has the form

$$f(x) = x + a(y^2 + y\varepsilon) + a' y^2 \varepsilon, \quad f(y - \varepsilon) = y - \varepsilon\,.$$

The ideals $(x, y - \varepsilon)$ and $(f(x), f(y - \varepsilon))$ are equal, as required. □

Lemma 5.10 *Let E be as in the previous proposition.*

(i) *For any automorphism σ of E and any invertible sheaf $\mathcal{Q}$ whose class is in* $\mathrm{Pic}^0 E$, *we have $\eta_{\mathcal{Q}^\sigma} = \sigma^{-1}\eta_{\mathcal{Q}}\sigma$.*

(ii) *If $(E, \sigma, \mathcal{L})$ is a regular triple and $\mathcal{Q}$ is the sheaf* (5.4), *then $\sigma\eta_{\mathcal{Q}} = \eta_{\mathcal{Q}}\sigma$.*

(iii) *Let $\eta = \eta_{\mathcal{Q}}$, and let $\mathcal{L}$ be an invertible sheaf on E of total degree d. Then $\mathcal{L}^\eta \approx \mathcal{L} \otimes \mathcal{Q}^{-d}$.*

Proof. The first assertion is routine, and the second follows from it because $\mathcal{Q}^\sigma \approx \mathcal{Q}$. To prove the third, we recall our assumption that the ground field k is algebraically closed. Consider first the case that E is a reduced divisor. In that case, we may write $\mathcal{L} \approx \mathcal{O}(p_1 + \ldots + p_r)$, where p_i are distinct smooth points of E. Then $\mathcal{L}^\eta = \mathcal{O}(\eta^{-1}p_1 + \ldots + \eta^{-1}p_d)$. By (5.9i), $\mathcal{O}(\eta^{-1}p_i) \approx \mathcal{O}(p_i) \otimes \mathcal{Q}^*$, so $\mathcal{L}^\eta \approx \mathcal{L} \otimes \mathcal{Q}^{*d}$, as required.

The case that E is not reduced will be treated by a specialization argument. We choose a one-parameter family of divisors E_S whose generic fibre is reduced, and we extend $\mathcal{Q}$ and $\mathcal{L}$ to the family. This is possible locally for the étale topology because, since E_S/S has relative dimension 1, $Pic^0 E_S/S$ is smooth. Having done this, we consider the invertible sheaf $\mathcal{N} = (\mathcal{L}^\eta)^{-1} \otimes \mathcal{L} \otimes \mathcal{Q}^{-d}$. It follows from (5.8ii) that the class of $\mathcal{N}$ in $\mathrm{Pic}\, E_S/S$ defines a section of $Pic^0 E_S/S$. By what has been shown, this section is zero above the generic point of S. Since $Pic^0 E_S/S$ is separated, it follows that the section is zero. Therefore $\mathcal{L}^\eta \approx \mathcal{L} \otimes \mathcal{Q}^{-d}$, as required.

Unfortunately, we do not have a reference for the fact that Pic^0 is separated. So we will sketch the verification here for the case that E_S is the linear pencil which is spanned by our divisor E and a generic divisor E'. This suffices for our purposes. The total space of such a family E_S will be smooth except at the points of the fibre E which correspond to intersection points $E \cap E'$. At such a point p, E_S has a rational double point of type A_{r-1}, where r is the multiplicity of the component of E containing p. Let $\pi: Z_S \to E_S$ denote the minimal resolution of singularities of E_S. Then one verifies that Z_S is a minimal model, and that the fibre Z over E has a component of multiplicity one. Moreover, $Pic^0 E_S/S \approx Pic^0 Z_S/S$. By [BLR, 9.5, Thm. 4] this group scheme is the connected component of the Néron model of $Pic\, Z_S/S$, which is separated. □

Recall that $A = A(\mathcal{T})$ has a canonical quotient ring $B = B(\mathcal{T})$, which is defined in terms of the triple $\mathcal{T}$ as follows: *Let* $\mathcal{B}_0 = \mathcal{O}_E$, and for $n > 0$, set

$$\mathcal{B}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}. \tag{5.11}$$

Then $B = \oplus B_n$, where $B_n = H^0(E, \mathcal{B}_n)$. The multiplication $B_m \times B_n \to B_{m+n}$ is given by $bc = b \otimes c^{\sigma^m}$, where the tensor product symbol is interpreted using the natural isomorphisms $\mathcal{B}_m \otimes \mathcal{B}_n^{\sigma^m} \cong \mathcal{B}_{m+n}$. If the triple is linear, then $A = B$, and if it is elliptic, then $B = A/gA$, where g is a normalizing element of degree $\imath r$, which is unique up to scalar factor. One of the important properties of the ring B is that the A-modules which are point modules are annihilated by g, i.e., they are B-modules (see [ATV, Sect. 3]).

Suppose that we are in the elliptic case. If E is reduced and irreducible, the ring B is a domain. However, B will not be a domain, and it needn't even be a prime ring, if E is reducible. To see this, suppose that $E = C + D$, where C, D are positive divisors. For suitable m, n there exist a non-zero section $\gamma \in H^0(E, \mathcal{B}_m)$ which vanishes on C, and a non-zero section $\delta \in H^0(E, \mathscr{B}_n)$ which vanishes on $\sigma^{-m}D$. Then $\gamma \otimes \delta^{\sigma^m}$ vanishes on E, hence it is the zero section of $\mathscr{B}_{m+n}$. Thus $\gamma\delta = 0$ in B.

Let us investigate this situation a little more closely. Let $\pi: A \to B$ be the canonical homomorphism, and let us write $\pi(\alpha) = \bar{\alpha}$. We will say that an element $\alpha \in A$ vanishes on a divisor $C \leqq E$ if $\bar{\alpha} = 0$ on C. Denote by I_C the subset of A of such elements. The asymmetry of multiplication makes I_C into a right, but not a left ideal: If $\gamma \in A_m$ vanishes on C and $\alpha \in A_n$, then $\gamma\alpha = \gamma \otimes \alpha^{\sigma^m}$ also vanishes on C, whereas $\bar{\alpha}\bar{\gamma} = \bar{\alpha} \otimes \bar{\gamma}^{\sigma^n}$ vanishes on $\sigma^n C$. On the other hand, if C is σ-invariant, then this computation shows that I_C is a two-sided ideal. In this case σ restricts to an automorphism of C, and we may use the triple $(C, \sigma_C, \mathscr{L}_C)$ to define a ring $B_C = B(C, \sigma_C, \mathscr{L}_C)$ by

$$B_C = \oplus H^0(C, \mathscr{L}_C \otimes \cdots \otimes \mathscr{L}_C^{\sigma^{n-1}}) = \oplus H^0(C, \mathscr{O}_C \otimes \mathscr{B}_n) \tag{5.12}$$

analogous to B. There is a canonical homomorphism $A \to B_C$ whose kernel is I_C. Whether or not C is invariant, (5.12) defines a right A-module B_C, and the canonical map $A \to B_C$ is a homomorphism of right A-modules, with kernel I_C.

We denote the total degree of the divisor C by c. So $c = \deg C$ if $r = 3$, and $c = c' + c''$ if $r = 2$ and C has bidegree (c', c''). The form (5.2) of the automorphism σ shows that if $C < E$ is σ-invariant and $r = 2$, then the bidegree of C is $(1, 1)$.

Proposition 5.13 *Let $E = C + D$, where C, D are positive divisors. Suppose that $r = 3$ or that $r = 2$ and $(c', c'') = (1, 1)$.*

(i) *The map $A \to B_C$ is surjective.*

(ii) *The Hilbert series of B_C is*

$$h_{B_C} = \begin{cases} (1 - t^c)/(1 - t)^3 & \text{if } r = 3 \\ (1 - t^c)/(1 - t)^3(1 + t) & \text{if } r = 2 \end{cases}.$$

(iii) *The space I_C of elements of A which vanish on C is a principal right ideal, generated by an element $\gamma \in A_c$. The element γ is unique up to constant factor.*

(iv) *If C is σ-invariant, then γ is normalizing, and I_C is the kernel of the canonical surjective homomorphism $A \to B_C$.*

Proof. The divisors C and D are numerically connected [ATV, 7.5], and have arithmetic genus 0. We restrict the tensor products $\mathscr{O}_D(-D \cdot C) \otimes \mathscr{B}_n$ to an irreducible component Z of D, and compute the degree of this invertible sheaf on Z, using the facts that $\mathscr{L}^{\sigma^i}$ is numerically equivalent to $\mathscr{L}$ [ATV, 7.9] and that if $r = 2$, then $\mathscr{L}^\sigma = pr_2^* \mathscr{O}_{\mathbb{P}^1}(1)$. The result is

Lemma 5.14 *With the above notation,* $\deg_Z \mathscr{O}_D(-D \cdot C) \otimes \mathscr{B}_n \geqq -1$ *for all $n > 0$.*

To prove (i), we tensor the exact sequence

$$0 \to \mathscr{O}_D(-D \cdot C) \to \mathscr{O}_E \to \mathscr{O}_C \to 0 . \tag{5.15}$$

on the right with $\mathscr{B}_n$. The lemma implies that $h^1(D, \mathscr{O}_D(-D \cdot C) \otimes \mathscr{B}_n) = 0$ for all $n > 0$, which shows that the map $B_n = H^0(E, \mathscr{B}_n) \to H^0(C, \mathscr{O}_C \otimes \mathscr{B}_n) = (\mathscr{B}_C)_n$ is surjective for $n > 0$. This proves (i), the surjectivity for $n = 0$ following from the fact [ATV, 7.9] that $H^0(C, \mathscr{O}_C) = k$. Part (ii) of the proposition is just a calculation, using the Riemann-Roch theorem on C.

To prove (iii), we note that the form of the Hilbert series for A and for B_C predicts that there is a non-zero element $\gamma \in I_C$ of degree c. Then B_C is a quotient of the right A-module $A/\gamma A$. Since A is a domain, γ is not a zero divisor, so the Hilbert series of $A/\gamma A$ can be computed from that of A. It is the same as the Hilbert series of B_C. Thus $\gamma A = I_C$. Similarly, if C is σ-invariant, then $A\gamma = I_C$, which proves (iv). □

Corollary 5.16 *If $C < E$ is a reduced divisor whose irreducible components form a single σ-orbit, then B_C is a prime ring. If C is irreducible and σ-invariant, then B_C is a domain.*

Proof. Suppose that C is reduced and that its irreducible components form a single σ-orbit, say of order s. Let $\alpha, \alpha' \in B_C$ be non-zero sections of degrees n, n' respectively. Then there is a component Z of C on which α does not vanish identically, and similarly α' does not vanish identically on some component, say $\sigma^i Z$. For sufficiently large k, there exists a section β of degree $ks + i - n$ which does not vanish identically on any component of C. Then $\alpha\beta\alpha' = \alpha \otimes \beta^{\sigma^n} \otimes \alpha'^{\sigma^{ks+i}}$ does not vanish on $\sigma^i Z$, hence it is not zero. This shows that B_C is a prime ring. If C is irreducible, then $\alpha\beta$ does not vanish identically, hence it is not zero, which shows that B_C is a domain in that case. □

Let us write

$$E = \sum_{i=1}^{m} n_i C_i \,, \tag{5.17}$$

where each C_i is a reduced divisor whose components form a single σ-orbit, and let c_i be the total degree of C_i. For each i, the above Proposition 5.13(iv) provides us with a normalizing element g_i of degree c_i which generates the kernel of the homomorphism $A \to B_i := B_{C_i}$.

Proposition 5.18 *Let g' denote the product of n_i copies of g_i, for $i = 1, \ldots, m$, taken in an arbitrary order. Then $g' = cg$ for some $c \in k^*$.*

Proof. Certainly g' is a homogeneous element of A, of the required degree $\imath r$, and since A is a domain, $g' \neq 0$. The image $\bar{g}'$ of g' in $B_{\imath r}$ is a section of $H^0(E, \mathscr{B}_{\imath r})$. Since $g_i = 0$ on C_i, it is immediately seen that $\bar{g}' = 0$ on E. By [ATV, 6.8], $g' = cg$ for some $c \in k^*$, as required. □

6 Line modules and their relation to modules of gk-dimension 1

Throughout this section, A will denote an elliptic regular algebra of dimension 3 corresponding to a triple $(E, \sigma, \mathscr{L})$, and X will denote $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, according as $r = 3$ or 2. As before, the term module will mean finite left or right graded A-module. Recall that $\varepsilon(M) = \imath e(M)$ (2.23).

We first describe some special right modules of gk-dimension 2. When $r = 3$, modules of the form $M = A/aA$, where a is a non-zero element of A_1, are in canonical bijective correspondence with lines ℓ in the projective space $X = \mathbb{P}^2 = \mathbb{P}(A_1)$. We will refer to such a module as a *line module*, and will denote the module corresponding to the line ℓ: $\{a = 0\}$ by $M_\ell := A/aA$. In order to extend this terminology to the case $r = 2$, we adopt the convention that a *line* in

$X = \mathbb{P}^1 \times \mathbb{P}^1$ will mean a set of the form $p \times \mathbb{P}^1$, where p: $\{a = 0\}$ is a point of $\mathbb{P}^1$. Then modules of the above form are in bijective correspondence with lines ℓ in X. We refer to them as *line modules* too, and we use the notation M_ℓ as before.

Proposition 6.1 *Let a be a non-zero element of A_1, and let $M = A/aA$ be the associated line module.*
(i) *The Hilbert series of M is $(1-t)/p_A(t)$, and $\varepsilon(M) = 1$.*
(ii) *The only automorphisms of M are scalars*: $\mathrm{Aut}_A(M) \approx k^*$.
(iii) *M is a critical module of gk-dimension* 2.

Proof. Since A is a domain, the only non-trivial assertion is that M is critical. To see this, let M' be a non-zero submodule of M such that M/M' has trivial socle. Then $pd(M) = 1$ and $pd(M/M') \leqq 2$, hence $pd(M') = 1$. Therefore the gk-dimension of M' is 2 (2.41). Also, $0 < \varepsilon(M') \leqq \varepsilon(M) = 1$, hence $\varepsilon(M) = \varepsilon(M')$, and this shows that $gk(M/M') \leqq 1$, as required. □

Proposition 6.2 *A module M is isomorphic to a shifted line module if and only if it is a Cohen-Macaulay module of gk-dimension* 2, *and* $\varepsilon(M) = 1$.

Proof. This is the case $n = 3$ of Corollary (2.43). □

We now consider modules of gk-dimension 1. As in Sect. 4, we denote $\mathrm{Ext}^q_A(N, A)$ by $E^q(N)$. We note the following corollary of Proposition 2.46 and Corollary 4.2:

Proposition 6.3 (i) *A module N of gk-dimension* 1 *is Cohen-Macaulay if and only if its socle is zero.*

(ii) *The map $N \rightsquigarrow N^\vee = E^2(N)$ is a duality between left and right Cohen-Macaulay modules of gk-dimension* 1.

Proposition 6.4 *Let N be a module of gk-dimension* 1. *The Hilbert series of N has the form*

$$h_N = \begin{cases} e/(1-t) + f(t) & \text{if } r = 3 \\ (e_0 + e_1 t)/(1-t^2) + f(t) & \text{if } r = 2 \end{cases}$$

for some $f(t) \in \mathbb{Z}[t, t^{-1}]$, where $\varepsilon(N) = e_0 + e_1$ if $r = 2$, and $\varepsilon(N) = e(N) = e$ if $r = 3$.

Proof. The Hilbert series has a pole of order 1 at $t = 1$, and it has the form $q_N(t)/p_A(t)$, where $p_A(t) = (1-t)^3$ or $(1-t)^2(1-t^2)$ according to the case (2.8). Therefore q_N has a zero of order 2 at $t = 1$, which implies that h_N has the form indicated. □

By the *tail* $N_{\geqq p}$ of a module N we mean the module defined by

$$(6.5) \qquad (N_{\geqq p})_n = \begin{cases} 0 & \text{if } n < p \\ N_n & \text{if } n \geqq p\,. \end{cases}$$

We will call two modules N and N' *equivalent* if their tails are isomorphic, for sufficiently large p. More precisely, we will call an *equivalence* from N to N' a class of isomorphisms $N_{\geqq p} \to N'_{\geqq p}$, where two such isomorphisms define the same equivalence if they agree on some tail $N_{\geqq q}$.

A module N of gk-dimension 1 will be called *normalized* if it is Cohen-Macaulay and if its Hilbert series has the form (6.4), with $f(t) = 0$. So, the Hilbert function of a normalized module is zero in negative degree. If $r = 3$, then it is the constant function $\dim_k N_n = e$ for $n \geqq 0$, while if $r = 2$, then it has the form $\dim_k N_n = e_0$ if n is even, and $\dim_k N_n = e_1$ if n is odd.

Proposition 6.6 *Let N, N' be modules of gk-dimension* 1.

(i) *An equivalence from N to N' induces an equivalence from $N'^\vee$ to $N^\vee$.*

(ii) *If N is Cohen-Macaulay, then N is contained in an equivalent module N'' which is a negative shift of a normalized module.*

(iii) *Every module N of gk-dimension* 1 *is equivalent to a normalized module, and this normalized module is unique up to unique isomorphism.*

(iv) *Suppose that N' is normalized. An equivalence from N to N' extends uniquely to a homomorphism φ: $N_{\geqq 0} \to N'$.*

(v) *Suppose that k is algebraically closed, that N is critical, and that N' is its normalization. The only maps $N_{\geqq 0} \to N'$ are constant multiples of the map φ of* (iv).

Proof. (i) Let $T = N/N_{\geqq p}$, which is a module of finite length. Taking Ext, we obtain an exact sequence

$$0 \longrightarrow N^\vee \longrightarrow (N_{\geqq p})^\vee \longrightarrow E^3(T)\,.$$

Since $E^3(T)$ has finite length, this provides the required isomorphism between the tails of the dual modules.

(ii) It is clear that $N_{\geqq p}$ is a positive shift of a normalized module, if $p \gg 0$. Hence $(N_{\geqq p})^\vee$ is a negative shift of a normalized module. (This follows from (2.35).) Since $(N_{\geqq p})^\vee \supset N^\vee$, this shows that $N^\vee$ is contained in a negative shift of a normalized module. By duality, the same is true of N.

(iii) We may assume that the socle of N is trivial, hence that N is Cohen-Macaulay. Let N'' be as in (ii). The required normalized module is $N''_{\geqq 0}$. Its uniqueness will follow from (iv).

(iv) By (i), an equivalence from N to N' gives us an equivalence from $N'^\vee$ to $N^\vee$. Let $N'' := ((N'^\vee)_{\geqq p})^\vee$. Dualizing the maps $(N'^\vee)_{\geqq p} \to N^\vee$ and $(N'^\vee)_{\geqq p} \to N'^\vee$ yields maps $\alpha: N \to N''$, and $\beta: N' \to N''$ which are uniquely determined by the original equivalence. Since N' is equivalent to N'' and since N' is normalized while N'' is a negative shift of a normalized module, N' is isomorphic to $N''_{\geqq 0}$. The map α required map is $\beta_{\geqq 0}^{-1}\alpha_{\geqq 0}: N_{\geqq 0} \to N'$.

(v) This follows from (iv) and (2.30vi). □

Note that there is a *normalized shift* operation on normalized modules of gk-dimension 1, defined by $N \rightsquigarrow N^+$, where $N_n^+ = N_{n+1}$ if $n \geqq 0$, i.e., $N^+ = N(1)_{\geqq 0}$ is the normalization of $N(1)$. The previous proposition allows us to define a negative normalized shift as well: N^- is the normalized module associated to $N(-1)$.

Proposition 6.7 (i) *Let φ: $M \to N$ be a surjective map from a line module to a Cohen-Macaulay module of gk-dimension* 1. *Let $\varepsilon = \varepsilon(N)$. Then* ker *$\varphi$ is isomorphic to the shift by $-\varepsilon$ of a line module, so we have an exact sequence*

$$(*) \qquad 0 \to M'(-\varepsilon) \to M \to N \to 0\,,$$

where M, M' are line modules. In this case, the minimal projective resolution of N has the form

$$(**)\qquad 0 \to A(-\varepsilon-1) \xrightarrow{\binom{b'}{-a'}} A(-1) \oplus A(-\varepsilon) \xrightarrow{(a,b)} A \to N \to 0\,,$$

where $\deg a = \deg a' = 1$ *and* $\deg b = \deg b' = \varepsilon$. *Thus* $N \approx A/(a,b)A$.

(ii) *Conversely, suppose given a complex of the form* (**), *such that* $N \approx \operatorname{coker}(a,b)$. *Assume that* $a \neq 0$, *that* $b \notin aA$, *and that* $(b', -a') \neq (0,0)$. *Then N is a Cohen-Macaulay module of gk-dimension* 1, *the complex is a minimal resolution of* N, *and* $\varepsilon(N) = \varepsilon$.

(iii) *Suppose that k is algebraically closed. Let N be a critical module of gk-dimension* 1, *and if* $r = 2$ *suppose that* $e_0 \geqq e_1$. *There is a module equivalent to N which fits into exact sequences of the form* (*), (**).

(iv) *Let N be as in* (i), *and assume that* $r = 2$. *Then* $e_0 = e_1 = m$ *if* $\varepsilon = 2m$, *while if* $\varepsilon = 2m+1$, *then* $e_0 = m+1$ *and* $e_1 = m$. *Hence* $0 \leqq e_0 - e_1 \leqq 1$.

(v) *Assume that* $r = 2$. *If N is any critical module of gk-dimension* 1, *then* $e_0 = e_1$ *if ε is even, and* $|e_0 - e_1| = 1$ *if ε is odd.*

Proof. (i) Say that $M = A/aA$, and let $K = \ker \varphi$. Then $pd\, M = 1$ and $pd\, N = 2$, hence $\operatorname{pd} K = 1$. Also, the Hilbert series of K is $h_K = h_M - h_N$, so K has gk-dimension 2 and multiplicity $e(K) = e(A)$. Proposition 6.2 tells us that K is isomorphic to a shift of a line module M'. Direct computation shows that the appropriate shift is $-\varepsilon$. This shows that there is an exact sequence of the form (*).

To construct the minimal resolution (**), we note that $M \approx A/aA$ for some $a \in A_1$, and the generator of M' is in degree ε. So $N \approx M/M' \approx A/(a,b)A$ for some $b \in A_\varepsilon$. This shows that the first three terms from the right in the minimal resolution of N are as indicated in (**). Since $pd\, N = 2$, there is only one more step. Computation of the Hilbert series using (*) predicts the second syzygy in degree $\varepsilon + 1$. Hence the resolution has the required form.

(ii) The complex provides an exact commutative diagram

$$(6.8)\qquad \begin{array}{ccccc} A(-\varepsilon-1) & \xrightarrow{a'} & A(-\varepsilon) & \to & A/a'A \\ \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \beta} \\ A(-1) & \xrightarrow{a} & A & \to & A/aA \\ \downarrow & & \downarrow & & \downarrow \\ A/b'A & \xrightarrow{\alpha} & A/bA & \to & N. \end{array}$$

By hypothesis, the two elements a', b' are not both zero. If $b' \neq 0$, then since A is a domain, $ab' \neq 0$, which implies that $a' \neq 0$. So $a' \neq 0$ in any case. Thus by (6.1), A/aA and $A/a'A$ are critical modules of gk-dimension 2. Since $b \notin aA$, $gk(N) \leqq 1$. Since $A/a'A$ is critical, the map β is injective. This being so, one sees that the diagram remains exact when zeros are placed around the periphery, which implies that the complex (**) is a resolution of N.

To prove (6.7iii), we use the following lemma.

Lemma 6.9 *Assume that k is algebraically closed. Let N be a normalized module of gk-dimension* 1, *and if* $r = 2$, *assume that* $e_0 \geqq e_1$. *For* $a \in A_1$ *let* $\rho_a\colon N_0 \to N_1$ *denote right multiplication by a. There exists a non-zero element* $a \in A_1$ *such that* $\ker \rho_a \neq 0$.

Proof. If $r = 2$ and $e_0 > e_1$, then $\dim N_0 > \dim N_1$, so the assertion is trivial. In the other cases, $\dim N_0 = \dim N_1$, and by choice of bases for N_i, we can represent ρ_a by a square matrix. Since ρ depends linearly on a, the matrix entries are linear functions of $a \in A_1$. So $\det \rho_a$ is a non-constant function of $\geqq 2$ variables, which, since k is algebraically closed, has non-trivial zeros. □

Now to prove (iii), let a critical module N of gk-dimension 1 be given. If $r = 2$, assume that $e_0 \geqq e_1$. We may replace N by the equivalent normalized module (6.6). Applying the Lemma, we find an element $a \in A_1$, such that $\ker \rho_a \neq 0$. Let $u \in \ker \rho_a$. The map Φ: $A \to N$ defined by $\Phi(x) = ux$ has aA in its kernel, hence it defines a map φ: $A/aA \to N$. Since N is critical, the cokernel of φ is of finite length, and so N is equivalent to $N' = \operatorname{im} \varphi$.

Part (iv) is proved by computing the Hilbert series of N, using the exact sequence (*). Part (v) follows from (iv) by applying (iii) to N or N^+. □

The previous proposition allows us to parametrize the quotients N of A which have a resolution of the form (**). Assume first that $\varepsilon > 1$ (which is automatic if $r = 2$). Let N be defined by (**). Let $L = (a, b)A$ be the right ideal of relations in N. Then L is determined by the two subspaces $L_1 \subset A_1$ and $L_\varepsilon \in A_\varepsilon$, which have dimensions 1 and $1 + \dim A_{\varepsilon-1}$ respectively. We can parametrize such a pair of subspaces by a point in the product $G \times G'$ of two Grassmannians. The pair will define an ideal L generated by two elements a, b of the required degrees provided that $L_1 A_{\varepsilon-1} \subset L_\varepsilon$. This is a Zariski closed condition on $G \times G'$. Also, the existence of the second syzygy is equivalent with the condition $\dim L_{\varepsilon+1} < \dim A_\varepsilon + \dim A_1$, which is also a closed condition in $G \times G'$. Proposition 6.8(ii) tells us that then $N = A/L$ has a resolution of the form (**).

If $\varepsilon = 1$, then $r = 3$, and the corresponding modules are point modules. There is a similar description in this case: Point modules are parametrized by the scheme of 2-dimensional subspaces L_1 of A_1 such that $\dim L_1 A_1 < 6$. Of course, we already know that the point modules are parametrized by the scheme E [ATV, Sect. 3].

Let $\mathscr{F}$ denote the functor defined by: $\mathscr{F}(S) =$ isomorphism classes of flat families of graded $A \otimes \mathcal{O}_S$-modules N, which are quotients of A and which have resolutions of the form (**) for a given value of ε. An analysis of the above description yields the part (i) of the following Proposition:

Proposition 6.10 (i) *The functor $\mathscr{F}$ defined above is represented by a closed subscheme F of a product of Grassmannians. Hence it is a proper scheme over k.* (ii) *Assume $\varepsilon > 1$. The subfunctor of $\mathscr{F}$ of those families which have resolutions of the form* (*), *with given line modules M, M', is represented by a closed subscheme Y of F, and Y is isomorphic to the projective space $\mathbb{P}(V^*)$, where*

$$V = (aA_\varepsilon \cap A_\varepsilon a')/(aA_{\varepsilon-1}a') \,. \tag{6.11}$$

Proof of (*ii*). A module N determines the element a appearing in (**) up to scalar factor, namely a generates L_1. Similarly, N determines a' as the corresponding element for the dual module $N^\vee$. Fixing M and M' amounts to fixing these elements projectively. Once they are fixed, the module N is determined by the elements b, b', which must satisfy the relation $ab' = ba' \in aA_\varepsilon \cap A_\varepsilon a'$. We can change b to $cb + ax$, where $c \in k^*$ and $x \in A_{\varepsilon-1}$, without changing N. These are the only allowable changes. Thus $Y \approx \mathbb{P}(V^*)$. □

Equivalence of modules (6.5) defines an equivalence relation on the scheme F of (6.10), which we describe here. We will need the description in the next section. It is convenient to introduce the following notation. Let N be a module of gk-dimension 1 whose socle is trivial, and let

$$N = N^0 \supset N^1 \supset \ldots \supset N^m \supset 0 \tag{6.12}$$

be a filtration whose successive quotients $\bar{N}^i = N^i/N^{i+1}$ are critical modules of gk-dimension 1. We will denote by $\mathrm{gr}(N)$ the associated graded module $\oplus \bar{N}^i$. The equivalence class of $\mathrm{gr}(N)$ is uniquely determined by the class of N.

Proposition 6.13 *Let N be a module with a resolution of the form (**). The locus of geometric points $z \in F$ such that* $\mathrm{gr}(N_z)$ *is equivalent to* $\mathrm{gr}(N)$ *is a Zariski closed subset Z of F.*

Proof. We will verify this in two steps. First we show that the locus Z is a constructible set, and then we use the valuative criterion to show that it is closed. To show constructibility, we show that if we are given a family M_S of modules of the type under consideration, parametrized by a scheme of finite type $S = \operatorname{Spec} R$, then there is another scheme S' of finite type over S, such that for any geometric point $s \in S$, M_s has the required property if and only if $s \in \operatorname{im}(S')$. To show this, we may choose an ordering of gr N, and we may assume that $M_S = A_S/(a,b)A_S$, where $A_S = A \otimes R$, $a \in (A_S)_1$, and $b \in (A_S)_\varepsilon$. We may also assume (see Proposition 6.7(i)) that the graded modules $\bar{M}_S^i$ will have the same form (**) as M_S, with different values of ε. So to describe a filtration, it suffices to give homogeneous elements $b = b^0, b^1, \ldots, b^m$ of suitable degrees, these elements being required to satisfy the relations described by (**) and by $b^i \in (a, b^{i+1})A_S$. This data is parametrized by a scheme S' of finite type over S, and we replace S by S'.

Next, the condition that $\bar{M}_S^i$ is equivalent to $\bar{N}^i$ is described by Proposition 6.6(iv). Let $\tilde{N}^i$ be the normalized module associated to $\bar{N}^i$. Then the equivalence defines a unique map

$$\varphi_S \colon \bar{M}_S^i \to \tilde{N}^i . \tag{6.14}$$

Since $\tilde{N}^i$ is critical and $e(\bar{M}_S^i) = e(\bar{N}^i) = e(\tilde{N}^i)$, any non-zero map is an equivalence. So gr M_S and gr N are equivalent if and only if $\operatorname{Hom}(\bar{M}_S^i, \tilde{N}^i)_0 \neq 0$ for all i. This is a constructible condition.

It remains to prove that the locus Z is closed, and to do this we let $S = \operatorname{Spec} R$, where $R = k[[t]]$. We denote the generic point of S by $\eta = \operatorname{Spec} K$, the associated geometric point by $\bar{\eta}$, and the closed geometric point by $\bar{s}_0$. The valuative criterion translates as follows: Let M_S be a family parametrized by S, such that $\mathrm{gr}(M_{\bar{\eta}})$ is equivalent to $\mathrm{gr}(N) \otimes \bar{K}$. Then $\mathrm{gr}(M_{\bar{s}_0})$ is equivalent to $\mathrm{gr}(N)$. It is permissible to make a finite extension of the field K, so we may assume that $\mathrm{gr}(M_\eta)$ is equivalent to $\mathrm{gr}(N) \otimes K$. The filtration which exhibits this equivalence extends uniquely to a flat filtration of M_S. So we obtain quotients $\bar{M}_s^i$, and we have to show that $\bar{M}_{s_0}^i$ is equivalent to $\bar{N}^i$. Proposition 6.7(i) reduces us to the case that N is critical. As before, let $\tilde{N}$ denote the associated normalized module. Then the equivalence of M_η and $N \otimes K$ defines a map

$$\varphi_\eta \colon M_\eta \to \tilde{N} \otimes K ,$$

which is unique up to multiplication by an element of K. The isomorphism $\mathrm{Hom}(M_\eta, \tilde{N}\otimes K) \approx \mathrm{Hom}(M_S, \tilde{N}\otimes R)\otimes K$, and the exact sequence

$$\mathrm{Hom}(M_S, \tilde{N}\otimes R) \xrightarrow{t} \mathrm{Hom}(M_S, \tilde{N}\otimes R) \to \mathrm{Hom}(M_{s_0}, \tilde{N})$$

show that φ_η can be adjusted by an element of K so that it extends to a map $\varphi_S \in \mathrm{Hom}(M_S, \tilde{N}\otimes R)$ with the property that $\varphi_{s_0} \neq 0$. This yields the required equivalence of M_{s_0} with N. □

The critical modules of gk-dimension one for which $e_0 \neq e_1$ are somewhat anomalous, though they exist. We will see examples in (8.43). Luckily, there are not too many. Suppose that over some finite extension k' of the ground field, the algebra $A' = A\otimes k'$ has a module N of gk-dimension 1, such that $e_0 \neq e_1$. Let $\varepsilon_{\min}$ denote the minimum value attained by $\varepsilon(N)$ for such modules.

Lemma 6.15 *Let N be a Cohen-Macaulay module of gk-dimension* 1. *Assume that $e_0 \neq e_1$ and $\varepsilon(N) = \varepsilon_{\min}$. Then N is critical.*

Proof. If N is not critical, there is an exact sequence $0 \to N' \to N \to N'' \to 0$ such that N', N'' have gk-dimension 1. If $e_0 \neq e_1$ for the module N, then the same is true for at least one of the modules N', N''. And since $\varepsilon(N) = \varepsilon(N') + \varepsilon(N'')$, $\varepsilon(N)$ can not be equal to $\varepsilon_{\min}$. □

Proposition 6.16 *Suppose that k is algebraically closed. There are only finitely many equivalence classes of modules of gk-dimension* 1 *with $e_0 \neq e_1$ and with $\varepsilon(N) = \varepsilon_{\min}$.*

Proof. Assume that there are infinitely many equivalence classes, and that $e_0 > e_1$. It is clear that the last hypothesis is not a restriction. Let $\mathscr{F}$ be the functor of isomorphism classes of flat families of modules of gk-dimension 1 which are generated in degree zero and which have a presentation of the form (**). We know by (6.7iii) that every equivalence class has a representative of this form. By Proposition 6.10, $\mathscr{F}$ is represented by a proper scheme F. Denote the universal family of modules over F by $\mathscr{N}$. Thus $\mathscr{N}$ is a quotient of the sheaf of algebras $A\otimes\mathcal{O}_F$. We have $\mathscr{N}_0 = \mathcal{O}_F$, and for each n, $\mathscr{N}_n$ is a locally free $\mathcal{O}_F$-module, whose rank is ε if n is sufficiently large.

Let φ denote the composed map $A \to H^0(F, A\otimes\mathcal{O}_F) \to H^0(F, \mathscr{N})$, and let $\mathfrak{a} = \ker\varphi$. For every point $x\in F$, the given map $A \to N_x = H^0(F, \mathscr{N}_x)$ factors through φ. By hypothesis, there are infinitely many such quotients which are non-equivalent, from which it follows that $gk(A/\mathfrak{a}) > 1$. On the other hand, $(A/\mathfrak{a})_n \subset H^0(F, \mathscr{N}_n)$, hence $H^0(F, \mathscr{N}_n)$ increases with n. Thus there is an integer n such that $\mathrm{rk}\,\mathscr{N}_n = \mathrm{rk}\,\mathscr{N}_{n+4} = \varepsilon$ but that $\mathscr{N}_n$ is not isomorphic to $\mathscr{N}_{n+4}$. This implies that right multiplication by the normalizing element g of A of degree 4 [ATV, 6.8] does not define not an isomorphism $\mathscr{N}_n \to \mathscr{N}_{n+4}$, hence this map does not have maximal rank everywhere on F. So there exists a point $x\in F$ such that N_x contains elements annihilated by g. Since N_x is critical by the previous lemma, g annihilates N_x, and hence N_x is a B-module, where $B = A/gA$. On the other hand, the equivalence classes of B-modules of gk-dimension one are in bijective correspondence with $\mathcal{O}_E$-modules which are finite over k, and since the simple $\mathcal{O}_E$-modules correspond to points of E, the critical B-modules of gk-dimension 1 are equivalent to point modules [ATV, 1.3]. This is a contradiction. □

From now on we assume that k is algebraically closed, and we specialize to the case that N is a point module, as defined in (1.1). A point module can be described as a normalized right module of gk-dimension 1 whose Hilbert series is $1/(1 - t)$, or equivalently, whose Hilbert function $\dim_k N_n$ is the constant function 1, for $n \geqq 0$.

Proposition 6.17 *A module N is a shifted point module if and only if it is Cohen-Macaulay, of gk-dimension and multiplicity* 1. *Such a module is critical.*

Proof. The fact that a shifted point module is critical and has multiplicity 1 follows directly from the definition. Conversely, let N be a module of gk-dimension 1, multiplicity 1, and socle zero. Then Proposition 6.4 shows that the Hilbert function $\dim N_n$ is the constant function 1 for large n, provided that $r = 3$ or that $r = 2$ and $e_0 = e_1 = 1$. In any case, if $r = 2$ then the invariants e_0 and e_1 of N satisfy the relations $e_0 + e_1 = 2$, and $e_i \geqq 0$. So the only other possibility is that one e_i is zero and the other one is 2. This would imply that infinitely many N_n are zero and infinitely many are not zero, which is impossible because A is generated in degree 1. If $N_n = 0$ for some n, then $N_m = 0$ for all $m \geqq n$. So the Hilbert function is the constant 1 for large n in every case. By Proposition 6.6(ii), N is contained in a negative shift of an equivalent normalized module N', and N' is a shifted point module. This implies that N is a shifted point module too. □

As we remarked above, point modules are parametrized by the scheme E. We will now describe the universal family $\mathcal{N}$ of point modules over E. Recall that the canonical quotient ring $B = A/gA$ is defined in terms of the triple $(E, \sigma, \mathcal{L})$ associated to A as follows: For $n \geqq 0$, set $\mathcal{B}_n = \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \ldots \otimes \mathcal{L}^{\sigma^{n-1}}$ as in (5.11), and set

$$\mathcal{B}_{-n} = \mathcal{L}^{*\sigma^{-1}} \otimes \ldots \otimes \mathcal{L}^{*\sigma^{-n}}. \tag{6.18}$$

Then $B = \bigoplus_{n \geqq 0} B_n$, where $B_n = H^0(E, \mathcal{B}_n)$. There is a functor

$$\Gamma_*\colon (\text{quasi-coherent } \mathcal{O}_E\text{-modules}) \to (\text{graded right } B\text{-modules})$$

defined by $\Gamma_*(M) = \bigoplus_{-\infty}^{\infty} \Gamma_n(M)$, where $\Gamma_n(M) = H^0(E, M \otimes \mathcal{B}_n)$. Thus $B = (\Gamma_*(\mathcal{O}_E))_{\geqq 0}$. The right action of B on $\Gamma_*(M)$ is obtained from the canonical isomorphisms $(M \otimes \mathcal{B}_m) \otimes \mathcal{B}_n^{\sigma^m} \simeq M \otimes \mathcal{B}_{m+n}$. These isomorphisms allow us to define maps

$$\Gamma_*(M \otimes \mathcal{B}_m) \otimes \Gamma_*(\mathcal{B}_n) \to \Gamma_*(M \otimes \mathcal{B}_{m+n})$$

analogous to $\mu_{m,n}$. In [AV] this functor is discussed more generally. It is shown there that if σ is an automorphism of a projective scheme E and if $\mathcal{L}$ is σ-ample in the sense defined below, then Γ_* defines an equivalence from the category of quasi-coherent $\mathcal{O}_E$-modules to the category of graded right B-modules, modulo the full subcategory of direct limits of right bounded modules.

Definition 6.19 An invertible sheaf $\mathcal{L}$ on E is called *σ-ample* if for every coherent $\mathcal{O}_E$-module M there exists an integer n_0 such that $H^q(E, M \otimes \mathcal{B}_n) = 0$ if $q > 0$ and $n \geqq n_0$, where $\mathcal{B}_n$ is defined as above.

Proposition 6.20 *Let σ be an automorphism of a projective scheme E, and let $\mathcal{L}$ be a σ-ample invertible sheaf on E which is generated by its sections. For any quasi-coherent $\mathcal{O}_E$-module M, the socle of the B-module $\Gamma_*(M)$ is zero.*

Proof. The multiplication map $\Gamma_*(M\otimes\mathscr{B}_m)\otimes\Gamma_*(\mathscr{B}_1)\to\Gamma_*(M\otimes\mathscr{B}_{m+1})$ does not annihilate any section of $M\otimes\mathscr{B}_m$ because $\mathscr{B}_1$ is generated by its sections. □

Proposition 1.5 of [AV] shows that

Corollary 6.21 *Let $(E,\sigma,\mathscr{L})$ be the triple associated to a regular algebra A of dimension 3. Then $\mathscr{L}$ is σ-ample.* □

Proposition 6.22 *The universal family of point modules over E is $\mathscr{N}=\bigoplus_{n\geq 0}\mathscr{B}_n$.*

Proof. Since $B=H^0(E,\mathscr{N})$, there is a natural structure of right B-module on $\mathscr{N}$, and $\mathscr{N}$ is made into an A-module using the canonical homomorphism $A\to B$. In this way, $\mathscr{N}$ becomes an $\mathcal{O}_E\otimes A$-module. Since $\mathscr{B}_n$ is an invertible sheaf on E for each n, this $\mathcal{O}_E\otimes A$-module structure makes $\mathscr{N}$ into a flat family of point modules. The equivalence of categories [ATV, 1.4] shows that $\mathscr{B}$ is the universal family. □

We now specialize to zero-dimensional families of point modules. Let $Z=\operatorname{Spec}R$ be an arbitrary zero-dimensional subscheme of E, where R is a finite k-algebra. We view $\mathcal{O}_Z$ as an $\mathcal{O}_E$-module, and we put

$$N_Z := (\Gamma_*(\mathcal{O}_Z))_{\geq 0}.$$

This is a graded B-module which we make into an A-module by means of the canonical map $A\to B$. If Z is a closed point p of E with residue field k, then N_Z is the corresponding point module. For arbitrary Z, $(N_Z)_m$ is a free R-module of rank 1 for each $m\geq 0$, and $(N_Z)_0=R$. It follows from Proposition 6.20 that N_Z is a normalized A-module of gk-dimension 1 and multiplicity $[R:k]$.

Proposition 6.23 *Let $Z=\operatorname{Spec}R$ be a zero-dimensional subscheme of E.*

(i) *Let $\varphi: A\to N_Z$ be an A-homomorphism of degree 0. Then* coker φ *has finite length if and only if $\varphi(1)$ is a unit in $(N_Z)_0=R$.*

(ii) *Let ℓ be the line $\{a=0\}$, where $a\in A_1$, and let K_Z be the annihilator in A of the element $1\in R\approx(N_Z)_0$. The following are equivalent:*

(a) $Z\subset\ell$,

(b) $a\in K_Z$,

(c) *N_Z is equivalent to a quotient of the line module $M:=A/aA$.*

When these conditions hold, $M'=K/aA$ is the unique submodule of M such that M/M' is equivalent to N_Z and has trivial socle.

Proof. (ii) Applying the functor Γ_* to the surjective map $\mathcal{O}_E\to\mathcal{O}_Z$, we obtain a map $B=\Gamma_*(\mathcal{O}_E)\to\Gamma_*(\mathcal{O}_Z)=N_Z$ which is surjective in high degree [AV, 3.7ii]. Let φ: $A\to N_Z$ denote the map obtained from this one by composition with the canonical map $A\to B$. So ker $\varphi=K_Z$, and hence N_Z is equivalent to $\varphi(A)=A/K_Z$. Then via the identification $R\approx(N_Z)_0$, $\varphi(1_A)=1_R$.

Now $Z\subset\ell$ if and only if $1_R\otimes a=0$ in $H^0(E,\mathcal{O}_Z\otimes\mathscr{L})$, i.e., if and only if $1_Ra=0$ in N_Z, and this is true if and only if $a\in K_Z$, in which case φ factors through the canonical surjection $A\to M$. When this is so, N_Z is equivalent to $M/(K_Z/aA)\approx A/K_Z$.

Conversely, suppose that N_Z is equivalent to M/M' for some M' such that M/M' has socle zero. By (6.6iv), M/M' is isomorphic to a submodule of N_Z, so there is a homomorphism ψ: $A\to N_Z$ factoring through $A\to M$ and surjective in high degree. Let $\psi(1) = u \in R$. Then $\psi = u\varphi$. Since ψ is surjective in some degree, $u \in R^*$. This implies that ψ and φ have the same kernel K_Z, which shows that $M' = K_Z/aA$ as claimed.

(i) This follows directly by applying the functor $\sim$ of [AV] to the corresponding map $\bar{\varphi}: B \to N_Z$. $\square$

Now let ℓ be a line which is not a component of E. Let S be the scheme-theoretic intersection $E \cap \ell$, so that by definition of $\mathscr{L}$(5.1), we have $\mathscr{L} \approx \mathcal{O}_E(\tilde{S})$. If p is a k-rational point of S, we denote by $S - p = Z$ the scheme obtained by deleting p from S. The scheme structure on Z is uniquely determined by the condition $Z \subset \ell$. Conversely, a subscheme Z of $S = E \cap \ell$ of length $r - 1$ determines a unique point p such that, as division on l, $S = Z + p$.

Proposition 6.24 *Let $p \in \ell \cap E$, and let $\varphi: M_\ell \to N_p$ denote a surjective map whose existence is guaranteed by Proposition* 6.23(ii). *Then* ker φ *is isomorphic to the shift by $-\imath$ of a line module. The line ℓ' which corresponds to this module is determined as follows:*

(i) *If ℓ is a component of E, then $\ell' = \sigma^\imath \ell$.*

(ii) *If ℓ is not a component of E, and if S, Z are as above, then ℓ' is the unique line containing $\sigma^\imath Z$.*

Proof. (ii) Write $M' = (\ker \varphi)(\imath)$. Proposition 6.7 tells us that $M' = M_{\ell'}$ for some line ℓ'. To determine ℓ', recall first that the shifts of N are given by the rule $N_p^+ = N_{\sigma p}$. Let S, Z be as above, and say that $S = \operatorname{Spec} R, Z = \operatorname{Spec} \bar{R}$. Since S is contained in a line, $\bar{R}$ is A-isomorphic as R-module to the kernel I of the projection $R \to k(p)$, and therefore N_Z and $\Gamma_*(I)$ are A-isomorphic. So we obtain a diagram of A-modules

$$\begin{array}{ccccccccc} 0 & \to & M'(-\imath) & \to & M_\ell & \to & N_p & \to & 0 \\ & & \downarrow^{\delta} & & \downarrow^{\psi} & & \| & & \\ 0 & \to & N_Z & \to & N_S & \to & N_p & \to & 0. \end{array} \tag{6.25}$$

The cokernel of δ has finite length because it is isomorphic to coker ψ. Shifting δ, we obtain a map $M' \to N_{\sigma^\imath Z}$ with finite cokernel. Hence ℓ' contains $\sigma^\imath Z$. The line ℓ' is unique because $\sigma^\imath Z$ has length 2 if $r = 3$ and 1 if $r = 2$. To prove (i), we choose a suitable subscheme S of ℓ of finite length, and argue as before. $\square$

Given a regular triple $(E, \sigma, \mathscr{L})$, we will set

$$\eta = \eta_{\mathscr{Q}}, \tag{6.26}$$

where $\mathscr{Q} = \mathscr{L}^{1-\sigma^{-\imath}}$ as before.

Lemma 6.27 *If $(E, \sigma, \mathscr{L})$ is a regular triple and $\mathscr{Q} = \mathscr{L}^{(1-\sigma^{-\imath})}$, then $\mathscr{L}^{(\sigma^{\imath r}\eta)} \approx \mathscr{L}$.*

Proof. We use (5.10iii) and the fact (5.4) that $\mathscr{Q}^\sigma \approx \mathscr{Q}$ to write

$$\mathscr{L}^{(\sigma^{\imath r}\eta)} \approx \mathscr{L}^{\sigma^{\imath r}} \otimes \mathscr{Q}^{-r} \approx \mathscr{L}^{(\sigma^{\imath r} - r + r\sigma^{-\imath})}.$$

It follows from (5.3) that this sheaf is isomorphic to $\mathscr{L}$. $\square$

The next proposition describes the action of conjugation by the normalizing element g on lines.

Proposition 6.28 *Let ℓ be the line $\{a = 0\}$, suppose that ℓ is not a component of E, and set $S = E \cap \ell$. Let $M = A/aA$ be the line module corresponding to ℓ.*

(i) *The scheme $\tilde{S} := \sigma^{\imath r}\eta S$ is contained in the line $\tilde{\ell}$ defined by the equation $\{g^{-1}ag = 0\}$.*

(ii) *The kernel of the canonical map $\varphi: M \to N_S$ is the module Mg, which is isomorphic to the shift by $-\imath r$ of the line module $M_{\tilde{\ell}}$.*

Proof. Since $\mathscr{L} \approx \mathscr{O}_E(S)$, we have $\mathscr{L}^{(\sigma^{\imath r}\eta)^{-1}} \approx \mathscr{O}_E(\tilde{S})$. Hence (6.27) implies that $\tilde{S}$ is contained in a line. Since N_S is a B-module, g annihilates N_S, and so $Mg \in \ker \varphi$. A consideration of the Hilbert functions shows that $Mg = \ker \varphi$. Writing $M = A/aA$, we find

$$(6.29) \qquad Mg = (A/aA)g = Ag/aAg = gA/agA \approx (A/g^{-1}agA)(-\imath r).$$

This identifies Mg as the shift of the line module $M_{\tilde{\ell}}$. It remains to show that $\tilde{\ell}$ is the line which contains $\tilde{S}$, and it suffices to show this for a generic line ℓ. Suppose for example that $r = 3$. Then if E is reduced, we may assume that S consists of 3 distinct points: $S = p_1 + p_2 + p_3$. Setting $p = p_1$, the diagram (6.25) and Proposition 5.9(i) identify M' as the line through the points $\eta\sigma p_1\, \sigma p_2, \sigma p_3$. We map M' to $N_{\sigma p_2}$, obtaining a kernel $M''(-1)$, where $M'' = M_{\ell''}$, and where ℓ'' is the line through the points $\sigma\eta\sigma p_1\, \eta\sigma^2 p_2, \sigma^2 p_3$. Then mapping M'' to $N_{\sigma^2 p_3}$, we obtain a kernel $M_{\ell'''}(-1)$, where ℓ''' is the line through $\sigma^2\eta\sigma p_1, \sigma\eta\sigma^2 p_2, \eta\sigma^3 p_3$. This line module is the shift of the kernel of φ, and since $\sigma\eta = \eta\sigma$, the three points form the scheme $\sigma^3\eta S$, as required. If E is a triple line, then S contains a single point p. Let $Z = S - p$. Then $\eta Z = Z$, by (5.9). Computing as above, one finds that $\tilde{\ell}$ is the unique line through $\sigma^3 Z = \sigma^3\eta Z$, as required. The remaining cases are treated in a similar way. □

7 Characterization of the algebras which are finite modules over their centers

In this section we consider a regular algebra A of dimension 3 which corresponds to a triple $(E, \sigma, \mathscr{L})$. We are going to prove that, when the automorphism σ is of finite order n, such an algebra is a finite module over its center:

Theorem 7.1 *Let A be an algebra corresponding to a regular triple $\mathscr{T} = (E, \sigma, \mathscr{L})$. Then A is a finite module over its center if and only if the automorphism σ has finite order.*

As may be expected, the case that the algebra is elliptic is the difficult one. We have not determined the rank of A over its center $Z(A)$ in all cases.

In addition to A, we will study its localization $\Lambda = A[g^{-1}]$, where g is the canonical normalizing element. This is a $\mathbb{Z}$-graded algebra, and we denote its degree zero part by Λ_0. As we have remarked in the introduction, Spec Λ_0 plays the role of the open complement of Proj B in Proj A. The structure of the ring Λ_0 is described by Theorem (7.3) below.

We denote by s_0 the smallest positive integer such that σ^{s_0} fixes the class $[\mathscr{L}]$ in Pic E, if such an integer exists, and we set $s_0 = \infty$ otherwise. If s_0 is finite, then the automorphism σ^{s_0} of E is compatible with an automorphism of $\mathbb{P} = \mathbb{P}(A_1)$. Some confusing factors of 2 will enter when $r = 2$. In order to handle them conveniently, we define

$$(7.2)\qquad s = \begin{cases} \frac{1}{2}s_0 & \text{if } r = 2 \text{ and } s_0 \text{ is even} \\ s_0 & \text{otherwise} \end{cases}$$

Thus $s_0 = s$ if $r = 3$. Note that s is the smallest positive integer such that s_0 divides rs.

We call our algebra Λ_0 *almost Azumaya* of rank p^2 over its center if $\Lambda_0/\mathfrak{m}$ is a central simple algebra of rank p^2 over its center for all but finitely many two-sided maximal ideals $\mathfrak{m}$.

Theorem 7.3 *Let A be the regular algebra determined by an elliptic triple $(E, \sigma, \mathcal{L})$ and let $\Lambda = A[g^{-1}]$. Let s_0,s be defined as above.*

(i) If $s_0 = \infty$, *then Λ_0 is a simple ring.*

(ii) *Assume that s_0 is finite.*

(a) *If $r = 3$ or if $r = 2$ and s_0 is even, then Λ_0 is an Azumaya algebra of rank s^2 over its center.*

(b) *If $r = 2$ and s_0 is odd, then Λ_0 is almost Azumaya algebra of rank s^2 over its center.*

Proposition 7.4 *Let A be a graded algebra generated in degree 1, and let g be a homogeneous normalizing element of A of positive degree d. The $\mathbb{Z}$-graded ring $\Lambda := A[g^{-1}]$ is strongly graded, i.e., $\Lambda_i\Lambda_j = \Lambda_{i+j}$ for every pair of integers i, j.*

Proof. This is a consequence of the fact that A is generated in degree 1. For any i, Λ_i is the direct limit of the k-vector spaces $g^{-n}A_{i+nd}$. Thus $\Lambda_i\Lambda_j = \bigcup_n g^{-2n}A_{i+nd}A_{j+nd}$. Since A is generated in degree 1, $A_iA_j = A_{i+j}$ for any $i,j \geqq 0$. Hence

$$\Lambda_i\Lambda_j = \bigcup_n g^{-2n}A_{i+j+2nd} = \Lambda_{i+j}. \qquad \square$$

Let (A-gr) and (Λ_0-mod) denote the categories of finite graded A-modules and of finite Λ_0-modules respectively. Since Spec Λ_0 is an "open subscheme of Proj A", these two categories are related. In one direction, we have the localization functor $(A\text{-gr}) \to (\Lambda_0\text{-mod})$ given by $M \rightsquigarrow M[g^{-1}]_0$. On the other hand, the Λ_0-module $V = M[g^{-1}]_0$ does not contain enough information to describe M. To recover M, we also need to know its formal completion along the "closed subscheme" Proj(A/gA). But this formal completion will be zero if M has gk-dimension 1 and is g-torsion free, and this leads us to the following proposition. Let us call an A-module $\tilde{N}$ *normalized* if it is generated by finitely many elements of degree 0, and if its Hilbert function is periodic. This extends the definition given above for regular algebras.

Proposition 7.5 *Let g be a homogeneous normalizing element of positive degree in a noetherian graded k-algebra A which is generated in degree* 1, *and let $\Lambda = A[g^{-1}]$. The following categories are equivalent*:

(i) *finite-dimensional Λ_0-modules V,*

(ii) *graded Λ-modules M with* dim $M_n < \infty$ *for all n,*

(iii) *normalized A-modules $\tilde{N}$ which are g-torsion free,*

(iv) *finitely generated graded A-modules N such that* dim N_n *is bounded, modulo g-torsion modules.*

Proof. The equivalence of the categories (i) and (ii) is a consequence of the fact that Λ is strongly graded (see [NV, Ch. A, Thm. I.3.4]). Let M be a graded Λ-module. Then multiplication by g is a bijective map $M_n \rightsquigarrow M_{n+d}$, where d is the degree of g. So the Hilbert function of M is periodic of period d, provided that it is defined, i.e., that $\dim M_n < \infty$ for all n. If so, then since multiplication by g is bijective, the A-module $M_{\geq 0}$ is finitely generated by $M_0 + \ldots + M_{d-1}$. Then the fact that A is generated in degree 1 shows that M_d generates $M_{\geq d}$. Shifting shows that $M_{\geq 0}$ is generated in degree 0, hence that it is a normalized module. Clearly $\tilde{N} \approx \tilde{N}[g^{-1}]_{\geq 0}$ if $\tilde{N}$ is normalized and g-torsion free. Thus the categories (ii) and (iii) are equivalent. The equivalence of (ii) and (iv) is a standard localization argument. □

The rest of this section is devoted to the proofs of Theorems 7.1 and 7.3.

Proposition 7.6 *Theorem* (7.1) *is true in the case that A is linear.*

Proof. This is the easy case. Assume that A is linear. Suppose first that $r = 3$, so that $E = \mathbb{P}^2$. From the description [ATV (6.8)] of A as $B(E, \sigma, \mathscr{L})$, it follows that the graded ring of fractions of A is an Ore extension of the form $K[t, t^{-1}; \sigma]$, where K is the function field of $\mathbb{P}^2$. If σ has infinite order then $K[t, t^{-1}; \sigma]$ is not finite over its center. This proves one half of (7.1). The other half is [ATV, 8.5]. If $r = 2$, then $s_0 = 2$ and $s = 1$. In this case, the description of the ring A shows that its Veronese subring $A\langle 2\rangle := \oplus A_{2n} \approx B(E, \sigma^2, \mathscr{L} \otimes \mathscr{L}^{\sigma})$ has a graded ring of fractions of the form $K[t, t^{-1}; \sigma^2]$, where K is the function field of $\mathbb{P}^1 \times \mathbb{P}^1$. The proof is completed as before. □

Some of the statements in the sequel remain true in the linear cases, and others have to be modified only slightly. However, since the proofs of Theorems 7.1 and 7.3 are fairly complicated, we will assume for the rest of the section that the algebras A under consideration are elliptic. We will also assume that the ground field is algebraically closed. It is clear that this is permissible.

By *extension of point modules* we mean a module having a finite filtration whose successive quotients are shifted point modules, i.e., are Cohen-Macaulay modules of gk-dimension and multiplicity 1.

Proposition 7.7 (i) *The critical B-modules of gk-dimension* 1, *where $B = A/gA$, are the shifted point modules.*

(ii) *Let N be an A-module of gk-dimension* 1 *with trivial socle. Then N is an extension of point modules if and only if it is annihilated by a power of g.*

Proof. The first assertion follows from Theorem (1.3) of [AV]. To prove the second one, we note that since the A-modules which are point modules are B-modules and $B = A/(g)$, they are annihilated by g. So an extension of point modules is annihilated by a power of g. Conversely, suppose that a power of g annihilates N, and consider a filtration (6.12) whose successive quotients $\bar{N}^i = N^i/N^{i+1}$ are critical modules of gk-dimension 1. Then g annihilates each $\bar{N}^i$, and so $\bar{N}^i$ is a B-module. Part (ii) now follows from (i). □

Proposition 7.8 *Let N be a critical module of gk-dimension* 1 *which is not a shifted point module, and let $\varepsilon = \varepsilon(N)$. Then σ^{ε} fixes the class of $\mathscr{L}$ in* Pic E.

Proof. By Proposition 6.7, we may replace N by a shift of an equivalent module for which there exist lines ℓ, ℓ' and an exact sequence

$$0 \to M'(-\varepsilon) \to M \to N \to 0,$$

where $M = M_\ell$, $M' = M_{\ell'}$. Let $S = E \cap \ell$, and let N_S be the corresponding family of point modules (6.23). Note that ℓ is not contained in E. If it were, then M would be a B-module. This would contradict the fact that N is not a B-module. By Proposition 6.23, there is a map $\varphi: M \to N_S$ whose cokernel has finite length. Since N_S has a filtration whose successive quotients are shifted point modules, there is no non-trivial map from N to N_S, and it follows from this that the induced map $M'(-\varepsilon) \to N_S$ has a cokernel of finite length. The shift of this map by ε is a map φ': $M' \to N_{S'}$, where $S' = \sigma^\varepsilon S$, and the cokernel of φ' has finite length too. Therefore $S' \subset \ell'$, by (6.23). So $\mathcal{O}_E(S) \approx \mathcal{L} \approx \mathcal{O}_E(S')$, and on the other hand, since $S' = \sigma^\varepsilon S$, $\mathcal{O}_E(S') \approx \mathcal{L}^{\sigma^{-\varepsilon}}$. □

Corollary 7.9 *If $s_0 = \infty$, Λ_0 has no non-zero finite dimensional representations.*

Proof. This follows from Propositions 7.8 and 7.5. □

Proof of Theorem 7.3 The simplicity of the ring Λ_0 when $s_0 = \infty$ follows from two facts: It has no finite dimensional representations, and its dimension is 2.

Assume that $s_0 = \infty$. The g-adic filtration on A induces a filtration on Λ and hence on Λ_0. One easily verifies for this filtration that gr Λ_0 is isomorphic to the subring $\bigoplus_n B_{\iota sn}$ of B. Hence $\mathrm{gk}(\Lambda_0) = 2$ [KL, Prop. 6.6]. Assume that Λ_0 is not simple and let J be a nonzero prime ideal in Λ_0. Then $\mathrm{gk}(\Lambda_0/J) \leq \mathrm{gk}(\Lambda_0) - 1 = 1$ [KL, Prop. 3.15]. Hence Λ_0/J is a polynomial identity ring [SSW], and so it has finite dimensional representations. This is a contradiction. □

We now turn to the proof of Theorem 7.3(ii). Since we have assumed that k is algebraically closed, we can apply Proposition 6.16. Assuming that s_0 is finite, we will show that there is a faithful family of irreducible representations of Λ_0 of dimension s^2 over k. Because of the equivalence of categories (7.5), it suffices to find a faithful family of critical A-modules N of gk-dimension 1 whose Hilbert series has the required property. In this equivalence, a normalized A-module N corresponds to the Λ_0-module N_0. With this in mind, it becomes clear that the requirement is $e(N) = s$ if $r = 3$, and $e_0(N) = s$ if $r = 2$ (see (6.4)). Proposition 6.7(v) shows that $\varepsilon(N) = \iota s$ implies this condition in either case. Also, we know by Proposition 6.7(iii) that every equivalence class of critical modules of gk-dimension 1 contains a quotient of a line module. So we look for such quotients. We will show if ℓ is not a component of E, then the line module M_ℓ has infinitely many inequivalent critical quotients N with $\varepsilon(N) = \iota s$.

By definition of s, $\mathcal{L}^{\sigma^{\iota s}}$ is isomorphic to $\mathcal{L}$. We fix an isomorphism u, thus obtaining a linear operator ρ on $H^0(E, \mathcal{L})$ defined by

$$a^\rho = u(a^{\sigma^{\iota s}}). \tag{7.10}$$

We denote by the same letter ρ the automorphism induced on $\mathbb{P} = \mathbb{P}(A_1)$, the one which is compatible with the automorphism $\sigma^{\iota s}$ of E. With this notation, if ℓ is the line $\{a = 0\}$, then $\rho\ell$ is the line $\{a' = 0\}$, where $a' = a^{\rho^{-1}}$.

Let us fix a line ℓ which is not a component of E, and let $\ell' = \rho\ell$. We fix this notation:

$$a \in A_1, a' = a^{\rho^{-1}}, \tag{7.11}$$

$$\ell:\{a = 0\}, \ell' = \rho\ell:\{a' = 0\},$$

$$M = M_\ell, M' = M_{\ell'}.$$

Lemma 7.12 (i) *M contains a submodule Q such that M/Q is an extension of point modules, and such that Q is isomorphic to $M'(-\imath s)$.*

(ii) *Let n be a positive integer. There are only finitely many submodules Q of M which are isomorphic to $M'(-\imath n)$ for some line module M', and such that M/Q is an extension of point modules.*

Proof. (i) Let $S = E \cap \ell$. Choose a point, say p, in the support of S. Set $Z = S - p$, $\ell_1 =$ line passing through $\sigma^{\imath}Z$, $S_1 = E \cap \ell_1$, and $p_1 =$ the unique point $S_1 - \sigma^{\imath}Z$. Also, let N_p be the point module corresponding to p and let M_1 be the line module corresponding to ℓ_1. As we know (6.24), there is a surjective map $M \to N_p$ whose kernel is isomorphic to $M_1(-\imath)$. We repeat the construction, replacing (ℓ, M, p) by (ℓ_1, M_1, p_1), and in this way we obtain a sequence of points $p, p_1, \ldots, p_s$ and of shifted line modules $M \supset M_1(-\imath) \supset \ldots \supset M_s(-\imath s)$. We define $Q_1 = M_s(-\imath s)$. At each step, ℓ_j is the line containing $\sigma^{\imath j}Z$, and M_j is the corresponding line module. Since ρ is the extension of $\sigma^{\imath s}$ to an automorphism of $\mathbb{P}(A_1)$, $\rho\ell$ is the line ℓ_s, which is the one containing $\sigma^{\imath s}Z$.

(ii) Let $Q \approx M'(-\imath n)$ be such a submodule. Then $N = M/Q$ is a module of gk-dimension 1 and multiplicity n, with trivial socle. Since it is generated in degree 0 and is an extension of point modules, N has a quotient N_0 which is a point module. The kernel Q_1 of the map $M \to N_0$ has the form $M_1(-\imath)$ and is uniquely determined by N_0 (6.23ii). Since ℓ is not a component of E, there are only finitely many choices for the modules N_0 and M_1. We replace M by $M_1 = Q_1(\imath)$ and proceed by induction. □

To simplify notation, we denote $\imath s$ by ε in the next two lemmas.

Lemma 7.13 *With $B = A/gA$ as before, we have $aB_\varepsilon = B_\varepsilon a'$.*

Proof. Let u be defined as above, so that $x^\rho = u(x^{\sigma^\varepsilon})$. Then for any $x, y \in B_1$ and $z \in B_{\varepsilon-1}$, we have $x^\rho zy = y^\rho zx$ in $B_{\varepsilon+1}$. This is because by definition of multiplication in B,

$$x^\rho zy = u(x^{\sigma^\varepsilon}) \otimes z^\sigma \otimes y^{\sigma^\varepsilon} = (u \otimes 1 \otimes 1)(x^{\sigma^\varepsilon} \otimes z^\sigma \otimes y^{\sigma^\varepsilon})\ .$$

The right side is symmetric in x and y. Since B_ε is generated by $B_1 B_{\varepsilon-1}$, it follows that $x^\rho B_\varepsilon \subset B_\varepsilon x$ for any x, hence that $aB_\varepsilon \subset B_\varepsilon a'$. Since the two spaces have the same dimension, $aB_\varepsilon = B_\varepsilon a'$. □

Remark. Let us define a morphism of triples $(E', \sigma', \mathscr{L}') \to (E, \sigma, \mathscr{L})$ to be a pair (f, u) consisting of a morphism of schemes $f: E' \to E$ such that $\sigma f = f\sigma'$, and a morphism of $\mathcal{O}_{E'}$-modules $u: f^* \mathscr{L} \to \mathscr{L}'$. Then the B-construction is made into a contravariant functor from the category of triples to the category of graded algebras in an obvious way. If we take $(E', \sigma', \mathscr{L}') = (E, \sigma, \mathscr{L})$, $f = \sigma^\varepsilon$, where $\varepsilon = \imath s$, and u is as in (7.10), we see that the linear operator ρ extends to an automorphism of B. And, since A is obtained as a quotient of the tensor algebra on $B_1 = H^0(E, \mathscr{L})$ using the defining relations of B of minimal degree, ρ extends to an automorphism of A as well. This situation also gives rise to an automorphism τ of $B\langle\varepsilon\rangle = B(E, \sigma^\varepsilon, \mathscr{B}_\varepsilon)$, where $\mathscr{B}_\varepsilon = \mathscr{L} \otimes \mathscr{L}^\sigma \otimes \ldots \otimes \mathscr{L}^{\sigma^{\varepsilon-1}}$ as before. We note that $\mathscr{B}_\varepsilon = \mathscr{B}_1 \otimes (\mathscr{B}_{\varepsilon-1})^\sigma = \mathscr{B}_{\varepsilon-1} \otimes \mathscr{B}_1^{\sigma^{\varepsilon-1}}$. The map τ is induced by the automorphism of triples (σ, v), where $v: \sigma^* \mathscr{B}_\varepsilon = (\mathscr{B}_{\varepsilon-1})^\sigma \otimes \mathscr{B}_1^{\sigma^\varepsilon} \to \mathscr{B}_1 \otimes (\mathscr{B}_{\varepsilon-1})^\sigma = \mathscr{B}_\varepsilon$ is defined by $a \otimes b \rightsquigarrow u(b) \otimes a$, for local sections a of $\mathscr{B}_{\varepsilon-1}$ and b of $\mathscr{B}_1^{\sigma^\varepsilon}$. Putting $a = z^\sigma$ and $b = y^{\sigma^\varepsilon}$, and interpreting the tensor products as multiplication in the algebra B, we have $\tau(zy) = \rho(y)z$ for all $y \in B_1$ and all $z \in B_{\varepsilon-1}$. Also, putting $z = y_1 \otimes y_2 \otimes \ldots \otimes y_{\varepsilon-1}$ and $y = y_\varepsilon$ in the above computation gives $\tau(y_1 \otimes y_2 \otimes \ldots \otimes y_\varepsilon) = \rho(y_\varepsilon) \otimes y_1 \otimes y_2 \otimes \ldots \otimes y_{\varepsilon-1}$ and iterating this cyclic permutation ε times, we find that the restriction of ρ to $B\langle\varepsilon\rangle$ is equal to τ^ε.

Lemma 7.14 *The vector space $V = (aA_\varepsilon \cap A_\varepsilon a')/(aA_{\varepsilon-1}a')$ has dimension at least* 2.

Proof. Let $a_\varepsilon = \dim A_\varepsilon$ denote the Hilbert function of A. By the previous lemma, $A_\varepsilon a' \subset aA_\varepsilon + gA_{\varepsilon-\imath r+1}$. Also, since g is a normalizing element of degree $\imath r$, $aA_\varepsilon \cap gA_{\varepsilon-\imath r+1} \supset aA_{\varepsilon-\imath r}g$. Thus

$$\dim(aA_\varepsilon + A_\varepsilon a') \leqq \dim(aA_\varepsilon + gA_{\varepsilon-\imath r+1}) \leqq a_\varepsilon + a_{\varepsilon-\imath r+1} - a_{\varepsilon-\imath r},$$

$$\dim(aA_\varepsilon \cap A_\varepsilon a') = 2a_\varepsilon - \dim(aA_\varepsilon + A_\varepsilon a') \geqq a_\varepsilon - a_{\varepsilon-\imath r+1} + a_{\varepsilon-\imath r},$$

and

$$\dim V \geqq a_\varepsilon - a_{\varepsilon-1} - a_{\varepsilon-\imath r+1} + a_{\varepsilon-\imath r}. \tag{7.15}$$

If $r = 3$, then $\imath = 1$ and $a_n - a_{n-1} - a_{n-2} + a_{n-3} = 2$ for all $n > 1$. So $\dim V \geqq 2$ as required. If $r = 2$, then $\imath = 2$. In that case

$$a_n - a_{n-1} - a_{n-3} + a_{n-4} = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even}. \end{cases}$$

Since $\varepsilon = 2s$ is even, $\dim V \geqq 2$ in this case as well. □

Lemma 7.16 *Let $N = M/Q$, where Q is isomorphic to $M'(-\imath s)$, with M, M' as above. Then one of the following possibilities occurs:*

(i) *N is critical.*
(ii) *N is an extension of point modules.*
(iii) *N is an extension of two critical modules having $e_0 \neq e_1$.*

The third possibility can arise only if $r = 2$ and s_0 is odd.

Proof. We recall (6.7) that $\varepsilon(N) = \imath s$. Assume that N is not of the first two types, and consider a filtration as in (6.12), whose successive quotients $\bar{N}^i$ are critical and of gk-dimension 1. Let $\bar{N}^i$ be one of these quotients, $e = e(\bar{N}^i)$, and $\varepsilon = \varepsilon(\bar{N}^i)$. Then $\varepsilon < \imath s$, and by Proposition 7.8, s divides ε. The only possibility is that $r = 2$, s_0 is odd, and $s_0 = s = \varepsilon$, because if s_0 is even, then $s_0 = \imath s$. Hence (6.7v) $e_0(\bar{N}^i) \neq e_1(\bar{N}^i)$. This being so, we must have $e_0(\bar{N}^j) \neq e_1(\bar{N}^j)$ for some $j \neq i$ as well, from which we deduce that $\varepsilon(\bar{N}^j) = s$ too. Since $\varepsilon(N) = e(gr(N)) = 2s$, $\bar{N}^i$ and $\bar{N}^j$ are the only two terms in $\mathrm{gr}(N)$, and we are in case (iii). □

Lemma 7.17 *With the notation of* (7.11), *there exist infinitely many equivalence classes* (*see* (6.5)) *of critical quotients N of M of gk-dimension* 1 *and with $\varepsilon(N) = \imath s$.*

Proof. We will show that there are infinitely many equivalence classes of critical quotients $N = M/Q$ in which Q is isomorphic to $M'(-\imath s)$. To do so, we go back to Corollary (6.10). The quotients M/Q such that $Q \approx M'(-\imath s)$ are parametrized by the projective space $Y = \mathbb{P}(V^*)$, where V is as in Lemma 7.14. By the previous lemma, Y has dimension at least 1. By Proposition 6.13, the quotients N with $\mathrm{gr}(N)$ in a given equivalence class are parametrized by a closed subset $Z \subset F$, and those for which $Q \approx M'(-\imath s)$ form the locus $Z \cap Y$, which is a closed subset of Y. Thus Y is decomposed into closed subsets Z. According to Lemma 7.12(ii), at least one, and at most finitely many points of Y correspond to quotients which are extensions of point modules. Thus Y is not covered by a single subset $Z \cap Y$. Since Y is irreducible, it follows that there are infinitely many of these subsets. On the other hand, Propositions 6.16 and 7.12(ii), combined with Lemma 7.16, show that only finitely many subsets correspond to classes of modules which are not critical. Hence there are infinitely many classes of critical quotients. □

Proof of Theorem 7.3 (ii) Propositions 6.16, 7.16 and 7.5 show that Λ_0 has only a finite number of representations of dimension $< s$, and that it has none unless $r = 2$ and s_0 is odd. So it suffices to prove that Λ_0 satisfies the identities of $s \times s$ matrices [Ro, 1.8.32]. Let $I = \cap\, \mathrm{ann}_A N$ be the intersection of the annihilators of the critical A-modules N of gk-dimension 1 and multiplicity s. Since line modules are critical and since a generic line module M has infinitely many equivalence classes of such modules N as quotients, I is contained in the intersection of the annihilators of the generic line modules, which is clearly zero. Thus $I = 0$. It follows from Proposition 7.5 that using a critical A-module N of gk-dimension 1, one obtains a family of irreducible representations $V_i = (N(i)_g)_0 = (N_g)_i$ of Λ_0. Then one finds

$$\bigcap_N \bigcap_i \mathrm{ann}_{\Lambda_0} V_i = \bigcap_N ((\mathrm{ann}_A N)[g^{-1}])_0 = ((\bigcap_N \mathrm{ann}_A N)[g^{-1}])_0 = 0,$$

as required. □

Proposition 7.18 *Let A be a regular algebra corresponding to an elliptic triple $(E, \sigma, \mathscr{L})$ such that σ has finite order. Conjugation by the normalizing element g is an automorphism of finite order of A. In other words, g^m is in the center of A for some positive integer m.*

Proof. Let n be the order of σ, and let φ denote conjugation by g on $A_1: ag = g\varphi(a)$. It suffices to show that φ^n acts trivially on the space $\mathbb{P}(A_1^*)$ of lines, i.e., that $\varphi^n(a) = c_a a$ for some $c_a \in k^*$. For, the rule $a \rightsquigarrow c_a$ defines a map $\mathbb{P}(A_1^*) \to k^*$. Since $\mathbb{P}(A_1^*)$ is proper, this map is constant. So there is a non-zero element $c \in k^*$ such that $ag^n = cg^n a$ for all $a \in A_1$. Then $ug^n = c^k g^n u$ for all $u \in A_k$ too. Putting $u = g$, we find that $c^{\imath r} = 1$, hence that $g^{\imath rn} \in Z(A)$, as required.

To show that φ^n acts trivially on lines in $\mathbb{P}(A_1^*)$, we apply Proposition 6.28, which identifies the action of φ on lines as $\sigma^{\imath r}\eta$. Since σ has finite order and since $\sigma\eta = \eta\sigma$, it remains to verify that η has finite order, or that $\mathscr{Q}$ has finite order in $\mathrm{Pic}\, E$. This is done using the relations $\sigma^n = 1$ and (5.3). We write

$$\sigma^{\imath n} - 1 = (\sigma^{\imath n - \imath} + \ldots + \sigma^{\imath} + 1)\sigma^{\imath}(1 - \sigma^{-\imath}).$$

Operation by this element annihilates the class of $\mathscr{L}$, hence $(\sigma^{\imath n - \imath} + \ldots + \sigma^{\imath} + 1)$ annihilates the class of $\mathscr{Q}$. Since $\mathscr{Q}^\sigma \approx \mathscr{Q}$, this shows that $\mathscr{Q}^n \approx \mathcal{O}_E$, as required. □

Since Theorem 7.3 has been proved, the next Proposition will complete the proof of Theorem 7.1.

Proposition 7.19 *Theorem* 7.3 *implies Theorem* 7.1.

Proof. It is clear from (7.3i) that A is not finite over its center if σ has infinite order. The point is to show that if σ has finite order and if Λ_0 is known to be almost Azumaya, then A is finite over its center. It follows from Theorem 7.3 and Proposition 7.18 that $\Lambda = A[g^{-1}]$ is a polynomial identity ring, and since A is a subring of Λ, it is a polynomial identity ring too.

Let $T(A)$ denote the trace ring of A [AmSm]. Since A is noetherian, $T(A)$ is a finite A-module which is finite over its center R. So it suffices to prove that $A = T(A)$. We introduce an auxiliary ring A', the reflexive hull of $T(A)$, considered as R-module. If R_0 is a polynomial subring of R over which R is finite, then A' is the bidual of $T(A)$ as R_0-module. It is characterized by these properties: $T(A) \subset A'$, A' satisfies the condition S_2 of Serre, and the support of $A'/T(A)$ in $\operatorname{Spec} R$ has dimension $\leqq 1$. Thus A' is an R-algebra, finite over its center R, and it is a finite left and right A-module as well. Note that $T(A)$ and A' are graded compatibly with the grading of A.

As R-module, A' has no non-trivial extension whose cokernel has dimension 1. Therefore Corollary 4.2(iv) shows that A' is a reflexive A-module too. It suffices to show that $A = A'$.

Lemma 7.20 *A'/A is annihilated by a power of the normalizing element g.*

Proof. We identify the graded ring $\Lambda' = A'[g^{-1}]$ as the reflexive hull of the trace ring $T(\Lambda)$. The ring Λ'_0 is a finite Λ_0-algebra, and wherever Λ_0 is Azumaya, Λ'_0 is locally a central extension. Now since A has finite global dimension so does Λ, and it follows from (7.4) and [NV, Ch. A, Thm. I.3.4] that Λ_0 has finite global dimension too. So wherever Λ_0 is Azumaya, its center is also of finite global dimension, hence is integrally closed. It follows that Λ'_0 is equal to Λ_0 at such points. We now use that fact that Λ_0 is almost Azumaya to conclude that the quotient Λ'_0/Λ_0 has finite length, hence by [NV, loc cit] that Λ'/Λ has gk-dimension one, unless it is zero. This in turn implies that $g^n(A'/A)$ has gk-dimension one if n is sufficiently large. On the other hand, since A' is reflexive, $pd(A') \leqq 1$, hence $pd(A'/A) \leqq 1$. By Theorem 4.1(iii), A'/A contains no submodule of gk-dimension $\leqq 1$. Thus $g^n(A'/A) = 0$ for large n, as required. □

To complete the proof of Proposition 7.19, we will use the factorization of the normalizing element given in (5.13). Assume that $A \neq A'$. Applying Lemma 7.20, we may choose a bimodule Q with $A \subset Q \subset A'$, such that $C = Q/A$ is non-zero, but is annihilated on the left by one of the normalizing elements g_i, say by g_1. Since the sequence

$$0 \to A \to Q \to C \to 0$$

does not split, Theorem 4.1(i) implies that $gk\,(C)$ is at least 2. Hence it is equal to 2. By Lemma 3.1, the gk-dimension of $A/(\operatorname{ann} C)$ is also equal to 2. Since $P = g_1 A$ is a prime ideal which annihilates C and $gk(A/P) = 2$, we conclude that P is the left annihilator of C.

Consider the exact sequence of bimodules

$$0 \to C \to g_1^{-1}A/A \to g_1^{-1}A/Q \to 0\,.$$

The ring A/P is a critical bimodule because it is prime, and therefore $g_1^{-1}A/A$ is also critical. It follows that $gk(g_1^{-1}A/Q) \leqq 1$. Hence $g_1^{-1}A$ is contained in the reflexive hull $\tilde{Q}$ of Q. But $\tilde{Q} \subset A'$ because A' is reflexive. Thus A' contains $A[g_1^{-1}]$, and since $A[g_1^{-1}]$ is not a finite A-module, this is a contradiction, which completes the proof of Proposition 7.19. □

8 Twisting a graded algebra by an automorphism, and determination of the algebras associated to non-reduced divisors

In this section we begin by describing a construction that twists a graded algebra using a (graded) automorphism. This construction allows us to identify some of our elliptic regular algebras as twists of a few standard ones. In particular, we will identify as twists all algebras whose associated elliptic curve is not reduced. We will see that in these cases, the ring Λ_0 is closely related to the Weyl algebra. Incidentally, in this section the letter s is not used as in (7.2), but is defined locally wherever it is used.

Let τ be an automorphism of a $\mathbb{Z}$-graded algebra A. We define a new graded algebra A_τ which we call the twist of A by τ. As a graded abelian group, A_τ is an isomorphic copy of A. The element of A_τ corresponding to $a \in A$ will be denoted by a_τ. The product of two homogeneous elements a_τ, b_τ of A_τ is defined to be $a_\tau b_\tau = (a b^{\tau^d})_\tau$, where $d = \deg a$. So if $x_1, \ldots, x_n$ are elements of A_1, then

$$(x_1 \ldots x_n)_\tau = (x_1)_\tau (x_2^{\tau^{-1}})_\tau \ldots (x_n^{\tau^{-n+1}})_\tau . \tag{8.1}$$

If $f = \Sigma a_{i_1 \ldots i_n} x_{i_1} \ldots x_{i_n}$ is a relation in A among elements of degree 1, then the corresponding relation

$$f_\tau = \Sigma a_{i_1 \ldots i_n} (x_{i_1})_\tau (x_{i_2}^{\tau^{-1}})_\tau \ldots (x_{i_n}^{\tau^{-n+1}})_\tau \tag{8.2}$$

holds in A_τ. For example, if $r = 3$ and A is the algebra corresponding to the linear triple $(\mathbb{P}^2, \sigma, \mathcal{O}_{\mathbb{P}^2}(1))$, then A is simply the twist by τ^{-1} of the polynomial ring $k[x, y, z]$ for a $\tau \in \mathrm{GL}_3(k)$ lifting $\sigma \in \mathrm{PGL}_3(k) = \mathrm{Aut}(\mathbb{P}_k^2)$ (see [ATV, 7.4′]).

Let M be a graded right A-module. Then an A_τ-module is defined in a similar way using an isomorphic copy M_τ of the graded abelian group M, with the multiplication rule $m_\tau b_\tau = (m b^{\tau^d})_\tau$, $d = \deg m$. If $\varphi \colon M \to N$ is an A-homomorphism, the corresponding map $\varphi_\tau \colon M_\tau \to N_\tau$ defined by $\varphi_\tau(m_\tau) = (\varphi(m))_\tau$ is an A_τ-homomorphism. This gives us a functor

$$F_\tau \colon (\text{gr-}A) \to (\text{gr-}A_\tau) . \tag{8.3}$$

Lemma 8.4 *Let ρ, τ be two commuting automorphisms of a graded ring A. Then $a_\tau \rightsquigarrow (\rho a)_\tau$ is an automorphism of A_τ, and $(A_\tau)_\rho$ is canonically isomorphic to $A_{\tau\rho}$. In particular, $(A_\tau)_{\tau^{-1}} \approx A$. If $u \in k^*$, and if φ_u denotes the automorphism which acts on A_n as multiplication by u^n, then $A_{\varphi_u \tau} = A_\tau$.*

Corollary 8.5 *The functor $F_\tau \colon (\text{gr-}A) \to (\text{gr-}A_\tau)$ defined by $M \rightsquigarrow M_\tau$ is an equivalence of categories.*

This is true because $F_\tau F_{\tau^{-1}} = F_{\tau^{-1}} F_\tau =$ identity.

Corollary 8.6 *Let τ be an automorphism of a regular algebra A of arbitrary global dimension. Then A_τ is a regular algebra of the same global dimension and the same gk-dimension.*

Proof. This follows from the fact that F_τ is an equivalence of categories and preserves Hilbert functions. □

Now let A be a regular algebra of dimension 3, determined by a regular triple $\mathscr{T} = (E, \sigma, \mathscr{L})$. Then an automorphism of A restricts to an invertible linear operator on A_1. Let us describe the conditions under which an invertible linear operator τ on A_1 extends to an automorphism of A. Let $\mathbb{P}$ be the space of hyperplanes in A_1. As in [ATV], we denote by Γ_m the subscheme of $(\mathbb{P})^m$ defined by the multilinearizations of the relations in A of degree m, and we recall that this scheme can be identified as follows. For $(p_1, \ldots, p_m) \in (\mathbb{P})^m$, set $P_i = (p_i, \ldots, p_{i+s-1})$, where $r + s = 5$. Then

$$\Gamma_m = \{(p_1, \ldots, p_m) \in (\mathbb{P})^m \mid P_i \in E \text{ and } \sigma(P_i) = P_{i+1} \text{ for all } i\} .$$

We will write Γ for Γ_s.

If τ is an invertible linear operator on A_1, we will denote the corresponding linear map $\mathbb{P} \to \mathbb{P}$ by the same symbol. Define τ' and τ'' by the rules

$$\tau' = \begin{cases} \tau & \text{if } r = 3 \\ (\tau, \tau) & \text{if } r = 2 \end{cases} \quad \text{and} \quad \tau'' = \begin{cases} (\tau, \tau) & \text{if } r = 3 \\ (\tau, \tau, \tau) & \text{if } r = 2 . \end{cases} \tag{8.7}$$

Proposition 8.8 *With the above notation, let τ be an invertible linear operator on A_1. Then τ extends to an automorphism of A if $\tau'(E) = E$ and $\sigma\tau' = \tau'\sigma$. Furthermore, if $\mathscr{T} = \mathscr{T}(A)$, then these conditions are also necessary.*

Proof. The conditions are equivalent with the single condition $\tau''\Gamma = \Gamma$. Hence τ'' preserves the space of multilinear forms vanishing on Γ. This implies that τ preserves the defining relations of A. If $\mathscr{T} = \mathscr{T}(A)$, then the defining relations determine Γ, and hence the conditions are also necessary. □

Proposition 8.9 *Let A be a regular algebra of the form $A(\mathscr{T})$, where $\mathscr{T} = (E, \sigma, \mathscr{L})$ is a regular triple, and let τ be an automorphism of A. As in Proposition 8.1, we also denote by τ the induced isomorphism on $\mathbb{P}$. Let $\mathscr{T}_\tau = (E, \tau'\sigma, \mathscr{L})$, where τ' is defined by (8.7). Then $A_\tau = A(\mathscr{T}_\tau)$.*

Proof. We already know that A_τ is regular (8.6). Let $\varphi = (1, \tau, \ldots, \tau^{s-1})$, and let $\Gamma' = \varphi(\Gamma)$, where $\Gamma = \Gamma(A)$ is the locus of zeros of the multilinearized defining relations $\{\tilde{f}_i\}$. It follows that Γ' is the locus of zeros of the set $\{\tilde{f}_{i_\tau}\}$ (see (8.2)). On the other hand, the locus of zeros of these polynomials $\tilde{f}_{i_\tau}$ is $\Gamma(A_\tau)$. Thus $\Gamma' = \Gamma(A_\tau)$.

Let $E' = pr_{1,\ldots,s-1}\Gamma'$. The map $\psi = (1, \ldots, \tau^{s-2})\colon E \to E'$ is an isomorphism, and since A_τ is regular, Γ' defines an automorphism σ' of E'. Thus A_τ is defined by the triple $\mathscr{T}' = (E', \sigma', \mathscr{L}')$, where $\mathscr{L}' = \psi_* \mathscr{L}$. We have a diagram of maps

$$\begin{array}{ccc} (p_1, \ldots, p_{s-1}) & \xrightarrow{\sigma} & (p_2, \ldots, p_s) \\ \downarrow \psi & & \downarrow \tau'\psi = \psi\tau' \\ (p_1, \tau p_2, \ldots, \tau^{s-2} p_{s-1}) & \xrightarrow{\sigma'} & (\tau p_2, \ldots, \tau^{s-1} p_s) \end{array} = \tau'(p_2, \ldots, \tau^{s-2} p_s), \tag{8.10}$$

which identifies σ' as $\psi\tau'\sigma\psi^{-1}$. So via the isomorphism ψ, $\mathscr{T}'$ is also isomorphic to the triple $\mathscr{T}_\tau = (E, \tau'\sigma, \mathscr{L})$. □

Proposition 8.11 *Let $\mathscr{T}$ be a regular elliptic triple and let $A = A(\mathscr{T})$. Let g be the canonical normalizing element in A, which is determined up to a scalar factor. Then g_τ is the canonical normalizing element in A_τ.*

Proof. The element g is characterized by the fact that $\tilde{g}$ vanishes on Γ_{s+1} but g is not in $I_s \otimes A_1 + A_1 \otimes I_s$, where I is the defining ideal of A. Then $\tilde{g}_\tau$ will vanish on $(\Gamma_{s+1})_\tau = (1, \ldots, \tau^{s+1})\Gamma_{s+1}$, and it will not be in $(I_\tau)_s \otimes A_1 + A_1 \otimes (I_\tau)_s$. □

The following Proposition, applied in the case $S = \{cg^n\}$, $c \in k^*$, $n \in \mathbb{Z}$, $n \geqq 0$, shows that the rings $\Lambda_0 = A[g^{-1}]_0$ are isomorphic for all of the twists A_τ. Hence, if (7.3) is true for an algebra A, it is true for every twist of A. We omit the proof.

Proposition 8.12 *Let A be a $\mathbb{Z}$-graded algebra with a graded automorphism τ. Let S be a homogeneous Ore set in A which is stable under τ and τ^{-1}, and define $S_\tau = \{s_\tau | s \in S\}$. Then τ extends uniquely to AS^{-1}, and $(AS^{-1})_\tau = A_\tau S_\tau^{-1}$. The map $as^{-1} \rightsquigarrow (as^{-1})_\tau = a_\tau(\tau^{-d}s)_\tau^{-1}$ defines a ring isomorphism $(AS^{-1})_0 \approx (A_\tau S_\tau^{-1})_0$ in degree zero.*

Remark 8.13 Let $r = 2$. Then the twist introduced in this section is not the same as the half twist introduced in [ATV, 7.4] for the linear case. A twist by τ as defined here is the same as a half twist by τ^2. Since square roots of 2×2 matrices need not exist in characteristic 2, the concept of a half twist is more general. However, we don't know how to extend the notion of a half twist to elliptic algebras.

Now, returning to our study of regular graded algebras of dimension 3, we first consider the case that $r = 3$. In this case we will give an explicit description, as a twist, of any elliptic algebra $A = A(E, \sigma, \mathscr{L})$ in which E contains a line stabilized by σ. Call the line C, and say that, as a divisor in $\mathbb{P}^2$, $E = C + D$. Thus D is a conic, which may be degenerate and which may contain C. We choose a basis $\{x, y, z\}$ for A_1 so that C is the locus $\{x = 0\}$, and we choose an element $f \in A_1^{\otimes 2}$ whose image in $\mathrm{Symm}^2(A_1)$ has D as zero set.

In order to be explicit about defining equations, we will list natural choices for the basis and for f when E has a triple point:

(8.14)

(a) $E = 3C$ is a triple line. We set $f = x^2$.

(b) $E = 2C + C'$, where $C' \neq C$. We choose y so that C' is the locus $\{y = 0\}$, and we set $f = xy$.

(c) $E = C + 2C'$, where $C' \neq C$. We choose y so that C' is the locus $\{y = 0\}$, and we set $f = y^2$.

(d) $E = C + C' + C''$, where C, C', C'' are distinct lines through a point. We choose the basis so that $C' = \{y = 0\}$, $C'' = \{x = y\}$, and we set $f = y(x - y)$.

Also, we denote the Weyl algebra by W:

(8.15) $$W = k[u, v]/(uv - vu - 1)\,.$$

Theorem 8.16 *Let A be an elliptic regular algebra with associated triple $(E, \sigma, \mathscr{L})$ and with $r = 3$. Assume that $E = C + D$, where C is a line stabilized by σ. Let x, y, z be a basis for A_1 such that the locus $\{x = 0\}$ is the line C.*

(i) *x is a normalizing element of A. If τ denotes the automorphism such that $xa^\tau = ax$, then $\tau\sigma$ operates trivially on the scheme D.*

(ii) *The element x_τ is central in the twist A_τ.*

(iii) *Suppose that x is central in A. Then A has defining relations of the form*

$$xy - yx = 0,\ xz - zx = 0,\ yz - zy = f,$$

where $f \in A_1^{\otimes 2}$ is an element whose image in $\mathrm{Symm}^2(A_1)$ *defines the divisor D. In each of the cases listed above* (8.14), *we may take for f the element indicated.*

(iv) *If x is central and if we are in one of the cases* (8.14), *then Λ_0 is isomorphic to a localization of the Weyl algebra W.*

Proof. (i) The fact that x is a normalizing element is shown in Proposition 5.13. It generates the kernel of the canonical homomorphism $A \to B(C, \sigma_C, \mathscr{L}_C)$.

Given a line ℓ which is not a component of E, let $p = C \cap \ell$ and let $Z = D \cap \ell$, so that as divisors on ℓ, $E \cap \ell = p + Z$. To show that $\tau\sigma$ operates trivially on D, we will show that for every such line ℓ, $\tau\sigma Z = Z$, or equivalently, that $\tau^{-1}\ell$ contains σZ.

Let ℓ' be the line containing σZ, and let M, M' be the line modules corresponding to ℓ, ℓ' respectively. According to Proposition 6.24, we have an exact sequence

$$\to M'(-1) \to M \to N_p \to 0\ ,$$

where N_p is the point module determined by p. On the other hand, if $a = 0$ is the equation for ℓ, then $N_p \approx A/(aA + xA) = A/(aA + Ax) = M/Mx$. Since $Mx \approx (A/a^\tau A)(-1) = M_{\tau^{-1}\ell}(-1)$ (see (6.29)), this shows that $\tau^{-1}\ell = \ell'$, hence that $\tau^{-1}\ell$ contains σZ, as required.

(ii) This is a direct calculation, using the relation $x^\tau = x$.

(iii) Suppose that x is in the center of A. Then two of the three defining relations are $xy - yx = 0$ and $xz - xz = 0$. Also, the automorphism τ is the identity, so (i) shows that σ operates trivially on D. As above, let $f \in A_1^{\otimes 2}$ be an element whose locus of zeros is D, and let f also denote its image in A_2. Then f is in the kernel of the canonical homomorphism $\pi: A \to B_D := B(D, \mathrm{id}, \mathscr{L}_D)$. By Proposition 5.13, f is a normalizing element which generates $\ker \pi$. Since B_D is commutative, $yz - zy \in \ker \pi$. Hence $yz - zy = cf$ for some $c \in k^*$, which means that $yz - zy - cf = 0$ is our third defining relation. The constant c is not zero because we have assumed that A is elliptic, so it may be absorbed into f. In the cases (8.14), the constant can be absorbed into one of the coordinates.

Note 8.17. If E is the union of the coordinate axes, then the third relation becomes $yz - zy = cyz$, or $\lambda yz = zy$. The constant c can not be eliminated in these cases, so there is 1-parameter family of algebras which are not related by twists.

(iv) This is a computation using the defining equations. We take $u = yx^{-1}$ in each case, and

$$v = \begin{cases} zx^{-1} & \text{in case (a)} \\ zy^{-1} & \text{in case (b)} \\ xzy^{-2} & \text{in case (c)} \\ xzy^{-1}(1-y)^{-1} & \text{in case (d)}. \end{cases}$$

Then

$$(8.18) \qquad \Lambda_0 = \begin{cases} W & \text{in case (a)} \\ W[u^{-1}] & \text{in case (b)} \\ W[u^{-1}] & \text{in case (c)} \\ W[(u-u^2)^{-1}] & \text{in case (d)}. \end{cases}$$

To verify this, we note that the normalizing element is represented by $g = xf$ (see Proposition 5.13 again). A priori, Λ_0 is generated by the set $\{mg^{-1}\}$, where m runs through the monomials of degree 3. However, f factors into linear factors, and in each of the cases one sees that Λ_0 is generated by the set $\{mb^{-1}\}$, where m is linear and b is a linear factor of g. Setting $u = yx^{-1}$ and $w = zx^{-1}$, we have

$$uw - wu = fx^{-2} = 1, u, u^2, u(1-u),$$

respectively, in the four cases. Putting $v = w, wu^{-1}, wu^{-2}$, or $wu^{-1}(1-u)^{-1}$ according to the case, we find indeed that $uv - vu = 1$ in each case, and one checks that Λ_0 is presented as indicated. □

We now consider the case that $r = 2$. In this case we will describe as a twist any elliptic algebra whose associated curve E is not reduced.

Lemma 8.19 *Assume that $r = 2$, and $(E, \sigma, \mathscr{L})$ be a regular triple such that E is not reduced. Then $E = 2C$, where either C is an irreducible divisor of bidegree $(1, 1)$, or else $C = (q \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times q)$ for some point $q \in \mathbb{P}^1 \times \mathbb{P}^1$.*

We omit the proof of this lemma. It follows from the fact that the automorphism σ has the form (5.2). □

Theorem 8.20 *Let A be an elliptic regular algebra with associated triple $(E, \sigma, \mathscr{L})$, such that $r = 2$, and that $E = 2C$, where C is an irreducible curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(1, 1)$. Then*

(i) *A is a twist of the enveloping algebra of the Heisenberg Lie algebra, and is defined by the relations*

$$[x, [x, y]] = x^2y - 2xyx + yx^2 = 0,$$
$$[y, [y, x]] = xy^2 - 2yxy + y^2x = 0.$$

(ii) *The characteristic of k is different from 2.*

(iii) *The ring Λ_0 is the ring of invariants $W^{\langle\varepsilon\rangle}$ in the Weyl algebra W (8.15) under the automorphism $\varepsilon(u) = -u$, $\varepsilon(v) = -v$.*

Before proceeding with the proof of this theorem, we will compute the group of automorphisms of the divisor $E = 2C$. We use the first projection to map C isomorphically to $\mathbb{P}^1$. Since C has bidegree $(1, 1)$, it is the graph of an automorphism, say φ, of $\mathbb{P}^1$. So C is the locus $\{(x, \varphi(x)\}$. If we apply the inverse automorphism to the second factor, C is transformed into the diagonal Δ of $\mathbb{P}^1 \times \mathbb{P}^1$, and E to the double diagonal. Thus E is canonically isomorphic to the double diagonal.

The inclusion of C as a closed subscheme of E is described by an exact sequence

$$0 \to I \to \mathcal{O}_E \to \mathcal{O}_C \to 0, \tag{8.21}$$

where I is a square-zero ideal which is canonically isomorphic to Ω^1_C. This sequence is split by the first projection: $\mathcal{O}_E = \mathcal{O}_C \oplus I$. So we may write a section of $\mathcal{O}_E$ on an open set U in the form $f + \alpha$, where $f \in \Gamma(U, \mathcal{O}_C)$ and $\alpha \in \Gamma(U, \Omega^1_C)$.

Proposition 8.22 (i) *The group $\mathrm{Aut}^0 E$ of automorphisms of E which induce the identity on C is the canonical semi-direct product of $\mathbf{G}_m$ by $\mathbf{G}_a$, isomorphic to the group of invertible matrices of the form $\theta = \begin{pmatrix} u & b \\ 0 & 1 \end{pmatrix}$, where the automorphism θ operates on $\mathcal{O}_E$ by $f + \alpha \rightsquigarrow f + (bdf + u\alpha)$.*

(ii) *The group of all automorphisms of E is the direct product*

$$\mathrm{Aut}\, E = (\mathrm{Aut}^0 E) \times (\mathrm{Aut}\, C) \approx (\mathrm{Aut}^0 E) \times \mathrm{PGL}_2.$$

Proof. (i) To determine the group of automorphisms of E, we note that $\mathrm{Aut}^0 E$ is the group of sections of the sheaf $Aut^0 E$ of local automorphisms. Its sections on an open set U are the automorphisms of U which are the identity on the underlying reduced scheme U_{red}. Moreover, the sheaf $Aut^0 E$ can be described by another short exact sequence. Denote by $Aut^1 E$ the subsheaf of $Aut^0 E$ of local automorphisms which act as the identity on I. Then we have a split exact sequence

$$0 \to Aut^1 E \to Aut^0 E \to Aut\, I \to 0 \,. \tag{8.23}$$

Also, $Aut\, I \approx \mathbf{G}_m$ and $Aut^1 E \approx Der(\mathcal{O}_C, I) \approx Hom(\Omega_C^1, \Omega_C^1) \approx \mathcal{O}_C$. Taking global sections gives us a split exact sequence

$$0 \to \mathbf{G}_a \to \mathrm{Aut}^0 E \to \mathbf{G}_m \to 0 \,. \tag{8.24}$$

A direct computation shows that the operation which describes the semi-direct product structure on $\mathrm{Aut}^0 E$ is the canonical operation of $\mathbf{G}_m$ on $\mathbf{G}_a$, i.e., that $\mathrm{Aut}^0 E$ is the required semi-direct product.

(ii) Since E is isomorphic to the double diagonal which is defined intrinsically in terms of C, the map $\mathrm{Aut}\, E \to \mathrm{Aut}\, C$ is a split surjection. So we have a split exact sequence

$$0 \to \mathrm{Aut}^0 E \to \mathrm{Aut}\, E \to \mathrm{Aut}\, C \to 0 \,.$$

Since PGL_2 is a simple group, its operation on $\mathrm{Aut}^0 E$ is trivial, and $\mathrm{Aut}\, E$ is a direct product as asserted.

Proof of Theorem 8.20 We decompose the automorphism σ appearing in our triple according to the product (8.22ii), say as $\sigma = (\theta, \tau)$, where $\tau \in \mathrm{PGL}_2$ and where θ, τ commute. The kernel of the map $A \to B_C$ defines a normalizing element of degree 2 in A, which we denote by f. Hence there is an automorphism μ such that

$$fa^\mu = af \,. \tag{8.25}$$

This automorphism μ defines a map $A_1 \to A_1$, and hence a map $\mathbb{P}(A_1^*) \to \mathbb{P}(A_1^*)$, which in turn defines a map from C to itself. We denote these maps by μ too.

Lemma 8.26 *We have $\mu = \tau^{-2}$ on C.*

Proof. Let $p \in C$ and let ℓ be the line through p. We will show that $\mu^{-1}\ell$ contains $\tau^2 p$. According to Proposition 6.24, there is an exact sequence

$$0 \to M_{\ell'}(-2) \to M_\ell \to N_p \to 0 \,,$$

where ℓ' contains $\tau^2 p$. On the other hand, we know that f^2 is the canonical normalizing element in A (5.2), and hence f^2 annihilates N_p. Since N_p is critical, this implies that f annihilates N_p. A computation similar to (6.29) shows that $M_{\ell'}(-2) = M_\ell f = M_{\mu^{-1}\ell}(-2)$. Hence $\tau^2 p \in \ell' = \mu^{-1}\ell$. □

Lemma 8.27 *An arbitrary lifting of the automorphism τ to GL_2 defines an automorphism of the algebra A.*

Proof. It is true that σ and τ commute. So according to Proposition 8.8, it remains to show that $\tau' = (\tau, \tau)$ sends E to itself. This is a set-theoretic problem, so it suffices to show that τ' sends C to C. Note that $\sigma|_C = \tau|_C$. Since C is the locus of points $(x, \varphi(x))$ for some $\varphi \in \mathrm{PGL}_2$, the action of τ' on C is $\tau'(x, \varphi(x)) = (\tau(x), \tau\varphi(x))$. So we must show that $\varphi\tau = \tau\varphi$. In fact, $\varphi = \tau$. To see this we use the fact that the automorphism σ has the form (5.2) again. It shows that σ acts as $\sigma(x, \varphi(x)) = (\varphi(x), u(x))$, and hence that $\varphi = \sigma|_C = \tau|_C$. □

This lemma allows us to replace our algebra by the twist $A_{\tau^{-1}}$, which is associated to the triple $(E, \theta, \mathscr{L})$, where $\sigma = (\theta, \tau)$ as above. This reduces us to the case that the automorphism $\sigma = \theta$ is purely infinitesimal, of the form described in Proposition 8.22(i). Then E becomes the double diagonal.

Since f is in the kernel of the map $A \to B_C$ and B_C is now commutative,

$$yx - xy = cf,$$

where, since A is three-dimensional, $c \neq 0$. Also, according to Lemma (8.26), $\mu = \mathrm{id}$, and hence

$$xf = fx, \quad yf = fy. \tag{8.28}$$

This leads to (8.20i). One now verifies easily that the triple corresponding to these equations is linear if char $k = 2$ and is a double diagonal otherwise.

The final step is to compute the ring Λ_0 for the enveloping algebra A. The canonical normalizing element is $g = (yx - xy)^2$, and $f = yx - xy$ is central. We form the ring $R = A[z]$, where z is a central variable whose square is f. The defining equations for R are $[x, z] = [y, z] = 0$, $[x, y] = z^2$. After a cyclic permutation of the variables, we obtain the regular algebra (8.16iii) for which E is a triple line. So $R[z^{-1}]_0$ is the Weyl algebra, and Λ_0 is the ring of invariants, as stated. □

We now discuss the special case of elliptic algebras in which $r = 2$ and $E = 2(\mathbb{P}^1 \times q) + 2(q \times \mathbb{P}^1)$. We will show by an explicit computation that if k is algebraically closed and of characteristic $\neq 2$, then there is a unique isomorphism class of regular algebras of this form, while in characteristic 2 they form a one-parameter family. This one-parameter family is the specialization to characteristic 2 of the algebras described in Theorem 8.20.

It will be convenient to introduce a local notation for the type of algebras arising in the discussion. For $a, c \in k$, let $A(a, c)$ denote the algebra defined by the relations

$$f_1 = xy^2 + y^2x + ay^3$$

$$f_2 = x^2y + yx^2 + a(xy^2 + yxy + y^2x) + (a^2 + c)y^3.$$

In the notation of [ArSch, p. 181], we have $Q = I$, $\alpha = \beta = 1$, and $w = w_2 + aw_1 + (a^2 + c)w_0$.) The matrix $\tilde{M}$ (see [ATV]) is, dropping the tildas on the coordinates,

$$\tilde{M} = \begin{pmatrix} y_1 y_2 & (x_1 + ay_1) y_2 \\ y_1(x_2 + ay_2) & (x_1 + ay_1)(x_2 + ay_2) + cy_1 y_2 \end{pmatrix}.$$

Since the four entries of $\tilde{M}$ have no common zero $((x_1, y_1), (x_2, y_2))$ in $\mathbb{P}^1 \times \mathbb{P}^1$, the algebra is regular for all $a, c \in k$. Since $\det \tilde{M} = cy_1^2 y_2^2$, $A(a, c)$ is linear if $c = 0$, and if $c \neq 0$, then A is elliptic and $E = 2(\mathbb{P}^1 \times \{q\} + \{q\} \times \mathbb{P}^1)$, where q is the point $y = 0$

in $\mathbb{P}^1$. In the latter case the changes of coordinates keeping E fixed are, up to a scalar, of the form $x \rightsquigarrow \alpha x + \beta$, $y \rightsquigarrow y$. This substitution gives an isomorphism

$$A(a, c) \approx A\left(\frac{a + 2\beta}{\alpha}, \frac{c}{\alpha^2}\right).$$

Thus, if k has characteristic different from 2, every elliptic algebra of the type we are considering is isomorphic to one of the form $A(0, c)$, and if $c \neq 0$ is a square in k, to $A(1, 0)$. On the other hand, if char $k = 2$, then the isomorphism classes of elliptic algebras A of this form are represented by the algebras $A(0, c)$, $c \in k^*/k^{*2}$, together with the one-parameter family $A(1, c)$, $c \in k^*$.

Theorem 8.29 *Let A be an elliptic regular algebra with associated triple $(E, \sigma, \mathscr{L})$, such that $r = 2$, and that $E = 2C$, where $C = (\mathbb{P}^1 \times q) + (q \times \mathbb{P}^1)$. Then with the above notation, $A \approx A(a, c)$ for some $a \in k$, $c \in k^*$, and Λ_0 is isomorphic to the Weyl algebra.*

Proof. We use the standard description of the algebras by means of the tensor

$$w = (x\ y)M\begin{pmatrix} x \\ y \end{pmatrix} = xf_1 + yf_2 = g_1 x + g_2 y \tag{8.30}$$

as in [ArSch, 2.3], so that (f_1, f_2) and (g_1, g_2) are two sets of defining equations for A. The variables x, y are understood to correspond to a choice of homogeneous coordinates $(\tilde{x}, \tilde{y})$ in $\mathbb{P}^1$, and we choose them so that q is the point $\tilde{y} = 0$. Then the defining equation for E in $\mathbb{P}^1 \times \mathbb{P}^1$ is $\tilde{y}_1^2 \tilde{y}_2^2 = 0$. We will drop the tildas from the coordinates from now on. As in [ATV, 5.4], we write

$$f_1 = a_1x^3 + a_2x^2y + a_3xyx + a_4xy^2 + a_5yx^2 + a_6yxy + a_7y^2x + a_8y^3,$$

$$f_2 = b_1x^3 + b_2x^2y + b_3xyx + b_4xy^2 + b_5yx^2 + b_6yxy + b_7y^2x + b_8y^3.$$

Then

$$g_1 = a_1x^3 + a_3x^2y + a_5xyx + a_7xy^2 + b_1yx^2 + b_3yxy + b_5y^2x + b_7y^3,$$

$$g_2 = a_2x^3 + a_4x^2y + a_6xyx + a_8xy^2 + b_2yx^2 + b_4yxy + b_6y^2x + b_8y^3.$$

Recall that $\tilde{f}_1$ and $\tilde{f}_2$ vanish on $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Also, the form (5.2) of the automorphism σ shows that Γ contains $q \times \mathbb{P}^1 \times q$. Therefore $a_1 = a_3 = b_1 = b_3 = 0$. Similarly, from $\tilde{g}_1$ and $\tilde{g}_2$ we obtain the relations $a_1 = a_5 = a_2 = a_6 = 0$. So the equations have the form

$$\begin{aligned} f_1 &= \phantom{b_2x^2y + {}} a_4xy^2 \phantom{{}+ b_5yx^2 + b_6yxy} + a_7y^2x + a_8y^3 \\ f_2 &= b_2x^2y + b_4xy^2 + b_5yx^2 + b_6yxy + b_7y^2x + b_8y^3, \\ g_1 &= \phantom{a_4x^2y + {}} a_7xy^2 \phantom{{}+ b_2yx^2 + b_4yxy} + b_5y^2x + b_7y^3 \\ g_2 &= a_4x^2y + a_8xy^2 + b_2yx^2 + b_4yxy + b_6y^2x + b_8y^3. \end{aligned} \tag{8.31}$$

The matrix $\tilde{M}$ is

(8.32)
$$\tilde{M}(x_1, y_1; x_2, y_2) = \begin{pmatrix} a_7y_1y_2 & a_4x_1y_2 + a_8y_1y_2 \\ b_5y_1x_2 + b_7y_1y_2 & b_2x_1x_2 + b_4x_1y_2 + b_6y_1x_2 + b_8y_1y_2 \end{pmatrix}.$$

Its determinant is

$$\det \tilde{M} = (a_7 b_2 - a_4 b_5) x_1 x_2 y_1 y_2 + (a_7 b_4 - a_4 b_7) x_1 y_1 y_2^2 + (a_7 b_6 - a_8 b_5) y_1^2 x_2 y_2 + (a_7 b_8 - a_8 b_7) y_1^2 y_2^2 . \tag{8.33}$$

On the other hand, $\det \tilde{M} = 0$ defines E, so $\det \tilde{M} = c y_1^2 y_2^2$ with $c = a_7 b_8 - a_8 b_7$, and the first three coefficients in this expression vanish. Since Γ is non-degenerate, rank $\tilde{M} \geqq 1$ everywhere. Applying this fact to the point $(x_i, y_i) = (1,0)$ shows that $\tilde{M}(1,0;1,0) \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. It follows that $b_2 \neq 0$, which by (8.31) implies that $a_4 \neq 0$ and $b_5 \neq 0$. We normalize a_4 to 1. Then the vanishing of the coefficients in the expansion of $\det \tilde{M}$ yields the relations

$$a_7 b_2 = b_5, a_7 b_4 = b_7, a_7 b_6 = a_8 b_5 . \tag{8.34}$$

Moreover, the form of the relations (8.31) shows that $g_1 = a_7 f_1$ and that $a_7 \neq 0$. Hence

$$b_5 = a_7^2, \; b_7 = a_7 a_8 . \tag{8.35}$$

Substituting into the Eqs. (8.34) yields the relations

$$b_2 = a_7, \; b_4 = a_8, \; b_6 = a_7 a_8 . \tag{8.36}$$

Next, the coefficients of $x^2 y$ in (8.31) show that $f_2 = b_2 g_2 + \gamma g_1$ for same $\gamma \in k$. This implies that $b_7 = b_2 b_6 + \gamma b_5$ and $b_8 = b_2 b_8 + \gamma b_7$. Using the above relations, we rewrite these as

$$a_8(1 - a_7) = \gamma a_7, \; b_8(1 - a_7) = \gamma b_7 . \tag{8.37}$$

The algebra is linear if and only if $a_7 b_8 = a_8 b_7$. Assume that E is elliptic. Then it follows that $\gamma = 0$ and $a_7 = 1$. Setting $a = a_8$ and $b = b_8$, the defining equations for A become

$$\begin{aligned} f_1 &= \quad xy^2 \quad + y^2 x + a y^3 \\ f_2 &= x^2 y + a x y^2 + y x^2 + a y x y + a y^2 x + b y^3 . \end{aligned} \tag{8.38}$$

Putting $c = b - a^2$, we have $c \neq 0$ and $A \approx A(a,c)$, as was to be shown.

The normalizing element g such that $B = A/gA$ is $g = y^4$ (see (5.13)). The relation $f_1 = 0$ can be written as

$$y^2 x y^{-2} = -x - ay , \tag{8.39}$$

which shows that y^2 is a normalizing element in A, and that y^4 is central. Conjugation by y is an automorphism θ of Λ whose order divides 4. (It is usually equal to 4.) The ring $\Lambda = \Sigma \Lambda_0 y^n$ is the Ore domain $\Lambda_0[y, y^{-1}; \theta_0]$, where $\theta_0 = \theta | \Lambda_0$.

Let $u = xy^{-1}$, $v = y^{-1}x$. Formula (8.39) shows that $\theta u = -v - a$, and $\theta v = u$. So $k[u,v]$ is a θ-stable subalgebra of Λ_0. Since $k[u,v,y,y^{-1}] = \Lambda$, we have $\Lambda_0 = k[u,v] = k[u,v,y,y^{-1}]_0$.

So far, we have used only the relation $f_1(x,y)=0$. Working out things like $(yx^2)y^{-3}=(y(xy^{-1})y^{-1})(y^2(xy^{-1})y^{-2})=(\theta u)(\theta^2 u)=(-v-a)(-u-a)=vu+au+av+a^2$, one finds that in Λ the relation $f_2(x,y)y^{-3}=0$ boils down to

(8.40) $$vu-uv=b-a^2=c\,,$$

Thus in the elliptic case, Λ_0 is isomorphic to the Weyl algebra, as required. This completes the proof of Theorem (8.29). □

It is interesting to interpret Theorems (7.1) and (7.3) for the rings we have described here. In each case the ring Λ_0 is closely related to the Weyl algebra, which is a simple ring if and only if k has characteristic zero. In fact, σ has infinite order in characteristic zero and finite order in characteristic p, a fact which can be checked directly, and which also follows from Theorem (7.3) and from the next proposition.

Proposition 8.41 *Let k be a field of characteristic $p\neq 0$.*

(i) *The Weyl algebra W (8.15) is an Azumaya algebra of rank p^2 over its center $Z=k[s,t]$, where $s=u^p$, $t=v^p$.*

(ii) *Assume that $p\neq 2$, and let ε be the automorphism defined by $\varepsilon(u)=-u$ and $\varepsilon(v)=-v$. The ring of invariants $W^{\langle\varepsilon\rangle}$ is a finite module over $Z^{\langle\varepsilon\rangle}$, and it is an Azumaya algebra of rank p^2 at all points of* Spec $Z^{\langle\varepsilon\rangle}$ *except at the image p_0 of the origin $s=t=0$.*

(iii) *Let $\mathfrak{m}_0$ be the maximal ideal of $Z^{\langle\varepsilon\rangle}$ corresponding to p_0. There are two maximal ideals P^0,P^1 of $W^{\langle\varepsilon\rangle}$ which contain $\mathfrak{m}_0$, and suitably numbered, we have $W^{\langle\varepsilon\rangle}/P^0\approx\mathcal{M}((p-1)/2)$, and $W^{\langle\varepsilon\rangle}/P^1\approx\mathcal{M}((p+1)/2)$, where $\mathcal{M}(n)$ denotes the algebra of $n\times n$ matrices over k.*

Proof. The first assertion is well known [Re]. Indeed, it is obvious that $k[s,t]$ is in the center of W and that W is a free $k[s,t]$-module of rank p^2. On the other hand, the usual trace argument shows that the equation $xy-yx=1$ has no solution in $\mathcal{M}(n)$ unless p divides n. Hence W has no representation of dimension $<p$. It follows that W is an Azumaya algebra of pi degree p, and hence that $Z=k[s,t]$ is its center.

To show that $W^{\langle\varepsilon\rangle}$ is almost Azumaya if the characteristic is not 2, we note that the automorphism ε induces the automorphism $s\rightsquigarrow -s$, $t\rightsquigarrow -t$ of Z. This automorphism has the origin as its only fixed point. It follows by descent that $W^{\langle\varepsilon\rangle}$ is locally an Azumaya algebra at all points of Spec Z except at p_0.

Direct computation shows that the matrix representation of W corresponding to the origin in Spec Z is

(8.42) $$u\rightsquigarrow\begin{bmatrix}0&1&&\\&&\ddots&\\&&&1\\&&&0\end{bmatrix},\quad v\rightsquigarrow\begin{bmatrix}0&&&&\\1&&&&\\&2&\ddots&&\\&&&p-1&0\end{bmatrix},$$

and ε is compatible with the automorphism $\bar\varepsilon$ of $\mathcal{M}(p)$ which acts on matrix units as $\bar\varepsilon(e_{ij})=(-1)^{i+j}e_{ij}$. Clearly

$$\mathcal{M}(p)^{\langle\varepsilon\rangle}=\sum_{i+j\text{ even}}ke_{ij},$$

and this ring decomposes as a sum of two subalgebras:

$$\left(\sum_{i,j\text{ even}} ke_{ij}\right)\oplus\left(\sum_{i,j\text{ odd}} ke_{ij}\right)\approx \mathscr{M}(\tfrac{1}{2}(p-1))\oplus\mathscr{M}(\tfrac{1}{2}(p+1))\,.$$

By semi-simplicity, the map $W^{\langle\varepsilon\rangle}\to\mathscr{M}(p)^{\langle\varepsilon\rangle}$ is surjective. This provides the required maximal ideals. Since the ranks add to p, there are no others. □

Note 8.43 The maximal ideals P^i provide examples of critical A-modules of gk-dimension 1 with $e_0 \neq e_1$ (see (6.15)). Let V denote the representation (8.42) on k^p. The decomposition $V = V^0 \oplus V^1$ which corresponds to the two maximal ideals is $V^0 = \sum_{i\text{ even}} k_i$, $V^1 = \sum_{i\text{ odd}} k_i$. The normalized A-module N^0 which corresponds to V^0 as in (7.8) is described as follows: We set $N_i = V^i$, reading the upper index modulo 2, and we let x, y act as u, v: $V^i \to V^{i+1}$. The module N^1 is the shift N^{0+} of N^0. □

References

[AmSm] Amitsur, S.A., Small, L.W.: Prime ideals in P.I. rings. J. Algebra **62**, 358–383 (1980)

[ArSch] Artin, M., Schelter, W.: Graded algebras of global dimension 3. Adv. Math. **66**, 171–216 (1987)

[ATV] Artin, M., Tate, J., Van den Bergh, M.: Some algebras associated to automorphisms of elliptic curves. The Grothendieck Festschrift, vol. 1, pp. 33–85, Boston Basel Stuttgart: Birkhäuser 1990

[AV] Artin, M., Van den Bergh, M.: Twisted homogeneous coordinate rings. J. Algebra **133**, 249–271 (1990)

[Bj] Björk, J.-E.: The Auslander condition on noetherian rings. Séminaire Dubreil-Malliavin 1987–8. Lect. Notes Math., vol. 1404, pp. 137–173, Berlin Heidelberg New York: Springer 1990

[BLR] Bosch, S., Lütkebohmert, W., Raynaud, M.: Néron models. Berlin Heidelberg New York: Springer 1990

[BS] Borevitch, Z.I., Shafarevitch, I.R.: Number theory. New York: Academic Press 1966

[EG] Evans, E.G., Griffith, P.: Syzygies. Lond. Math. Soc. Lect. Note Ser. vol. 106, Cambridge: Cambridge University Press 1986

[KL] Krause, G.R., Lenagan, T.H.: Growth of Algebras and Gelfand-Kirillov Dimension. Res. Notes Math. vol. 116, Boston: Pitman 1985

[NV] Nastacescu, C., Van Oystaeyen, F.: Graded ring theory, p. 16. Amsterdam: North Holland 1982

[OF] Odeskii, A.B., Feigin, B.L.: Sklyanin algebras associated to elliptic curves. (Manuscript)

[Ra] Ramras, R.: Maximal orders over regular rings of dimension 2. Trans. Am. Math. Soc. **142**, 457–474 (1969)

[Re] Revoy, M.P.: Algèbres de Weyl en charactéristique p. C.R. Acad. Sci. Sér. A**276**, 225–227 (1973)

[Ro] Rowen, L.: Polynomial identities in ring theory. New York London: Academic Press 1980

[Sn] Snider, R.L.: Noncommutative regular local rings of dimension 3. Proc. Am. Math. Soc. **104**, 49–50 (1988)

[SSW] Small, L.W., Stafford, J.T., Warfield, R.B.: Affine algebras of Gelfand-Kirillov dimension one are PI. Math. Proc. Camb. Philos. Soc. **97**, 407–414 (1985)

[Staf] Stafford, J.T.: Noetherian full quotient rings. Proc. Lond. Math. Soc. **44**, 385–404 (1982)

[Stan] Stanley, R.P.: Generating functions. In: Studies in Combinatorics. MAA Stud. Math., vol. 17, pp. 100–141, Washington: MAA, Inc. 1978

[VdB] Van den Bergh, M.: Regular algebras of dimension 3. Séminaire Dubreil-Malliavin 1986. Lect. Notes Math., vol. 1296, pp. 228–234, Berlin Heidelberg New York: Springer 1987